Three Instanton Computations in Gauge Theory And String Theory

Christopher Edward Beasley

A Dissertation

Presented to the Faculty

of Princeton University

in Candidacy for the Degree

of Doctor of Philosophy

Recommended for Acceptance

by the Department of

Physics

May, 2005

Abstract

We employ a variety of ideas from geometry and topology to perform three new instanton computations in gauge theory and string theory.

First, we consider supersymmetric QCD with gauge group $SU(N_c)$ and with N_f flavors. In this theory, it is well known that instantons generate a superpotential if $N_f = N_c - 1$ and deform the moduli space of supersymmetric vacua if $N_f = N_c$. We extend these results to supersymmetric QCD with $N_f > N_c$ flavors, for which we show that instantons generate a hierarchy of new, multi-fermion F-terms in the effective action.

Second, we revisit the question of which Calabi-Yau compactifications of the heterotic string are stable under worldsheet instanton corrections to the effective space-time superpotential. For instance, compactifications described by $(0, 2)$ linear sigma models are believed to be stable, suggesting a remarkable cancellation among the instanton effects in these theories. We show that this cancellation follows directly from a residue theorem, whose proof relies only upon the right-moving worldsheet supersymmetries and suitable compactness properties of the $(0, 2)$ linear sigma model. We also extend this residue theorem to a new class of "half-linear" sigma models. Using these half-linear models, we show that heterotic compactifications on the quintic hypersurface in $\mathbb{CP}^4$ for which the gauge bundle pulls back from a bundle on $\mathbb{CP}^4$ are stable.

Third, we study Chern-Simons gauge theory on a Seifert manifold M (the total space of a nontrivial circle bundle over a Riemann surface). When M is a Seifert manifold, Lawrence and Rozansky have shown from the exact solution of Chern-Simons theory that the partition function has a remarkably simple structure and can be rewritten entirely as a sum of local "instanton" contributions from the flat connections on M. We explain how this empirical fact follows from the technique of non-abelian localization as applied to the Chern-Simons path integral. In the process, we show that the partition function of Chern-Simons theory on M admits a topological interpretation in terms of the equivariant cohomology of the moduli space of flat connections on M.

Acknowledgements

I would first and foremost like to thank my advisor, Edward Witten, for generously sharing his ideas and his time with me and, in particular, for suggesting the projects that form the basis of this thesis. It has been a great opportunity for me to work with Edward, from whom I have learned much, and this thesis is based entirely on our joint work.

I would also like to thank especially my undergraduate advisor, Ronen Plesser, who first taught me about field theory, string theory, and (most importantly) what it means to conduct research. My work with Ronen, along with his constant encouragement, were decisive factors in my choice to pursue a career in physics.

I take pleasure in thanking Steve Gubser, Igor Klebanov, Alexander Polyakov, Kostas Skenderis, Erik Verlinde, and Herman Verlinde for teaching me many things about fields and strings. I also thank Nissan Itzhaki and Leonardo Rastelli for supervising the Princeton string theory student seminar, which has always been a lot of fun. I am particularly grateful to Greg Moore and Nathan Seiberg for helpful comments on the material in Chapter 2 of this thesis.

In addition, I would like to thank Sujay Ashok, Aaron Bergman, Eleonora Dell'Aquila, Anatoly Dymarsky, Tamar Friedmann, Josh Friess, Chris Herzog, Joanna Karczmarek, Justin Khoury, Liat Maoz, John McGreevy, Indrajit Mitra, Sameer Murthy, Peter Ouyang, John Pearson, Mukund Rangamani, Slava Rychkov, Natalia Saulina, David Shih, Peter Svrček, and Martijn Wijnholt for many stimulating conversations about my research and theirs. For a variety of conversations about physics and mathematics in general, I would also like to thank Sasha Baitine, Latham Boyle, Davide Gaiotto, Pedro Goldbaum, Justin

Kinney, Subroto Mukerjee, David Olson, Nikhil Padmanabhan, Kumar Raman, Sri Raghu, and Emil Yuzbashyan.

For no reason other than that they are my first friends, I would like to acknowledge Charles, Clay, Dan, Hodges, Hubbard, Kulmer, Lindgren, Micah, and Norm. Last but not least, I would like to thank my dad for explaining to me once upon a time about baseballs and flashlights on moving trains.

This thesis is dedicated to my parents.

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Chapter 1

Introduction

Perturbation theory is a useful tool to describe quantitatively the behavior of weakly-coupled, interacting quantum systems. However, even at weak coupling, certain important and intrinsically quantum phenomena cannot be seen in perturbation theory.

As a simple example, consider a single bosonic particle of unit mass which moves along the real axis in the double-well potential $U(x)$ in Figure 1.1. We assume that the potential

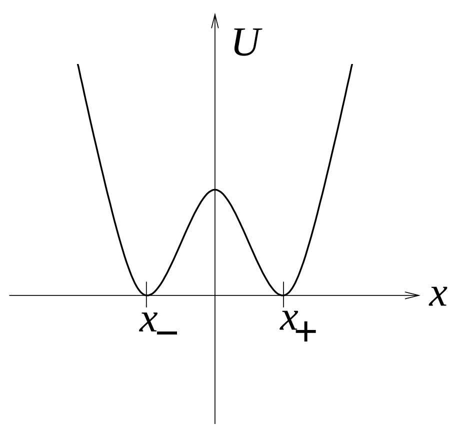


Figure 1.1: *Double-well potential*

is invariant under the reflection $x \to -x$, so that $U(x) = U(-x)$, and in particular the Taylor expansions of $U(x)$ about the minima labelled x_+ and x_- in Figure 1.1 coincide,

$$U(\delta x_\pm) \, = \, \frac{1}{2}\omega^2 \, \delta x_\pm^2 + \cdots , \qquad \delta x_\pm = \pm \left(x - x_\pm\right) \, . \tag{1.0.1}$$

If we treat the higher order terms which we have omitted from (1.0.1) as perturbations, then to all orders in $\hbar$ the quantum mechanical system possesses a pair of degenerate groundstates whose wavefunctions are localized in the respective wells of $U(x)$. We denote this pair of states by $|\Omega_\pm\rangle$.

On the other hand, one can prove by quite general arguments (see for instance Lecture 1 of [1]) that this quantum system actually has a unique groundstate $|0\rangle$, which must arise from the symmetric, reflection-invariant combination $|\Omega_+\rangle + |\Omega_-\rangle$. Consequently, effects which are non-perturbative in $\hbar$ and which are associated to quantum tunneling between the wells necessarily lift the degeneracy of the system, which is an important qualitative effect that cannot be seen at any order in perturbation theory.

As was explained long ago by Polyakov [2] and later amplified by Coleman in Chapter 7 of his beautiful lectures [3] (on which the following exposition is based), the non-perturbative effects which lift the degeneracy between the states $|\Omega_+\rangle$ and $|\Omega_-\rangle$ can be understood — and even computed — semi-classically as *instanton* effects. Such instanton effects are broadly the subject of this thesis, and as we illustrate here with this toy model, the corresponding semi-classical analysis of quantum systems can provide powerful insight into otherwise intractable, non-perturbative phenomena. These semi-classical ideas become particularly important in the context of string theory or M-theory, for which a complete, non-perturbative definition of the theory is generally lacking.

1.1 What Is an Instanton?

The essential idea of any instanton computation is simply to apply the stationary phase approximation (or the method of steepest descent) to compute the Feynman path integral for a given quantum system in its semi-classical limit. Although at first glance one might not expect to learn much by using such a crude description of the path integral, the stationary phase approximation can actually capture quite non-trivial information about the quantum system.

As a concrete example, let us continue to consider the toy model of a particle moving in the double-well potential of Figure 1.1. We wish to compute the splitting in energy between the groundstate $|0\rangle$ and the first excited state $|1\rangle$ to leading order in $\hbar$. We denote this splitting by $\Delta E = E_1 - E_0$, and our goal is to show directly that ΔE is non-zero. This computation turns out to be very illustrative of the general features of any instanton computation, so we present it in detail. (The reader who wishes to proceed to matters more immediately related to the thesis can skip to Section 2.)

Setting Up the Instanton Computation

We first recall how to extract the splitting ΔE from the Feynman path integral description of this quantum system. As is standard, we denote by $|x'\rangle$ the position eigenstate associated to a given point x' on the real axis. Then to extract the splitting ΔE from the path integral, we consider the Euclidean Green's function describing the propagation of the particle from an initial point x_i to a final point x_f over a Euclidean time interval τ. This propagator has the following path integral description,

$$\langle x_f| \exp\left(-\frac{\tau}{\hbar}H\right)|x_i\rangle \;=\; \int_{x(-\tau/2)=x_i}^{x(\tau/2)=x_f} \mathcal{D}x(t)\, \exp\left(-\frac{1}{\hbar}\,S_E\right). \tag{1.1.1}$$

Here H is the Hamiltonian associated to the potential $U(x)$, S_E is the Euclidean action,

$$S_E \;=\; \int_{-\tau/2}^{\tau/2} dt\, \left[\frac{1}{2}\dot{x}^2 + U(x)\right], \tag{1.1.2}$$

and the path integral is formally defined as an integral over all paths $x(t)$ starting at x_i at time $t = -\frac{\tau}{2}$ and ending at x_f at time $t = \frac{\tau}{2}$.

If we know the propagator in (1.1.1), then we can use its expansion over all eigenmodes $|n\rangle$ of H to study the spectrum of energies E_n, since

$$\langle x_f| \exp\left(-\frac{\tau}{\hbar}H\right)|x_i\rangle \;=\; \sum_{n\geq 0} \exp\left(-\frac{\tau}{\hbar}E_n\right)\langle x_f|n\rangle\langle n|x_i\rangle\,. \tag{1.1.3}$$

To extract the splitting $\Delta E = E_1 - E_0$ from the expression in (1.1.3), we note that the groundstate $|0\rangle$ arises from the reflection-symmetric linear combination of the two perturbative states $|\Omega_\pm\rangle$, and hence the excited state $|1\rangle$ arises from the anti-symmetric linear combination of these states. In particular, the anti-symmetric state $|a\rangle$,

$$|a\rangle \;=\; |x_+\rangle - |x_-\rangle\,, \tag{1.1.4}$$

is orthogonal to $|0\rangle$ but not to $|1\rangle$. Similarly, the symmetric state $|s\rangle$,

$$|s\rangle \;=\; |x_+\rangle + |x_-\rangle\,, \tag{1.1.5}$$

is orthogonal to $|1\rangle$ but not to $|0\rangle$. So we see from (1.1.3) that when τ is large,

$$\frac{\langle a|\exp\left(-\tau H/\hbar\right)|a\rangle}{\langle s|\exp\left(-\tau H/\hbar\right)|s\rangle} \;=\; c\,\exp\left(-\frac{\tau}{\hbar}\Delta E\right) + \cdots\,. \tag{1.1.6}$$

Here c is an irrelevant constant that arises from the ratio $|\langle a|1\rangle|^2 \cdot |\langle s|0\rangle|^{-2}$, and the ellipses in (1.1.6) denote terms which are exponentially suppressed relative to the given term at large τ.

Finally, since the system is reflection-invariant, we note that

$$\begin{aligned} \langle x_+|\exp\left(-\frac{\tau}{\hbar}H\right)|x_+\rangle \;&=\; \langle x_-|\exp\left(-\frac{\tau}{\hbar}H\right)|x_-\rangle\,,\\ \langle x_+|\exp\left(-\frac{\tau}{\hbar}H\right)|x_-\rangle \;&=\; \langle x_-|\exp\left(-\frac{\tau}{\hbar}H\right)|x_+\rangle\,. \end{aligned} \tag{1.1.7}$$

Also, as we will show next, the propagators from x_+ to x_- and vice versa in the second line of (1.1.7) are exponentially suppressed at small $\hbar$ relative to the propagators from x_+ and from x_- back to themselves in the first line of (1.1.7). With these facts, we can approximate the left side of (1.1.6) as

$$\frac{\langle a|\exp\left(-\tau H/\hbar\right)|a\rangle}{\langle s|\exp\left(-\tau H/\hbar\right)|s\rangle} \;=\; 1 - \frac{\langle x_+|\exp\left(-\tau H/\hbar\right)|x_-\rangle}{\langle x_-|\exp\left(-\tau H/\hbar\right)|x_-\rangle} + \cdots\,, \tag{1.1.8}$$

where we again drop terms exponentially suppressed at large τ relative to those appearing explicitly in (1.1.8). From (1.1.6) and (1.1.8), we take logarithms to conclude that, to leading order in τ,

$$\frac{\langle x_+|\exp\left(-\tau H/\hbar\right)|x_-\rangle}{\langle x_-|\exp\left(-\tau H/\hbar\right)|x_-\rangle} \;=\; \frac{\tau}{\hbar}\,\Delta E \,+\, \mathcal{O}(\tau^0)\,. \tag{1.1.9}$$

From the path integrals which represent the Green's functions in (1.1.9), we extract ΔE. So we set

$$\begin{aligned} Z_{+-}(\tau) \;&=\; \langle x_+|\exp\left(-\tau H/\hbar\right)|x_-\rangle \;=\; \int_{x(-\tau/2)=x_-}^{x(\tau/2)=x_+} \mathcal{D}x(t)\,\exp\left(-\frac{1}{\hbar}S_E\right)\,,\\ Z_{--}(\tau) \;&=\; \langle x_-|\exp\left(-\tau H/\hbar\right)|x_-\rangle \;=\; \int_{x(-\tau/2)=x_-}^{x(\tau/2)=x_-} \mathcal{D}x(t)\,\exp\left(-\frac{1}{\hbar}S_E\right)\,. \end{aligned} \tag{1.1.10}$$

An Instanton Computation for the Double-Well Potential

So far, we have only described in (1.1.9) how to extract the splitting ΔE from the propagators of the system. We now compute these propagators semi-classically from their path integral description in (1.1.10). This computation serves as a canonical example of an instanton computation.

As we have mentioned, the basic idea of the instanton computation is to apply the method of steepest descent to compute semi-classically the path integrals appearing in (1.1.10). In this approximation, the leading contributions to the path integral come from critical points of the Euclidean action S_E, which here correspond to trajectories that satisfy

$$0 = \frac{\delta S_E}{\delta x} = -\ddot{x} + \frac{dU}{dx} \,. \tag{1.1.11}$$

This equation (1.1.11) is the equation of motion for a particle that moves in the inverted potential $-U(x)$ shown in Figure 1.2.

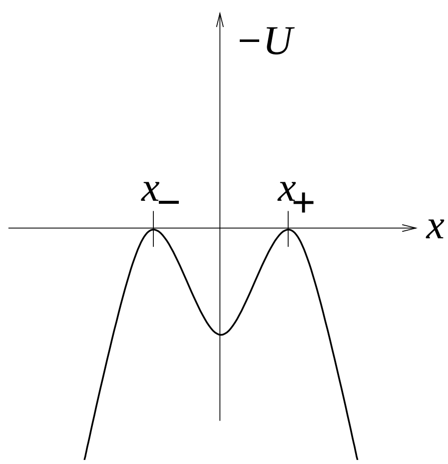


Figure 1.2: *Inverted double-well potential*

For instance, if we consider the path integral over trajectories $x(t)$ that satisfy the trivial boundary conditions $x(-\tau/2) = x(\tau/2) = x_-$, then the leading semi-classical contribution to the path integral $Z_{--}(\tau)$ in (1.1.10) comes from the constant trajectory $x(t) = x_-$.

As a much more interesting case, we consider the path integral $Z_{+-}(\tau)$ over paths with the boundary conditions $x(-\tau/2) = x_-$ and $x(\tau/2) = x_+$. Then the leading semi-classical contribution to $Z_{+-}(\tau)$ comes from the obvious classical trajectory by which the particle "rolls" from x_- to x_+ in Figure 1.2. When τ is very large, the energy $E' = \frac{1}{2}\dot{x}^2 - U(x)$ of the particle moving in the potential $-U(x)$ is nearly zero, and such trajectories appear as in Figure 1.3.

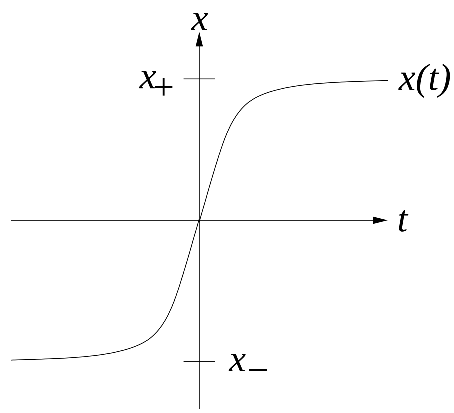


Figure 1.3: *An instanton solution in the double-well potential*

The important feature of the classical trajectory in Figure 1.3 is that it has a kink which is localized in time. Analytically, this fact follows from the observation that a trajectory with $E' = \frac{1}{2}\dot{x}^2 - U(x) = 0$ satisfies

$$\dot{x} \;=\; \sqrt{2U(x)}\,, \tag{1.1.12}$$

and near the minima $x_\pm$, the potential is well-approximated by

$$U(\delta x_\pm) \;\approx\; \frac{1}{2}\omega^2\,\delta x_\pm^2\,, \qquad \delta x_\pm \;=\; \pm\,(x - x_\pm)\;. \tag{1.1.13}$$

Hence for x near x_+, we see from (1.1.12) and (1.1.13) that the trajectory $x(t)$ approaches the fixed point x_+ exponentially fast,

$$(x_+ - x) \;\approx\; \exp\left(-\omega t\right), \tag{1.1.14}$$

and similarly for x_-. Because the kink is thus localized in an instant of time, with a width $\sim 1/\omega$ much less than τ, this classical solution is an "instanton".

Another very important distinguishing feature of the instanton solution is the following. If we consider the trivial trajectory $x(t) = x_-$, then we immediately see that $S_E = 0$ for this classical solution, and hence it attains the absolute minimum of the (manifestly positive) Euclidean action. In contrast, the instanton solution arises from a higher critical point of S_E with non-zero, finite action. More precisely, if we consider the solution depicted in Figure 1.3 and we recall the relation (1.1.12), then we compute the Euclidean action I_0 of this instanton as

$$I_0 \approx \int_{-\infty}^{\infty} dt \left[\frac{1}{2}\dot{x}^2 + U(x)\right] = \int_{-\infty}^{\infty} dt \left(\frac{dx}{dt}\right)^2 = \int_{x_-}^{x_+} dx \sqrt{2U(x)}\,. \tag{1.1.15}$$

To explain why the finite action I_0 of the instanton solution is an essential feature, let us recall more precisely how we apply the method of steepest descent to approximate the path integrals $Z_{+-}(\tau)$ and $Z_{--}(\tau)$ in (1.1.10). For each classical trajectory $x_0(t)$, we parametrize a neighborhood of this trajectory in the space of all paths as

$$x(t) = x_0(t) + \sum_m c_m\, x_m(t)\,. \tag{1.1.16}$$

Here the fluctuating modes $x_m(t)$ are a complete set of orthonormal functions which satisfy the boundary conditions $x_m(\pm\tau/2) = 0$, and the parameters c_m are coefficients which specify an arbitrary such fluctuation about the classical trajectory $x_0(t)$.

We now expand the action $S_E[x(t)]$ as a functional of the path $x(t)$ to quadratic order about $x_0(t)$, so that

$$S_E[x(t)] \approx S_E[x_0(t)] + \frac{1}{2}\left(\frac{\delta^2 S_E[x_0(t)]}{\delta x_m \delta x_n}\right) c_m c_n\,. \tag{1.1.17}$$

Because $x_0(t)$ is a critical point of the functional $S_E[x]$, no linear term appears in (1.1.17).

To obtain the leading semi-classical behavior of the path integral, we just evaluate the integral over fluctuations about $x_0(t)$ in the Gaussian approximation, so that

$$\begin{aligned} Z(\tau)\Big|_{x_0(t)} &\approx Z_0 \int_{x(-\tau/2)=x_i}^{x(\tau/2)=x_f} \left[\frac{dc}{\sqrt{2\pi\hbar}}\right] \exp\left[-\frac{1}{\hbar}\left(S_E[x_0(t)] + \frac{1}{2}\left(\frac{\delta^2 S_E[x_0(t)]}{\delta x_m \delta x_n}\right) c_m c_n\right)\right], \\ &= Z_0 \exp\left(-\frac{1}{\hbar} S_E[x_0(t)]\right) \left[\det\left(\frac{\delta^2 S_E[x_0(t)]}{\delta x_m \delta x_n}\right)\right]^{-\frac{1}{2}}. \end{aligned} \tag{1.1.18}$$

Here Z_0 is a normalization constant that generally appears when we relate the path integral measure $\mathcal{D}x(t)$, which is defined implicity by the relation (1.1.1), to the natural measure that appears in the Gaussian integral above,

$$\left[\frac{dc}{\sqrt{2\pi\hbar}}\right] \equiv \prod_m \left(\frac{dc_m}{\sqrt{2\pi\hbar}}\right) . \tag{1.1.19}$$

So we see from (1.1.18) that the leading contribution from the classical trajectory $x_0(t)$ to the path integral is just determined by its classical action $S_E[x_0(t)]$, and at the next, one-loop order, a determinant arises from the Gaussian integral over the variables c_m parametrizing the quantum fluctuations about $x_0(t)$. For simplicity in writing (1.1.18), we assume here that the Hessian matrix of second derivatives of S_E is positive-definite, implying that $x_0(t)$ is an isolated, locally stable critical point of S_E. In particular, since the trivial path $x_0(t) = x_-$ with $S_E = 0$ makes the leading contribution to $Z_{--}(\tau)$, and since the non-trivial instanton solution in Figure 1.3 with $S_E \neq 0$ makes the leading contribution to $Z_{+-}(\tau)$, we immediately conclude that $Z_{+-}(\tau)$ is exponentially suppressed relative to $Z_{--}(\tau)$ when $\hbar$ is small.

On the other hand, $Z_{+-}(\tau)$ is non-zero, and this fact fundamentally leads to the presence of the non-perturbative splitting ΔE. To prove this assertion, we just apply the semi-classical result (1.1.18) to compute $Z_{--}(\tau)$ and $Z_{+-}(\tau)$ and hence to compute ΔE.

First, from (1.1.18) we can immediately write a formula for $Z_{--}(\tau)$ to leading order in $\hbar$. In this context, an interesting general observation to make is that, besides the contribution from the constant solution, $Z_{--}(\tau)$ also receives semi-classical contributions from non-trivial, multi-instanton solutions which appear as in Figure 1.4. In this figure, we consider time scales τ much longer than the width $\sim 1/\omega$ of a single instanton, so the individual instantons appear as sharp jumps from one well to the other.

However, we only wish to compute here $Z_{--}(\tau)$ to leading order in $\hbar$. If the multi-instanton solution has N instantons (or kinks), then clearly the classical action of this solution is NI_0, where I_0 is the action for a single instanton in (1.1.15). Such multi-instantons consequently make contributions to $Z_{--}(\tau)$ of order $\exp(-NI_0/\hbar)$, and these contributions are exponentially suppressed relative to the contribution from the constant trajectory $x(t) = x_-$ of vanishing action.

Thus, the leading contribution to $Z_{--}(\tau)$ comes from the constant trajectory, and to

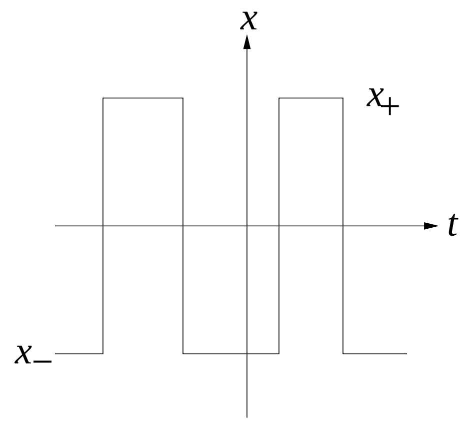


Figure 1.4: *A multi-instanton solution*

evaluate its contribution to the path integral we must consider the one-loop determinant describing fluctuations about $x(t) = x_-$. Expanding the potential $U(x)$ to quadratic order about x_-, we can formally express this determinant as

$$\det\left(\frac{\delta^2 S_E[x_0]}{\delta x_m \delta x_n}\right) \;=\; \det\left(-\frac{d^2}{dt^2} + \omega^2\right). \tag{1.1.20}$$

Hence, to leading order in $\hbar$,

$$Z_{--}(\tau) \;=\; Z_0 \left[\det\left(-\frac{d^2}{dt^2} + \omega^2\right)\right]^{-\frac{1}{2}}. \tag{1.1.21}$$

Similarly, the leading semi-classical contribution to $Z_{+-}(\tau)$ comes from the instanton solution in Figure 1.3. However, unlike the constant solution which represents an isolated critical point of S_E, this instanton solution is actually a member of a one-parameter family of critical points of S_E. To explain this fact, we let $\widetilde{x}(t)$ denote the instanton solution in Figure 1.3, which crosses the point $x = 0$ at time $t = 0$. Because the classical equation of motion (1.1.11) is invariant under time-translation, when τ is large then for any constant t_0 we can define another instanton solution by $\widetilde{x}(t; t_0) \equiv \widetilde{x}(t - t_0)$.

The parameter t_0, which represents the "position" of the kink in time, is thus a collective coordinate for the instanton solution, and when we compute the instanton contribution to $Z_{+-}(\tau)$ we must integrate over this coordinate t_0. All instantons in this family make the same contribution to $Z_{+-}(\tau)$, so the integral over t_0 simply contributes the overall factor

$$\int_{-\tau/2}^{\tau/2} \frac{dt_0}{\sqrt{2\pi\hbar}} \;=\; \frac{\tau}{\sqrt{2\pi\hbar}}\,. \tag{1.1.22}$$

As is usual in soliton computations of this sort, the factor $1/\sqrt{\hbar}$ appears in the measure for each collective coordinate. This factor arises directly from the corresponding factor in the measure on the parameters c_m in (1.1.19).

Another factor that we must also consider is the one-loop determinant associated to quantum fluctuations about the instanton solution $\widetilde{x}(t)$. Formally, this one-loop determinant is given by

$$\det{}'\left(\frac{\delta^2 S_E[\widetilde{x}(t)]}{\delta x_m \delta x_n}\right) \;=\; \det{}'\left(-\frac{d^2}{dt^2} + \frac{d^2U}{dx^2}\bigg|_{\widetilde{x}(t)}\right). \tag{1.1.23}$$

Here the notation "det′" indicates that the zero-mode associated to the collective coordinate t_0 of the instanton is omitted from the determinant, since otherwise the determinant vanishes.

As we indicate, the second derivative of the potential $U(x)$ must be evaluated on the instanton solution $\widetilde{x}(t)$ and hence is time-dependent. However, since τ is large, the solution $\widetilde{x}(t)$ is essentially constant and equal to x_- or x_+ for the vast majority of the time. As a result, the determinant in (1.1.23) for the non-trivial instanton solution essentially reduces to the corresponding determinant for the trivial solution in (1.1.20), up to a multiplicative correction to account for the small time interval containing the kink,

$$\det{}'\left(\frac{\delta^2 S_E[x_0]}{\delta x_m \delta x_n}\right) \;=\; K \, \det\left(-\frac{d^2}{dt^2} + \omega^2\right). \tag{1.1.24}$$

This equation (1.1.24) defines the (dimensionful) constant K, which is independent of τ for large τ.

Assembling these factors (1.1.22) and (1.1.24), we compute $Z_{+-}(\tau)$ at leading order in $\hbar$ to be

$$Z_{+-}(\tau) \;=\; Z_0 \, \frac{\tau}{\sqrt{2\pi\hbar K}} \, \exp\left(-\frac{I_0}{\hbar}\right) \left[\det\left(-\frac{d^2}{dt^2} + \omega^2\right)\right]^{-\frac{1}{2}}. \tag{1.1.25}$$

From (1.1.21) and (1.1.25), the ratio of $Z_{+-}(\tau)$ to $Z_{--}(\tau)$ is given for small $\hbar$ and large τ by

$$\frac{Z_{+-}(\tau)}{Z_{--}(\tau)} \;=\; \frac{\tau}{\sqrt{2\pi\hbar K}} \, \exp\left(-\frac{I_0}{\hbar}\right). \tag{1.1.26}$$

Hence from (1.1.9) and (1.1.10) we compute the splitting between the groundstate and the first excited state to be

$$\Delta E = \sqrt{\frac{\hbar}{2\pi K}}\, \exp\left(-\frac{I_0}{\hbar}\right). \tag{1.1.27}$$

In particular, ΔE is non-zero, and the leading contribution to the splitting arises semi-classically from the instanton solution.

1.2 An Overview of the Thesis

In the rest of the thesis, we perform three new instanton computations in gauge theory and string theory. The basic philosophy of an instanton computation in these quantum systems with infinitely-many degrees of freedom is exactly the same as in the single-particle toy model: we simply compute a path integral by summing over the semi-classical contributions from suitable classical solutions.

More precisely, the three theories in which we perform instanton computations are $\mathcal{N} = 1$ supersymmetric QCD, heterotic string theory compactified to four dimensions on a Calabi-Yau threefold, and Chern-Simons gauge theory on a three-manifold. We devote Chapters 2, 3, and 4 of the thesis to the study of these theories respectively, and in each chapter we provide an introduction to our work therein. However, we find it useful to include here a brief overview of the main results in each chapter.

The material in Chapters 2, 3, and 4 of this thesis is based on joint work with Edward Witten and has appeared in [4–6].

1.2.1 Instantons in Supersymmetric QCD

In Chapter 2, we consider supersymmetric QCD (or SQCD) with gauge group $SU(N_c)$ and with N_f flavors, each flavor being a massless chiral multiplet transforming in the direct sum of the fundamental plus the anti-fundamental representations of the gauge group. Our main result is to show that, in the regime $N_f > N_c$, supersymmetric instantons generate a hierarchy of new, multi-fermion F-terms in the low-energy effective action of this theory. Among other interactions, such F-terms describe effective vertices for $2(N_f - N_c) + 4$ fermions, and hence the name.

These new instanton effects are quite subtle, as they do not change the classical geometry of the moduli space of supersymmetric vacua and do not qualitatively alter the low-energy physics. Nonetheless, they naturally generalize two famous and much more drastic instanton effects which occur in SQCD with $N_f = N_c - 1$ and with $N_f = N_c$ flavors. In the remainder of this section, we briefly review these two basic examples.

The Affleck-Dine-Seiberg Superpotential

We begin by considering SQCD with $N_f = N_c - 1$ flavors. As shown by Affleck, Dine, and Seiberg [7], supersymmetric instantons in this theory generate a superpotential which completely lifts the moduli space and dynamically breaks supersymmetry. These instantons in SQCD naturally generalize the self-dual or anti-self-dual solutions of pure Yang-Mills theory first considered by Belavin, Polyakov, Schwartz, and Tyupkin [8], and the Affleck-Dine-Seiberg computation is a supersymmetric extension of the foundational instanton computation by 't Hooft [9] in non-supersymmetric QCD.

We will not review the details of the Affleck-Dine-Seiberg instanton computation here. However, we will review the most important feature of this computation, which is the fact that the form of any superpotential term in SQCD with $N_f < N_c$ flavors is completely determined by the symmetries of the theory and holomorphy. We later apply this general analysis to the multi-fermion F-terms we consider in Chapter 2.

We first introduce the notation Q^i_a and $\widetilde{Q}^a_i$ to denote the quark and the anti-quark chiral superfields in SQCD, where $a = 1, \ldots, N_c$ is a color index and $i = 1, \ldots, N_f$ is a flavor index. In the regime $N_f < N_c$, all gauge-invariant chiral operators that can be made from the superfields Q^i_a and $\widetilde{Q}^a_i$ are functions of the composite meson superfields M^i_j,

$$M^i_j = Q^i_a \widetilde{Q}^a_j \,. \tag{1.2.1}$$

In particular, no chiral baryons or anti-baryons are present in SQCD in the regime $N_f < N_c$. Any superpotential W that could be generated is consequently a function of the mesons

M^i_j.

Of course, the superpotential W cannot be an arbitrary function of M^i_j but must respect the global symmetries of SQCD. Besides the $SU(N_c)$ gauge symmetry, SQCD with N_f flavors has a large group of non-anomalous global symmetries, which is given by

$$SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_{\mathcal{R}} \,. \tag{1.2.2}$$

Here $U(1)_B$ is a baryon number symmetry and $U(1)_{\mathcal{R}}$ is an $\mathcal{R}$-symmetry. The superpotential W must be invariant under the subgroup $SU(N_f) \times SU(N_f) \times U(1)_B$, and W must have charge $+2$ under the $\mathcal{R}$-symmetry. Furthermore, SQCD has an anomalous axial $U(1)_A$ symmetry. This anomalous symmetry imposes an additional selection rule on the form of W once we consider the standard holomorphic coupling scale Λ to transform under its action. We summarize below the action of these symmetries on the fields Q^i_a, $\widetilde{Q}^a_i$, and the coupling Λ.

	$SU(N_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_A$	$U(1)_{\mathcal{R}}$
Q^i_a	$\mathbf{N_c}$	$\mathbf{N_f}$	$\mathbf{1}$	1	1	$1 - \frac{N_c}{N_f}$
$\widetilde{Q}^a_i$	$\overline{\mathbf{N}}_\mathbf{c}$	$\mathbf{1}$	$\mathbf{N_f}$	-1	1	$1 - \frac{N_c}{N_f}$
$\Lambda^{3N_c-N_f}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	$2N_f$	0

(1.2.3)

The quantity $\Lambda^{3N_c-N_f}$ that appears in (1.2.3) is particularly natural, since it is precisely the instanton counting parameter in SQCD with gauge group $SU(N_c)$ and with N_f flavors. That is, $\Lambda^{3N_c-N_f}$ plays a role analogous to the classical factor $\exp\left(-I_0/\hbar\right)$ in the toy model we considered earlier.

To fix the form of W, we observe that the only $SU(N_f) \times SU(N_f)$ invariant function of M^i_j is the determinant $\det M$, or a power thereof. The condition that W have $\mathcal{R}$-charge $+2$ determines the power of $\det M$ that can appear, and the condition that W be invariant under the axial symmetry determines the corresponding power of Λ. As a result, any non-perturbative superpotential in SQCD with $N_f < N_c$ flavors necessarily takes the form

$$W = \left[\frac{\Lambda^{3N_c-N_f}}{\det M}\right]^{\frac{1}{N_c-N_f}} \,. \tag{1.2.4}$$

In particular, for the case $N_f = N_c - 1$, the Affleck-Dine-Seiberg superpotential is

$$W = \frac{\Lambda^{3N_c - N_f}}{\det M}, \qquad N_f = N_c - 1. \tag{1.2.5}$$

We note that in precisely this case an integral power of the instanton-counting parameter $\Lambda^{3N_c - N_f}$ appears in (1.2.4). Indeed, the superpotential (1.2.5) depends on a single power of this parameter. Hence this superpotential can be generated by a single instanton, and the main result of [7] is to show that a single instanton does make a non-zero contribution to it.

By considering a general supersymmetric mass deformation of SQCD, one can also deduce that the superpotential for general $N_f < N_c$ in (1.2.4) is generated non-perturbatively, though not by simple instanton effects except in the special case $N_f = N_c - 1$. The superpotential in (1.2.4) has a drastic effect on the infrared structure of SQCD with $N_f < N_c$ flavors. For any theory with global $\mathcal{N} = 1$ supersymmetry, a supersymmetric vacuum must be a critical point of W, at which $dW = 0$. Yet the Affleck-Dine-Seiberg superpotential in (1.2.4) has no critical points away from $M \to \infty$. Thus, the superpotential lifts completely the moduli space of supersymmetric vacua and dynamically breaks supersymmetry.

The Complex Structure Deformation of the Moduli Space

The non-perturbative superpotential in (1.2.4) is only generated in SQCD in the regime $N_f < N_c$, and one can prove, again using symmetries and holomorphy, that no superpotential can be generated in SQCD when $N_f \geq N_c$. For instance, if we consider the expression for W in (1.2.4), then for $N_f < N_c$ we see that this expression vanishes in the weak-coupling limit $\Lambda \to 0$ as it must; for $N_f \geq N_c$, this expression does not vanish as $\Lambda \to 0$ and hence cannot be generated.

However, as observed by Seiberg [10] (see also [11] for a related analysis), a very interesting and somewhat more subtle instanton effect still occurs in SQCD with $N_f = N_c$ flavors. In this case, the instanton does not generate a superpotential in the low-energy effective action, but it does deform the complex structure of the classical moduli space of supersymmetric vacua. This result is essential for our work in Chapter 2, so we briefly review it here.

To distinguish the classical from the quantum moduli space, we introduce the notation $\mathcal{M}_{cl}$ for the classical moduli space of supersymmetric vacua, and we let $\mathcal{M}$ denote the exact, quantum moduli space. In general, the classical moduli space $\mathcal{M}_{cl}$ of SQCD is parametrized by the expectation values of the chiral, composite meson and baryon operators in this theory. However, these expectation values are not arbitrary but satisfy a set of classical constraints that follow immediately from the definition of the composite mesons and baryons in terms of the quarks Q^i_a and the anti-quarks $\widetilde{Q}^a_i$.

The classical moduli space of the theory with $N_f = N_c$ flavors is particularly simple, as it can be described with only a single constraint. Besides the mesons M^i_j in (1.2.1), SQCD with $N_f = N_c$ flavors possesses a single baryon B and a single anti-baryon $\widetilde{B}$, which are given by

$$B \,=\, \epsilon_{i_1\cdots i_{N_f}}\,\epsilon^{a_1\cdots a_{N_c}}\,Q^{i_1}_{a_1}\cdots Q^{i_{N_f}}_{a_{N_c}}\,, \qquad \widetilde{B} \,=\, \epsilon^{i_1\cdots i_{N_f}}\,\epsilon_{a_1\cdots a_{N_c}}\,\widetilde{Q}^{a_1}_{i_1}\cdots\widetilde{Q}^{a_{N_c}}_{i_{N_f}}\,. \tag{1.2.6}$$

Here the ϵ-tensors denote the standard invariant, anti-symmetric tensors of the special unitary group.

The expectation values of the mesons M^i_j, the baryon B, and the anti-baryon $\widetilde{B}$ are not arbitrary but satisfy the obvious classical constraint,

$$\det M - B\,\widetilde{B} \,=\, 0\,. \tag{1.2.7}$$

Hence the classical moduli space $\mathcal{M}_{cl}$ is a hypersurface parametrized by all expectation values of M^i_j, B, and $\widetilde{B}$ consistent with the constraint (1.2.7).

As shown by Seiberg [10], the classical constraint (1.2.7) is modified in the full quantum theory to become

$$\det M - B\,\widetilde{B} \,=\, \Lambda^{2N_c}\,. \tag{1.2.8}$$

The form of this deformation is fixed completely by the global symmetries of SQCD and by dimensional analysis. The deformation in (1.2.8) does not alter the asymptotic structure of the moduli space far from the origin, where the theory is weakly-coupled, but it drastically alters the structure at the origin. In particular, the deformation (1.2.8) removes the conical singularity at the origin of $\mathcal{M}_{cl}$ which is associated to the unbroken classical gauge symmetry at that point, and the quantum moduli space $\mathcal{M}$ defined by the constraint (1.2.8) is smooth. Physically, the disappearance of the singularity in the classical moduli space is associated to confinement, since in a confining vacuum massless gluons cannot be seen.

Schematically, this deformation from $\mathcal{M}_{cl}$ to $\mathcal{M}$ appears as in Figure 1.5.

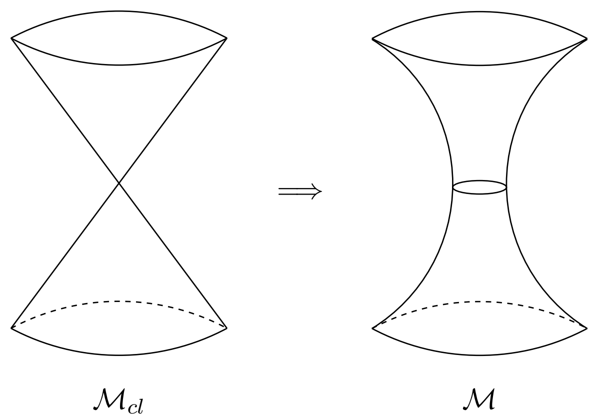


Figure 1.5: *Quantum deformation of the SQCD moduli space*

As Seiberg noted, the deformation from $\mathcal{M}_{cl}$ to $\mathcal{M}$ can also be understood as a one-instanton effect. This interpretation is suggested by the fact that the deformation in (1.2.8) depends on precisely one power of the instanton-counting parameter Λ^{2N_c} appropriate for SQCD with $N_f = N_c$ flavors. We check this fact directly in Chapter 2, where we actually perform an instanton computation to demonstrate this quantum effect.

1.2.2 Worldsheet Instantons in Heterotic String Theory

In Chapter 3, we study heterotic string theory compactified to four-dimensional Minkowski space on a Calabi-Yau threefold X, with a stable, holomorphic gauge bundle E over X. With these conditions on X and E, the background preserves $\mathcal{N} = 1$ supersymmetry in four dimensions, and the worldsheet sigma model generically has $(0, 2)$ supersymmetry.

Associated to the choices of X and E are various continuous parameters which describe the classical moduli space of the heterotic compactification. For instance, we must choose a complex structure on X to specify it as a complex manifold, and we must choose a Kähler class on X to fix its Ricci-flat metric. This Kähler class is naturally complexified once we choose a background configuration for the heterotic B-field. Finally, we must choose the complex structure on the bundle E. In the four-dimensional, low-energy effective action that describes the heterotic compactification, these complex moduli associated to the pair (X, E) are respresented by a set of light chiral superfields which are singlets under the unbroken gauge group.

