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Thermodynamics of BTZ black hole and entanglement entropy

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Abstract. The BTZ black Hole is (2+1) dimensional black hole solution asymptotic to anti-de-Sitter space-time. We study the discretized quantum scalar fields in background of non-rotating BTZ black hole space-time and construct the entanglement thermodynamics for massless scalar field. The behavior of the entanglement energy is understood by red shift factor caused by the curved background. The entanglement thermodynamics is compared with the black hole thermodynamics.

1. Introduction

Black hole is region of space time where gravitational pull is so strong that even light can not escape. The event horizon is defined as the null surface in space time. The total area of the event horizon never decreases in any physical process. This is analogous to entropy in thermodynamics, which never decreases in a physical process. Thus the black holes can be assigned entropy proportional to the area of black hole horizon. One can ask the question about microscopic degrees of freedom responsible for entropy as nothing comes out of black holes. The quantum mechanical treatment predicts that the black holes radiates and the radiation carries no information as it is thermal in nature.

In this paper we carry the investigation of entanglement thermodynamics of BTZ black hole in three dimensional space-time with negative cosmological constant [1, 2]. This solution can be embedded in string theory and a microscopic counting of degrees of freedom responsible for black hole entropy is also known [3]. The near horizon geometry of BTZ black hole is anti-de-Sitter space time and the machinery of AdS/CFT correspondence can also be applied. Thus BTZ black hole can be used as a laboratory to study the quantum aspects of black holes. There are several attempts to understand the microscopic origin of black hole entropy and its relation with horizon area [2–14].

In this paper, we investigate the entanglement thermodynamics of scalar fields in the background of BTZ black hole space-time. The entanglement entropy $S_{\rm ent}$ is always proportional to the area A of the event horizon of the black hole as expected, $S_{\rm ent} = C_S\, A/a$. Here 'a' is cut-off length and C_s is model dependent coefficient, which we have calculated numerically in this paper. Entanglement entropy $S_{\rm ent}$ is independent of the position of an observer once a quantum state is fixed [7, 10, 14]. We also calculate the entanglement energy of the BTZ black hole. The behavior of the entanglement energy is understood by the redshift factor caused by curved background. Because of the gravitational redshift, these energies are related as $E = \sqrt{-g_{tt}}\, E_{\rm ent}$.

2. Entanglement entropy of scalar field in non-rotating BTZ black hole

The BTZ black hole is a solution of Einstein's gravity in (2+1) dimensions with a negative cosmological constant ($\frac{1}{l^2}$). The metric of non rotating BTZ black hole can be written as [1];

$$ds^2 = -(-M + r^2/l^2)dt^2 + \frac{1}{(-M + r^2/l^2)}dr^2 + r^2 d\phi^2. \tag{1}$$

The horizon of the black hole is located at $r_+ = \sqrt{M}\, l$. The proper length from the horizon is defined as, $\rho = \int_{r_+}^{r} (r'^2/l^2 - M)^{-1/2}\, dr' = l \log[(\sqrt{u^2 + M} + u)/r_+]$, where we have defined $u = \sqrt{r^2/l^2 - M}$. The metric can be rewritten in term of proper distance as:

$$ds^2 = -u^2 dt^2 + d\rho^2 + r_+^2 \left(\frac{u^2}{M} + 1\right) d\phi^2. \tag{2}$$

Let us consider a massless scalar field in BTZ black hole space-time. Using the separation of variable as appropriate for the cylindrical symmetry of the system; $\Phi(t, \rho, \phi) = \sum_m \Phi_m(t, \rho)\, e^{im\phi}$, and defining a new variable, $\psi_m(t, \rho) = \left[\left\{r_+ \sqrt{(u^2/M) + 1}\right\}/u\right]^{1/2} \Phi_m(t, \rho)$, the Hamiltonian of the system can be written as,

$$H = \frac{1}{2}\int d\rho\, \pi_m^2(\rho) + \frac{1}{2}\int d\rho\, d\rho'\, \psi_m(\rho)\, V_m(\rho,\, \rho')\, \psi_m(\rho'),$$

where

$$\psi_m(\rho) V(\rho,\, \rho') \psi_m(\rho') = \left[u\sqrt{u^2/M + 1}\, \left\{\partial_\rho \left(\sqrt{\frac{u}{\sqrt{(u^2/M + 1)}}}\psi_m\right)\right\}^2 + m^2 \frac{u^2}{r_+^2\, (u^2/M + 1)}\psi_m^2\right]. \tag{3}$$

The system can be discretized as $\rho \to (A - 1/2)a$ and $\delta(\rho - \rho') \to \delta_{AB}/a$, where A,B=1,2 ....N and a is cut-off length. The corresponding Hamiltonian of the discretized system can be obtained by the replacement: $\psi_m(\rho) \to q^A,\ \pi_m(\rho) \to p_A/a,\ V(\rho, \rho') \to V_{AB}/a^2$.

