Duality structures for representation categories of vertex operator algebras and the Feigin–Fuchs boson

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Abstract

Huang, Lepowsky and Zhang developed a representation theory for vertex operator algebras that endows suitably chosen module categories with the structures of braided monoidal categories. Included in the theory is a functor which assigns to discretely strongly graded modules a contragredient module, obtained as a gradewise dual. In this paper, we show that this gradewise dual endows the representation category with the structure of a ribbon Grothendieck-Verdier category. This duality structure is more general than that of a rigid monoidal category; in contrast to rigidity, it naturally accommodates the fact that a vertex operator algebra and its gradewise dual need not be isomorphic as modules and that the tensor product of modules over vertex operator algebras need not be exact. We develop criteria which allow the detection of ribbon Grothendieck-Verdier equivalences and use them to explore ribbon Grothendieck-Verdier structures in the example of the rank n Heisenberg vertex operator algebra or chiral free boson on a not necessarily full rank even lattice with arbitrary choice of conformal vector. We show that these categories are equivalent, as ribbon Grothendieck-Verdier categories, to certain categories of graded vector spaces and categories of modules over a certain quasi-Hopf algebra.

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1 Introduction

Vertex operator algebras are algebraic structures with numerous applications in mathematical physics, representation theory, geometry and combinatorics. For any algebraic structure, it is important to first select a "sensible" category of representations and then to understand the structure this representation category naturally carries. In the case of vertex operator algebras, the consensus expectation is that a sensible category of representations should admit the structure of a braided tensor category. For a large class of choices of vertex operator algebra module categories a good tensor product theory that includes a braiding has been found in by Huang, Lepowsky and Zhang in the long series of papers [1]. In Theorem 2.11 we record a list of sufficient conditions, collated from [1], for a module category to admit these structures and specialise these in Corollary 2.14 for easier application to the categories considered in Sect. 3.3. Categories satisfying these conditions include $\mathbb{N}$ gradable modules over C_2-cofinite vertex operator algebras (this covers all rational theories and also all logarithmic C_2-cofinite theories such as the $\mathcal{W}_{p,q}$ triplet models [2]) as well as certain module categories with infinitely many inequivalent simple modules such as Heisenberg or bosonic ghost module categories to name but a few [3–7].

Contragredient representations appear for many algebraic structures. They lead, in many cases where the representation category is a monoidal category, to the notion of *rigidity*. Recall that an object is called rigid, if it has both left and right duals, each of which comes with evaluation and coevaluation morphisms that satisfy the usual zig-zag relations. A category is called rigid, if every object is rigid. This is a property: any left dual or right dual is unique up to unique isomorphism.

Rigidity as categorical formalisation of duality is widely used: it applies to the category of finite-dimensional vector spaces and to categories of finite-dimensional modules over finite-dimensional (weak quasi) Hopf algebras. Duals are also widely used in quantum topology, since they lead to a powerful graphical calculus which allows, for suitable tensor categories, for the construction of invariants of knots, links and manifolds.

On the other hand, the notion of rigidity has severe limitations. They already become apparent when one considers Hopf algebroids: Hopf algebroids are interesting algebraic structures with the desirable property that any tensor category that is finite as a linear category (that is, it is equivalent to the category of finite dimensional left modules over some finite dimensional associative algebra) can be realised as the representation category of a finite-dimensional Hopf algebroid [8, Theorem 7.6]. However, the natural notion of duality for Hopf algebroids is *not* rigidity, see [9, 10], where it was shown that the natural duality structure is that of a Grothendieck-Verdier category.

A *Grothendieck-Verdier category* is a monoidal category $\mathscr{C}$, with a distinguished object K, called the dualising object, such that for any pair of objects $X, Y \in \mathscr{C}$, there are natural isomorphisms

$$\mathrm{Hom}\,(- \otimes Y, K) \xrightarrow{\cong} \mathrm{Hom}\,(-, DY), \tag{1.1}$$

where D is a contravariant equivalence of categories. In the context of vertex operator algebra module categories the dualising object K should be thought of as the gradewise

dual of the vertex operator algebra, seen as a module over itself, and D as the functor which assigns to any object its gradewise dual and to any morphism its transpose. This seemingly simple definition of a Grothendieck-Verdier category has important consequences, for example, it guarantees the existence of internal Homs for all objects $X, Z \in \mathscr{C}$, by providing the explicit formula

$$\underline{\mathrm{Hom}}(X, Z) \cong D(X \otimes D^{-1}Z), \tag{1.2}$$

and implies that the tensor product of $\mathscr{C}$ is right exact, if the category is abelian. Intriguingly, every Grothendieck-Verdier category is also endowed with a second tensor product $X \bullet Y = D^{-1}(DY \otimes DX)$ which turns out to be left exact [11, 12], again, if the category is abelian. The two tensor products $\bullet$ and $\otimes$ should be considered on an equal footing. It remains to be discovered what the full implication of these two tensor products is for vertex operator algebras and conformal field theories. Rigid categories are examples of Grothendieck-Verdier categories, where the tensor unit is a dualising object, though the tensor unit being a dualising object does not imply that the category is rigid in general.

The notion of a Grothendieck-Verdier category (no rigidity assumed) is nicely compatible with additional structure on the category $\mathscr{C}$, for example, a braiding, and it is possible to introduce notions of a balancing and a twist. It is thus not surprising that this structure has surfaced in numerous disparate places: Grothendieck-Verdier categories are also known as $*$-autonomous categories [13], however, in this paper, we use the more recent terminology of Grothendieck-Verdier categories [11, 12, 14]. The main insight of this paper is that the notion of a ribbon Grothendieck-Verdier category is the natural duality structure on tensor categories of modules of vertex operator algebras to which the HLZ-theory of tensor products applies. This is a very welcome insight. For example, the tensor product of a rigid abelian tensor category is necessarily exact, however, this can, in general, not be expected to be true for representation categories of vertex operator algebras. Indeed, the $\mathcal{W}_{2,3}$ triplet model provides just such a counter example [15].

It should be emphasised that the structure of a Grothendieck-Verdier category naturally appears in many fields of mathematics. Grothendieck-Verdier structures are linked to the appearance of dualising sheaves see, for example, [16] for a recent discussion of dualising sheaves; further cyclic algebras over the framed little disc operad with values in the bicategory of finite linear categories are ribbon Grothendieck-Verdier categories [14]; Grothendieck-Verdier structures are referred to as the (categorical semantics) of the multiplicative fragment of linear logic (MLL) [17]; and Grothendieck-Verdier categories have also been used to describe categorical structures on categories of topological vector spaces [13, Appendix].

The purpose of this paper is three-fold: first, we show that categories of representations of vertex operator algebras to which the HLZ theory of tensor products applies are naturally ribbon Grothendieck-Verdier categories. Second, we provide tools to compare categories of different algebraic origin as Grothendieck-Verdier categories. Finally, we provide first simple, yet instructive applications of these general results by considering vertex operator algebras based on Feigin–Fuchs bosons. These are also interesting building blocks for the description of more general classes of vertex

operator algebras of recent interest, for example, of ghost systems [3, 18] or the triplet models [2] and their higher rank generalisations [19].

Let us comment on the importance of these results: in the HLZ theory of tensor products, an important role is played by the contragredient dual, that is, a grade-wise dual. It is known that this dual does not, in general, provide the structure of rigidity on the representation category. Moreover, examples show that it is *not* natural to require the contragredient dual of the vertex operator algebra, that is, of the monoidal unit of the tensor category, to be isomorphic to the monoidal unit. Indeed, in the Grothendieck-Verdier structure, the contragredient dual of the monoidal unit has an important independent conceptual role as a dualising object. To the best of the authors' knowledge, this is the first paper explicitly observing that vertex operator algebra module categories admit Grothendieck-Verdier structures, however, consequences of Grothendieck-Verdier duality structures for vertex operator algebra module categories have been observed in the past. For example, in [15, Display (3.19) and Theorem 3.10] it was noted that internal homs for the $c = 0$ triplet algebra satisfy the formula (1.2) above. Further, in [20, second paragraph above Main Theorem 1 and end of Section 5.1] it was noted that, if the vertex operator algebra is self-contragredient as a module over itself (hence the vertex operator algebra is a dualising object), then internal homs exist. Consequences of the Grothendieck-Verdier structure arising when the vertex operator algebra is self-contragredient were also crucial to recent results [21] relating C_2-cofiniteness and rigidity. It is most gratifying to see Grothendieck-Verdier structure explain and generalise such phenomena and that the deep and general HLZ theory of tensor product finds its natural categorical counterpart in general ribbon Grothendieck-Verdier categories.

We expect that our results will enable much future research. Vertex operator algebras are notoriously intricate algebraic structures. Thus for many constructions, in particular, the construction of full local conformal field theories from chiral conformal field theories, it is therefore desirable to work, as far as possible, in terms of the appropriate categorical structures. The structure of a ribbon Grothendieck-Verdier category is rich enough to give us confidence that such a theory can be developed.

We now summarise the main results and how this paper is structured. In Sect. 2 we give an overview of the categorical notions required for this paper including ribbon Grothendieck-Verdier structures and HLZ tensor product theory. The two main general results are:

- In Theorem 2.12, we state precise conditions which ensure that a representation category $\mathscr{C}$ of a vertex operator algebra V is a ribbon Grothendieck-Verdier category with dualising object V' the contragredient of the vertex algebra as a module over itself and the functor of taking contragredients as the dualising functor.
- In Lemma 2.15 and the subsequent Corollary 2.16, we establish explicit ways to set up equivalences of ribbon Grothendieck-Verdier categories.

In Sect. 3 we then turn to chiral free bosons which are partially compactified, on a lattice (which can have non-maximal rank) and a non-degenerate bilinear form of indefinite signature. For the conformal structure, we admit the possibility of Feigin–Fuchs bosons, that is, we consider conformal vectors of the form

$$\omega_\gamma = \frac{1}{2} \sum_i \alpha^i_{-1} \alpha^{i*}_{-1} |0\rangle + \gamma_{-2}|0\rangle, \quad \gamma \in \mathfrak{h}_\mathbb{C}, \tag{1.3}$$

where the first summand is the standard Sugawara formula for a conformal vector and the second is a deformation by a derivative of (a linear combination of) any of the conformal weight 1 generating free boson fields. This leads to the following set of input data which we collect in the form of *bosonic lattice data*, see Definition 3.1: a finite dimensional real vector space $\mathfrak{h}$ with a non-degenerate symmetric real-valued bilinear form $\langle -, - \rangle$, an even lattice $\Lambda \subset \mathfrak{h}$ and a distinguished element $\xi \in \Lambda^*/\Lambda$ (where Λ^* is the subgroup of $\mathfrak{h}$ that pairs integrally with Λ) which describes the duality structure of the Feigin–Fuchs boson. From these data, we construct three different algebraic structures:

- A ribbon Grothendieck-Verdier category of graded vector spaces, extending a classical construction of Eilenberg-MacLane [22] and Joyal-Street [23] for braided categories, see Proposition 3.5.
- A lattice vertex operator algebra built from Heisenberg vertex algebras and a category of lattice vertex operator algebra modules to which the HLZ tensor product theory applies so that is a ribbon Grothendieck-Verdier category.
- A quasi-Hopf algebra that is possibly infinite dimensional, together with a representation category that is a ribbon Grothendieck-Verdier category.

Then the two Theorem 3.12 and Part (1) assert that, given a set of bosonic lattice data, these three categories are all equivalent as ribbon Grothendieck-Verdier categories. The associativity and braiding structures of these categories all do not depend on the distinguished element $\xi \in \Lambda^*/\Lambda$. The role of this element is to determine the dualising object and twist of the ribbon Grothendieck-Verdier structure, and in the case of the vertex operator algebra module category the conformal structure.

In Sect. 4, the final section, we explore the characters of simple lattice vertex operator algebra modules and explore their transformation under the modular group $SL(2, \mathbb{Z})$, and compute the generalisation of the Verlinde formula conjectured by the standard module formalism [24–31]. We observe that this conjectured Verlinde formula correctly predicts the multiplicities of simple modules in the tensor product, which do not depend on the distinguished element ξ. In contradistinction, the formulae of the modular S and T matrices show shifts that take into account the non-trivial Grothendieck-Verdier structure of the Feigin–Fuchs boson. It remains an interesting problem to explain these shifts in terms of a systematic theory of traces for pivotal Grothendieck-Verdier categories.

2 The categorical framework

In this section we review the notion of a ribbon Grothendieck-Verdier category. This notion includes rigid ribbon categories, but is more general. We show that this notion is the natural duality structure on tensor categories of modules of vertex operator algebras. We assume a basic familiarity with tensor categories and with vertex operator algebras, referring readers unfamiliar with these to [32] or [33], respectively. The

notion of a ribbon Grothendieck-Verdier category and the relevant aspects of the HLZ-theory of tensor products will be reviewed.

2.1 Grothendieck–Verdier categories

Duality, in particular, in the form a rigidity, plays an important role in quantum topology, a subject intimately linked to vertex algebras and their representation categories. The tensor product of a rigid abelian tensor category is necessarily exact. This can, in general, not be expected to be true for representation categories of vertex algebras, which are monoidal categories. Indeed, the $\mathcal{W}_{2,3}$ triplet model provides just such a counter example [15]. Rigidity is a property; it is actually a special case of a more general duality *structure*. Categories with such a structure are called $*$-autonomous categories [13] or, more recently, Grothendieck–Verdier categories [11, 12, 14]. In this paper, we will see that this duality structure is indeed very naturally realised on vertex operator algebra module categories to which the HLZ-tensor product theory applies. In fact, since these categories are naturally braided and have a canonical identification of the bidual with the original module, they admit a pivotal structure. This pivotal structure is equivalent to the existence of a ribbon structure [11, Cor 9.3] (which has a prominent manifestation in the context of vertex operator algebras). So we are naturally lead to study these module categories as ribbon Grothendieck-Verdier categories.

Definition 2.1 A *Grothendieck-Verdier category* is a monoidal category $\mathscr{C}$, together with a distinguished object $K \in \mathscr{C}$, *called the dualising object* satisfying the following conditions.

(1) For any object $Y \in \mathscr{C}$, the contravariant functor $\mathrm{Hom}\,(- \otimes Y, K)$ is representable, that is, there exists an object $DY \in \mathscr{C}$ such that there is a natural isomorphism

$$\mathrm{Hom}\,(- \otimes Y, K) \stackrel{\cong}{\Longrightarrow} \mathrm{Hom}\,(-, DY). \tag{2.1}$$

By Yoneda's Lemma there therefore exists a unique (up to natural isomorphism) contravariant functor D, called *the dualising functor*, which assigns to every $Y \in \mathscr{C}$ the representing object DY, that is $D(Y) = DY$.
(2) The contravariant functor D above is an anti-equivalence.

If in addition the category $\mathscr{C}$ is braided, then it is called a *braided Grothendieck-Verdier category*.

Remark An immediate consequence of the above definition of Grothendieck-Verdier categories is the existence of a natural isomorphism in two variables

$$\mathrm{Hom}\,(-_1 \otimes -_2, K) \stackrel{\cong}{\Longrightarrow} \mathrm{Hom}\,(-_1, D(-_2)), \tag{2.2}$$

where the subscripts indicate that the ordering of the variables is preserved, of the contravariant functors $\mathrm{Hom}\,(- \otimes -, K)$ and $\mathrm{Hom}\,(-, D(-))$.

Note that the choice of a dualising object K is structure, as there can be many inequivalent choices.

Proposition 2.2 (Boyarchenko-Drinfeld [11, Proposition 1.3]) *Let $\mathscr{C}$ be a Grothendieck-Verdier category with dualising object K and corresponding dualising functor D.*

(1) *For any invertible object U, the objects $D(U) \cong K \otimes U^{-1}$ and $D^{-1}U \cong U^{-1} \otimes K$ are again dualising objects in $\mathscr{C}$.*
(2) *The functors $U \mapsto D(U) \cong K \otimes U^{-1}$ and $U \mapsto D^{-1}U \cong U^{-1} \otimes K$ are anti-equivalences between the full subcategory of invertible objects $U \in \mathscr{C}$ and the full subcategory of dualising objects.*
(3) *If $U \in \mathscr{C}$ is invertible then so is D^2U and one has a canonical isomorphism $K \otimes U^{-1} \cong (D^2U)^{-1} \otimes K$.*

Proposition 2.3 *Let $\mathscr{C}$ be a Grothendieck-Verdier category with dualising object $K_{\mathscr{C}}$ and let $\mathscr{D}$ be a monoidal category. For any monoidal equivalence $F : \mathscr{C} \to \mathscr{D}$, the object $F(K_{\mathscr{C}})$ is a dualising object for $\mathscr{D}$. Thus $\mathscr{D}$ admits a Grothendieck-Verdier structure. In particular, if $\mathscr{D}$ has already been endowed with a dualising object $K_{\mathscr{D}}$, then $F(K_{\mathscr{C}})$ and $K_{\mathscr{D}}$ differ by tensoring with an invertible object.*

Proof Let F^{-1} be a quasi-inverse of the equivalence F. For $X, Y \in \mathscr{D}$, consider

$$
\begin{aligned}
\mathrm{Hom}_{\mathscr{D}}(X \otimes Y, FK_{\mathscr{C}}) &\xrightarrow{\cong} \mathrm{Hom}_{\mathscr{D}}(FF^{-1}X \otimes FF^{-1}Y, FK_{\mathscr{C}}) \\
&\xrightarrow{\cong} \mathrm{Hom}_{\mathscr{D}}(F(F^{-1}X \otimes F^{-1}Y), FK_{\mathscr{C}}) \\
&\xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}(F^{-1}X \otimes F^{-1}Y, K_{\mathscr{C}}) \\
&\xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}(F^{-1}X, D_{\mathscr{C}}F^{-1}Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{D}}(X, FD_{\mathscr{C}}F^{-1}Y),
\end{aligned}
\tag{2.3}
$$

where the second bijection follows from the monoidal structure on F and the forth uses the definition property of the dualising object $K_{\mathscr{C}}$. This implies that $FK_{\mathscr{C}}$ is a dualising object in $\mathscr{D}$ with corresponding dualising functor $F \circ D_{\mathscr{C}} \circ F^{-1}$. Finally, if $\mathscr{D}$ was already endowed with a dualising object $K_{\mathscr{D}}$, then $FK_{\mathscr{C}}$ and $K_{\mathscr{D}}$ differing by tensoring with an invertible object is an immediate consequence of Proposition 2.2.(2). $\square$

Proposition 2.3 shows that monoidal equivalences transport dualising objects, thus allowing their comparison. In particular, in the notation of this proposition the pair of Grothendieck-Verdier categories $\mathscr{C}, \mathscr{D}$ have equivalent Grothendieck-Verdier structures if and only if $FK_{\mathscr{C}} \cong K_{\mathscr{D}}$.

Braided Grothendieck-Verdier categories can admit ribbon twists that are compatible with the Grothendieck-Verdier structure (see [11, Sec 8] for a more detailed account). To make this precise we recall the definition of a twist on a braided monoidal category. Let $\mathscr{C}$ be a braided monoidal category with braiding c, then the identity functor with monoidal structure given by the double braiding $J_{\mathscr{C}} = (\mathrm{id}_{\mathscr{C}}, \mathrm{id}_{\mathbf{1}}, c \circ c)$ is a braided monoidal auto-equivalence called the *Joyal-Street equivalence*. A twist on a

braided monoidal category is a monoidal isomorphism $\theta : \mathrm{id}_{\mathscr{C}} \to J_{\mathscr{C}}$. Explicitly, this means that the twist θ obeys

$$\theta_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}}, \quad \text{and} \quad \theta_{X \otimes Y} = c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y). \tag{2.4}$$

If θ is a twist on a Grothendieck–Verdier category $\mathscr{C}$, then

$$\theta_X^D = D^{-1}(\theta_{DX})$$

is also a twist on $\mathscr{C}$. By [11, Proposition 7.3], this is an involution on the set of twists. The fixed points under this involution are relevant for representation categories of conformal vertex operator algebras.

Definition 2.4 A *ribbon Grothendieck-Verdier category* is a braided category $\mathscr{C}$ with a twist θ such that

$$D(\theta_X) = \theta_{DX}, \quad \forall X \in \mathscr{C}. \tag{2.5}$$

Combining all of the notions above, we are lead to the following natural definition of an equivalence of ribbon Grothendieck-Verdier categories.

Definition 2.5 Let $\mathscr{C}$ and $\mathscr{D}$ be ribbon Grothendieck-Verdier categories with respective dualising objects $K_{\mathscr{C}}$ and $K_{\mathscr{D}}$, and twists $\theta_{\mathscr{C}}$ and $\theta_{\mathscr{D}}$. A *ribbon Grothendieck-Verdier equivalence* is a braided monoidal equivalence satisfying

- equivalence of dualising objects: $F K_{\mathscr{C}} \cong K_{\mathscr{D}}$,
- equivalence of twists: $F(\theta_{\mathscr{C}}) = \theta_{\mathscr{D}}$.

Example An immediate source of examples is provided by rigid tensor categories with a different choice of dualizing object, see Proposition 2.2. This includes the examples associated to lattices in the present article.

A natural source of generically non-rigid GV categories is given by finite dimensional bimodules over finite dimensional associative unital algebra A over any field $\mathbb{K}$, with monoidal product given by the tensor product over A and the canonical choice of dualising object given by the vector space dual $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$. If A is in addition commutative, then the monoidal subcategory of left modules is also a GV category. For a more detailed discussion see [34].

A rather different source of examples comes from logic [17], where the two tensor products are to be thought of as the binary operations AND and OR and the dualising functor as NOT.

2.2 Huang–Lepowsky–Zhang tensor categories

The complete reference for tensor structures arising from vertex operator algebras and intertwining operators is the series of papers [1] by Huang, Lepowsky and Zhang. Due to the series admirably operating in great generality, while also providing many

technical details, it can be perceived as intimidatingly long. There are therefore a number of articles in the literature, which include helpful reviews highlighting different aspects of the series relevant to different types of vertex operator algebras and choices of module category [35–37]. Here we give our own overview with an emphasis on the results necessary for later sections. There will be three types of grading appearing below, whose relative importance might not be immediately clear for readers unfamiliar with the theory. There is the conformal grading by generalised eigenvalues of the Virasoro L_0 operator and an additional grading by two abelian groups $A \leq B$, with A grading the vertex operator algebra and B its modules. The latter two gradings have an analogy in the setting of a simple finite dimensional Lie algebra, where A is the root lattice (which grades the Lie algebra) and B is the dual of the Cartan subalgebra (which grades general weight modules). The definitions below assume that the reader is already familiar with the definition of vertex operator algebras and jump directly to setting notational conventions and addressing important subtleties in the definitions of modules, which are required for formulating and understanding tensor products. We refer readers unfamiliar with vertex operator algebras to the first five chapters of Frenkel and Ben-Zvi's book [38].

Definition 2.6 Let V be a vertex operator algebra and B an abelian group with subgroup A.

- The vertex operator algebra V is said to be A-graded, if in addition to the grading by conformal weights V admits a decomposition into homogenous spaces $V^{(\gamma)}$, $\gamma \in A$ such that

  (1) $V = \bigoplus_{\gamma \in A} V^{(\gamma)}$.
  (2) For any $\alpha, \beta \in A$ and any $v \in V^{(\alpha)}$, $w \in V^{(\beta)}$

$$Y(v, z)w = \sum_{n \in \mathbb{Z}} v_n w z^{-n-1} \in V^{(\alpha+\beta)}[\![z, z^{-1}]\!], \tag{2.6}$$

  where $Y : V \otimes V \to V[\![z, z^{-1}]\!]$ denotes the field map of V. It is common to suppress the vector w in $v_n w$ so that v_n is to be thought of as an endomorphism of V and $Y(v, z)$ as an End V-valued power series.

- A *weak V-module* is a vector space M together with a field map

$$Y_M : V \otimes M \to M[\![z, z^{-1}]\!]$$
$$v \mapsto Y_M(v, z)m = \sum_{n \in \mathbb{Z}} v_n m z^{-n-1}$$

  satisfying

  (1) *Lower truncation*: For all $v \in V$ and $m \in M$, $v_n m = 0$ for sufficiently large $n \in \mathbb{Z}$.
  (2) *Vacuum property*: $Y_M(\mathbf{1}, z) = \mathrm{id}_M$, where $\mathbf{1} \in V$ is the vacuum vector.

(3) *L_{-1} derivation*: For any $v \in V$

$$Y_M(L_{-1}v, z) = \frac{\mathrm{d}}{\mathrm{d}z}Y_M(v, z). \tag{2.7}$$

(4) *Jacobi identity*: For any $u, v \in V$, $m \in M$,

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_M(u, z_1)Y_M(v, z_2)m$$
$$= z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)Y_M(v, z_2)Y_M(u, z_1)m$$
$$+ z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_M(Y(u, z_0)v, z_2)m, \tag{2.8}$$

where δ denotes the algebraic delta distribution, that is the formal power series

$$\delta\left(\frac{z_2 - z_1}{z_0}\right) = \sum_{\substack{r \in \mathbb{Z} \\ s \geq 0}} \binom{r}{s}(-1)^s z_1^s z_2^{r-s} z_0^{-r}. \tag{2.9}$$

If in addition there is a B-grading on the weak module $M = \bigoplus_{\beta \in B} M^{(\beta)}$, then M is a *B-graded weak V-module*, if the following condition is satisfied.

(5) *Grading compatibility*: For all $\alpha \in A$, $v \in V^{(\alpha)}$, $\beta \in B$,

$$Y_M(v, z)M^{(\beta)} \subset M^{(\alpha+\beta)}[\![z, z^{-1}]\!]. \tag{2.10}$$

- A *B-graded generalised V-module M* is a weak V-module that is graded by generalised L_0 eigenvalues that in addition is also B-graded, that is, $M = \bigoplus_{h \in \mathbb{C}, \beta \in B} M_h^{(\beta)}$, where

$$M_h^{(\beta)} = \{m \in M^{(\beta)} \ : \ \exists n \in \mathbb{N}, \ (L_0 - h)^n m = 0\}, \qquad h \in \mathbb{C}, \beta \in B. \tag{2.11}$$

The elements of $M_h^{(\beta)}$ are called *doubly homogeneous vectors*. Note that *B-graded generalised V-modules* together with module homomorphisms form an abelian category.

- A *B-graded generalised V-module M* is called *lower bounded* if for each $\beta \in B$, $M_h^{(\beta)} = 0$ for $\mathrm{Re}\, h$ sufficiently negative.
- A *strongly B-graded generalised V-module M* is a B-graded generalised V-module whose simultaneous homogeneous spaces $M_h^{(\beta)}$ are all finite dimensional and for fixed $h \in \mathbb{C}$ and $\beta \in B$, $M_{h+k}^{(\beta)} = 0$ for sufficiently negative $k \in \mathbb{Z}$. Such a module is called *discretely strongly graded* if all non-zero homogeneous spaces have real conformal weight and for any $h \in \mathbb{R}$, $\beta \in B$, the space $\bigoplus_{\tilde{h} \in \mathbb{R}, \tilde{h} \leq h} M_{\tilde{h}}^{(\beta)}$ is finite dimensional.