We have described the classical moduli space of this compactification, but as in the example of SQCD with $N_f < N_c$ flavors, quantum effects can drastically change the picture. Just as in SQCD, instantons can contribute to a superpotential for the singlet chiral superfields that lifts flat directions on the moduli space and, in the case of a theory with local $\mathcal{N} = 1$ supersymmetry, generates a non-zero cosmological constant.

The instantons which we consider in Chapter 3 are worldsheet instantons, and one nice feature of these instantons is that they can be studied perturbatively in the string genus expansion. As shown by Dine, Seiberg, Wen, and Witten [12], the supersymmetric worldsheet instantons which can contribute to a superpotential are those described by (nontrivial) holomorphic maps $\Phi : \Sigma \longrightarrow X$ from the string worldsheet Σ to the Calabi-Yau target space X. More specifically, non-renormalization theorems imply that worldsheet instanton contributions to the background superpotential can only arise at string tree level, meaning that the string worldsheet Σ has genus zero. Thus, if X contains a genus zero holomorphic curve C, worldsheet instantons which wrap C can potentially contribute to a superpotential for the moduli of the compactification.

In the simplest case that C is a smooth, isolated, genus zero holomorphic curve, then the leading instanton contribution from C to the background superpotential takes the form

$$W(C) = \exp\left(-\frac{A(C)}{2\pi\alpha'} + i\int_C B\right) \times \textit{(1-Loop)}\,, \tag{1.2.9}$$

where $A(C)$ denotes the area of C in the Calabi-Yau metric on X, B is the heterotic B-field, and α' is the string tension. The argument of the exponential in $W(C)$ is just the classical Euclidean action of a string worldsheet which wraps once about C, and we have not made the one-loop determinants that multiply this classical factor explicit. (We will be quite explicit about these one-loop factors in Chapter 3.) The important feature of this expression for $W(C)$ is simply that such a contribution to the superpotential clearly induces a potential for the Kähler modulus of X, since this Kähler modulus determines the area of C appearing in (1.2.9). So if worldsheet instantons make a non-zero contribution to the superpotential, that contribution has the drastic effect of lifting the Kähler modulus of the compactification and destabilizing the model.

In this context, a natural question to ask is whether any $\mathcal{N} = 1$ supersymmetric, Calabi-Yau compactifications of the heterotic string are actually stable against worldsheet instanton corrections. Certainly some more or less trivially stable examples are known. For instance, if $E = TX$, then the worldsheet supersymmetry algebra is enhanced from $(0, 2)$ to $(2, 2)$ supersymmetry, and the two extra left-moving supersymmetries cause the one-loop determinants that appear in $W(C)$ to vanish. Hence in such a model, the superpotential contribution from each holomorphic curve C identically vanishes.

On the other hand, Silverstein and Witten [13] (and later Basu and Sethi [14]) have argued that general compactifications described by $(0, 2)$ linear sigma models are stable against worldsheet instanton corrections. In such models, the Calabi-Yau threefold X has many holomorphic curves which make contributions to the superpotential, and the result of [13] implies that a miraculous cancellation must occur among all the contributions from the individual curves.

Our main result in Chapter 3 is to directly explain this cancellation among instanton effects as a consequence of a residue theorem, whose proof relies only upon the right-moving worldsheet supersymmetries and suitable compactness properties of the $(0, 2)$ linear sigma model. We also extend our residue theorem to a new class of "half-linear" sigma models. Using these half-linear models, we show for instance that heterotic compactifications on the quintic hypersurface in $\mathbb{CP}^4$ for which the gauge bundle pulls back from a bundle on $\mathbb{CP}^4$ are stable. Finally, we apply similar ideas to compute the superpotential contributions from families of membrane instantons in M-theory compactifications on manifolds of G_2 holonomy.

1.2.3 Chern-Simons Theory and Localization

In Chapter 4, we study Chern-Simons gauge theory on a three-manifold M. This theory is described by the following action for the gauge field A,

$$CS(A) \,=\, \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A{\wedge}dA + \frac{2}{3}A{\wedge}A{\wedge}A\right)\,. \tag{1.2.10}$$

Here k is an integer, the level of the theory, and Tr is a suitably normalized, negative-definite, invariant quadratic form on the Lie algebra of the gauge group. The classical solutions of this gauge theory are simply the flat connections on M, and the Chern-Simons action is notable for the fact that, unlike the Yang-Mills action, no metric on M is necessary to define it. Hence Chern-Simons gauge theory serves as a sterling example of a topological quantum field theory.

A remarkable fact about Chern-Simons theory is that it is exactly solvable, as shown by Witten in [15]. This solution relies on deep connections between Chern-Simons theory on M and two-dimensional rational conformal field theory [16]. On the other hand, the exact solution of Chern-Simons theory bears little apparent relation to the standard computational techniques of perturbative field theory, and a long-standing mathematical puzzle has been to understand how the exact solution of Chern-Simons theory is related to its perturbative expansion.

One elegant result in this direction has recently been obtained by Lawrence and Rozansky [17]. In the special case that the three-manifold M is a Seifert manifold (the total space of a nontrivial circle bundle over a Riemann surface), these authors have shown from the exact solution of Chern-Simons theory that the partition function has a remarkably simple structure and can be rewritten entirely as a sum of local "instanton" contributions from the flat connections on M.

In Chapter 4, we explain how this empirical fact follows from the semi-classical technique of non-abelian localization as applied to the Chern-Simons path integral. In the process, we show that the partition function of Chern-Simons theory on M admits a topological interpretation in terms of the equivariant cohomology of the moduli space of flat connections on M.

Abelian Localization and the Duistermaat-Heckman Formula

In Chapter 4, we provide a self-contained review of non-abelian localization, a technique introduced by Witten [18] to evaluate exactly a certain class of symplectic integrals. This localization technique is essentially a sophisticated version of the stationary phase approximation, and in this sense we can rephrase our main result as the statement that, when M is a Seifert manifold, the stationary phase approximation to the Chern-Simons path integral is exact.

Before we delve into the technical analysis of Chapter 4, we pause to present here a simple example of localization. As a model for the path integral, we consider a finite-dimensional integral defined in terms of the following data. We let X be a compact symplectic manifold, with symplectic form Ω. We assume that the group $U(1)$ acts on X in a Hamiltonian fashion, with Hamiltonian (or moment map) μ. Finally, we let V be a vector field on X that generates the action of $U(1)$. By definition, the Hamiltonian μ associated to V satisfies

$$\iota_V \Omega \,=\, d\mu\,, \tag{1.2.11}$$

where ι_V is the interior product operator which acts on forms by contraction with V.

Using this data, we consider the following symplectic integral over X,

$$Z(t) \,=\, \int_X \exp\left(\Omega \,-\, it\mu\right). \tag{1.2.12}$$

Here t is a parameter, and the term $\exp(\Omega)$ is to be interpreted by expanding the exponential in series and picking out the term of proper degree to integrate over X. If X has dimension $2n$, then this term will be $\Omega^n/n!$, the usual symplectic measure on X.

A classic result of Duistermaat and Heckman [19] states that the stationary phase approximation to $Z(t)$ is exact. Thus all contributions to $Z(t)$ arise from the critical points of the Hamiltonian μ, at which $d\mu = 0$. From the relation (1.2.11) and the non-degeneracy of

Ω, we see that the generating vector field V vanishes at precisely these critical points, which are the fixed points of the $U(1)$ action on X. Thus, the Duistermaat-Heckman formula very broadly asserts that $Z(t)$ can be written as a sum over local contributions from the fixed points of the $U(1)$ action,

$$Z(t) \,=\, \sum_p Z_p(t)\,. \tag{1.2.13}$$

For simplicity in writing (1.2.13), we assume that all the fixed points p are isolated, and the sum is a finite sum over these points. By definition, $Z_p(t)$ is the local contribution to $Z(t)$ from the point p as evaluated in the stationary phase approximation. As beautifully explained by Atiyah and Bott [20], the result of Duistermaat and Heckman is best understood as an example of abelian localization, since the formula (1.2.13) fundamentally asserts that all contributions to $Z(t)$ arise locally from fixed points of $U(1)$.

To make the formula (1.2.13) more explicit, we need to evaluate the local quantities $Z_p(t)$. This computation is a toy model for the more involved computations in Chapter 4, so we find it useful to present here. Our exposition follows the very elegant discussion in §7 of [20].

We first consider the local action of $U(1)$ near a given fixed point p, and to leading order we need only consider the action on the tangent space $T \equiv T_pX$ at p. Since p is an isolated fixed point, $U(1)$ acts freely on the vector space T, which therefore decomposes into two-dimensional, irreducible represenations of $U(1)$. We write

$$T \,=\, \bigoplus_{j=1}^{n} T_j\,, \tag{1.2.14}$$

where $n = \frac{1}{2}\dim X$ and where each T_j is an irreducible representation of $U(1)$ associated to an integer charge $q_j \neq 0$.

Without loss, we assume that the generating vector field V takes the form

$$V \,=\, \sum_{j=1}^{n} q_j \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right)\,. \tag{1.2.15}$$

Here (x_j, y_j) are real coordinates on each representation T_j which are adapted to the action of V as above. By exponentiation, these coordinates on T_j extend to coordinates on a neighborhood of p in X.

To evaluate $Z_p(t)$, we must determine local expressions for the symplectic form Ω and the Hamiltonian μ at p. Because Ω is invariant under $U(1)$, we can assume that the pairs (x_j, y_j) determine canonical coordinates at p, and Ω takes the local form

$$\Omega \,=\, \sum_{j=1}^{n} dx_j \wedge dy_j + \cdots\,. \tag{1.2.16}$$

Here we drop higher order terms in Ω which vanish at p; these terms are not relevant when we evaluate the integral in the leading, Gaussian approximation. Given the local form of V and of Ω at p, the Hamiltonian relation (1.2.11) then implies that μ takes the local form

$$\mu \,=\, \mu(p) \,-\, \sum_{j=1}^{n} q_j \left(\frac{1}{2}x_j^2 \,+\, \frac{1}{2}y_j^2\right) \,+\, \cdots\,, \tag{1.2.17}$$

where again we drop higher order terms in (x_j, y_j).

Thus, at leading order near p, we evaluate $Z_p(t)$ in the Gaussian approximation as

$$\begin{aligned} Z_p(t) \,&=\, \exp\left[-it\mu(p)\right] \prod_{j=1}^{n} \int dx_j \wedge dy_j\, \exp\left[itq_j \left(\frac{1}{2}x_j^2 \,+\, \frac{1}{2}y_j^2\right)\right]\,,\\ &=\, \exp\left[-it\mu(p)\right] \left(\frac{2\pi}{it}\right)^n \frac{1}{e_p}\,, \qquad e_p \,=\, \prod_{j=1}^{n} q_j\,. \end{aligned} \tag{1.2.18}$$

Hence if $U(1)$ acts with isolated fixed points on X, we can explicitly evaluate $Z(t)$ by summing the local expression in (1.2.18) over the fixed points.

In Chapter 4, a large part of the analysis is devoted to performing a similar semi-classical computation to evaluate the local contributions to the Chern-Simons path integral from flat connections on M. In the example here, the interesting dependence of $Z_p(t)$ on the local geometry near p is encoded in the one-loop factor e_p. This factor has a very interesting topological interpretation, as it is the equivariant Euler class of T interpreted as a $U(1)$-equivariant bundle over p. We will explain this statement in Section 5.3 of Chapter 4, where equivariant Euler classes also appear naturally in the non-abelian localization formula.

An Simple Example of the Duistermaat-Heckman Formula

Finally, we give a simple example of the Duistermaat-Heckman formula. In our example, we take X to be $\mathbb{CP}^1$, which we endow with the standard angular coordinates (θ, ϕ), where θ

runs from 0 to π and ϕ runs from 0 to 2π. We also choose the standard symplectic structure on $\mathbb{CP}^1$, with symplectic form $\Omega = d(\cos\theta)\wedge d\phi$.

Finally, we consider the $U(1)$ action on X which is generated by the vector field $V = \partial/\partial\phi$. This $U(1)$ action has two fixed points, at the poles $\theta = 0$ and $\theta = \pi$. Since $\iota_V \Omega = -d(\cos\theta)$, we see that, up to an arbitrary additive constant, the Hamiltonian for this action is

$$\mu \,=\, -\cos\theta\,. \tag{1.2.19}$$

The Duistermaat-Heckman integral $Z(t)$ is thus given explicitly by

$$\begin{aligned} Z(t) \,&=\, \int d(\cos\theta)\wedge d\phi \, \exp\left(it\cos\theta\right), \\ &=\, \frac{2\pi}{it} \left[\exp\left(it\right) \,-\, \exp\left(-it\right)\right]\,. \end{aligned} \tag{1.2.20}$$

Of course, this integral is an elementary integral, and we evaluate it directly in passing to the second line of (1.2.20). As implied by the Duistermaat-Heckman formula, $Z(t)$ is the sum of two terms, each of the form in (1.2.18), which represent the local contributions from the two fixed points at $\theta = 0$ and $\theta = \pi$. A relative sign arises between these contributions because in one case the local $U(1)$ action at the fixed point is right-handed with respect to the orientation induced by Ω and in the other case it is left-handed.

Chapter 2

New Instanton Effects in Supersymmetric QCD

2.1 Introduction

Supersymmetric QCD with gauge group $SU(N_c)$ and with N_f massless flavors is among the simplest and most studied of four dimensional $\mathcal{N} = 1$ supersymmetric gauge theories. In particular, due to holomorphy and the large amount of symmetry this theory possesses, many properties of its low energy vacuum structure are amenable to exact analysis. Yet the theory still displays a wealth of interesting non-perturbative phenomena, including the generation of a superpotential [7,21–24], a deformation of the complex structure of the moduli space of supersymmetric vacua [10] (see also [11] for a related analysis), and electric-magnetic duality [25]. Although we briefly reviewed a few aspects of this story in Chapter 1, we refer the reader to [26,27] for a more complete account.

As we did explain in Chapter 1, some of these nonperturbative phenomena, such the generation of a superpotential when $N_f = N_c - 1$ and the deformation of the moduli space when $N_f = N_c$, can be understood semi-classically as instanton effects. On the other hand, SQCD with $N_f > N_c$ flavors possesses supersymmetric instantons in precise analogy to the instantons for $N_f = N_c - 1$ and $N_f = N_c$, yet the instantons for $N_f > N_c$ neither generate a superpotential nor deform the classical moduli space. As a result, one might ask, "So what are these instantons good for?" In this chapter, our purpose is to answer this question. To simplify the technical analysis, we focus on the case $N_c = 2$, but we believe that our results generalize for arbitrary N_c.

Thus, in $SU(N_c)$ SQCD with $N_f > N_c$ flavors, we find that instantons generate a new class of F-terms in the low-energy effective action of the theory. Among other interactions, these F-terms generate vertices with $2(N_f - N_c) + 4$ fermions, and for this reason we call them "multi-fermion" F-terms. These new F-terms fit into a hierarchy which naturally generalizes the superpotential that appears in the theory with $N_f = N_c - 1$ flavors and the four-fermion (or two-derivative) F-term that describes the complex structure deformation in the theory with $N_f = N_c$ flavors. In fact, we were originally motivated to consider such multi-fermion F-terms precisely as a means to describe systematically a general deformation of the moduli space.

One reason that these instanton effects have gone unnoticed for so long is that for $N_f > N_c$ the multi-fermion F-terms have no effect on the classical geometry of the moduli space and no qualitative effect on the physics. (However, if we turn on supersymmetric bare masses for some of the flavors, then these F-terms induce the usual instanton effects in the cases $N_f = N_c$ and $N_f = N_c - 1$.) As a result, such F-terms have not been much considered in the context of $\mathcal{N} = 1$ supersymmetric theories in four dimensions. However, despite the novelty of the multi-fermion F-terms in four dimensions, these F-terms are closely related to well-known chiral operators that appear in the context of two-dimensional, $\mathcal{N} = (2, 2)$ supersymmetric sigma models. In the two-dimensional context, these operators are the generators of the (c, c) chiral ring, or equivalently the ring of local observables of the topological B model.

Because the multi-fermion F-terms themselves are novel, we begin our discussion in Section 2 with some general remarks on multi-fermion F-terms in $\mathcal{N} = 1$ supersymmetric effective actions.

We next specialize in Section 3 to the case of SQCD with gauge group $SU(2)$. We show that the symmetries of $SU(2)$ SQCD suffice to fix uniquely the possible form of any multi-fermion F-term correction to the effective action. Here we exploit the fact that, in the special case of SQCD with gauge group $SU(2)$, the flavor symmetry is enhanced because both the quarks and the anti-quarks transform in the fundamental representation of the gauge group.

Finally, we show in Section 4 that these multi-fermion F-terms are indeed generated in the effective action of SQCD. We do this in three ways, each of which casts a different light on the origin of these unusual F-terms. First, we perform a direct instanton computation as in [7] to show that the multi-fermion F-terms are generated. Second, in the special case that $N_c = 2$, $N_f = N_c + 1 = 3$, we show that the multi-fermion F-terms arise from a tree-level Feynman diagram computation in the Seiberg dual description of the theory. Third, we consider the supersymmetric mass deformation of SQCD, and we show that the multi-fermion F-terms give rise by renormalization group flow to the standard superpotential in the theory with $N_f = N_c - 1$ flavors. As mentioned above, we believe these analyses generalize from the case $N_c = 2$ to arbitrary N_c.

This chapter of the thesis is based on [4].

2.2 General Remarks on Multi-Fermion F-Terms

In this section, we describe the general structure of multi-fermion F-terms in $\mathcal{N} = 1$ supersymmetric effective actions. However, before discussing generalities, we motivate our study of these interactions by considering a very specific and well-known example: the complex structure deformation of the moduli space $\mathcal{M}$ of vacua that occurs in $SU(2)$ SQCD with two flavors. As in Chapter 1, we write $\mathcal{M}_{cl}$ for the classical moduli space of supersymmetric vacua, and we write $\mathcal{M}$ for the exact quantum moduli space.

2.2.1 Example: $SU(2)$ SQCD With Four Doublets

As we apply extensively later, for the gauge group $SU(2)$ the fundamental and the anti-fundamental representations coincide, so that $SU(2)$ gauge theory with $N_f = n$ flavors is described more symmetrically as a theory with $2n$ doublets, or equivalently with chiral multiplets transforming as $2n$ copies of the two-dimensional representation $\mathbf{2}$. To avoid the perhaps confusing terminology of "flavors" for $SU(2)$, we just write the number of doublets as $2n$. (Because of a global anomaly, the number of chiral doublets in $SU(2)$ gauge theory must be even [28].)

To establish notation for the rest of the chapter, we combine the matter fields into one chiral multiplet,

$$Q^i_a \;=\; q^i_a \,+\, \theta\psi^i_a \,+\, \cdots , \tag{2.2.1}$$

with $a = 1, 2$ being the color index for the $\mathbf{2}$ of the $SU(2)$ gauge symmetry, and $i = 1, \ldots, 2n$ being the flavor index for the $\mathbf{2n}$ of the global $SU(2n)$ flavor symmetry. Of course, we have indicated in (2.2.1) the component expansion of Q^i_a, including a scalar field q^i_a and a Weyl fermion ψ^i_a.

We also introduce the gauge invariant, composite meson chiral superfield M^{ij}, given by

$$M^{ij} = \epsilon^{ab}\, Q^i_a Q^j_b. \tag{2.2.2}$$

The meson M^{ij} is clearly anti-symmetric in the flavor indices i and j and so transforms in the skew representation $\wedge^2(\mathbf{2n})$ of $SU(2n)$.

Using the mesons M^{ij}, we can succinctly describe the classical moduli space $\mathcal{M}_{cl}$ of supersymmetric vacua as being parametrized by arbitrary expectation values of M^{ij} subject to the constraint

$$M{\wedge}M \;=\; 0\,, \tag{2.2.3}$$

or more explicitly,

$$\epsilon_{i_1 j_1 i_2 j_2 \cdots i_n j_n}\, M^{i_1 j_1} M^{i_2 j_2} \;=\; 0\,. \tag{2.2.4}$$

This system of quadratic equations (2.2.3) simply enforces the condition that

$$\mathrm{rank}(M) \le 2\,, \tag{2.2.5}$$

as follows from the definition (2.2.2) of M^{ij} as the skew product of two quark superfields.

Now, if the number of doublets is $2n = 4$, the classical constraint (2.2.3) reduces to a single quadratic equation

$$\epsilon_{i_1 j_1 i_2 j_2} M^{i_1 j_1} M^{i_2 j_2} = 0 \tag{2.2.6}$$

which must be satisfied by M^{ij}. Upon introducing suitable complex linear combinations m^I, $I = 1, \ldots, 6$, of the six independent components M^{ij}, $i, j = 1, \ldots, 4$, so as to diagonalize the nondegenerate quadratic form that appears on the left hand side of (2.2.6), the classical equation (2.2.6) becomes

$$\sum_{I=1}^{6} \left(m^I \right)^2 = 0 \, . \tag{2.2.7}$$

The classical moduli space $\mathcal{M}_{cl}$ is thus smooth away from the origin. Its singularity at the origin is a signal of the unbroken gauge symmetry. The m^I transform in the vector representation of the $SU(4)$ or $SO(6)$ flavor symmetry of the $SU(2)$ gauge theory with four doublets.

The classical moduli space $\mathcal{M}_{cl}$ whose structure we have just reviewed is deformed in the quantum theory [10] and does not coincide with the quantum moduli space of vacua $\mathcal{M}$. To describe this deformation, we introduce the usual holomorphic coupling scale Λ. Then, in the quantum theory, the moduli space $\mathcal{M}$ is described by the modified constraint

$$M \wedge M \;=\; \Lambda^4 \, , \tag{2.2.8}$$

or equivalently, with $\epsilon \sim \Lambda^4$,

$$\sum_{I=1}^{6} \left(m^I \right)^2 = \epsilon \, . \tag{2.2.9}$$

Up to a multiplicative constant, the form of the deformation (2.2.8) is determined completely by the $SU(4)$ flavor symmetry and dimensional analysis. Of course, as a result of the deformation, the singularity of $\mathcal{M}_{cl}$ at the origin is removed and $\mathcal{M}$ is a smooth complex manifold.

Representing the Deformation in the Effective Action

But precisely how does the geometric deformation (2.2.8) appear physically as a quantum correction to the effective action of SQCD?

In this very simple example, one way to implement the quantum deformation in the low energy effective theory is to introduce a massive field Σ and a superpotential W into the effective action,

$$W = \Sigma\left(M \wedge M - \Lambda^4\right) , \tag{2.2.10}$$

which thus takes the form

$$S \,=\, \int d^4x\, d^4\theta\; K\Big(M, \overline{M}; \Sigma, \overline{\Sigma}\Big) \,+\, \int d^4x\, d^2\theta\; W \,+\, c.c., \tag{2.2.11}$$

where K is the Kähler potential. At a critical point of W, we find that $\Sigma = 0$ and $M \wedge M = \Lambda^4$, so the quantum moduli space is reproduced by this model. In this description, the quantum correction to the effective action is clearly an F-term, being a correction $\Delta W = -\Lambda^4\Sigma$ to the effective superpotential. In the weak-coupling limit $\Lambda \to 0$, this term vanishes and the constraint reduces to the classical one $M \wedge M = 0$.

The description we have just given is useful for this particular example, but it is an extrinsic rather than an intrinsic description of the deformation. In the extrinsic description of the classical moduli space $\mathcal{M}_{cl}$ and its deformation $\mathcal{M}$, we use a linear sigma model to describe these spaces in terms of unconstrained linear fields Σ and m^I, $I = 1, \ldots, 6$, together with a superpotential. However, as is usual for linear sigma models, not all of the linear fields are massless at generic points (away from the origin). In this example, of the seven total fields, we see that in the generic vacuum two, namely Σ and a linear combination of the m^I, are massive, while five components of m^I are massless and parametrize intrinsically the moduli space. Obviously, our deformation $\Delta W = -\Lambda^4\Sigma$ involves the massive fields. In an analogous but different example, the moduli space might not admit such a simple, linear sigma model description. For this reason, we want to describe the deformation intrinsically in a low-energy effective action constructed only from the massless fields.

To find such an intrinsic description, we could just integrate out the massive fields in the linear sigma model to convert ΔW into an effective interaction for massless fields only. In doing so, we work modulo D-terms and attempt to determine what F-terms are generated. This computation is both simple and instructive and we will perform it, along with an analogous computation in the theory with six doublets, in Section 4.2.

However, we can alternatively use supersymmetry (and a bit of geometry) to determine what F-terms are possible on $\mathcal{M}_{cl}$. At least away from the origin of $\mathcal{M}_{cl}$, the low-energy effective action of this theory is intrinsically described as an $\mathcal{N}=1$ supersymmetric, nonlinear sigma model governing maps $\phi : M^4 \longrightarrow \mathcal{M}_{cl}$ from Minkowski space M^4 to $\mathcal{M}_{cl}$. From this perspective, the perturbative effective action is the usual sigma model action,

$$S = \int d^4x\, d^4\theta\; K\Big(\Phi^i, \overline{\Phi}^{\bar{i}}\Big)\,. \tag{2.2.12}$$

Here Φ^i and $\overline{\Phi}^{\bar{i}}$ are chiral and anti-chiral superfields whose lowest components ϕ^i and $\overline{\phi}^{\bar{i}}$ are local holomorphic and anti-holomorphic coordinates on $\mathcal{M}_{cl}$, and K is again the Kähler potential associated to some Kähler metric $ds^2 = g_{i\bar{i}}\, d\phi^i d\overline{\phi}^{\bar{i}}$ on $\mathcal{M}_{cl}$. (In this discussion, 'i' is not a flavor index but an index parametrizing local coordinates on $\mathcal{M}_{cl}$.) The reason that we consider a sigma model whose target is $\mathcal{M}_{cl}$ is that this is the low energy structure in perturbation theory. We want to know how this description may be modified by instantons, in other words, what F-term on $\mathcal{M}_{cl}$ may be induced by instantons.

Of course, we also know the quantum effective action: it is the same nonlinear sigma model but with target space $\mathcal{M}$, as opposed to $\mathcal{M}_{cl}$, and in this language the F-term must describe the complex structure deformation of $\mathcal{M}_{cl}$ into $\mathcal{M}$. So let us discuss what terms in the effective action of an $\mathcal{N}=1$ sigma model with a given target (in our case, $\mathcal{M}_{cl}$) describe a deformation of the complex structure of the target. We have already described this deformation extrinsically, as a modification of the algebraic equations which define the target. To describe the deformation intrinsically, we instead consider it as a modification of the $\overline{\partial}$ operator of the target.

In general, a deformation of the complex structure on $\mathcal{M}_{cl}$ is described as a change in the $\overline{\partial}$ operator on $\mathcal{M}_{cl}$ of the form

$$\overline{\partial}_{\bar{j}} \longmapsto \overline{\partial}_{\bar{j}} + \omega_{\bar{j}}^{\;i}\,\partial_i\,. \tag{2.2.13}$$

Here $\omega_{\bar{j}}^{\;i}$ is a representative of a Dolbeault cohomology class in $H^1(\mathcal{M}_{cl}, T\mathcal{M}_{cl})$, whose elements parametrize infinitesimal deformations of $\mathcal{M}_{cl}$. We use standard notation, with $T\mathcal{M}_{cl}$ and $\Omega^1_{\mathcal{M}_{cl}}$ denoting the holomorphic tangent and cotangent bundles of $\mathcal{M}_{cl}$.

We can equally well represent the change (2.2.13) in the $\overline{\partial}$ operator on $\mathcal{M}_{cl}$ as a change in the dual basis of holomorphic one-forms $d\phi^i$,

$$d\phi^i \longmapsto d\phi^i - \omega_{\bar{j}}{}^i \, d\overline{\phi}^{\bar{j}} \, . \tag{2.2.14}$$

As a result, under the deformation, the metric on $\mathcal{M}_{cl}$ changes as

$$g_{i\bar{i}} \, d\phi^i d\overline{\phi}^{\bar{i}} \longmapsto g_{i\bar{i}} \left(d\phi^i - \omega_{\bar{j}}{}^i \, d\overline{\phi}^{\bar{j}} \right) d\overline{\phi}^{\bar{i}} \, , \tag{2.2.15}$$

so that, upon deforming $\mathcal{M}_{cl}$, the metric picks up a component of type $(0,2)$ when written in the original holomorphic and anti-holomorphic coordinates. (Of course, there is also a complex conjugate term of type (2,0).)

Since we know how the metric on $\mathcal{M}_{cl}$ changes when $\mathcal{M}_{cl}$ is deformed, we can immediately deduce that the corresponding correction to the sigma model action is generally of the form

$$\delta S = \int d^4x \, d^2\theta \, \omega_{\bar{i}\,\bar{j}} \, \overline{D} \, \overline{\Phi}^{\bar{i}} \cdot \overline{D} \, \overline{\Phi}^{\bar{j}} = \int d^4x \, \omega_{\bar{i}\,\bar{j}} \, d\overline{\phi}^{\bar{i}} \, d\overline{\phi}^{\bar{j}} + \cdots \, , \tag{2.2.16}$$

with

$$\omega_{\bar{i}\,\bar{j}} = \frac{1}{2} \left(g_{i\bar{i}} \, \omega_{\bar{j}}{}^i + g_{i\bar{j}} \, \omega_{\bar{i}}{}^i \right) \, . \tag{2.2.17}$$

Here $\overline{D} \equiv \overline{D}_{\dot{\alpha}}$ is the usual spinor covariant derivative on superspace, and we have introduced the shorthand notation "$\cdot$" for the contraction of spinor indices (so for any two spinors η and ζ, $\eta \cdot \zeta$ is shorthand for $\eta_{\dot{\alpha}} \zeta^{\dot{\alpha}}$). We have also performed the fermionic integral over θ in (2.2.16), from which we see that the leading bosonic term reproduces the correction to the metric in (2.2.15).

Of course, the most important property of δS — and the primary motivation for this work — is the fact that δS is an F-term. But δS is not a correction to the superpotential — it generates terms with two derivatives of bosons, or with four fermions. Because of the latter contribution, δS is a special case of what we call a multi-fermion F-term.

In contrast to a superpotential interaction, a deformation of the complex structure (of a smooth complex manifold, such as $\mathcal{M}_{cl}$ with the origin removed) is trivial locally. So locally on $\mathcal{M}_{cl}$, it must be possible to write δS in the form $\int d^4\theta(\ldots)$. As will become clear, this cannot be done globally on $\mathcal{M}_{cl}$, and it cannot be done even locally in a way that respects the $SU(4)$ flavor symmetry. In that sense, δS is a non-trivial F-term.

We also note that this F-term is not manifestly supersymmetric, since the operator $\mathcal{O}_\omega = \omega_{\bar{i}\bar{j}} \overline{D}\overline{\Phi}^{\bar{i}} \cdot \overline{D}\overline{\Phi}^{\bar{j}}$ is not manifestly chiral. Rather, the chirality of $\mathcal{O}_\omega$ in the on-shell supersymmetry algebra determined by the unperturbed sigma model action S follows from the fact that $\omega_{\bar{j}}^{\;i}$ is annihilated by $\overline{\partial}$.

In Section 2.2, we discuss more systematically the basic properties of multi-fermion F-terms such as δS.

Computing δS in SQCD

We have described in general what sort of term in the low energy effective action of an $\mathcal{N} = 1$ sigma model describes the deformation of the complex structure of the target. We will now be more explicit for $SU(2)$ gauge theory with four doublets.

For this purpose, we reconsider the extrinsic, algebraic description of the deformation of $\mathcal{M}_{cl}$, using the coordinates m^I. Rather than considering the deformation as a change in the classical constraint equation

$$\sum_I (m^I)^2 = 0 \tag{2.2.18}$$

to a quantum constraint

$$\sum_I (m^I)^2 = \epsilon, \tag{2.2.19}$$

we want to provide a description in which the target space remains the same (away from the origin and perturbatively in ϵ) but a new interaction is generated.

To obtain this description, we first make a non-holomorphic change of variables, such that away from the origin the quantum constraint (2.2.19) is converted to the classical constraint (2.2.18). Explicitly, when the old coordinates m^I satisfy the quantum constraint (2.2.19), the new coordinates

$$\widetilde{m}^I = m^I - \frac{\epsilon}{2} \frac{\delta^I_{\bar{J}} \overline{m}^{\bar{J}}}{\overline{m}m} \tag{2.2.20}$$

obey the classical constraint (2.2.18) to first order in ϵ. (We could work beyond first order, but this is not necessary.) Here $\overline{m}m = \sum_{I=1}^6 |m^I|^2$, and in describing $\widetilde{m}^I$ we introduce the tensor $\delta^I_{\overline{J}}$ constructed from the $SO(6)$ invariant tensors δ^{IJ} and $\delta_{\overline{I}\overline{J}}$; in the language of $SU(4)$, these tensors would be respectively $\epsilon^{ij}{}_{\overline{k}\overline{l}}$, ϵ^{ijkl}, and $(\delta_{i\bar{i}}\,\delta_{j\bar{j}} - \delta_{j\bar{i}}\,\delta_{i\bar{j}})$.

Thus, when the original coordinates m^I satisfy the quantum constraint, the new coordinates $\widetilde{m}^I$ satisfy the classical constraint, at least to leading order in ϵ,

$$\sum_{I=1}^{6} \left(\widetilde{m}^I\right)^2 \;=\; \mathcal{O}(\epsilon^2)\,. \tag{2.2.21}$$

The new coordinates $\widetilde{m}^I$ are obviously not holomorphic in the old complex structure on $\mathcal{M}_{cl}$, but we can find a new complex structure in which they are holomorphic. In other words, we correct the $\overline{\partial}$ operator as in (2.2.13) so that the new operator annihilates the new coordinates $\widetilde{m}^I$. So we impose the condition

$$\left(\frac{\partial}{\partial \overline{m}^{\overline{J}}} \,+\, \omega_{\overline{J}}{}^{I}\,\frac{\partial}{\partial m^I}\right)\widetilde{m}^K \;=\; 0\,. \tag{2.2.22}$$

From this equation, we can directly solve for the tensor $\omega_{\overline{J}}{}^{I}$ in terms of the components m^I of M. We find, again to leading order in ϵ, that

$$\omega_{\overline{J}}{}^{I} \;=\; \frac{\epsilon}{2}\left(\frac{\delta^I_{\overline{J}}}{\overline{m}m} - \frac{\overline{m}^I m_{\overline{J}}}{(\overline{m}m)^2} - \frac{m^I \overline{m}_{\overline{J}}}{(\overline{m}m)^2}\right), \tag{2.2.23}$$

with indices raised and lowered with δ^{IJ} and $\delta_{\overline{I}\overline{J}}$ as appropriate.

In this expression, only the first two terms in (2.2.23) arise directly from solving the equation (2.2.22). In fact, the last term in the expression for $\omega_{\overline{J}}{}^{I}\, d\overline{m}^{\overline{J}}\,\partial/\partial m^I$ vanishes identically when we restrict to $\mathcal{M}$, as on $\mathcal{M}$ we have the relation

$$0 = \sum_{I=1}^{6} \overline{m}^{\overline{I}} d\overline{m}^{\overline{I}} = \frac{1}{2} d\left(\sum_I (\overline{m}^{\overline{I}})^2\right). \tag{2.2.24}$$

We have included this trivial term in $\omega_{\overline{J}}{}^{I}$ just so that, upon lowering one index with the Kähler metric, the tensor $\omega_{\overline{I}\,\overline{J}}$ is manifestly symmetric.

Of course, we do not actually know the Kähler metric g on $\mathcal{M}_{cl}$, as appears implicitly in determining δS by converting the section ω of $\overline{\Omega}^1_{\mathcal{M}_{cl}} \otimes T\mathcal{M}_{cl}$ to a section of $\overline{\Omega}^1_{\mathcal{M}_{cl}} \otimes \overline{\Omega}^1_{\mathcal{M}_{cl}}$, as in (2.2.16) and (2.2.17). By symmetry, we do know that this metric must equal the metric on $\mathcal{M}_{cl}$ induced from the Euclidean metric times a function of $\overline{m}m$, and asymptotically for large $\overline{m}m$ the metric must reduce to the classical metric describing canonical kinetic terms for underlying quarks in the ultraviolet regime of SQCD.

All of our expressions for the multi-fermion F-terms depend on the metric g. However, this dependence is irrelevant in the sense that the fundamental holomorphic object ω which represents a class in $H^1(\mathcal{M}, T\mathcal{M})$ and determines the existence of the multi-fermion F term does not depend on a choice of Kähler metric. Of course, the metric is known asymptotically, near infinity on $\mathcal{M}$, where it can be determined from the underlying classical field theory and asymptotic freedom.

We will now give a concrete formula for δS. Because of the dependence on g, we can present this formula in various ways. The most general approach, which also leads to the simplest expressions, is simply to leave g implicit, absorbing it into the index structure of $\omega_{\overline{I}\,\overline{J}}$ as we did in (2.2.17). This means that we simply use an unknown Kähler metric in raising and lowering indices. With this convention understood, from (2.2.16), (2.2.17), and (2.2.23), we see that δS takes the form

$$\delta S \;=\; \int d^4x\, d^2\theta\, \frac{\epsilon}{2} \left(\frac{\delta^{IJ}}{\overline{m}m} \;-\; \frac{\overline{m}^I\, m^J}{(\overline{m}m)^2} \;-\; \frac{m^I\, \overline{m}^J}{(\overline{m}m)^2} \right) \overline{D}\,\overline{m}_I \cdot \overline{D}\,\overline{m}_J\,. \tag{2.2.25}$$

Alternatively, this expression (2.2.25) is what results if we assume that g is the flat metric, so that we simply raise and lower indices with the Kronecker delta.

On the other hand, because the mesons m^I and $\overline{m}^{\overline{I}}$ most naturally (that is in the classical theory) have dimension 2, the metric $g_{I\overline{I}}$ most naturally has dimension -2 (so that $ds^2 = g_{I\overline{I}} dm^I dm^{\overline{I}}$ has dimension two). As a result, the dimensional analysis of our expression in (2.2.25) is not transparent. Asymptotically on $\mathcal{M}$, the Kähler potential is known to be asymptotic to $K = \sqrt{\overline{m}m}$. With this knowledge, we can make the asymptotic form of the interaction more precise. In doing so, it is convenient to also make dimensional analysis manifest by simply using the Kronecker delta $\delta_{I\overline{I}}$ to raise and lower indices on m and $\overline{m}$, while writing factors of $\sqrt{\overline{m}m}$ explicitly. In this case, all components of m and $\overline{m}$ with indices up or down have dimension two. The asymptotic form of the interaction δS

then becomes

$$\delta S \,=\, \int d^4x\, d^2\theta\, \frac{\epsilon}{2\sqrt{\overline{m}m}} \left(\frac{\delta^{IJ}}{\overline{m}m} - \frac{\overline{m}^I\, m^J}{(\overline{m}m)^2} - \frac{m^I\, \overline{m}^J}{(\overline{m}m)^2} \right) \overline{D}\overline{m}_I \cdot \overline{D}\overline{m}_J\,. \tag{2.2.26}$$

Recalling that $\epsilon \sim \Lambda^4$, one can check directly that the naive dimensional analysis holds.

In the rest of the chapter, we will mainly follow the first convention, as in (2.2.25), so that g appears only implicitly.

In terms of the components M^{ij} of M written using $SU(4)$ flavor indices, as we will use in Section 3, the expression (2.2.25) becomes

$$\begin{aligned}\delta S \,=\, \int d^4x\, d^2\theta\, \Lambda^4 & \left(\frac{\epsilon^{i_1 j_1 i_2 j_2}}{\overline{M}M} - \frac{\epsilon^{i_1 j_1 kl}\, M^{i_2 j_2}\, \overline{M}_{kl}}{2\left(\overline{M}M\right)^2} - \frac{\epsilon^{i_2 j_2 kl}\, M^{i_1 j_1}\, \overline{M}_{kl}}{2\left(\overline{M}M\right)^2} \right) \times \\ & \times\, \overline{D}\overline{M}_{i_1 j_1} \cdot \overline{D}\overline{M}_{i_2 j_2}\,.\end{aligned} \tag{2.2.27}$$

Here we take $\overline{M}M \equiv \frac{1}{2}\sum_{ij} \overline{M}_{ij}M^{ij}$. (The factor of 1/2 is included so that if the only nonzero components of M^{ij} are $M^{12} = -M^{21} = 1$, then $\overline{M}M = 1$. The factors of 1/2 in (2.2.27) relative to (2.2.25) arise from this convention and lead to the simple formula below.)

For future reference, we observe that up to a constant factor the expression in (2.2.27) can be written more compactly as

$$\delta S \,=\, \Lambda^4 \int d^4x\, d^2\theta\, \left(\overline{M}M\right)^{-2} \epsilon^{i_1 j_1 i_2 j_2}\, \overline{M}_{i_1 j_1} \left(M^{kl}\, \overline{D}\overline{M}_{i_2 k} \cdot \overline{D}\overline{M}_{l j_2} \right)\,. \tag{2.2.28}$$

In Section 3, we will show that this form (2.2.28) of the F-term is completely determined by symmetry and furthermore extends naturally to the case of $SU(2)$ SQCD with $n > 2$ flavors.

2.2.2 Multi-Fermion F-terms

Our description of the complex structure deformation in SQCD by means of a multi-fermion F-term may seem perverse, as the algebraic description of the deformation in (2.2.8) is so much simpler than (2.2.28). However, by phrasing this deformation as a multi-fermion F-term in an effective four-dimensional $\mathcal{N} = 1$ supersymmetric sigma model, we can see an immediate generalization to F-terms of even higher order.