The discretized Hamiltonian resembles to the set of N coupled harmonic oscillators [5, 6],

$$H = \sum_{A,B=1}^{N} \Big[\frac{1}{2a}\delta^{AB} p_A\, p_B + \frac{1}{2} V_{AB}\, q_m^A\, q^B\Big] \tag{4}$$

To calculate the entanglement entropy, we divide the system into two sub-systems and define,

$$(W)_{AB} = \sqrt{a\, V_{AB}} = \begin{pmatrix} A_{ab} & B_{a\beta} \\ (B^T)_{\alpha b} & D^{(2)}_{\alpha\beta} \end{pmatrix},$$

and the indices 'a,b..' and 'α, β...' refer to two subsystems.

The entanglement entropy of the system is given by $S = -tr[\rho \log \rho]$, where ρ is the density matrix. The density matrix of the system is defined as,

$$\rho(q, q') = \int \prod_{i=1}^{N} dq_i \psi(q_1, ...q_n; q_{n+1}, ...q_N)\psi^\star(q_1, ...q_n; q_{n+1}, ...q_N). \tag{5}$$

The corresponding density matrix can be calculated as,

$$\rho_0(q, q') = \exp\Big[-(q^T D' q + q'^T D' q')/2 + q^T (D - D')q\Big] \tag{6}$$

where $D' = D - \frac{1}{2}B^T A^{-1} B$.

The entanglement entropy can be written as $S_{\rm ent} = \sum_{i=1}^{N-n_B} S_i$, where $S_i = -\mu_i(1 - \mu_i)^{-1} \ln \mu_i - \ln(1 - \mu_i)$ and $\mu_i := \lambda_i^{-1}(\sqrt{1 + \lambda_i} - 1)^2$. The λ_i are the eigenvalues of the W_{AB} matrix. Using these relations, we evaluate the entanglement entropy of the system.

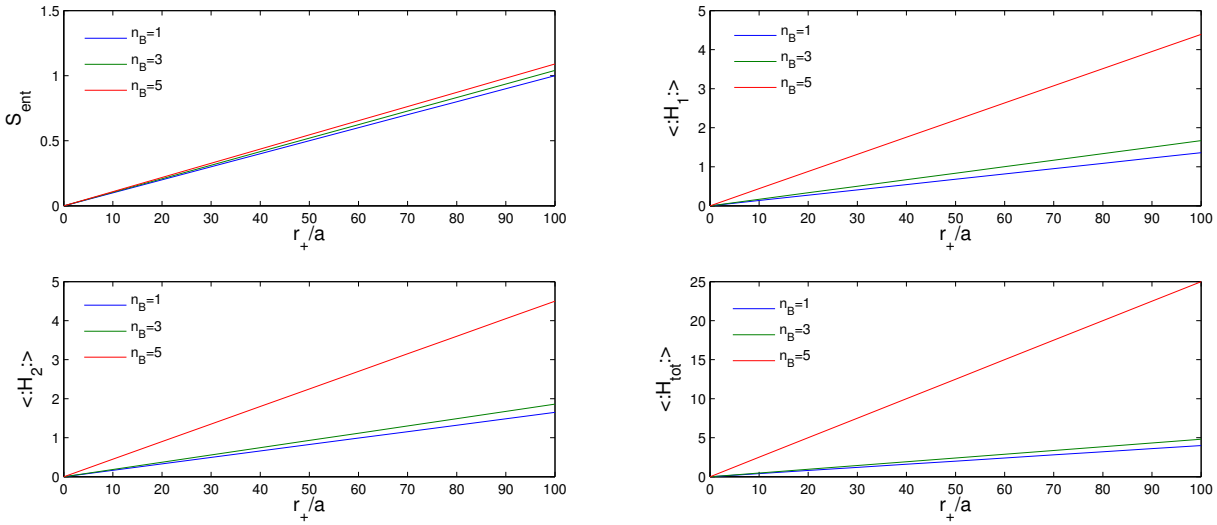

Figure 1. The numerical calculation for $S_{\rm ent}, H_1$ (upper panel), $H_2, H_{\rm tot}$ (lower panel) of the scalar field in BTZ space-time, for n_B =1,3,5 is shown as a functions of r_+/a with $N = 100$ and $m = 0$.