- A strongly B-graded generalised V-module M is called *graded C_1-cofinite* if for any $\beta \in B$ the space

$$C_1(M)^{(\beta)} = \operatorname{span}_{\mathbb{C}} \{v_{-1}m \in M^{(\beta)} \ : \ v \in V_h, h > 0, m \in M\} \quad (2.12)$$

has finite codimension in $M^{(\beta)}$.

Remark We abbreviate B-graded generalised V-module as B-graded V-module, or when the abelian group B is obvious from context as V-module. For the specific vertex operator algebras to be considered below, we will mainly be interested in discretely strongly graded modules which are in addition graded C_1-cofinite with respect to a suitable choice of vertex operator subalgebra.

Proposition 2.7 *(Huang–Lepowsky–Zhang [1, Part I, Theorem 2.34]) Let $A \leq B$ be abelian groups, V an A-graded vertex operator algebra, let M be a B-graded weak V-module and define the vector spaces*

$$M' = \bigoplus_{b \in B, h \in \mathbb{C}} \left(M_h^{(\beta)}\right)^*, \quad \left(M_h^{(\beta)}\right)^* = \operatorname{Hom}_{\mathbb{C}}\left(M_h^{(\beta)}, \mathbb{C}\right). \quad (2.13)$$

If M is strongly B-graded, then the canonical linear isomorphisms identifying a finite dimensional vector space with its double dual extends to a canonical linear isomorphism $M \cong M''$ of bigraded vector spaces. If, in addition, M is discretely strongly B-graded, then M' is also a discretely strongly B-graded with field map $Y_{M'}$ uniquely characterised by

$$\langle Y_{M'}(v, z)\phi, m \rangle = \langle \phi, Y_M^{\operatorname{opp}}(v, z)m \rangle, \quad v \in V, \ \phi \in M', m \in M, \quad (2.14)$$

where Y_M^{opp} is the opposed field map

$$Y_M^{\operatorname{opp}}(v, z) = Y_M\left(e^{zL_1}\left(-z^{-2}\right)^{L_0}v, z^{-1}\right). \quad (2.15)$$

The module M' is called the contragredient *of M. Opposing the field map is involutive, that is, $Y_M^{\operatorname{opp}\ \operatorname{opp}} = Y_M$, hence the canonical linear isomorphism $M \cong M''$ above is an isomorphism of V-modules.*

Note that by (2.15) the opposed field map depends on the conformal (or at least the Möbius) structure on the vertex operator algebra, that is, the actions of the Virasoro L_0 and L_1 operators enter explicitly. Note further that the opposed field map can be used to define an action of V on M' (or even the full vector space dual M^*) for any weak module M, however, in general the lower truncation axiom need not hold and thus the terms in the Jacobi identity need not converge. There are numerous boundedness conditions on conformal weights which are sufficient for module structures on M'. Here we shall only consider discrete strong gradation, as this is also a sufficient condition for tensor product structures in module categories to be considered below.

Definition 2.8 Let $A \leq B$ be abelian groups, V an A-graded vertex operator algebra and let M_1, M_2, M_3 be B-graded weak V-modules. Denote by $M_3\{z\}\left[\log z\right]$ the space of formal power series in z and $\log z$ with coefficients in M_3, where the exponents of z can be arbitrary complex numbers and with only finitely many $\log z$ terms for any fixed exponent of z. A *logarithmic intertwining operator of type* $\binom{M_3}{M_1, M_2}$ is a linear map

$$\mathcal{Y} : M_1 \otimes M_2 \to M_3\{z\}\left[\log z\right],$$
$$m_1 \otimes m_2 \mapsto \mathcal{Y}(m_1, z)m_2 = \sum_{\substack{s \geq 0 \\ t \in \mathbb{C}}} (m_1)_{t,s} m_2 z^{-t-1}(\log z)^s \qquad (2.16)$$

where $(m_1)_{t,s} \in \mathrm{Hom}_{\mathbb{C}}(M_2, M_3)$ and hence $(m_1)_{t,s} m_2 \in M_3$, satisfying the following properties.

(1) *Lower truncation*: For any $m_i \in M_i$, $i = 1, 2$, and $s \geq 0$

$$(m_1)_{t+k,s} m_2 = 0 \qquad (2.17)$$

for sufficiently large $k \in \mathbb{Z}$.

(2) L_{-1} *derivation*: For any $m_i \in M_i$, $i = 1, 2$,

$$\mathcal{Y}(L_{-1}m_1, z)m_2 = \frac{\mathrm{d}}{\mathrm{d}z}\mathcal{Y}(m_1, z)m_2. \qquad (2.18)$$

(3) *Jacobi identity*: For any $v \in V$, $m_i \in M_i$, $i = 1, 2$,

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_{M_3}(v, z_1)\mathcal{Y}(m_1, z_2)m_2$$
$$= z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) \mathcal{Y}(m_1, z_2)Y_{M_2}(v, z_1)m_2$$
$$+ z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) \mathcal{Y}\left(Y_{M_1}(v, z_0)m_1, z_2\right)m_2. \qquad (2.19)$$

The intertwining operator $\mathcal{Y}$ is called *grading compatible* if addition to the conditions above it also satisfies the following condition.

(4) *Grading compatibility*: For any $\beta_1, \beta_2 \in B$, $m_1 \in M_1^{(\beta_1)}$

$$\mathcal{Y}(m_1, z)M_2^{(\beta_2)} \subset M_3^{(\beta_1+\beta_2)}\{z\}\left[\log z\right]. \qquad (2.20)$$

The conditions above are all linear and so we denote by

$$\mathrm{I}\binom{M_3}{M_1, M_2}, \qquad \mathrm{Gr}\binom{M_3}{M_1, M_2}, \qquad (2.21)$$

respectively, the vector space of all logarithmic intertwining operators of type $\binom{M_3}{M_1,M_2}$ and the subspace of all grading compatible ones.

Note that if, as will be the case in Sect. 3.3, the B-grading corresponds to eigenvalues of zero modes of certain vectors in V of conformal weight 1, then the Jacobi identity implies that all logarithmic intertwining operators are grading compatible. Intertwining operators admit a dualisation analogous to the opposed field map in Proposition 2.7 which will prove crucial to showing the existence of Grothendieck-Verdier structures on vertex operator algebra module categories.

Theorem 2.9 (Huang–Lepowsky–Zhang [1, Part II Proposition 3.46]) *Let M_1, M_2, M_3 be strongly graded generalised modules over some vertex operator algebra V. Then there exists a natural linear isomorphism $D : \mathrm{I}\binom{M_3}{M_1,M_2} \to \mathrm{I}\binom{M_2'}{M_1,M_3'}$, which on any intertwining operator $\mathcal{Y} \in \mathrm{I}\binom{M_3}{M_1,M_2}$ evaluates as*

$$
\langle D(\mathcal{Y})(m_1, x)m_3', m_2 \rangle_{M_2} = \left\langle m_3', \mathcal{Y}\left(e^{xL_1} e^{\mathrm{i}\pi L_0}\left(x^{-L_0}\right)^2 m_1, x^{-1} \right) m_2 \right\rangle_{M_3},
$$
$$
m_1 \in M_1, \ m_2 \in M_2, \ m_3' \in M_3', \tag{2.22}
$$

where the subscript indicates which module the pairings are evaluated in. The isomorphism D preserves grading and hence restricts to a natural isomorphism $\mathrm{Gr}\binom{M_3}{M_1,M_2} \to \mathrm{Gr}\binom{M_2'}{M_1,M_3'}$.

The Jacobi identity for intertwining operators implies that intertwining operators are essentially maps from a pair M_1, M_2 of modules to a third module M_3, which are bilinear in the action of the vertex operator algebra V. It therefore makes sense to ask if there exists some universal tensor product module through which all intertwining operators factor. That is, given some choice of V module category $\mathscr{C}$ and two modules $M_1, M_2 \in \mathscr{C}$, does there exist a module $M_1 \boxtimes M_2 \in \mathscr{C}$ together with an intertwining operator $\mathcal{Y}_{M_1,M_2} \in V\binom{M_1 \boxtimes M_2}{M_1,M_2}$ such that for any $R \in \mathscr{C}$ and intertwining operator $\mathcal{Y} \in V\binom{R}{M_1,M_2}$ there exists a unique module map $f \in \mathrm{Hom}_{\mathscr{C}}(M_1 \boxtimes M_2, R)$ such that $\mathcal{Y} = f \circ \mathcal{Y}_{M_1,M_2}$? That is, such that the diagram

$$
\begin{array}{ccc}
M_1 \otimes M_2 & \xrightarrow{\ \mathcal{Y}_{M_1,M_2}\ } & M_1 \boxtimes M_2 \{z\}\left[\log z\right] \\
& \searrow{\scriptstyle \mathcal{Y}} & \downarrow{\scriptstyle \exists! f} \\
& & R\{z\}\left[\log z\right]
\end{array} \tag{2.23}
$$

commutes. Assuming that $M_1 \boxtimes M_2 \in \mathscr{C}$ exists for all pairs of modules in $\mathscr{C}$, $- \boxtimes -$ becomes a bifunctor after defining the following evaluation on pairs of morphisms. For $M_1, N_1, M_2, N_2 \in \mathscr{C}$ and morphisms $f_1 : M_1 \to N_1$, $f_2 : M_2 \to N_2$, the tensor product morphism $f_1 \boxtimes f_2 : M_1 \boxtimes M_2 \to N_1 \boxtimes N_2$ is the unique morphism, characterised by the universal property (2.23), such that $\mathcal{Y}_{N_1,N_2} \circ (f_1 \otimes f_2) = (f_1 \boxtimes f_2) \circ \mathcal{Y}_{M_1,M_2}$. This characterisation of tensor products (also called fusion products) of

vertex operator algebra modules via a universal property is conceptually very clear and allows us to construct maps out of tensor products in terms of vertex operators; this will be used frequently in the sequel. However, it does not provide an actual construction, nor does it guarantee existence. We sketch some of the ideas of the construction of $M_1 \boxtimes M_2$ here but refer to the original source [1, Part IV] and the review [36] for details. The fusion product of two modules M_1, M_2 can be constructed inside $(M_1 \otimes M_2)^*$, the full vector space dual of the complex tensor product. While $(M_1 \otimes M_2)^*$ is not a V-module, it is possible to move the action of V on the tensor factors M_1, M_2 to $(M_1 \otimes M_2)^*$ using a (generalisation) of the opposed field map. This leads to the consideration of the subspace COMP $(M_1, M_2) \subset (M_1 \otimes M_2)^*$ consisting of all linear functionals on which the evaluation of a field from V has only finitely many singular terms and on which the transported action of fields from either tensor factor agrees, that is, linear functionals compatible with the lower truncation property of Definition 2.6 and the vertex operator algebra version of bilinearity (see [1, Equation (5.141)] or [36, Equation (6.19)]). The name COMP refers to the compatibility of the actions V on the two tensor factors transported to $(M_1 \otimes M_2)^*$. It was shown in [1, Part IV, Theorem 5.48] that COMP (M_1, M_2) is a weak V module and should morally be thought of as the contragredient of the fusion product $M_1 \boxtimes M_2$. The subspace COMP (M_1, M_2) is however usually too large to be contained in the category $\mathscr{C}$ one is considering. For example, it generally contains vectors which are not finite sums of homogeneous vectors. Under suitable conditions on $\mathscr{C}$ (such as those in Theorem 2.11) one can construct the subspace $M_1 \boxdot M_2 \subset$ COMP (M_1, M_2) consisting of the sum of all images of module maps from objects in $\mathscr{C}$ into COMP (M_1, M_2). The contragredient $(M_1 \boxdot M_2)'$ is then the fusion product module satisfying the universal property (2.23). Being constructed as equaliser (and hence left exact) we expect that $M_1 \boxdot M_2$ can be interpreted as the second tensor product $- \bullet -$ (referred to in Sect. 1) that all GV categories are naturally equipped with.

Proposition 2.10 (Huang–Lepowsky–Zhang [1, Part VIII, Section 12]) *Let $A \leq B$ be abelian groups, let V be an A-graded vertex operator algebra and $\mathscr{C}$ a choice of category of V-modules (that is a subcategory of the category of all B-graded V-modules) containing V as an object such that the following conditions hold.*

(1) *For any $M_1, M_2 \in \mathscr{C}$ there exist $M_1 \boxtimes M_2 \in \mathscr{C}$ and $\mathcal{Y}_{M_1,M_2} \in \mathrm{Gr}\binom{M_1 \boxtimes M_2}{M_1,M_2}$ such that the universal property (2.23) holds.*
(2) *For any $M_1, M_2, M_3 \in \mathscr{C}$, there is a family of isomorphisms $A_{M_1,M_2,M_3}^{x_1,x_2}$: $(M_1 \boxtimes M_2) \boxtimes M_3 \rightarrow M_1 \boxtimes (M_2 \boxtimes M_3)$ depending on complex variables x_1, x_2 that is functorial in M_1, M_2, M_3. Further, for $m_i \in M_i$, $x_1, x_2 \in \mathbb{C}$, $|x_1| > |x_2| > |x_1 - x_2| > 0$, the expressions*

$$\mathcal{Y}_{M_1,M_2 \boxtimes M_3}(m_1, x_1) \mathcal{Y}_{M_2,M_3}(m_2, x_2) m_3,$$
$$\mathcal{Y}_{M_1 \boxtimes M_2, M_3}\big(\mathcal{Y}_{M_1,M_2}(m_1, x_1 - x_2) m_2, x_2\big) m_3, \qquad (2.24)$$

converge absolutely for any choice of branch of logarithm for x_1, x_2 in the algebraic completions of $M_1 \boxtimes (M_2 \boxtimes M_3)$ and $(M_1 \boxtimes M_2) \boxtimes M_3$, respectively. Finally,

$$\overline{A}_{M_1,M_2,M_3}^{x_1,x_2}\big(\mathcal{Y}_{M_1,M_2\boxtimes M_3}(m_1,x_1)\mathcal{Y}_{M_2,M_3}(m_2,x_2)m_3\big)$$
$$= \mathcal{Y}_{M_1\boxtimes M_2,M_3}\big(\mathcal{Y}_{M_1,M_2}(m_1,x_1-x_2)m_2,x_2\big)m_3, \qquad (2.25)$$

where $\overline{A}_{M_1,M_2,M_3}^{x_1,x_2}$ is the natural extension of $A_{M_1,M_2,M_3}^{x_1,x_2}$ to algebraic completions. Then $\mathcal{C}$ is a braided monoidal category with the vertex operator algebra V as the monoidal unit, whose structure isomorphisms are uniquely characterised by the following.

- For $M \in \mathcal{C}$ the unit morphisms are uniquely characterised by

$$\ell_M\big(\mathcal{Y}_{V,M}(v,z)m\big) = Y_M(v,z)m$$
$$r_M\big(\mathcal{Y}_{M,V}(m,z)v\big) = e^{zL_{-1}}Y_M(v,-z)m, \qquad (2.26)$$

  where Y_M is the field map of V acting on the module M, $v \in V$ and $m \in M$.
- For $M_1, M_2 \in \mathcal{C}$ the braiding isomorphism $c_{M_1,M_2} : M_1 \boxtimes M_2 \to M_2 \boxtimes M_1$ is uniquely characterised by

$$c_{M_1,M_2}\big(\mathcal{Y}_{M_1,M_2}(m_1,z)m_2\big) = e^{zL_{-1}}\mathcal{Y}_{M_2,M_1}\big(m_2,e^{i\pi}z\big)m_1, \qquad (2.27)$$

  where $m_1 \in M_1$ and $m_2 \in M_2$.
- There is a twist morphism $\theta_{M_1} = e^{2\pi i L_0}|_{M_1}$, $M_1 \in \mathcal{C}$, which satisfies $\theta_V = \mathrm{id}_V$ and for any $M_2 \in \mathcal{C}$ also satisfies the balancing equation

$$\theta_{M_1\boxtimes M_2} = c_{M_2,M_1} \circ c_{M_1,M_2} \circ \big(\theta_{M_1} \boxtimes \theta_{M_2}\big). \qquad (2.28)$$

- For $i = 1, 2, 3$, $M_i \in \mathcal{C}$, $m_i \in M_i$, $x_1, x_2 \in \mathbb{R}$, $x_1 > x_2 > x_1 - x_2 > 0$, the associativity isomorphism $A_{M_1,M_2,M_3} : M_1 \boxtimes (M_2 \boxtimes M_3) \to (M_1 \boxtimes M_2) \boxtimes M_3$ is given by the isomorphism of Part (2) above, where the branches of any logarithms are chosen such that the arguments for $x_1, x_2, x_1 - x_2$ are all 0.

Proving that a choice of vertex operator algebra module category admits the braided tensor structure of Proposition 2.10 is a highly non-trivial task. Fortunately a number of sufficient conditions were identified in [1], which we quote and summarise in the following theorem.

Theorem 2.11 (Huang–Lepowsky–Zhang) *Let $A \leq B$ be abelian groups and let V be an A-graded vertex operator algebra. Then the following conditions on a choice of B-graded module category $\mathcal{C}$ are sufficient for $\mathcal{C}$ to have the braided monoidal structure induced from intertwining operators described in Proposition 2.10.*

(1) *The vertex operator algebra V is an object in $\mathcal{C}$ and all objects of $\mathcal{C}$ are strongly B-graded. For any $M_1, M_2 \in \mathcal{C}$ the logarithmic intertwining operator $\mathcal{Y}_{M_1,M_2}$ satisfying the universal property in the definition of the tensor product of M_1 and M_2 is grading compatible (hence all logarithmic intertwining operators are grading compatible) [1, Part III, Assumption 4.1].*
(2) *$\mathcal{C}$ is a full subcategory of the category of strongly B-graded modules and is closed under the contragredient functor and under taking finite direct sums [1, Part IV, Assumption 5.30].*

(3) *For any object in $\mathscr{C}$ all conformal weights are real and the non-semisimple part of L_0 acts nilpotently, that is, there is a uniform bound on the size of Jordan blocks for any given module though there need not be global bound for the entire category* [1, Part V, Assumption 7.11].

(4) *$\mathscr{C}$ is closed under images of module homomorphisms* [1, Part VI, Assumption 10.1.7].

(5) *The convergence and extension properties for either products or iterates of intertwining operators holds* [1, Part VII, Theorem 11.4].

(6) *For any objects $M_1, M_2 \in \mathscr{C}$, let M_v be the V-module generated by a B-homogeneous generalised L_0 eigenvector $v \in \mathrm{COMP}\,(M_1, M_2)$. If M_v is lower bounded then M_v is strongly graded and an object in $\mathscr{C}$* [39, Theorem 3.1].

Remark The above sufficient conditions are in a sense the weakest known conditions for a vertex operator algebra module category admitting a braided monoidal structure. However, they can in practice be difficult to verify (especially Conditions (5) and (6)). Other more restrictive and hence more tractable sets of conditions are therefore also commonly considered in the literature. The most famous set arguably being:

- The vertex operator algebra V is C_2-cofinite, all L_0 eigenspaces are finite dimensional, the only non-zero eigenspaces have non-negative integral L_0 eigenvalue, $\dim V_0 = 1$ and $V \cong V'$.
- The category $\mathscr{C}$ of all admissible (also known as $\mathbb{N}$ gradable) modules is semisimple.

If the above conditions hold, then $\mathscr{C}$ is a modular tensor category and the much celebrated Verlinde formula holds [40]. A weaker set of sufficient conditions only requires the vertex operator algebra to be C_2-cofinite and the category to be the category of admissible modules without any assumptions on semisimplicity (C_2-cofiniteness, however, still guarantees that the category is finite). Comparatively few general results are known when the C_2-cofiniteness condition is not satisfied, that is, there is currently no known general choice of module category for a general vertex operator algebra that admits the braided monoidal structure of 2.10. However, a recent flurry of new insights appears to be changing this at last [3, 5–7, 37, 41] for certain families of non-C_2-cofinite vertex operator algebras. In particular, the categories of modules considered in [41] are not rigid, with the vertex operator algebra not simple as a module over itself.

With the braided monoidal properties described in Proposition 2.10 and the sufficient conditions of Theorem 2.11 in hand, we can now connect these structures with the Grothendieck-Verdier structures introduced in Sect. 2.1.

Theorem 2.12 *Let V be a vertex operator algebra and $\mathscr{C}$ a choice of category of V-modules which contains V as an object, is closed under taking contragredients and which satisfies the two conditions (and hence also the conclusions) of Proposition 2.10. Then $\mathscr{C}$ is a ribbon Grothendieck-Verdier category with dualising object V' (the contragredient of the vertex operator algebra as a module over itself), dualising functor given by the taking of contragredients, and with twist $\theta = e^{2\pi i L_0}$.*

Proof Let $X, Y, Z \in \mathscr{C}$ and recall the linear isomorphism $A : \mathrm{I}\binom{Z}{X,Y} \to \mathrm{I}\binom{Y'}{X,Z'}$ of Theorem 2.9. When working with a strong grading consider instead the restriction

$A : \mathrm{Gr} \left({}^{Z}_{X,Y} \right) \rightarrow \mathrm{Gr} \left({}^{Y'}_{X,Z'} \right)$. Since category $\mathscr{C}$ is closed under contragredients we therefore also have a natural isomorphism $\mathrm{Hom} \left(X \boxtimes Y, Z \right) \xRightarrow{\cong} \mathrm{Hom} \left(X \boxtimes Z', Y' \right)$. Setting $Z = V'$, we have

$$\mathrm{Hom} \left(X \boxtimes Y, V' \right) \xRightarrow{\cong} \mathrm{Hom} \left(X \boxtimes V'', Y' \right) \xRightarrow{\cong} \mathrm{Hom} \left(X \boxtimes V, Y' \right) \xRightarrow{\cong} \mathrm{Hom} \left(X, Y' \right) \tag{2.29}$$

where we have made use of the canonical isomorphism $V'' \simeq V$ of Proposition 2.7 and the left unit isomorphism $V \boxtimes Y \simeq Y$. This proves that $\mathscr{C}$ is a braided Grothendieck-Verdier category with dualising object V'. Next we show that the twist $\theta_M = \mathrm{e}^{2\pi \mathrm{i} L_0}|_M$ and the contragredient functor satisfy the compatibility condition (2.5). From formula (2.15) for the opposed field map one sees that $L_0^{\mathrm{opp}} = L_0$ and hence for any module $M \in \mathscr{C}$, $(\theta_M)' = \theta_{M'}$. Thus $\mathscr{C}$ is ribbon Grothendieck-Verdier. $\qquad\square$

Remark Note that in Theorem 2.12 we do not require V' and V to be isomorphic as V-modules. Indeed, V' plays the important structural role of a dualising object. Note also that Theorem 2.12 and the remark preceeding it together imply that categories of admissible modules over C_2-cofinite vertex operator algebras are a source of finite ribbon Grothendieck-Verdier categories.

The convergence and extension property of Theorem 2.11.(5) is a technical condition on the analytic properties of intertwining operators, whose details we shall not state here. Instead we give sufficient conditions for the convergence and extension property to hold.

Theorem 2.13 (Allen-0Wood [3, Theorem 5.7]) *Let $A \leq B$ be abelian groups, let V be an A-graded vertex operator algebra and let $\overline{V}$ be a vertex subalgebra of $V^{(0)}$. Further, let M_i, $i = 0, 1, 2, 3, 4$ be B-graded V-modules. Finally let $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$ and $\mathcal{Y}_4$ be logarithmic grading compatible intertwining operators of types $\left({}^{M_0}_{M_1, M_4} \right)$, $\left({}^{M_4}_{M_2, M_3} \right)$, $\left({}^{M_0}_{M_4, M_3} \right)$ and $\left({}^{M_4}_{M_1, M_2} \right)$ respectively. If the modules M_i, $i = 0, 1, 2, 3$ (note $i = 4$ is excluded) are discretely strongly graded, and graded C_1-cofinite as $\overline{V}$-modules, then $\mathcal{Y}_1, \mathcal{Y}_2$ satisfy the convergence and extension property for products and $\mathcal{Y}_3, \mathcal{Y}_4$ satisfy the convergence and extension property for iterates.*

Choosing the module category to be abelian and combining Theorem 2.11 and 2.13 we obtain the following simplified sufficient conditions.

Corollary 2.14 *Let $A \leq B$ be abelian groups, let V be an A-graded vertex operator algebra and let $\mathscr{C}$ be an abelian full subcategory of the category of all B-graded V-modules, which contains V. Then the following conditions are sufficient for $\mathscr{C}$ to have the ribbon Grothendieck-Verdier category structures induced from intertwining operators described in Proposition 2.10 and Theorem 2.12.*

(1) *All objects in $\mathscr{C}$ are discretely strongly graded and $\mathscr{C}$ is closed under taking contragredients.*
(2) *The non-semisimple part of L_0 acts nilpotently on any object in $\mathscr{C}$.*

(3) *There exists a vertex operator subalgebra $\overline{V} \subset V^{(0)}$ such that all objects in $\mathscr{C}$ are graded C_1-cofinite as $\overline{V}$ modules.*
(4) *For any objects $M_1, M_2 \in \mathscr{C}$, every lower bounded submodule of $\mathrm{COMP}\,(M_1, M_2)$ that is finitely generated by doubly homogeneous vectors is an object in $\mathscr{C}$.*

Proof The category $\mathscr{C}$ being abelian, containing V and Condition (1) together imply the Conditions (1), (2) and (4) of Theorem 2.11. Condition (2) together with strong gradation implies Condition (3) of Theorem 2.11. Condition (3) is sufficient for the application of Theorem 2.13, which in turn implies Condition (5) of Theorem 2.11. Finally, Condition (4) implies (6) of Theorem 2.11.   □

We have ordered the conditions in Corollary 2.14 by how difficult they are to verify in practice. Note in particular that Conditions (1) – (3) are merely properties of the types of modules one wishes to consider and make no reference to tensor products.

2.3 Functors involving vertex operator algebra module categories

The monoidal structures of vertex operator algebra module categories are a consequence of the properties of intertwining operators. The following lemma illustrates how monoidal functors from linear braided monoidal categories to vertex operator algebra module categories interact with intertwining operators.

Lemma 2.15 *Let V be a vertex algebra with choice of module category $(\mathscr{C}, \boxtimes, l, r, \alpha, c)$, admitting the braided monoidal structure induced from intertwining operators described in Proposition 2.10. Let $(\mathscr{D}, \otimes, l, r, \alpha, c)$ be a linear braided monoidal category, $G : \mathscr{D} \to \mathscr{C}$ a $\mathbb{C}$-linear abelian functor and φ_0 a choice of morphism $\varphi_0 : V \to G(1_{\mathscr{D}})$. Then the following are equivalent.*

(1) *There exists a natural transformation $\varphi_2 : G(-) \boxtimes G(-) \to G(- \otimes -)$ such that $(G, \varphi_0, \varphi_2)$ is a braided lax monoidal functor (lax here means that φ_0 and φ_2 are not required to be isomorphisms).*
(2) *There exists a family of linear maps*

$$G^T : Hom_{\mathscr{D}}(M \otimes N, P) \to \mathrm{I}\binom{G(P)}{G(M), G(N)},$$
$$f \mapsto G^T_f(z), \tag{2.30}$$

for all $M, N, P \in \mathscr{D}$, satisfying the following conditions.