To introduce this generalization, we begin by recalling that a four-dimensional sigma model with $\mathcal{N}=1$ supersymmetry can be dimensionally reduced to a two-dimensional sigma model with $\mathcal{N}=(2,2)$ supersymmetry. Under this reduction, chiral operators in one sigma model map naturally to chiral operators in the other. As it turns out, the multi-fermion F-terms that we introduce in four dimensions have better-known analogs in two dimensions.

In two dimensions, rings of chiral operators have been much studied [29–32] in the context of string theory and correspond to the rings of local observables in the topological A- and B-models. In fact — with the superpotential being a typical example — F-terms in four dimensions reduce to chiral observables of the B-model in two dimensions. These chiral operators in the B-model arise geometrically in one-to-one correspondence with elements of the Dolbeault cohomology groups $H^p(\mathcal{M}, \wedge^q T\mathcal{M})$.

Motivated by the general B-model observables, to construct multi-fermion F-terms we begin with a section ω of the bundle $\overline{\Omega}^p_{\mathcal{M}} \otimes \overline{\Omega}^p_{\mathcal{M}}$. (Lorentz-invariance imposes the requirement that we consider only the B-model observables for $p=q$ above.) In components, ω is given by a tensor $\omega_{\bar{i}_1\cdots\bar{i}_p\,\bar{j}_1\cdots\bar{j}_p}$ that is antisymmetric in the $\bar{i}_k$ and also in the $\bar{j}_k$. Given such a tensor, we construct a possible term in the effective action that generalizes what we found in (2.2.16):

$$\begin{aligned} \delta S &= \int d^4x\, d^2\theta\; \omega_{\bar{i}_1\cdots\bar{i}_p\,\bar{j}_1\cdots\bar{j}_p} \left(\overline{D}\,\overline{\Phi}^{\bar{i}_1}\cdot\overline{D}\,\overline{\Phi}^{\bar{j}_1}\right)\cdots\left(\overline{D}\,\overline{\Phi}^{\bar{i}_p}\cdot\overline{D}\,\overline{\Phi}^{\bar{j}_p}\right)\,, \\ &\equiv \int d^4x\, d^2\theta\; \mathcal{O}_\omega\,. \end{aligned} \tag{2.2.29}$$

To achieve Lorentz invariance, spinor indices are contracted here. To denote these contractions, we recall our abbreviation

$$\left(\overline{D}\,\overline{\Phi}^{\bar{i}_1}\cdot\overline{D}\,\overline{\Phi}^{\bar{j}_1}\right) \equiv \left(\overline{D}_{\dot\alpha}\overline{\Phi}^{\bar{i}_1}\,\overline{D}^{\dot\alpha}\overline{\Phi}^{\bar{j}_1}\right)\,. \tag{2.2.30}$$

Furthermore, given the form of this operator, we can assume that ω is symmetric under the overall exchange of i's and j's.

Supersymmetry of $\mathcal{O}_\omega$

The interaction δS is not manifestly supersymmetric. For it to be supersymmetric, $\mathcal{O}_\omega$ must be chiral, that is, annihilated by the anti-chiral supersymmetries $\overline{Q}_{\dot\alpha}$. And even if δS

is supersymmetric, it may represent a trivial F-term. Though we write (2.2.29) in the form $\int d^2\theta(\ldots)$, it may be that δS can be alternatively written $\int d^4\theta(\ldots)$, in other words as a D-term. This will be so if it is possible to write $\mathcal{O}_\omega = \{\overline{Q}_{\dot\alpha}, [\overline{Q}^{\dot\alpha}, V]\}$ for some V. In this case, $\mathcal{O}_\omega$ is trivially chiral and $\delta S = \int d^4x\, d^4\theta\, V$.

To describe the chirality condition on $\mathcal{O}_\omega$, which will be no surprise from experience with the two-dimensional B-model, we first note that we can use the Kähler metric $g_{i\bar{i}}$ on $\mathcal{M}$ to raise either set of $\bar{i}$ or $\bar{j}$ indices on ω. The raised indices become holomorphic, so upon raising the indices, ω becomes interpreted as a section of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ in two distinct ways. By our assumption on the symmetry of ω, we find the same section of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ either way.

We now consider the action of the anti-chiral supercharges $\overline{Q}_{\dot\alpha}$ in the on-shell supersymmetry algebra of the unperturbed sigma model, so that we consider for simplicity only the linearized supersymmetry constraint on δS. Under the action of $\overline{Q}_{\dot\alpha}$, the component fields ϕ^i and ψ^i_β of Φ^i and the component fields $\overline{\phi}^{\bar{i}}$ and $\overline{\psi}^{\bar{i}}_{\dot\beta}$ of $\overline{\Phi}^{\bar{i}}$ transform as

$$\begin{aligned} \delta_{\dot\alpha}\phi^i &= 0\,, & \delta_{\dot\alpha}\overline{\phi}^{\bar{i}} &= \overline{\psi}^{\bar{i}}_{\dot\alpha}\,, \\ \delta_{\dot\alpha}\psi^i_\beta &= i\,\partial_{\dot\alpha\beta}\phi^i\,, & \delta_{\dot\alpha}\overline{\psi}^{\bar{i}}_{\dot\beta} &= -\Gamma^{\bar{i}}_{\bar{j}\,\bar{k}}\,\overline{\psi}^{\bar{j}}_{\dot\alpha}\,\overline{\psi}^{\bar{k}}_{\dot\beta}\,. \end{aligned} \tag{2.2.31}$$

Here Γ is the connection associated to the Kähler metric $g_{i\bar{i}}$ on $\mathcal{M}$. So long as we consider only the action of a single supercharge, we can without loss set Γ to zero by a suitable coordinate choice on $\mathcal{M}$.

By using the metric to interpret each set of anti-chiral fermions $\overline{\psi}^{\bar{i}}_{\dot\beta}$ for $\dot\beta = 1, 2$ as alternatively anti-holomorphic one-forms $d\overline{\phi}^{\bar{i}}$ or holomorphic tangent vectors $\partial/\partial\phi^i$, we see directly from (2.2.31) that the action of each of the two supercharges $\overline{Q}_{\dot\alpha}$ on $\mathcal{O}_\omega$ corresponds to the action of $\overline{\partial}$ on ω when ω is regarded as a section of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ in either of the two possible ways. Thus, the chirality constraint on $\mathcal{O}_\omega$ is simply the condition that ω be annihilated by $\overline{\partial}$. This result is familiar in the B-model.

Cohomology of $\mathcal{O}_\omega$

We must also impose an equivalence relation on the space of operators $\mathcal{O}_\omega$, such that $\mathcal{O}_\omega$ is considered trivial if δS is equivalent to a D-term. The condition we will get is closely related to the reduction to $\overline{\partial}$ cohomology in the B-model.

As a simple example, any perturbative correction δK to the Kähler form can be trivially rewritten as an F-term correction upon performing half the integral over superspace:

$$\begin{aligned}\int d^4x\, d^4\theta\; \delta K &= \int d^4x\, d^2\theta\; \overline{D}^2 \delta K\,,\\ &= \int d^4x\, d^2\theta\; \nabla_{\bar{i}}\nabla_{\bar{j}}\delta K\, \left(\overline{D}\,\overline{\Phi}^{\bar{i}} \cdot \overline{D}\,\overline{\Phi}^{\bar{j}}\right).\end{aligned} \tag{2.2.32}$$

In the second line, we have introduced the covariant derivative ∇ associated to the connection Γ in (2.2.31), and we have explicitly rewritten the chiral integrand in the form of an operator $\mathcal{O}_\omega$, with

$$\omega_{\bar{i}\,\bar{j}} = \nabla_{\bar{i}}\nabla_{\bar{j}}\delta K\,. \tag{2.2.33}$$

Even more generally, we must consider possible corrections to the effective action which involve integrals over three quarters of superspace and are of the form

$$\begin{aligned}\delta S \;&=\; \int d^4x\, d^2\theta\, d\overline{\theta}_{\dot{\alpha}}\; \xi_{\bar{i}_2\cdots\bar{i}_p\,\bar{j}_1\cdots\bar{j}_p}\; \overline{D}^{\dot{\alpha}}\overline{\Phi}^{\bar{j}_1}\, \left(\overline{D}\,\overline{\Phi}^{\bar{i}_2}\cdot\overline{D}\,\overline{\Phi}^{\bar{j}_2}\right)\cdots\left(\overline{D}\,\overline{\Phi}^{\bar{i}_p}\cdot\overline{D}\,\overline{\Phi}^{\bar{j}_p}\right)\,,\\ &\equiv\; \int d^4x\, d^2\theta\, d\overline{\theta}_{\dot{\alpha}}\; {\mathcal{O}_\xi}^{\dot{\alpha}}\,,\\ &=\; \int d^4x\, d^2\theta\; \nabla_{\bar{i}_1}\xi_{\bar{i}_2\cdots\bar{i}_p\,\bar{j}_1\cdots\bar{j}_p}\; \left(\overline{D}\,\overline{\Phi}^{\bar{i}_1}\cdot\overline{D}\,\overline{\Phi}^{\bar{j}_1}\right)\cdots\left(\overline{D}\,\overline{\Phi}^{\bar{i}_p}\cdot\overline{D}\,\overline{\Phi}^{\bar{j}_p}\right)\,.\end{aligned} \tag{2.2.34}$$

Here ξ is a section of $\overline{\Omega}^{p-1}_{\mathcal{M}} \otimes \overline{\Omega}^{p}_{\mathcal{M}}$. We do not know of any actual examples of operators of this type that can be written as integrals over three quarters of superspace but not over all of superspace, but we must still allow for this possibility.

Because the correction in (2.2.34) has the same form as the F-term in (2.2.29), we must consider the chiral operators $\mathcal{O}_\omega$ as defined up to the equivalence

$$\mathcal{O}_\omega \;\sim\; \mathcal{O}_\omega \,+\, \left\{\overline{Q}_{\dot{\alpha}}, {\mathcal{O}_\xi}^{\dot{\alpha}}\right\}\,. \tag{2.2.35}$$

Mathematically, this equivalence becomes an equivalence relation on sections of $\overline{\Omega}^{p}_{\mathcal{M}} \otimes \overline{\Omega}^{p}_{\mathcal{M}}$,

$$\omega_{\bar{i}_1\cdots\bar{i}_p\,\bar{j}_1\cdots\bar{j}_p} \;\sim\; \omega_{\bar{i}_1\cdots\bar{i}_p\,\bar{j}_1\cdots\bar{j}_p} \,+\, \nabla_{[\bar{i}_1}\xi_{\bar{i}_2\cdots\bar{i}_p]\,\bar{j}_1\cdots\bar{j}_p} \,+\, \left(\bar{i}_k \leftrightarrow \bar{j}_k\right)\,. \tag{2.2.36}$$

As we indicate, the term involving ξ is to be symmetrized like ω under the exchange of all pairs $\bar{i}_k \leftrightarrow \bar{j}_k$.

Because of this symmetrization, the equivalence relation implied by (2.2.36) on sections of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ is not the same as the usual equivalence relation in Dolbeault cohomology. Furthermore, since the corrections (2.2.34) arise from an integral only over three quarters of superspace, they are not supersymmetric unless we impose the (nontrivial) condition that $\overline{Q}^2$ annihilate the operator $\mathcal{O}_\xi{}^{\dot\alpha}$, which implies a corresponding constraint on the sections ξ which appear in (2.2.34) and (2.2.36).

We are unaware of a more standard mathematical description of this sort of cohomology, specific to the bundles $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ on an arbitrary Kähler manifold, and we will not comment further on its general structure. Luckily, symmetries alone will suffice in Section 3 to show that the operators $\mathcal{O}_\omega$ which we consider for SQCD cannot be written as integrals over three-fourths of superspace, much less all of it.

2.2.3 Adding a Superpotential to the Sigma Model

Although we are most interested in SQCD with massless flavors, a useful technique to study this theory is to consider instead SQCD with massive flavors and to ask how various observables depend upon the mass parameters. Because these mass parameters appear in a superpotential, holomorphy serves as a powerful tool to constrain their appearance in the effective action. In Section 4, we will apply exactly this technique as one way to compute the multi-fermion F-terms in SQCD.

More generally, we can consider adding any background superpotential W to the basic sigma model action,

$$S = \int d^4x\, d^4\theta\; K(\Phi^i, \overline{\Phi}^{\bar{i}}) + \int d^4x\, d^2\theta\; W(\Phi^i) + c.c. \tag{2.2.37}$$

Because of the superpotential, the on-shell supersymmetry algebra of the sigma model is altered, and hence the chirality condition on $\mathcal{O}_\omega$ is also altered. This fact is fundamental to our study of the mass deformation of SQCD in Section 4, so we pause to explain it here in the general setting.

In the new action (2.2.37), the on-shell variations under $\overline{Q}_{\dot\alpha}$ of the component fields ϕ^i,

$\overline{\phi}^{\bar{i}}$, ψ^i_β, and $\overline{\psi}^{\bar{i}}_{\dot{\beta}}$ are now given by

$$\begin{aligned} \delta_{\dot{\alpha}}\phi^i &= 0\,, & \delta_{\dot{\alpha}}\overline{\phi}^{\bar{i}} &= \overline{\psi}^{\bar{i}}_{\dot{\alpha}}\,, \\ \delta_{\dot{\alpha}}\psi^i_\beta &= i\,\partial_{\dot{\alpha}\beta}\phi^i\,, & \delta_{\dot{\alpha}}\overline{\psi}^{\bar{i}}_{\dot{\beta}} &= -\Gamma^{\bar{i}}_{\bar{j}\,\bar{k}}\,\overline{\psi}^{\bar{j}}_{\dot{\alpha}}\,\overline{\psi}^{\bar{k}}_{\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}\,g^{\bar{i}i}\partial_i W\,. \end{aligned} \tag{2.2.38}$$

Because of the appearance of the one-form dW in the variation of $\overline{\psi}^{\bar{i}}_{\dot{\beta}}$ in (2.2.38), the action of the supercharges $\overline{Q}_{\dot{\alpha}}$ on $\mathcal{O}_\omega$ is no longer given geometrically by the action of $\overline{\partial}$ on ω. Instead, when $\overline{\psi}^{\bar{i}}_{\dot{\beta}}$ is interpreted as a holomorphic tangent vector $\partial/\partial\phi^i$, the term involving W corresponds geometrically to the interior product of $\partial/\partial\phi^i$ with the holomorphic one-form dW. So the $\overline{\partial}$ operator is now generalized to the operator

$$\delta = \overline{\partial} + \iota_{dW}\,, \tag{2.2.39}$$

acting on sections of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$. Here ι_{dW} denotes the operator on $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ which acts by the interior product with the one-form dW. (In other words, ι_{dW} acts by removing $\overline{\psi}$ and replacing it with dW.) We note that because W is holomorphic, $\delta^2 = 0$. Thus, the first order chirality condition on the operator $\mathcal{O}_\omega$ becomes the requirement that δ annihilate ω.

A nice mathematical discussion of the cohomology theory associated to δ is given by Liu in [33], and applications to string theory are discussed in Chapter 3 of the thesis.

When ω is a section of $\overline{\Omega}^1_{\mathcal{M}} \otimes T\mathcal{M}$, then the modified chirality condition has a very direct geometric interpretation. In this case, the condition that $\delta\omega = 0$ implies that ω is annihilated separately by both the operators $\overline{\partial}$ and ι_{dW}. The latter condition implies that

$$\omega_{\bar{j}}^{\;i}\,\partial_i W = 0\,. \tag{2.2.40}$$

Since W is holomorphic, this condition is then equivalent to the condition that

$$\left(\overline{\partial}_{\bar{j}} + \omega_{\bar{j}}^{\;i}\,\partial_i\right)W = 0\,, \tag{2.2.41}$$

implying that the deformation of $\overline{\partial}$ represented by ω must preserve the holomorphy of W. More generally, if it is possible to modify W to a function $W + \Delta W$ that is holomorphic in the deformed complex structure, then $\omega + \Delta W$ is annihilated by δ.

2.3 Multi-Fermion F-Terms in $SU(2)$ SQCD

Up to this point, we have discussed general properties of multi-fermion F-terms in an arbitrary $\mathcal{N} = 1$ sigma model. We now specialize our analysis to the particular case of SQCD. Our main goal in the rest of the chapter, concentrating mainly on the example of gauge group $SU(2)$, is to show that multi-fermion F-terms are generated in the effective action of SQCD.

To this end, we begin in this section by analyzing the constraints imposed by symmetries and holomorphy on the form of any multi-fermion F-term corrections in $SU(2)$ SQCD. The case of SQCD with gauge group $SU(2)$ is particularly simple due to the enhancement of the flavor symmetry. In this case, we fix the form of the operators $\mathcal{O}_\omega$ uniquely, and we demonstrate that they are nontrivial in the cohomology of $\overline{Q}_{\dot{\alpha}}$.

In the general case of SQCD with gauge group $SU(N_c)$ and $N_f > N_c$ flavors, a similar analysis to determine the form of the operators $\mathcal{O}_\omega$ appears to be more complicated, since the geometry of the moduli space $\mathcal{M}$ itself is more complicated. However, the direct instanton computation of Section 4.1 shows that such interactions arise for all N_c and $N_f \geq N_c - 1$. The other derivations in Section 4 generalize in spirit.

In the case of $SU(2)$ SQCD with $N_f = n$ flavors, we have already described algebraically the classical moduli space $\mathcal{M}$ as being parametrized by the mesons M^{ij}, subject to the system of quadratic equations $M \wedge M = 0$. This description of $\mathcal{M}$ has the virtue of being very succinct. However, we now give another description of $\mathcal{M}$ which makes its symmetry more apparent and consequently enables us to determine immediately the chiral operators $\mathcal{O}_\omega$ which arise from cohomology classes on $\mathcal{M}$.

2.3.1 More About the Geometry of $\mathcal{M}$

Since symmetries are of the utmost importance, we first review the symmetries of $SU(2)$ SQCD with $N_f = n$ flavors. Besides the $SU(2)$ color and $SU(2n)$ flavor symmetries, this gauge theory also possesses a non-anomalous $U(1)$ $\mathcal{R}$-symmetry as well as an anomalous $U(1)$ axial symmetry. Under these symmetries, the quark superfields Q^i_a, the mesons M^{ij}, and the holomorphic coupling scale Λ transform as follows:

$$\begin{array}{ccccc} & SU(2)_c & SU(2n) & U(1)_A & U(1)_{\mathcal{R}} \\ Q^i_a & \mathbf{2} & \mathbf{2n} & 1 & 1-\frac{2}{n} \\ M^{ij} & \mathbf{1} & \wedge^2(\mathbf{2n}) & 2 & 2\left(1-\frac{2}{n}\right) \\ \Lambda^{6-n} & \mathbf{1} & \mathbf{1} & 2n & 0\,. \end{array} \tag{2.3.1}$$

Here Λ^{6-n} is the standard instanton counting parameter. (In this one place, we denote the gauge group as $SU(2)_c$, to distinguish it from an unbroken $SU(2)$ flavor group that will appear momentarily.)

We now describe $\mathcal{M}$ by considering the pattern of symmetry breaking around a fixed supersymmetric vacuum. Up to the action of the symmetries, any solution of the usual D-term equations takes the form

$$Q^i_a = \begin{pmatrix} v & 0 \\ 0 & v \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \equiv v\,\widehat{\delta}^i_a\,, \tag{2.3.2}$$

with v being an arbitrary complex number.

So long as v is non-zero, the expectation value of Q^i_a in (2.3.2) breaks the symmetry group in (2.3.1) down to a subgroup

$$SU(2) \times SU(2n-2) \times U(1)'_A \times U(1)'_{\mathcal{R}}\,. \tag{2.3.3}$$

The unbroken $SU(2) \times SU(2n-2)$ factor arises in the obvious way, and the unbroken $U(1)$ axial and $\mathcal{R}$-symmetries arise from linear combinations of the corresponding generators in (2.3.1) with the diagonal flavor generator in the center of the subgroup

$$S(U(2) \times U(2n-2)) \subset SU(2n)\,. \tag{2.3.4}$$

Of course, the gauge group is completely Higgsed, and the massless fluctuations of the quarks Q^i_a about the point (2.3.2) decompose into two irreducible representations of the unbroken symmetry group (2.3.3), with

$$\begin{array}{ccccc} & SU(2) & SU(2n-2) & U(1)'_A & U(1)'_{\mathcal{R}} \\ \Phi^s_c & \mathbf{2} & \mathbf{2n-2} & \frac{n}{n-1} & \frac{n-2}{n-1} \\ \Phi & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \Lambda^{6-n} & \mathbf{1} & \mathbf{1} & 2n & 0\,. \end{array} \tag{2.3.5}$$

Here the singlet Φ describes a rescaling of M; the other fluctuations transform as an irreducible representation Φ^s_c of the unbroken symmetry, where $c = 1, 2$ is an index labelling the $\mathbf{2}$ of the unbroken $SU(2)$ and $s = 3, \ldots, 2n$ is now an index labelling the $\mathbf{2n-2}$ of $SU(2n-2)$. Throughout the chapter, we will apply the convention that c, d, e, f refer to indices $1, 2$ of the unbroken $SU(2)$, that s, t, u, v refer to indices $3, \ldots, 2n$ of the unbroken $SU(2n-2)$, and that i, j, k, l run over all indices $1, \ldots, 2n$ of the full $SU(2n)$ flavor symmetry. These massless fluctuations Φ and Φ^s_c represent local coordinates on $\mathcal{M}$, such as were used in section 2. Finally, for future reference in Section 3.2 we have included in (2.3.5) the charges of Λ^{6-n}, which are identical to those in (2.3.1).

Because any solution of the D-term equations can be brought to the form (2.3.2) using the $SU(2) \times SU(2n)$ symmetry of SQCD, we see that the $SU(2n)$ flavor symmetry acts transitively on the quotient of $\mathcal{M}$ minus the origin by the $\mathbb{C}^*$ action which scales v. We thus set $\widetilde{\mathcal{M}} = \mathcal{M} - \{0\}$, and we let B be this quotient of $\widetilde{\mathcal{M}}$ by $\mathbb{C}^*$.

Furthermore, our description of the symmetry breaking pattern in (2.3.3) is equivalent to the geometric observation that, at any non-zero v, the subgroup of $SU(2n)$ which stabilizes the point corresponding to $Q^i_a = v\,\widehat{\delta}^i_a$ on $\mathcal{M}$ is $S(U(2) \times U(2n-2))$. Thus, we can describe B as a homogeneous (and in fact symmetric) space,

$$B = SU(2n)/S(U(2) \times U(2n-2))\,. \tag{2.3.6}$$

To incorporate the value of v into our description of $\mathcal{M}$, we observe that the $\mathbb{C}^*$ action which scales v is the complexification of the $U(1)_A$ symmetry in (2.3.1). This symmetry corresponds to the action of the central $U(1)$ which lies in the stabilizer subgroup $S(U(2) \times U(2n-2))$ and whose generator mixes with the generator of $U(1)_A$ under the symmetry breaking. Associated to this $U(1)$ generator in $S(U(2) \times U(2n-2))$ is a corresponding homogeneous line bundle $\mathcal{L}$ — and hence a $\mathbb{C}^*$ bundle — over B. To specify $\mathcal{L}$, we simply note that the singlet field Φ transforms as a section of $\mathcal{L}$ and has charge $+2$ under the original $U(1)_A$ symmetry, as Φ describes the rescaling of M^{ij}.

So, if we excise the singularity at the origin of $\mathcal{M}$, then $\widetilde{\mathcal{M}}$ can be globally described as this $\mathbb{C}^*$ bundle over the base B,

$$\mathbb{C}^* \longrightarrow \widetilde{\mathcal{M}} \stackrel{\pi}{\longrightarrow} B\,. \tag{2.3.7}$$

A direct relationship now exists between the algebraic description of $\mathcal{M}$ in (2.2.3) and the intrinsic description of $\mathcal{M}$ in (2.3.7). To describe this relation, we consider the mesons M^{ij} modulo overall scaling, corresponding to the $\mathbb{C}^*$ action generated by $U(1)_A$. Then the equations $M \wedge M = 0$ are the classical Plücker relations [34] which describe the Grassmannian $Gr(2, 2n)$ of complex two planes in $\mathbb{C}^{2n}$ as an algebraic subvariety of the projective space parametrized by M^{ij}.

On the other hand, this Grassmannian can also be described as a quotient,

$$Gr(2, 2n) = U(2n)/(U(2) \times U(2n-2))\,, \tag{2.3.8}$$

which is equivalent to our description in (2.3.6) of the base B. Thus, the $\mathbb{C}^*$ bundle over B in (2.3.7) is simply the bundle associated to the affine cone over the Grassmannian $Gr(2, 2n)$ with its Plücker embedding in projective space. Equivalently, the line bundle $\mathcal{L}$ arises as the pullback from the degree one bundle $\mathcal{O}(1)$ on projective space.

2.3.2 The New F-Terms

With our thorough discussion of the symmetries of SQCD, we can immediately derive the form of any multi-fermion F-terms that might appear on $\mathcal{M}$. We perform our analysis in two steps: first locally, and then globally.

Local Analysis

Locally, we construct the chiral operator $\mathcal{O}_\omega$ from the massless fluctuations described by Φ^s_c and Φ about the vacuum $Q^i_a = \widehat{\delta}^i_a$. Thus, in terms of the section ω of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$, we only consider ω as restricted to the tangent space of $\mathcal{M}$ at this point.

Now, the operator $\mathcal{O}_\omega$ must be invariant under the symmetries $SU(2) \times SU(2n-2) \times U(1)'_A$ and must have charge $+2$ under $U(1)'_{\mathcal{R}}$ in (2.3.5). Furthermore, since we are only considering the corresponding section ω as restricted to the tangent space of a point in $\mathcal{M}$, we must construct $\mathcal{O}_\omega$ completely from the fermionic fields $\overline{D}_{\dot\alpha}\overline{\Phi}$ and $\overline{D}_{\dot\alpha}\overline{\Phi}^c_s$ which represent either one-forms or (by raising an index) tangent vectors to $\mathcal{M}$. From (2.3.5) we see that $\overline{D}_{\dot\alpha}\overline{\Phi}$ and $\overline{D}_{\dot\alpha}\overline{\Phi}^c_s$ have respective charges $+1$ and $+1/(n-1)$ under $U(1)'_{\mathcal{R}}$. So just to make an operator of $U(1)'_{\mathcal{R}}$ charge $+2$, we require that it contain either two copies of $\overline{D}_{\dot\alpha}\overline{\Phi}$, or one copy of $\overline{D}_{\dot\alpha}\overline{\Phi}$ and $n-1$ copies of $\overline{D}_{\dot\alpha}\overline{\Phi}^c_s$, or $2(n-1)$ copies of $\overline{D}_{\dot\alpha}\overline{\Phi}^c_s$.

We can immediately rule out the first possibility, necessarily of the form $\overline{D}\,\overline{\Phi}\cdot\overline{D}\,\overline{\Phi}$, since from (2.3.5) this operator is not charged under $U(1)'_A$ and hence is not multiplied by any power of Λ, contradicting the fact that our operator must vanish in the appropriate weak coupling limit as well as the fact that we expect it to be generated by instantons. (A more detailed study shows that there are no non-trivial chiral operators of this type.) On the other hand, since the only tensors of $SU(2)\times SU(2n-2)$ which we can use to make invariants out of the fields $\overline{D}_{\dot\alpha}\overline{\Phi}^c_s$ are the anti-symmetric tensors ϵ_{cd} and $\epsilon^{s_1t_1\cdots s_pt_p}$ with $p=n-1$, we cannot make an invariant operator from one copy of $\overline{D}_{\dot\alpha}\overline{\Phi}$ and only $n-1$ copies of $\overline{D}_{\dot\alpha}\overline{\Phi}^c_s$.

We are left to consider the operator $\mathcal{O}_\omega$ which is made from $2(n-1)$ copies of $\overline{D}_{\dot\alpha}\,\overline{\Phi}^c_s$, of the form

$$\Lambda^{6-n}\,\epsilon^{s_1t_1\cdots s_pt_p}\,\epsilon_{c_1d_1}\cdots\epsilon_{c_pd_p}\,\left(\overline{D}\,\overline{\Phi}^{c_1}_{s_1}\cdot\overline{D}\,\overline{\Phi}^{d_1}_{t_1}\right)\cdots\left(\overline{D}\,\overline{\Phi}^{c_p}_{s_p}\cdot\overline{D}\,\overline{\Phi}^{d_p}_{t_p}\right)\,, \quad p=n-1\,. \qquad (2.3.9)$$

This operator is invariant under $SU(2)\times SU(2n-2)$ and carries charge $+2$ under $U(1)'_{\mathcal{R}}$. The pattern of contractions of spinor indices is fixed by the fact that each expression in parentheses must be antisymmetric under exchanges of both the pairs (c,d) and (s,t) and must also obey Fermi statistics.

Also, we see from (2.3.5) that each fermion appearing in $\mathcal{O}_\omega$ carries charge $-n/(n-1)$ under $U(1)'_A$, so the fermionic part of $\mathcal{O}_\omega$ carries axial charge $-2n$. The fact that $\mathcal{O}_\omega$ must be invariant under the axial symmetry then fixes the dependence on Λ. In particular, we see that the operator in (2.3.9) involves a single power of the instanton counting parameter

Λ^{6-n} and so could arise as a one-instanton effect.

So the local form of $\mathcal{O}_\omega$ is fixed completely by the symmetries, and moreover $\mathcal{O}_\omega$ has the correct dependence on Λ to be generated by instantons. Furthermore, in terms of the section ω of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$, we see that the parameter p is related to the number of flavors n by $p = n - 1$. This fact is a special case of the relation $p = N_f - N_c + 1$ which must hold in $SU(N_c)$ SQCD with N_f flavors. In the direct instanton computation in Section 4, this relation follows most immediately by counting fermion zero modes in the instanton background.

A Geometric Remark on Pullbacks From B

Because $\mathcal{O}_\omega$ only involves $\overline{D}\,\overline{\Phi}^c_s$ and not the singlet $\overline{D}\,\overline{\Phi}$, the section ω has only components along the base B, with no legs along the $\mathbb{C}^*$ fiber. Naively, one might have concluded that ω then arises as the pullback from a section of $\overline{\Omega}^{n-1}_B \otimes \wedge^{n-1} TB$ on B. Actually, the dependence of $\mathcal{O}_\omega$ on scaling of the quark superfields means that it is a pullback from a section of $\overline{\Omega}^{n-1}_B \otimes \wedge^{n-1} TB \otimes \mathcal{L}^k$ for some k. (There is an irrelevant subtlety here. Because of the nontrivial exact sequence $0 \to TF \to T\mathcal{M} \to TB \to 0$, where TF is the tangent space to the fibers of $\mathcal{M} \to B$, a section of TB cannot literally be pulled back to a section of $\mathcal{M}$. However, our concern is really with cohomology, and because the cohomology of TF is trivial, we can ignore this subtlety.)

In fact, the degree of the twist by $\mathcal{L}$ is fixed to be $k = -n$. Indeed, as we noted above, the fermionic part of $\mathcal{O}_\omega$ carries $U(1)'_A$ charge $-2n$. As $U(1)'_A$ differs from $U(1)_A$ by a generator of $SU(2n)$ under which $\mathcal{O}_\omega$ is invariant, this means that, if we omit the factor of Λ^{6-n} from (2.3.9), then $\mathcal{O}_\omega$ has $U(1)_A$ charge $-2n$. Since the basic meson field M has $U(1)_A$ charge 2, this means that $\mathcal{O}_\omega$ transforms as M^{-n} and ω can be regarded as a section of $\overline{\Omega}^{n-1}_B \otimes \wedge^{n-1} TB \otimes \mathcal{L}^{-n}$.

Consider a general scaling $M \to \lambda M$, $\overline{M} \to \overline{\lambda}\,\overline{M}$, for $\lambda \in \mathbb{C}^*$. Under this scaling, $\omega \to \lambda^{-n} \overline{\lambda}^0 \omega = \lambda^{-n} \omega$. The fact that the exponent of $\overline{\lambda}$ is zero is implied by the fact that $\overline{\partial}\omega = 0$, and the fact that the exponent of λ is $-n$ is equivalent to the fact that ω is a section of $\overline{\Omega}^{n-1}_B \otimes \wedge^{n-1} TB \otimes \mathcal{L}^{-n}$. We apply these observations when we write a global expression for $\mathcal{O}_\omega$.

Chirality and Cohomology of $\mathcal{O}_\omega$

Let us now check that $\mathcal{O}_\omega$ is chiral — annihilated by $\overline{Q}_{\dot\alpha}$ — and moreover represents a nontrivial $\overline{Q}_{\dot\alpha}$ cohomology class. This check follows directly from symmetries.

We recall that the chirality condition on $\mathcal{O}_\omega$ is equivalent to the geometric condition that $\overline{\partial}$ annihilate ω. Because $\mathcal{O}_\omega$ is a pullback from B, we can consider just the action of the $\overline{\partial}$ operator along B on ω, considered as a section of $\overline{\Omega}^p_B \otimes \wedge^p TB \otimes \mathcal{L}^{-n}$. Because both the $\overline{\partial}$ operator on B and ω are singlets under the action of $SU(2)\times SU(2n-2)$, the section $\overline{\partial}\omega$ of $\Omega^{p+1}_B \otimes \wedge^p TB \otimes \mathcal{L}^{-n}$ must also be a singlet. But no (nontrivial) invariant section of $\Omega^{p+1}_B \otimes \wedge^p TB \otimes \mathcal{L}^{-n}$ exists; such a section would be constructed from an $SU(2)$ singlet made from the tensor product of $2p+1$ $\mathbf{2}$'s. So the $\overline{\partial}$ operator on B necessarily annihilates ω.

A similar argument based upon symmetries also shows that $\mathcal{O}_\omega$ cannot be written in the form $\{\overline{Q}_{\dot\alpha}, {\mathcal{O}_\xi}^{\dot\alpha}\}$ in a way that respects the flavor symmetry. Indeed, invariant sections of $\overline{\Omega}^{p-1}_B \otimes \wedge^p TB \otimes \mathcal{L}^{-n}$ and $\overline{\Omega}^p_B \otimes \wedge^{p-1} TB \otimes \mathcal{L}^{-n}$ do not exist, since one cannot make an $SU(2)$ invariant from $2p-1$ $\mathbf{2}$'s.

Global Analysis

Our expression in (2.3.9) is only a local expression for $\mathcal{O}_\omega$, but because the $SU(2n)$ flavor symmetry acts transitively on $\mathcal{M}$, this local expression suffices to determine a global expression for $\mathcal{O}_\omega$. In order to write such an expression using the mesons M^{ij}, we observe that the local tensors $\epsilon^{s_1 t_1 \cdots s_p t_p}$ and ϵ_{cd} in (2.3.9) extend globally to tensors on $\mathcal{M}$ given by $\epsilon^{i_1 j_1 \cdots i_n j_n}\, \overline{M}_{i_1 j_1}$ and M^{kl}. Then $\mathcal{O}_\omega$ must take the global form

$$\mathcal{O}_\omega \;=\; \Lambda^{6-n}\, F(\overline{M}M)\, \epsilon^{i_1 j_1 \cdots i_n j_n}\, \overline{M}_{i_1 j_1}\, \mathcal{O}_{i_2 j_2} \cdots \mathcal{O}_{i_n j_n}\,, \tag{2.3.10}$$

with

$$\mathcal{O}_{ij} \;\equiv\; M^{kl}\, \overline{D}\,\overline{M}_{ik} \cdot \overline{D}\,\overline{M}_{lj}\,, \qquad \overline{M}M \;\equiv\; \frac{1}{2}\,\overline{M}_{ij} M^{ij}\,. \tag{2.3.11}$$

Of course, we employ the usual summation convention in writing $\overline{M}M$ as in (2.3.11), using the Kähler metric g on $\mathcal{M}$ to raise and lower indices throughout.

In writing $\mathcal{O}_\omega$, we have also included as a prefactor an invariant function $F(\overline{M}M)$ on $\mathcal{M}$ which is not directly determined by the local expression in (2.3.9). The function $F(\overline{M}M)$ is, however, determined by dimensional analysis and also, as we will now discuss, by requiring $\mathcal{O}_\omega$ to be chiral.

The chirality condition on $\mathcal{O}_\omega$ is most naturally expressed as the condition that the corresponding section ω of $\overline{\Omega}^{n-1}_{\mathcal{M}} \otimes \wedge^{n-1} T\mathcal{M}$ be annihilated by $\overline{\partial}$. Explicitly, the section ω which determines the operator $\mathcal{O}_\omega$ in (2.3.10) is given globally by

$$\omega \;=\; F(\overline{M}M)\, \epsilon^{i_1 j_1 \cdots i_n j_n} \overline{M}_{i_1 j_1} \left(M^{k_2 l_2}\, d\overline{M}_{i_2 k_2} \frac{\partial}{\partial M^{l_2 j_2}} \right) \cdots \left(M^{k_n l_n}\, d\overline{M}_{i_n k_n} \frac{\partial}{\partial M^{l_n j_n}} \right) . \tag{2.3.12}$$

In order that ω be annihilated by $\overline{\partial}$, we have already observed that it must be invariant under the scaling $\overline{M} \to \overline{\lambda}\,\overline{M}$. Furthermore, in order that ω arise from a section of the bundle $\overline{\Omega}^{n-1} \otimes \wedge^{n-1} TB \otimes \mathcal{L}^{-n}$, we have also observed that it must transform under the scaling $M \to \lambda M$ as $\omega \to \lambda^{-n}\omega$.

However, if we ignore $F(\overline{M}M)$, we see that ω in (2.3.12) otherwise scales with degree n in $\overline{\lambda}$ and with degree zero in λ. Thus, we set $F(\overline{M}M) = (\overline{M}M)^{-n}$ to ensure that ω scales as M^{-n}. So we must set

$$\mathcal{O}_\omega \;=\; \Lambda^{6-n} \left(\overline{M}M\right)^{-n} \epsilon^{i_1 j_1 \cdots i_n j_n}\, \overline{M}_{i_1 j_1}\, \mathcal{O}_{i_2 j_2} \cdots \mathcal{O}_{i_n j_n} \, . \tag{2.3.13}$$

This expression directly generalizes our previous formula (2.2.28) in the special case $n = 2$.

Let us also make a remark about the global form of $\mathcal{O}_\omega$, or equivalently ω in (2.3.12). In this expression, the components M^{ij} of M which appear are just affine coordinates on a vector space in which $\mathcal{M}$ is embedded, and it must be that only the components of $\partial/\partial M^{ij}$ and $d\overline{M}^{ij}$ which represent tangent and cotangent vectors to $\mathcal{M}$ itself appear in (2.3.12). To check this condition, we can without loss consider the point of $\mathcal{M}$ at which $M^{ij} = \widehat{\epsilon}^{ij}$. (We recall that the nonzero components of $\widehat{\epsilon}$ are $\widehat{\epsilon}^{12} = -\widehat{\epsilon}^{21} = 1$.) Then the holomorphic tangent space to $\mathcal{M}$ at this point is spanned by vectors $\partial/\partial M^{ij}$ for which both $i, j = 1, 2$, corresponding to the singlet Φ, or for which $i = 1, 2$ and $j > 2$, corresponding to Φ^s_c in the representation $(\mathbf{2}, \mathbf{2n-2})$.

In particular, the vector $\partial/\partial M^{ij}$ for which both $i, j > 2$ is not a tangent vector to $\mathcal{M}$ at this point. So in order for (2.3.12) to be well defined as a section of $\overline{\Omega}^{n-1}_{\mathcal{M}} \otimes \wedge^{n-1} T\mathcal{M}$, such components of $d\overline{M}_{ij}$ and $\partial/\partial M^{ij}$ with both $i, j > 2$ must not appear. However, upon substituting $M^{ij} = \hat{\epsilon}^{ij}$ into (2.3.12), we see that the factors of $\overline{M}_{i_1 j_1}$ and M^{kl} ensure that these unwanted components do not appear, and the expression in (2.3.12) is a section of $\overline{\Omega}^{n-1}_{\mathcal{M}} \otimes \wedge^{n-1} T\mathcal{M}$ as claimed.

Like (2.2.25), (2.3.13) is written in terms of an arbitrary unknown Kähler metric on $\mathcal{M}$. As in (2.2.26), we can make the asymptotic behavior more explicit, since we know the asymptotic form of the Kähler metric. In writing this formula, just as in (2.2.26), we use Kronecker deltas to raise and lower indices on M (so all components of M and $\overline{M}$ with index up or down have dimension two), and write all factors of $\overline{M}M$ explicitly. With this understood, the asymptotic form of the interaction is

$$\Lambda^{6-n} \left(\overline{M}M\right)^{-(3n-1)/2} \epsilon^{i_1 j_1 \cdots i_n j_n} \, \overline{M}_{i_1 j_1} \, \mathcal{O}_{i_2 j_2} \cdots \mathcal{O}_{i_n j_n}. \tag{2.3.14}$$

2.4 Computing The Multi-Fermion F-Terms

Although symmetries suffice to fix the form of the F-term correction in SQCD uniquely, we must still check that it is actually generated. So in this section, we provide three computations which show this.

2.4.1 A Direct Instanton Computation

Since instanton effects are the subject of this thesis, we first generate the F-terms directly by a one-instanton computation which generalizes the classic one-instanton computation [7,23,24] of the superpotential in the theory with $N_f = N_c - 1$ flavors.

