CHIRAL ANOMALY

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I. Introduction

The history of the chiral anomaly is as old as the history of the modern field theory. It began in the evaluation of the process[1)] $\pi^o \to \gamma\gamma$, which still provides the strongest evidence for the physical relevance of the phenomenon called "anomaly".[2~5)] The anomaly in general is regarded as a consequence of the breakdown of the classical symmetry by the quantization procedure. From this view point, it is natural to associate the anomaly with the functional Jacobian factor in the path integral.[6,7)] In the operator formulation, on the other hand, the anomaly is generally accompanied by the anomalous commutators, since the commutators specify the symmetry properties in the canonical formalism.[4,8)] Quite recently it has also been shown that one can construct the non-Abelian anomaly by a powerful differential geometrical method,[9)] which helped to revive our interest in the chiral anomaly. In this last approach, the chiral anomaly is understood as a result of the generalized Atiyah-Singer index theorem.[10)] In my present talk, I concentrate on the path integral formulation. The anomalous commutator is going to be discussed by Professor L. Faddeev.[11)]

II. Non-Abelian Anomaly

It is sometimes convenient to distinguish two kinds of chiral anomalies, namely, the U(1) anomaly and the non-Abelian anomaly. To be specific, for the chiral SU(n) theory specified by

$$\mathcal{L} = \bar{\psi} i\gamma^\mu D_\mu \psi \tag{1}$$

with

$$\begin{aligned} D_\mu &= \partial_\mu - i\left(\frac{1-\gamma_5}{2}\right)A_\mu - i\left(\frac{1+\gamma_5}{2}\right)B_\mu \\ A_\mu &= A^a_\mu T^a \ , \quad B_\mu = B^a_\mu T^a \\ [T^a, T^b] &= if^{abc}T^c \end{aligned} \tag{2}$$

the chiral U(1) anomaly takes the form

$$\begin{aligned} \partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) = &\frac{i}{16\pi^2}\mathrm{Tr}[\tilde{F}^{\mu\nu}(A)F_{\mu\nu}(A) \\ &+ \tilde{F}^{\mu\nu}(B)F_{\mu\nu}(B)] \end{aligned} \tag{3}$$

and the non-Abelian anomaly[2)]

$$\begin{aligned} D_\mu\left(\bar{\psi}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)T^a\psi\right) = &-\frac{i}{24\pi^2}\mathrm{Tr}T^a\varepsilon^{\mu\nu\alpha\beta}\partial_\mu[A_\nu\partial_\alpha A_\beta \\ &- \frac{i}{2}A_\nu A_\alpha A_\beta]. \end{aligned} \tag{4}$$

This form of the anomaly (4) satisfies the Wess-Zumino condition.[3)] In practice, however, the distinction of these two kinds of anomalies becomes less definite. For example, the $\pi^o \to \gamma\gamma$ process could be regarded either as the U(1) anomaly or as the non-Abelian anomaly. In this respect, we note that the non-Abelian anomaly can also be written as[6)]

$$D_\mu\left(\bar{\psi}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)T^a\psi\right) = -\frac{i}{16\pi^2}\mathrm{Tr}[T^a\tilde{F}^{\mu\nu}(A)F_{\mu\nu}(A)] \tag{5}$$

depending on the specification[12)] of det $\not{D}$ or composite current operator $\bar{\psi}\gamma^\mu(\frac{1-\gamma_5}{2})T^a\psi$. This problem has also been clarified by Bardeen and Zumino.[13)] Both of (4) and (5) give rise to the same anomaly-free condition, and they have important applications in different physical contexts.[12,13)] The evaluation of the non-Abelian anomaly is equivalent to the extraction of the phase (or anomaly) factor $A(\alpha)$ from det $\not{D}$ under, for example, an infinitesimal axial gauge transformation:

$$e^{iY(\alpha)}\det \not{D} = e^{iA(\alpha)}\det \not{D} \tag{6}$$

where $Y(\alpha)$ is the Wess-Zumino generator[3)] of axial gauge transformation with a gauge parameter $\alpha(x) \equiv \alpha^a(x)T^a$. From this view point, the anomaly factor (4) corresponds to a non-linear realization of the non-Abelian gauge group in the one-dimensional anomaly factor $A(\alpha)$, whereas (5) corresponds to a linear realization of the one-parameter group $\exp[itY(\alpha)]$ in the anomaly factor $A(\alpha)$, where t is a real parameter with the gauge parameter $\alpha(x)$ kept fixed.[12)] This latter linear realization of the particular element $Y(\alpha)$ spoils the Wess-Zumino condition.