- *Functoriality: For any $M, M', N, N', P, P' \in \mathscr{D}$ and any $g : M' \to M$, $h : N' \to N$, $k : P \to P'$, we have*

$$G^T_{k \circ f \circ (g \otimes h)}(z) = G(k) \circ G^T_f(z) \circ (G(g) \otimes_{\mathbb{C}} G(h)), \tag{2.31}$$

where $\otimes_{\mathbb{C}}$ denotes the tensor product of complex vector spaces and linear maps.

- *Unitality: For any $N \in \mathscr{D}$,*

$$G^T_{l_N}(\varphi_0(v), z)n = Y_{G(N)}(v, z)n, \tag{2.32}$$

  where $G^T_{l_N}(\varphi_0(v), z)n$ is the evaluation of $G^T_{l_N}(z)$ at $\varphi_0(v) \otimes n$ and where $Y_{G(N)}$ is the vertex operator algebra field map on the module $G(N)$.

- *Skew symmetry: For any $M, N \in \mathscr{D}$ and $m \in G(M)$, $n \in G(N)$,*

$$G^T_{c_{N,M}}(n, z)m = e^{zL_{-1}} G^T_{\mathrm{id}_{M \otimes N}}\left(m, e^{\mathrm{i}\pi} z\right) n. \tag{2.33}$$

- *Associativity: For any $M, N, P \in \mathscr{D}, m \in G(M), n \in G(N), p \in G(P)$ and $x_1, x_2 \in \mathbb{C}$ such that $|x_1| > |x_2| > 0$ and $|x_2| > |x_1 - x_2| > 0$,*

$$G^T_{\alpha_{M,N,P}}(m, x_1)G^T_{\mathrm{id}_{N \otimes P}}(n, x_2)p = G^T_{\mathrm{id}_{(M \otimes N) \otimes P}}\left(G^T_{\mathrm{id}_{M \otimes N}}(m, x_1 - x_2)n, x_2\right)p, \tag{2.34}$$

  where both sides of the equality are to be seen as elements of the algebraic completion of $G((M \otimes N) \otimes P)$ and the associativity map $\alpha_{M,N,P}$ is to be seen as an element of $\mathrm{Hom}\left(M \otimes (N \otimes P), (M \otimes N) \otimes P\right)$ so that $G^T_{\alpha_{M,N,P}}(z)$ is an intertwiner of type $\binom{G((M \otimes N) \otimes P)}{G(M), G(N \otimes P)}$.

The linear maps G^T and the natural transformation φ_2 uniquely characterise each other through the equality $G^T_{\mathrm{id}_{M \otimes N}}(z) = \varphi_2(M, N) \circ \mathcal{Y}_{G(M), G(N)}(-, z)$, where $M, N \in \mathscr{D}$ and where $\mathcal{Y}_{G(M), G(N)}$ is the intertwining operator of the universal property (2.23) characterising $G(M) \boxtimes G(N)$.

Remark By the functoriality condition above, the linear maps G^T are completely determined by their values on $\mathrm{id}_{M \otimes N} \in \mathrm{Hom}_{\mathscr{D}}(M \otimes N, M \otimes N)$, for $P = M \otimes N$. If $G^T_{\mathrm{id}_{M \otimes N}} \in \mathrm{Gr}\binom{G(M \otimes N)}{G(M), G(N)}$ for all $M, N \in \mathscr{D}$, then all G^T_f will be graded intertwining operators, since all morphisms in $\mathscr{C}$ preserve the grading. Further, note that for each of the equations (2.32), (2.33), (2.34), the left-hand sides and right-hand sides are respectively in the same space of intertwining operators. If these spaces of intertwining operators are finite dimensional, then it is sufficient to verify the equation for only a finite number of coefficients. In particular, if the intertwining operator space is one dimensional then it is sufficient to compare the leading coefficients.

Proof Note that if the family of maps G^T in (2.30) exists, then since $\mathrm{id}_{M \otimes N} \in \mathrm{Hom}_{\mathscr{D}}(M \otimes N, M \otimes N)$, it follows that $G^T_{\mathrm{id}_{M \otimes N}}$ is an intertwining operator of type $\binom{G(M \otimes N)}{G(M), G(N)}$. Let $\mathcal{Y}_{G(M), G(N)}$ be the universal intertwining operator of type $\binom{G(M) \boxtimes G(N)}{G(M), G(N)}$ coming from universal property (2.23) characterising $G(M) \boxtimes G(N)$. This universal property further implies the existence and uniqueness of a family of morphisms $\varphi_2(M, N) \in \mathrm{Hom}_{\mathscr{C}}(G(M) \boxtimes G(N), G(M \otimes N))$ satisfying $G^T_{\mathrm{id}_{M \otimes N}}(z) = \varphi_2(M, N) \circ \mathcal{Y}_{G(M), G(N)}(-, z)$. Conversely given a family of morphisms $\varphi_2(M, N) : G(M) \boxtimes G(N) \to G(M \otimes N)$, we can define $G^T_{\mathrm{id}_{M \otimes N}}$ via $G^T_{\mathrm{id}_{M \otimes N}}(z) = \varphi_2(M, N) \circ \mathcal{Y}_{G(M), G(N)}$.

We show the logical equivalence of Assertions (1) and (2) by respectively showing the equivalence of naturality of φ_2 and functoriality of G^T; the left unit square constraint for φ_2 commuting and the unitality of G^T; the braiding square constraint for φ_2 commuting and the skew symmetry of G^T; and the associativity hexagon constraint for φ_2 commuting and the associativity of G^T. Note that the right unit square constraint does not need to be verified, since it is implied by the left unit and braiding.

Assume G^T is functorial, and $g : M' \to M, h : N' \to N$, then

$$
\begin{aligned}
(G(g \otimes h)) &\circ \varphi_2(M', N') \circ \mathcal{Y}_{G(M),G(N)} \\
&= G(g \otimes h) \circ G^T_{\mathrm{id}_{M' \otimes N'}} = G^T_{g \otimes h} = G^T_{\mathrm{id}_{M \otimes N}} \circ G(g) \otimes_{\mathbb{C}} G(h) \\
&= \varphi_2(M, N) \circ \mathcal{Y}_{G(M),G(N)} \circ G(g) \otimes_{\mathbb{C}} G(h) \\
&= \varphi_2(M, N) \circ (G(g) \boxtimes G(h)) \mathcal{Y}_{G(M),G(N)}.
\end{aligned}
\tag{2.35}
$$

Thus $(G(g \otimes h)) \circ \varphi_2(M', N') = \varphi_2(M, N) \circ (G(g) \boxtimes G(h))$ and hence φ_2 is natural.

Conversely, assume φ_2 is natural. As noted above, We first define G^T on identity morphisms $\mathrm{id}_{M \otimes N}$ by $G^T_{\mathrm{id}_{M \otimes N}} = \varphi_2(M, N) \circ \mathcal{Y}_{G(M),G(N)}$ and extend functorially, that is for $f \in \mathrm{Hom}_{\mathscr{D}}(M \otimes N, P), g : M' \to M, h : N' \to N, k : P \to P'$,

$$
G^T_{k \circ f \circ (g \otimes h)}(z) = G(k) \circ G(f) \circ G^T_{\mathrm{id}_{M \otimes N}}(z) \circ (G(g) \otimes_{\mathbb{C}} G(h)).
\tag{2.36}
$$

This is well defined if and only if $G(g \otimes h) \circ G^T_{\mathrm{id}_{M' \otimes N'}}(z) = G^T_{\mathrm{id}_{M \otimes N}}(z) \circ G(g) \otimes_{\mathbb{C}} G(h)$. Consider

$$
\begin{aligned}
G(g \otimes h) \circ G^T_{\mathrm{id}_{M' \otimes N'}}(z) &= G(g \otimes h) \circ \varphi_2(M', N') \circ \mathcal{Y}_{G(M'),G(N')}(-, z) \\
&= \varphi_2(M, N) \circ (G(g) \boxtimes G(h)) \circ \mathcal{Y}_{G(M'),G(N')}(-, z) \\
&= \varphi_2(M, N) \circ \mathcal{Y}_{G(M),G(N)}(-, z) \circ G(g) \otimes_{\mathbb{C}} G(h), \quad (2.37)
\end{aligned}
$$

where the second equality uses that φ_2 is natural and the third uses the definition of the tensor product of morphisms in $\mathscr{C}$. Hence the formula (2.36) is well defined and G^T is functorial. For the remainder of the proof we will assume that φ_2 natural and hence also that G^T is functorial.

We next show the logical equivalence of the left unit constraint for $(G, \varphi_0, \varphi_2)$ and the unitality of G^T. Consider the following squares.

$$
\begin{array}{ccc}
V \otimes_{\mathbb{C}} G(M) \xrightarrow{\mathcal{Y}_{V,G(M)}} V \boxtimes G(M) \xrightarrow{l_{G(M)}} G(M) \\
\downarrow{\varphi_0 \otimes_{\mathbb{C}} \mathrm{id}_{G(M)}} \qquad \downarrow{\varphi_0 \boxtimes \mathrm{id}_{G(M)}} \qquad \uparrow{G(l_M)} \\
G(1_{\mathscr{D}}) \otimes_{\mathbb{C}} G(M) \xrightarrow{\mathcal{Y}_{G(1_{\mathscr{D}}),G(M)}} G(1_{\mathscr{D}}) \boxtimes G(M) \xrightarrow{\varphi_2(1_{\mathscr{D}},M)} G(1_{\mathscr{D}} \otimes M)
\end{array}
\tag{2.38}
$$

Note that we have suppressed formal variables in the images of intertwining operators for visual clarity. The left square commutes by the definition (see Theorem 2.12) of how the functor $\boxtimes$ is evaluated on pairs of morphisms in $\mathscr{C}$. Consider the two sequences of equalities

$$G(l_M) \circ \varphi_2(1_{\mathscr{D}}, M) \circ \big(\varphi_0 \boxtimes \mathrm{id}_{G(M)}\big)\big(\mathcal{Y}_{V,G(M)}(v,z)m\big)$$
$$= G(l_M) \circ \varphi_2(1_{\mathscr{D}}, M)\big(\mathcal{Y}_{G(1_{\mathscr{D}}),G(M)}(\varphi_0(v),z)m\big)$$
$$= G(l_M) \circ G^T_{\mathrm{id}_{1_{\mathscr{D}} \otimes M}}(\varphi_0(v),z)m = G^T_{l_M}(\varphi_0(v),z)m, \tag{2.39a}$$
$$l_{G(M)}\big(\mathcal{Y}_{V,G(M)}(v,z)m\big) = Y_{G(M)}(v,z)m. \tag{2.39b}$$

The first equality of (2.39a) follows from the definition of $\boxtimes$ evaluated on pair of morphisms, the second from the identity relating G^T and φ_2, the third from the functoriality of G^T, while (2.39b) is the defining property of left unit morphisms in $\mathscr{C}$. If we assume that G^T is unital, then the last expressions of (2.39a) and (2.39b) are equal and hence the first terms must be too, implying the commutativity of the right square in (2.38). Conversely, if we assume that the right square in (2.38) commutes, then the first expressions of (2.39a) and (2.39b) are equal and hence the last terms must be too. Thus G^T is unital.

We next show the logical equivalence of the braiding constraint for $(G, \varphi_0, \varphi_2)$ and the skew symmetry of G^T. Consider the following squares.

$$
\begin{array}{ccccc}
G(M) \otimes_{\mathbb{C}} G(N) & \xrightarrow{\mathcal{Y}_{G(M),G(N)}} & G(M) \boxtimes G(N) & \xrightarrow{\varphi_2(M,N)} & G(M \otimes N) \\
\Big\downarrow{\scriptstyle P} & & \Big\downarrow{\scriptstyle c_{G(M),G(N)}} & & \Big\downarrow{\scriptstyle G(c_{M,N})} \\
G(N) \otimes_{\mathbb{C}} G(M) & \xrightarrow{\mathcal{Y}_{G(N),G(M)}} & G(N) \boxtimes G(M) & \xrightarrow{\varphi_2(N,M)} & G(N \otimes M)
\end{array}
\tag{2.40}
$$

where P is the tensor flip. We have again suppressed formal variables. The left square commutes by the definition (see Theorem 2.12) of braiding for intertwining operators. Consider the two sequences of equalities

$$\varphi_2(N,M) \circ c_{G(M),G(N)}\big(\mathcal{Y}_{G(M),G(N)}(m,z,n)\big)$$
$$= \varphi_2(N,M)\Big(e^{zL_{-1}}\mathcal{Y}_{G(N),G(M)}\big(n, e^{i\pi}z, m\big)\Big)$$
$$= e^{zL_{-1}}\varphi_2(N,M)\Big(\mathcal{Y}_{G(N),G(M)}\big(n, e^{i\pi}z, m\big)\Big)e^{zL_{-1}}G^T_{\mathrm{id}_{N \otimes M}}\big(n, e^{i\pi}z\big)m, \tag{2.41a}$$
$$G(c_{M,N}) \circ \varphi_2(M,N) \circ \mathcal{Y}_{G(M),G(N)}(m,z)n$$
$$= G(c_{M,N}) \circ G^T_{\mathrm{id}_{M \otimes N}}(m,z)n = G^T_{c_{M,N}}(m,z)n. \tag{2.41b}$$

As for the unitality argument above, the equalities follow from the defining properties of the tensor structures in $\mathscr{C}$ and the functoriality of G^T or naturality of φ_2. Note that $\varphi_2(N, M)$ is a module map and hence commutes with L_{-1}. If we assume that G^T is skew symmetric, then the last expressions of (2.41a) and (2.41b) are equal and hence the first are too. Thus the right square in (2.40) commutes. Conversely, if the right square in (2.40) commutes, then the first expressions of (2.41a) and (2.41b) are equal and hence the last are too, implying the skew symmetry of G^T.

Finally we show the equivalence of the associativity hexagon condition for φ_2, and the associativity of G^T. Consider the following triangle and hexagon.

$$\small G(M) \otimes_{\mathbb{C}} G(N) \otimes_{\mathbb{C}} G(P) \xrightarrow{\;\mathcal{Y}(\,)\mathcal{Y}(\,)\;} G(M) \boxtimes (G(N) \boxtimes G(P)) \xrightarrow{\;\mathrm{id}\,\boxtimes\varphi_2\;} G(M) \boxtimes G(N \otimes P) \xrightarrow{\;\varphi_2\;} G(M \otimes (N \otimes P))$$

(2.42)

Here $\mathcal{Y}(\,)\mathcal{Y}(\,)$ and $\mathcal{Y}(\mathcal{Y}(\,))$ denote the obvious product and iterate of intertwining operators and we have suppressed the objects labelling the natural transformations $\varphi_2, \alpha_{\mathscr{C}}, \alpha_{\mathscr{D}}$. The left triangle commutes by the definition (see Theorem 2.12) of associativity for intertwining operators. Let $m \in G(M), n \in G(N), p \in G(P), x_1, x_2 \in \mathbb{C}$, $|x_1| > |x_2| > 0$, $|x_2| > |x_1 - x_2| > 0$ and consider the two sequences of equalities

$$G(\alpha_{M,N,P}) \circ \varphi_2(M, N \otimes P) \circ \big(\mathrm{id}_{G(M)} \boxtimes \varphi_2(N, P)\big) \circ$$
$$\qquad \mathcal{Y}_{G(M),G(N)\boxtimes G(P)}(m, x_1) \mathcal{Y}_{G(N),G(P)}(n, x_2) p$$
$$\quad = G(\alpha_{M,N,P}) \circ \varphi_2(M, N \otimes P) \circ \mathcal{Y}_{G(M),G(N\otimes P)}(m, x_1) \varphi_2(N, P) \circ$$
$$\qquad \mathcal{Y}_{G(N),G(P)}(n, x_2) p$$
$$\quad = G(\alpha_{M,N,P}) \circ G^T_{\mathrm{id}_{M\otimes(N\otimes P)}}(m, x_1) G^T_{\mathrm{id}_{N\otimes P}}(n, x_2) p$$
$$\quad = G^T_{\alpha_{M,N,P}}(m, x_1) G^T_{\mathrm{id}_{N\otimes P}}(n, x_2) p, \tag{2.43a}$$

$$\varphi_2(M \otimes N, P) \circ \big(\varphi_2(M, N) \boxtimes \mathrm{id}_{G(P)}\big) \circ \alpha_{G(M),G(N),G(P)} \circ$$
$$\qquad \mathcal{Y}_{G(M),G(N)\boxtimes G(P)}(m, x_1) \mathcal{Y}_{G(N),G(P)}(n, x_2) p$$
$$\quad = \varphi_2(M \otimes N, P) \circ \big(\varphi_2(M, N) \boxtimes \mathrm{id}_{G(P)}\big) \circ$$
$$\qquad \mathcal{Y}_{G(M\otimes N),G(P)}\big(\mathcal{Y}_{G(M),G(N)}(m, x_1 - x_2)n, x_2\big) p$$
$$\quad = \varphi_2(M \otimes N, P) \circ \mathcal{Y}_{G(M\otimes N),G(P)}\big(\varphi_2(M, N) \circ \mathcal{Y}_{G(M),G(N)}(m, x_1 - x_2)n, x_2\big) p$$
$$\quad = G^T_{\mathrm{id}_{(M\otimes N)\otimes P}}\Big(G^T_{\mathrm{id}_{M\otimes N}}(m, x_1 - x_2)n, x_2\Big) p. \tag{2.43b}$$

As with the arguments for the previous commutative diagrams, the equivalence of G^T being associative and the hexagon in (2.42) commuting follows by recognising the equality of either the first or last terms of (2.43a) and (2.43b).

Assertions (1) and (2) are therefore equivalent. $\qquad\square$

Corollary 2.16 *Let $\mathscr{C}$, $\mathscr{D}$, G and φ_0 be as in Lemma 2.15. Further, assume φ_0 is an isomorphism and that there exists a natural transformation $\varphi_2 : G(M) \boxtimes G(N) \to G(M \otimes N)$ such that $(G, \varphi_0, \varphi_2)$ is a braided monoidal functor. Then φ_2 is a natural isomorphism (equivalently $(G, \varphi_0, \varphi_2)$ is a strong braided monoidal functor) if either of the following sets of sufficient conditions are satisfied.*

(1) *The unique morphism $f_{M,N} \in \mathrm{Hom}_{\mathscr{C}}(G(M) \boxtimes G(N), G(M \otimes N))$ satisfying $G^T_{\mathrm{id}_{M\otimes N}}(z) = f_{M,N} \circ \mathcal{Y}_{G(M),G(N)}(-, z)$ is an isomorphism for all $M, N \in \mathscr{D}$.*
(2) *For all $M, N \in \mathscr{D}$, the objects $G(M) \boxtimes G(N)$ and $G(M \otimes N)$ are isomorphic, and $G^T_{\mathrm{id}_{M\otimes N}}(z)$ is a surjective intertwining operator.*

If in addition $\mathscr{D}$ is ribbon Grothendieck-Verdier with dualising object $K_{\mathscr{D}}$ and twist $\theta_{\mathscr{D}}$, then a braided monoidal equivalence $(G, \varphi_0, \varphi_2)$ is a ribbon Grothendieck-Verdier equivalence, if and only if

$$G(K_{\mathscr{D}}) \cong V' \quad and \quad G(\theta_{\mathscr{D}}) = e^{2\pi i L_0}|_{G(-)}. \tag{2.44}$$

3 The (chiral) free boson in three guises

While the previous section was very general, we now change gears and consider a specific family of monoidal categories, typically called free bosons in the context of vertex operator algebras. We will, however, consider these free bosons in slightly greater generality than is usually done in the literature (by allowing for different choices of conformal structures) and we will show that they admit ribbon Grothendieck-Verdier structures.

3.1 Lattice data for free bosons

Throughout this section, we will make frequent use of certain linear algebraic and lattice data, whose structure we record here.

Definition 3.1 A set of *bosonic lattice data* is a quadruple $(\mathfrak{h}, \langle -, - \rangle, \Lambda, \xi)$, where

- $\mathfrak{h}$ is a finite dimensional real vector space,
- $\langle -, - \rangle$ is a non-degenerate symmetric real-valued bilinear form on $\mathfrak{h}$,
- $\Lambda \subset \mathfrak{h}$ is a lattice (that is, a discreet subgroup of $\mathfrak{h}$), which is even with respect to $\langle -, - \rangle$,
- $\xi \in \Lambda^*/\Lambda$ is a distinguished element called the *Feigin–Fuchs boson*, where $\Lambda^* = \{\mu \in \mathfrak{h} \mid \langle \mu, \Lambda \rangle \subset \mathbb{Z}\}$.

Note also that we are not assuming that, $\langle -, - \rangle$ is positive definite, that Λ is non-trivial nor that $\langle -, - \rangle$ restricted to Λ is non-degenerate. Further, if Λ is not full rank, then Λ^* will not be discreet. For any set of bosonic lattice data we can always choose a section $s : \Lambda^*/\Lambda \to \Lambda^*$, that is $\forall \rho \in \Lambda^*/\Lambda$, $s(\rho) \in \rho$ or in other words a map which chooses a representative for each coset. Note that s will generally only be a set theoretic section and not a group homomorphism. Additionally, we will always assume that $s(\Lambda) = 0 \in \Lambda^*$. From the section s we construct the associated 2-cocycle $k : \Lambda^*/\Lambda \times \Lambda^*/\Lambda \to \Lambda$, $k(\mu, \nu) = s(\mu + \nu) - s(\mu) - s(\nu)$, $\mu, \nu \in \Lambda^*$, which encodes the failure of s to be a group homomorphism. Finally, let $\varepsilon : \Lambda \times \Lambda \to \mathbb{C}^\times$ be a normalised 2-cocycle with commutator function $C(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$, that is, ε satisfies the following conditions, for $\alpha, \beta, \gamma \in \Lambda$.

$$\varepsilon(\alpha, 0) = \varepsilon(0, \alpha) = 1, \quad \varepsilon(\beta, \gamma)\varepsilon(\alpha + \beta, \gamma)^{-1}\varepsilon(\alpha, \beta + \gamma)\varepsilon(\alpha, \beta)^{-1} = 1,$$
$$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha)^{-1} = (-1)^{\langle \alpha, \beta \rangle}. \tag{3.1}$$

An example of such a 2-cocycle can be constructed from any ordered $\mathbb{Z}$-basis $\{\alpha_i\}$ of Λ by defining ε to be the group homomorphism uniquely characterised by

$$\varepsilon(\alpha_i, \alpha_j) = \begin{cases} (-1)^{\langle \alpha_i, \alpha_j \rangle} & \text{if } i < j \\ 1 & \text{if } i \geq j \end{cases} \tag{3.2}$$

Note that in general ε need not be a homomorphism, however, since Λ is an abelian group, all choices of 2-cocycle are cohomologous. The section and 2-cocycles will always be denoted s, k and ε, respectively, for any set of bosonic lattice data. They will be required for giving explicit formulae for certain structures such as braiding and associativity isomorphisms.

Each set of bosonic lattice data will allow us to define a category of graded vectors spaces, a vertex operator algebra module category and a quasi-Hopf algebra module category all with a natural choice of ribbon Grothendieck-Verdier structure determined by these bosonic lattice data. Any such triple of categories will be shown to be ribbon Grothendieck-Verdier equivalent provided their bosonic lattice data are equal. Different choices of section s and 2-cocycle ε will yield equivalent categories, hence s and ε are not data.

Lemma 3.2 *Let $(\mathfrak{h}, \langle -, - \rangle, \Lambda, \xi)$ be a set of bosonic lattice data.*

(1) Let $\Lambda^{\perp} = \{\mu \in \mathfrak{h} : \langle \mu, \Lambda \rangle = 0\}$. There exists a finitely generated free abelian subgroup $\Gamma \subset \Lambda^*$ such that $\Lambda^* = \Lambda^{\perp} \oplus \Gamma$, where $\oplus$ is the (internal) direct sum of $\mathbb{Z}$-modules.

(2) Let $\Lambda^{\circ} = \{v \in \Lambda : \langle v, w \rangle = 0, \forall w \in \Lambda\}$. The subgroup Γ from above can be chosen in such a way that there exists a vector subspace $V \subset \Lambda^{\perp}$ and free finitely generated groups $F, D \subset \Gamma$ such that all of the following hold.

- *As an abelian group, Λ^* admits a direct sum decomposition*

$$\Lambda^* = V \oplus \operatorname{span}_{\mathbb{R}}\{\Lambda^{\circ}\} \oplus F \oplus D. \tag{3.3}$$

- *The three subgroups V, $\operatorname{span}_{\mathbb{R}}\{\Lambda^{\circ}\} \oplus F$ and D are mutually orthogonal.*
- *The restriction of $\langle -, - \rangle$ to each of the three subgroups V, $\operatorname{span}_{\mathbb{R}}\{\Lambda^{\circ}\} \oplus F$ and D individually is non-degenerate.*
- *The restriction of $\langle -, - \rangle$ to $\operatorname{span}_{\mathbb{R}}\{\Lambda^{\circ}\}$ and F individually is trivial.*

Proof (1) If Γ exists, then it must be isomorphic to the quotient $\Lambda^*/\Lambda^{\perp}$, we therefore first need to show that $\Lambda^*/\Lambda^{\perp}$ is freely finitely generated. By definition $\Lambda^{\perp}$ is the kernel of the surjective group homomorphism

$$\begin{aligned} \psi : \Lambda^* &\to \operatorname{Hom}(\Lambda, \mathbb{Z}), \\ \kappa &\mapsto \langle \kappa, - \rangle|_{\Lambda}. \end{aligned} \tag{3.4}$$

Hence $\Lambda^*/\Lambda^{\perp} \cong \operatorname{Hom}(\Lambda, \mathbb{Z}) \cong \Lambda$ which is freely finitely generated. Thus $\Lambda^* \to \Lambda^*/\Lambda^{\perp}$ is a surjective homomorphism onto a free finitely generated $\mathbb{Z}$ module with $\Lambda^{\perp}$ as its kernel, hence $\Lambda^{\perp}$ admits a free finitely generated direct sum complement in Λ^*.

(2) Note that Λ/Λ° is torsion free. This can be seen by contradiction. If there was an element $t \in \Lambda \backslash \Lambda^\circ$ such that $kt \in \Lambda^\circ$ for some non-zero $k \in \mathbb{Z}$, then $0 = \langle kt, w \rangle = k \langle t, w \rangle$ for all $w \in \Lambda$, but this would imply $t \in \Lambda^\circ$. Since Λ/Λ° is torsion free, Λ° admits a direct sum complement Λ^c in Λ. The directness of the sum $\Lambda = \Lambda^\circ \oplus \Lambda^c$ implies that $\langle -, - \rangle$ restricted to Λ^c is non-degenerate. Note that the subgroup Γ from above can be chosen such that $\Lambda^c \subset \Gamma$. Define $D = \Gamma \cap \text{span}_{\mathbb{R}}\{\Lambda^c\}$, then the restriction of $\langle -, - \rangle$ to D is non-degenerate, because it is non-degenerate on Λ^c. Next define $F = \{t \in \Gamma \ : \ \langle t, f \rangle = 0, \forall f \in D\}$, $V = \{v \in \Lambda^\perp \ : \ \langle v, t \rangle = 0, \forall t \in F\}$ and $W = \{w \in \Lambda^\perp \ : \ \langle w, v \rangle = 0, \forall v \in V\}$.