The most basic, and most illuminating, feature of this instanton computation is that it directly explains how the relation $p = n - 1$ arises in the $SU(2)$ theory with $N_f = n$ flavors. This relation arises from counting fermion zero modes in the instanton background, and the same counting implies that, in the $SU(N_c)$ theory, we must have $p = N_f - N_c + 1$.

Very briefly, before we review the details of the instanton computation, we will explain the counting of fermion zero modes that controls the structure of the F-term. We thus recall that, in the one-instanton background, we find at leading order $2N_c$ gaugino zero modes and $2N_f$ quark zero modes. However, beyond leading order, the Yukawa couplings pair $2(N_c - 1)$ of the gaugino and quark zero modes, and these modes are lifted. As a result, two gaugino zero modes and $2(N_f - N_c + 1)$ quark zero modes remain. The two gaugino zero modes that remain are generated by exact global supersymmetries. Thus, if we consider the general form of the multi-fermion F-term in (2.2.29), the two gaugino zero modes are associated to the fermionic collective coordinates θ^α that appear in the integral over superspace, and the $2(N_f - N_c + 1)$ quark zero modes must be absorbed by the chiral operator $\mathcal{O}_\omega$ itself. So $p = N_f - N_c + 1$.

We now present the details of the instanton computation in the case of $SU(2)$ SQCD. As described above, this computation should generalize directly to the case of $SU(N_c)$ SQCD, though one must consider a more involved integral over the collective coordinates of the instanton.

Following closely the computation of Affleck, Dine, and Seiberg [7], we work on the Higgs branch of SQCD, under the assumption that the classical quark vacuum expectation value, $Q^i_a = v\,\widehat{\delta}^i_a$, is large and the effective gauge coupling $g^2(v)$ is small. In this regime, the approximate instanton equations are valid,

$$D^\mu F_{\mu\nu} \,=\, 0\,, \quad D^2 q^i_a \,=\, 0\,, \tag{2.4.1}$$

where we recall that q^i_a is the scalar component of Q^i_a. In a one-instanton background, the solution of (2.4.1) for q^i_a with boundary condition fixed by its classical expectation value is given by

$$q^i_a = \frac{\sigma^i_{\mu a}\, x^\mu\, v}{\sqrt{\rho^2 + x^2}}\,. \tag{2.4.2}$$

Here $\sigma_\mu = (1, -i\sigma^A)$, with σ^A the Pauli matrices, are the usual quaternion representatives. Also, x^μ is a coordinate on $\mathbb{R}^4$, and ρ is the scale of the instanton solution. The classical action for this instanton background is

$$S_0 = \frac{1}{g^2}\left(8\pi^2 + 4\pi^2\rho^2|v|^2\right) . \tag{2.4.3}$$

When $|v|^2 \neq 0$, instantons of large size are exponentially suppressed by this classical action, and the integral over the scale ρ will be convergent.

We must now consider what sort of correlation function to compute in order to probe for the multi-fermion F-term determined by the operator $\mathcal{O}_\omega$ in (2.3.13). For this purpose, we recall the chiral superfields Φ and Φ_c^s which we introduced in Section 3 to describe massless fluctuations of the quark superfields around the Higgs vacuum. Introducing components for these fields,

$$\begin{aligned} \Phi &= \phi + \theta\chi + \dots\,, \\ \Phi_c^s &= \phi_c^s + \theta\chi_c^s + \dots\,, \end{aligned} \tag{2.4.4}$$

we see that among the various interactions which arise from the multi-fermion F-term is an effective interaction for $2n$ fermions of the form

$$\frac{\Lambda^{6-n}}{v^4\,|v|^{2(n-1)}} \int d^4x\; \epsilon^{s_1 t_1 \cdots s_p t_p}\, \epsilon_{c_1 d_1} \cdots \epsilon_{c_p d_p}\, \chi\cdot\chi\, \left(\overline{\chi}^{c_1}_{s_1}\cdot\overline{\chi}^{d_1}_{t_1}\right)\cdots\left(\overline{\chi}^{c_p}_{s_p}\cdot\overline{\chi}^{d_p}_{t_p}\right)\,, \quad p = n-1\,. \tag{2.4.5}$$

We have included the dependence of this interaction on v and $\overline{v}$. This dependence can either be checked directly, or it can be deduced from requirement that the interaction transforms as λ^{-n} under $M \to \lambda M$, $\overline{M} \to \overline{\lambda}\,\overline{M}$, as discussed in Section 3.

To probe for the presence of the F-term, we thus compute in the instanton background the correlation function

$$\left\langle \overline{\chi}\cdot\overline{\chi}\left(\chi^{s_1}_{c_1}\cdot\chi^{t_1}_{d_1}\right)\cdots\left(\chi^{s_p}_{c_p}\cdot\chi^{t_p}_{d_p}\right)\right\rangle . \tag{2.4.6}$$

(Because the correlator includes external legs with massless propagators, the fermions conjugate to those in the effective vertex appear.) This computation as usual has two pieces: a one-loop integral over fluctuating modes in the instanton background and an integral over zero modes. Because the instanton background is supersymmetric to leading order, the one-loop integral over quantum fluctuations is trivial and contributes only a factor of unity. So the important integral to consider is the integral over zero modes.

Bosonic Zero Modes

As usual, in the instanton background we have eight bosonic zero modes. Four zero modes are associated to the collective coordinate x_0 for the location of the instanton in $\mathbb{R}^4$. One zero mode is associated to the scale ρ of the instanton. Finally, three zero modes arise from global $SU(2)$ gauge transformations and are associated to a collective coordinate h on $SU(2)$.

Fermionic Zero Modes

Much more important than the bosonic zero modes are the fermionic zero modes. We have already discussed the counting of these modes generally, but now we review the details.

First, we have two gaugino zero modes which arise from the action of the chiral supercharges Q_α and which take the form

$$\lambda_\alpha^{SS\,A\,[\beta]} = \frac{\rho^2\,\sigma^{A\,\beta}_{\ \ \alpha}}{(\rho^2+x^2)^2}\,. \tag{2.4.7}$$

Here SS stands for global supersymmetry, A labels the adjoint representation of $SU(2)$, α is a spinor index, and β simply labels the two zero modes. Since we will not try to compute the absolute normalization of our interaction, we have not bothered to normalize the zero modes.

Second, at leading order in g^2, we have an additional $2n+2$ fermion zero modes. Two of these extra zero modes are gaugino zero modes associated to the action of the superconformal generators $x_\beta^{\dot\beta}Q^\beta$, of the form

$$\lambda_\alpha^{SC\,A\,[\dot\beta]} = \frac{\rho\,x_\beta^{\dot\beta}\,\sigma^{A\,\beta}_{\ \ \alpha}}{(\rho^2+x^2)^2}\,. \tag{2.4.8}$$

The other $2n$ zero modes arise from the $2n$ fermion doublets and are of the form

$$\psi^i_{\alpha a[j]} = \frac{\rho\,\delta^i_j\,h_a^b\,\epsilon_{\alpha b}}{(\rho^2+x^2)^{3/2}}\,. \tag{2.4.9}$$

Again, j is just an index that labels the zero modes. We have also included explicitly the dependence of these modes on the element h_a^b of $SU(2)$ parametrizing global gauge transformations. We could also have included this collective coordinate in (2.4.7) and

(2.4.8), but any dependence of the gaugino zero modes on h will drop out immediately in our computation.

These $2n$ zero modes transform in the representation $\mathbf{2n}$ of the flavor group $SU(2n)$. After giving expectations to the quark superfields, $SU(2n)$ is broken to $SU(2) \times SU(2n-2)$ (where in an instanton field, $SU(2)$ must be combined with a rotation). Under the subgroup, the zero modes of ψ transform as $(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2n-2})$. The superconformal zero modes similarly transform as $(\mathbf{2}, \mathbf{1})$.

Yukawa Interactions

The zero modes in (2.4.7), (2.4.8), and (2.4.9) are simply zero modes of the $\not{D}$ operator in the instanton background. However, to perform the instanton computation, we must go beyond leading order and consider the effect of the Yukawa couplings in SQCD. These couplings of course take the form

$$\int d^4x \; \overline{q}_i^a \; \left(\psi_b^i \cdot \lambda_a^b\right) \; . \tag{2.4.10}$$

On the Higgs branch, with q satisfying (2.4.2), this interaction pairs the two superconformal zero modes λ^{SC} with the two zero modes of the quarks that transform the same way, which are those with $i = 1, 2$ in (2.4.9) (and which we have denoted χ in (2.4.4)). As a result, when we compute the correlator (2.4.6), these fermion zero-modes can be absorbed by pulling down two copies of the Yukawa interaction (2.4.10) from the SQCD action, which contributes a factor proportional to $\overline{v}^2$ to the correlator.

We are then left with the two gaugino zero modes λ^{SS} and the other $2n-2$ quark zero modes appearing in (2.4.9). Of course, these $2n-2$ quark zero modes are absorbed directly by the massless fermions χ_c^s appearing in the correlator (2.4.6). But what of the zero modes λ^{SS}?

To answer this question, we recall that another very important, qualitative effect of the Yukawa coupling (2.4.10) is that it alters the form of the zero modes λ^{SS} to include components also involving the fermion $\overline{\chi}$. Specifically, to first order in ρv, the relevant equations of motion are

$$\not{D}\,\lambda \,=\, 0\,, \qquad \overline{\not{D}}\,\overline{\psi} \,=\, \sqrt{2}\,\overline{q}\cdot\lambda\,, \tag{2.4.11}$$

which have solution

$$\lambda \,=\, \lambda^{SS}\,, \qquad \overline{\psi}^{\,i\,[\beta]}_{\dot\alpha\,a} \,=\, \frac{1}{4\pi}\,\overline{\not{D}}^{[\beta]}_{\dot\alpha}\,\overline{q}^{\,i}_{a}\,, \tag{2.4.12}$$

with $\overline{q}$ as in (2.4.2). Simply by symmetry, the massless components of $\overline{\psi}^{\,i\,[\beta]}_{\dot\alpha\,a}$ which mix with λ^{SS} must correspond to the singlet $\overline{\chi}$. Thus, the two supersymmetric zero modes λ^{SS} are absorbed by the two fermions $\overline{\chi}$ which appear in the correlator (2.4.6).

The classical wavefunction of $\overline{\chi}$ can be explicitly evaluated in the instanton background from (2.4.12), and far from the instanton location x_0 the wavefunction takes the form

$$\overline{\chi}^{[\beta]}_{\dot\alpha}(x) \,=\, \overline{v}\rho^2 S^{\beta}_{\dot\alpha}(x, x_0)\,, \tag{2.4.13}$$

where $S^{\beta}_{\dot\alpha}(x, x_0)$ is the free fermion propagator.

Computing the Correlator

We are now prepared to compute the fermion correlator (2.4.6) in the instanton background. Using the classical wavefunctions (2.4.9) and (2.4.13) for the fermion zero modes, we see that

$$\begin{aligned}
&\left\langle \overline{\chi}\cdot\overline{\chi}\left(\chi^{s_1}_{c_1}\cdot\chi^{t_1}_{d_1}\right)\cdots\left(\chi^{s_p}_{c_p}\cdot\chi^{t_p}_{d_p}\right)\right\rangle \,=\\
&\overline{v}^4\,\Lambda^{6-n}\int d^4x_0\,d\rho\,d\mu\,\rho^{2n+5}\,\exp(-4\pi^2\rho^2|v|^2/g^2)\,\epsilon^{s_1t_1\cdots s_pt_p}\,\left(h^{e_1}_{c_1}h^{f_1}_{d_1}\,\epsilon_{e_1f_1}\right)\cdots\left(h^{e_p}_{c_p}h^{f_p}_{d_p}\,\epsilon_{e_pf_p}\right)\times\\
&\times\;\left(S(y_1-x_0)\cdot S(y_2-x_0)\right)\cdots\left(S(y_{2n-1}-x_0)\cdot S(y_{2n}-x_0)\right).
\end{aligned} \tag{2.4.14}$$

In this expression, $y_1, \ldots, y_{2n}$ are the positions of the $2n$ fermions in $\mathbb{R}^4$, which are assumed to be far from the position x_0 of the instanton. We then make use of the fact that, in this limit, the classical wavefunctions (2.4.9) of the fermions χ^s_c have the correct asymptotic behavior so that the correlator can be written using the free fermion propagator S. In computing the amputated vertex, we would simply drop these factors and the integration over the position x_0 of the instanton.

Besides the factor d^4x_0, the bosonic measure also includes a factor $d\rho\,\rho^{2n+5}$ and a factor $d\mu$, which represents the invariant Haar measure on $SU(2)$. We have determined the power of ρ that appears simply by dimensional analysis.

Thus, since a prefactor of $\overline{v}^4$ appears from the fermion zero modes, the Gaussian integral over ρ then produces the correct dependence on v and $\overline{v}$ as in (2.4.5). We have not been careful about factors of the gauge coupling g^2 which also appear in the integration measure and upon performing the Gaussian integral. By holomorphy, any explicit dependence of the correlator on g^2 should be absorbed into a wavefunction renormalization of the external legs.

The only integral left to consider is the group integral over $SU(2)$, which takes the form

$$I^{d_1\,d_2\cdots\,d_{2p}}_{c_1\,c_2\cdots\,c_{2p}} = \int d\mu\, h^{d_1}_{c_1}\, h^{d_2}_{c_2} \cdots h^{d_{2p}}_{c_{2p}}\,. \tag{2.4.15}$$

This integral is manifestly non-zero. The $SU(2)\times SU(2)$ symmetry implies that

$$I^{d_1\,d_2\cdots\,d_{2p}}_{c_1\,c_2\cdots\,c_{2p}} \propto \epsilon^{d_1 d_2}\,\epsilon_{c_1 c_2}\cdots\epsilon^{d_{2p-1}d_{2p}}\,\epsilon_{c_{2p-1}c_{2p}} + (permutations)\,. \tag{2.4.16}$$

Here the first term on the right hand side must be symmetrized under the exchanges of indices corresponding to exchanges between the factors of h in (2.4.15). These symmetries arise in the effective interaction (2.4.5) from the permutation symmetries of the fermions. Thus, upon substituting (2.4.16) into (2.4.14), we produce the effective interaction which arises from the multi-fermion F-term.

2.4.2 A Computation in the Seiberg Dual With Six Doublets

In many examples of duality, non-perturbative effects in the direct theory become classical effects in the dual theory. In this section we show, at least in the $SU(2)$ theory with $2n = 6$ doublets, how the multi-fermion F-term which we have now computed non-perturbatively in the direct description of SQCD can also be computed classically, at tree-level, in the Seiberg dual [10,25] description.

As promised in Section 2, we also revisit here the deformation of complex structure that occurs in the theory with four doublets. In particular, we reproduce the effective interaction in (2.2.28) by integrating out the massive fields in the linear sigma model with superpotential $W = \Sigma(M \wedge M - \Lambda^4)$ which describes the deformation. Since this computation is exactly the same in spirit as our classical computation in the Seiberg dual of the theory with six doublets, we describe both computations together.

The Seiberg dual of $SU(2)$ SQCD with six doublets is distinguished by the fact that the dual gauge group is trivial, and hence this theory is especially simple. In particular, the elementary degrees of freedom in the dual theory are described entirely by the mesonic fields M^{ij}, with Wess-Zumino action

$$S = \frac{1}{\mu^2} \int d^4x\, d^4\theta\, \overline{M} M + \int d^4x\, d^2\theta\, \Lambda^{-3}\, M \wedge M \wedge M + c.c. \tag{2.4.17}$$

We have included the canonical kinetic terms in S, with an arbitrary scale μ that appears so that, by convention, M has dimension two. Using a different kinetic term for M would not affect the computation of F-terms.

The cubic superpotential plays an interesting role in this theory. As shown by Seiberg [10], this potential appears nonperturbatively in the electric theory, but in the dual theory it arises at tree level. In either case, the F-term equations which follow from this superpotential are simply the classical Plücker relations $M \wedge M = 0$ that enforce the condition $\mathrm{rank}(M) \leq 2$, which is necessary to describe $\mathcal{M}$.

In the special case $n = 3$, the multi-fermion F-term in (2.3.13) takes the explicit form

$$\begin{aligned} \delta S = \frac{1}{\mu^4} \int d^4x\, d^2\theta\, \Lambda^3\, (\overline{M} M)^{-3}\, \epsilon^{i_1 j_1 i_2 j_2 i_3 j_3}\, \overline{M}_{i_1 j_1} \times \\ \times \left(M^{kl}\, \overline{D}\,\overline{M}_{i_2 k} \cdot \overline{D}\,\overline{M}_{l j_2} \right) \left(M^{k' l'}\, \overline{D}\,\overline{M}_{i_3 k'} \cdot \overline{D}\,\overline{M}_{l' j_3} \right) . \end{aligned} \tag{2.4.18}$$

We will generate this effective interaction in the most naive way possible. We simply observe that, when we expand the Wess-Zumino model around a generic point on $\mathcal{M}$, the cubic superpotential induces a mass for some components of M. We then integrate out these massive modes at tree level in a Feynman diagram computation to generate (2.4.18).

At this point, one might immediately protest that we are making the quixotic proposal to generate an F-term in perturbation theory and in blatant violation of standard non-renormalization theorems. However, these non-renormalization theorems have only been considered for conventional F-terms which describe superpotentials, and the multi-fermion F-terms we consider evade them in an interesting way.

The essential point here is that the multi-fermion F-terms arise from cohomology classes on $\mathcal{M}$. Whenever we perform a perturbative computation around some vacuum on $\mathcal{M}$, we are only working in a small neighborhood of that point, and in that neighborhood any operator $\mathcal{O}_\omega$ which represents a positive degree cohomology class of $\overline{Q}_{\dot\alpha}$ becomes $\overline{Q}_{\dot\alpha}$-trivial. As a result, though globally on $\mathcal{M}$ the multi-fermion F-terms cannot be written as D-terms, they can be written as D-terms if we expand in fluctuations around a given vacuum. These D-terms can then be directly generated in perturbation theory.

As a simple and highly relevant example, we consider the F-term at hand in (2.4.18). We expand (2.4.18) around some point with $\langle M^{ij}\rangle \neq 0$. With no loss of generality, we can assume that the only nonzero component of $\langle M^{ij}\rangle$ is $\langle M^{12}\rangle$. In expanding around this particular vacuum, we apply our standard convention that c, d, e, f refer to indices $1, 2$, s, t, u, v refer to indices $3, \ldots, 6$, and i, j, k, l run over all indices $1, \ldots, 6$. From (2.4.18), we generate a series of interactions among the fluctuating fields δM, one interaction being

$$\begin{aligned} \delta S \;=\; & \frac{1}{\mu^4} \int d^4x \, d^2\theta \; \Lambda^3 \, \langle \overline{M} M\rangle^{-3} \, \langle \overline{M}_{12}\rangle \times \\ & \times \, \epsilon^{s_1 t_1 s_2 t_2} \left(\delta M^{cd} \, \overline{D}\, \delta\overline{M}_{s_1 c} \cdot \overline{D}\, \delta\overline{M}_{d t_1}\right) \left(\delta M^{ef} \, \overline{D}\, \delta\overline{M}_{s_2 e} \cdot \overline{D}\, \delta\overline{M}_{f t_2}\right) . \end{aligned} \tag{2.4.19}$$

Of course, the effective fermion interaction (2.4.5) which we considered in the instanton computation is one of the terms that arises from (2.4.19).

By definition, if $\delta\overline{M}_{ij}$ is massless, then the basic equation of motion (2.2.38) for $\delta\overline{M}_{ij}$ takes the form $\overline{D}^2\, \delta\overline{M}_{ij} \;=\; \mathcal{O}(\delta M^2)$. Since only massless fluctuations appear in the effective interaction (2.4.19), we can immediately integrate this F-term into a D-term at leading order,

$$\begin{aligned} \delta S \;=\; & \frac{1}{\mu^4} \int d^4x \, d^4\theta \; \Lambda^3 \, \langle \overline{M} M\rangle^{-3} \, \langle \overline{M}_{12}\rangle \times \\ & \times \, \epsilon^{s_1 t_1 s_2 t_2} \left(\delta M^{cd} \, \delta\overline{M}_{s_1 c} \, \delta\overline{M}_{d t_1}\right) \left(\delta M^{ef} \, \overline{D}\, \delta\overline{M}_{s_2 e} \cdot \overline{D}\, \delta\overline{M}_{f t_2}\right) . \end{aligned} \tag{2.4.20}$$

We have used the fact that to this order, two $\overline{D}$'s cannot act on the same $\delta\overline{M}$, and none can act on δM.

In the case of the theory with $n = 2$, the same observations imply that the analogous part of the F-term in (2.2.28) can be rewritten locally as the simple D-term below,

$$\delta S \;=\; \frac{1}{\mu^2}\int d^4x\, d^4\theta\; \Lambda^4\, \langle \overline{M} M\rangle^{-1}\, \epsilon^{cd}\epsilon^{st}\, \delta\overline{M}_{sc}\delta\overline{M}_{dt}\,. \tag{2.4.21}$$

Here again we expand around a vacuum in which the nonvanishing part of $\langle M\rangle$ is $\langle M^{12}\rangle$, and $c, d = 1, 2$ while $s, t = 3, 4$.

Thus, the appearance of these unusual F-terms is signaled by the perturbative appearance of the D-terms in (2.4.20) and (2.4.21), which we must now compute. As in the instanton computation, we could compute some particular component of this superspace interaction. However, we are in a situation perfectly suited for a manifestly supersymmetric computation using the formalism of super Feynman diagrams.

Evaluating a Super Feynman Diagram

We will not review here the basic derivation of Feynman rules in superspace, for which we recommend Section 6.3 of [35]. In general, superspace Feynman rules can be derived by standard path integral manipulations just as for ordinary Feynman rules, and for the sake of brevity we will only state the super Feynman rules that we need for our very simple, tree-level computations.

In the case of the theory with $2n = 6$, we begin by expanding the tree-level Wess-Zumino action in fluctuations δM about the vacuum, so that

$$S \;=\; \frac{1}{\mu^2}\int d^4x\, d^4\theta\; \delta\overline{M}\delta M \;+\; \int d^4x\, d^2\theta\, (3\,\lambda\,\langle M\rangle{\wedge}\delta M{\wedge}\delta M \;+\; \lambda\,\delta M{\wedge}\delta M{\wedge}\delta M) + c.c., \tag{2.4.22}$$

where for convenience we introduce the abbreviation

$$\lambda \;\equiv\; \Lambda^{-3}\,. \tag{2.4.23}$$

We will not be concerned with constants here, and we simply absorb the numerical factor of 3 in (2.4.22) into $\langle M\rangle$. We will also suppress the appearance of the mass scale μ in all expressions that follow, since its appearance is trivially fixed at the end of the computation by dimensional analysis.

Of course, we similarly expand the sigma model action in the theory with $n = 2$,

$$S \;=\; \frac{1}{\mu^2}\int d^4x\, d^4\theta\; \delta\overline{M}\delta M + \int d^4x\, d^2\theta\, (2\,\langle M\rangle\wedge\delta M\,\delta\Sigma \,+\, \delta M\wedge\delta M\,\delta\Sigma \,-\, \varepsilon\,\delta\Sigma) + c.c. + \cdots\,, \tag{2.4.24}$$

where the ellipses indicate kinetic terms and a mass term for the fluctuations of the auxiliary field Σ. As above, we ignore constants, and we abbreviate

$$\varepsilon \,\equiv\, \Lambda^4\,. \tag{2.4.25}$$

The most important terms in (2.4.24) for our computation are simply the linear source term for $\delta\Sigma$ which represents the deformation as well as the mass term mixing δM and $\delta\Sigma$.

Propagators

In the vacuum with only $\langle M^{12}\rangle \neq 0$, we want to get an effective interaction for the massless fields by integrating out the massive fields M^{st}, $s, t = 3, \ldots, 2n$.

These fields have standard superspace propagators, which may be either chiral or non-chiral. We indicate these propagators below, in the theory with $n = 3$,

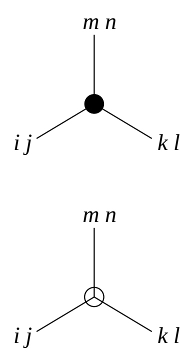
$$\begin{aligned}
\delta\overline{M}_{st} \;\text{———}\; \delta M^{uv} \quad &= \quad \delta^{uv}_{st}\,/\left(p^2 + \overline{\lambda}\lambda\langle\overline{M}M\rangle\right)\,,\\
\delta M^{st} \;\overset{D^2}{\text{- - - - - -}}\; \delta M^{uv} \quad &= \quad \overline{\lambda}\,\langle\overline{M}_{12}\rangle\,\epsilon^{stuv}\,D^2\,/\,p^2\left(p^2 + \overline{\lambda}\lambda\langle\overline{M}M\rangle\right)\,,\\
\delta\overline{M}_{st} \;\overset{\overline{D}^2}{\text{- - - - - -}}\; \delta\overline{M}_{uv} \quad &= \quad \lambda\,\langle M^{12}\rangle\,\epsilon_{stuv}\,\overline{D}^2\,/\,p^2\left(p^2 + \overline{\lambda}\lambda\langle\overline{M}M\rangle\right)\,.
\end{aligned} \tag{2.4.26}$$

In writing the non-chiral propagator, we use the standard notation $\delta^{uv}_{st} = \delta^u_s\delta^v_t - \delta^u_t\delta^v_s$. We have also suppressed a superspace delta function $\delta^4(\theta-\theta')$ which accompanies these propagators. Finally, we note the superspace derivatives D^2 and $\overline{D}^2$ which appear in the chiral and anti-chiral propagators. These factors arise ubiquitously in supergraph computations when chiral integrals over half of superspace are rewritten as non-chiral integrals over the full superspace.

In the theory with $n = 2$, similar propagators appear for the appropriate linear combinations of $\delta\Sigma$ and δM, for which the mass squared is again proportional to $\langle\overline{M}M\rangle$. (If a separate mass term $m\Sigma^2$ for Σ is also present, this statement remains true in the classical limit that $\langle \overline{M} M \rangle$ is large.)

Vertices

In the theory with $n = 3$, the cubic superpotential gives rise to cubic vertices for chiral and anti-chiral interactions, as we distinguish in Figure 2.1. We have written these interactions in an $SU(6)$ symmetric fashion, though of course each chiral and anti-chiral vertex decomposes under the unbroken $SU(2) \times SU(4)$ symmetry to give various interactions between the massive and massless components of M, which we leave implicit. Each superspace vertex comes with a factor of $\int d^4\theta$, and the delta functions from the propagators simply ensure that the overall diagram has precisely one factor of $\int d^4\theta$, as we expect.

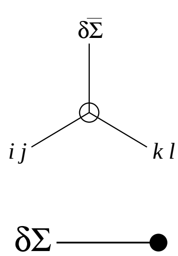


$$= \lambda\, \epsilon_{ijklmn}\,,$$

$$= \overline{\lambda}\, \epsilon^{ijklmn}\,.$$

Figure 2.1: *Vertices for* $n = 3$

In the corresponding theory with $n = 2$, we require a similar cubic vertex arising from the interaction $\delta\overline{M} \wedge \delta\overline{M}\, \delta\overline{\Sigma}$ as well as the chiral source term $\varepsilon\, \delta\Sigma$, as shown in Figure 2.2. Again, we leave the obvious decomposition under $SU(2) \times SU(2)$ implicit.



$$= \epsilon^{ijkl}\,,$$

$$= \varepsilon\,.$$

Figure 2.2: *Vertices for* $n = 2$

Last, we recall the rule that if a chiral vertex has N internal legs (external legs don't count), then $N-1$ of those legs appear with a factor of $\overline{D}^2$ attached. Briefly, if $J(x,\theta)$ is the chiral source introduced as usual to derive Feynman rules, then the functional derivative of J satisfies $\delta J(x,\theta)/\delta J(x',\theta') = \overline{D}^2\,\delta^4(x-x')\,\delta^4(\theta-\theta')$. So N factors of $\overline{D}^2$ appear from these derivatives, but one factor of $\overline{D}^2$ is used to write $\int d^2\theta\,\overline{D}^2 = \int d^4\theta$, as mentioned above.

With these rules in hand, we can immediately generate the interactions in (2.4.20) and (2.4.21). First, in the simpler case of $n=2$, we immediately evaluate the simple diagram in Figure 2.3 at zero momentum to produce the effective interaction

$$\int d^4x\, d^4\theta\; \epsilon^{cd}\epsilon^{st}\,\delta\overline{M}_{sc}\delta\overline{M}_{dt}\,\frac{\varepsilon}{\langle\overline{M}M\rangle}\,, \tag{2.4.27}$$

as in (2.4.21).

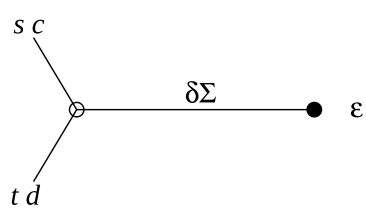


Figure 2.3: *Two-point super Feynman diagram*

For the theory with $n=3$, we consider the slightly more involved diagram in Figure 2.4. We note that the D^2 operator in this diagram arises from the central chiral propagator, and the two $\overline{D}^2$ operators arise from the two chiral vertices.

At first sight, one might worry about the spurious pole at zero momentum that appears to arise from the extra factor of p^2 appearing in the central chiral propagator, as in (2.4.26). Physically, since we only integrate out massive fields, we do not expect to find any pole at zero momentum.

However, we can integrate by parts to move one of the $\overline{D}^2$ operators onto the central chiral propagator to form $\overline{D}^2D^2$. Since $\overline{D}^2D^2 = p^2$ when acting on a chiral field, this factor of $\overline{D}^2D^2$ cancels against the extra factor of p^2 in the denominator of the chiral propagator.

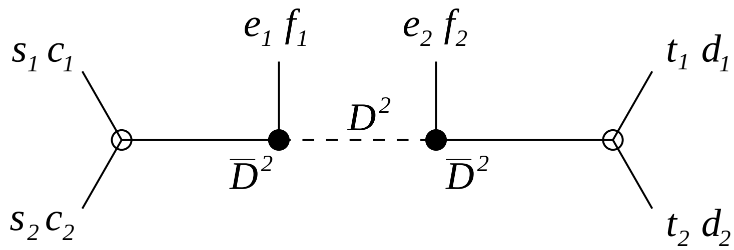


Figure 2.4: *Six-point super Feynman diagram*

Thus, the diagram is well defined in the limit of zero momentum, and we evaluate it in this limit to reproduce the D-term (2.4.20). We also note that once we cancel the factor of $\overline{D}^2 D^2$, we are left with only one factor of $\overline{D}^2$, which acts on the external anti-chiral legs just as in the interaction (2.4.20).

So at zero momentum, the remainder of our computation is a trivial matter of algebra. We find that this diagram produces the effective interaction

$$\int d^4x\, d^4\theta\; \delta\overline{M}_{s_1c_1}\, \delta\overline{M}_{s_2c_2} \left(\overline{D}\delta\overline{M}_{t_1d_1} \cdot \overline{D}\delta\overline{M}_{t_2d_2}\right) \delta M^{e_1f_1}\, \delta M^{e_2f_2} \times$$
$$\times \frac{\overline{\lambda}\, \epsilon^{c_1c_2} \epsilon^{s_1s_2uv}}{\overline{\lambda}\lambda\, \langle \overline{M}M \rangle} \cdot \lambda\, \epsilon_{uvu'v'} \epsilon_{e_1f_1} \cdot \frac{\overline{\lambda}\, \epsilon^{u'v'wx} \langle \overline{M}_{12} \rangle}{\overline{\lambda}\lambda\, \langle \overline{M}M \rangle} \cdot \lambda\, \epsilon_{wxw'x'} \epsilon_{e_2f_2} \cdot \frac{\overline{\lambda}\, \epsilon^{w'x't_1t_2} \epsilon^{d_1d_2}}{\overline{\lambda}\lambda\, \langle \overline{M}M \rangle}\,. \tag{2.4.28}$$

The tensor on the second line of (2.4.28) is then proportional to

$$\lambda^{-1}\, \langle \overline{M}M \rangle^{-3}\, \langle \overline{M}_{12} \rangle\; \delta^{c_1c_2}_{e_1f_1}\; \delta^{d_1d_2}_{e_2f_2}\; \epsilon^{s_1s_2t_1t_2}\,, \tag{2.4.29}$$

which has precisely the form required to produce the F-term. The $\overline{\lambda}$'s have happily canceled, ensuring the requisite holomorphy.

N. Seiberg has pointed out the following interpretation of the $1/\lambda$ factor. As the meson superfield M has dimension two in the classical theory, the dependence on Λ of the superpotential interaction in the Wess-Zumino model (2.4.17) is fixed by dimensional analysis to be $\int d^4x\, d^2\theta\, \Lambda^{-3}\, M \wedge M \wedge M$. Thus, the abstract coupling λ is a multiple of Λ^{-3} in SQCD, as in (2.4.23), and the multi-fermion F-term interaction, being proportional to λ^{-1} in the Seiberg dual description, is proportional to Λ^3 in the original SQCD description. But

Λ^3 is the standard instanton factor for $SU(2)$ with six doublets, and the direct instanton computation of Section 4.1 did, accordingly, give a result proportional to Λ^3.

2.4.3 Mass Deformation And Renormalization Group Flow

For our final computation, we perturb $SU(2)$ SQCD with $2n$ massless doublets by adding a tree-level superpotential which gives a mass to some of the n flavors,

$$W \,=\, m_{ij}\, M^{ij}\,. \tag{2.4.30}$$

As usual, we assign charges to the mass parameters m_{ij} under the symmetries of the massless theory so that W is formally invariant,

$$\begin{array}{ccccc} & SU(2) & SU(2n) & U(1)_A & U(1)_{\mathcal{R}} \\ m_{ij} & \mathbf{1} & \wedge^2(\overline{\mathbf{2n}}) & -2 & -2\left(1-\frac{2}{n}\right)\,. \end{array} \tag{2.4.31}$$

The whole computation will be performed on B.

As we observed in general in Section 2.3, the tree-level superpotential alters the on-shell supersymmetry algebra of the theory. Consequently, the operator $\mathcal{O}_\omega \equiv \mathcal{O}_\omega^{(n)}$ in (2.3.13) which is chiral in SQCD with $2n$ massless doublets is no longer chiral when some of those doublets become massive.

Physically, we expect that there is instead some deformation $\widetilde{\mathcal{O}}_\omega$ of this operator, depending holomorphically on m_{ij}, which is chiral in the massive theory and which reduces to $\mathcal{O}_\omega^{(n)}$ upon setting m_{ij} to zero.

On the other hand, if we give very large masses to k of the flavors and integrate them out, we also expect that $\widetilde{\mathcal{O}}_\omega$ must reduce to the operator $\mathcal{O}_\omega^{(n-k)}$ appropriate for the massless theory with $n-k$ flavors. In particular, upon integrating out all but one flavor, $\widetilde{\mathcal{O}}_\omega$ should reproduce the well known nonperturbative superpotential,

$$W = \frac{\Lambda^5}{M}\,. \tag{2.4.32}$$

Here $M = M^{12}$ is the only independent component of the 2×2 antisymmetric matrix M^{ij}.

We now compute $\widetilde{\mathcal{O}}_\omega$, which will be uniquely determined from $\mathcal{O}_\omega^{(n)}$ by supersymmetry and will have the properties above. Since we know already from the work of [7] that the superpotential (2.4.32) is generated, we will thus show that the F-term involving $\mathcal{O}_\omega^{(n)}$ is generated in the massless theory with n flavors. Finally, we remark that this sort of analysis extends, at least in spirit, directly to the general case of $SU(N_c)$ SQCD with $N_f > N_c$ flavors and might be successfully applied there.

As before, we use $\mathcal{M}$ to denote the moduli space of the massless theory, and we recall that $\mathcal{M}$ is a complex cone over the Grassmannian $B = SU(2n)/S(U(2)\times U(2n-2))$. Then our problem of constructing $\widetilde{\mathcal{O}}_\omega$ is equivalent to the geometric problem of finding a tensor $\widetilde{\omega}$, which is generally an inhomogeneous sum of sections of $\overline{\Omega}^p_{\mathcal{M}} \otimes \wedge^p T\mathcal{M}$ for various p, such that $\widetilde{\omega}$ satisfies the supersymmetry condition,

$$\left(\overline{\partial} + \iota_{dW}\right)\widetilde{\omega} \,=\, 0\,, \tag{2.4.33}$$

and in the massless limit reduces to our former tensor ω.

Preliminaries

As in Section 3, the important analysis of $\widetilde{\omega}$ is the local analysis on B near the point corresponding to $M^{ij} = \widehat{\epsilon}^{ij}$. However, we first find it useful to revisit our construction of the simpler tensor ω in greater detail and in a manner which immediately generalizes to the construction of $\widetilde{\omega}$.

Let us recall our construction of ω in Section 3. We begin by picking a point P on B, for concreteness corresponding to the point $M^{ij} = \widehat{\epsilon}^{ij}$ on $\mathcal{M}$. By an overall scaling of the M^{ij}, we can set $M^{12} = 1$, and then we take complex coordinates on B (in a neighborhood of the point P) to be simply the off-diagonal matrix elements $\phi^s_c = \epsilon_{cd} M^{sd}$, $c = 1, 2$, $s = 3, \ldots, 2n$. These complex coordinates are the usual complex coordinates on the Grassmannian, and they transform as in (2.3.5) under the action of the unbroken $S(U(2)\times U(2n-2))$ symmetry group at P. The matrix elements M^{ij}, $i, j > 2$, are determined implicitly in terms of the ϕ^s_c by the equation $M \wedge M = 0$. We will not need the explicit form of these matrix elements; because we work locally at P, the only important fact is that they are of order $(\phi^s_c)^2$.

In (2.3.13), we determined the form of the multi-fermion F-term:

$$\mathcal{O}_\omega \,=\, \Lambda^{6-n}\left(\overline{M}M\right)^{-n}\epsilon^{i_1 j_1\cdots i_n j_n}\,\overline{M}_{i_1 j_1}\,\mathcal{O}_{i_2 j_2}\cdots\mathcal{O}_{i_n j_n}\,, \tag{2.4.34}$$

where

$$\mathcal{O}_{ij} \equiv M^{kl}\,\overline{D}\,\overline{M}_{ik}\cdot\overline{D}\,\overline{M}_{lj}\,, \qquad \overline{M}M = \frac{1}{2}\overline{M}_{ij}M^{ij}\,. \tag{2.4.35}$$

In that discussion, we used an argument based on symmetries to prove that $\overline{\partial}\omega = 0$. As a prelude to including the superpotential deformation, we will here demonstrate this more explicitly.

Since ω is invariant under the action of $SU(2n)$ on the homogeneous space B, it suffices to show that $\overline{\partial}\omega = 0$ at the point P. Furthermore, to evaluate this derivative at P we need only describe ω up to terms of order ϕ^2. Once we recognize this fact, we can immediately see why ω is annihilated by $\overline{\partial}$. Thus, we note that $\overline{M}M = 1 + \mathcal{O}(\phi^2)$, so working up to terms of order ϕ^2 allows us to set $\overline{M}M = 1$ in (2.4.34). Furthermore, examining (2.4.34) and (2.4.35), we see that up to terms of order ϕ^2, we can replace the explicit factor of $\overline{M}_{i_1j_1}$ in (2.4.34) by $\widehat{\epsilon}_{i_1j_1}$, so that all factors of $\mathcal{O}_{i_kj_k}$ have $i_k, j_k > 2$. (We also observed this fact at the end of Section 3.2.) Further, for $i_k, j_k > 2$, we can take $\mathcal{O}_{i_kj_k} = \epsilon^{cd}\overline{D}\,\overline{M}_{i_kc}\cdot\overline{D}\,\overline{M}_{dj_k}$, again up to terms of order ϕ^2. So, up to terms vanishing to second order at P, ω takes the particularly simple form

$$\omega = \epsilon^{s_1t_1s_2t_2\ldots s_nt_n}\left(\epsilon_{c_1d_1}d\overline{\phi}^{c_1}_{s_1}\frac{\partial}{\partial\phi^{t_1}_{d_1}}\right)\cdots\left(\epsilon_{c_nd_n}d\overline{\phi}^{c_n}_{s_n}\frac{\partial}{\partial\phi^{t_n}_{d_n}}\right). \tag{2.4.36}$$

Now the fact that $\overline{\partial}\omega = 0$ at P is manifest: all terms in ω have constant coefficients and are trivially annihilated by $\overline{\partial}$.

The benefit of this approach is that we can now conveniently understand the generalization with the superpotential turned on. We claim that the generalization of $\mathcal{O}_\omega$ is simply

$$\begin{aligned}\widetilde{\mathcal{O}}_\omega &= \Lambda^{6-n}\left(\overline{M}M\right)^{-n}\epsilon^{i_1j_1\cdots i_nj_n}\,\overline{M}_{i_1j_1}\,\widetilde{\mathcal{O}}_{i_2j_2}\cdots\widetilde{\mathcal{O}}_{i_nj_n}\,,\\ \widetilde{\mathcal{O}}_{ij} &\equiv M^{kl}\,\overline{D}\,\overline{M}_{ik}\cdot\overline{D}\,\overline{M}_{lj} - \left(\overline{M}M\right)m_{ij}\,.\end{aligned} \tag{2.4.37}$$

This certainly reduces to $\mathcal{O}_\omega$ at $m = 0$; we just have to prove that it is chiral. In other words, we need to show that the object $\widetilde{\omega}$, obtained from ω by replacing each $\mathcal{O}_{ij}$ by $\widetilde{\mathcal{O}}_{ij}$, is annihilated by $\overline{\partial}+\iota_{dW}$. It suffices to do the computation at the point $P \in B$ with $M^{ij} = \widehat{\epsilon}^{ij}$ since, as we will make no particular assumption about the form of the mass matrix m_{ij}, the computation would proceed in the same way at any other point.