3. Numerical estimation

3.1. Entanglement entropy

We are interested in the numerical estimation of the entanglement entropy of the scalar fields in the background of BTZ black hole for the above mentioned system. The result for the entanglement entropy of the discretized system is; $S_{\rm ent} \simeq 0.01\,(r_+/a)$, where the proportionality constant is evaluated numerically for $n_B = 1$. The results are shown in Fig.1 for different values of n_B. The entropy is proportional to r_+, which in turn is proportional to horizon area.

3.2. Entanglement energy

We also calculate the entanglement energy of the scalar fields in the background of BTZ black hole. For the subsystem '1' the entanglement energy is given by,

$$\langle : H_1 : \rangle = \frac{1}{4a}\left[aV^{(1)}_{ab}(\tilde{A})^{ab} + A_{ab}\delta^{ab} - 2w^{(1)}_{ab}\delta^{ab}\right] \tag{7}$$

Using this, we estimate the numerical value of the entanglement energy. $\langle : H_1 : \rangle \simeq 0.014\left(\frac{r_+}{a}\right)$, The results are shown in Fig.1 for different values of n_B.

Similarly for the subsystem '2' the entanglement energy is given by,

$$\langle : H_2 : \rangle = \frac{1}{4a}\left[aV^{(2)}_{\alpha\beta}(\tilde{D})^{\alpha\beta} + D_{\alpha\beta}\delta^{ab} - 2w^{(1)}_{\alpha\beta}\delta^{\alpha\beta}\right] \tag{8}$$

Using this, we estimate the numerical value of the entanglement energy.$\langle : H_2 : \rangle \simeq 0.019\left(\frac{r_+}{a}\right)$. The results are shown in Fig.1 for different values of n_B.

We also estimate the total energy of the system ,

$$\langle : H_{tot} : \rangle_{\rho'} = \frac{1}{4a}Tr[aVM^{-1} - W] = \frac{1}{4}Tr[V(M^{-1} - W^{-1})] = -\frac{1}{2}[V^T_{int}\tilde{B}], \tag{9}$$

which is given by;$\langle : H_{tot} : \rangle_{\rho'} \simeq .045\left(\frac{r_+}{a}\right)$, The results for total energy are shown in Fig.1 for different values of n_B.

The entanglement entropy and energy of the BTZ black hole looks like as $S_{\rm ent} \propto A$ and $E_{\rm ent} \propto A$, where $A = 2\pi r_+$ is area of the black hole and $r_+ = \sqrt{M}l$. The corresponding entanglement entropy and energy of the thermal AdS space-time are given as, $S_{\rm ent} \propto A$ and $E_{\rm ent} \propto A^2$, where A is area of the slice of radius r. Thus the discrepancy between entanglement energy of the BTZ black hole and thermal AdS space time. This discrepancy can be removed by consideration of the redshift factor.

The entropy of black hole has the same behavior as that measured by the observer at infinity, since the degree of freedom are independent of n_B. The results for the entanglement energy is dependent on n_B and one can try to understand this is in terms of red shift of energy $\sqrt{-g_{tt}}E_{\rm ent}$ in curved spacetime [7], as measured by an observer who is at proper distance $n_B\, a$ from the horizon. The entanglement energy is redshifted by the factor $\rho\, r_+$, where $\sqrt{-g_{tt}} = \rho\, r_+$, and ρ is the proper distance from the horizon correspond to stationary observer. The gravitational effect restores the agreement between entanglement thermodynamics and black hole thermodynamics.

4. Conclusion

In this paper, we have calculated the entanglement entropy and energy of quantum scalar fields in BTZ black hole space-time. The model is very similar to a bunch of harmonic oscillators for free fields in curved space-time but seem to capture area law of black hole thermodynamic. The results can be compared with corresponding quantum field entropy in Schwarzschild black hole space-times in four dimensions [7]. The discrepancy between BTZ Black hole thermodynamics and thermal AdS space-time has been removed by gravitational redshift factor. It would be interesting to generalize these results for Fermionic degrees of freedom.

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