We show that $\Gamma = F \oplus D$. Let $\{f_i\}_{i=1}^{\text{rk}D}$ be a $\mathbb{Z}$-basis of D. Since $\langle -, - \rangle$ is non-degenerate on D, there exists an $\mathbb{R}$-basis $\{f^i\}_{i=1}^{\text{rk}D}$ of $\text{span}_{\mathbb{R}}\{D\}$, which is dual to $\{f_i\}_{i=1}^{\text{rk}D}$, that is $\langle f_i, f^j \rangle = \delta_{i,j}$. Note that this implies that the f^i basis elements pair integrally with any element in D. Consider $v \in \Gamma$, then

$$\tilde{v} = \sum_{i=1}^{\text{rk}D} \left\langle v, f^i \right\rangle f_i \in D \tag{3.5}$$

and for any f^j we have

$$\left\langle v - \tilde{v}, f^j \right\rangle = \left\langle v, f^j \right\rangle - \left\langle v, f^j \right\rangle = 0. \tag{3.6}$$

Since all elements of D are $\mathbb{R}$-linear combinations of the $\mathbb{R}$-basis elements f^i, this implies $v - \tilde{v} \in F$ and hence $v \in F + D$. Next consider $v \in F \cap D$, then $\langle v, f \rangle = 0$ for all $f \in D$, but $\langle -, - \rangle$ is non-degenerate on D, hence $v = 0$ and $\Gamma = F \oplus D$.

Note that D is orthogonal to F by construction and to $\Lambda^\perp$ since $D \subset \text{span}_{\mathbb{R}}\{\Lambda\}$. Thus V is orthogonal to D, F and W and so $\langle -, - \rangle$ must be non-degenerate on V in order to be non-degenerate on $\mathfrak{h}$. By a similar argument as for D and F, we therefore have that $\Lambda^\perp = V \oplus W$.

By construction Λ° is orthogonal to V (because it is orthogonal to $\Lambda^\perp$) and also $\Lambda^\circ \subset \Lambda^\perp$. Hence $\text{span}_{\mathbb{R}}\{\Lambda^\circ\} \subset W$. A brief counting of dimensions and ranks reveals $\text{rk}\Lambda^\circ = \text{rk}\Lambda - \text{rk}D = \dim W = \text{rk}F$, implying that $\text{span}_{\mathbb{R}}\{\Lambda^\circ\} = W$. Finally, by construction F is orthogonal to V and Γ hence, by the non-degeneracy of $\langle -, - \rangle$ on $\mathfrak{h}$, F must pair non-trivially with W. $\qquad \square$

Remark The quotient group Λ^*/Λ will feature prominently below. The decomposition in Part (2), after observing that $\Lambda = \Lambda^\circ \oplus D \cap \Lambda$, implies the decomposition

$$\Lambda^*/\Lambda = V \oplus \frac{\text{span}_{\mathbb{R}}\{\Lambda^\circ\}}{\Lambda^\circ} \oplus F \oplus \frac{D}{D \cap \Lambda}. \tag{3.7}$$

Thus Λ^*/Λ decomposes into an abelian Lie group with a vector space part V and a compact part $\text{span}_{\mathbb{R}}\{\Lambda^\circ\}/\Lambda^\circ$, and a finitely generated group with a free part F and a finite part $D/D \cap \Lambda$.

Example We have the following natural examples to consider.

(1) An empty lattice: $\mathfrak{h} = \mathbb{R}^n$ and $\Lambda = \{0\}$. In the decomposition of Lemma 3.2, we have $\Lambda^* = \mathfrak{h} = V$, $\Gamma = \Lambda^\circ = F = D = \{0\}$. Hence $\Lambda^*/\Lambda \cong \mathfrak{h}$ and ξ can be any element in $\mathfrak{h}$. In this case there is only one choice of section s, the canonical identification of $\mathfrak{h}/\{0\}$ with $\mathfrak{h}$, and $k = 0$.

(2) A full rank lattice: $\mathfrak{h} = \mathbb{R}^n$ and Λ a rank n even integral lattice. In the decomposition of Lemma 3.2, $\Lambda^\perp = \{0\}$ and so $\Lambda^* = \Gamma = D$ is finitely generated. Further, Λ^*/Λ is a finite group whose order is equal to the determinant (up to a sign) of the Gram matrix of the pairing in any choice of $\mathbb{Z}$-basis of Λ. For any given $\mathbb{Z}$-basis $\{e_i\}$ of Λ (which is also an $\mathbb{R}$-basis of $\mathbb{R}^n$), there is a distinguished choice of a section s constructed as follows. Any class in $\mathbb{R}^n/\Lambda$ has a unique representative with an expansion in the basis $\{e_i\}$ whose coefficients all lie in the interval $[0, 1)$. This defines a section $s : \mathbb{R}^n/\Lambda \to \mathbb{R}^n$, which induces a section $s : \Lambda^*/\Lambda \to \Lambda^*$ by restriction.

(3) Half rank indefinite lattice: $\mathfrak{h} = \mathbb{R}^2$ with pairing $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_2 + x_2 y_1$ and lattice $\Lambda = \{(0, m) : m \in \mathbb{Z}\}$. Then, in the decomposition of Lemma 3.2, $\Lambda^\circ = \Lambda$, $\Lambda^\perp = \text{span}_\mathbb{R}\{\Lambda^\circ\} = \{(0, x) : x \in \mathbb{R}\}$, $\Lambda^* = \{(m, x) : m \in \mathbb{Z}, x \in \mathbb{R}\} \cong \mathbb{Z} \times \mathbb{R}$, $\Lambda^*/\Lambda \cong \mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, $V = D = \{0\}$ and $F = \{(0, m) : m \in \mathbb{Z}\}$. Since the pairing is trivial when restricted to Λ, we can choose the 2-cocycle to be trivial, that is, $\varepsilon = 1$. We choose $\xi = (1, 0 + \mathbb{Z})$ for the Feigin–Fuchs boson, as this a convenient choice for the free field realisations of bosonic ghost systems. See [42] for an example. Finally, we can define a choice of section s by $s(x, y + \mathbb{Z}) = (x, \tilde{y})$, where $\tilde{y}$ is the unique representative of the coset $y + \mathbb{Z}$ in the interval $[0, 1)$.

3.2 Categories of vector spaces graded by abelian groups

Definition 3.3 Let Vect_G denote the category of finite dimensional complex vector spaces graded by an abelian group G, whose morphisms are all grade preserving linear maps. This category is semisimple with the isomorphism classes of simple objects represented by the one dimensional vector spaces $\mathbb{C}_g$ which are $\mathbb{C}$ at grade $g \in G$ and trivial at other grades.

Note that if G is not finite, then objects in Vect_G will have only finitely many non-trivial homogenous spaces. We define a tensor product bifunctor on Vect_G by asserting

$$(M \otimes N)_g = \bigoplus_{h \in G} M_{g-h} \otimes_\mathbb{C} N_h, \qquad g \in G, \ M, N \in \mathsf{Vect}_G, \tag{3.8}$$

where $\otimes_\mathbb{C}$ is the tensor product of complex vectors spaces and having the tensor product of morphisms be that of linear maps. Further the unit morphisms of vector spaces then also define unit morphisms for the tensor functor $\otimes$ on Vect_G. The associativity and braiding isomorphisms can then be defined on tensor products of the simple objects $\mathbb{C}_g$ to be scalar multiples of the vector space associator and tensor flip respectively. We shall denote these scalar multiples as F and Ω below.

Theorem 3.4 (Eilenberg and MacLane [22], Joyal and Street [23]) *Let G be an abelian group and Vect_G the category of finite dimensional G graded complex vector spaces with tensor functor and unit isomorphisms defined above. Then the braiding and associativity morphisms on Vect_G are in bijection with normalised abelian 3-cocycles (F, Ω), that is pairs of maps $F : G \times G \times G \to \mathbb{C}^\times$ and $\Omega : G \times G \to \mathbb{C}^\times$ characterised by the relations*

$$F(g + h, k, l)F(g, h, k + l) = F(g, h, k)F(g, h + k, l)F(h, k, l), \quad g, h, k, l \in G$$
$$F(h, k, g)^{-1}\Omega(g, h + k)F(g, h, k)^{-1} = \Omega(g, k)F(h, g, k)^{-1}\Omega(g, h),$$
$$F(k, g, h)\Omega(g + h, k)F(g, h, k) = \Omega(g, k)F(g, k, h)\Omega(h, k), \tag{3.9}$$

and additionally requiring that both maps evaluate to $1 \in \mathbb{C}^\times$ if any argument is $0 \in G$. Inequivalent associativity and braiding structures are parametrised by the cohomology classes of the third abelian group cohomology $H_{ab}^3(G, \mathbb{C}^\times)$. The cohomology classes $\omega = [(F, \Omega)] \in H_{ab}^3(G, \mathbb{C}^\times)$ are uniquely characterised by their trace $(\mathrm{tr}\,\omega)(g) = \Omega(g, g) = q(g)$, which yields a quadratic form $q : G \to \mathbb{C}^\times$.

Due to the above theorem, we denote by Vect_G^q the equivalence class of braided tensor categories with structure characterised by q, and by $\mathsf{Vect}_G^{(F,\Omega)}$ the specific representative category whose associativity and braiding structures corresponds to the abelian 3-cocycle (F, Ω).

Proposition 3.5 *Let (F, Ω) be an abelian 3-cocycle and consider the braided tensor category $\mathsf{Vect}_G^{(F,\Omega)}$.*

(1) *For any $h \in G$, $K = \mathbb{C}_h$ is a dualising object and hence endows $\mathsf{Vect}_G^{(F,\Omega)}$ with the structure of a Grothendieck-Verdier category.*
(2) *The dualising functor corresponding to the choice of dualising object $K = \mathbb{C}_h$, $h \in G$ is characterised by $D(M)_g \simeq (M_{h-g})^*$, $g \in G$, $M \in \mathsf{Vect}_G^{(F,\Omega)}$, where $*$ denotes the ordinary vector space dual.*
(3) *Every dualising object of $\mathsf{Vect}_G^{(F,\Omega)}$ is isomorphic to one of the simple objects $\mathbb{C}_h$ for some $h \in G$. Denote by $\mathsf{Vect}_G^{(F,\Omega,h)}$ the Grothendieck-Verdier category constructed from $\mathsf{Vect}_G^{(F,\Omega)}$ with dualising object $K = \mathbb{C}_{2h}$.*
(4) *The Grothendieck-Verdier category $\mathsf{Vect}_G^{(F,\Omega,h)}$ admits a twist θ by defining*

$$\theta_Q|_{M_g} = Q(g)\,\mathrm{id}_{M_g}, \quad Q(g) = \frac{\Omega(g - h, g - h)}{\Omega(-h, -h)}, \quad M \in \mathsf{Vect}_G^{(F,\Omega,h)}, \ g \in G. \tag{3.10}$$

Proof We first show Part (1). This can be computed directly by comparing the dimensions of morphism spaces or one can note the following. The category $\mathsf{Vect}_G^{(F,\Omega,h)}$ is known to be rigid (the rigid dual of any simple object $\mathbb{C}_h$, $h \in G$ is $C_h^\vee \cong \mathbb{C}_{-h}$ and the evaluation and coevaluation maps are those of vector spaces) and hence the unit object $\mathbb{C}_0$ is dualising. The simple modules $\mathbb{C}_h$, $h \in G$ are all invertible. Thus, by Proposition 2.2.(2), $\mathbb{C}_0 \otimes (\mathbb{C}_h)^{-1} \cong \mathbb{C}_{-h}$ is also dualising. Proposition 2.2.(2) also

immediately implies Part (3). Part (2) follows by noting that the proposed formula for D satisfies the defining relation (2.1) for the dualising object $K = \mathbb{C}_h$. Part (4) follows from the fact that the given formula for $Q(g)$ satisfies the relations implied by (2.4) and (2.5). □

Remark Observe that the choice of $\mathbb{C}_{2h}$ as dualising object in $\mathsf{Vect}_G^{(F,\Omega,h)}$ excludes those simple objects not labelled by the double of a group element in G. This is not an oversight; while every simple object in $\mathsf{Vect}_G^{(F,\Omega)}$ is a valid choice of dualising object (this follows from the category being rigid, hence the tensor unit is dualising, and by Proposition 2.2, we can shift by invertible objects), simple objects not labelled by the double of a group element need not admit a twist which satisfies $D(\theta) = \theta_{D(-)}$. Fortunately, the vertex operator algebraic constructions to be discussed below will always yield dualising objects that admit twists and make a preferred choice of twist.

Functions of the form Q in Equation (3.10) are called weak quadratic forms centred at h. It is interesting to note that (at least in the special case of G being a finite group) the Grothendieck-Verdier ribbon twists on $\mathsf{Vect}_G^{(F,\Omega,h)}$ are in bijection with such weak quadratic forms centred at h, as was shown in Zetzsche's Masters thesis [43, Theorem 4.2.2]. This classification of ribbon Grothendieck-Verdier structures by weak quadratic forms is a generalisation of the classification of rigid braided tensor structures by quadratic forms, which corresponds to the special case $h = 0$ for the dualising object. Note also that for $h = 0$ the category is ribbon.

Let $\Psi = (\mathfrak{h}, \langle -, - \rangle, \Lambda, \xi)$ be a set of bosonic lattice data and recall the decomposition $\Lambda^* = \Lambda^\perp \oplus \Gamma$ of Lemma 3.2.(1). We specialise the results of Theorem 3.4 and Proposition 3.5 using Ψ. We choose the abelian group to be $G = \Lambda^*/\Lambda$ and the quadratic form to be

$$q(\alpha) = e^{i\pi \langle s(\alpha), s(\alpha) \rangle}, \quad \alpha \in \Lambda^*/\Lambda, \tag{3.11}$$

which defines the equivalence class of braided monoidal categories $\mathsf{Vect}_{\Lambda^*/\Lambda}^q$. Note that this choice of quadratic form is independent of the choice of section s due to Λ being even. Note further that $\langle s(\alpha), s(\alpha) \rangle$ need not be integral and so we have chosen $e^{i\pi}$ as a specific branch of logarithm for -1. The section s then allows us to realise a representative $\mathsf{Vect}_{\Lambda^*/\Lambda}^{(F,\Omega)}$ of $\mathsf{Vect}_{\Lambda^*/\Lambda}^q$, by defining the abelian 3-cocycle to be

$$\Omega(\alpha, \beta) = e^{i\pi \langle s(\alpha), s(\beta) \rangle},$$

$$F(\alpha, \beta, \gamma) = (-1)^{\langle s(\alpha), k(\beta, \gamma) \rangle} \frac{\varepsilon(k(\alpha, \beta), k(\alpha + \beta, \gamma))}{\varepsilon(k(\beta, \gamma), k(\alpha, \beta + \gamma))} \quad \alpha, \beta, \gamma \in \Lambda^*/\Lambda. \tag{3.12}$$

Note that the abelian 3-cocycle does depend on the choice of section s, however, all choices of s yield the same trace and hence yield equivalent braided monoidal structures. Similarly, different choices of the 2-cocycle ε will yield equivalent associators. Finally, every $\xi \in \Lambda^*/\Lambda$ yields a ribbon Grothendieck-Verdier category $\mathsf{Vect}_{\Lambda^*/\Lambda}^{(F,\Omega,\xi)}$ with dualising object $\mathbb{C}_{2\xi}$ and with ribbon twist $\theta|_{M_\alpha} = Q(\alpha) \, \mathrm{id}_{M_\alpha}$, $M \in \mathsf{Vect}_{\Lambda^*/\Lambda}^{(F,\Omega,\xi)}$, $\alpha \in \Lambda^*/\Lambda$, given by

$$Q(\alpha) = e^{i\pi[\langle s(\alpha-\xi), s(\alpha-\xi)\rangle - \langle s(-\xi), s(-\xi)\rangle]}$$
$$= e^{i\pi[\langle s(\alpha)-s(\xi), s(\alpha)-s(\xi)\rangle - \langle s(-\xi), s(-\xi)\rangle]}$$
$$= e^{i\pi\langle s(\alpha), s(\alpha)+2s(-\xi)\rangle}$$
$$= e^{i\pi\langle s(\alpha), s(\alpha)-2s(\xi)\rangle}, \tag{3.13}$$

where we have used that the lattice Λ is even. As with the quadratic form, the weak quadratic form Q, which characterises the twist, is independent of the choice of section due to Λ being even. We denote the ribbon Grothendieck-Verdier category constructed above by $\mathsf{Vect}(\Psi)$.

Example Recall the half rank lattice example at the end of Sect. 3.1. In the notation and conventions introduced there, we have the ribbon Grothendieck-Verdier structure defined by the abelian 3-cocycle, trace and twist

$$\Omega((x_1, x_2 + \mathbb{Z}), (y_1, y_2 + \mathbb{Z})) = e^{i\pi(x_1\tilde{y}_2 + \tilde{x}_2 y_1)},$$
$$F((x_1, x_2 + \mathbb{Z}), (y_1, y_2 + \mathbb{Z}), (z_1, z_2 + \mathbb{Z})) = (-1)^{x_1\left(\widetilde{y_2+z_2} - \tilde{y}_2 - \tilde{z}_2\right)},$$
$$q(x_1, x_2 + \mathbb{Z}) = e^{i2\pi x_1 \tilde{x}_2}, \quad Q(x_1, x_2 + \mathbb{Z}) = e^{i2\pi(x_1-1)\tilde{x}_2}. \tag{3.14}$$

3.3 Categories of Heisenberg and lattice vertex operator algebra modules

We largely follow the book [33] for describing lattice vertex operator algebras and their properties, but make some adjustments to notation to accommodate the needs of this paper. Let $\Psi = (\mathfrak{h}, \langle -, -\rangle, \Lambda, \xi)$ be a set of bosonic lattice data. Treating $\mathfrak{h}$ as a real abelian Lie algebra, let $\widehat{\mathfrak{h}} = \mathfrak{h}_\mathbb{C} \otimes \mathbb{C}\left[t, t^{-1}\right] \oplus \mathbb{C}\mathbf{1}$ be the affinisation of $\mathfrak{h}_\mathbb{C}$ (the complexification of $\mathfrak{h}$ with the bilinear form extended in the obvious way) with respect to the bilinear form $\langle -, -\rangle$. This is called the Heisenberg Lie algebra (at level 1). For $\alpha \in \mathfrak{h}_\mathbb{C}$ and $n \in \mathbb{Z}$ denote $\alpha_n = \alpha \otimes t^n$, then we have

$$[\alpha_n, \beta_m] = n\langle \alpha, \beta\rangle \delta_{n,-m}\mathbf{1}, \quad \alpha_n, \beta_m \in \widehat{\mathfrak{h}}, \tag{3.15}$$

with $\mathbf{1}$ central and always taken to act as scalar multiplication by 1 in modules. We choose the triangular decomposition $\widehat{\mathfrak{h}} = \widehat{\mathfrak{h}}_- \oplus \widehat{\mathfrak{h}}_0 \oplus \widehat{\mathfrak{h}}_+$ with $\widehat{\mathfrak{h}}_0 = \mathfrak{h}_\mathbb{C} \otimes 1 \oplus \mathbb{C}\mathbf{1}$ and $\widehat{\mathfrak{h}}_\pm = \mathrm{span}_\mathbb{C}\{\alpha_n : \alpha \in \mathfrak{h}_\mathbb{C}, \pm n > 0\}$. The highest weight modules with respect to this decomposition ($\widehat{\mathfrak{h}}_-$ acting freely, $\widehat{\mathfrak{h}}_+$ nilpotently and $\widehat{\mathfrak{h}}_0$ semisimply) are called Fock spaces

$$\mathcal{F}_\lambda = \mathrm{Ind}_{\widehat{\mathfrak{h}}_+ \oplus \widehat{\mathfrak{h}}_0}^{\widehat{\mathfrak{h}}} \mathbb{C}|\lambda\rangle, \quad \lambda \in \mathfrak{h}_\mathbb{C}, \tag{3.16}$$

where

$$\widehat{\mathfrak{h}}_+|\lambda\rangle = 0, \quad \mathbf{1}|\lambda\rangle = |\lambda\rangle, \quad \alpha_0|\lambda\rangle = \langle \alpha, \lambda\rangle|\lambda\rangle, \quad \alpha \in \mathfrak{h}_\mathbb{C}, \tag{3.17}$$

and $\widehat{\mathfrak{h}}_-$ acts freely. In sequel, any reference to a Fock space $\mathcal{F}_\lambda$ will assume the explicit choice of highest weight vector $|\lambda\rangle$ given in (3.16). This explicit choice of

highest weight vector will be required for giving explicit normalisations of intertwining operators. For the lattice vertex operator algebras and modules to be considered in this section, we shall mostly focus on real weights, that is, $\lambda \in \mathfrak{h}$. For any coset $\mu \in \Lambda^*/\Lambda$ we define the lattice Fock space

$$\mathbb{F}_\mu = \bigoplus_{\nu \in \mu} \mathcal{F}_\nu. \tag{3.18}$$

Proposition 3.6 *The Fock space $\mathcal{F}_0$ admits the structure of a vertex operator algebra uniquely characterised by the choice of field map*

$$Y(\alpha_{-1}|0\rangle, z) = \alpha(z) = \sum_{n=0}^\infty \alpha_n z^{-n-1}, \quad \alpha \in \mathfrak{h}_\mathbb{C}, \tag{3.19}$$

and choice of conformal vector

$$\omega_\gamma = \frac{1}{2} \sum_i \alpha_{-1}^i \alpha_{-1}^{i*}|0\rangle + \gamma_{-2}|0\rangle, \quad \gamma \in \mathfrak{h}_\mathbb{C}, \tag{3.20}$$

where $\{\alpha^i\}_{i=1}^{\dim \mathfrak{h}}$ and $\{\alpha^{j*}\}_{j=1}^{\dim \mathfrak{h}}$ are any dual choices of basis of $\mathfrak{h}_\mathbb{C}$. We denote this vertex operator algebra by $V(\gamma)$. For any $\alpha, \beta \in \mathfrak{h}_\mathbb{C}$, the operator product expansions of the corresponding fields $\alpha(z), \beta(z)$ amongst themselves and with the conformal field $T_\gamma(z) = Y(\omega_\gamma, z)$ are

$$\alpha(z)\beta(w) \sim \frac{\langle \alpha, \beta \rangle}{(z-w)^2}, \quad T_\gamma(z)\alpha(w) \sim \frac{-2\langle \gamma, \alpha \rangle}{(z-w)^3} + \frac{\alpha(w)}{(z-w)^2} + \frac{\partial \alpha(w)}{z-w}, \tag{3.21}$$

and the central charge determined by ω_γ is

$$c_\gamma = \dim \mathfrak{h} - 12\langle \gamma, \gamma \rangle. \tag{3.22}$$

Any choice of basis of $\mathfrak{h}_\mathbb{C}$ is a set of strong generators of $V(\gamma)$. For any $\alpha \in \mathfrak{h}_\mathbb{C}$, the Fock space $\mathcal{F}_\alpha$ is a module over $V(\gamma)$ with field map Y_α characterised by the same formula (3.19) as the field map of $V(\gamma)$ acting on itself.

Let $\mathbb{C}[\mathfrak{h}_\mathbb{C}]$ be the group algebra of $\mathfrak{h}_\mathbb{C}$ seen as an abelian group under addition and denote the basis element corresponding to any group element $\alpha \in \mathfrak{h}_\mathbb{C}$ by e^α. To each such basis vector we assign a linear map e^α, called a shift operator,

$$\begin{aligned} e^\alpha : \mathcal{F}_\gamma &\to \mathcal{F}_{\alpha+\gamma}, \\ p|\gamma\rangle &\mapsto p|\alpha + \gamma\rangle, \end{aligned} \tag{3.23}$$

where $p \in \mathsf{U}(\widehat{\mathfrak{h}}_-)$. Further let

$$
E^{\pm}(\alpha, x) = \exp\left(\mp \sum_{n=1}^{\infty} \frac{\alpha_{\pm n}}{n} x^{\mp n}\right), \, U(p, \alpha, x) = E^-(\alpha, x) Y(p, x) E^+(\alpha, x),
$$
$$
\alpha \in \mathfrak{h}_{\mathbb{C}}, \; p \in \mathsf{U}(\widehat{\mathfrak{h}}_-),
\tag{3.24}
$$

where $Y(p, z)$ is the series of Heisenberg algebra valued coefficients obtained by expanding the field map $Y(p|0\rangle, z)$ in the vertex operator algebra $\mathsf{V}(\gamma)$. Then we define linear maps $I_{\mu,\nu} : \mathcal{F}_\mu \otimes \mathcal{F}_\nu \to \mathcal{F}_{\mu+\nu}[\![z, z^{-1}]\!] z^{\langle \mu, \nu \rangle}$, for $\mu, \nu \in \mathfrak{h}_{\mathbb{C}}$ by

$$
p|\mu\rangle \otimes q|\nu\rangle \mapsto I_{\mu,\nu}(p|\mu\rangle, z) q|\nu\rangle = z^{\langle \mu, \nu \rangle} e^\mu U(p, \alpha, z) q|\nu\rangle, \qquad p, q \in \mathsf{U}(\widehat{\mathfrak{h}}_-).
\tag{3.25}
$$

The linear maps $I_{\mu,\nu}$ are generally known as (chiral) vertex operators in theoretical physics literature and are called untwisted vertex operators in [33].

Proposition 3.7 *Let $\mu, \nu, \rho, \in \mathfrak{h}_{\mathbb{C}}$, then*

$$
\dim \mathrm{I} \begin{pmatrix} \mathcal{F}_\rho \\ \mathcal{F}_\mu, \mathcal{F}_\nu \end{pmatrix} = \begin{cases} 1 & \rho = \mu + \nu \\ 0 & \rho \neq \mu + \nu \end{cases}
\tag{3.26}
$$

and $I_{\mu,\nu}$ is an intertwining operator of type $\begin{pmatrix} \mathcal{F}_{\mu+\nu} \\ \mathcal{F}_\mu, \mathcal{F}_\nu \end{pmatrix}$.

Lattice vertex operator algebras are constructed from Heisenberg vertex operator algebras by taking the underlying vector space to be a sum over Fock spaces whose weights lie in a lattice. The field maps for vectors lying in Fock spaces with non-zero weight are then constructed from the untwisted intertwining operators $I_{\mu,\nu}$ above. As can be seen from the definitions of modules and intertwining operators, and the unit isomorphism conditions (2.26), the field maps encoding the action of a vertex operator algebra on its modules are a special case of an intertwining operator with a canonical choice of normalisation. General intertwining operators, however, have no obvious choice of normalisation. So in order to extend a Heisenberg vertex operator algebra to a lattice vertex operator algebra, one needs to specify normalisations. These normalisations need to be compatible with the vacuum, skew-symmetry and associativity properties of vertex operator algebras, which implies that they satisfy the defining properties of the 2-cocycles ε in (3.1). As previously noted all choices of 2-cocycle are cohomologous and hence give rise to isomorphic lattice vertex operator algebras [44, Chapter 5].