So as before, we want to write out a simple formula for $\widetilde{\omega}$ that is valid near P to order ϕ^2. To this order, the explicit factors of $\overline{M}_{i_1 j_1}$ in $\widetilde{\mathcal{O}}_\omega$ and of M^{kl} in $\widetilde{\mathcal{O}}_{ij}$ can be replaced by $\widehat{\epsilon}_{i_1 j_1}$ and $\widehat{\epsilon}^{kl}$. Since in $\widetilde{\mathcal{O}}_{ij}$, the indices i, j are then in the range $3, \ldots, 2n$ the mass matrix m_{ij} can be replaced by μ_{ij}, its orthogonal projection onto the part with $i, j > 2$. We write Π for the projector onto components of m_{ij} with $i, j > 2$ and will describe Π more explicitly momentarily.

The net result is that up to terms of order ϕ^2, $\widetilde{\omega}$ is described near P by a simple generalization of (2.4.36),

$$\begin{aligned} \widetilde{\omega} \,&=\, \epsilon^{s_1 t_1 \cdots s_p t_p}\, \widetilde{\omega}_{s_1 t_1} \cdots \widetilde{\omega}_{s_p t_p}\,, \quad p = n-1\,, \\ \widetilde{\omega}_{st} &= \epsilon_{cd} \left(d\overline{\phi}^c_s \frac{\partial}{\partial \phi^t_d} + \frac{\partial}{\partial \phi^s_c} d\overline{\phi}^d_t \right) - \mu_{st}\,. \end{aligned} \tag{2.4.38}$$

The virtue of factorizing $\widetilde{\omega}$ in this way is as we will see each factor $\widetilde{\omega}_{s_i t_i}$ is separately annihilated by $\overline{\partial} + \iota_{dW}$. Also, in the expression for $\widetilde{\omega}_{st}$ in the second line of (2.4.38), we have explicitly indicated the two terms that arise from the contraction of spinor indices on $\overline{D}_{\dot{\alpha}}$ in (2.4.37), since we will try to be careful about factors of two in the following.

Let us first evaluate $\iota_{dW}(\widetilde{\omega}_{st})$. The contraction operator ι_{dW} trivially annihilates μ_{st} (because the latter is a zero-form). As $W = m_{ij} M^{ij}$, we have $dW = m_{ij} dM^{ij}$. So the effect of contraction with dW is just to map $\partial / \partial \phi^s_c$ to μ^c_s, the projection of the mass matrix m to terms that transform like $\partial / \partial \phi^s_c$ (in other words, as $(\mathbf{2}, \overline{\mathbf{2n-2}})$) under the subgroup of the symmetry group that leaves fixed the point $P \in B$. Hence we have

$$\iota_{dW} \widetilde{\omega}_{st} \,=\, \epsilon_{cd} \left(d\overline{\phi}^c_s\, \mu^d_t \,+\, \mu^c_s\, d\overline{\phi}^d_t \right)\,. \tag{2.4.39}$$

It remains to evaluate $\overline{\partial}(-\mu_{st})$. This is nonzero because of the projection in the definition of μ_{st}. As we will show,

$$\overline{\partial} \mu_{st} \,=\, \epsilon_{cd} \left(d\overline{\phi}^c_s\, \mu^d_t \,+\, \mu^c_s\, d\overline{\phi}^d_t \right)\,. \tag{2.4.40}$$

From (2.4.39) and (2.4.40), we then see directly that $\overline{\partial} + \iota_{dW}$ annihilates $\widetilde{\omega}_{st}$ and hence $\widetilde{\omega}$ at the point P on B.

To derive the formula (2.4.40) for $\overline{\partial} \mu_{st}$, we begin by considering the projection Π of the mass matrix m onto its components which transform in the representation $\wedge^2 \left(\overline{\mathbf{2n-2}}\right)$. We can directly write a global formula for this projection,

$$\begin{aligned}\Pi(m)_{ij} \;=\; m_{ij} \,+\, \left(\overline{M}M\right)^{-1}\left(m_{ik}M^{kl}\overline{M}_{lj} - m_{jk}M^{kl}\overline{M}_{li}\right) \,+\\ +\, \left(\overline{M}M\right)^{-2}\left(\overline{M}_{ik}M^{kl}m_{lp}M^{pq}\overline{M}_{qj}\right)\,.\end{aligned} \tag{2.4.41}$$

Upon substituting $M^{ij} = \widehat{\epsilon}^{ij}$ and using repeatedly that $\widehat{\epsilon}^{kl}\widehat{\epsilon}_{lj} = -\widehat{\delta}^k_j$ (explaining the signs above), one can check that the second and third terms of (2.4.41) subtract the components of m transforming in the representations $\mathbf{1}$ and $(\mathbf{2}, \overline{\mathbf{2n-2}})$ under $SU(2)\times SU(2n-2)$ at P, leaving only the components in $\wedge^2\left(\overline{\mathbf{2n-2}}\right)$. Since the formula (2.4.41) for Π is invariant, it is correct globally on B.

Because the action of $\overline{\partial}$ commutes with pullback, we can now act with $\overline{\partial}$ directly on (2.4.41) as an unconstrained expression in the ambient vector space (or projective space) parametrized by M^{ij}. We then pull this expression back to $\mathcal{M}$ by dropping all terms which involve the one-forms $d\overline{M}_{ij}$ with both indices $i, j > 2$.

To evaluate $\overline{\partial}\mu$ at $\phi = 0$, we can discard all terms proportional to ϕ, and in particular to components M^{ij} or $\overline{M}_{ij}$ with i or j bigger than 2. Terms that survive at $\phi = 0$ only arise from the action of $\overline{\partial}$ on the second term of (2.4.41), with the expression

$$\left(\overline{M}M\right)^{-1}\left(m_{ik}M^{kl}d\overline{M}_{lj} - m_{jk}M^{kl}d\overline{M}_{li}\right)\,, \quad i, j > 2\,. \tag{2.4.42}$$

From this global expression (2.4.42) we immediately deduce the local formula (2.4.40) upon setting $M^{ij} = \widehat{\epsilon}^{ij}$ and identifying $m_{ik}M^{kl}d\overline{M}_{lj}$ as representing locally $\epsilon_{cd}\,\mu^c_s\,d\overline{\phi}^d_t$ at P. We remark that the relative sign between the two terms in (2.4.40) and (2.4.42) arises from a rearrangement of flavor indices in passing from (2.4.42) to (2.4.40).

Finally, although we have thus far only considered the special case that $W = m_{ij}M^{ij}$, if we now consider the case of a general superpotential deformation of SQCD, then our construction of $\widetilde{\omega}$ immediately generalizes upon substituting everywhere $\partial W/\partial M^{ij}$ for m_{ij}. The only important property of m which we used was the fact that it is annihilated by $\overline{\partial}$, which is always true for dW.

Renormalization Group Flow

To conclude, we consider how $\widetilde{\mathcal{O}}_\omega$ in (2.4.37) behaves under renormalization group flow.

If we expand $\widetilde{\mathcal{O}}_\omega$ as a polynomial in m, then the term of degree k in m is given by

$$\begin{aligned}\mathcal{O}_\omega^{(n-k)} &= (-1)^k \binom{n-1}{k} \Lambda^{6-n} \left(\overline{M}M\right)^{-(n-k)} \epsilon^{i_1 j_1 \cdots i_n j_n} \times \\ &\quad \times \overline{M}_{i_1 j_1}\, m_{i_2 j_2} \cdots m_{i_{k+1} j_{k+1}}\, \mathcal{O}_{i_{k+2} j_{k+2}} \cdots \mathcal{O}_{i_n j_n}\,, \\ \mathcal{O}_{ij} &\equiv M^{kl}\, \overline{D}\,\overline{M}_{ik} \cdot \overline{D}\,\overline{M}_{lj}\,.\end{aligned} \tag{2.4.43}$$

This operator $\mathcal{O}_\omega^{(n-k)}$ has the same form as the operator in (2.3.13) which appears in the theory with $n-k$ massless flavors.

We consider the limit in which k flavors have masses $m \gg \Lambda$. To integrate out these flavors, we restrict to the sublocus of $\mathcal{M}$ describing supersymmetric vacua in the massive theory, so that $m_{ik}M^{kj} = 0$ for all i, j (as follows from the F-term equations), and we simply omit from the operator $\widetilde{\mathcal{O}}_\omega$ any terms which involve the heavy quarks. The operator to which $\widetilde{\mathcal{O}}_\omega$ flows in the infrared is thus $\mathcal{O}_\omega^{(n-k)}$ in (2.4.43).

In particular, we can consider flowing to the theory with only one flavor. The operator to which $\widetilde{\mathcal{O}}_\omega$ flows is then given by

$$\mathcal{O}_\omega^{(1)} = (-1)^{(n-1)}\, \Lambda^{6-n} \left(\overline{M}M\right)^{-1} \epsilon^{i_1 j_1 \cdots i_n j_n}\, \overline{M}_{i_1 j_1}\, m_{i_2 j_2} \cdots m_{i_n j_n}\,. \tag{2.4.44}$$

As we see, $\mathcal{O}_\omega^{(1)}$ involves no fermions at all and represents a function on $\mathcal{M}$. Of course, this function is not holomorphic on all of $\mathcal{M}$.

However, if we restrict $\mathcal{O}_\omega^{(1)}$ to the sublocus of $\mathcal{M}$ describing supersymmetric vacua, then $\mathcal{O}_\omega^{(1)}$ is holomorphic. Indeed, this locus can be described by a single massless meson M, so the matrix structure disappears and $\overline{M}$ cancels out. On this locus, $\mathcal{O}_\omega^{(1)}$ can be written in terms of M as

$$\mathcal{O}_\omega^{(1)} = (-1)^{(n-1)}\, \Lambda^{6-n}\, \epsilon^{i_1 j_1 \cdots i_{n-1} j_{n-1}}\, m_{i_1 j_1} \cdots m_{i_{n-1} j_{n-1}}\, \frac{2}{M}\,. \tag{2.4.45}$$

In this expression, the Pfaffian of the rank $2(n-1)$ minor of m appears, and an extra factor of two arises from the contraction of indices of $\overline{M}_{i_1 j_1}$. So, once ultraviolet and infrared scales are matched, $\mathcal{O}_\omega^{(1)}$ reproduces the nonperturbative superpotential in (2.4.32).

Chapter 3

Residues and Worldsheet Instantons

3.1 Introduction

String theory backgrounds which preserve only $\mathcal{N} = 1$ supersymmetry in four dimensions are of great interest both from a theoretical and a phenomenological perspective. A textbook way to obtain such a background is to compactify either the $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$ heterotic string on a Calabi-Yau threefold X with a stable, holomorphic gauge bundle E. One might suppose that these compactifications, which admit a completely perturbative string description, would be a natural starting point from which to study the moduli space of $\mathcal{N} = 1$ backgrounds of string theory.

However, in fact we know very little about which pairs (X, E) give rise to consistent heterotic backgrounds, even in string perturbation theory. The issue, of course, is that models described by generic X and E, even though they may satisfy the classical equations of motion to all orders in α', are destabilized non-perturbatively by world-sheet instantons [12]. These instantons, arising from world-sheets which wrap rational (i.e. holomorphic, genus zero) curves in X, can each contribute to a background superpotential W which lifts the Kähler moduli of X and generates a cosmological constant. So one might think that the only stable $\mathcal{N} = 1$ heterotic compactifications would arise from very special choices of X and E — for instance corresponding to world-sheet theories with $(2,2)$ supersymmetry or the $(0,2)$ models studied by Distler and Greene [36,37] — for which *each* world-sheet instanton simply cannot contribute to W.

In this light, the result of [13] that there are no non-perturbative contributions to W that destabilize compactifications described by $(0,2)$ linear sigma models [38,39] is somewhat surprising. This result does not rely upon any consideration of world-sheet instantons and instead follows from simple facts about the linear sigma model. One simply observes that W must always be a holomorphic section of a complex line-bundle of strictly negative curvature over the moduli space of the low-energy effective theory, which is naturally a compact Kähler manifold in the case of a linear sigma model. The compactness of the moduli space implies that W must have a pole somewhere on the moduli space or else vanish identically. However, the linear sigma model, being a two-dimensional, super-renormalizable gauge theory, can only become singular when the target space becomes non-compact, as some bosonic field develops a dangerous, unsuppressed zero-mode. In computing the linear sigma model correlators which describe the couplings of gauge-singlet fields in the effective theory and so probe for a background W, one finds that, after suitably twisting the model, no boson has a dangerous zero-mode. So W has no poles on the moduli space and thus vanishes.

Now, Calabi-Yau compactifications which are described by $(0,2)$ linear sigma models are certainly not generic — but nor are they so special that each world-sheet instanton simply does not contribute to W. So from the world-sheet perspective, the stability of $(0,2)$ linear sigma models implies in these compactifications a remarkable cancellation among the contributions to W from world-sheet instantons wrapped on rational curves in each homology class of X.

For instance, the analysis of [13] was applied in most detail to the simple case that X is a quintic hypersurface in $\mathbb{CP}^4$ and E is a deformation of the holomorphic tangent bundle TX, corresponding to a deformation off the locus of $(2,2)$ supersymmetric world-sheet theories.

In this case, the linear sigma model result implies that, contrary to one's naive expectation, the world-sheet instanton contributions to W from the 2875 lines on the generic quintic sum to zero.

Our main goal in this chapter of the thesis is to understand, from the world-sheet perspective, the source of this remarkable cancellation among instantons. In the process, we will introduce a new $(0,2)$ "half-linear" sigma model and show that heterotic compactifications described by these models form another class of stable $\mathcal{N}=1$ string backgrounds. For instance, using the half-linear model we show that heterotic compactifications on the quintic hypersurface in $\mathbb{CP}^4$ for which the gauge bundle pulls back from a bundle on $\mathbb{CP}^4$ are stable.

More generally, just as for the linear models, the half-linear models can be used to describe compactification on any Calabi-Yau threefold X which is a complete-intersection in a compact toric variety Y. However, in the half-linear models the bundle E on X is now any stable, holomorphic bundle which pulls back from a bundle on Y. In particular, E need not be a "monad" bundle on X, the sort most naturally described in the linear sigma model. (Technically, a monad bundle is one which admits a description as the cohomology of a complex $A \to B \to C$ of three bundles A, B, and C on X.) Conversely, however, there are also monad bundles on X (including obvious ones such as its tangent bundle) that do not pull back (at least in any obvious way) from a holomorphic bundle on Y. So we will also develop a version of the vanishing argument adapted to linear models and monad bundles on X.

3.1.1 A Brief Sketch of the Idea

Our essential idea can be motivated by considering the actual form of the instanton contributions to W in the simple case that the string world-sheet wraps once about an isolated rational curve C embedded in X. Actually, the most direct and elegant way [40,41] in this case to derive the instanton contribution to W is to evaluate the partition function of the worldvolume theory on a single $D1$-brane wrapped on C in the Type I theory, which is the dual description [42] of a world-sheet instanton in the $Spin(32)/\mathbb{Z}_2$ heterotic theory. (As explained in [41], the derivation of W from the Type I theory most directly applies to the $Spin(32)/\mathbb{Z}_2$ heterotic theory, but holomorphy and gauge-invariance allow us to interpret the answer for the $E_8 \times E_8$ heterotic theory as well.) Holomorphy allows us to evaluate this partition function at one-loop, so the instanton contribution to W from C is just

$$W(C) = \exp\left(-\frac{A(C)}{2\pi\alpha'} + i\int_C B\right) \frac{\mathrm{Pfaff}'(\mathcal{D}_F)}{\sqrt{\det'(\mathcal{D}_B)}}\,. \tag{3.1.1}$$

Here the exponential factor in $W(C)$ represents the classical action of the $D1$-brane. We have written this action in heterotic units, so that $A(C)$ is the area of C in the heterotic string metric on X, α' is the heterotic string tension, and B is the heterotic B-field.

The other factor in $W(C)$ arises from the one-loop integral over the fluctuations of the bosons and fermions living on the worldvolume of the $D1$-brane. $\mathcal{D}_B$ and $\mathcal{D}_F$ are thus the respective kinetic operators of the worldvolume bosons and fermions, and the "prime" in $\det'(\mathcal{D}_B)$ and $\mathrm{Pfaff}'(\mathcal{D}_F)$ indicates that these expressions are to be evaluated only after omitting the zero-modes associated to the bulk symmetries which are broken by the $D1$-brane. Four bosonic zero-modes associated to the broken translational symmetries in $\mathbb{R}^4$ and two right-moving fermionic zero-modes associated to the broken supersymmetries arise in this fashion.

The complex structure moduli of X and E are described by chiral superfields in the low-energy, effective $\mathcal{N}=1$ theory, and $W(C)$ must depend holomorphically on these fields. Unfortunately, our simple expression (3.1.1) for $W(C)$ is not manifestly holomorphic. To get a manifestly holomorphic expression for $W(C)$, we must use the fact that the two supersymmetries left unbroken by the $D1$-brane imply a cancellation between the contributions of the right-moving fermionic modes to $\mathrm{Pfaff}'(\mathcal{D}_F)$ and the contributions of the right-moving bosonic modes to $\det'(\mathcal{D}_B)$.

To make this cancellation explicit, we write $W(C)$ solely in terms of the left-moving bosonic and fermionic modes. By convention, the kinetic operator of a left-moving fermion on C will be a $\overline{\partial}$ operator, while the kinetic operator for a right-moving fermion will be a ∂ operator. Thus, since the left-moving worldvolume fermions transform as sections of the left-moving spin bundle $S_- = \mathcal{O}(-1)$ on C tensored with the gauge bundle E as restricted to C, their contribution to $\mathrm{Pfaff}'(\mathcal{D}_F)$ is just the Pfaffian of the $\overline{\partial}$ operator coupled to $E \otimes \mathcal{O}(-1) \equiv E(-1)$, which we denote $\overline{\partial}_{E(-1)}$. Here $\mathcal{O}(n)$ is the usual notation for the complex line-bundle of degree n on projective space. In particular, $\mathcal{O}$ is the trivial complex bundle of rank one.

Similarly, in the formula (3.1.1) for $W(C)$, we have written the boson kinetic operator $\mathcal{D}_B$ as a real operator acting on the eight real bosons representing the normal directions to C in $\mathbb{R}^4 \times X$. Since C, X, and $\mathbb{R}^4$ all have complex structures, we can equally well group the eight real bosons into four complex bosons taking values in the complex normal bundle N to C in $\mathbb{R}^4 \times X$. When C is isolated in X, N is isomorphic to $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, the first two summands representing the normal directions in $\mathbb{R}^4$ and the last two summands representing the normal directions in X. Thus, the contribution of the non-zero left-moving bosonic modes to $\det'(\mathcal{D}_B)$ just arises from the $\overline{\partial}$ operator on C coupled to the normal bundle N.

So, cancelling out the right-moving modes from $W(C)$ in (3.1.1), we have

$$W(C) = \exp\left(-\frac{A(C)}{2\pi\alpha'} + i\int_C B\right) \frac{\mathrm{Pfaff}\left(\overline{\partial}_{E(-1)}(C)\right)}{\left(\det'\overline{\partial}_{\mathcal{O}}\right)^2 \left(\det\overline{\partial}_{\mathcal{O}(-1)}(C)\right)^2}\,. \tag{3.1.2}$$

This expression for $W(C)$ is now manifestly holomorphic. Specifying a $\overline{\partial}$ operator on either X or E is equivalent to specifying its complex structure, so the operators $\overline{\partial}_{E(-1)}(C)$ and $\overline{\partial}_{\mathcal{O}(-1)}(C)$ themselves depend holomorphically on the complex structure moduli of X and E. We have also emphasized in (3.1.2) that the way in which the complex structure moduli of X and E appear in these $\overline{\partial}$ operators itself depends upon which curve C in X that the instanton wraps. In fact, at least when X is elliptically fibered, one can derive very explicit expressions in given examples for the dependence of $W(C)$ on the moduli of E and X [43,44], although we will not be needing such detailed expressions here.

Thus, in the case that X is a generic quintic in $\mathbb{CP}^4$ and E is a deformation of TX, the vanishing of W implies as a corollary that, summing $W(C)$ over the 2875 lines on X,

$$\sum_{i=1}^{2875} \frac{\exp\left(i\int_{C_i} B\right)\,\mathrm{Pfaff}\left(\overline{\partial}_{E(-1)}(C_i)\right)}{\left(\det \overline{\partial}_{\mathcal{O}(-1)}(C_i)\right)^2} = 0\,. \tag{3.1.3}$$

In this expression, we have dropped from $W(C)$ an overall factor of $\exp\left(\frac{-A(C)}{2\pi\alpha'}\right)$, which is constant for curves on X of given degree, and a factor of $(\det'\overline{\partial}_{\mathcal{O}})^{-2}$, which is simply constant.

One is very much also tempted to drop from (3.1.3) the factor of $\exp\left(i\int_{C_i} B\right)$, which is at least "morally" constant on curves of given degree. However, as reviewed in [41], because only the product of $\exp\left(i\int_{C_i} B\right)$ and the fermion Pfaffian is even well-defined, we must technically include in (3.1.3) this factor involving B so that the full expression makes sense. Nevertheless, our interest in (3.1.3) resides in the holomorphic dependence of this formula on the complex structure moduli of X and E, and we will not dwell here on the subtleties of the heterotic B-field.

At first sight, the formula (3.1.3) might seem like an exotic mathematical prediction derived only indirectly from the underlying $(0,2)$ linear sigma model. But in fact, this sort of formula has a clear precedent from algebraic geometry, in the form of a residue theorem.

To derive the simplest example of such a residue theorem, suppose that ω is a meromorphic one-form on $\mathbb{CP}^1$ with simple poles at points P_i, $i = 1, \ldots, N$. Letting z be a holomorphic coordinate on $\mathbb{CP}^1$, we can express ω as

$$\omega = \frac{g(z)dz}{f(z)}\,, \tag{3.1.4}$$

where f and g are polynomials in z, f having non-degenerate zeroes at the points $P_1, \ldots, P_N$. Without loss, we assume that ω does not have a pole at $z = \infty$, so that the degrees of f and g satisfy

$$\deg g \leq \deg f - 2\,. \tag{3.1.5}$$

As usual, we then define the residue of ω at each point P_i, denoted $\mathrm{Res}_{P_i}(\omega)$, by integrating ω over a small contour γ_i about the point P_i,

$$\mathrm{Res}_{P_i}(\omega) = \frac{1}{2\pi i}\oint_{\gamma_i} \omega \;=\; \frac{g(P_i)}{\partial f/\partial z(P_i)}\,. \tag{3.1.6}$$

We now obtain a residue theorem simply by considering the sum of contours

$$\Gamma = \gamma_1 + \cdots + \gamma_N \,. \tag{3.1.7}$$

Since Γ is contractible, we have

$$0 = \frac{1}{2\pi i} \oint_\Gamma \omega = \sum_{i=1}^{N} \mathrm{Res}_{P_i}(\omega) = \sum_{i=1}^{N} \frac{g(P_i)}{\partial f/\partial z(P_i)} \,. \tag{3.1.8}$$

So the residue theorem simply states that the sum of the residues of ω is zero.

Comparing (3.1.3) to (3.1.8), we can already see a vague similarity between these two formulae, with the Pfaffian in (3.1.3) being a holomorphic function analogous to g in (3.1.8), and the bosonic determinant in (3.1.3) being analogous to $\partial f/\partial z$ in (3.1.8). Our main goal in this chapter is to make the correspondence between these formulae precise, showing directly that the instanton contributions to W vanish in suitable models due to an infinite-dimensional generalization of the simple one-dimensional residue theorem above.

3.1.2 The Plan of the Chapter

Our plan for this chapter is as follows. In Section 2, we start by generalizing the one-dimensional residue theorem to finitely many dimensions. Although standard mathematical approaches exist for studying multi-dimensional residues, as for instance in [34,45], we will take a more physical approach by studying a certain supersymmetric, finite-dimensional integral. This integral is a natural abstraction of the path-integral over the right-moving world-sheet fields on the heterotic string, and from it we easily prove a very general, multi-dimensional residue theorem.

At the end of Section 2, we also describe precisely how the partition function of the worldvolume theory on a supersymmetric $D1$-brane can be interpreted as a residue. Unfortunately, although the $D1$-brane formalism provides a very elegant description of the superpotential contribution from any single instanton, the world-sheet description of the heterotic string turns out to be better for proving vanishing results for the sums of these instanton contributions.

So in Section 3, we apply our analysis from Section 2 to the heterotic world-sheet theory itself. This analysis most directly generalizes to the “half-linear” class of heterotic compactifications, for which X is a complete-intersection in a compact toric variety Y and the gauge bundle E on X pulls back from a bundle on Y. For these compactifications, the vanishing of the instanton contributions to W follows from essentially the same argument as we use in Section 2 to deduce the multi-dimensional residue theorems. We also show how this argument can be applied to the $(0, 2)$ linear sigma models to prove directly formulae such as (3.1.3).

Very recently, Basu and Sethi [14] have also given another argument for the stability of $(0, 2)$ linear sigma models. Their argument focuses on showing the absence of corrections to the world-sheet superpotential.

Finally, in Section 4 we consider the $\mathcal{N} = 1$ compactification of M-theory on a manifold X of G_2 holonomy. Using ideas very similar to those in Sections 2 and 3, we extend the results of [46] by computing the superpotential contribution from membranes which wrap a continuous family of supersymmetric three-cycles in X.

This chapter of the thesis is based on [5].

3.2 Residues and Supersymmetry

Rather than trying to generalize immediately from the one-dimensional residue theorem to an infinite-dimensional residue theorem which is applicable to the heterotic string, we will warm up with the simpler generalization to residue theorems in only a finite number of dimensions. Our strategy is to consider a finite-dimensional, supersymmetric integral on an arbitrary compact, complex manifold M. The finite-dimensional residue theorem then follows from the supersymmetry, which allows us to localize the integral to a sum of terms generalizing the one-dimensional residues, and from the compactness of M, which leads to the vanishing of the integral and hence the sum. After we obtain this result, we will indicate some easy generalizations of it which also have relevance to the heterotic models we introduce in Section 3. Finally, we describe precisely how the partition function of the worldvolume theory on supersymmetric $D1$-brane can be interpreted as a residue.

Standard mathematical approaches to multi-dimensional residues and residue theorems can be found in [34] and [45]. Mathematical discussions somewhat more related to our approach via supersymmetry are given in [47], [48], and [49].

3.2.1 A Finite-Dimensional Integral

We now introduce the finite-dimensional, supersymmetric integral that is central to our study of residues and which serves as a model for the path-integral over the world-sheet fields of the heterotic string. Since the supersymmetry in our integral is essential, we will begin by specifying how it acts on the variables of integration.

As mentioned above, we perform the integral over a compact, complex manifold M, having (complex) dimension n. So the bosonic variables of integration will be the local holomorphic and anti-holomorphic coordinates z^i and $z^{\bar{i}} \equiv \overline{z^i}$ on M.

We also introduce a set of anti-commuting, fermionic coordinates $\theta^{\bar{i}}$ and χ^α. Here the fermions $\theta^{\bar{i}}$ transform as coordinates on the anti-holomorphic tangent bundle $\overline{TM}$, and the fermions χ^α transform as coordinates on a holomorphic vector bundle V, of rank r, over M. The bundle V is completely arbitrary and should be considered, like M, as part of the defining data for our integral.

Besides specifying V itself, we must now also choose a global holomorphic section s of V. We need this section s simply to define an interesting supersymmetry transformation for the fermions χ^α, since none of the other variables of integration have anything to do with V. So under the supersymmetry δ, the bosonic and fermionic variables transform as

$$\begin{aligned} \delta z^i &= 0\,, \qquad \delta z^{\bar{i}} = \theta^{\bar{i}}\,, \\ \delta\chi^\alpha &= s^\alpha\,, \quad\ \delta\theta^{\bar{i}} = 0\,. \end{aligned} \tag{3.2.1}$$

Note that since s is holomorphic, $\delta^2 = 0$, the most important property of δ.

The supersymmetric integral which we consider takes the general form

$$Z = \int_M g\, d\mu\, \exp\left(-t\, S\right), \tag{3.2.2}$$

where t is a positive real parameter representing the "coupling constant" for Z, S is a finite-dimensional "action" which we will soon present, and

$$g\,d\mu \equiv g(z)\,d^n z\,d^n\overline{z}\,d^n\theta\,d^r\chi \tag{3.2.3}$$

is the measure. Locally, g is a function which represents the particular choice of measure for Z, and to ensure that the measure respects the supersymmetry, g must be holomorphic.

The fact that we have to worry about the measure for Z may seem slightly odd, since in many supersymmetric integrals, one can make a canonical choice of measure (up to normalization). The point is that, under any change of variables, the resulting Jacobians for the bosonic variables are cancelled by the fermionic Jacobians for their superpartners.

In the case of $d\mu$ above, such a cancellation occurs between the anti-holomorphic bosons $z^{\bar{i}}$ and their superpartners $\theta^{\bar{i}}$. So the factor $d^n\overline{z}\,d^n\theta$ appearing in $d\mu$ indeed represents a canonical choice of measure for these variables.

On the other hand, the bosonic variables z^i and the fermionic variables χ^α are unrelated by supersymmetry, which means that we really must choose the factor $g(z)\,d^n z\,d^r\chi$ appearing in (3.2.3). Globally, g is not a function but transforms as a holomorphic section of the line-bundle $\Omega^n_M \otimes \wedge^r V$ on M, where Ω^n_M denotes as usual the canonical bundle of holomorphic n-forms on M and $\wedge^r V$ is the top exterior power of V. Since we generally have no preferred choice of such a section, we must interpret our choice of g as another part of the input data needed to specify Z.

We must, of course, also specify the action S for the integrand of Z. We first choose a positive-definite, hermitian metric $h_{\overline{\alpha}\alpha}$ on V. Then we consider a δ-trivial action,

$$S = \delta\left(h_{\overline{\alpha}\alpha}\, s^{\overline{\alpha}}\chi^\alpha\right)\,, \tag{3.2.4}$$

or expanding,

$$S = h_{\overline{\alpha}\alpha}s^{\overline{\alpha}}s^\alpha + h_{\overline{\alpha}\alpha}D_{\overline{j}}s^{\overline{\alpha}}\,\theta^{\overline{j}}\chi^\alpha\,. \tag{3.2.5}$$

Here $D_{\overline{j}}$ is the covariant derivative associated to the canonical connection arising from the metric $h_{\overline{\alpha}\alpha}$ on V. Recall that the canonical connection [34] is the unique connection on V

for which $h_{\overline{\alpha}\alpha}$ is covariantly constant and for which $D_{\overline{j}} = \partial_{\overline{j}}$ when acting on a holomorphic frame of V.

One easy consequence of the fact that S is δ-trivial is that S is obviously supersymmetric, $\delta S = 0$. A deeper consequence of the fact that S is δ-trivial is that the integral Z is formally independent of the real parameter t and the metric $h_{\overline{\alpha}\alpha}$ on V which we introduced. For instance, the invariance of Z under changes in t is derived by first observing that

$$\frac{dZ}{dt} = -\int_M g\, d\mu\, S \exp\left(-t\, S\right) = -\langle S\rangle\,. \tag{3.2.6}$$

However, if $\mathcal{O}$ is any function of the variables z^i, $z^{\overline{i}}$, χ^α, and $\theta^{\overline{i}}$, then

$$\langle\delta\mathcal{O}\rangle = \int_M g\, d\mu\, \delta\mathcal{O}\, \exp\left(-t\, S\right) = 0\,, \tag{3.2.7}$$

which in the language of topological field theory is the decoupling of BRST-trivial observables [50,51]. Since the action S is of the form $\delta\mathcal{O}$, we deduce immediately that $dZ/dt = 0$. The invariance of Z under deformations of the metric $h_{\overline{\alpha}\alpha}$ follows by the same argument.

Finally, we observe that S is invariant under a ghost number symmetry, under which the anti-commuting variables χ^α and $\theta^{\overline{i}}$ carry charges -1 and $+1$ respectively, and δ itself carries charge $+1$. Since the measure $d\mu$ thus carries ghost number

$$\dim M - \text{rank } V = n - r\,, \tag{3.2.8}$$

Z vanishes identically unless $n = r$. So, if we wish to use Z to prove a residue theorem, we must assume that $\dim M = \text{rank } V$.

3.2.2 A Residue Theorem

As is familiar from the study of other topological models, we can prove an interesting theorem by using the fact that Z is independent of t and then evaluating Z for $t \to \infty$ and $t = 0$. Sometimes, a formal statement such as "Z is independent of t" could fail to hold if the convergence of Z were sufficiently poor. See [13] for a nice demonstration of such a failure in the context of the linear sigma model. However, because here Z is an integral over a compact manifold M, the convergence of Z is assured, even when $t = 0$, and there are no difficulties with the formal statements above.

Evaluating Z when $t = 0$ is easy. Then

$$Z = \int_M g\, d\mu\, 1\; = 0\,, \tag{3.2.9}$$

since neither χ^α nor $\theta^{\bar{i}}$ appear in the integrand above.

Evaluating Z for $t \to \infty$, we see from the action S in (3.2.5) that only points in a neighborhood of the vanishing locus L of the section s contribute to Z. In general, L will consist of several disconnected components C, and Z must have an expression

$$Z = \sum_{C \subset L} Z(C)\,, \tag{3.2.10}$$

where $Z(C)$ denotes the local contribution to Z from the component C. So (3.2.9) and (3.2.10) imply as a very general vanishing theorem that

$$\sum_{C \subset L} Z(C) = 0\,. \tag{3.2.11}$$

The power of this approach is that the vanishing theorem (3.2.11) does not rely on any specific behavior of the section s of V. In the simplest case, s has simple zeroes on a set of isolated points of M. But we can equally well consider the case that s has degenerate zeroes at some points, or even that s vanishes over some components of positive dimension. In order to translate (3.2.11) into a more explicit formula, along the lines of the one-dimensional residue theorem (3.1.8), we must simply evaluate the expression $Z(C)$ for each case.

Multi-dimensional residues

To make contact with the one-dimensional residue theorem (3.1.8), we will consider at first only the easiest case that s vanishes in a non-degenerate fasion on a set of isolated points P of M.

Recall that the requirement that s vanish non-degenerately at a point P is simply the condition that the Jacobian $\det(ds)$ be non-vanishing at P,

$$\det(ds)(P) = \det\Big(\frac{\partial(s^1, \cdots, s^n)}{\partial(z^1, \cdots, z^n)}\Big)(P) \neq 0. \tag{3.2.12}$$

In this case, the contribution $Z(P)$ from P can be evaluated exactly using the Gaussian approximation to Z near this point, and we easily see that

$$Z(P) = \frac{g(P)}{\det{(ds)(P)}}\,. \tag{3.2.13}$$

In this expression, we suppress overall factors of π that arise from the Gaussian integration. Thus, the vanishing result (3.2.11) becomes

$$\sum_{P \in L} \frac{g(P)}{\det{(ds)(P)}} = 0\,. \tag{3.2.14}$$

This expression represents a natural generalization of the one-dimensional residue theorem (3.1.8).

To sharpen the correspondence between the formula (3.2.14) and a multi-dimensional residue theorem, we consider the particular case that the bundle V is a direct sum of n line bundles,

$$V = \mathcal{O}(D_1) \oplus \cdots \oplus \mathcal{O}(D_n)\,, \tag{3.2.15}$$

which are associated to n irreducible, effective divisors $D_1, \ldots, D_n$ intersecting transversely at isolated points P in M.

To describe the appropriate section s of V for this case, we note that each divisor D_i is determined as the vanishing locus of a holomorphic section s_i of the line-bundle $\mathcal{O}(D_i)$. Then we simply take s to be the direct sum of the s_i, so that s has components

$$s = (s_1, \ldots, s_n)\,. \tag{3.2.16}$$

We note that the section s vanishes non-degenerately at each point $P \in D_1 \cap \cdots \cap D_n$, so our simple expression for $Z(P)$ in (3.2.13) is valid.

In this case, we can now give a very nice geometric interpretation of the local contribution $Z(P)$ from each point $P \in D_1 \cap \cdots \cap D_n$. Near P, we can trivialize all the line bundles $\mathcal{O}(D_i)$ as well as the canonical bundle of M. Upon doing so, we can regard g as an ordinary holomorphic function that is nonzero at P, and the s_i as holomorphic functions that vanish on D_i. Now we can define a meromorphic n-form ω that generalizes the one-dimensional expression (3.1.4),

$$\omega = \frac{g\, dz^1 \wedge \cdots \wedge dz^n}{s_1 \cdots s_n}\,. \tag{3.2.17}$$

Given the meromorphic n-form ω, and a real n-cycle γ that links in a suitable way the locus of its poles, we can naturally define an n-dimensional residue $\mathrm{Res}_P(\omega) = (1/2\pi i)^n \int_\gamma \omega$ that will generalize the usual one-dimensional residue. We let γ be the real n-cycle determined by

$$|s_i| = \epsilon\,, \quad i = 1, \ldots, n\,, \tag{3.2.18}$$

where ϵ is a small parameter. Technically, we must also orient γ, which we do by the condition $d(\arg s_1) \wedge \cdots \wedge d(\arg s_n) \geq 0$.

On γ, ω is holomorphic, so we can define

$$\mathrm{Res}_P(\omega) = \left(\frac{1}{2\pi i}\right)^n \int_\gamma \omega\,. \tag{3.2.19}$$

Since $d\omega = 0$ on a neighborhood of γ, this definition only depends on the homology class of γ and in particular does not depend the parameter ϵ above.

The residue $\mathrm{Res}_P(\omega)$ can be then be evaluated by a change of variables and the iterative application of Cauchy's theorem. We find

$$\mathrm{Res}_P(\omega) = \frac{g(P)}{\det\,(ds)(P)}\,, \tag{3.2.20}$$

generalizing the one-dimensional expression in (3.1.6). Of course, $\mathrm{Res}_P(\omega)$ agrees precisely with $Z(P)$ for the special choices of V and s above, so our main result (3.2.11) is properly interpreted as a generalized, multi-dimensional residue theorem.

A quick example

Before proceeding further, we will give a quick example of the residue theorem.

For our example, we take $M = \mathbb{CP}^2$ and $V = TM$, the holomorphic tangent bundle. If we let $[X_0 : X_1 : X_2]$ be homogeneous coordinates on M, then any holomorphic section s of V takes the form

$$s = a_0\, X_0 \frac{\partial}{\partial X_0} + a_1\, X_1 \frac{\partial}{\partial X_1} + a_2\, X_2 \frac{\partial}{\partial X_2}\,, \tag{3.2.21}$$

where (a_0, a_1, a_2) are complex coefficients parametrizing s. Because $[X_0 : X_1 : X_2]$ are only homogeneous coordinates, the coefficients (a_0, a_1, a_2) are only defined up to the addition of a multiple of $(1, 1, 1)$, which describes the zero section of V. If (a_0, a_1, a_2) are generic coefficients, then s vanishes non-degenerately at the three points $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$, and $P_3 = [0 : 0 : 1]$ of M.

Since $V = TM$, the measure $d\mu$ is a section of the trivial bundle $\mathcal{O} = \Omega^n_M \otimes \wedge^n TM$. Consequently, in this example we do have a canonical measure for Z and g is a constant.

Now in the patch where $X_0 \neq 0$, with local coordinates (z^1, z^2), s takes the form

$$s = (a_1 - a_0)\, z^1 \frac{\partial}{\partial z^1} + (a_2 - a_0)\, z^2 \frac{\partial}{\partial z^2}\,, \tag{3.2.22}$$

and so the residual contribution from P_1 to Z is

$$Z(P_1) = \frac{1}{(a_1 - a_0)(a_2 - a_0)}\,. \tag{3.2.23}$$

Similar contributions from the points P_2 and P_3 are

$$Z(P_2) = \frac{1}{(a_0 - a_1)(a_2 - a_1)}\,, \quad Z(P_3) = \frac{1}{(a_0 - a_2)(a_1 - a_2)}\,. \tag{3.2.24}$$

The residue theorem then simply states that $Z(P_1) + Z(P_2) + Z(P_3) = 0$, as one can verify directly.

3.2.3 Generalizations

The ghost number symmetry preserved by S implies that Z trivially vanishes unless the condition $\operatorname{rank} V = \dim M$ holds. So if we wish to study bundles V such that $\operatorname{rank} V \neq \dim M$, we should consider not Z itself but expectation values $\langle \mathcal{O} \rangle$,

$$\langle \mathcal{O} \rangle = \int_M g\, d\mu\, \mathcal{O}\, \exp\left(-tS\right), \tag{3.2.25}$$

where $\mathcal{O}$ is any function of z^i, $z^{\bar{i}}$, χ^α, and $\theta^{\bar{i}}$ which satisfies $\delta \mathcal{O} = 0$. Of course, $\mathcal{O}$ must also have ghost number $n - r$ if $\langle \mathcal{O} \rangle$ is to be any more interesting that Z itself.

Globally, functions $\mathcal{O}$ of z^i, $z^{\bar{i}}$, χ^α, and $\theta^{\bar{i}}$ are elements of the complex

$$\bigoplus_{(p,q)} A^{(0,q)}(M) \otimes \wedge^p V^*\,. \tag{3.2.26}$$

Here $A^{(0,q)}(M)$ is the bundle of smooth $(0,q)$ forms on M, and V^* is the holomorphic bundle dual to V. A function homogeneous and q^{th} order in $\theta^{\bar{i}}$ is a $(0,q)$-form on M, while a function homogeneous and p^{th} order in χ^α is a section of $\wedge^p V^*$. We will often refer to an element of $A^{(0,q)}(M) \otimes \wedge^p V^*$ for fixed (p,q) as having "type" (p,q).