I now want to comment on the point which becomes relevant in the practical applications of non-Abelian anomalies to chiral dynamics. Perturbatively, the anomaly (4) is obtained by the regularization (in the limit $m \to \infty$)[12)]

$$\mathcal{L} = \bar{\psi} i[\not{D} - \not{D}\frac{\partial^2}{m^2}]\psi \tag{7}$$

which preserves the global chiral U(n) x U(n) symmetry but breaks all the local symmetry. This regularization clearly shows that the anomaly written in the form of (4) emphasizes the global chiral symmetry, which is indispensable in pion physics. On the other hand, if one wants to incorporate electro-weak interactions into chiral dynamics,[14] one has to impose the gauge covariance on all the gauge vertices, whether they are vector or axial-vector. This situation is treated by the regularization[12]

$$\mathcal{L} = \bar{\psi} i[\not{D} - \frac{\tilde{\not{D}}\tilde{\not{D}}^{\dagger}\tilde{\not{D}}}{m^2}]\psi \qquad (8)$$

where $\tilde{\not{D}}$ contains only the electro-weak fields whereas $\not{D}$ contains all the fields including hadronic source fields. This regularization (8) gives rise to the "covariant" anomaly (5) for purely gauge vertices and to the "consistent" anomaly (4) for purely hadronic vertices; for general cases, the anomaly factors become covariant under electro-weak gauge transformations.[12] The existence of the regulator (8) shows that the two distinct requirements, namely, the Wess-Zumino condition on non-gauge vertices and the gauge covariance on electro-weak vertices can be consistently imposed in the level of quark diagram calculation.

The discussions of the actual applications of the path integral method and non-Abelian anomalies to quantum chromodynamics are beyond the scope of my present talk, and they will be covered by Professor A. Slavnov in these Proceedings.

III. Gravitational Anomaly[F1)]

Alvarez-Gaumé and Witten[15] recently discovered the anomaly in Einstein's general coordinate transformation for certain chiral fields in 4k+2 dimensional space-time. The local Lorentz anomaly was also evaluated.[16~18] An interesting aspect of the gravitational anomaly is that the local Lorentz anomaly, although it appears in a well-defined manner,[16,18] can be largely subtracted away by local counter terms. A powerful mathematical method[13,19] indicates that the cancellation of general coordinate anomaly alone already ensures the anomaly-free gravitational theory. In the lowest non-trivial order in gravitational interactions, it is also possible to show this by using the non-linear realization of the general coordinate transformation.[20]

I now want to comment on several points which are relevant in the practical calculation of gravitational anomalies. To begin with, it is convenient to write the Einstein equation for quantized matter fields (by taking the variation with respect to the metric)

$$R_{\mu\nu} - \text{[illegible]} - \text{[illegible]} T_{\mu\nu}> . \qquad (9)$$

From this equation, one obtains the constraints on the symmetric part $T^S_{\mu\nu}$ and the anti-symmetric part $T^A_{\mu\nu}$ of the energy-momentum tensor as

$$\begin{aligned} &<T^A_{\mu\nu}> = 0 \\ &D^{\mu}<T^S_{\mu\nu}> = 0 . \end{aligned} \qquad (10)$$

The first condition in (10) indicates the vanishing local Lorentz anomaly and the second relation in (10) implies the absence of the anomaly for a certain combination of general coordinate and local Lorentz transformations.[18] This decomposition (10) clarifies the physical meaning of the anomaly-free condition in the theory with higher spin fields. As for the self-dual anti-symmetric tensor field which appears in certain higher dimensional theories,[15] one can evaluate the anomaly from the chiral Dirac-Kähler fermion (or bi-spinor field) defined by the Lagrangian[18]

$$\mathcal{L} = \bar{\psi}^{\alpha} i\gamma^{\mu}[\partial_{\mu} - \frac{i}{2}A^{mn}_{\mu}(\sigma^{mn} + \tau^{mn})](\frac{1-\gamma_5}{2})\psi_{\alpha} \qquad (11)$$

where τ^{mn} acts on the extra spinor index of ψ_{α}. This procedure is convenient to specify thhe path integral measure, for example. Another interesting aspect of gravitational anomalies is that the different field representation, for example, ψ_{μ} or ψ_a for J = 3/2 (where a stands for the local Lorentz index) gives rise to the different Lorentz anomaly, but the anomaly for the symmetric part $T^S_{\mu\nu}$ is independent of the field representation.[18] As for this peculiar behavior of the Lorentz anomaly, the remarkable analysis of Bardeen and Zumino[13] suggests that the representation dependence of the Lorentz anomaly can also be compensated for by local counter terms, and thus the field representation independence is expected to hold provided that the general coordinate anomaly is canceled in the theory.

Note added

In my oral presentation at the Conference, I stated a stronger constraint which could arise from the local Lorentz anomaly. The interesting question is whether higher symmetries such as local supersymmetry constrain the allowed form of counter terms when one attempts to subtract local Lorentz anomalies.

As for an interesting application of the chiral anomaly, I just mention the contribution to this Conference by K. Ishikawa,[21] who discussed the connection of the chiral anomaly with the quantized Hall effect.

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Footnote

F1) The gravitational anomaly here is meant for the non-Abelian-type gravitational anomaly. As for the U(1)-type gravitational anomaly, see T. Kimura, Prog. Theor. Phys. 42 (1969) 1191; R. Delbourgo and A. Salam, Phys. Lett. 40B (1972) 381; T. Eguchi and P. G. O. Freund, Phys. Rev. Lett. 37 (1976) 1251; N. K. Nielsen, G. T. Grisaru, H. Römer and P. van Nieuwenhuizen, Nucl. Phys. B140 (1978) 477; R. Endo and T. Kimura, Prog. Theor. Phys. 63 (1980) 683.