Proposition 3.8 *Let $\widetilde{\xi}$ be a choice of representative of ξ.*

(1) *The lattice Fock space $\mathbb{F}_\Lambda = \bigoplus_{\alpha \in \Lambda} \mathcal{F}_\alpha$ admits the structure of a vertex operator algebra, uniquely characterised by the choice of field map*

$$
Y|_{\mathcal{F}_\mu \otimes \mathcal{F}_\nu} = \varepsilon(\mu, \nu) I_{\mu,\nu}, \quad \mu, \nu \in \Lambda,
\tag{3.27}
$$

(note that on $\mathcal{F}_0$ this specialises to the field map of the Heisenberg vertex operator algebra) and choice of conformal vector

$$\omega_{\widetilde{\xi}} = \frac{1}{2} \sum_i \alpha^i_{-1} \alpha^{i*}_{-1} |0\rangle + \widetilde{\xi}_{-2}|0\rangle, \quad \widetilde{\xi} \in \Lambda^*, \tag{3.28}$$

where $\{\alpha^i\}_{i=1}^{\dim \mathfrak{h}}$ and $\{\alpha^{j*}\}_{j=1}^{\dim \mathfrak{h}}$ are any dual choices of basis of $\mathfrak{h}_{\mathbb{C}}$. We denote this vertex operator algebra by $V(\widetilde{\xi}, \Lambda)$. The central charge determined by $\omega_{\widetilde{\xi}}$ is

$$c_{\widetilde{\xi}} = \dim \mathfrak{h} - 12\langle \widetilde{\xi}, \widetilde{\xi} \rangle. \tag{3.29}$$

(2) *The zero modes of $Y(\alpha_{-1}|0\rangle, z)$, $\alpha \in \Lambda$ furnish $V(\widetilde{\xi}, \Lambda)$ with a Λ-grading.*
(3) *For any $\rho \in \Lambda^*/\Lambda$, the lattice Fock space $\mathbb{F}_\rho$, as defined in (3.18), equipped with the field map*

$$Y_{\mathbb{F}_\rho}\big|_{\mathcal{F}_\mu \otimes \mathcal{F}_{s(\rho)+\nu}} = \varepsilon(\mu, \nu) I_{\mu, s(\rho)+\nu}, \quad \mu, \nu \in \Lambda, \tag{3.30}$$

is a simple discreetly strongly Λ^*-graded generalised $V(\widetilde{\xi}, \Lambda)$ module, with semisimple L_0 action. The conformal weight of the highest weight vector $|\mu\rangle$ of a Fock space direct summand $\mathcal{F}_\mu$, $\mu \in \rho$ is

$$h_\mu = \frac{1}{2}\langle \mu, \mu - 2\widetilde{\xi} \rangle. \tag{3.31}$$

(4) *Every lattice Fock space $\mathbb{F}_\rho$, $\rho \in \Lambda^*/\Lambda$, is graded C_1-cofinite as a module over the Heisenberg vertex operator algebra $V(\widetilde{\xi})$.*

Proof (1) The existence of the vertex algebra structure on $\mathbb{F}_\Lambda$ was shown in [45, Theorem 3.6, Remark 3.7]. Note that this vertex algebra structure is also unique in the sense that all choices of normalised 2-cocycles are cohomologous and yield isomorphic vertex algebras. The restriction of $\widetilde{\xi}$ to Λ^* is equivalent to requiring that the grading of $\mathbb{F}_\Lambda$ be integral.

(2) This follows by construction.

(3) That the lattice Fock space $\mathbb{F}_\rho$ is a module follows from [45, Theorem 3.6]. Each doubly homogeneous space of $\mathbb{F}_\rho$ is just an L_0 eigenspace of one of the underlying Fock spaces $\mathcal{F}_\mu$, $\mu \in \rho$. Since these eigenspaces are all finite dimensional, the doubly homogeneous spaces are too. Formula (3.31) follows by direct computation and implies that all conformal weights are real and that the Fock spaces $\mathcal{F}_\mu$ are discreetly strongly graded. Hence the $\mathbb{F}_\rho$ are also discreetly strongly graded.

(4) The Λ^* homogeneous spaces of lattice Fock spaces are just the ordinary Fock spaces. These are all C_1-cofinite over $V(\widetilde{\xi})$ because the C_1 subspace has codimension 1. $\square$

Remark Note that the conformal structure of $V(\widetilde{\xi}, \Lambda)$ genuinely depends on the choice of vector $\widetilde{\xi} \in \Lambda^*$ rather than its coset $\xi = \widetilde{\xi} + \Lambda \in \Lambda^*/\Lambda$. For example, shifting

$\widetilde{\xi}$ by some $\alpha \in \Lambda$ will generally give a different central charge. It will also shift the conformal weight of any lattice module by some integer. However, the ribbon Grothendieck-Verdier structure of the module category to be defined below will only depend on the coset ξ (specifically, the dualising object and the twist depend on ξ, the associativity and braiding isomorphisms do not), rather than a choice of representative of this coset.

Definition 3.9 For any set of bosonic lattice data $\Psi = (\mathfrak{h}, \langle -, - \rangle, \Lambda, \xi)$ and a representative $\widetilde{\xi} \in \xi$, let $\mathsf{VM}(\Psi)$ be the full subcategory of generalised Λ^*-graded $\mathsf{V}(\widetilde{\xi}, \Lambda)$-modules whose objects are finitely generated, with $\widehat{\mathfrak{h}}_+$ acting locally nilpotently and $\mathfrak{h}$ acting semisimply with real eigenvalues.

Proposition 3.10 *The category $\mathsf{VM}(\Psi)$ is linear, abelian and semisimple (in particular the L_0 operator never has Jordan blocks). The lattice Fock spaces $\mathbb{F}_\mu$, $\mu \in \Lambda^*/\Lambda$ form a complete set of mutually inequivalent representatives of isomorphism classes of simple objects. Further, the category $\mathsf{VM}(\Psi)$ satisfies all of the conditions of Corollary 2.14, and therefore admits the braided monoidal structure of Proposition 2.10 and the ribbon Grothendieck-Verdier structure of Theorem 2.12.*

Proof The category $\mathsf{VM}(\Psi)$ is clearly linear and abelian by construction. We first show semisimplicity. Let $M \in \mathsf{VM}(\Psi)$ be indecomposable. Since $\mathfrak{h}$ is required to act semisimply and real, M must be $\mathfrak{h}$ graded. Further, in order for M to be a $\mathsf{V}(\widetilde{\xi}, \Lambda)$-module all fields in $\mathsf{V}(\widetilde{\xi}, \Lambda)$ must have integral exponents when expanded on M. Hence M is Λ^* graded and its Λ^* homogeneous spaces are modules over $\mathsf{V}(\widetilde{\xi})$ by restriction. Since $\mathsf{V}(\widetilde{\xi}, \Lambda)$ is Λ-graded, homogenous spaces of M corresponding to elements in Λ^* which are in different cosets of Λ cannot mix under the action of $\mathsf{V}(\widetilde{\xi}, \Lambda)$. Since M is indecomposable the weights of non-zero Λ^* homogeneous spaces of M must all lie in the same Λ coset. Local nilpotence of $\widehat{\mathfrak{h}}_+$ and semisimple action of $\mathfrak{h}$ then implies by an algebraic version of the Stone-von Neumann theorem [46, Prop 3.6] that each Λ^* homogeneous space of M is a semisimple $\mathsf{V}(\widetilde{\xi})$ module and a possibly infinite direct sum of Fock spaces. So assume there exists a direct sum decomposition $M^{(\mu)} = A \oplus B$ of the homogeneous space of weight $\mu \in \Lambda^*$ into non-zero but not necessarily simple $\mathsf{V}(\widetilde{\xi})$ modules A, B. Then the $\mathsf{V}(\widetilde{\xi}, \Lambda)$ submodules of M generated by A and B would intersect trivially and hence provide a direct sum decomposition of M, contradicting indecomposability. Thus every non-trivial homogeneous space of M is isomorphic to a single Fock space of the same weight. The module M is therefore isomorphic to a lattice Fock space and hence simple. Further, lattice Fock spaces form a complete set of mutually inequivalent simple objects. Here we implicitly use the uniqueness of module structures on lattice Fock spaces which was shown in [45, Proposition 4.2].

Since the contragredient of a lattice Fock space is again a lattice Fock space (though generally of different weight), the first three conditions of Corollary 2.14 clearly hold and so we only need to verify the fourth. Consider two lattice Fock spaces $\mathbb{F}_\mu, \mathbb{F}_\nu$, $\mu, \nu \in \Lambda^*/\Lambda$ and let M be a finitely generated lower bounded submodule of $\mathrm{COMP}(\mathbb{F}_\mu, \mathbb{F}_\nu)$. We need to verify that M is an object in $\mathsf{VM}(\Psi)$. Since $\mathsf{VM}(\Psi)$ is closed under contragredients, this is equivalent to M' being in $\mathsf{VM}(\Psi)$. By [1, Part IV, Proposition 5.24], $M \subset \mathrm{COMP}(\mathbb{F}_\mu, \mathbb{F}_\nu)$ implies the existence of a surjective

intertwining operator $\mathcal{Y}$ of type $\binom{M'}{\mathbb{F}_\mu, \mathbb{F}_\nu}$, we show that the image of any such intertwining operator must be an object in $\mathsf{VM}(\Psi)$. By assumption M' is finitely generated and hence we need only verify that $\mathfrak{h}$ acts semisimply and $\widehat{\mathfrak{h}}_+$ locally nilpotently. Assume $m_\mu \in \mathbb{F}_\mu, m_\nu \in \mathbb{F}_\nu$, the Jacobi identity for intertwining operators implies for any $\alpha \in \mathfrak{h}_\mathbb{C}$ and $n \geq 1$

$$\alpha_0 \mathcal{Y}(m_\mu, x)m_\nu = \mathcal{Y}(m_\mu, x)\alpha_0 m_\nu + \mathcal{Y}(\alpha_0 m_\mu, x)m\nu,$$

$$(\alpha_n - x\alpha_{n-1})\mathcal{Y}(m_\mu, x)m_\nu = \mathcal{Y}(m_\mu, x)(\alpha_n - x\alpha_{n-1})m_\nu$$
$$+ \sum_{t=0}^{n} \binom{t-n}{t}(-1)^t x^{n-t-1}\mathcal{Y}(\alpha_{t+1}m_\mu, x)m_\nu, \quad n \geq 1.$$

The first equality shows that the semisimplicity of α_0 on m_μ and m_ν implies the semisimplicity of α_0 on the image of $\mathcal{Y}$. The second equality shows that the nilpotency of $\widehat{\mathfrak{h}}_+$ on m_μ and m_ν implies the local nilpotency of $\widehat{\mathfrak{h}}_+$ on the image of $\mathcal{Y}$. Thus all conditions of Corollary 2.14 are satisfied, hence intertwining operators equip $\mathsf{VM}(\Psi)$ with the braided monoidal structures of Proposition 2.10. Hence Theorem 2.12 applies and thus $\mathsf{VM}(\Psi)$ admits a ribbon Grothendieck-Verdier structure. □

Recall again that we are not assuming the lattice Λ to be non-zero and so the above considerations capture the ordinary free boson without a lattice by setting $\Lambda = \{0\}$. Henceforth all references to $\mathsf{VM}(\Psi)$ are to be understood as including the braided monoidal and ribbon Grothendieck-Verdier structures provided in Proposition 3.10.

Proposition 3.11 *Let Ψ be a set of bosonic lattice data and let (Ω, F) be the abelian 3-cocycle constructed from Ψ by the formulae (3.12). Since $\mathsf{VM}(\Psi)$ is semisimple its structure isomorphisms are uniquely determined by their values on simple modules. Consider the lattice Fock spaces $\mathbb{F}_\mu, \mathbb{F}_\nu, \mathbb{F}_\rho, \mu, \nu, \rho \in \Lambda^*/\Lambda$.*

(1) *For any two lattice Fock spaces $\mathbb{F}_\mu, \mathbb{F}_\nu$ a choice of fusion product is given by*

$$\mathbb{F}_\mu \boxtimes \mathbb{F}_\nu = \mathbb{F}_{\mu+\nu}, \tag{3.32}$$

with corresponding universal intertwining operator

$$\mathcal{Y}_{\mathbb{F}_\mu, \mathbb{F}_\nu}\big|_{\mathcal{F}_{s(\mu)+\alpha_1} \otimes \mathcal{F}_{s(\nu)+\alpha_2}}$$
$$= (-1)^{\langle s(\mu), \alpha_2 \rangle} \varepsilon(\alpha_1, \alpha_2) \varepsilon(\alpha_1 + \alpha_2, k(\mu, \nu)) I_{s(\mu)+\alpha_1, s(\nu)+\alpha_2}, \quad \alpha_1, \alpha_2 \in \Lambda. \tag{3.33}$$

(2) *The braiding isomorphism $c_{\mu,\nu} : \mathbb{F}_\mu \boxtimes \mathbb{F}_\nu \to \mathbb{F}_\nu \boxtimes \mathbb{F}_\mu$ is given by*

$$c_{\mu,\nu} = e^{i\pi \langle s(\mu), s(\nu) \rangle} \mathrm{id}_{\mathbb{F}_{\mu+\nu}} = \Omega(\mu, \nu) \mathrm{id}_{\mathbb{F}_{\mu+\nu}}. \tag{3.34}$$

(3) *The associativity isomorphism $A_{\mu,\nu,\rho} : \mathbb{F}_\mu \boxtimes \left(\mathbb{F}_\nu \boxtimes \mathbb{F}_\rho \right) \rightarrow \left(\mathbb{F}_\mu \boxtimes \mathbb{F}_\nu \right) \boxtimes \mathbb{F}_\rho$ is given by*

$$A_{\mu,\nu,\rho} = (-1)^{\langle s(\mu), k(\nu,\rho) \rangle} \frac{\varepsilon(k(\mu,\nu), k(\mu+\nu,\rho))}{\varepsilon(k(\nu,\rho), k(\mu,\nu+\rho))} \, \mathrm{id}_{\mathbb{F}_{\mu+\nu+\rho}} = F(\mu,\nu,\rho) \, \mathrm{id}_{\mathbb{F}_{\mu+\nu+\rho}}. \tag{3.35}$$

(4) *The contragredient of a lattice Fock space is*

$$\mathbb{F}'_\rho = \mathbb{F}_{2\xi-\rho}, \qquad \rho \in \Lambda^*/\Lambda, \tag{3.36}$$

and hence the dualising object is $\mathbb{F}_{2\xi+\Lambda}$.

(5) *The twist isomorphism is given by*

$$\theta_{\mathbb{F}_\rho} = e^{\pi \mathrm{i} \langle s(\rho), s(\rho) - 2\widetilde{\xi} \rangle} \, \mathrm{id}_{\mathbb{F}_\rho}, \qquad \rho \in \Lambda^*/\Lambda. \tag{3.37}$$

Note that Λ being even guarantees that the above twist formula is independent of the choice of section s.

Proof Parts (2)–(5) follow by simple computations from the explicit formulae for intertwining operators in Part (1).

(1) The lattice intertwining operator formulae (3.33) were proved in [33] in the context of full rank even lattices, however, the arguments showing that these formulae satisfy the intertwining operator axioms, such as the Jacobi identity, do not depend on the lattice being full rank. See also [47] for detailed descriptions on how to compute with Heisenberg intertwining operators.

(2) Since the lattice Fock spaces are simple modules, the braiding isomorphism is determined by comparing the leading terms of $\mathcal{Y}_{\mathbb{F}_\mu, \mathbb{F}_\nu}(|s(\mu) + \alpha_1\rangle, z)|s(\nu) + \alpha_2\rangle$ and $e^{zL_{-1}} \mathcal{Y}_{\mathbb{F}_\mu, \mathbb{F}_\nu}\left(|s(\nu) + \alpha_2\rangle, e^{\mathrm{i}\pi} z\right)|s(\mu) + \alpha_1\rangle$, where $\mu, \nu \in \Lambda^*/\Lambda$ and $\alpha_1, \alpha_2 \in \Lambda$. These are

$$\begin{aligned}
&e^{zL_{-1}} \mathcal{Y}_{\mathbb{F}_\nu, \mathbb{F}_\mu}\left(|s(\nu) + \alpha_2\rangle, e^{\mathrm{i}\pi} z\right)|s(\mu) + \alpha_1\rangle \\
&= (e^{\mathrm{i}\pi} z)^{\langle s(\mu) + \alpha_1, s(\nu) + \alpha_2 \rangle} \varepsilon(\alpha_2, \alpha_1)(-1)^{\langle s(\nu), \alpha_1 \rangle} (|s(\mu) + \alpha_1 + s(\nu) + \alpha_2\rangle + \mathcal{O}(z)), \\
&\mathcal{Y}_{\mathbb{F}_\mu, \mathbb{F}_\nu}(|s(\mu) + \alpha_1\rangle, z)|s(\nu) + \alpha_2\rangle \\
&= z^{\langle s(\mu) + \alpha_1, s(\nu) + \alpha_2 \rangle} \varepsilon(\alpha_1, \alpha_2)(-1)^{\langle s(\mu), \alpha_2 \rangle} (|s(\mu) + \alpha_1 + s(\nu) + \alpha_2\rangle + \mathcal{O}(z)).
\end{aligned} \tag{3.38}$$

Comparing the leading terms we obtain

$$\frac{\varepsilon(\alpha_2, \alpha_1)}{\varepsilon(\alpha_1, \alpha_2)} e^{\mathrm{i}\pi \langle s(\mu) + \alpha_1, s(\nu) + \alpha_2 \rangle} (-1)^{\langle s(\nu), \alpha_1 \rangle} (-1)^{-\langle s(\mu), \alpha_2 \rangle} = e^{\mathrm{i}\pi \langle s(\mu), s(\nu) \rangle} \tag{3.39}$$

and hence $c_{\mu,\nu} = e^{\mathrm{i}\pi \langle s(\mu), s(\nu) \rangle} \, \mathrm{id}_{\mathbb{F}_{\mu+\nu}}$.

(3) As with the braiding isomorphisms, since the lattice Fock spaces are simple modules, thus the associativity isomorphisms are determined by comparing the leading terms. Let $\mu, \nu, \rho \in \Lambda^*/\Lambda$ and $\alpha_1, \alpha_2, \alpha_3 \in \Lambda$ and consider

$$
\mathcal{Y}_{\mathbb{F}_\mu, \mathbb{F}_{\nu+\rho}}(|s(\mu)+\alpha_1\rangle, x_1) \mathcal{Y}_{\mathbb{F}_\nu, \mathbb{F}_\rho}(|s(\nu)+\alpha_2\rangle, x_2)|s(\rho)+\alpha_3\rangle
$$
$$
= (-1)^{\langle s(\mu), k(\nu,\rho)+\alpha_2+\alpha_3\rangle} \varepsilon(\alpha_1, \alpha_2+\alpha_3+k(\nu,\rho)) \varepsilon(\alpha_1+\alpha_2+\alpha_3+k(\nu,\rho), k(\mu,\nu+\rho))
$$
$$
(-1)^{\langle s(\nu), \alpha_3\rangle} \varepsilon(\alpha_2, \alpha_3) \varepsilon(\alpha_2+\alpha_3, k(\nu,\rho))
$$
$$
(x_1-x_2)^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2\rangle} x_1^{\langle s(\mu)+\alpha_1, s(\rho)+\alpha_3\rangle}
$$
$$
x_2^{\langle s(\nu)+\alpha_2, s(\rho)+\alpha_3\rangle} (|s(\mu)+s(\nu)+s(\rho)+\alpha_1+\alpha_2+\alpha_3\rangle + \mathcal{O}(z))
$$
$$
\mathcal{Y}_{\mathbb{F}_{\mu+\nu}, \mathbb{F}_\rho}\left(\mathcal{Y}_{\mathbb{F}_\mu, \mathbb{F}_\nu}(|s(\mu)+\alpha_1\rangle, x_1-x_2)|s(\nu)+\alpha_2\rangle, x_2\right)|s(\rho)+\alpha_3\rangle
$$
$$
= (-1)^{\langle s(\mu), \alpha_2\rangle} \varepsilon(\alpha_1, \alpha_2) \varepsilon(\alpha_1+\alpha_2, k(\mu,\nu))
$$
$$
(-1)^{\langle s(\mu+\nu), \alpha_3\rangle} \varepsilon(\alpha_1+\alpha_2+k(\mu,\nu), \alpha_3) \varepsilon(\alpha_1+\alpha_2+\alpha_3+k(\mu,\nu), k(\mu+\nu,\rho))
$$
$$
(x_1-x_2)^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2\rangle} x_2^{\langle s(\mu)+s(\nu)+\alpha_1+\alpha_2, s(\rho)+\alpha_3\rangle}
$$
$$
\left(\frac{x_1}{x_2}\right)^{\langle s(\mu)+\alpha_1, s(\rho)+\alpha_3\rangle} (|s(\mu)+s(\nu)+s(\rho)+\alpha_1+\alpha_2+\alpha_3\rangle + \mathcal{O}(z)). \tag{3.40}
$$

The ratio of the x_1 dependent factors is

$$
\frac{(x_1-x_2)^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2\rangle} x_2^{\langle s(\mu)+s(\nu)+\alpha_1+\alpha_2, s(\rho)+\alpha_3\rangle} \left(\frac{x_1}{x_2}\right)^{\langle s(\mu)+\alpha_1, s(\rho)+\alpha_3\rangle}}{(x_1-x_2)^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2\rangle} x_1^{\langle s(\mu)+\alpha_1, s(\rho)+\alpha_3\rangle} x_2^{\langle s(\nu)+\alpha_2, s(\rho)+\alpha_3\rangle}} = 1 \tag{3.41}
$$

and the associativity isomorphism is scalar multiplication by

$$
\frac{(-1)^{\langle s(\mu), \alpha_2\rangle} \varepsilon(\alpha_1, \alpha_2) \varepsilon(\alpha_1+\alpha_2, k(\mu,\nu)) (-1)^{\langle s(\mu+\nu), \alpha_3\rangle} \varepsilon(\alpha_1+\alpha_2+k(\mu,\nu), \alpha_3) \varepsilon(\alpha_1+\alpha_2+\alpha_3+k(\mu,\nu), k(\mu+\nu,\rho))}{(-1)^{\langle s(\mu), k(\nu,\rho)+\alpha_2+\alpha_3\rangle} \varepsilon(\alpha_1, \alpha_2+\alpha_3+k(\nu,\rho)) \varepsilon(\alpha_1+\alpha_2+\alpha_3+k(\nu,\rho), k(\mu,\nu+\rho)) (-1)^{\langle s(\nu), \alpha_3\rangle} \varepsilon(\alpha_2, \alpha_3) \varepsilon(\alpha_2+\alpha_3, k(\nu,\rho))}
$$
$$
= (-1)^{\langle s(\mu), k(\nu,\rho)\rangle} \frac{\varepsilon(k(\mu,\nu), k(\mu+\nu,\rho))}{\varepsilon(k(\nu,\rho), k(\mu,\nu+\rho))} \tag{3.42}
$$

(4) The Heisenberg weight of $\mathbb{F}'_\rho$ is determined by computing the opposed field map of $\alpha_{-1}|0\rangle, \alpha \in \mathfrak{h}_\mathbb{C}$. This is given by

$$
Y(\alpha_{-1}|0\rangle, z)^{\mathrm{opp}} = Y(e^{zL_1}(-z^{-2})^{L_0}\alpha_{-1}|0\rangle, z^{-1})
$$
$$
= -z^{-2} Y(\alpha_{-1}|0\rangle, z^{-1}) + z^{-1} 2\langle \widetilde{\xi}, \alpha\rangle Y(|0\rangle, z^{-1})
$$
$$
= -z^{-2} Y(\alpha_{-1}|0\rangle, z^{-1}) + z^{-1} 2\langle \widetilde{\xi}, \alpha\rangle \mathrm{id}, \tag{3.43}
$$

for any $\alpha \in \mathfrak{h}_\mathbb{C}$. This implies that the Heisenberg weight of $\mathbb{F}'_\rho$ is $2\widetilde{\xi} - \rho$.

(5) The formula for the twist isomorphism follows immediately from the conformal weight of Fock space highest weight vectors.

□

We prepare some notation in order to use Lemma 2.15 to show that $\mathsf{Vect}(\Psi)$ and $\mathsf{VM}(\Psi)$ are equivalent as ribbon Grothendieck-Verdier categories. To any object in

$\mathsf{Vect}(\Psi)$ we can associate an object in $\mathsf{VM}(\Psi)$ by the following induction construction. Let $M = \bigoplus_{\alpha \in \Lambda^*/\Lambda} M_\alpha$ be a decomposition into homogeneous spaces and consider the vector space $M \otimes \mathbb{C}[\Lambda]$ and endow it with the structure of an $\widehat{\mathfrak{h}}_{\geq} = \widehat{\mathfrak{h}}_0 \oplus \widehat{\mathfrak{h}}_+$ module by defining

$$\widehat{\mathfrak{h}}_+ \cdot M \otimes \mathbb{C}[\Lambda] = 0, \quad \alpha \cdot m_\rho \otimes \beta = \langle \alpha, s(\rho) + \beta \rangle \, \mathrm{id}, \quad \alpha \in \mathfrak{h}_{\mathbb{C}}, \ \rho \in \Lambda^*/\Lambda, \ \beta \in \Lambda. \tag{3.44}$$

Further induce $M \otimes \mathbb{C}[\Lambda]$ to a module over $\widehat{\mathfrak{h}}$ by defining

$$\mathbb{F}[M] = \mathrm{Ind}_{\widehat{\mathfrak{h}}_{\geq}}^{\widehat{\mathfrak{h}}} M \otimes \mathbb{C}[\Lambda]. \tag{3.45}$$

Next, define the action of the shift operators e^γ, $\gamma \in \Lambda$ on $M \otimes \mathbb{C}[\Lambda]$ to be

$$\mathrm{e}^\gamma m \otimes \mathrm{e}^\delta = m \otimes \mathrm{e}^{\gamma+\delta}, \quad m \in M, \ \gamma, \delta \in \Lambda \tag{3.46}$$

and extend to all of $\mathbb{F}[M]$ to obtain a well defined action of the obvious analogue of untwisted vertex operators (3.25) (with the first of the two indices parametrising weights in Λ) and hence also the field map (3.30) by defining

$$Y_{\mathbb{F}[M]}(u|\alpha_1\rangle, z)v \cdot m \otimes \mathrm{e}^{\alpha_2} = \varepsilon(\alpha_1, \alpha_2)(-1)^{\langle s(\Lambda), \alpha_2 \rangle} z^{\langle \alpha_1, s(\mu)+\alpha_2 \rangle} \mathrm{e}^{\alpha_1} U(\alpha_1, u, z) \, v \cdot m \otimes \mathrm{e}^{\alpha_2}, \tag{3.47}$$

for $\alpha_1, \alpha_2 \in \Lambda$, $\mu \in \Lambda^*/\Lambda$, $m \in M_\mu$, $u, v \in \mathsf{U}(\widehat{\mathfrak{h}}_-)$ and where $U(\alpha_1, u, z)$ is the Heisenberg algebra valued series (3.24). Thus $\mathbb{F}[M]$ has the structure of a $\mathsf{V}(\widetilde{\xi}, \Lambda)$ module, with decomposition into lattice Fock spaces given by

$$\mathbb{F}[M] \cong \bigoplus_{\rho \in \Lambda^*/\Lambda} \dim(M_\rho) \mathbb{F}_\rho. \tag{3.48}$$

To define intertwining operators for the modules constructed above, we shall need the following auxiliary linear maps which for any $M, N, P \in \mathsf{Vect}(\Psi)$ and $f \in \mathrm{Hom}(M \otimes N, P)$ are defined to be

$$\begin{aligned} f_m : \mathbb{F}[N] &\to \mathbb{F}[P], \qquad m \in M, \ n \in N, \ u \in \mathsf{U}(\widehat{\mathfrak{h}}_-), \ \gamma \in \Lambda. \\ u \cdot n \otimes \mathrm{e}^\gamma &\mapsto u \cdot f(m \otimes n) \otimes \mathrm{e}^\gamma \end{aligned} \tag{3.49}$$

Theorem 3.12 *Let Ψ be a set of bosonic lattice data, $\mathsf{Vect}(\Psi)$ be the associated ribbon Grothendieck-Verdier category from the previous section and $\mathsf{VM}(\Psi)$ the module category of the lattice vertex operator algebra described above. Further, let $G : \mathsf{Vect}(\Psi) \to \mathsf{VM}(\Psi)$ be the functor which assigns to any $M \in \mathsf{Vect}(\Psi)$ the object $G(M) = \mathbb{F}[M]$ from (3.45) with the obvious extension to morphisms. Consider the following maps.*

- *Let $\varphi_0 : \mathsf{V}(\widetilde{\xi}, \Lambda) \mapsto G(\mathbb{C}_0)$ be the module map uniquely characterised by $\varphi_0(|0\rangle) = 1_0 \otimes e^0$, where $1_0 \in \mathbb{C}_0$.*

- *For $M, N, P \in \mathsf{Vect}(\Psi)$, $f \in \mathrm{Hom}\,(M \otimes N, P)$, $\mu, \nu \in \Lambda^*/\Lambda$, $m \in M_\mu$, $n \in N_\nu$, $\alpha_1, \alpha_2 \in \Lambda$ and $u, v \in \mathsf{U}(\widehat{\mathfrak{h}}_-)$ define G^T by*

$$
\begin{aligned}
&G_f^T\big(u \cdot m \otimes e^{\alpha_1}, z\big)v \cdot n \otimes e^{\alpha_2} \\
&= (-1)^{\langle s(\mu), \alpha_2 \rangle} \varepsilon(\alpha_1, \alpha_2)\varepsilon(\alpha_1 + \alpha_2, k(\mu, \nu))z^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2 \rangle} f_m e^{\alpha_1} U(u, s(\mu) \\
&\quad +\alpha_1, z)v \cdot n \otimes e^{\alpha_2}.
\end{aligned} \tag{3.50}
$$

Then φ_0 and G^T satisfy the conditions of Lemma 2.15 and hence endow G with the structure of a braided monoidal functor. The functor G with this choice of monoidal structure is an equivalence of ribbon Grothendieck-Verdier categories. In particular, for $\xi = 0$, the functor G is a ribbon equivalence.