The supersymmetry transformation δ acts on elements of this complex as

$$D = \theta^{\bar{i}} \frac{\partial}{\partial z^{\bar{i}}} + s^\alpha \frac{\partial}{\partial \chi^\alpha} \,. \tag{3.2.27}$$

More intrinsically, we can identify D with the operator

$$D = \overline{\partial} + \iota_s \,, \tag{3.2.28}$$

where $\overline{\partial}$ is the usual Dolbeault operator on M and ι_s acts on sections of $\wedge^p V^*$ by the interior product with s. The action of D on this complex has certainly been considered before in the mathematical literature, for instance in [47,48,49], though mostly for the case $V = TM$.

Since $\langle \delta \mathcal{O} \rangle = 0$ for any $\mathcal{O}$, the interesting observables $\mathcal{O}$ correspond to nontrivial elements of the cohomology of D. In general, what can we say about this cohomology?

Without placing additional conditions on M, V, and s, in fact we cannot say much. (However, see [49] for a nice discussion of the easiest case that $V = TM$ and s has zeroes at isolated points. In this case, the cohomology of D is isomorphic to $H^0(M, \mathcal{O}/\mathcal{I})$, where $\mathcal{I}$ is the ideal sheaf associated to s.) Nonetheless, we do have a systematic procedure to compute the D-cohomology, using a spectral sequence (see [52] for a clear introduction to spectral sequences).

In physical terms, we want to solve the equation $\delta \mathcal{O} = 0$, and the spectral sequence is essentially a perturbative way to do this, really by following one's nose. So to construct an $\mathcal{O}$ which satisfies $\delta \mathcal{O} = 0$, we start with an "order-zero" trial solution $\mathcal{O}^{(0)}$, of type (p,q), which satisfies $\overline{\partial} \mathcal{O}^{(0)} = 0$. (If we wished, we could equally well start with $\mathcal{O}^{(0)}$ satsifying $\iota_s \mathcal{O}^{(0)} = 0$ and reverse the roles of $\overline{\partial}$ and ι_s above. We find it convenient to do this in Section 3.) If $\mathcal{O}^{(0)}$ also happens to satisfy $\iota_s \mathcal{O}^{(0)} = 0$, then $\mathcal{O} = \mathcal{O}^{(0)}$, but generally $\iota_s \mathcal{O}^{(0)} \neq 0$.

To correct for this discrepancy, we then try to solve

$$\iota_s \mathcal{O}^{(0)} + \overline{\partial} \mathcal{O}^{(1)} = 0 \,, \tag{3.2.29}$$

to determine the "first-order" correction $\mathcal{O}^{(1)}$. We consider $\mathcal{O}^{(1)}$ as a correction to $\mathcal{O}^{(0)}$ in a very definite sense, since although $\mathcal{O}^{(0)}$ is of type (p,q), $\mathcal{O}^{(1)}$ is of type $(p-1,q-1)$. Thus, if we continue to solve iteratively

$$\iota_s \mathcal{O}^{(n)} + \overline{\partial}\mathcal{O}^{(n+1)} = 0\,, \tag{3.2.30}$$

we will either find an obstruction, or the procedure will terminate after a finite number of steps with $\mathcal{O} = \mathcal{O}^{(0)} + \mathcal{O}^{(1)} + \cdots$ satisfying $\delta\mathcal{O} = 0$. We will find this little procedure useful when constructing heterotic models in Section 3.

What sort of results, analogous to the generalized residue theorem (3.2.11), do we then obtain by considering the expectations $\langle\mathcal{O}\rangle$ of nontrivial observables $\mathcal{O}$? Evaluating $\langle\mathcal{O}\rangle$ at $t=0$ now yields

$$\langle\mathcal{O}\rangle = \int_M g\, d\mu\, \mathcal{O}\,, \tag{3.2.31}$$

which need not vanish if $\mathcal{O}$ carries the proper ghost number. Evaluating $\langle\mathcal{O}\rangle$ in the limit $t\to\infty$, we again see that $\langle\mathcal{O}\rangle$ can be expressed as a sum of local contributions from each of the components C of the vanishing locus L of s,

$$\langle\mathcal{O}\rangle = \sum_{C\subset L} \langle\mathcal{O}\rangle(C)\,. \tag{3.2.32}$$

So for instance, again in the case that s vanishes non-degenerately over isolated points P of M and $\mathcal{O}$ has ghost number zero,

$$\int_M g\, d\mu\, \mathcal{O} = \sum_{P\in L} \frac{g(P)\,\mathcal{O}(P)}{\det(ds)(P)}\,. \tag{3.2.33}$$

In the above expression, we must interpret the integral over M as picking out the component of $\mathcal{O}$ of type (n,n) and the evaluation at P as picking out the component of $\mathcal{O}$ of type $(0,0)$.

We can also consider the under-determined case, for which $\operatorname{rank} V < \dim M$, as well as the over-determined case, for which $\operatorname{rank} V > \dim M$. In the under-determined case, the components C of L will generically be complex submanifolds of dimension $n-r$ in M. We assume that s vanishes in a non-degenerate fashion on each C, which means that the Jacobian $\det(ds|_N)$ of s with respect to the normal directions to C in M is non-vanishing along C. Then the local contribution of C to $\langle\mathcal{O}\rangle$ is

$$\langle\mathcal{O}\rangle(C) = \int_C \frac{g\, d\mu\, \mathcal{O}}{\det\left(ds|_N\right)}\,. \tag{3.2.34}$$

In the above, $g\, d\mu \,/\, \det\left(ds|_N\right)$ determines an element of Ω^{n-r} on C, and thus only the component of $\mathcal{O}$ of type $(0, n-r)$ now contributes to the integral over C.

As we shall see in Section 3, the case of direct relevance to the heterotic string is actually the over-determined case, rank $V > \dim M$. In this case, generically $s \neq 0$, and upon taking $t \to \infty$ we immediately conclude that $\langle\mathcal{O}\rangle = 0$ for all $\mathcal{O}$.

However, if s is not generic, then the locus $s = 0$ need not be empty, and we can get a non-trivial result with a non-trivial $\mathcal{O}$, which in the simplest case has degree $(r-n, 0)$. This situation actually occurs in the half-twisted heterotic string, for which $\mathcal{O}$ is the exponential of a fermion bilinear and $r-n$ is infinite. If such an $\mathcal{O}$ is present, then the section s cannot vary freely, since the supersymmetry condition $(\overline{\partial} + \iota_s)\mathcal{O} = 0$ must be preserved. Hence for a suitable $\mathcal{O}$, it can be natural to consider a section s having zeroes with non-trivial residues. For instance, if s again vanishes non-degenerately at an isolated point P of M, now meaning that the matrix $ds = \left(\partial s^\alpha / \partial z^i\right)$ has full rank at P, then the local contribution from P to $\langle\mathcal{O}\rangle$ is

$$\langle\mathcal{O}\rangle(P) = \left(\frac{g\, d\mu\, \mathcal{O}}{ds}\right)(P) \equiv \left(\frac{g\, \epsilon_{i_1 \cdots i_q} \epsilon^{\alpha_1 \cdots \alpha_p}\, \mathcal{O}_{\alpha_{q+1} \cdots \alpha_p}}{\partial_{i_1} s^{\alpha_1} \cdots \partial_{i_q} s^{\alpha_q}}\right)(P)\,. \tag{3.2.35}$$

Evidently, in such an example with isolated zeroes of s, only the component of type $(r-n, 0)$ of $\mathcal{O}$ contributes to $\langle\mathcal{O}\rangle(P)$.

3.2.4 The $D1$-brane Partition Function as a Residue

Our discussion of multi-dimensional residues now allows us to make precise the manner in which the partition function of a supersymmetric $D1$-brane can be interpreted as a residue. We have already seen in the introduction a strong formal similarity between expressions such as (3.1.3) and (3.1.8) which suggests this interpretation. To check this idea, though, we must examine to what extent the worldvolume theory on a supersymmetric $D1$-brane actually generalizes our finite-dimensional model which produces the residues.

At first glance, one might be worried by the following fact. If we consider the bosonic action for a $D1$-brane which wraps an arbitrary, not necessarily holomorphic, surface Σ in the Calabi-Yau threefold X, then this action is just the area $A[\Sigma]$ of the surface. So if the $D1$-brane action were literally to be the obvious generalization of the action (3.2.5) of the finite-dimensional model, then $A[\Sigma]$ would have to admit a representation as the norm-squared of a suitable holomorphic section s over the space $\mathcal{M}$ of immersed surfaces in X. But $A[\Sigma]$ presumably does not admit such a representation, and it is not even obvious that the space $\mathcal{M}$, which should play the role of the complex manifold M in the finite-dimensional model, admits a complex structure.

Thus, as far as we know, the full $D1$-brane worldvolume action does not fit into the simple structure of the finite-dimensional model. As a result, we cannot hope to use the $D1$-brane formalism to prove vanishing results such as (3.1.3). Physically, the difficulty in using the $D1$-brane formalism to prove the vanishing results is that the $D1$-brane worldvolume description becomes more complicated when the brane is "off-shell", i.e. not supersymmetric. We do not believe that these off-shell complications are really essential, but we also do not know how to eliminate them in the $D1$-brane framework. (We remark parenthetically that $D5$-branes can be put in a gauge-invariant version of this framework.) When we deduce these vanishing results in Section 3, we will use instead approaches based on linear and half-linear sigma models, which are more closely related to the finite-dimensional model.

Yet to discuss the superpotential contribution from a $D1$-brane which wraps an isolated holomorphic curve C in X requires considerably less than the full worldvolume action. Since we evaluate the partition function at one-loop, we only need to discuss fluctuations about the holomorphic curve up to quadratic order in the action. Considering the worldvolume theory only to this order, we can nicely fit it into the framework of the finite-dimensional model. In particular, the second variation of $A[\Sigma]$ away from a minimum corresponding to a holomorphic curve indeed appears as the norm-squared of a suitable section s and the contribution to the superpotential is indeed a residue.

Geometrically, the approach of working only to quadratic order in the supersymmetric $D1$-brane action corresponds to linearizing the space $\mathcal{M}$ over the point corresponding to the given holomorphic curve C in X. The linearization possesses the requisite complex structure.

We now give a thorough discussion of how this approximation to the $D1$-brane action fits into the framework of the finite-dimensional model. As we have indicated, our identification of the supersymmetric $D1$-brane partition function as a residue is of more conceptual than practical interest here, not only because of the off-shell complications but also because of the lack of compactness in the $D1$-brane approach. However, in Section 4 we will apply similar ideas to study the superpotential contributions from continuous families of membrane instantons in M-theory compactifications on manifolds of G_2 holonomy.

To proceed, we begin with the general observation [53] that whenever a brane wraps a supersymmetric cycle, then the worldvolume theory on the brane is automatically twisted, implying the existence of at least one scalar supercharge. The existence of a scalar supercharge on the $D1$-brane worldvolume is crucial if we are to interpret the worldvolume theory in analogy to the finite-dimensional model, with its scalar supersymmetry generator δ.

We focus our attention on the sector of the $D1$-brane worldvolume theory describing fluctuations of the brane in X, as opposed to the trivial sector describing fluctuations in $\mathbb{R}^4$. When the $D1$-brane wraps a holomorphic curve C in X, the worldvolume bosons x^i and $x^{\bar{i}}$ which describe fluctuations of the brane in X transform as coordinates on the holomorphic normal bundle N and anti-holomorphic normal bundle $\overline{N}$ of C in X. The worldvolume theory also possesses fermions $\psi_{\dot{\alpha},i}$ which transform as right-moving Weyl fermions on $\mathbb{R}^4$, as indicated by the $\dot{\alpha}$ index, and as coordinates on the dual bundle N^* of N. Equivalently, using the hermitian metric $g_{\bar{i}i}$ on X, we can regard these fermions as transforming in the anti-holomorphic normal bundle $\overline{N}$. The twisted model has two scalar supercharges, described in detail later, which relate the worldvolume fields $(x^{\bar{i}}, \psi^{\bar{i}}_{\dot{\alpha}})$.

Now, in the finite-dimensional model, the supersymmetry transformations as well as the form of the action are determined by the holomorphic section s of V. So what are the analogues of s and V for the $D1$-brane?

As has already been observed in [54,55], for a variety of supersymmetric compactifications of string and M-theory, the supersymmetric brane configurations can be characterized as the critical points of a "superpotential" Ψ, suitably interpreted as a function on the space of arbitrary brane configurations. (This idea has also been discussed lately in a mathematical context in [56].) For the $D1$-brane, if δ is the exterior derivative on the space $\mathcal{M}$ of brane configurations, then $\delta\Psi$ is a one-form that vanishes at the point corresponding to a holomorphic curve C, and moreover $\delta\Psi$ is holomorphic once we linearize in a neighborhood of C. So a natural guess is to take V to be the holomorphic cotangent bundle $T^*\mathcal{M}$ and $s = \delta\Psi$.

To check that this identification is correct, we must describe Ψ explicitly. For argument's sake, we start by defining Ψ on surfaces Σ which are homologically trivial in X — although we note that any holomorphic surface, being calibrated by the Kähler form on X, actually resides in a nontrivial homology class. In any case, $\Psi(\Sigma)$ is defined for a homologically trivial surface Σ by

$$\Psi(\Sigma) = \frac{1}{6}\int_B \Omega\,, \tag{3.2.36}$$

where B is a bounding three-cycle for Σ and Ω is the holomorphic three-form on X. The factor of $\frac{1}{6}$ is simply to cancel some constants that would otherwise appear in later formulae. If $H_3(X,\mathbb{Z}) \neq 0$, as is always the case when X has complex structure moduli, then $\Psi(\Sigma)$ generally depends on the class of B and is defined only up to an additive constant.

Now, if Σ is a surface representing a nontrivial homology class in X, then a bounding three-cycle B does not exist. To define $\Psi(\Sigma)$ in this case, for each class in $H_2(X,\mathbb{Z})$ we choose a particular representative Σ_0. Then, if Σ lies in the same class as Σ_0, a bounding three-cycle B exists for $\Sigma - \Sigma_0$. That is, the boundary of B has two components, one of which is Σ and the other is Σ_0, considered with opposite orientation. So now we set

$$\Psi(\Sigma) - \Psi(\Sigma_0) = \frac{1}{6}\int_B \Omega\,. \tag{3.2.37}$$

In this case, the additive constant in Ψ also depends on the representative Σ_0 as well as the class of B.

The fact that Ψ is only defined up to an additive constant does not concern us, as this constant does not affect the location of the critical points, for which $\delta\Psi = 0$. Explicitly, in terms of holomorphic coordinates x^i on X,

$$\delta\Psi(\Sigma) = \frac{1}{2}\int_\Sigma \Omega_{ijk}\,\delta x^i\,dx^j\wedge dx^k\,. \tag{3.2.38}$$

So $\delta\Psi = 0$ precisely for those surfaces Σ on which the $(2,0)$-form $\Omega_{ijk}\,dx^j\wedge dx^k$ is equal to zero. If Σ is holomorphic in X, then any $(2,0)$-form vanishes when restricted to Σ, so $\delta\Psi$ vanishes when Σ is a holomorphic curve C. Because Ω_{ijk} is everywhere nonzero, holomorphy of Σ is necessary as well as sufficient for vanishing of $\delta\Psi$.

While $\delta\Psi$ vanishes at the point corresponding to C, we also need the linear behavior near this point. For this, we pick local complex coordinates on X consisting of a parameter z that is a local complex coordinate on C as well as two local coordinates y^i of the normal bundle N. We write ϵ_{ij} for Ω_{zij}. In (3.2.38), we take δx^i to be a displacement of one of the y^i, since otherwise we are not moving Σ away from C at all. So we will write δy^i for δx^i. Evaluated on Σ, we have $dx^j\wedge dx^k = dz\wedge d\overline{z}\,(\partial_z x^j\,\partial_{\overline{z}}x^k - \partial_{\overline{z}}x^j\partial_z x^k)$. Because of the antisymmetry in j and k (or because $\partial_{\overline{z}}z = 0$), we cannot set both x^j and x^k equal to z. To linearize $\delta\Psi$ around C, we set one of them, say x^j, to z, and the other to y^k. So we get

$$\delta\Psi = \int_C \epsilon_{ij}\,\delta y^i\overline{\partial}y^j + \ldots, \tag{3.2.39}$$

where the ellipses indicate that higher order terms have been dropped. From this, we can also deduce that to quadratic order,

$$\Psi = c + \frac{1}{2}\int_C \epsilon_{ij}\,y^i\overline{\partial}y^j, \tag{3.2.40}$$

where c is an integration constant.

In particular, we see from (3.2.40) that when evaluated on C,

$$\frac{\delta^2\Psi}{\delta y^j(z)\delta y^i(z')}\Big|_C = \epsilon_{ij}\,\partial_{\overline{z}}\delta(z,z')\,, \tag{3.2.41}$$

where, more intrinsically, $\partial_{\overline{z}}$ represents the $\overline{\partial}$ operator acting on sections of N.

Since Ψ functions like a superpotential, the unbroken worldvolume supersymmetries in the linearized theory can be very simply expressed in terms of Ψ. Under the twisted supercharges $\overline{Q}_{\dot\alpha}$, the transformations of the fields y^i, $y^{\bar i}$, and $\psi_{\dot\alpha,i}$ take the usual form

$$\begin{aligned} \delta_{\dot\alpha} y^i = 0\,, &\quad \delta_{\dot\alpha} y^{\bar i} = \psi^{\bar i}_{\dot\alpha}\,, \\ \delta_{\dot\alpha} \psi_{\dot\beta, i} &= \epsilon_{\dot\alpha\dot\beta}\, \frac{\delta\Psi}{\delta y^i}\,. \end{aligned} \tag{3.2.42}$$

Since Ψ is holomorphic, in the sense that $\delta\Psi/\delta y^{\bar i} = 0$, these supersymmetry transformations satisfy $\{\delta_{\dot\alpha}, \delta_{\dot\beta}\} = 0$ as required. Obviously these worldvolume supersymmetries are unbroken when $\delta\Psi/\delta y^i = 0$, which we have already observed is the proper condition for the $D1$-brane to be supersymmetric. Further, taking $s = \delta\Psi$, we see that (3.2.42) represents an $N = 2$ generalization of the supersymmetry transformations (3.2.1) in the finite-dimensional model.

The worldvolume action which describes to leading order the fluctuations of a $D1$-brane which wraps a holomorphic curve C in X takes a very simple form when written in terms of Ψ. Just as for the finite-dimensional action (3.2.4),

$$\begin{aligned} S &= \frac{1}{4} \int_C \epsilon^{\dot\alpha\dot\beta}\, \delta_{\dot\beta} \left(\omega\, g^{\bar i i}\, \frac{\delta\overline{\Psi}}{\delta y^{\bar i}}\, \psi_{\dot\alpha,i} \right) \\ &= \int_C \omega \left(\frac{1}{2} g^{\bar i i} \frac{\delta\overline{\Psi}}{\delta y^{\bar i}} \frac{\delta\Psi}{\delta y^i} + \frac{1}{4} \epsilon^{\dot\alpha\dot\beta} g^{\bar i i} \frac{D^2\overline{\Psi}}{D y^{\bar j} D y^{\bar i}} \psi^{\bar j}_{\dot\beta} \psi_{\dot\alpha,i} \right)\,. \end{aligned} \tag{3.2.43}$$

Here D is the covariant derivative with respect to the metric $g_{i\bar i}$ on X, and ω is the Kähler form on X which restricts to the volume form on C. We also note from (3.2.38) that $\delta\Psi/\delta y^i$ is actually a two-form on C, and we have implicitly used the induced metric to dualize $\delta\Psi/\delta y^i$ to a scalar on C.

The action S is to be interpreted by expanding to quadratic order in the normal fluctuations y^i and $y^{\bar i}$ about the given holomorphic curve C, so that

$$S = \int_C \omega \left(\frac{1}{2} g^{\bar i i} \frac{D^2\overline{\Psi}}{D y^{\bar j} D y^{\bar i}} \frac{D^2\Psi}{D y^i D y^j} y^{\bar j} y^j + \frac{1}{4} \epsilon^{\dot\alpha\dot\beta} g^{\bar i i} \frac{D^2\overline{\Psi}}{D y^{\bar j} D y^{\bar i}} \psi^{\bar j}_{\dot\beta} \psi_{\dot\alpha,i} \right)\,. \tag{3.2.44}$$

Using (3.2.41), we can write S more explicitly as

$$S = \frac{1}{2} \int_C \omega \left(g_{\bar i i}\, \partial_z y^{\bar i}\, \partial_{\bar z} y^i + \frac{1}{2} \epsilon^{\dot\alpha\dot\beta} \epsilon_{\bar i\bar j}\, \psi^{\bar j}_{\dot\beta} \partial_z \psi^{\bar i}_{\dot\alpha} \right)\,. \tag{3.2.45}$$

In the above, we have assumed that Ω is normalized on X so that $g^{\bar{j}j}\bar{\epsilon}_{\bar{j}\bar{i}}\epsilon_{ji} = g_{\bar{i}i}$. This action is just the free action for fluctuations which we implicitly used in the Introduction when we evaluated the partition function.

More geometrically, we can identify the complex linear space $\mathcal{M}$ describing fluctuations of the $D1$-brane about C with the space of sections of N. Our formula for S simply reflects the classic fact [57,58] that, given a section y^i, the second derivative of the area functional $A[\Sigma_t]$ along the one-parameter family of surfaces Σ_t determined by y^i, evaluated at $\Sigma_0 = C$, is just

$$\frac{d^2}{dt^2} A[\Sigma_t]\Big|_{t=0} = \frac{1}{2} \int_C \omega \, \left|\partial_{\bar{z}} y^i\right|^2 \,, \tag{3.2.46}$$

which appears as the bosonic term in (3.2.45). This formula indicates that holomorphic curves are always area-minimizing in X, and only holomorphic deformations of a holomorphic curve can preserve its area.

Finally, to make contact with the finite-dimensional model, we can evaluate the partition function $Z(C)$ of a $D1$-brane wrapped on C exactly as we evaluated the contribution to the finite-dimensional integral from an isolated, non-degenerate zero of s in (3.2.13). We find that

$$\begin{aligned} Z(C) &= \int_{\mathcal{M}} \mathrm{Pfaff}\left(\overline{\partial}_{E(-1)}\right) d\mu \, e^{-S} \,, \\ &= \frac{\mathrm{Pfaff}\left(\overline{\partial}_{E(-1)}\right)(C)}{\det\left(\delta^2\Psi/\delta y^j \delta y^i\right)(C)} \,. \end{aligned} \tag{3.2.47}$$

Here $d\mu = \mathcal{D}y^i \, \mathcal{D}y^{\bar{i}} \, \epsilon^{\dot{\alpha}\dot{\beta}} \, \mathcal{D}\psi^{\bar{j}}_{\dot{\beta}} \, \mathcal{D}\psi_{\dot{\alpha},j}$ is the naive path-integral measure, and the Pfaffian factor produced by the left-moving bundle fermions is directly analogous to the section g, since both are required for the path-integral measure to be well-defined. Recalling from (3.2.41) that $\delta^2\Psi/\delta y^j \delta y^i$ represents the $\overline{\partial}$ operator acting on sections of $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, we see that $Z(C)$ indeed agrees with the summand in the expression (3.1.3).

3.3 A Residue Theorem for the Heterotic String

We now extend our investigation of residues in Section 2 to the heterotic string itself. (Because the left-moving world-sheet fermions play only an auxiliary role in our analysis, we will not need to distinguish between the $E_8 \times E_8$ and $Spin(32)/\mathbb{Z}_2$ heterotic strings.) Our goal is to prove a residue theorem, precisely analogous to the theorems we derived in Section 2, for the vanishing of world-sheet instanton contributions to W. A very useful tool in our analysis is the twisted version of the heterotic world-sheet theory, as it is the twisted theory that directly generalizes the finite-dimensional model we introduced in Section 2. Thus, we begin this section with a short reminder of what it means to twist [59-62] the heterotic world-sheet theory, and we explain how this theory is related to the finite-dimensional model of Section 2.

3.3.1 Preliminary Remarks on Twisting

The twisted heterotic world-sheet theory is simply a version of the physical (untwisted) heterotic theory in which the right-moving world-sheet fermions are assigned unconventional spins. To describe the twisting, we first recall that the world-sheet theory contains complex bosons ϕ^i and $\phi^{\bar{i}} \equiv \overline{\phi^i}$ which describe sigma model maps $\Phi : \Sigma \to X$ from the world-sheet Σ to a Kähler target space X. In the physical theory, the superpartners of ϕ^i and $\phi^{\bar{i}}$ are right-moving fermions ψ^i and $\psi^{\bar{i}}$, which transform as sections of the bundles $\overline{K}^{\frac{1}{2}} \otimes \Phi^*(TX)$ and $\overline{K}^{\frac{1}{2}} \otimes \Phi^*(\overline{TX})$ respectively. Here, TX denotes the holomorphic tangent bundle of X, and $\overline{K}$ denotes the anti-canonical bundle of Σ. The anti-canonical bundle can be explicitly described as the line-bundle of $(0,1)$ forms on Σ, and from this description we see that $\overline{K}^{\frac{1}{2}}$ is a right-moving spin-bundle on Σ. Then in the twisted theory, we simply take ψ^i to transform as a section of $\overline{K} \otimes \Phi^*(TX)$ and $\psi^{\bar{i}}$ to transform as a section of $\Phi^*(\overline{TX})$.

One way to interpret the twist is that we shift the right-moving world-sheet stress tensor $T_{\overline{zz}}$ by

$$T_{\overline{zz}} \to \widetilde{T}_{\overline{zz}} = T_{\overline{zz}} + \frac{1}{2} \partial_{\bar{z}} j_{\bar{z}} \,, \tag{3.3.1}$$

where $j_{\bar{z}}$ is the world-sheet $U(1)$ current present in the right-moving $N = 2$ algebra. Upon twisting, one of the two right-moving world-sheet supersymmetry generators becomes a nilpotent scalar Q, which we interpret as a $BRST$-operator on the world-sheet. The decoupling of Q-trivial states from the correlation functions of Q-invariant operators then greatly simplifies the twisted theory. In particular, though the twisted heterotic theory is not topological, all correlation functions of Q-invariant operators in the twisted theory vary holomorphically on the world-sheet, because the twisted stress-tensor $\widetilde{T}_{\overline{z}\overline{z}}$ is Q-trivial.

We now explain how the general framework of Section 2 applies to the twisted world-sheet theory. Instead of performing an integral over a finite-dimensional complex manifold M, we now perform a path-integral over the infinite-dimensional complex manifold $\mathcal{M}$ which is the space of all sigma model maps $\Phi : \Sigma \to X$. The world-sheet bosons ϕ^i themselves provide local holomorphic coordinates on $\mathcal{M}$ and play the same role as the holomorphic coordinates z^i on M. In addition, the fermions $\psi^{\bar{i}}$, as sections of $\Phi^*(\overline{TM})$, are coordinates on $\overline{T\mathcal{M}}$ and correspond to the anti-commuting coordinates $\theta^{\bar{i}}$ in Section 2. Finally, we interpret the fermions $\psi^i_{\bar{z}}$, whichtransform as sections of $\overline{K} \otimes \Phi^*(TX)$, as anti-commuting coordinates on a holomorphic bundle $\mathcal{V}$ over $\mathcal{M}$, so that these fermions play the same role as the fermionic coordinates χ^α on V in Section 2. (We have slightly changed notation $\psi^i \to \psi^i_{\bar{z}}$ to remind ourselves that $\psi^i_{\bar{z}}$ now transforms as a $(0, 1)$ form on Σ.) In particular, on world-sheets for which $\overline{K}$ is trivial, we can identify the bundle $\mathcal{V}$ on $\mathcal{M}$ as the holomorphic tangent bundle $T\mathcal{M}$.

Under Q, the world-sheet fields transform as

$$\begin{aligned} \delta\phi^i &= 0\,, & \delta\phi^{\bar{i}} &= \psi^{\bar{i}}\,, \\ \delta\psi^i_{\bar{z}} &= \partial_{\bar{z}}\phi^i\,, & \delta\psi^{\bar{i}} &= 0\,. \end{aligned} \tag{3.3.2}$$

Comparing (3.3.2) to (3.2.1), we see that the action of Q is precisely analogous to the supersymmetry transformation in the finite-dimensional model. Further, we see that $\partial_{\bar{z}}\phi^i$ is the holomorphic section of $\mathcal{V}$ corresponding to the section s of V in Section 2.

The sigma model action for the world-sheet fields can now be written as

$$S = \int_\Sigma d^2z\, \delta\left(g_{\bar{i}i}\, \partial_z\phi^{\bar{i}}\, \psi^i_{\bar{z}}\right) + \cdots\,, \tag{3.3.3}$$

where $g_{\bar{i}i}$ is the Kähler metric on X. The Q-trivial expression above is a direct generalization of the action (3.2.4) which we considered in Section 2. Just as the finite-dimensional integral localizes on the set where $s = 0$, so the twisted path-integral localizes on sigma model maps satisfying $\partial_{\bar{z}}\phi^i = 0$. Such maps, being holomorphic, are either constant or represent world-sheet instantons.

The "$\cdots$" appearing in S represents the additional terms in the sigma model action which are not Q-trivial (but of course are Q-closed). More precisely, these additional terms arise either from a purely topological expression which is the integral of the complexified Kähler class of X over Σ or from the kinetic terms of the left-moving bundle fermions.

Both of these sorts of terms admit an easy interpretation in light of the results of Section 2. First, if we restrict the world-sheet path-integral to the sector describing maps whose images lie in a fixed homology class of X, the topological term in S is constant and can be ignored. Second, if we also only consider world-sheet correlation functions which do *not* involving the left-moving bundle fermions, then at least for isolated world-sheet instantons, the only role of the bundle fermions is to produce the Pfaffian factor that appears in the Introduction. As we have already observed in the contex of the $D1$-brane, like the section g in the finite-dimensional model, this Pfaffian factor can be interpreted as defining a suitable measure for the path-integral over the modes of ϕ^i, $\phi^{\bar{i}}$, $\psi^i_{\bar{z}}$, and $\psi^{\bar{i}}$.

Finally we remark that, although the physical and twisted theories are generally very different, some quantities in the physical theory can be computed using the twisted theory. In particular, as long as K is the trivial bundle on Σ, correlation functions computed on Σ in the twisted theory agree with those computed on Σ in the physical theory. For instance, if Σ is a cylinder with Ramond sector ground-states at each end, then correlation functions on Σ compute the Yukawa couplings arising from the superpotential W in the low-energy effective theory. In this fashion we can use the twisted theory to probe for a background W.

3.3.2 The Half-Linear Heterotic String

Our proof of the residue theorem in Section 2 only relies upon the fact that the integral Z is invariant under a nilpotent supersymmetry and the fact that the space M over which we integrate is compact. We wish to generalize this residue theorem to apply to world-sheet correlators in the twisted heterotic theory, so we must consider models for which both of these crucial facts hold. Since the action of the BRST-operator Q on the world-sheet naturally generalizes the supersymmetry transformation of Section 2, the first fact holds for an arbitrary $(0,2)$ compactification. However, as regards the second fact, the space $\mathcal{M}$ of sigma model maps is certainly not compact, and so to generalize the finite-dimensional residue theorem from Section 2 to a vanishing result for W of the form (3.1.3), we must look for heterotic models with some special sort of compactness.

The vanishing result of [13] naturally suggests that we start by considering the linear sigma models. Indeed, the compactness of the moduli spaces of X and E is an essential ingredient in the analysis of [13].

Moreover, the linear sigma models possess another sort of compactness not present in an arbitrary heterotic compactification. As discussed extensively in [63] and [64], in compactifications for which continuous families of world-sheet instantons exist, the instanton moduli spaces of the linear sigma model provide natural compactifications of the instanton moduli spaces of the corresponding nonlinear sigma model. The compactness of these instanton moduli spaces turns out to be the essential ingredient in our proof of a residue theorem for the heterotic string.

However, we do not really have to consider the linear sigma models themselves to exploit the fact that the instanton moduli spaces of the corresponding nonlinear sigma models have natural compactifications. We find it technically simpler, in fact, to discuss a class of half-linear heterotic models. These models are like the linear models in that X is a complete-intersection Calabi-Yau in a compact toric variety Y. Unlike the linear models, the gauge bundle E on X is any bundle which satisfies the usual consistency conditions on X and also pulls back from a bundle on Y. Thus E must generally be described in a nonlinear fashion.

So in the remainder of this section, we first introduce the half-linear models and demonstrate that the finite-dimensional residue theorems of Section 2 naturally generalize to formulae of the form (3.1.3). We then return to the linear sigma models themselves and give a direct proof of the vanishing of instanton contributions to W. For concreteness, we shall throughout this section consider only the case that X is the quintic hypersurface in $Y = \mathbb{CP}^4$.

Half-linear fields

We start by specifying the field content of the half-linear model. The world-sheet bosons and the right-moving world-sheet fermions are the usual fields which describe twisted non-linear sigma model maps $\Phi : \Sigma \to Y$. For the case $Y = \mathbb{CP}^4$, the model has four complex bosons ϕ^i and $\phi^{\bar{i}}$ which represent *local* holomorphic and anti-holomorphic coordinates on Y (as opposed to the global homogeneous coordinates on Y that would appear in the corresponding linear sigma model). Since the half-linear model is twisted, the right-moving supersymmetry associates to the bosons ϕ^i and $\phi^{\bar{i}}$ corresponding fermions $\psi^i_{\bar{z}}$ and $\psi^{\bar{i}}$, transforming on Σ as sections of $\overline{K} \otimes \Phi^*(TY)$ and $\Phi^*(\overline{TY})$ respectively.

As for the left-moving sector of the world-sheet, the bundle E on Y is represented in the usual nonlinear fashion by a set of thirty-two left-moving fermions λ^a coupled to the pull back of E to Σ. We assume that E satisfies the standard topological conditions for anomaly-cancellation and stability on X. Thus, E satisfies $p_1(E)/2 = c_2(TX)$ (and, if the structure group of E reduces to a subgroup with $U(1)$ factors, there are restrictions on the corresponding first Chern classes).

However, the field content of the half-linear model, as it stands, cannot be correct. As in the linear sigma model, to localize the half-linear model from Y onto X, we must introduce a potential $J(\phi^i)$ on the world-sheet. Geometrically, J transforms as a holomorphic section of the line-bundle $\mathcal{O}(5)$ on Y. Supersymmetry requires that J couple to the right-moving fermions as well as the bosons, but we currently have no way to couple J to these fermions.

A more fundamental problem is that, although we choose the bundle E so as to cancel sigma model anomalies on X, the half-linear model on Y is currently anomalous as $p_1(E)/2 \neq c_2(TY)$.

We can elegantly fix both of these problems by adding a pair of left-moving fermions to the model. These fermions, which we denote by χ_z and $\overline{\chi}$, transform on the world-sheet as sections of $K \otimes \Phi^*(\mathcal{O}(-5))$ and $\Phi^*(\mathcal{O}(5))$. Thus we can directly include the required

Yukawa terms for J in the model.

As for the anomalies, since χ_z and $\overline{\chi}$ are also "twisted" in the sense of having non-standard world-sheet spins, they cancel the excess left-moving central charge from the new boson. Also, upon adding χ_z and $\overline{\chi}$ to the left-moving sector of the model, we cancel the sigma model anomalies, since near X, the adjunction formula implies that TY splits as a smooth bundle into the sum $TX \oplus \mathcal{O}(5)$. Explicitly, relative to the model on X, the half-linear model on Y has an additional pair of twisted, right-moving fermions which arise from the directions in the normal bundle $\mathcal{O}(5)$ to X in Y. These fermions transform as sections of $\overline{K} \otimes \Phi^*(\mathcal{O}(5))$ and $\Phi^*(\mathcal{O}(-5))$. Since χ_z and $\overline{\chi}$ transform as the complex conjugates of these two fermions, they cancel the corresponding anomalies.

Half-linear supersymmetry

In the half-linear model, the action of the scalar supercharge Q slightly generalizes (3.3.2), due to the transformations of the left-moving fermions χ_z and $\overline{\chi}$ — the other left-moving fermions are invariant. So Q acts as

$$\begin{aligned} &\delta\phi^i = 0\,, \qquad \delta\phi^{\bar{i}} = \psi^{\bar{i}}\,, \\ &\delta\psi^i_{\bar{z}} = \partial_{\bar{z}}\phi^i\,, \quad \delta\psi^{\bar{i}} = 0\,, \\ &\delta\overline{\chi} = J(\phi^i)\,, \\ &\delta\chi_z = 0\,. \end{aligned} \tag{3.3.4}$$

As we have mentioned, J is locally a quintic polynomial in the holomorphic coordinates ϕ^i and globally a holomorphic section of $\mathcal{O}(5)$ on Y. Of course, J represents the data needed to determine X as a hypersurface in Y.

We see from the action of Q that the fermions $\psi^i_{\bar{z}}$, $\overline{\chi}$, and χ_z in the half-linear model can all be identified as the analogues of the fermions χ^α in the finite-dimensional model. In this basis, $s = (\partial_{\bar{z}}\phi^i, J(\phi^i), 0)$. So, if we construct an action for the half-linear model analogous to (3.2.4) in the finite-dimensional model, the half-linear model will localize on sigma model maps Φ satisfying

$$\partial_{\bar{z}}\phi^i = J(\phi^i) = 0\,. \tag{3.3.5}$$

The first condition requires that Φ be holomorphic, and the second condition requires that the image of Φ lie in the subset $J = 0$ of Y, which can be identified with X. So the half-linear model localizes on world-sheet instantons in X.

The half-linear action

To complete our description of the half-linear model, we must finally specify its world-sheet action S.

First, in complete analogy to the action of the finite-dimensional integral, S includes the terms

$$\begin{aligned} S_0 &= t \int_\Sigma d^2z \; \delta \left(g_{\bar{i}i} \, \partial_z \phi^{\bar{i}} \psi^i_{\bar{z}} + \overline{J} \, \overline{\chi} \right) \\ &= t \int_\Sigma d^2z \left(g_{\bar{i}i} \, \partial_z \phi^{\bar{i}} \, \partial_{\bar{z}} \phi^i + g_{\bar{i}i} \, D_z \psi^{\bar{i}} \, \psi^i_{\bar{z}} + \overline{J} J + \psi^{\bar{i}} D_{\bar{i}} \overline{J} \, \overline{\chi} \right) . \end{aligned} \tag{3.3.6}$$

Here t is a coupling parameter as in Section 2, and $g_{\bar{i}i}$ is a Kähler metric on Y. Because S_0 is Q-exact, quantities which we compute in the half-linear model are unchanged under deformations of t, $g_{\bar{i}i}$, and $\overline{J}$. We also observe parenthetically that, since the expression $\overline{J} \, \overline{\chi}$ transforms as a smooth section of the trivial bundle on Y, we do not actually need to specify a hermitian bundle metric on $\mathcal{O}(5)$ to make sense of this expression.

The action S_E for the left-moving fermions λ^a which describe E is the standard action, which we record for completeness below,

$$S_E = \int_\Sigma d^2z \, \left(\lambda_a D_{\bar{z}} \lambda^a + F_{\bar{i}ib}{}^{a} \, \lambda_a \lambda^b \psi^{\bar{i}} \psi^i_{\bar{z}} \right) , \tag{3.3.7}$$

where

$$\lambda_a D_{\bar{z}} \lambda^a = \lambda_a \partial_{\bar{z}} \lambda^a + \lambda_a \, \partial_{\bar{z}} \phi^i A^a_{ib} \, \lambda^b \, . \tag{3.3.8}$$

In the above, A^a_{ib} is a holomorphic connection on E, having components only of type $(1,0)$ on Y, and $F_{\bar{i}ib}{}^{a}$ is the curvature of this connection. Since S_E is the usual action for the left-moving bundle fermions, and since Q acts in the usual way (3.3.2) on the fields appearing in S_E, this action is clearly Q-invariant.

A more nontrivial fact is that we can also write a Q-invariant action for the fermions χ_z and $\overline{\chi}$. Abstractly, the presence of χ_z and $\overline{\chi}$ in the half-linear model implies that we are dealing with the over-determined case $\dim M < \text{rank } V$ discussed in Section 2.3. So we must add some Q-invariant observable $\mathcal{O}$ involving χ_z and $\overline{\chi}$ to the action if we wish to compute something nontrivial in the half-linear model.

Physically, this Q-invariant observable $\mathcal{O}$ must introduce a kinetic term $D_{\overline{z}}\overline{\chi}\,\chi_z$ for χ_z and $\overline{\chi}$. To find a Q-invariant extension of this kinetic term, we follow the philosophy of Section 2.3 and attempt to solve $\delta\mathcal{O} = 0$ perturbatively. We begin by noting that the expression $\mathcal{O}^{(0)}$,

$$\begin{aligned} \mathcal{O}^{(0)} &= D_{\overline{z}}\overline{\chi}\,\chi_z - \psi^i_{\overline{z}}\,D_i J\,\chi_z \\ &= \left(\partial_{\overline{z}} + \partial_{\overline{z}}\phi^i A_i\right)\overline{\chi}\,\chi_z - \psi^i_{\overline{z}}\,(\partial_i + A_i) J\,\chi_z\,, \end{aligned} \tag{3.3.9}$$

is trivially invariant under the variations of $\overline{\chi}$ and $\psi^i_{\overline{z}}$. That is, in analogy to the finite-dimensional model, $\iota_s\mathcal{O}^{(0)} = 0$.