The equivalence of $\mathsf{VM}(\Psi)$ and $\mathsf{Vect}(\Psi)$ as braided tensor categories is well known [33] in the special case of positive definite even full rank lattices. Here we use the opportunity to illustrate the application of Lemma 2.15 and to show the equivalence of the ribbon Grothendieck-Verdier structures as well.

Proof We prove the theorem by showing that φ_0 and the family of linear maps G^T of (3.50) satisfy the conditions of Lemma 2.15.(2) and Corollary 2.16.(2). We first show the functoriality of G^T. For any $M, M', N, N', P, P' \in \mathsf{Vect}(\Psi)$, $\mu, \nu \in \Lambda^*/\Lambda$, $m \in M'_\mu, n \in N'_\nu, \alpha_1, \alpha_2 \in \Lambda$ and $u, v \in \mathsf{U}(\widehat{\mathfrak{h}}_-)$ consider

$$
\begin{aligned}
&G_{k \circ f \circ (g \otimes h)}^T\big(u \cdot m \otimes e^{\alpha_1}; z\big)v \cdot n \otimes e^{\alpha_2} \\
&= z^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2 \rangle}(k \circ f \circ (g \otimes h))_m e^{\alpha_1} U(s(\mu)+\alpha_1, u, z))\, v \cdot n \otimes e^{\alpha_2} \\
&= G(k)z^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2 \rangle} f_{g(m)} e^{\alpha_1} U(s(\mu)+\alpha_1, u, z)v \cdot h(n) \otimes e^{\alpha_2} \\
&= G(k) \circ G_f^T\big(G(g)u \cdot m \otimes e^{\alpha_1}, z\big)G(h)v \cdot n \otimes e^{\alpha_2},
\end{aligned} \tag{3.51}
$$

where the second and third equalities follow from the definition of the f_m notation in (3.49). Thus G^T is functorial.

Next we show the unitality of G^T. For any $N \in \mathsf{Vect}(\Psi)$, $\nu \in \Lambda^*/\Lambda$, $n \in N_\nu$, $\alpha_1, \alpha_2 \in \Lambda$ and $u, v \in \mathsf{U}(\widehat{\mathfrak{h}}_-)$ consider

$$
\begin{aligned}
&G_{l_N}^T(\varphi_0(u|\alpha_1\rangle), z)v \cdot n \otimes e^{\alpha_2} \\
&= G_{l_N}^T\big(u \cdot 1_0 \otimes e^{\alpha_1}, z\big)v \cdot n \otimes e^{\alpha_2} \\
&= z^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2 \rangle}(l_N)_{1_0} e^{\alpha_1} U(s(\mu)+\alpha_1, u, z)\, v \cdot n \otimes e^{\alpha_2} \\
&= z^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2 \rangle} e^{\alpha_1} U(s(\mu)+\alpha_1, u, z)\, v \cdot n \otimes e^{\alpha_2} \\
&= Y_{G(N)}(u|\alpha_1\rangle, z)v \cdot n \otimes e^{\alpha_2},
\end{aligned} \tag{3.52}
$$

where in the third identity we have used that $l_N(1_0 \otimes n) = n$. Thus G^T is unital.

Next we show the skew symmetry of G^T. For any $M, N \in \mathsf{Vect}(\Psi)$, $\mu, \nu \in \Lambda^*/\Lambda$, $m \in M_\mu, n \in N_\nu, \alpha_1, \alpha_2 \in \Lambda$ and $u, v \in \mathsf{U}(\widehat{\mathfrak{h}}_-)$ consider

$$G^T_{c_{N,M}}\big(v \cdot n \otimes e^{\alpha_2}, z\big)u \cdot m \otimes e^{\alpha_1}$$

$$= z^{\langle s(v)+\alpha_2, s(\mu)+\alpha_1\rangle}\big(c_{N,M}\big)_n e^{\alpha_2} U(s(v)+\alpha_2, u, z)\, v \cdot n \otimes e^{\alpha_1}$$

$$= e^{i\pi\langle s(v), s(\mu)\rangle} z^{\langle s(v)+\alpha_2, s(\mu)+\alpha_1\rangle}\big(P_{N\otimes M}\big)_n e^{\alpha_2} U(s(v)+\alpha_2, u, z)\, v \cdot n \otimes e^{\alpha_1}$$

$$= e^{i\pi\langle s(v), s(\mu)\rangle} G^T_{P_{N\otimes M}}\big(v \cdot n \otimes e^{\alpha_2}, z\big)u \cdot m \otimes e^{\alpha_1}$$

$$= e^{zL_{-1}} G^T_{\mathrm{id}_{M\otimes N}}\Big(u \cdot m \otimes e^{\alpha_1}, e^{i\pi}z\Big) v \cdot n \otimes e^{\alpha_2}, \tag{3.53}$$

where $P_{N\otimes M} : N \otimes M \to M \otimes N$ is the standard tensor flip of (graded) vector spaces and where in the fourth identity we have used the well known behaviour of untwisted vertex operators (3.25) with respect to L_{-1}. Thus G^T is skew symmetric.

Next we show G^T is associative. For any M, N, $P \in \mathsf{Vect}(\Psi)$, $m \in M$, $n \in N$, $p \in P$, $\mu, v, \rho \in \Lambda^*/\Lambda$, $\alpha_1, \alpha_2, \alpha_3 \in \Lambda$, $u, v, w \in U(\widehat{\mathfrak{h}}_-)$ and $x_1, x_2 \in \mathbb{C}$ such that $|x_1| > |x_2| > 0$ and $|x_2| > |x_1 - x_2| > 0$, consider

$$G^T_{\alpha_{M,N,P}}\big(u \cdot m \otimes e^{\alpha_1}, x_1\big) G^T_{\mathrm{id}_{N\otimes P}}\big(v \cdot n \otimes e^{\alpha_2}, x_2\big) w \cdot p \otimes e^{\alpha_3}$$

$$= x_1^{\langle s(\mu)+\alpha_1, s(v+\rho)-k(v,\rho)+\alpha_2+\alpha_3\rangle} x_2^{\langle s(v)+\alpha_2, s(\rho)+\alpha_3\rangle}\big(\alpha_{M,N,P}\big)_m e^{\alpha_1} U(s(v)$$
$$+ \alpha_1, u, x_1)(\mathrm{id}_{N\otimes P})_n e^{\alpha_2} U(s(v)+\alpha_2, v, x_2)\, w \cdot p \otimes e^{\alpha_3}$$

$$= (-1)^{\langle s(\mu), k(v,\rho)\rangle} \frac{\varepsilon(k(\mu, v), k(\mu+v, \rho))}{\varepsilon(k(v, \rho), k(\mu, v+\rho))} x_1^{\langle s(\mu)+\alpha_1, s(v+\rho)-k(v,\rho)+\alpha_2+\alpha_3\rangle}$$
$$x_2^{\langle s(v)+\alpha_2, s(\rho)+\alpha_3\rangle}\big(\mathrm{id}_{M\otimes(N\otimes P)}\big)_m e^{\alpha_1} U(s(\mu)+\alpha_1, u, x_1)$$
$$\cdot (\mathrm{id}_{N\otimes P})_n e^{\alpha_2} U(s(v)+\alpha_2, v, x_2)\, w \cdot p \otimes e^{\alpha_3}$$

$$= G^T_{\mathrm{id}_{(M\otimes N)\otimes P}}\Big(G^T_{\mathrm{id}_{M\otimes N}}\big(u \cdot m \otimes e^{\alpha_1}, x_1 - x_2\big) v \cdot n \otimes e^{\alpha_2}, x_2\Big) w \cdot p \otimes e^{\alpha_3}, \tag{3.54}$$

where in the third equality we have used the well known behaviour of untwisted vertex operators (3.25), see for example [33, Section 12] or [47]. Thus G^T is associative.

The intertwining operators $G^T_{\mathrm{id}_{M\otimes N}}(z)$ are surjective by construction for any M, $N \in \mathsf{Vect}(\Psi)$. Hence, by Corollary 2.16, the functor G with the monoidal structure constructed from G^T is a braided monoidal equivalence.

The equivalence of the ribbon Grothendieck-Verdier structures then follows from noting that the dualising objects are isomorphic, that is,

$$G(\mathbb{C}_{2\xi}) \cong \dim(\mathbb{C}_{2\xi})\, \mathbb{F}_{2\xi} = \mathbb{F}_{2\xi}, \tag{3.55}$$

and that the twists are equivalent, that is, for any $\mu \in \Lambda^*/\Lambda$

$$G(\theta_{\mathbb{C}_\mu}) = e^{\pi i\langle s(\mu), s(\mu)-2s(\xi)\rangle}\, \mathrm{id}_{G(\mathbb{C}_\mu)} = e^{\pi i\langle s(\mu), s(\mu)-2\widetilde{\xi}\rangle}\, \mathrm{id}_{G(\mathbb{C}_\mu)}$$
$$= e^{2\pi iL_0}|_{G(\mathbb{C}_\mu)} = \theta_{G(\mathbb{C}_\mu)}, \tag{3.56}$$

where in the second equality we have used that $s(\xi)$ and $\widetilde{\xi}$ differ at most by an element in Λ. $\qquad\square$

Example Recall the half rank lattice example at the end of Sect. 3.1. In the notation and conventions introduced there, we choose $\widetilde{\xi} = (1, 0)$ as a representative of ξ. Further let $\alpha = (1, 0) \in \mathbb{R}^2$ and $\beta = (0, 1) \in \mathbb{R}^2$, then vertex operator algebra structure on $\mathcal{F}_0$ is strongly generated by the fields corresponding to α, β, whose defining operator product expansions are

$$\alpha(z)\alpha(w) \sim 0 \sim \beta(z)\beta(w), \qquad \alpha(z)\beta(w) \sim \frac{1}{(z - w)^2}. \tag{3.57}$$

The choice of element $\widetilde{\xi}$ defines the conformal vector and central charge

$$\omega_{\widetilde{\xi}} = \alpha_{-1}\beta_{-1}|0\rangle + \widetilde{\xi}_{-2}|0\rangle, \qquad c_{\widetilde{\xi}} = 2. \tag{3.58}$$

Further, β generates the lattice Λ and the Fock spaces with weights in Λ have generating highest weight vectors of conformal weights

$$h_{n\beta} = \frac{1}{2}\langle(0, n), (-2, n)\rangle = -n. \tag{3.59}$$

3.4 Categories of quasi-Hopf algebra modules

We want to also give a realisation of the ribbon Grothendieck-Verdier category $\mathsf{Vect}(\Psi)$ as representations of a quasi-triangular quasi-Hopf algebra H_Λ. Strictly speaking, this is not possible, firstly because it necessitates the appearance of infinite sums in both the product and coproduct relations and secondly because it also requires the R-matrix and coassociator Φ to be infinite sums. However, to make contact to existing literature we will now proceed as in [48, 49] for the unrolled quantum group: To resolve the first issue, the authors of [48, 49] introduce additional elements and restrict themselves to a subcategory of modules where certain infinite sum relations between the actions of these elements hold. To resolve the second issue, they give formulae for R and Φ involving infinite sums, and argue that as exponential sums these converge on any finite-dimensional module. In sequel, "quasi-triangular quasi-Hopf algebra" shall be meant it in this generalised sense.

For any set of bosonic lattice data $\Psi = (\mathfrak{h}, \langle-, -\rangle, \Lambda, \xi)$, let $\mathsf{U}(\Lambda^\perp)$ denote the universal enveloping algebra (or symmetric algebra) of the complexification $\Lambda^\perp_{\mathbb{C}}$ of the vector space $\Lambda^\perp$ seen as an abelian Lie algebra and $\mathbb{C}[\Lambda^*/\Lambda^\perp]$ the group algebra of the abelian group $\Lambda^*/\Lambda^\perp$. These associative algebras both admit well known Hopf algebra structures by defining the elements of $\Lambda^\perp_{\mathbb{C}}$ to be primitive and those of $\Lambda^*/\Lambda^\perp$ to be group like, that is

$$
\begin{aligned}
\Delta(\mu) &= \mu \otimes 1 + 1 \otimes \mu, & \epsilon(\mu) &= 0, & s(\mu) &= -\mu, & \mu &\in \Lambda^\perp_{\mathbb{C}}, \\
\Delta(K_\nu) &= K_\nu \otimes K_\nu, & \epsilon(K_\nu) &= 1, & s(K_\nu) &= K_{-\nu} = K_\nu^{-1}, & \nu &\in \Lambda^*/\Lambda^\perp,
\end{aligned}
\tag{3.60}
$$

where K_ν is the basis element of $\mathbb{C}[\Lambda^*/\Lambda^\perp]$ corresponding to $\nu \in \Lambda^*/\Lambda^\perp$. We can define a Hopf algebra

$$H_\Lambda = \mathsf{U}\left(\Lambda^\perp\right) \otimes \mathbb{C}[\Lambda^*/(\Lambda^\perp + \Lambda)], \tag{3.61}$$

where the Hopf algebra structures are those inherited from the two tensor factors. Now we will, as discussed in the first paragraph, introduce a subcategory equivalent to $\mathsf{Vect}(\Psi)$ and then an alternative nontrivial coassociator together with an R-matrix on H_Λ: Every object M in $\mathsf{Vect}(\Psi)$ can be given the structure of an H_Λ module by defining the representation $\rho_M : H_\Lambda \to \mathrm{End}\, M$ on homogeneous spaces by

$$\rho_M(\mu)|_{M_\alpha} = \langle \mu, s(\alpha) \rangle \, \mathrm{id}_{M_\alpha}, \quad \rho_M(K_\nu)|_{M_\alpha} = \mathrm{e}^{2\pi i \langle \nu, s(\alpha) \rangle} \, \mathrm{id}_{M_\alpha},$$
$$\alpha \in \Lambda^*/\Lambda, \ \mu \in \Lambda^\perp_\mathbb{C}, \ \nu \in \Lambda^*/(\Lambda^\perp + \Lambda). \tag{3.62}$$

Note that the above formulae do not depend on the choice of section s. We can therefore interpret $\mathsf{Vect}(\Psi)$ as a category of representations of the group Λ^*. For $\nu \in \Lambda^\perp + \Lambda$ we have $\rho_M(K_\nu)|_{M_\alpha} = \mathrm{id}_{M_\alpha}$, hence the objects of $\mathsf{Vect}(\Psi)$ also can be interpreted as representations of the quotient group $\Lambda^*/\Lambda^\perp + \Lambda$. Since H_Λ is a Hopf algebra, there is of course a natural representation on tensor products of objects $M, N \in \mathsf{Vect}(\Psi)$ given by $\rho_{M \otimes N} = (\rho_M \otimes \rho_N) \circ \Delta$. This representation defines a Λ^*/Λ-grading on $M \otimes N$, which is the same as the Λ^*/Λ-grading that $M \otimes N$ inherits as a tensor product of graded vector spaces. Now that we have recast $\mathsf{Vect}(\Psi)$, as an abelian category, as a category of modules over H_Λ, it is interesting to see if we can capture the braided monoidal, Grothendieck-Verdier and ribbon structures of $\mathsf{Vect}(\Psi)$ in Hopf algebraic terms by specifying an R-matrix, coassociator and ribbon element. To do so, we recall the decomposition $\Lambda^* = \Lambda^\perp \oplus \Gamma$ of Λ^* in Lemma 3.2.(1). We define formal operators in terms of their action on the objects of $\mathsf{Vect}(\Psi)$ (though they could also be thought of as lying in suitable completions of tensor powers of H_Λ). Let $\{\mu_i\}_{i=1}^{\dim \Lambda^\perp}$ be an $\mathbb{R}$-basis of $\Lambda^\perp$ and let $\{\nu_j\}_{j=1}^{\mathrm{rk}\Lambda}$ be a $\mathbb{Z}$-basis of Γ. Since the real span of Λ^* is $\mathfrak{h}$, $\{\mu_i, \nu_j\}$ is an $\mathbb{R}$ basis of $\mathfrak{h}$. Hence there exists a dual basis $\{\mu^i, \nu^j\}$. Let $\log_s K_\nu$, $\nu \in \Lambda^*/\Lambda^\perp$ be the formal operator, depending on the section s, defined on the homogeneous spaces of an object $M \in \mathsf{Vect}(\Psi)$ to act as

$$\log_s(K_\nu)|_{M_\alpha} = \langle \nu, s(\alpha) \rangle \, \mathrm{id}_{M_\alpha}, \qquad \alpha \in \Lambda^*/\Lambda. \tag{3.63}$$

Further, consider the $\mathfrak{h}$ valued operators

$$X = \sum_{i=1}^{\dim \Lambda^\perp} \mu^i \otimes \mu_i, \qquad \log_s K = \sum_{j=1}^{\mathrm{rk}\Lambda} \nu^j \otimes \log_s K_{\nu_j}, \tag{3.64}$$

which define maps $M \to \mathfrak{h}_{\mathbb{C}} \otimes M$ by the action

$$X|_{M_\alpha} = \sum_{i=1}^{\dim \Lambda^\perp} \mu^i \langle \mu_i, s(\alpha) \rangle \otimes \mathrm{id}_{M_\alpha}, \qquad \log_s K|_{M_\alpha} = \sum_{j=1}^{\mathrm{rk}\Lambda} \nu^j \langle \nu_j, s(\alpha) \rangle \otimes \mathrm{id}_{M_\alpha}.$$

$$(3.65)$$

So for any function $f : (\Lambda^*)^n \to \mathbb{C}^\times$, $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \Lambda^*/\Lambda$, we define the linear operator

$$f(X_1 + \log_s K_1, \ldots, X_n + \log_s K_n)|_{M_{\alpha_1} \otimes \cdots \otimes M_{\alpha_n}} = f(s(\alpha_1), \ldots, s(\alpha_n)) \, \mathrm{id}_{M_{\alpha_1} \otimes \cdots \otimes M_{\alpha_n}}.$$

$$(3.66)$$

Then we define the following ribbon element $r : M \to M$, R-matrix $R : M \otimes N \to M \otimes N$ and coassociator $\Phi : M \otimes (N \otimes P) \to M \otimes (N \otimes P)$, whose names will be justified by Part (1) of Theorem 3.13.

$$r = \exp\left[-\pi i \langle X_1 + \log_s K_1, X_1 + \log_s K_1 - 2s(\xi) \rangle\right],$$
$$R = \exp\left[i\pi \left(\langle X_1 + \log_s K_1, X_2 + \log_s K_2 \rangle\right)\right],$$
$$\Phi = \exp\left[i\pi \langle X_1 + \log_s K_1, \log_s K_2 + \log_s K_3 - \log_s K_{2\otimes 3} \rangle\right]$$
$$\cdot \frac{\varepsilon\left(k(X_1 + \log_s K_1, X_2 + \log_s K_2), k(X_1 + \log_s K_1 + X_2 + \log_s K_2, X_3 + \log_s K_3)\right)}{\varepsilon\left(k(X_2 + \log_s K_2, X_3 + \log_s K_3), k(X_1 + \log_s K_1, X_2 + \log_s K_2 + X_3 + \log_s K_3)\right)},$$

$$(3.67)$$

where $\log_s(K)_{2\otimes 3}$ is to be evaluated after the $N \otimes P$ tensor product has been evaluated, that is, $\log_s(K_\nu)|_{N_\alpha \otimes P_\gamma} = \langle \nu, s(\alpha + \gamma) \rangle \, \mathrm{id}_{N_\alpha \otimes P_\gamma}$. Note that this is not a ribbon element in the usual sense, but it implements the Grothendieck–Verdier structure. The convergence of these infinite sums acting on finite-dimensional vector spaces is obvious, because the exponential series converges for every matrix.

We have thus defined, in the sense of the first paragraph of this section, a quasi-triangular quasi-Hopf algebra H_Λ, which we will call the *lattice quasi-Hopf algebra* of Λ. The following theorem holds then as intended by the construction.

Theorem 3.13 (1) *Let HM(Ψ) be the category of finite-dimensional H_Λ modules M, where H_μ acts diagonalisably on M with real eigenvalues and K_ν acts diagonalisably on M. Then this category is equivalent to Vect(Ψ) as a braided tensor category.*

(2) *The ribbon element in (3.67) equips HM(Ψ) with the structure of a ribbon Grothendieck-Verdier category with twist θ given by $\theta_M = r^{-1}$ and we have an equivalence of ribbon Grothendieck-Verdier categories. All future references to HM(Ψ) will include the ribbon Grothendieck-Verdier structure given here.*

Proof Let $F : \mathsf{Vect}(\Psi) \to \mathsf{HM}(\Psi)$ be the functor which equips the vector space $M \in \mathsf{Vect}(\Psi)$ with the H_Λ action defined by the representation ρ_M, that is,

$$F : M \mapsto (M, \rho_M),$$

$$(3.68)$$

and which is the identity on morphisms. By formula (3.62) this lands in the subcategory asserted in the theorem. Let the isomorphism $\varphi_0 : (\mathbb{C}_0, \rho_{\mathbb{C}_0}) \to F(\mathbb{C}_0) = (\mathbb{C}_0, \rho_{\mathbb{C}_0})$ be the identity map $\mathrm{id}_{\mathbb{C}_0}$ on the tensor unit $\mathbb{C}_0$. Let $\varphi_2 : F(-) \otimes F(-) \to F(- \otimes -)$ be the natural transformation given by

$$\varphi_2((M, \rho_M), (N, \rho_N)) = \mathrm{id}_{M \otimes N} . \tag{3.69}$$

Then $(F, \varphi_0, \varphi_2)$ is an equivalence of braided tensor categories, basically by construction: We can use the proposed tensor functor F to map the twist, braiding and associativity isomorphisms from $\mathsf{Vect}(\Psi)$ to $\mathsf{HM}(\Psi)$. If the images of these structure morphisms match the evaluations of the formal operators (3.67), it then automatically follows that these operators satisfy the defining properties of ribbon elements, R-matrices and coassociators and that $(F, \varphi_0, \varphi_2)$ is an equivalence of ribbon Grothendieck-Verdier categories. Let $\eta, \kappa, \tau \in \Lambda^*/\Lambda$ and $M, N, P \in \mathsf{HM}(\Psi)$, then

$$r|_{M_\eta} = \exp\left[-\pi \mathrm{i}\left\langle \sum_{i=1}^{\dim \Lambda^\perp} \mu^i \langle \mu_i, s(\eta) \rangle + \sum_{j=1}^{\mathrm{rk}\Lambda} \nu^j \langle \nu_j, s(\eta) \rangle, \sum_{k=1}^{\dim \Lambda} \mu^k \langle \mu_k, s(\eta) \rangle + \sum_{l=1}^{\mathrm{rk}\Lambda} \nu^l \langle \nu_l, s(\eta) \rangle - 2s(\xi) \right\rangle\right]$$
$$= \mathrm{e}^{-\pi \mathrm{i}\langle s(\eta), s(\eta) - 2s(\xi) \rangle}\, \mathrm{id}_{M_\eta} \tag{3.70}$$

and similarly,

$$R|_{M_\eta \otimes N_\kappa} = \mathrm{e}^{\mathrm{i}\pi \langle s(\eta), s(\kappa) \rangle}\, \mathrm{id}_{M_\eta \otimes M_\kappa},$$
$$\Phi|_{M_\eta \otimes N_\kappa \otimes P_\tau} = (-1)^{\langle s(\eta), k(\kappa, \tau) \rangle} \frac{\varepsilon(k(\eta, \kappa), k(\eta + \kappa, \tau))}{\varepsilon(k(\kappa, \tau), k(\eta, \kappa + \tau))}\, \mathrm{id}_{M_\eta \otimes N_\kappa \otimes P_\tau} . \tag{3.71}$$

Therefore the ribbon element, R matrix and coassociator evaluate exactly as the twist, braiding isomorphisms and associativity isomorphism in $\mathsf{Vect}(\Psi)$ do and the theorem follows. The equivalence of the Grothendieck-Verdier structures then follows by noting that $\mathbb{C}_{2\xi}$ is the dualising object for both categories. □

Example Recall the examples from the end of Sect. 3.1.