In the expression for $\mathcal{O}^{(0)}$ above, we have introduced the canonical holomorphic connection A_i on the line-bundle $\mathcal{O}(5)$ on Y. Because A_i depends on $\phi^{\bar{i}}$ as well as ϕ^i, we have that

$$\delta\mathcal{O}^{(0)} = \overline{\partial}\mathcal{O}^{(0)} \neq 0\,. \tag{3.3.10}$$

Rather,

$$\overline{\partial}\mathcal{O}^{(0)} = F_{\bar{i}i}\,\psi^{\bar{i}}\,\partial_{\overline{z}}\phi^i\,\overline{\chi}\,\chi_z - F_{\bar{i}i}\,\psi^{\bar{i}}\psi^i_{\overline{z}}\,J\,\chi_z\,, \tag{3.3.11}$$

where $F_{\bar{i}i}$ is the curvature of A_i. However, introducing $\mathcal{O}^{(1)}$,

$$\mathcal{O}^{(1)} = F_{\bar{i}i}\,\psi^{\bar{i}}\psi^i_{\overline{z}}\,\overline{\chi}\,\chi_z\,, \tag{3.3.12}$$

we easily see that

$$\overline{\partial}\mathcal{O}^{(0)} + \iota_s\mathcal{O}^{(1)} = 0\,. \tag{3.3.13}$$

Because A_i is a holomorphic connection on $\mathcal{O}(5)$, the curvature satisfies $\overline{\partial}F_{\bar{i}i} = 0$, so that $\overline{\partial}\mathcal{O}^{(1)} = 0$. Consequently, $\mathcal{O} = \mathcal{O}^{(0)} + \mathcal{O}^{(1)}$ is Q-invariant (but not Q-trivial).

Thus, we can add kinetic terms for χ_z and $\overline{\chi}$ to the action S by including

$$\begin{aligned} S_\chi &= \int_\Sigma d^2z\,\mathcal{O} = \int_\Sigma d^2z\,\left(\mathcal{O}^{(0)} + \mathcal{O}^{(1)}\right) \\ &= \int_\Sigma d^2z\,\left(D_{\overline{z}}\overline{\chi}\,\chi_z - \psi^i_{\overline{z}}\,D_i J\,\chi_z + F_{\bar{i}i}\,\psi^{\bar{i}}\psi^i_{\overline{z}}\,\overline{\chi}\,\chi_z\right)\,. \end{aligned} \tag{3.3.14}$$

Finally, we include in S the purely topological term which describes the action of the world-sheet instanton itself,

$$S_{top} = \int_{\Sigma} \Phi^*(\omega_{\mathbb{C}})\,, \tag{3.3.15}$$

where $\omega_{\mathbb{C}}$ is the complexified Kähler class of Y. This term simply reproduces the exponential factor in (3.1.2), but we include it for completeness.

Thus, the action for the half-linear model is

$$S = S_0 + S_E + S_\chi + S_{top}\,. \tag{3.3.16}$$

3.3.3 A Half-Linear Residue Theorem

We now show in the half-linear model that world-sheet instanton contributions to the superpotential W vanish by a residue theorem precisely analogous to the finite-dimensional residue theorem of Section 2.

Before we discuss a residue theorem for the half-linear model, though, we must first demonstrate the general fact that the half-linear model on Y is equivalent to the usual twisted non-linear sigma model on X. Only then does the residue theorem for the half-linear model imply the vanishing of the instanton contributions to W in the non-linear sigma model.

Relative to the non-linear model on X, the half-linear model on Y possesses additional world-sheet degrees of freedom described by the left-moving fermions χ_z, $\overline{\chi}$, and the complex boson and associated right-moving fermions describing fluctuations of the world-sheet normal to X in Y. We will denote these normal fields simply by ϕ, $\overline{\phi}$, $\psi_{\overline{z}}$, and $\overline{\psi}$, suppressing indices associated to the tangent bundle TY.

The additional world-sheet fields present in the half-linear model on Y relative to the non-linear model on X are all massive due to the terms in the action involving J and $\overline{J}$. For instance, the normal bosons ϕ and $\overline{\phi}$ gain a mass from the $\overline{J}J$ term that appears in the Q-trivial action S_0,

$$t\int_{\Sigma} d^2z\,\delta\left(\overline{J}\overline{\chi}\right) = t\int_{\Sigma} d^2z\,\left(\overline{J}J + \psi^{\overline{i}} D_{\overline{i}}\overline{J}\,\overline{\chi}\right)\,. \tag{3.3.17}$$

Similarly, the fermions $\psi_{\overline{z}}$, $\overline{\psi}$, χ_z, and $\overline{\chi}$ all gain masses from the $\overline{DJ}$ term in (3.3.17) and the conjugate term appearing in S_χ in (3.3.14).

The mass terms for ϕ, $\overline{\phi}$, $\psi_{\overline{z}}$, $\overline{\psi}$, χ_z, and $\overline{\chi}$ thus appear in the half-linear action as

$$\int_\Sigma d^2z \left(\overline{\phi}\,\overline{DJ}DJ\,\phi + \overline{\psi}\,\overline{DJ}\,\overline{\chi} - \psi_{\overline{z}}\,DJ\,\chi_z\right) , \tag{3.3.18}$$

where DJ is the (holomorphic) normal derivative of J along X, and we have absorbed the coupling t in (3.3.17) into $\overline{J}$. Because we assume that X is a non-singular quintic hypersurface, DJ is everywhere non-vanishing on X and consequently transforms in the trivial line-bundle on X. Also, because J is holomorphic and vanishes on X, the $\overline{\partial}$ operator of X acts on DJ as $\overline{\partial}DJ = [\overline{\partial}, D]J = 0$, so that DJ is holomorphic on X. As such, once we choose a non-vanishing holomorphic section of the trivial bundle on X, a choice which we must make in defining the fermionic measure of the path-integral, we can regard DJ as merely a constant mass parameter for the normal modes.

As we have already remarked, since $\overline{J}$ only appears in the half-linear model through the Q-trivial terms in (3.3.17), the half-linear model is invariant under deformations of $\overline{J}$. Scaling $\overline{J}$ by a large constant, the massive world-volume fields in (3.3.18) all acquire arbitrarily large masses. As such, we can integrate out these massive world-sheet fields at one-loop with arbitrary precision. From (3.3.18), we see that the one-loop contributions from the massive modes of ϕ, $\overline{\phi}$, $\psi_{\overline{z}}$, $\overline{\psi}$, χ_z, and $\overline{\chi}$ all cancel but for a finite, anomalous factor associated to the index of the $\overline{\partial}$ operator acting on the pull back of the normal bundle N to the world-sheet. This one-loop contribution can be absorbed into a renormalization of the string coupling constant and the Kähler class of X and is not relevant for the vanishing argument. Finally, upon integrating out the massive fields, we set them to zero in the half-linear action and in all observables, so that the half-linear model on Y clearly localizes to the non-linear model on X.

For completeness, we give in this paragraph a brief description of the renormalization. Massive modes with nonzero momentum cancel in the path integral, so the renormalization comes from the constant modes. The constant modes contribute a factor

$$\frac{1}{(\overline{DJ}DJ)^{n_1}}\,\overline{DJ}^{n_2}\,DJ^{n_3}\,, \tag{3.3.19}$$

where n_1 formally denotes the number of modes of ϕ, n_2 the number of modes of $\overline{\chi}$, and n_3 the number of modes of χ_z. Since ϕ and $\overline{\chi}$ both transform in the pull back $\Phi^*(N)$ of the normal bundle $N = \mathcal{O}(5)$ to the world-sheet, n_1 equals n_2. However, χ_z transforms in the bundle $K \otimes \Phi^*(N^*)$, and thus the difference $n_1 - n_3$ is equal to the index of the $\overline{\partial}$ operator on the world-sheet acting on the bundle $\Phi^*(N)$. If the world-sheet is a Riemann surface Σ of genus g and Φ is a map of degree d into X, then the index theorem (or simply the Riemann-Roch theorem) implies that $n_1 - n_3 = 5d + 1 - g$. Thus, in this situation the one-loop contribution of the massive modes is a factor $(1/DJ)^{(5d+1-g)}$. In the non-linear model on X, this one-loop contribution can be written as

$$\left(\frac{1}{DJ}\right)^{(5d+1-g)} = \exp\left[-\frac{1}{2\pi}\int_\Sigma \log\left(DJ\right)\,\left(5\Phi^*(\omega) + \frac{1}{2}R\right)\right], \tag{3.3.20}$$

where $\Phi^*(\omega)$ is the pull back of the Kähler class from X, which we assume is normalized to satisfy $\int_\Sigma \Phi^*(\omega) = 2\pi d$, and R is the world-sheet curvature, which satisfies $\int_\Sigma R = 4\pi(1-g)$. The expression in (3.3.20) manifestly represents the renormalization of the Kähler class of X and the string coupling constant upon integrating out the massive modes.

Having shown that the half-linear model on Y is equivalent to the non-linear sigma model on X, we now establish a residue theorem for the half-linear model which implies the vanishing of world-sheet instanton contributions to the superpotential W.

The half-linear model is a closer cousin to the usual world-sheet CFT description of the heterotic string than to the dual $D1$-brane description which we explored in Section 2.4. As such, in neither the world-sheet CFT nor the half-linear model can we compute W directly. Rather, because of the presence of three right-moving fermion zero-modes arising from fluctuations tangent to the world-sheet, we must indirectly probe for W by computing a cubic correlator of vertex operators on the world-sheet. In the terminology of Section 2.3, the half-linear model describes the over-determined case rank $\mathcal{V} > \dim \mathcal{M}$, due to the presence of the fermions χ_z and $\overline{\chi}$ in the model, but the section $s = (\partial_{\overline{z}}\phi^i, J(\phi^i), 0)$ still vanishes over a locus on $\mathcal{M}$ of complex dimension three, due to the $SL(2,\mathbb{C})$ action on the world-sheet. So we must insert a suitable observable $\mathcal{O}$, the cubic correlator of vertex operators, to compute something non-trivial.

The easiest way to probe for W is to compute the correlator $\langle RRR\rangle$, where R is the vertex operator for the (unique) Kähler modulus of Y. Explicitly,

$$R = \omega_{\bar{i}i}\partial_z\phi^i\psi^{\bar{i}}\,, \tag{3.3.21}$$

where $\omega_{\bar{i}i}$ is a harmonic representative of the Kähler class of Y, implying that R is Q-invariant. Since the half-linear model arises from a sigma model on Y (and only restricts to X when $\overline{J}$ is large), we must consider operators such as R which are actually defined on Y. Note as claimed that each of the three right-moving fermion zero-modes from $SL(2,\mathbb{C})$ can be soaked up with the fermion $\psi^{\bar{i}}$ that appears in R.

Of course, the Kähler class of Y determines by restriction the Kähler class of X and thus the radius of the compactification. The only dependence of W on this Kähler modulus is through the exponential factor $\exp\left(-\int_\Sigma \Phi^*(\omega)\right)$ arising from the classical action of the instanton itself. If we let $\mathcal{R}$ be the $\mathcal{N}=1$ chiral field in the low-energy effective theory associated to the Kähler modulus, then the correlator $\langle RRR\rangle$ computes the third derivative $\partial^3_{\mathcal{R}}W$ of W with respect to $\mathcal{R}$. Thus, given the simple exponential dependence of W on $\mathcal{R}$, the vanishing of W is equivalent to the vanishing of the correlator $\langle RRR\rangle$.

In the case of the finite-dimensional model in Section 2, we deduced a residue theorem by taking $t=0$. Although we have already interpreted the half-linear model as being formally analogous to the finite-dimensional model, unlike the case of the finite-dimensional model, we cannot simply take $t=0$ in the half-linear model to deduce that $\langle RRR\rangle$ vanishes. Clearly with no exponential suppression of the fluctuating modes in the half-linear model, the half-linear path-integral ceases to be defined.

However, in localizing the half-linear model on Y to the non-linear model on X, we have already used the fact that the $BRST$-invariance of the half-linear model implies that the model is formally independent of $\overline{J}$ as well as t. So rather than taking $t=0$, we consider taking $\overline{J}=0$ instead.

When $\overline{J} = 0$, the half-linear model no longer localizes on instantons contained in X. Instead, after integrating out at weak coupling all fluctuating modes of the fields, the half-linear model localizes onto the moduli space of instantons in Y.

If we restrict attention to a given instanton sector of degree d holomorphic maps Φ from $\Sigma = \mathbb{CP}^1$ to Y, then the moduli space of these instantons has a natural compactification to $\mathbb{CP}^{5d+4}$. Because of this compactness, the half-linear path-integral over each instanton sector can be defined even when $\overline{J} = 0$. Thus, the correlator $\langle RRR \rangle$ can be computed either at large $\overline{J}$, where it is proportional to W as computed in the non-linear model on X, or it can be computed at $\overline{J} = 0$, where we will easily see that it vanishes order by order in d. Morally speaking, the vanishing of the instanton contribution to the superpotential follows by applying the residue theorem of section 2 to the compact manifold $\mathbb{CP}^{5d+4}$. Rather than invoking this theorem (which could lead one to worry about singularities in $\mathbb{CP}^{5d+4}$), we will imitate its proof and just look at what happens at $\overline{J} = 0$.

We now review in detail how $\mathbb{CP}^{5d+4}$ arises as a compactification of the moduli space of degree d instantons in Y, following [38,63,64]. In fact, even though we focus here on the case $Y = \mathbb{CP}^4$, the existence of such a compactification generalizes whenever Y is a compact toric variety, as already applied in [63,64].

We first introduce homogeneous coordinates $[\Phi^0 : \cdots : \Phi^4]$ on Y and homogeneous coordinates $[U : V]$ on Σ. In terms of the homogeneous coordinates, any degree d holomorphic map $\Phi : \Sigma \to Y$ is specified by a set of homogeneous, degree d polynomials $\{p^0(U,V), \ldots, p^4(U,V)\}$,

$$\begin{aligned} \Phi^0 &= p^0(U,V) = a^0_0\, U^d + a^0_1\, U^{d-1} V + \cdots + a^0_d\, V^d\,, \\ &\;\vdots \\ \Phi^4 &= p^4(U,V) = a^4_0\, U^d + a^4_1\, U^{d-1} V + \cdots + a^4_d\, V^d\,. \end{aligned} \tag{3.3.22}$$

Each polynomial p^i is determined by its $d+1$ coefficients $(a^i_0, \ldots, a^i_d)$, and the space of polynomials $\{p^0, \ldots, p^4\}$ can be parametrized by these coefficients as $\mathbb{C}^{5(d+1)}$. Since the coordinates $[\Phi^0 : \cdots : \Phi^4]$ are merely homogeneous coordinates on Y, defined only up to scaling, an overall scaling of $\{p^0, \ldots, p^4\}$ does not affect the map Φ. Subtracting from

$\mathbb{C}^{5(d+1)}$ the point at the origin which does not describe an actual map into Y and then taking the quotient by the overall scaling, we find the projective space $\mathbb{CP}^{5d+4}$.

The only subtlety in this example is that, as just observed, $\Phi^0 = \cdots = \Phi^4 = 0$ does not correspond to any point in Y, so that the moduli space of instantons of degree d on Y is actually the subset of the parameter space $\mathbb{CP}^{5d+4}$ for which the polynomials $p^0, \ldots, p^4$ have no common zeroes on Σ. The polynomials which do have at least one common zero appear as an algebraic locus of codimension four in $\mathbb{CP}^{5d+4}$, since we must tune one complex parameter in any four of $p^0, \ldots, p^4$ to reach this locus. Thus, the moduli space of "true" instantons in Y is a complicated but nonetheless dense, open subset of $\mathbb{CP}^{5d+4}$. In particular, $\mathbb{CP}^{5d+4}$ gives a natural compactification of the true moduli space.

We now consider evaluating the correlator $\langle RRR \rangle$ in the half-linear model with $\overline{J} = 0$. In this case, if we consider the contribution to the correlator from the topological sector of degree d world-sheet maps, we must integrate over the moduli space of degree d instantons in Y described globally above.

This integral over the instanton moduli space is actually a supersymmetric integral just as in Section 2, since both the world-sheet bosons and fermions possess zero-modes when $\overline{J} = 0$. As our global discussion above implies, the bosons ϕ^i, $\phi^{\bar{i}}$, and their superpartners $\psi^{\bar{i}}$ all have $5d+4$ zero-modes. Of these $5d+4$ zero-modes, three zero-modes arise from the $SL(2, C)$ action on $\mathbb{CP}^1$ and are immediately soaked up by the cubic correlator. The other $5d+1$ zero-modes represent the non-trivial holomorphic deformations of degree d rational curves in Y. The left-moving fermion $\overline{\chi}$ also has $5d+1$ zero-modes, which arise from holomorphic sections of the bundle $\Phi^* (\mathcal{O}(5)) = \mathcal{O}(5d)$. Neither the right-moving fermions $\psi^i_{\bar{z}}$ nor the left-moving fermion χ_z have any zero-modes in the instanton background.

When $\overline{J}$ is non-vanishing, these $5d+1$ interesting modes of ϕ^i, $\phi^{\bar{i}}$, $\psi^{\bar{i}}$, and $\overline{\chi}$ enter the half-linear model action through the Q-trivial terms involving $\overline{J}$ in (3.3.17) and through the four-fermion interactions in (3.3.7) and (3.3.14). In the weak coupling limit $t \to \infty$, the four-fermion interactions are irrelevant, since they always involve the fermions $\psi^i_{\bar{z}}$ which have no zero-modes. So the only way to absorb the zero-modes of $\psi^{\bar{i}}$ and $\overline{\chi}$ is through the quadratic mass terms that arise from $\overline{J}$.

In fact, if we consider integrating out all of the fluctuating modes at weak coupling, to reduce the half-linear path-integral to a finite-dimensional supersymmetric integral over these $5d+1$ modes, then the Q-trivial terms involving $\overline{J}$ in (3.3.17) implicitly represent the same finite-dimensional action (3.2.5) which we considered in Section 2. In this case, the modes of $\psi^{\bar{i}}$ represent the fermionic coordinates $\theta^{\bar{i}}$, the modes of $\overline{\chi}$ represent the bundle fermions χ^α, and J implicitly determines a holomorphic section s of a rank $5d+1$ bundle over the moduli space of instantons in Y which vanishes precisely over those instantons contained in X.

Just as in Section 2, once we set $\overline{J}$ to zero, then the $5d+1$ fermion zero-modes of $\psi^{\bar{i}}$ and $\overline{\chi}$ cannot be absorbed when computing the correlator $\langle RRR \rangle$. Hence, $\langle RRR \rangle$ vanishes order by order for each sector of degree d maps. Finally, since our vanishing result follows exactly as the residue theorem in Section 2, we naturally interpret it as a residue theorem for instanton contributions to W.

3.3.4 Extension to the Linear Sigma Model

Just as in the finite-dimensional case, the argument for the vanishing of the instanton contributions to W in the half-linear model relies only upon the right-moving world-sheet supersymmetries and suitable compactness. These ingredients are also present in the $(0,2)$ linear sigma models themselves, so we should also be able to give a similar, direct argument for the vanishing of instanton contributions to W in these models. The reason for doing so is that the linear sigma model version of the argument applies to a somewhat different class of models — bundles constructed in a simple way from polynomials, but which do not necessarily extend over $Y = \mathbf{CP}^4$.

We now present just such an argument. Although the gist of the vanishing argument for the linear sigma model is exactly the same as for the half-linear model, we must present the details of the argument in a slightly different way, since the specifics of the linear model and the half-linear model are very different. Nonetheless, the fact that the general argument does extend from the half-linear to the linear model, despite the obvious differences between these world-sheet theories, indicates that this argument is robust.

As in the previous section, we once more focus on the case that X is a quintic hypersurface in $Y = \mathbb{CP}^4$. However, we now assume that the bundle E on X is a deformation of the holomorphic tangent bundle TX, which corresponds in the linear sigma model to a deformation away from the locus of theories with $(2,2)$ world-sheet supersymmetry. Since neither TX nor E pulls back (in any obvious way) from any bundle on Y, the compactifications cannot necessarily be described by the half-linear model.

Background

We must recall a few facts about the $(0,2)$ linear sigma model which describes heterotic compactification on X with gauge bundle E. Useful background can be found in [13], [38], and [39]. We will be rather brief in our description of the linear sigma model, both because this material is well-known and also because the vanishing argument which we present does not rely on many details of the model.

We first recall the field content for this model. (We ignore the decoupled current algebra degrees of freedom which represent the unbroken space-time gauge group.) On the $(2,2)$ locus itself, the linear sigma model which describes a quintic X in $\mathbb{CP}^4$ is a two-dimensional $U(1)$ gauge theory with five chiral superfields S^i, $i = 1, \ldots, 5$, of charge $+1$ and one chiral superfield P of charge -5.

Once this model is deformed away from the $(2,2)$ locus, the $(2,2)$ gauge multiplet decomposes into a $(0,2)$ gauge multiplet and a neutral $(0,2)$ chiral multiplet. Similarly, each $(2,2)$ chiral multiplet decomposes into a $(0,2)$ chiral multiplet and a $(0,2)$ Fermi multiplet. We denote again by S^i and P the corresponding $(0,2)$ chiral superfields, with components (s^i, ψ_+^i) and (p, ψ_+^0), and by Ψ_-^i and Ψ^0 the associated Fermi superfields, with components ψ_-^i and ψ_-^0.

The action of the $(0,2)$ model contains many interactions, but the only interactions relevant to our vanishing argument arise from the $(0,2)$ superpotential. Recall that these interactions can be written as integrals over half of $(0,2)$ superspace, in the form

$$S_J = \frac{1}{\sqrt{2}} \int_\Sigma d\theta^+ \left(\Psi^i_- J_i + \Psi^0_- J_0\right) + \text{h.c.} \tag{3.3.23}$$

In general, J_i and J_0 are holomorphic functions of the chiral fields S^i and P. More specifically, J_i and J_0 take the form

$$\begin{aligned} J_i &= P\left(\frac{\partial F}{\partial S^i} + F_i\right), \quad i = 1, \ldots, 5\,, \\ J_0 &= F\,, \end{aligned} \tag{3.3.24}$$

where $F = F(S^i)$ is a quintic polynomial in the S^i which determines X as a hypersurface in Y, and the F_i are quartic polynomials in the S^i that are assumed to satisfy $S^i F_i = 0$ and which determine E as a deformation of TX.

In terms of the component fields, the $(0,2)$ superpotential (3.3.23) leads to a bosonic potential U,

$$U = \sum_{i=0}^{5} |J_i|^2\,, \tag{3.3.25}$$

and Yukawa interactions of the form

$$\psi^i_- \psi^j_+ \frac{\partial J_i}{\partial S^j} + \psi^i_- \psi^0_+ \frac{\partial J_i}{\partial P} + \text{h.c.} \tag{3.3.26}$$

The superpotential (3.3.23) also preserves a right-moving $U(1)$ $\mathcal{R}$-symmetry, under which the lowest components of S^i and Ψ^i_- carry charge $+\frac{1}{5}$, and the lowest components of P and Ψ^0_- are neutral.

The vanishing theorem

The first step in our vanishing argument is to twist the $(0,2)$ linear sigma model so that the supersymmetry generator usually denoted $\overline{Q}_+$ becomes a scalar, exactly as described in [13]. Under this twisting, the world-sheet spin of each field is shifted by $-\frac{1}{2}J_R + \frac{1}{10}Q$, where J_R is the $\mathcal{R}$-symmetry generator and Q is the gauge-symmetry generator. Since the gauge current corresponding to Q is of the form $\{\overline{Q}_+, \ldots\}$, the fact that $\frac{1}{10}Q$ appears in the twist is irrelevant and is merely for convenience, so that upon twisting all fields have integral or half-integral world-sheet spins.

Upon twisting the model, the spins of the bosons s^i are unaffected, but the boson p now has spin $-\frac{1}{2}$ and transforms as a section of $K^{\frac{1}{2}} \otimes \mathcal{L}^{-5}$ on $\Sigma = \mathbb{CP}^1$. Here K is the canonical bundle on Σ as earlier, and $\mathcal{L} = \mathcal{O}(d)$ is the line-bundle on Σ associated to a degree d instanton configuration in the gauge field. Also, just as in the half-twisted model, the fermions ψ_+^i and $\psi_+^{\bar{i}}$, for $i = 1, \ldots, 5$, now have spins $+1$ and 0 and transform as sections of $\overline{K} \otimes \mathcal{L}$ and $\overline{\mathcal{L}}$. Finally, the left-moving fermions ψ_-^i and $\psi_-^{\bar{i}}$ are unaffected by the twisting and transform as sections of $K^{\frac{1}{2}} \otimes \mathcal{L}$ and $K^{\frac{1}{2}} \otimes \overline{\mathcal{L}}$.

To proceed with the argument, we must compute the linear sigma model correlator analogous to $\langle RRR \rangle$ in the half-linear model. As explained in [13], the linear sigma model representative of the vertex operator R describing deformations of the Kähler class of Y (hence also X) is λ_-, the left-moving gaugino. This fact can be motivated by observing that, since the Kähler class of Y is represented in the linear model by a Fayet-Iliopolous D-term, the linear sigma model representative for R must come from the $(0,2)$ gauge multiplet. The supersymmetry and $\mathcal{R}$-symmetry then determine this representative to be λ_-. So we must compute the instanton contributions to $\langle \lambda_- \lambda_- \lambda_- \rangle$ in the linear sigma model.

As in the half-linear model, the twisted linear model is formally invariant under deformations of $\overline{J}_i$, $i = 0, \ldots, 5$, so we consider taking $\overline{J}_i = 0$. At first glance, one might worry that this deformation would be singular in the linear model, since at least in the untwisted theory, the boson p has an unbounded zero-mode which only receives a mass from the potential term U in (3.3.25). However, because p has spin $-\frac{1}{2}$ in the twisted theory, this dangerous zero-mode is not present. This observation was also central to the vanishing argument of [13], so we certainly expect it to play a role in our argument as well. Thus, we can compute $\langle \lambda_- \lambda_- \lambda_- \rangle$ in the theory with $\overline{J}_i = 0$, provided we perform the twist.

In the half-linear model, once we performed the analogous deformation by taking $\overline{J} = 0$, we easily saw that the correlator $\langle RRR \rangle$ vanished due to the presence of excess fermion zero-modes which could no longer be absorbed through world-sheet interactions. We will now argue that the correlator $\langle \lambda_- \lambda_- \lambda_- \rangle$ vanishes when $\overline{J}_i = 0$ in the linear sigma model, again due to excess fermion zero-modes.

The relevant zero-modes arise from the fermions $\psi_+^{\bar{i}}$ and ψ_-^i, for $i = 1, \ldots, 5$. In the background of a degree d instanton, each fermion $\psi_+^{\bar{i}}$ has $d+1$ zero-modes, and each fermion ψ_-^i has d zero-modes (and the conjugate partners of these fermions have no zero-modes).

To show that these fermion zero-modes cannot be absorbed in computing the correlator $\langle \lambda_- \lambda_- \lambda_- \rangle$ with $\overline{J}_i = 0$, we first make a few general remarks about the computation of $\langle \lambda_- \lambda_- \lambda_- \rangle$ even when the $\overline{J}_i$ are not assumed to vanish. First, since all kinetic terms in the linear model are $\overline{Q}_+$-trivial, we can by a field rescaling assume that the couplings appearing in J_i and $\overline{J}_i$ are arbitrarily small. Hence we can compute $\langle \lambda_- \lambda_- \lambda_- \rangle$ perturbatively in J_i and $\overline{J}_i$.

As a special case of our vanishing result, we now observe that $\langle \lambda_- \lambda_- \lambda_- \rangle$ trivially vanishes when $J_i = \overline{J}_i = 0$. In this case, the model with no superpotential describes, instead of X, the total space of the line-bundle $\mathcal{O}(-5)$ over $\mathbb{CP}^4$. As such, the model possesses a classical global symmetry which rotates the fiber of this space leaving fixed the base. Under this symmetry, the superfields S^i and Ψ_-^i, for $i = 1, \ldots, 5$, transform with charge $+1$ while all other fields are uncharged. In particular, the gaugino λ_- is uncharged, which distinguishes this global symmetry from the $\mathcal{R}$-symmetry.

The fermion zero-modes we discussed above are relevant precisely because they cause this classical symmetry to be anomalous. Due to these zero-modes, regardless of the degree d, the path-integral measure transforms with net charge $+5$ under this symmetry. This anomaly immediately implies that $\langle \lambda_- \lambda_- \lambda_- \rangle$ vanishes in the theory with no superpotential. For instance, computing $\langle \lambda_- \lambda_- \lambda_- \rangle$ perturbatively at weak coupling, all interactions respect the classical symmetry and so there is no way to absorb the fermion zero-modes by pulling down fermion interaction terms from the action. This fact is why the detailed structure of the linear model is largely irrelevant for our argument.

We now consider the general case that J_i and $\overline{J}_i$ are non-zero. Since the superpotential breaks the classical symmetry we used above, the fermion Yukawa terms involving J_i and $\overline{J}_i$ in (3.3.26) are candidates to soak up the zero-modes of $\psi_+^{\bar{i}}$ and ψ_-^i above. However, whatever interaction terms we bring down from the action to soak up the fermion zero-modes, the anomaly implies that these terms must carry net charge -5 to cancel the charge of the measure. We now observe from (3.3.26) that the interactions involving J_i all carry charge $+5$ and those involving $\overline{J}_i$ carry charge -5.

Thus, when $\overline{J}_i$ vanishes, the zero-modes of the fermions $\psi_+^{\bar{i}}$ and ψ_-^i cannot be absorbed, since perturbation theory in the J_i can only bring down interactions of positive charge. This observation merely reflects the fact that the twisted fermions $\psi_+^{\bar{i}}$, which give rise to the anomaly, appear in the Yukawa couplings involving $\overline{J}_i$, not J_i, in (3.3.26). Thus $\langle\lambda_-\lambda_-\lambda_-\rangle$ vanishes in an arbitrary degree d instanton background, and instantons in the linear sigma model do not contribute to the space-time superpotential.

Of course, when $\overline{J}_i$ is non-zero, the linear model sums over individual instantons in X, and the contribution of each instanton should generically be non-zero. Our argument is consistent with this fact, since insertions of the Yukawa couplings involving both J_i and $\overline{J}_i$ can carry the proper charge to absorb the zero-modes.

This vanishing argument is at its heart very similar to the vanishing argument of [13] that we reviewed in the Introduction. A key fact there is that W transforms as a section of a line-bundle of strictly negative curvature on the moduli space of the low-energy effective theory. Now in the context of the present argument, the complex coefficients which define the quintic polynomial F and the quartic polynomials F_i, and thus appear as couplings in the J_i, can be considered as projective coordinates on the moduli space of complex structures of X and E.

Our perturbative argument above can be rephrased as a selection rule for the dependence of the correlator $\langle\lambda_-\lambda_-\lambda_-\rangle$ on these coefficients. This selection rule follows from formally assigning the complex coefficients appearing in F and F_i charge -5 under the anomalous symmetry, so that formally the J_i are uncharged. The anomaly implies that the correlator $\langle\lambda_-\lambda_-\lambda_-\rangle$, as a function of these coefficients, transforms homogeneously with charge $+5$. As a result, the selection rule implies that $\langle\lambda_-\lambda_-\lambda_-\rangle$ (and hence W) must transform as a section of a line-bundle of strictly negative curvature over the complex structure moduli space which these coefficients parametrize. In this language, our vanishing theorem follows simply because, when $\overline{J}_i$ vanishes, a perturbative calculation of $\langle \lambda_- \lambda_- \lambda_- \rangle$ in terms of the J_i can only produce a polynomial in the complex coefficients, which has negative charge under the anomalous symmetry and does not have the required pole on the complex structure moduli space.

3.4 Families of Membrane Instantons

The vanishing result which we derived for world-sheet instanton contributions to the superpotential is a manifestation of the rigidity inherent in holomorphic objects. As an interesting contrast to this result, we now consider how M-theory membranes which wrap a continuous family of supersymmetric three-cycles in a manifold X of G_2 holonomy contribute to the superpotential. The approach which we take here is very similar to our discussion of $D1$-brane contributions to the superpotential at the end of Section 2. We note that the superpotential contribution from an isolated membrane in X has already been thoroughly discussed in [46].

Just as in the case of a $D1$-brane which wraps a holomorphic curve, the worldvolume theory on a membrane which wraps a supersymmetric three-cycle on X is naturally twisted. Unlike the case of the $D1$-brane though, in the case of a supersymmetric membrane, the sector of the worldvolume theory describing fluctuations in X is topological, as opposed to holomorphic, in character. This fact could hardly be otherwise, since X is not a complex manifold, but it represents a key distinction between $D1$-brane and membrane instantons.

Thus, if $\mathcal{C}$ represents a continuous family of supersymmetric membrane configurations within the space $\mathcal{M}$ of all membrane configurations in X, then the contribution to the superpotential from the family $\mathcal{C}$ only depends upon topological data associated to $\mathcal{C}$. Our main result here is to show that the contribution of the family $\mathcal{C}$ to the superpotential is proportional to the Euler character $\chi(\mathcal{C})$ of $\mathcal{C}$.

Our analysis ignores singularities. We suspect that it remains valid even if some of the membrane instantons parametrized by $\mathcal{C}$ are singular, as long as $\mathcal{C}$ itself is smooth.

The membrane worldvolume theory

Just as in the Introduction, the most elegant way to determine the superpotential contribution from a membrane instanton (or a family of such instantons) is to compute the partition function of the membrane worldvolume theory. The structure of this theory is largely determined by supersymmetry. More specifically, it is determined by the requirement that only supersymmetric membrane configurations contribute to the partition function. So we begin by recalling a few facts about supersymmetric three-cycles in X.

To describe which three-cycles in X are supersymmetric, we first recall that X, as a manifold of G_2 holonomy, possesses a canonical, covariantly constant three-form ϕ. Then, as emphasized generally in [65], the supersymmetric three-cycles are those which are calibrated by ϕ and hence are of minimal volume within each homology class. That is, if Σ is a supersymmetric three-cycle, then the calibration condition states that on Σ,

$$\phi|_\Sigma = \mathrm{vol}|_\Sigma \,, \tag{3.4.1}$$

where $\mathrm{vol} = \frac{1}{7}\phi\wedge\star\phi$ is the volume form associated to the metric on X.

Just as for supersymmetric $D1$-brane configurations, the supersymmetric membrane configurations in X can be characterized as the critical points of a superpotential Ψ on $\mathcal{M}$. Ψ is defined in a manner precisely analogous to the superpotential for $D1$-brane configurations in a Calabi-Yau threefold. Thus, we define $\Psi(\Sigma)$ for any three-cycle Σ by

$$\Psi(\Sigma) - \Psi(\Sigma_0) = \frac{1}{12}\int_B \star\phi\,. \tag{3.4.2}$$

Here $\star\phi$ is the four-form on X dual to ϕ, Σ_0 is a fixed representative in the homology class of Σ, and B is a four-cycle bounding $\Sigma - \Sigma_0$. Again, $\Psi(\Sigma)$ is defined only up to an additive constant, depending on the choices of Σ_0 and B.

But again, the fact that Ψ is only defined up to an additive constant does not concern us, as this constant does not affect the location of the critical points, for which $\delta\Psi = 0$. In terms of local coordinates x^i, $i = 1, \ldots, 7$, on X,

$$\delta\Psi(\Sigma) = \frac{1}{3}\int_\Sigma \star\phi_{ijkl}\,\delta x^i\, dx^j\wedge dx^k\wedge dx^l\,. \tag{3.4.3}$$

Thus, $\delta\Psi(\Sigma) = 0$ when $\star\phi_{ijkl}\, dx^j \wedge dx^k \wedge dx^l = 0$ on Σ. As observed in [66], this condition is equivalent to the condition (3.4.1) that Σ be calibrated by ϕ. So the critical points of Ψ correspond to supersymmetric three-cycles in X.

Thus, $\delta\Psi$ is a one-form on the space $\mathcal{M}$ of arbitrary membrane configurations in X which vanishes precisely over the supersymmetric configurations. So in this sense, $\delta\Psi$ plays much the same role as the section s we introduced in Section 2, and we expect the action of the worldvolume theory on a supersymmetric membrane to be expressed in terms of $\delta\Psi$, much as the action (3.2.5) is expressed in terms of s.

Unlike s, though, $\delta\Psi$ is not holomorphic, and the space $\mathcal{M}$ of membrane configurations is not even complex, even on-shell. As a result, the supersymmetry algebra on the membrane worldvolume takes a form slightly different from the supersymmetry (3.2.1) considered in Section 2.

We focus on the sector of the worldvolume theory which describes fluctuations of the membrane in X. As explicitly demonstrated in [46], this sector is automatically twisted when the membrane wraps a supersymmetric cycle Σ. Normal fluctuations of the membrane in X are described on the worldvolume by four real bosons y^i, $i = 1, \ldots, 4$, taking values in the (real) normal bundle N of Σ in X. Associated to these four bosons are four fermions $\psi^i_{\dot\alpha}$ also taking values in N and transforming as right-moving Weyl fermions in $\mathbb{R}^4$, as indicated by the $\dot\alpha$ index.

The worldvolume theory on the supersymmetric membrane then possesses two scalar supercharges $\overline{Q}_{\dot\alpha}$. The action of these supercharges on the worldvolume fields can be neatly summarized by introducing $(0|2)$ superfields Y^i, where

$$Y^i = y^i + \theta^{\dot\alpha}\psi^i_{\dot\alpha} + \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\theta^{\dot\alpha}\theta^{\dot\beta}F^i\,. \tag{3.4.4}$$

In defining the superfield Y^i, we have introduced an auxiliary boson F^i taking values in N. Even though the membrane worldvolume is three-dimensional, the appropriate superspace is only the $(0|2)$ superspace because, just as for the $D1$-brane, we regard the bosonic fields y^i as being an infinite set of tangential coordinates to the membrane configuration space $\mathcal{M}$ at the point corresponding to a given supersymmetric membrane configuration.

In the $(0|2)$ superspace, the action of the supercharges $\overline{Q}_{\dot{\alpha}}$ is exceedingly simple. Namely, the supercharges $\overline{Q}_{\dot{\alpha}}$ act as the fermionic derivatives $\partial_{\dot{\alpha}}$, corresponding to the component transformations

$$\delta_{\dot{\alpha}} y^i = \psi^i_{\dot{\alpha}} \, , \quad \delta_{\dot{\alpha}} \psi^i_{\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}} F^i \, , \quad \delta_{\dot{\alpha}} F^i = 0 \, . \tag{3.4.5}$$

We note that $\{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\}$ trivially vanishes.

The supersymmetry algebra, along with the requirement that the membrane partition function localize on configurations for which $\delta\Psi = 0$, determines the form of the worldvolume action on a supersymmetric membrane. As for the $D1$-brane, this action is really the leading order action for fluctuations around a supersymmetric configuration — but given the topological nature of the membrane worldvolume theory, the leading order action certainly suffices to determine the partition function.

When written in terms of the $(0|2)$ superspace, the membrane worldvolume action thus appears as

$$\begin{aligned} S &= \int_\Sigma d^2\theta \, \phi \left(\frac{1}{2} g_{ij}(Y) \, \epsilon^{\dot{\alpha}\dot{\beta}} \partial_{\dot{\beta}} Y^i \partial_{\dot{\alpha}} Y^j + \Psi(Y) \right) \\ &= \int_\Sigma \phi \left(\frac{1}{2} g^{ij} \frac{\delta\Psi}{\delta y^i} \frac{\delta\Psi}{\delta y^j} + 2 \frac{D^2 \Psi}{D y^i D y^j} (\psi^i \psi^j) + R_{ikjl} (\psi^i \psi^j)(\psi^k \psi^l) \right) . \end{aligned} \tag{3.4.6}$$

In this expression, g_{ij} is the metric on X, R_{ikjl} is the curvature, and the canonical three-form ϕ appears simply to represent the volume-form on the supersymmetric three-cycle Σ. We also note from (3.4.3) that $\delta\Psi/\delta y^i$ is actually a three-form on Σ, and so we have implicitly used the induced metric to dualize $\delta\Psi/\delta y^i$ to a scalar above. Finally, we have used the shorthand $(\psi^i \psi^j)$ to indicate the $SU(2)$ singlet combination $\frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}} \psi^i_{\dot{\beta}} \psi^j_{\dot{\alpha}}$, and in passing to the second line of (3.4.6) we integrated out the auxiliary bosons F^i.

The membrane worldvolume action (3.4.6) has a very familiar look. Formally, we can interpret this action as the reduction to $0+0$ dimensions of the standard supersymmetric quantum mechanics [67] on the membrane configuration space $\mathcal{M}$, with Morse function Ψ. As is well known, the partition function of supersymmetric quantum mechanics on a finite-dimensional Riemannian manifold M computes the Euler class $\chi(M)$ of M. Thus, our claim that the membrane partition function is proportional to the Euler class $\chi(\mathcal{C})$ of the family $\mathcal{C}$ follows almost immediately now, though we still discuss this result in detail below.