(1) If Λ is the trivial lattice, then $\Lambda^\perp = \Lambda^* = \mathfrak{h}$ and in this case the lattice quasi-Hopf algebra is the universal enveloping algebra $\mathsf{U}(\Lambda^\perp)$ of the complexification of $\Lambda^\perp$.
(2) If the lattice Λ is full rank, then $\Lambda^\perp$ is trivial and Λ^*/Λ is a finite group. In this case the lattice quasi-Hopf algebra is just the group algebra $\mathbb{C}[\Lambda^*]$.
(3) Finally, in the half rank example $\Lambda^* \cong \mathbb{Z} \times \mathbb{R}$ and so the lattice quasi-Hopf algebra is a tensor product of the $\mathbb{Z}$-group algebra and the universal enveloping algebra of the abelian one-dimensional Lie algebra $\mathfrak{gl}(1)$. Further, the modules defined by the action (3.62) descend to modules over the group $U(1) \times \mathbb{Z}$. Explicitly we can give the lattice quasi-Hopf algebra as

$$H_\Lambda = \mathbb{C}[X, K, K^{-1}], \qquad K^{\pm 1} K^{\mp 1} = 1,$$
$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad S(X) = -X,$$

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \qquad S(K^{\pm 1}) = K^{\mp 1}. \tag{3.72}$$

The action on the module $\mathbb{C}_{x_1, x_2 + \mathbb{Z}}$ is then given by

$$\rho_{\mathbb{C}_{x_1, x_2 + \mathbb{Z}}}(X) = x_1 \, \mathrm{id}_{\mathbb{C}_{x_1, x_2 + \mathbb{Z}}}, \qquad \rho_{\mathbb{C}_{x_1, x_2 + \mathbb{Z}}}(K) = \mathrm{e}^{2\pi \mathrm{i} \tilde{x}_2} \, \mathrm{id}_{\mathbb{C}_{x_1, x_2 + \mathbb{Z}}}. \tag{3.73}$$

The ribbon element, R-matrix and coassociator for this choice of data are then

$$
\begin{aligned}
r &= \exp\left(-2\mathrm{i}\pi\left(X_1 \log_s K_1 - \log_s K_1\right)\right) \\
R &= \exp\left[\mathrm{i}\pi\left(\log K_1 \otimes X_2 + X_1 \otimes \log K_2\right)\right], \\
\Phi &= \exp\left[\mathrm{i}\pi\left(X_1 \otimes \log K_2 \otimes \mathrm{id} + X_1 \otimes \mathrm{id} \otimes \log K_3 - X_1 \otimes \log K_{2 \otimes 3}\right)\right],
\end{aligned}
\tag{3.74}
$$

where $\log K$ acts as $\tilde{x}_2$ on $\mathbb{C}_{x_1, x_2 + \mathbb{Z}}$.

3.5 Simple Current Extensions

The process of extending a vertex operator algebra by (tensor powers of) modules whose tensor product is invertible (such extentions are called *simple current extentions*) has a long history in the conformal field theory and vertex operator algebra literature for both finite order extensions [50] and more recently also infinite ones [37, 51]. At a categorical level, extensions (not necessarily the simple current type) correspond to algebra objects in a braided monoidal category [35, 52, 53]. In particular, algebra objects in categories of graded vectors spaces and their connections to vertex operator algebras and conformal field theory have been studied in [54]. Let $\Psi_i = (\mathfrak{h}, \langle -, - \rangle, \Lambda_i, \xi_i)$ for $i = 1, 2$ be two sets of bosonic lattice data. Then by Theorem 3.12 and Part (1) we have two triples of ribbon Grothendieck-Verdier equivalent categories

$$\mathsf{Vect}(\Psi_i) \;\cong\; \mathsf{VM}(\Psi_i) \;\cong\; \mathsf{HM}(\Psi_i), \quad i = 1, 2. \tag{3.75}$$

We will show that if $\Lambda_1 \subset \Lambda_2$ and $\xi_1 \subset \xi_2$, we can find an algebra object A in the direct sum completion $\mathsf{Vect}(\Psi_1)_\oplus$ such that the module category for A is equivalent to $\mathsf{Vect}(\Psi_2)$. Transferring the algebra object A to $\mathsf{VM}(\Psi_1)_\oplus$ then yields the simple current extension of $\mathsf{V}(\tilde{\xi}, \Lambda_1)$ to $\mathsf{V}(\tilde{\xi}, \Lambda_2)$, if we choose the same representative $\tilde{\xi}$ for both ξ_1 and ξ_2. Finally, we will pose the problem of constructing H_{Λ_2} from H_{Λ_1}.

Proposition 3.14 *Let Ψ_1, Ψ_2 be two sets of bosonic lattice data, satisfying $\Lambda_1 \subset \Lambda_2$ and $\xi_1 \subset \xi_2$. Let $\sigma : \Lambda_2/\Lambda_1 \otimes \Lambda_2/\Lambda_1 \to \mathbb{C}^\times$ satisfy*

$$
\begin{aligned}
&\sigma(\lambda, \Lambda_1) = \sigma(\Lambda_1, \lambda) = 1, \qquad \sigma(\lambda_1, \lambda_2)\sigma(\lambda_2, \lambda_1)^{-1} = \Omega(\lambda_1, \lambda_2), \\
&\sigma(\lambda_2, \lambda_3)\sigma(\lambda_1 + \lambda_2, \lambda_3)^{-1}\sigma(\lambda_1, \lambda_2 + \lambda_3)\sigma(\lambda_1, \lambda_2)^{-1} = F(\lambda_1, \lambda_2, \lambda_3),
\end{aligned}
\tag{3.76}
$$

where (F, Ω) is the abelian 3-cocycle associated to Ψ_1. Taking $\Lambda_2/\Lambda_1 \subset \Lambda_1^*/\Lambda_1$ as a subgroup, we define the triple $(A, \mu : A \otimes A \to A, \eta : \mathbb{C}_{\Lambda_1} \to A)$ by

$$A = \bigoplus_{\lambda \in \Lambda_2/\Lambda_1} \mathbb{C}_\lambda, \qquad \mu|_{\mathbb{C}_{\lambda_1} \otimes \mathbb{C}_{\lambda_2}} = \sigma(\lambda_1, \lambda_2) J_{\lambda_1, \lambda_2}, \qquad \eta = \mathrm{id}_{\mathbb{C}_{\Lambda_1}}, \quad (3.77)$$

where J_{λ_1, λ_2} is the canonical identification $\mathbb{C}_{\lambda_1} \otimes \mathbb{C}_{\lambda_2} \cong \mathbb{C}_{\lambda_1 + \lambda_2}$. Then

(1) *(A, μ, η) defines an associative commutative algebra with trivial twist and a unique unit (that is, $\dim \mathrm{Hom}\,(\mathbb{C}_0, A) = 1$, also called the haploid condition), in $\mathsf{Vect}(\Psi_1)_\oplus$.*

(2) *The category of local A-modules A-$\mathsf{Mod}^{loc}(\mathsf{Vect}(\Psi_1)_\oplus)$ (also called dyslectic modules) is a ribbon Grothendieck-Verdier category and is equivalent to $\mathsf{Vect}(\Psi_2)$. Thus the images of A under the functors G and F in Theorem 3.12 and Part (1) define equivalent algebras $A_V = G(A)$ and $A_H = F(A)$ in $\mathsf{VM}(\Psi_1)_\oplus$ and $\mathsf{HM}(\Psi_1)_\oplus$, respectively. Hence we have the sequence*

$$\mathsf{Vect}(\Psi_2) \cong A_V\text{-}\mathsf{Mod}^{loc}\big(\mathsf{VM}(\Psi_1)_\oplus\big) \cong \mathsf{VM}(\Psi_2)$$
$$\cong A_H\text{-}\mathsf{Mod}^{loc}\big(\mathsf{HM}(\Psi_1)_\oplus\big) \cong \mathsf{HM}(\Psi_2). \qquad (3.78)$$

of ribbon Grothendieck-Verdier equivalences.

(3) *Let $\widetilde{\xi} \in \xi_1 \subset \xi_2$ be a choice of representative for both ξ_1 and ξ_2. The algebra object $A_V = G(A)$ admits the structure of a vertex operator algebra via the field map $Y = G_\mu^T$, with vacuum and conformal vectors given by the images of the vacuum and conformal vectors in $V(\widetilde{\xi}, \Lambda_1)$ under the tensor structure map φ_0. Further, this vertex operator algebra is isomorphic to $V(\widetilde{\xi}, \Lambda_2)$.*

Proof Denote by s_i the respective sections of the bosonic lattice data Ψ_i. The 2-cocycles k and ε shall only be needed for Ψ_1 and will hence not be given an index, to reduce notational clutter.

(1) The conditions (3.76) are equivalent to the constraints imposed on μ and η by the definition of an associative unital commutative algebra [32][Definitions 7.8.1 and 8.8.1]. Unitality is implied by the first relation, commutativity by the second and associativity by the third.

The haploid or uniqueness of the unit property follows from A containing $\mathbb{C}_{\Lambda_1}$ only once as a direct summand and $\dim \mathrm{Hom}\,(\mathbb{C}_{\Lambda_1}, \mathbb{C}_{\Lambda_1}) = 1$.

The algebra having trivial twist follows by direct computation. On each summand of A, the twist evaluates to $\theta(\lambda) = \mathrm{e}^{\mathrm{i}\pi \langle s_1(\lambda), s_1(\lambda) - 2s_1(\xi_1)\rangle}$, $\lambda_2 \in \Lambda_2/\Lambda_1$. Since Λ_2 is even, $s_1(\lambda) \in \Lambda_2$ and $s_1(\xi_1) \in \xi_2 \subset \Lambda_2^*$, we have $\langle s_1(\lambda), s_1(\lambda)\rangle$, $2\langle s_1(\lambda), s_1(\xi_2)\rangle \in 2\mathbb{Z}$ and hence the twist is trivial.

(2) Let A-$\mathsf{Mod}(\mathsf{Vect}(\Psi_1)_\oplus)$ be the category of all A-modules in $\mathsf{Vect}(\Psi_1)_\oplus$. Combining [52, Theorem 1.6], which asserts that induction and restriction are adjoint, exact and injective on morphisms, and that induction is a tensor functor with the semisimplicity of $\mathsf{Vect}(\Psi_1)_\oplus$, we can quickly deduce that A-$\mathsf{Mod}(\mathsf{Vect}(\Psi_1)_\oplus)$ is also semisimple and that every simple object in A-$\mathsf{Mod}(\mathsf{Vect}(\Psi_1)_\oplus)$ is the induction of a simple object in $\mathsf{Vect}(\Psi_1)$. We denote the simple modules induced from

the $\mathbb{C}_\alpha$, $\alpha \in \Lambda_1^*/\Lambda_1$ by

$$N_\alpha = A \otimes \mathbb{C}_\alpha \cong \bigoplus_{\lambda \in \Lambda_2/\Lambda_1} \mathbb{C}_{\lambda+\alpha}. \tag{3.79}$$

Let A-Mod$^{\mathrm{loc}}$(Vect(Ψ_1)) be the full subcategory of local modules, that is, all objects which have trivial double braiding with the algebra A. For one of the N_α above this means that for all $\lambda \in \Lambda_2/\Lambda_1$, we require that

$$\Omega(\lambda, \alpha)\Omega(\alpha, \lambda) = \mathrm{e}^{2\pi \mathrm{i}\langle s_1(\lambda), s_1(\alpha)\rangle} = 1, \quad \text{or equivalently} \quad \langle s_1(\lambda), s_1(\alpha)\rangle \in \mathbb{Z}. \tag{3.80}$$

By assumption $s_1(\alpha) \in \Lambda_1^*$. If $s_1(\alpha) \in \Lambda_2^*$, then the above condition is satisfied for all $\lambda \in \Lambda_2/\Lambda_1$. Conversely, if $s_1(\alpha) \notin \Lambda_2^*$ then there exits a $\mu \in \Lambda_2$ such that $\langle \mu, s_1(\alpha)\rangle \notin \mathbb{Z}$. But then $s_1(\alpha)$ would pair non-integrally with every representative of the Λ_1 coset of μ and hence the above condition cannot be satisfied. Therefore $\alpha \in \Lambda_2^*/\Lambda_1$ exhausts all labels for simple objects in A-Mod$^{\mathrm{loc}}$(Vect(Ψ_1)). Two induced simple modules N_α, N_β are isomorphic if and only if their labels differ by a coset in Λ_2/Λ_1. Therefore the isomorphism classes of simple modules are labelled by the elements of the quotient group $(\Lambda_2^*/\Lambda_1)/(\Lambda_2/\Lambda_1) \cong \Lambda_2^*/\Lambda_2$. This implies that A-Mod$^{\mathrm{loc}}$(Vect$(\Psi_1)_\oplus$) and Vect(Ψ_2) are equivalent as abelian categories. By [52, Theorem 1.10] or [55, Theorem 2.5], A-Mod$^{\mathrm{loc}}$(Vect$(\Psi_1)_\oplus$) is braided monoidal with the braiding descending from Vect(Ψ_1). Further, from [52, Theorem 1.6] one can deduce that $N_\alpha \otimes_A N_\beta \cong N_{\alpha+\beta}$. Thus A-Mod$^{\mathrm{loc}}$(Vect$(\Psi_1)_\oplus$) also has the same tensor product as Vect(Ψ_2), hence the braiding and associativity isomorphisms are characterised by abelian 3-cocycles for the group Λ_2^*/Λ_2. To conclude equivalence as braided monoidal categories it is therefore sufficient for the trace of the abelian 3-cocycles of A-Mod$^{\mathrm{loc}}$(Vect$(\Psi_1)_\oplus$) and Vect(Ψ_2) to be equal. Let Ω_i, $i = 1, 2$ be the respective braidings associated to Ψ_i, then for $\alpha \in \Lambda_2^*/\Lambda_1$ we need to compare $\Omega_1(\alpha, \alpha)$ and $\Omega_2(\alpha + \Lambda_2, \alpha + \Lambda_2)$. Recall that $s_1(\alpha) \in \Lambda_2^*$ and hence $s_2(\alpha + \Lambda_2) - s_1(\alpha) = \kappa \in \Lambda_2$, so

$$\begin{aligned} \Omega_2(\alpha + \Lambda_2, \alpha + \Lambda_2) &= \mathrm{e}^{\mathrm{i}\pi \langle s_2(\alpha+\Lambda_2), s_2(\alpha+\Lambda_2)\rangle} = \mathrm{e}^{\mathrm{i}\pi \langle s_1(\alpha)+\kappa, s_1(\alpha)+\kappa\rangle} \\ &= \mathrm{e}^{\mathrm{i}\pi \langle s_1(\alpha), s_1(\alpha)\rangle} = \Omega_1(\alpha, \alpha), \end{aligned} \tag{3.81}$$

where the third equality follows from Λ_2 being even. Thus A-Mod$^{\mathrm{loc}}$(Vect$(\Psi_1)_\oplus$) and Vect(Ψ_2) are equivalent as braided monoidal categories. Grothendieck-Verdier equivalence follows by noting that the induction of the dualising object N_{ξ_1} has $\xi_1 + \Lambda_2 = \xi_2$ as its label and is hence equivalent to the dualising object of Vect(Ψ_2). Finally, ribbon equivalence follows by comparing the twist scalars θ_1, θ_2 in both categories. We denote $s_2(\xi_2) - s_1(\xi_1) = \tau \in \Lambda_2$ and consider for any $\alpha \in \Lambda_2^*/\Lambda_1$

$$\theta_2(\alpha + \Lambda_2) = \mathrm{e}^{\mathrm{i}\pi \langle s_2(\alpha+\Lambda_2), s_2(\alpha+\Lambda_2)-2s_2(\xi_2)\rangle} = \mathrm{e}^{\mathrm{i}\pi \langle s_1(\alpha)+\kappa, s_1(\alpha)+\kappa-2s_1(\xi_1)-2\tau\rangle}$$

$$= e^{i\pi \langle s_1(\alpha), s_1(\alpha) - 2s_1(\xi_1) \rangle} = \theta_1(\alpha), \tag{3.82}$$

where we have again used the Λ_2 is even. Thus $A\text{-Mod}^{\text{loc}}(\text{Vect}(\Psi_1)_\oplus)$ and $\text{Vect}(\Psi_2)$ are ribbon Grothendieck-Verdier equivalent.

(3) As a module over the Heisenberg algebra A_V decomposes as follows.

$$A_V = G(A) = \bigoplus_{\lambda \in \Lambda_2/\Lambda_1} G(M_\lambda) \cong \bigoplus_{\lambda \in \Lambda_2/\Lambda_1} \mathbb{F}_\lambda \cong \bigoplus_{\lambda \in \Lambda_2/\Lambda_1} \bigoplus_{\alpha \in \Lambda_1} \mathcal{F}_{s_1(\lambda)+\alpha} = \bigoplus_{\lambda \in \Lambda_2} \mathcal{F}_\lambda, \tag{3.83}$$

which is isomorphic to the vector space on which $V(\widetilde{\xi}, \Lambda_2)$ is defined. We need to verify that $Y = G_\mu^T$ is indeed a field map. We show this by comparing G_μ^T to the field map of $V(\widetilde{\xi}, \Lambda_2)$. Consider $\lambda_1, \lambda_2 \in \Lambda_2/\Lambda_1$, $\alpha_1, \alpha_2 \in \Lambda_1$, then $G_\mu^T|_{(G(\mathbb{C}_{\lambda_1}) \otimes e^{\alpha_1}) \otimes (G(\mathbb{C}_{\lambda_2}) \otimes e^{\alpha_2})}$ is essentially an untwisted vertex operator of the form (3.25) up to a scaling factor of

$$(-1)^{\langle s_1(\lambda_1), \alpha_2 \rangle} \varepsilon(\alpha_1, \alpha_2) \varepsilon(\alpha_1 + \alpha_2, k(\lambda_1, \lambda_2)) \sigma(\lambda_1, \lambda_2). \tag{3.84}$$

Therefore G_μ^T defines a vertex operator algebra structure if and only if

$$\begin{aligned} \tau(\gamma, \delta) = (-1)^{\langle s_1(\gamma + \Lambda_1), \delta - s_1(\delta + \Lambda_1) \rangle} &\varepsilon(\gamma - s_1(\gamma + \Lambda_1), \delta - s_1(\delta + \Lambda_1)) \\ &\varepsilon(\gamma - s_1(\gamma + \Lambda_1) + \delta - s_1(\delta + \Lambda_1), k(\gamma + \Lambda_1, \delta + \Lambda_1)) \\ &\sigma(\gamma + \Lambda_1, \delta + \Lambda_1), \quad \gamma, \delta \in \Lambda_2 \end{aligned} \tag{3.85}$$

satisfies the 2-cocycle conditions of (3.1) for Λ_2. Since all 2-cocycles for Λ_2 are cohomologous, the vertex operator algebra structure defined by G_μ^T is isomorphic to that of $V(\widetilde{\xi}, \Lambda_2)$.

$\square$

Problem 3.15 Consider a quasitriangular (quasi-)Hopf algebra H over a field $\Bbbk$ and a commutative algebra A_H in the braided tensor category H-Mod. It is an interesting and seemingly nontrivial problem to construct, under suitable assumptions, from H, A_H a quasitriangular (quasi-)Hopf algebra, whose module category is ribbon Grothendieck-Verdier equivalent to $A_H\text{-Mod}^{\text{loc}}(H\text{-Mod})$, the braided tensor category of local A_H-modules in H-Mod.

For a Hopf algebra H and an algebra A_H in H-Mod it is proven in [56, 57], under suitable assumptions, that A_H-modules in H-Mod are free as A_H-modules in vector spaces. They also provide a construction for an algebra H', whose module category is equivalent to the category of A_H-modules, by taking A_H-invariants. In the setting of our initial question this implies that H' should indeed have the structure of a quasi-Hopf algebra, and a suitable quotient should be the quasitriangular (quasi-)Hopf algebra in question. However, it is not clear how precisely the coproduct of H' can be derived from the coproduct of H.

A first class of examples, which includes the examples covered in the present article, are the *simple current extensions*, which we now describe in the Hopf algebra setting:

Consider a quasi-triangular (quasi-)Hopf algebra H over a field $\Bbbk$ with R-matrix written as $R = \sum_i R_1^{(i)} \otimes R_2^{(i)}$, $R_1^{(i)}, R_2^{(i)} \in H$, and a group Γ of 1-dimensional characters $\phi : H \to \Bbbk$ such that $\sum_{i,j} \phi(R_2^{(i)} R_1^{(j)}) \psi(R_1^{(i)} R_2^{(j)}) = 1$ for all $\phi, \psi \in \Gamma$. Each such character ϕ defines a one-dimensional module $\Bbbk_\phi$ on which $h \in H$ acts as $\phi(h)$ id. Then the object $A_H = \bigoplus_{\phi \in \Gamma} \Bbbk_\phi$ can be endowed with the structure of a commutative algebra in H-Mod using the multiplication in Γ and the 2-cocycle $\sigma(\phi, \psi) = \sum_i \phi(R_1^{(i)}) \psi(R_2^{(i)})$ (this is a 2-cocycle because it satisfies the pentagon identity $\sigma(\phi * \rho, \psi) = \sigma(\phi, \psi)\sigma(\rho, \psi)$, $\phi, \rho, \psi \in \Gamma$, where $*$ is convolution). For quantum groups, the respective construction of a quasi-Hopf algebras has been dubbed *uprolling* [58–61].

4 The impact of Grothendieck-Verdier structure on characters and modular transformations

We conclude this paper with a final section giving observations on the modular properties of lattice module characters. Ideally one would want to extract from ribbon Grothendieck-Verdier categories some analogue of the rich structures enjoyed by modular tensor categories such as a generalisation of the mapping class group action and the Verlinde formula. In particular this would require some notion of categorical trace. The tools and understanding required for this have, however, not yet been developed and we hope to return to this in the future. To support this future research, we record here character formulae and their modular transformation properties and show that they admit a naive generalisation of the Verlinde formula in the sense of the standard module formalism [24].

Let $\Psi = (\mathfrak{h}, \langle -, - \rangle, \Lambda, \xi)$ be a set of bosonic lattice data as defined at the beginning of Sect. 3.1 and $\widetilde{\xi}$ a choice of representative of ξ.

Proposition 4.1 *Let $\zeta \in \mathfrak{h}_{\mathbb{C}}$, $\tau \in \mathbb{H}_+$, $q = e^{2\pi i \tau}$ and $\gamma \in \Lambda^*/\Lambda$. Then the character of the lattice Fock space $\mathbb{F}_\gamma$ as a $V(\widetilde{\xi}, \Lambda)$ module is*

$$\chi_\gamma^{\widetilde{\xi}}(\zeta, \tau) = Tr_{\mathbb{F}_\gamma} e^{2\pi i \langle \zeta, - \rangle} q^{L_0 - c/24} = \sum_{\lambda \in \Lambda} e^{2\pi i \langle \zeta, s(\gamma) + \lambda \rangle} \frac{q^{\frac{1}{2}\left(\langle s(\gamma) + \lambda - \widetilde{\xi}, s(\gamma) + \lambda - \widetilde{\xi} \rangle - \frac{\dim \mathfrak{h}}{12}\right)}}{\eta(\tau)^{\dim \mathfrak{h}}}. \tag{4.1}$$

Note that since the sum on the right-hand side of (4.1) ranges over the entire lattice Λ it does not depend on the choice of representative $s(\gamma)$ of the coset γ and it only depends on the choice of representative $\widetilde{\xi}$ of the coset ξ by a global factor coming from the first exponential.

Proof The argument of the sum is the well known character formula for non-lattice Fock spaces $\mathcal{F}_\lambda$. Hence the character for the lattice Fock space is just the sum of the characters of the $\mathcal{F}_\lambda$ summed over all $\lambda \in \Lambda$. □

Recall the decomposition $\Lambda^* = V \oplus \text{span}_{\mathbb{R}}\{\Lambda^\circ\} \oplus F \oplus D$ in Lemma 3.2.(2).

Theorem 4.2 *The T-transformation of lattice module characters is*

$$T\{\chi_\gamma^{\widetilde{\xi}}(\zeta, \tau)\} = \chi_\gamma^{\widetilde{\xi}}(\zeta, \tau + 1) = e^{\pi i \left(\langle \gamma - \xi, \gamma - \xi \rangle - \frac{\dim \mathfrak{h}}{12} \right)} \chi_\gamma^{\widetilde{\xi}}(\zeta, \tau). \tag{4.2}$$

If $\langle -, - \rangle$ restricted to the groups V and D is positive definite, then the S-transformation is

$$S\{\chi_\gamma^{\widetilde{\xi}}(\zeta, \tau)\} = \chi_\gamma^{\widetilde{\xi}}\left(\frac{\zeta}{\tau}, \frac{-1}{\tau} \right)$$

$$= \left(\frac{|\tau|}{i\tau} \right)^{rk\Lambda^\circ} \frac{e^{2\pi i \left(\frac{\|\zeta\|^2}{2\tau} + \frac{\langle \widetilde{\xi}, \zeta \rangle}{\tau} - \langle \widetilde{\xi}, \zeta \rangle \right)}}{\sqrt{|D/(D \cap \Lambda)|}}$$

$$\int_{\Lambda^*/\Lambda} e^{-2\pi i \langle \gamma - \xi, \mu - \xi \rangle} \chi_\mu^{\widetilde{\xi}}(\zeta, \tau) d(\mu), \tag{4.3}$$

where $\int_{\Lambda^*/\Lambda} d(\mu) = \int_V d(v) \cdot \sum_{f \in F} \int_{\text{span}_\mathbb{R}\{\Lambda^\circ\}/\Lambda^\circ} d(v) \cdot \sum_{t \in D/D \cap \Lambda}$, where the integrals over V and $\text{span}_\mathbb{R}\{\Lambda^\circ\}/\Lambda^\circ$ are to be expanded as follows.

- *Pick any $\mathbb{R}$-basis $\{e_i\}_{i=1}^{\dim V}$ of V and let $\det \langle -, - \rangle|_V$ be the determinant of the Gram matrix for the pairing $\langle -, - \rangle$ restricted to V in this basis. For any $v \in V$ denote its expansion in the basis $\{e_i\}_{i=1}^{\dim V}$ by $v = \sum_i v_i e_i$, then*

$$\int_V \psi(v) d(v) = \sqrt{\det \langle -, - \rangle|_V} \int_{\mathbb{R}^{\dim V}} \psi\left(\sum_i v_i e_i \right) dv_1 \cdots dv_{\dim V}. \tag{4.4}$$

- *Pick any $\mathbb{Z}$-basis $\{g_i\}_{i=1}^{rk\Lambda^\circ}$ of Λ° (which will also be an $\mathbb{R}$-basis for $\text{span}_\mathbb{R}\{\Lambda^\circ\}$) and any $\mathbb{Z}$-basis $\{h_i\}_{i=1}^{rk\Lambda^\circ}$ of F. For any $v \in \text{span}_\mathbb{R}\{\Lambda^\circ\}$ and $f \in F$ denote their expansion in their respective bases by $v = \sum_i v_i g_i$ and $f = \sum_i f_i h_i$, then*

$$\sum_{f \in F} \int_{\text{span}_\mathbb{R}\{\Lambda^\circ\}/\Lambda^\circ} \psi(f, v) d(v)$$

$$= \sum_{\substack{f_k \in \mathbb{Z} \\ 1 \le k \le rk\Lambda^\circ}} \int_{[0,1]^{rk\Lambda^\circ}} \psi\left(\sum_j f_j h_j, \sum_i v_i g_i \right) dv_1 \cdots dv_{rk\Lambda^\circ}. \tag{4.5}$$

Note that the S-transformation does not depend of the choice of the groups D and V in the decomposition of Λ^*, as any change of choice will be compensated for by the determinants. Note further that, due to Λ being integral and even, the expressions $e^{i\pi \langle \gamma - \xi, \gamma - \xi \rangle}$ and $e^{-2\pi i \langle \gamma - \xi, \mu - \xi \rangle}$ do not depend on the elements of Λ^* chosen to represent $\gamma - \xi$, $\mu - \xi$ to evaluate the pairings. Thus (up to the phase factor of S) the modular transformation formulae depend on ξ (which is also the datum that characterises the ribbon Grothendieck-Verdier structures, that is the dualising object and twist) but not on the choice of representative $\widetilde{\xi}$. The above theorem is most easily

proved by decomposing the characters in a manner compatible with the decomposition of Λ^* in Lemma 3.2.(2).

Lemma 4.3 *Given $\gamma \in \Lambda^*$ and the decomposition $\Lambda^* = V \oplus \mathrm{span}_{\mathbb{R}}\{\Lambda^\circ\} \oplus F \oplus D$ of Lemma 3.2.(2), let $\gamma_V, \gamma_\circ, \gamma_F, \gamma_D$ be the components of the representative $s(\gamma) = \gamma_V + \gamma_\circ + \gamma_F + \gamma_D$ in the summands of Λ^* and let $\widetilde{\xi} = \widetilde{\xi}_V + \widetilde{\xi}_\circ + \widetilde{\xi}_F + \widetilde{\xi}_D$ be the analogous decomposition for $\widetilde{\xi}$. The character $\chi_\gamma^{\widetilde{\xi}}$, $\gamma \in \Lambda^*/\Lambda$ admits the following factorisation with contributions from V, $\mathrm{span}_{\mathbb{R}}\{\Lambda^\circ\} \oplus F$, D.*

$$\chi_\gamma^{\widetilde{\xi}}(\zeta, \tau) = \chi_\gamma^{\widetilde{\xi}, V}(\zeta, \tau) \cdot \chi_\gamma^{\widetilde{\xi}, \circ}(\zeta, \tau) \cdot \chi_\gamma^{\widetilde{\xi}, D}(\zeta, \tau) \tag{4.6}$$

where

$$\chi_\gamma^{\widetilde{\xi}, V}(\zeta, \tau) = \frac{e^{2\pi i \langle \zeta_V, \gamma_V \rangle} q^{\frac{1}{2} \|\gamma_V - \widetilde{\xi}_V\|^2}}{\eta(\tau)^{\dim V}},$$

$$\chi_\gamma^{\widetilde{\xi}, \circ}(\zeta, \tau) = \frac{e^{2\pi i \langle \zeta_\circ, \gamma_F \rangle} q^{\langle \widetilde{\xi}_\circ, \widetilde{\xi}_F - \gamma_F \rangle}}{\eta(\tau)^{2 \mathrm{rk}\Lambda^\circ}} \sum_{f \in F} e^{2\pi i \langle \gamma_\circ, f \rangle} \delta_F\big(\zeta_F + \tau(\gamma_F - \widetilde{\xi}_F) - f\big),$$

$$\chi_\gamma^{\widetilde{\xi}, D}(\zeta, \tau) = \sum_{\kappa \in \Lambda \cap D} \frac{e^{2\pi i \langle \zeta_D, \kappa + \gamma_D \rangle} q^{\frac{1}{2} \|\gamma_D + \kappa - \widetilde{\xi}_D\|^2}}{\eta(\tau)^{\mathrm{rk} D}}, \tag{4.7}$$

and where $\delta_F(x)$, is the lattice δ-distribution on F, that is,

$$\delta_F(x) = \prod_{j=1}^{\mathrm{rk}\Lambda^\circ} \delta\big(\langle a_j, x \rangle\big), \quad x \in \mathrm{span}_{\mathbb{R}}\{F\}, \ \{a_j\} \text{ any } \mathbb{Z}\text{-basis of } \Lambda^\circ. \tag{4.8}$$

Proof Then result follows by direct calculation and using the fact the the three subgroups V, $\mathrm{span}_{\mathbb{R}}\{\Lambda^\circ\} \oplus F$, D are mutually orthogonal. The only complication is the Λ°, F contribution, which we sketch here. Recall that Λ° and F are orthogonal to themselves but pair crosswise. Let $\{a_i\}_{i=1}^{\dim \Lambda^\circ}$ be a $\mathbb{Z}$-basis of Λ° and let $\{a^i\}_{i=1}^{\dim \Lambda^\circ}$ be its dual in F.

$$\sum_{\kappa \in \Lambda^\circ} \frac{e^{2\pi i \langle \zeta_\circ + \zeta_F, \gamma_\circ + \gamma_F + \kappa \rangle} q^{\frac{1}{2} \|\gamma_\circ + \gamma_F + \kappa - \widetilde{\xi}_\circ - \widetilde{\xi}_F\|^2}}{\eta(\tau)^{2\mathrm{rk}\Lambda^*}}$$

$$= \frac{e^{2\pi i \langle \zeta_\circ, \gamma_F \rangle} e^{2\pi i \langle \zeta_F, \gamma_\circ \rangle}}{\eta(\tau)^{2\mathrm{rk}\Lambda^*}} q^{\langle \gamma_\circ - \widetilde{\xi}_\circ, \gamma_F - \widetilde{\xi}_F \rangle} \sum_{\kappa \in \Lambda^\circ} e^{2\pi i \langle \zeta_F + \tau(\gamma_F - \widetilde{\xi}), \kappa \rangle}$$

$$= \frac{e^{2\pi i \langle \zeta_\circ, \gamma_F \rangle} e^{2\pi i \langle \zeta_F, \gamma_\circ \rangle}}{\eta(\tau)^{2\mathrm{rk}\Lambda^*}} q^{\langle \gamma_\circ - \widetilde{\xi}_\circ, \gamma_F - \widetilde{\xi}_F \rangle} \sum_{\substack{n_i \in \mathbb{Z} \\ 1 \leq i \leq \mathrm{rk}\Lambda^\circ}} e^{2\pi i n_i \langle \zeta_F + \tau(\gamma_F - \widetilde{\xi}), a_i \rangle}$$

$$= \frac{e^{2\pi i \langle \zeta_\circ, \gamma_F \rangle} e^{2\pi i \langle \zeta_F, \gamma_\circ \rangle}}{\eta(\tau)^{2\mathrm{rk}\Lambda^*}} q^{\langle \gamma_\circ - \widetilde{\xi}_\circ, \gamma_F - \widetilde{\xi}_F \rangle} \sum_{\substack{n_i \in \mathbb{Z} \\ 1 \leq i \leq \mathrm{rk}\Lambda^\circ}} \delta\big(\langle \zeta_F + \tau(\gamma_F - \widetilde{\xi}), a_i \rangle - n_i\big)$$

$$
= \frac{e^{2\pi i\langle\zeta_\circ,\gamma_F\rangle}e^{2\pi i\langle\zeta_F,\gamma_\circ\rangle}}{\eta(\tau)^{2\mathrm{rk}\Lambda^*}}q^{\langle\gamma_\circ-\widetilde{\xi}_\circ,\gamma_F-\widetilde{\xi}_F\rangle}\sum_{\substack{n_i\in\mathbb{Z}\\1\leq i\leq\mathrm{rk}\Lambda^\circ}}\delta_F\left(\zeta_F+\tau(\gamma_F-\widetilde{\xi}_F)-n_ia^i\right)
$$

$$
= \frac{e^{2\pi i\langle\zeta_\circ,\gamma_F\rangle}e^{2\pi i\langle\zeta_F,\gamma_\circ\rangle}}{\eta(\tau)^{2\mathrm{rk}\Lambda^*}}q^{\langle\gamma_\circ-\widetilde{\xi}_\circ,\gamma_F-\widetilde{\xi}_F\rangle}\sum_{f\in F}\delta_F\left(\zeta_F+\tau(\gamma_F-\widetilde{\xi}_F)-f\right)
$$

$$
= \frac{e^{2\pi i\langle\zeta_\circ,\gamma_F\rangle}q^{\langle\widetilde{\xi}_\circ,\widetilde{\xi}_F-\gamma_F\rangle}}{\eta(\tau)^{2\mathrm{rk}\Lambda^\circ}}\sum_{f\in F}e^{2\pi i\langle\gamma_\circ,f\rangle}\delta_F\left(\zeta_F+\tau(\gamma_F-\widetilde{\xi}_F)-f\right), \tag{4.9}
$$

where the third identity follows from the Dirac comb identity

$$
\sum_{k\in\mathbb{Z}}\delta(x-k)=\sum_{n\in\mathbb{Z}}e^{2\pi inx}, \tag{4.10}
$$

and the final identity is obtained by using the δ distributions to substitute $\tau(\gamma_F-\widetilde{\xi}_F)=f-\zeta_F$ in the exponent of q. $\qquad\square$

Proof of Theorem 4.2 The T-transformation expression is immediate. The S-transformation formulae follow from computing the S-transformations of the three factors in Lemma 4.3. All three cases boil down to repeated evaluation of Gaussian integrals, that is, the well known identity

$$
\int_{\mathbb{R}^n}e^{-\frac{1}{2}\sum_{i,j=1}^n x_iA_{i,j}x_j+\sum_{i=1}^n B_ix_i}dx_1\cdots dx_n=\sqrt{\frac{(2\pi)^n}{\det A}}e^{\frac{1}{2}B^TA^{-1}B}, \tag{4.11}
$$

where A is a symmetric positive matrix and B is a (possibly complex) n-vector.

We first determine the S transformation of the V part, $\chi_\gamma^{\widetilde{\xi},V}$. Fix an $\mathbb{R}$-basis $\{e_i\}_{i=1}^{\dim V}$ of V and consider

$$
\int_V e^{-2\pi i\langle\gamma_V-\widetilde{\xi}_V,v-\widetilde{\xi}_V\rangle}\chi_v^{\widetilde{\xi},V}(\zeta,\tau)d(v)
$$

$$
= \int_V e^{-2\pi i\langle\gamma_V-\widetilde{\xi}_V,v-\widetilde{\xi}_V\rangle}\frac{e^{2\pi i\langle v,\zeta_V\rangle}q^{\frac{1}{2}\|v-\widetilde{\xi}_V\|^2}}{\eta(\tau)^{\dim V}}d(v)
$$

$$
= \frac{e^{\pi i\tau\|\widetilde{\xi}_V\|^2}e^{2\pi i\langle\widetilde{\xi}_V,\gamma_V-\widetilde{\xi}_V\rangle}}{\eta(\tau)^{\dim V}}\sqrt{\det\langle-,-\rangle|_V}\int_{\mathbb{R}^{\dim V}}e^{\pi i\tau\langle v,v\rangle}e^{2\pi i\langle v,\zeta_V-\gamma_V+\widetilde{\xi}_V-\tau\widetilde{\xi}_V\rangle}dv_1\cdots dv_n
$$

$$
= \frac{e^{\pi i\tau\|\widetilde{\xi}_V\|^2}e^{2\pi i\langle\widetilde{\xi}_V,\gamma_V-\widetilde{\xi}_V\rangle}}{\eta(\tau)^{\dim V}\sqrt{-2\pi i\tau}^{\dim V}}\sqrt{\det\langle-,-\rangle|_V}\int_{\mathbb{R}^{\dim V}}e^{-\frac{1}{2}\langle v,v\rangle}e^{\frac{2\pi i}{\sqrt{-2\pi i\tau}}\langle v,\zeta_V-\gamma_V+\widetilde{\xi}_V-\tau\widetilde{\xi}_V\rangle}dv
$$

$$
= e^{2\pi i\langle\zeta_V/\tau,\widetilde{\xi}_V\tau-\zeta_V/2-\widetilde{\xi}_V\rangle}\frac{e^{2\pi i\langle\zeta_V/\tau,\gamma_V\rangle}e^{-\pi i\|\gamma_V-\widetilde{\xi}_V\|^2/\tau}}{\sqrt{-i\tau}^{\dim V}\eta(\tau)^{\dim V}}
$$

$$
= e^{-2\pi i\left(\frac{\|\zeta_V\|^2}{2\tau}+\frac{\langle\widetilde{\xi}_V,\zeta_V\rangle}{\tau}-\langle\widetilde{\xi}_V,\zeta_V\rangle\right)}\chi_\gamma^{\widetilde{\xi},V}\left(\frac{\zeta}{\tau},\frac{-1}{\tau}\right), \tag{4.12}
$$

where in the third equality we have rescaled all integration variables to absorb a factor of $\sqrt{2\pi}$ and in the fourth equality we have used the Gaussian integral formula (4.11) with A equal to the Gram matrix of $\langle -, - \rangle|_V$.

The D part, $\chi_\gamma^{\widetilde{\xi}, D}$, is a lattice θ-function (for the lattice $D \cap \Lambda$). The modular transformation properties of lattice θ-functions are well known. In particular, the S-transformation can be determined by combining the Gaussian integral formula (4.11) with the Dirac comb, that is, the δ-distribution identity

$$\sum_{k \in \mathbb{Z}} \delta(x - k) = \sum_{n \in \mathbb{Z}} e^{2\pi i n x}. \tag{4.13}$$

Note that $\sqrt{\det \langle -, - \rangle|_D} = \sqrt{|D/D \cap \Lambda|^{-1}}$ and consider

$$\chi_\gamma^{\widetilde{\xi}, D}\left(\frac{\zeta}{\tau}, \frac{-1}{\tau}\right) = \sum_{\kappa \in \Lambda \cap D} \frac{e^{2\pi i \langle \zeta/\tau, \gamma_D + \kappa \rangle} e^{2\pi i \frac{-1}{2\tau} \langle \gamma_D - \widetilde{\xi}_D + \kappa, \gamma_D - \widetilde{\xi}_D + \kappa \rangle}}{\eta(\tau)^{\mathrm{rk} D}}$$

$$= \sum_{\kappa \in \Lambda \cap D} e^{2\pi i \left(\frac{\|\zeta_D\|^2}{2\tau} + \frac{\langle \widetilde{\xi}_D, \zeta_D \rangle}{\tau} - \langle \widetilde{\xi}_D, \zeta_D \rangle \right)} \sqrt{\det \langle -, - \rangle|_D}$$

$$\int_{\mathrm{span}_\mathbb{R}\{D\}} e^{-2\pi i \langle \gamma_D - \widetilde{\xi}_D + \kappa, \nu - \widetilde{\xi}_D \rangle} \frac{e^{2\pi i \langle \zeta_D, \nu \rangle} e^{2\pi i \frac{\tau}{2} \langle \nu - \widetilde{\xi}_D, \nu - \widetilde{\xi}_D \rangle}}{\eta(\tau)^{\mathrm{rk} D}} \, d(\nu)$$

$$= \frac{e^{2\pi i \left(\frac{\|\zeta_D\|^2}{2\tau} + \frac{\langle \widetilde{\xi}_D, \zeta_D \rangle}{\tau} - \langle \widetilde{\xi}_D, \zeta_D \rangle \right)}}{\sqrt{|D/D \cap \Lambda|}}$$

$$\int_{\mathrm{span}_\mathbb{R}\{D\}} e^{-2\pi i \langle \gamma_D - \widetilde{\xi}_D, \nu - \widetilde{\xi}_D \rangle} \frac{e^{2\pi i \langle \zeta_D, \nu \rangle} e^{2\pi i \frac{\tau}{2} \langle \nu - \widetilde{\xi}_D, \nu - \widetilde{\xi}_D \rangle}}{\eta(\tau)^{\mathrm{rk} D}} \sum_{\kappa \in \Lambda \cap D} e^{-2\pi i \langle \kappa, \nu - \widetilde{\xi}_D \rangle} d(\nu)$$

$$= \frac{e^{2\pi i \left(\frac{\|\zeta_D\|^2}{2\tau} + \frac{\langle \widetilde{\xi}_D, \zeta_D \rangle}{\tau} - \langle \widetilde{\xi}_D, \zeta_D \rangle \right)}}{\sqrt{|D/D \cap \Lambda|}}$$

$$\int_{\mathrm{span}_\mathbb{R}\{D\}} e^{-2\pi i \langle \gamma_D - \widetilde{\xi}_D, \nu - \widetilde{\xi}_D \rangle} \frac{e^{2\pi i \langle \zeta_D, \nu \rangle} e^{2\pi i \frac{\tau}{2} \langle \nu - \widetilde{\xi}_D, \nu - \widetilde{\xi}_D \rangle}}{\eta(\tau)^{\mathrm{rk} D}}$$

$$\sum_{k \in D} \delta_D(\nu - k) d(\nu)$$

$$= \frac{e^{2\pi i \left(\frac{\|\zeta_D\|^2}{2\tau} + \frac{\langle \widetilde{\xi}_D, \zeta_D \rangle}{\tau} - \langle \widetilde{\xi}_D, \zeta_D \rangle \right)}}{\sqrt{|D/D \cap \Lambda|}}$$

$$\sum_{k \in D} e^{-2\pi i \langle \gamma_D - \widetilde{\xi}_D, \nu - \widetilde{\xi}_D \rangle} \frac{e^{2\pi i \langle \zeta_D, k \rangle} e^{2\pi i \frac{\tau}{2} \langle k - \widetilde{\xi}_D, k - \widetilde{\xi}_D \rangle}}{\eta(\tau)^{\mathrm{rk} D}}$$

$$= \frac{e^{2\pi i \left(\frac{\|\zeta_D\|^2}{2\tau} + \frac{\langle \widetilde{\xi}_D, \zeta_D \rangle}{\tau} - \langle \widetilde{\xi}_D, \zeta_D \rangle \right)}}{\sqrt{|D/D \cap \Lambda|}}$$

$$\sum_{t \in D/\Lambda \cap D} e^{-2\pi i \langle \gamma_D - \widetilde{\xi}, t - \widetilde{\xi} \rangle} \sum_{\kappa \in t} \frac{e^{\langle \zeta_D, \kappa \rangle} e^{2\pi i \frac{\tau}{2} \|\kappa - \widetilde{\xi}_D\|^2}}{\eta(\tau)^{\mathrm{rk} D}}$$

$$= \frac{e^{2\pi i\left(\frac{\|\zeta_D\|^2}{2\tau} + \frac{\langle \widetilde{\xi}_D, \zeta_D\rangle}{\tau} - \langle \widetilde{\xi}_D, \zeta_D\rangle\right)}}{\sqrt{|D/D \cap \Lambda|}} \sum_{t\in D/\Lambda\cap D} e^{-2\pi i\langle \gamma_D - \widetilde{\xi}, t - \widetilde{\xi}\rangle} \chi_t^{\widetilde{\xi}, D}(\zeta, \tau) \quad (4.14)$$

Finally, we consider the Λ°, F part, $\chi_\gamma^{\widetilde{\xi},\circ}$. Note first that, for $\gamma_F \in F$ and $\gamma_\circ \in$ span$_\mathbb{R}\{\Lambda^\circ\}$

$$\chi_{\gamma_F+\gamma_\circ}^{\widetilde{\xi},\circ}\left(\frac{\zeta}{\tau}, \frac{-1}{\tau}\right) = \frac{e^{2\pi i\langle \zeta_\circ/\tau, \gamma_F\rangle} e^{-2\pi i\langle \widetilde{\xi}_\circ, \widetilde{\xi}_F - \gamma_F\rangle/\tau}}{\eta(\tau)^{2\text{rk}\Lambda^\circ}\sqrt{-i\tau}^{2\text{rk}\Lambda^\circ}}$$

$$\sum_{f\in F} e^{2\pi i\langle \gamma_\circ, f\rangle} \delta_F(\zeta_F/\tau - (\gamma_F - \widetilde{\xi}_F)/\tau - f)$$

$$= \frac{|\tau|^{\text{rk}\Lambda^\circ}}{(-i\tau)^{\text{rk}\Lambda^\circ}} \frac{e^{2\pi i\langle \zeta_\circ/\tau, \gamma_F\rangle}}{\eta(\tau)^{2\text{rk}\Lambda^\circ}} e^{-2\pi i\langle \widetilde{\xi}_\circ, \widetilde{\xi}_F - \gamma_F\rangle/\tau}$$

$$\sum_{f\in F} e^{2\pi i\langle \gamma_\circ, f\rangle} \delta_F(\zeta_F + \widetilde{\xi}_F - \gamma_F - \tau f), \qquad (4.15)$$

where the second equality follows from the scaling behaviour of δ-distributions. Let $\{a_i\}_{i=1}^{\dim \Lambda^\circ}$ be a $\mathbb{Z}$-basis of $\Lambda^\circ \cap \Lambda$ and let $\{a^i\}_{i=1}^{\dim \Lambda^\circ}$ be its dual in F. Then compare the above to

$$\sum_{k\in F} \int_{\text{span}_\mathbb{R}\{\Lambda^\circ\}/\Lambda^\circ} e^{-2\pi i\langle \gamma_F+\gamma_\circ-\widetilde{\xi}_F-\widetilde{\xi}_\circ, k+\nu-\widetilde{\xi}_F-\widetilde{\xi}_\circ\rangle} \chi_{k+\nu}^{\widetilde{\xi},\circ}(\zeta, \tau) d(\nu)$$

$$= \sum_{k\in F} \int_{[0,1]^{\text{rk}\Lambda^\circ}} e^{-2\pi i\langle \gamma_F+\gamma_\circ-\widetilde{\xi}_F-\widetilde{\xi}_\circ, k+\nu-\widetilde{\xi}_F-\widetilde{\xi}_\circ\rangle} \frac{e^{2\pi i\langle \zeta_\circ, k\rangle} q^{\langle \widetilde{\xi}_\circ, \widetilde{\xi}_F-k\rangle}}{\eta(\tau)^{2\text{rk}\Lambda^\circ}}$$

$$\sum_{f\in F} e^{2\pi i\langle \nu_\circ, f\rangle} \delta_F(\zeta_F + \tau(k - \widetilde{\xi}_F) - f) d\nu_1 \cdots d\nu_{\text{rk}\Lambda^\circ}$$

$$= \sum_{k\in F} e^{-2\pi i\langle \gamma_F+\gamma_\circ-\widetilde{\xi}_F-\widetilde{\xi}_\circ, k-\widetilde{\xi}_F-\widetilde{\xi}_\circ\rangle} \frac{e^{2\pi i\langle \zeta_\circ, k\rangle} q^{\langle \widetilde{\xi}_\circ, \widetilde{\xi}_F-k\rangle}}{\eta(\tau)^{2\text{rk}\Lambda^\circ}}$$

$$\sum_{f\in F} \delta_F(\zeta_F + \tau(k - \widetilde{\xi}_F) - f) \int_{[0,1]^{\text{rk}\Lambda^\circ}} e^{2\pi i\langle \nu, \widetilde{\xi}_F-\gamma_F+f\rangle} d\nu_1 \cdots d\nu_{\text{rk}\Lambda^\circ}$$

$$= \sum_{k\in F} e^{-2\pi i\langle \gamma_F+\gamma_\circ-\widetilde{\xi}_F-\widetilde{\xi}_\circ, k-\widetilde{\xi}_F-\widetilde{\xi}_\circ\rangle} \frac{e^{2\pi i\langle \zeta_\circ, k\rangle} q^{\langle \widetilde{\xi}_\circ, \widetilde{\xi}_F-k\rangle}}{\eta(\tau)^{2\text{rk}\Lambda^\circ}} \delta_F(\zeta_F + \tau(k - \widetilde{\xi}_F) + \widetilde{\xi}_F - \gamma_F)$$

$$= \sum_{k\in F} e^{-2\pi i\langle \gamma_F+\gamma_\circ-\widetilde{\xi}_F-\widetilde{\xi}_\circ, -k-\widetilde{\xi}_\circ\rangle} \frac{e^{2\pi i\langle \zeta_\circ, \widetilde{\xi}_F-k\rangle} q^{\langle \widetilde{\xi}_\circ, k\rangle}}{\eta(\tau)^{2\text{rk}\Lambda^\circ}} \delta_F(\zeta_F + \widetilde{\xi}_F - \gamma_F - \tau k)$$

$$= e^{2\pi i\langle \zeta_F+\zeta_\circ, \widetilde{\xi}_f+\widetilde{\xi}_\circ\rangle} e^{-2\pi i\left(\frac{\|\zeta_F+\zeta_\circ\|^2}{2\tau} + \frac{\langle \zeta_F+\zeta_\circ, \widetilde{\xi}_F+\widetilde{\xi}_\circ\rangle}{\tau}\right)} \frac{e^{\frac{2\pi i}{\tau}\langle \gamma_F, \zeta_\circ\rangle} e^{\frac{2\pi i}{\tau}\langle \widetilde{\xi}_\circ, \gamma_F-\widetilde{\xi}_F\rangle}}{\eta(\tau)^{2\text{rk}\Lambda^\circ}}$$

$$\sum_{k\in F} e^{2\pi i\langle \gamma_\circ, k\rangle} \delta_F(\zeta_F + \widetilde{\xi}_F - \gamma_F - \tau k)$$

$$= e^{2\pi i \langle \zeta_F + \zeta_\circ, \widetilde{\xi}_F + \widetilde{\xi}_\circ \rangle} e^{-2\pi i \left(\frac{\|\zeta_F + \zeta_\circ\|^2}{2\tau} + \frac{\langle \zeta_F + \zeta_\circ, \widetilde{\xi}_F + \widetilde{\xi}_\circ \rangle}{\tau} \right)} \left(\frac{-i\tau}{|\tau|} \right)^{\mathrm{rk}\Lambda^\circ} \chi_{\gamma_F + \gamma_\circ}^{\widetilde{\xi}, \circ} \left(\frac{\zeta}{\tau}, \frac{-1}{\tau} \right) \quad (4.16)$$

$$\square$$

Remark Armed with the above *S*-transformations, we can now propose a Verlinde formula following the standard module formalism of [24] by setting

$$N_{\lambda,\mu}^\rho = \int_{\Lambda^*/\Lambda} \frac{S_{\lambda,\kappa} S_{\mu,\kappa} \overline{S_{\rho,\kappa}}}{S_{0,\kappa}} d\kappa, \qquad \lambda, \mu, \rho \in \Lambda^*/\Lambda \tag{4.17}$$

and asking, if

$$\mathbb{F}_\lambda \boxtimes \mathbb{F}_\mu \cong \int_{\Lambda^*/\Lambda} N_{\lambda,\mu}^\rho \mathbb{F}_\rho \, d\rho \tag{4.18}$$

holds. Indeed a quick calculation reveals that

$$N_{\lambda,\mu}^\rho = \det \langle -, - \rangle|_D \int_{\Lambda^*/\Lambda} e^{-2\pi i \langle \lambda + \mu - \rho, \kappa - \widetilde{\xi} \rangle} d\kappa = \int_{\Lambda^*/\Lambda} e^{-2\pi i \langle \lambda + \mu - \rho, \kappa \rangle} d\kappa$$

$$= \delta(\lambda_V + \mu_V - \rho_v) \delta_{\lambda_F + \mu_F, \rho_F} \left(\sum_{\phi \in \Lambda^\circ} \delta_\Lambda^\circ (\lambda_\circ + \mu_\circ - \rho_\circ - \phi) \right) \delta_{\lambda_D + \mu_D = \rho_D},$$
$$\tag{4.19}$$

which are of course precisely the fusion multiplicities of

$$\mathbb{F}_\lambda \boxtimes \mathbb{F}_\mu = \mathbb{F}_{\lambda + \mu}. \tag{4.20}$$

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