We can also compare the form of the membrane worldvolume theory to the form of the $D1$-brane worldvolume theory (or more generally to the holomorphic models we considered in Section 2). Upon integrating out the auxiliary bosons F^i, the supersymmetries on the membrane worldvolume act as

$$\begin{aligned} \delta_{\dot\alpha}\, y^i &= \psi^i_{\dot\alpha}\,, \\ \delta_{\dot\alpha}\, \psi^i_{\dot\beta} &= -\Gamma^i_{jk}\psi^j_{\dot\alpha}\psi^k_{\dot\beta} + \frac{1}{2}\epsilon_{\dot\alpha\dot\beta} g^{ij}\,\frac{\delta\Psi}{\delta y^j}\,. \end{aligned} \tag{3.4.7}$$

In the above, Γ^i_{jk} is the usual torsion-free affine connection associated to the metric g_{ij} on X; this connection must appear so that the fermions $\psi^i_{\dot\alpha}$ transform covariantly under reparametrizations of the y^i. Comparing the supersymmetries (3.4.7) and action (3.4.6) of the membrane worldvolume theory to the general supersymmetry (3.2.1) and action (3.2.5) from Section 2, we see that the membrane worldvolume theory is just a real, $N = 2$ version of the holomorphic models relevant for world-sheet instantons which we considered earlier. Clearly the one-form $\delta\Psi$ on $\mathcal{M}$ plays exactly the same role as the holomorphic section s on the complex manifold M, and from (3.4.6) we see that at weak coupling the membrane partition function localizes on the zeroes of $\delta\Psi$. We also note that the $N = 2$ supersymmetry present in the membrane worldvolume theory determines a canonical choice for the measure of the membrane partition function, as all bosons are paired by supersymmetry with all fermions in (3.4.5). So there is no analogue here of the section g which was necessary to define a measure for the holomorphic models.

The membrane partition function

Our simple description of the membrane worldvolume theory allows us to easily evaluate the membrane partition function, even in the degenerate case that the membranes wrap a continuous family of supersymmetric three-cycles in X.

We first observe that, because $\overline{Q}_{\dot\alpha} = \partial_{\dot\alpha}$, the worldvolume action (3.4.6) is evidently $\overline{Q}_{\dot\alpha}$-trivial. As a result, the membrane partition function Z is clearly topological in character.

In particular, Z is unchanged if we multiply $\phi \to t\,\phi$, so that taking t to be large we can evaluate Z at weak coupling. Furthermore, Z is unchanged under deformations of the metric g_{jk} and even the one-form $\delta\Psi$. This latter observation is in clear contrast to the holomorphic models in Section 2, which were unchanged under deformations of $\overline{s}$ but certainly depended upon s.

Thus, we suppose that X contains a continuous family of supersymmetric three-cycles. Then the vanishing locus of $\delta\Psi$ on $\mathcal{M}$ contains a component $\mathcal{C}$ of positive dimension representing this continuous family. To evaluate Z for membranes which wrap three-cycles in $\mathcal{C}$, we simply make a generic deformation of $\delta\Psi$, which is small in the sense that $\delta\Psi$ still grows sufficiently fast away from $\mathcal{C}$ so that Z is defined. Under such a deformation, we lift the degeneracy of $\delta\Psi$, which now has a finite set of isolated zeroes on $\mathcal{C}$.

At weak coupling, we can directly evaluate the contribution to Z from each non-degenerate zero of $\delta\Psi$ as a one-loop integral over the fluctuating bosons and fermions. Generally speaking, if P is such a zero, then the contribution to Z from P takes the form

$$Z_P = Z(\mathcal{N}) \cdot Z(\mathcal{C})_P \,, \tag{3.4.8}$$

where $Z(\mathcal{N})$ represents the one-loop integral over modes normal to $\mathcal{C}$, and $Z(\mathcal{C})_P$ represents the one-loop integral over the finite number of modes tangent to $\mathcal{C}$ at P. Because of the topological invariance of Z, the factor $Z(\mathcal{N})$ in (3.4.8) does not depend on P, so that

$$Z = Z(\mathcal{N}) \cdot \sum_P Z(\mathcal{C})_P \,. \tag{3.4.9}$$

Clearly the second factor in (3.4.9) captures the interesting dependence of the superpotential on $\mathcal{C}$. In the Gaussian approximation, we can express the contribution $Z(\mathcal{C})_P$ from each point P as

$$Z(\mathcal{C})_P = \frac{\det_{\mathcal{C}}\left(\partial_i\partial_j\Psi\right)(P)}{\left|\det_{\mathcal{C}}\left(\partial_i\partial_j\Psi\right)(P)\right|} = \pm 1 \,, \tag{3.4.10}$$

where the subscript $\mathcal{C}$ indicates that the determinants are only evaluated over the modes tangent to $\mathcal{C}$. Geometrically, we recognize the expression (3.4.10) as the index of the vector field $\nabla\Psi$ (projected onto $T\mathcal{C}$) at the point P, where it vanishes. Thus,

$$\sum_P Z(\mathcal{C})_P = \chi(\mathcal{C})\,, \tag{3.4.11}$$

and Z is proportional to the Euler character $\chi(\mathcal{C})$ of $\mathcal{C}$ as claimed. We could also derive this result, without explicitly deforming $\delta\Psi$ to lift its degeneracy, by using the four-fermion interaction in (3.4.6) to absorb the fermion zero-modes tangent to $\mathcal{C}$, producing the Chern-Weil representation of the Euler character.

Finally, we remark that the factor $Z(\mathcal{N})$, studied in [46] for the case of an isolated membrane instanton, is simply the formal generalization of (3.4.10) from the phase of a determinant on the tangential directions of $\mathcal{C}$ to the normal directions. $Z(\mathcal{N})$ can thus be expressed as the sign of the Dirac operator acting on the membrane worldvolume spinors multiplied by a factor coming from the C-field.

Chapter 4

Non-Abelian Localization For Chern-Simons Theory

4.1 Introduction

Chern-Simons gauge theory is remarkable for the deep connections it bears to an array of otherwise disparate topics in mathematics and physics. For instance, Chern-Simons theory is intimately related to the theory of knot invariants and the topology of three-manifolds [15,68], to two-dimensional rational conformal field theory [16] via a holographic correspondence, to three-dimensional quantum gravity [69–71], to the open string field theory of the topological A-model [72], and via a large N duality to the Gromov-Witten theory of non-compact Calabi-Yau threefolds [73–77].

Of course, Chern-Simons theory is also a topological gauge theory, though of a very exotic sort. In the case of a more conventional topological gauge theory such as topological Yang-Mills theory on a Riemann surface or on a four-manifold (for a review of both topics, see [51]), the theory can be fundamentally interpreted in terms of the cohomology ring of some classical moduli space of connections. In this sense, such gauge theories are themselves essentially classical. In contrast, Chern-Simons theory is intrinsically a quantum theory, and it is exotic precisely because it does not admit a general mathematical interpretation in terms of the cohomology of some classical moduli space of connections.

Yet if we consider Chern-Simons theory not on a general three-manifold M but only on three-manifolds which are of a simple sort and which perhaps carry additional geometric structure, then we might expect Chern-Simons theory itself to simplify. In particular, we might hope that the theory in this case admits a more conventional mathematical interpretation in terms of the cohomology of some classical moduli space of connections.

For instance, in the very special case that M is just the product of S^1 and a Riemann surface Σ, so that $M = S^1 \times \Sigma$, then the partition function Z of Chern-Simons theory on M does have a well-known topological interpretation. In this case, Z is the dimension of the Chern-Simons Hilbert space, obtained from canonical quantization on $\mathbb{R} \times \Sigma$. In turn, this Hilbert space can be interpreted geometrically as the space of global holomorphic sections of a certain line bundle over the moduli space $\mathcal{M}_0$ of flat connections on Σ.

If we consider for simplicity Chern-Simons theory with gauge group $G = SU(r+1)$ at level k, then the relevant line bundle over $\mathcal{M}_0$ is the k-th power of a universal determinant line $\mathcal{L}$ on $\mathcal{M}_0$. Of course, the moduli space $\mathcal{M}_0$ is singular at the points corresponding to the reducible flat connections on Σ. However, suitably interpreted, the index theorem in combination with the Kodaira vanishing theorem for the higher cohomology of $\mathcal{L}^k$ still yields a topological expression for Z,

$$Z(k) \,=\, \dim H^0(\mathcal{M}_0, \mathcal{L}^k) \,=\, \chi(\mathcal{M}_0, \mathcal{L}^k) \,=\, \int_{\mathcal{M}_0} \exp\left(k\,\Omega'\right) \mathrm{Td}(\mathcal{M}_0)\,, \tag{4.1.1}$$

where $\Omega' = c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$ and $\mathrm{Td}(\mathcal{M}_0)$ is the Todd class of $\mathcal{M}_0$.

In this chapter, we show that the Chern-Simons partition function has an analogous topological interpretation on a related but much broader class of three-manifolds. Specifically, we consider the case that M is a Seifert manifold, so that M can be succinctly described as the total space of a *nontrivial* circle bundle over a Riemann surface Σ,

$$S^1 \longrightarrow M \xrightarrow{\pi} \Sigma\,, \tag{4.1.2}$$

where, as we later explain, Σ is generally allowed to have orbifold points and the circle bundle is allowed to be a corresponding orbifold bundle.

In this case, our fundamental result is to reinterpret the Chern-Simons partition function as a topological quantity determined entirely by a suitable equivariant cohomology ring on the moduli space of flat connections on M. Because the moduli space of flat connections on M is directly related to the moduli space of solutions of the Yang-Mills equation on Σ, our result implies that Chern-Simons theory on M can be also be interpreted as a two-dimensional topological theory on Σ akin, in a way which we make precise, to two-dimensional Yang-Mills theory. This two-dimensional interpretation of Chern-Simons theory on M has also been noted recently by Aganagic and collaborators in [78], where the theory is identified with a q-deformed version of two-dimensional Yang-Mills theory.

Of course, physical Yang-Mills theory on a Riemann surface Σ also has a well-known topological interpretation in terms of intersection theory on the moduli space $\mathcal{M}_0$ of flat connections on Σ. This interpretation follows from the technique of non-abelian localization, as applied to the Yang-Mills path integral [18]. In an analogous fashion, we arrrive at our new interpretation of Chern-Simons theory by applying non-abelian localization to the Chern-Simons path integral,

$$Z(k) \;=\; \int \mathcal{D}A\, \exp\left[i\frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right]. \tag{4.1.3}$$

As we recall in Section 4, non-abelian localization provides a method for computing symplectic integrals of the canonical form

$$Z(\epsilon) \;=\; \int_X \exp\left[\Omega - \frac{1}{2\epsilon}\,(\mu,\mu)\right]. \tag{4.1.4}$$

Here X is an arbitrary symplectic manifold with symplectic form Ω. We assume that a Lie group H acts on X in a Hamiltonian fashion, with moment map $\mu : X \to \mathfrak{h}^*$, where $\mathfrak{h}^*$ is the dual of the Lie algebra $\mathfrak{h}$ of H. Finally, $(\cdot,\cdot)$ is an invariant quadratic form on $\mathfrak{h}$ and dually on $\mathfrak{h}^*$ which we use to define the action $S = \frac{1}{2}(\mu,\mu)$, and ϵ is a coupling parameter.

As we briefly review in Section 2, the path integral of Yang-Mills theory on a Riemann surface immediately takes the canonical form in (4.1.4), where the affine space of all connections on a fixed principal bundle plays the role of X and where the group of gauge transformations plays the role of H. In contrast, the path integral (4.1.3) of Chern-Simons theory on a Seifert manifold is not manifestly of this required form. Nonetheless, in Section 3 we show that this path integral can be cast into the form (4.1.4) for which non-abelian localization applies. More abstractly, we show that Chern-Simons theory on a Seifert manifold has a symplectic interpretation generalizing the classic interpretation due to Atiyah and Bott [79] of two-dimensional Yang-Mills theory.

Because the path integral of Chern-Simons theory on a Seifert manifold M assumes the canonical form (4.1.4), we deduce as an immediate corollary that the path integral localizes on critical points of the Chern-Simons action, which are the flat connections on M. In fact, this observation has been made previously by Lawrence and Rozansky [17,80] (and later generalized by Mariño in [81]) as an entirely empirical statement deduced from the known formula for the exact partition function.

Considering $SU(2)$ Chern-Simons theory on a Seifert homology sphere M, Lawrence and Rozansky managed to recast the known formula for $Z(k)$, which initially involves an unwieldy sum over the integrable representations of an $SU(2)$ WZW model at level k, into a simple sum of contour integrals and residues which can be formally identified with the contributions from the flat connections on M in the stationary phase approximation to the path integral.

A very simple example of a Seifert manifold is S^3, by virtue of the Hopf fibration over $\mathbb{CP}^1$. The result of Lawrence and Rozansky in the case of $SU(2)$ Chern-Simons theory on S^3 then amounts to rewriting the well-known expression for $Z(k)$ as below,

$$Z(k) \;=\; \sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi}{k+2}\right) \;=\; \frac{1}{2\pi i}\,\mathrm{e}^{-\frac{i\pi}{k+2}}\int_{-\infty}^{+\infty}\!dx\;\sinh^2\left(\frac{1}{2}\mathrm{e}^{\frac{i\pi}{4}}\,x\right)\,\exp\left(-\frac{(k+2)}{8\pi}x^2\right). \tag{4.1.5}$$

We note that, when the hyperbolic sine is expressed as a sum of exponentials, the integral in (4.1.5) becomes a sum of elementary Gaussian integrals which conspire to produce the standard expression for $Z(k)$. Because the only flat connection on S^3 is the trivial connection, the integral over x in (4.1.5) is to be identified with the stationary phase contribution from the trivial connection to the path integral.

So one immediate application of our work here is to provide an underlying mathematical explanation for the phenomenological results in [17,80,81]. In fact, we will apply localization to the Chern-Simons path integral to derive directly the expression of Lawrence and Rozansky in (4.1.5) for the partition function on S^3. One amusing aspect of this computation is that we will see the famous shift in the level $k \to k+2$.

In order to perform concrete computations in Chern-Simons theory using localization, we must have a thorough understanding of the local symplectic geometry near each flat connection. As we will see, this local geometry shares important features with the local geometry near the higher, unstable critical points of Yang-Mills theory on a Riemann surface.

Thus, as a warmup for our computations in Chern-Simons theory, we begin in Section 4 by discussing localization for Yang-Mills theory. We first review the computation in [18] of the contribution to the path integral from flat Yang-Mills connections, corresponding to the stable minima of the Yang-Mills action, and then we extend this result to compute precisely the contributions from the higher, unstable critical points as well. Localization at the unstable critical points of Yang-Mills theory has been studied previously in the physics literature by Blau and Thompson [82] and (most recently) in the mathematics literature by Woodward [83], but we find it useful to supplement these references with another discussion more along the lines of [18]. Of course, the roots of our work on localization trace back to the beautiful equivariant interpretation by Atiyah and Bott [20] of the Duistermaat-Heckman formula [19].

In Section 5 we then apply localization to perform path integral computations in Chern-Simons theory on a Seifert manifold. As mentioned above, these computations depend on the nature of the local symplectic geometry near each critical point, and for illustration we consider two extreme cases.

First, we consider localization at the trivial connection on a Seifert homology sphere. In this case, the first homology group of M is zero, $H_1(M, \mathbb{Z}) = 0$, and the trivial connection is an isolated flat connection. On the other hand, all constant gauge transformations on M fix the trivial connection, and this large isotropy group, isomorphic to the gauge group G itself, plays an important role in the localization. Here we directly derive a formula found by Lawrence and Rozansky in [17] and generalized by Mariño in [81].

Second, we consider localization on a smooth component of the moduli space of flat connections. Such a component consists of irreducible connections, for which the isotropy group arises solely from the center of G. In this case, we derive a formula originally obtained by Rozansky in [80] by again working empirically from the known formula for the partition function.

Finally, although we will not elaborate on this perspective here, one of the original motivations for our study of localization in Chern-Simons theory was to place computations in this theory into a theoretical framework analogous to the framework of abelian localization in the topological A-model of open and closed strings (see Chapter 9 of [84] for a nice mathematical review of abelian localization in the closed string A-model).

This chapter of the thesis is based on [6].

4.2 The Symplectic Geometry of Yang-Mills Theory on a Riemann Surface

A central theme of this chapter is the close relationship between Chern-Simons theory on a Seifert manifold M and Yang-Mills theory on the associated Riemann surface Σ. Thus, as a prelude to our discussion of the path integral of Chern-Simons theory on M, we begin by recalling how the path integral of Yang-Mills theory on Σ can be understood as a symplectic integral of the canonical form (4.1.4).

In fact, we start by considering the path integral of Yang-Mills theory on a compact Riemannian manifold Σ of *arbitrary* dimension, so that

$$\begin{aligned} Z(\epsilon) &= \frac{1}{\mathrm{Vol}(\mathcal{G}(P))}\left(\frac{1}{2\pi\epsilon}\right)^{\Delta_{\mathcal{G}(P)}/2}\int_{\mathcal{A}(P)}\mathcal{D}A\,\exp\left[\frac{1}{2\epsilon}\int_\Sigma \mathrm{Tr}\left(F_A\wedge\star F_A\right)\right],\\ \Delta_{\mathcal{G}(P)} &= \dim\mathcal{G}(P)\,. \end{aligned} \tag{4.2.1}$$

Here $F_A = dA + A\wedge A$ is the curvature of the connection A. We assume that the Yang-Mills gauge group G is compact, connected, and simple. If $G = SU(r+1)$, then "Tr" in (4.2.1) denotes the trace in the fundamental representation. With our conventions, A is an anti-hermitian element of the Lie algebra of $SU(r+1)$, so that the trace determines a negative-definite quadratic form. For more general G, "Tr" denotes the unique invariant, negative-definite quadratic form on the Lie algebra $\mathfrak{g}$ of G which is normalized so that, for simply-connected G, the Chern-Simons level k in (4.1.3) obeys the conventional integral quantization. Of course, the parameter ϵ is related to the Yang-Mills coupling g via $\epsilon = g^2$.

In order to define Z formally, we fix a principal G-bundle P over Σ. Then the space $\mathcal{A}(P)$ over which we integrate is the space of connections on P. The group $\mathcal{G}(P)$ of gauge transformations acts on $\mathcal{A}(P)$, and we have normalized Z in (4.2.1) by dividing by the volume of $\mathcal{G}(P)$ and a formal power of ϵ. As we review in Section 4, this normalization of Z is the natural normalization when Σ is a Riemann surface and we apply non-abelian localization to compute Z.

The space $\mathcal{A}(P)$ is an affine space, which means that, if we choose a particular basepoint A_0 in $\mathcal{A}(P)$, then we can identify $\mathcal{A}(P)$ with its tangent space at A_0. This tangent space is the vector space of sections of the bundle $\Omega^1_\Sigma \otimes \mathrm{ad}(P)$ of one-forms on Σ taking values in the adjoint bundle associated to P. In other words, an arbitrary connection A on P can be written as $A = A_0 + \eta$ for some section η of $\Omega^1_\Sigma \otimes \mathrm{ad}(P)$.

Of course, to discuss an integral over $\mathcal{A}(P)$ even formally, we must also discuss the measure $\mathcal{D}A$ that appears in (4.2.1). Because the space $\mathcal{A}(P)$ is affine, we can define $\mathcal{D}A$ up to an overall multiplicative constant by taking any translation-invariant measure on $\mathcal{A}(P)$.

In general, the Yang-Mills action is only defined once we choose a metric on Σ, which induces a corresponding duality operator $\star$, as appears in (4.2.1). This duality operator $\star$ induces a metric on $\mathcal{A}(P)$ such that if η is any tangent vector to $\mathcal{A}(P)$, then the norm of η is defined by

$$(\eta, \eta) \;=\; -\int_\Sigma \mathrm{Tr}\,(\eta\wedge\star\eta)\;. \tag{4.2.2}$$

Thus, a convenient way to represent the path integral measure and to fix its normalization is to take $\mathcal{D}A$ to be the Riemannian measure induced by the metric (4.2.2) on $\mathcal{A}(P)$. We also use the operator $\star$ to define a similar invariant metric on $\mathcal{G}(P)$, which formally determines the volume of $\mathcal{G}(P)$.

Although we generally require a metric on Σ to define physical Yang-Mills theory, when Σ is a Riemann surface we actually need much less geometric structure to define the theory. In this case, to define the Yang-Mills action in (4.2.1) we only require a duality operator $\star$ which relates the zero-forms and the two-forms on Σ. In turn, to define such an operator we require only a symplectic structure with associated symplectic form ω on Σ, so that $\star$ is defined by $\star 1 = \omega$.

The symplectic form ω is invariant under all area-preserving diffeomorphisms of Σ, and this large group acts as a symmetry of two-dimensional Yang-Mills theory. More precisely, this symmetry group is "large" in the sense that its complexification is the full group of orientation-preserving diffeomorphisms of Σ [85]. This fact is fundamentally responsible for the topological nature of two-dimensional Yang-Mills theory.

Furthermore, when Σ is a Riemann surface, the affine space $\mathcal{A}(P)$ acquires additional geometric structure. First, $\mathcal{A}(P)$ has a natural symplectic form Ω. If η and ξ are any two tangent vectors to $\mathcal{A}(P)$, then Ω is defined by

$$\Omega(\eta, \xi) = -\int_\Sigma \mathrm{Tr}\,(\eta \wedge \xi)\ . \tag{4.2.3}$$

Clearly Ω is closed and non-degenerate. Second, $\mathcal{A}(P)$ has a natural complex structure. This complex structure is associated to the duality operator $\star$ itself, since $\star^2 = -1$ when acting on the tangent space of $\mathcal{A}(P)$. Finally, the metric on $\mathcal{A}(P)$ is manifestly Kahler with respect to this symplectic form and complex structure, since we see that the metric defined by (4.2.2) can be rewritten as $\Omega(\,\cdot\,, \star\,\cdot\,)$.

An important consequence of the fact that the metric on $\mathcal{A}(P)$ is Kahler when Σ is a Riemann surface is that the Riemannian measure $\mathcal{D}A$ on $\mathcal{A}(P)$ is actually the same as the symplectic measure defined by Ω. If X is a symplectic manifold of dimension $2n$ with symplectic form Ω, then the symplectic measure on X is given by the top-form $\Omega^n/n!$. This measure can be represented uniformly for X of arbitrary dimension by the expression $\exp(\Omega)$, where we implicitly pick out from the series expansion of the exponential the term which is of top degree on X. Consequently, because the Riemannian and the symplectic measures on $\mathcal{A}(P)$ agree, we can formally replace $\mathcal{D}A$ in the Yang-Mills path integral (4.2.1) by the expression $\exp(\Omega)$, as in the canonical symplectic integal (4.1.4). This natural symplectic measure on $\mathcal{A}(P)$ makes no reference to the metric on Σ.

The Yang-Mills Action as the Square of the Moment Map

Of course, as an affine space, $\mathcal{A}(P)$ is pretty boring. What makes Yang-Mills theory interesting is the fact that $\mathcal{A}(P)$ is acted on by the group $\mathcal{G}(P)$ of gauge transformations. In fact, another special consequence of considering Yang-Mills theory on a Riemann surface is that the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$ is Hamiltonian with respect to the symplectic form Ω.

To recall what the Hamiltonian condition implies, we consider the general situation that a connected Lie group H with Lie algebra $\mathfrak{h}$ acts on a symplectic manifold X preserving the symplectic form Ω. The action of H on X is then Hamiltonian when there exists an algebra homomorphism from $\mathfrak{h}$ to the algebra of functions on X under the Poisson bracket. The Poisson bracket of functions f and g on X is given by $\{f, g\} = -V_f(g)$, where V_f is the Hamiltonian vector field associated to f. This vector field is determined by the relation $df = \iota_{V_f}\Omega$, where ι_{V_f} is the interior product with V_f. More explicitly, in local canonical coordinates on X, the components of V_f are determined by f as $V_f^m = -(\Omega^{-1})^{mn}\,\partial_n f$, where Ω^{-1} is an "inverse" to Ω that arises by considering the symplectic form as an isomorphism $\Omega : TM \to T^*M$ with inverse $\Omega^{-1} : T^*M \to TM$. In coordinates, Ω^{-1} is defined by $(\Omega^{-1})^{lm}\,\Omega_{mn} = \delta^l_n$, and $\{f, g\} = \Omega_{mn} V_f^m V_g^n$. The algebra homomorphism from the Lie algebra $\mathfrak{h}$ to the algebra of functions on X under the Poisson bracket is then specified by a moment map $\mu : X \longrightarrow \mathfrak{h}^*$, under which an element ϕ of $\mathfrak{h}$ is sent to the function $\langle \mu, \phi \rangle$ on X, where $\langle \cdot, \cdot \rangle$ is the dual pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$.

The moment map by definition satisfies the relation

$$d\langle \mu, \phi \rangle = \iota_{V(\phi)}\Omega\,, \tag{4.2.4}$$

where $V(\phi)$ is the vector field on X which is generated by the infinitesimal action of ϕ. In terms of μ, the Hamiltonian condition then becomes the condition that μ also satisfy

$$\{\langle \mu, \phi \rangle, \langle \mu, \psi \rangle\} = \langle \mu, [\phi, \psi] \rangle\,. \tag{4.2.5}$$

Geometrically, the equation (4.2.5) is an infinitesimal expression of the condition that the moment map μ commute with the action of H on X and the coadjoint action of H on $\mathfrak{h}^*$.

Returning from this abstract discussion to the case of Yang-Mills theory on Σ, we first consider the moment map for the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$, as originally discussed in [79]. Elements of the Lie algebra of $\mathcal{G}(P)$ are represented by sections of the adjoint bundle $\mathrm{ad}(P)$ on Σ, so if ϕ is such a section then the corresponding vector field $V(\phi)$ on $\mathcal{A}(P)$ is given as usual by

$$V(\phi) \,=\, d_A\phi \,=\, d\phi + [A, \phi]\,. \tag{4.2.6}$$

We then compute directly using (4.2.3),

$$\iota_{V(\phi)}\Omega \,=\, -\int_\Sigma \mathrm{Tr}\,(d_A\phi\wedge\delta A) \,=\, \int_\Sigma \mathrm{Tr}\,(\phi\, d_A\delta A) \,=\, \delta\int_\Sigma \mathrm{Tr}\,(F_A\phi)\;. \tag{4.2.7}$$

Here we write δ for the exterior derivative acting on $\mathcal{A}(P)$, so that, for instance, δA is regarded as a one form on $\mathcal{A}(P)$. Thus, the relation (4.2.4) determines, up to an additive constant, that the moment map μ for the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$ is

$$\mu = F_A\,. \tag{4.2.8}$$

Here we regard F_A, being a section of $\Omega^2_\Sigma \otimes \mathrm{ad}(P)$, as an element of the dual of the Lie algebra of $\mathcal{G}(P)$.

One can then check directly that μ in (4.2.8) satisfies the condition (4.2.5) that it arise from a Lie algebra homomorphism, and this condition fixes the arbitrary additive constant that could otherwise appear in μ to be zero. Thus, $\mathcal{G}(P)$ acts in a Hamiltonian fashion on $\mathcal{A}(P)$ with moment map given by $\mu = F_A$. In particular, if we introduce the obvious positive-definite, invariant quadratic form on the Lie algebra of $\mathcal{G}(P)$, defined by

$$(\phi, \phi) \,=\, -\int_\Sigma \mathrm{Tr}\,(\phi\wedge\star\phi)\;, \tag{4.2.9}$$

then the Yang-Mills action S is proportional to the square of the moment map,

$$S \,=\, -\frac{1}{2}\int_\Sigma \mathrm{Tr}\,(F_A\wedge\star F_A) \,=\, \frac{1}{2}\,(\mu, \mu)\;. \tag{4.2.10}$$

As a result, the path integral of Yang-Mills theory on Σ can be recast completely in terms of the symplectic data associated to the Hamiltonian action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$,

$$Z(\epsilon) \,=\, \frac{1}{\mathrm{Vol}(\mathcal{G}(P))} \left(\frac{1}{2\pi\epsilon}\right)^{\Delta_{\mathcal{G}(P)}/2} \int_{\mathcal{A}(P)} \exp\left[\Omega - \frac{1}{2\epsilon}\,(\mu,\mu)\right], \tag{4.2.11}$$

precisely as in (4.1.4).

4.3 The Symplectic Geometry of Chern-Simons Theory on a Seifert Manifold

In this section, we explain how the path integral of Chern-Simons theory on a Seifert manifold can be recast as a symplectic integral of the canonical form (4.1.4) which is suitable for non-abelian localization. More generally, we explain some beautiful facts about the symplectic geometry of Chern-Simons theory on a Seifert manifold.

To set up notation, we consider Chern-Simons theory on a three-manifold M with compact, connected, simply-connected, and simple gauge group G. With these assumptions, any principal G-bundle P on M is necessarily trivial, and we denote by $\mathcal{A}$ the affine space of connections on the trivial bundle. We denote by $\mathcal{G}$ the group of gauge transformations acting on $\mathcal{A}$.

We begin with the Chern-Simons path integral,

$$\begin{aligned} Z(\epsilon) &= \frac{1}{\mathrm{Vol}(\mathcal{G})} \left(\frac{1}{2\pi\epsilon}\right)^{\Delta_{\mathcal{G}}} \int_{\mathcal{A}} \mathcal{D}A\, \exp\left[\frac{i}{2\epsilon}\int_M \mathrm{Tr}\left(A{\wedge}dA + \frac{2}{3}A{\wedge}A{\wedge}A\right)\right], \\ \epsilon &= \frac{2\pi}{k}\,, \qquad \Delta_{\mathcal{G}} = \dim\mathcal{G}\,. \end{aligned} \tag{4.3.1}$$

We have introduced a coupling parameter ϵ by analogy to the canonical integral in (4.1.4), and we have included a number of formal factors in Z. First, we have the measure $\mathcal{D}A$ on $\mathcal{A}$, which we define up to norm as a translation-invariant measure on $\mathcal{A}$. As is standard, we have also divided the path integral by the volume of the gauge group $\mathcal{G}$. Finally, to be fastidious, we have normalized Z by a formal power of ϵ which, as in (4.2.1), will be natural in defining Z by localization.

4.3.1 A New Formulation of Chern-Simons Theory

At the moment, we make no assumption about the three-manifold M. However, if M is an S^1 bundle over a Riemann surface Σ, or an orbifold thereof, then to reduce Chern-Simons theory on M to a topological theory on Σ we must eventually decouple one of the three components of the gauge field A. This observation motivates the following general reformulation of Chern-Simons theory, which proves to be key to the rest of the chapter.

In order to decouple one of the components of A, we begin by choosing a one-dimensional subbundle of the cotangent bundle T^*M of M. Locally on M, this choice can be represented by the choice of an everywhere non-zero one-form κ, so that the subbundle of T^*M consists of all one-forms proportional to κ. However, if t is any non-zero function, then clearly κ and $t\,\kappa$ generate the same subbundle in T^*M. Thus, our choice of a one-dimensional subbundle of T^*M corresponds locally to the choice of an equivalence class of one-forms under the relation

$$\kappa \sim t\,\kappa\,. \tag{4.3.2}$$

We note that the representative one-form κ which generates the subbundle need only be defined locally on M. Globally, the subbundle might or might not be generated by a non-zero one-form which is defined everywhere on M; this condition depends upon whether the sign of κ can be consistently defined under (4.3.2) and thus whether the subbundle is orientable or not.

We now attempt to decouple one of the three components of A. Specifically, our goal is to reformulate Chern-Simons theory on M as a theory which respects a new local symmetry under which A varies as

$$\delta A \,=\, \sigma\kappa\,. \tag{4.3.3}$$

Here σ is an *arbitrary* section of the bundle $\Omega^0_M \otimes \mathfrak{g}$ of Lie algebra-valued functions on M.

The Chern-Simons action certainly does not respect the local "shift" symmetry in (4.3.3). However, we can trivially introduce this shift symmetry into Chern-Simons theory if we simultaneously introduce a new scalar field Φ on M which transforms like A in the adjoint representation of the gauge group. Under the shift symmetry, Φ transforms as

$$\delta\Phi \,=\, \sigma\,. \tag{4.3.4}$$

Now, if κ in (4.3.3) is scaled by a non-zero function t so that $\kappa \to t\,\kappa$, then this rescaling can be absorbed into the arbitrary section σ which also appears in (4.3.3) so that the transformation law for A is well-defined. However, from the transformation (4.3.4) of Φ under the same symmetry, we see that because we absorb t into σ we must postulate an inverse scaling of Φ, so that $\Phi \to t^{-1}\Phi$. As a result, although κ is only locally defined up to scale, the product $\kappa\,\Phi$ is well-defined on M.

The only extension of the Chern-Simons action which now incorporates both Φ and the shift symmetry is the Chern-Simons functional $CS(\,\cdot\,)$ of the shift invariant combination $A - \kappa\,\Phi$. Thus, we consider the theory with action

$$S(A,\Phi) \,=\, CS(A - \kappa\,\Phi)\,, \tag{4.3.5}$$

or more explicitly,

$$S(A,\Phi) \,=\, CS(A) - \int_M \Big[2\kappa\wedge\mathrm{Tr}(\Phi F_A) - \kappa\wedge d\kappa\,\mathrm{Tr}(\Phi^2)\Big]\,. \tag{4.3.6}$$

To proceed, we play the usual game used to derive field theory dualities by path integral manipulations, as for T-duality in two dimensions [86,87] or abelian S-duality in four dimensions [88]. We have introduced a new degree of freedom, namely Φ, into Chern-Simons theory, and we have simultaneously enlarged the symmetry group of the theory so that this degree of freedom is completely gauge trivial. As a result, we can either use the shift symmetry (4.3.4) to gauge Φ away, in which case we recover the usual description of Chern-Simons theory, or we can integrate Φ out, in which case we obtain a new description of Chern-Simons theory which respects the action of the shift symmetry (4.3.3) on A.

A Contact Structure on M

Hitherto, we have supposed that the one-dimensional subbundle of T^*M represented by κ is arbitrary, but at this point we must impose an important geometric condition on this subbundle. From the action $S(A, \Phi)$ in (4.3.6), we see that the term quadratic in Φ is multiplied by the local three-form $\kappa \wedge d\kappa$. In order for this quadratic term to be everywhere non-degenerate on M, so that we can easily perform the path integral over Φ, we require that $\kappa \wedge d\kappa$ is also everywhere non-zero on M.

Although κ itself is only defined locally and up to rescaling by a non-zero function t, the condition that $\kappa \wedge d\kappa \neq 0$ pointwise on M is a globally well-defined condition on the subbundle generated by κ. For when κ scales as $\kappa \to t\,\kappa$ for any non-zero function t, we easily see that $\kappa \wedge d\kappa$ also scales as $\kappa \wedge d\kappa \to t^2\, \kappa \wedge d\kappa$. Thus, the condition that $\kappa \wedge d\kappa \neq 0$ is preserved under arbitrary rescalings of κ.

The structure which we thus introduce on M is the choice of a one-dimensional subbundle of T^*M for which any local generator κ satisfies $\kappa \wedge d\kappa \neq 0$ at each point of M. This geometric structure, which appears so naturally here, is known as a contact structure [89–91]. More generally, on an arbitrary manifold M of odd dimension $2n+1$, a contact structure on M is defined as a one-dimensional subbundle of T^*M for which the local generator κ satisfies $\kappa \wedge (d\kappa)^n \neq 0$ everywhere on M.

In many ways, a contact structure is the analogue of a symplectic structure for manifolds of odd dimension. The fact that we must choose a contact structure on M for our reformulation of Chern-Simons theory is thus closely related to the fact, mentioned previously, that we must choose a symplectic structure on the Riemann surface Σ in order to define Yang-Mills theory on Σ.

We will say a bit more about contact structures on Seifert manifolds later, but for now, we just observe that, by a classic theorem of Martinet [92], any compact, orientable three-manifold possesses a contact structure. (We note that, because $\kappa \wedge d\kappa \to t^2\, \kappa \wedge d\kappa$ under a local rescaling of κ and because t^2 is always positive, the sign of the local three-form $\kappa \wedge d\kappa$ is well-defined. So any three-manifold with a contact structure is necessarily orientable.)

Path Integral Manipulations

Thus, we choose a contact structure on the three-manifold M, and we consider the theory defined by the path integral

$$\begin{aligned} Z(\epsilon) \;=\; & \frac{1}{\mathrm{Vol}(\mathcal{G})}\,\frac{1}{\mathrm{Vol}(\mathcal{S})}\,\left(\frac{1}{2\pi\epsilon}\right)^{\Delta_{\mathcal{G}}} \times \\ & \times \int \mathcal{D}A\,\mathcal{D}\Phi\;\exp\left[\frac{i}{2\epsilon}\left(CS\,(A) - \int_M 2\kappa\wedge\mathrm{Tr}\,(\Phi F_A) \;+\; \int_M \kappa\wedge d\kappa\,\mathrm{Tr}\left(\Phi^2\right)\right)\right]. \end{aligned} \tag{4.3.7}$$

Here the measure $\mathcal{D}\Phi$ is defined independently of any metric on M by the invariant, positive-definite quadratic form

$$(\Phi,\Phi) = -\int_M \kappa\wedge d\kappa\,\mathrm{Tr}\left(\Phi^2\right)\,, \tag{4.3.8}$$

which is invariant under the scaling $\kappa \to t\,\kappa$, $\Phi \to t^{-1}\,\Phi$. We similarly use this quadratic form to define formally the volume of the group $\mathcal{S}$ of shift symmetries, as appears in the normalization of (4.3.7).

Using the shift symmetry (4.3.4), we can fix $\Phi = 0$ trivially, with unit Jacobian, and the resulting group integral over $\mathcal{S}$ produces a factor of $\mathrm{Vol}(\mathcal{S})$ to cancel the corresponding factor in the normalization of $Z(\epsilon)$. Hence, the new theory defined by (4.3.7) is fully equivalent to Chern-Simons theory.

On the other hand, because the field Φ appears only quadratically in the action (4.3.6), we can also perform the path integral over Φ directly. Upon integrating out Φ, the new action $S(A)$ for the gauge field becomes

$$S(A) \;=\; \int_M \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \;-\; \int_M \frac{1}{\kappa\wedge d\kappa}\mathrm{Tr}\Big[(\kappa\wedge F_A)^2\Big]\,. \tag{4.3.9}$$

We find it convenient to abuse notation slightly by writing "$1/\kappa\wedge d\kappa$" in (4.3.9). To explain this notation precisely, we observe that, as $\kappa\wedge d\kappa$ is nonvanishing, we can always write $\kappa\wedge F_A = \varphi\,\kappa\wedge d\kappa$ for some function φ on M taking values in the Lie algebra $\mathfrak{g}$. Thus, we set $\kappa\wedge F_A/\kappa\wedge d\kappa = \varphi$, and the second term in $S(A)$ becomes $\int_M \kappa\wedge\mathrm{Tr}\,(F_A\varphi)$. As our notation in (4.3.9) suggests, this term is invariant under the transformation $\kappa \to t\,\kappa$, since φ transforms as $\varphi \to t^{-1}\,\varphi$.

By construction, the new action $S(A)$ in (4.3.9) is invariant under the action of the shift symmetry (4.3.3) on A. We can directly check this invariance once we note that, under the shift symmetry, the expression $\kappa\wedge F_A$ transforms as

$$\kappa\wedge F_A \;\longrightarrow\; \kappa\wedge F_A + \sigma\,\kappa\wedge d\kappa\,. \tag{4.3.10}$$

The partition function $Z(\epsilon)$ now takes the form

$$Z(\epsilon) = \frac{1}{\mathrm{Vol}(\mathcal{G})} \frac{1}{\mathrm{Vol}(\mathcal{S})} \left(\frac{-i}{2\pi\epsilon}\right)^{\Delta_{\mathcal{G}}/2} \times \\ \times \int_{\mathcal{A}} \mathcal{D}A \, \exp\left[\frac{i}{2\epsilon}\left(\int_M \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) - \int_M \frac{1}{\kappa\wedge d\kappa}\mathrm{Tr}\left[(\kappa\wedge F_A)^2\right]\right)\right], \tag{4.3.11}$$

where the Gaussian integral over Φ cancels some factors of $2\pi\epsilon$ in the normalization of Z. As is standard, in integrating over Φ we assume that the integration contour has been slightly rotated off the real axis, effectively giving ϵ a small imaginary part, to regulate the oscillatory Gaussian integral. Thus, the theory described by the path integral (4.3.11) is fully equivalent to Chern-Simons theory, but now one component of A manifestly decouples.

4.3.2 Contact Structures on Seifert Manifolds

Our reformulation of Chern-Simons theory in (4.3.11) applies to any three-manifold M with a specified contact structure. However, in order to apply non-abelian localization to Chern-Simons theory on M, we require that M has additional symmetry.

Specifically, we require that M admits a locally-free $U(1)$ action, which means that the generating vector field on M associated to the infinitesimal action of $U(1)$ is nowhere vanishing. A free $U(1)$ action on M clearly satisfies this condition, but more generally it is satisfied by any $U(1)$ action such that no point on M is fixed by all of $U(1)$ (at such a point the generating vector field would vanish). Such an action need not be free, since some points on M could be fixed by a cyclic subgroup of $U(1)$. The class of three-manifolds which admit a $U(1)$ action of this sort are precisely the Seifert manifolds [93].

To proceed further to a symplectic description of Chern-Simons theory, we now restrict attention to the case that M is a Seifert manifold. We first review a few basic facts about such manifolds, for which a complete reference is [93].

M Admits a Free $U(1)$ Action

For simplicity, we begin by assuming that the three-manifold M admits a free $U(1)$

action. In this case, M is the total space of a circle bundle over a Riemann surface Σ,

$$S^1 \longrightarrow M \xrightarrow{\pi} \Sigma\,, \tag{4.3.12}$$

and the free $U(1)$ action simply arises from rotations in the fiber of (4.3.12). The topology of M is completely determined by the genus g of Σ and the degree n of the bundle. Assuming that the bundle is nontrivial, we can always arrange by a suitable choice of orientation for M that $n \geq 1$.

At this point, one might wonder why we restrict attention to the case of nontrivial bundles over Σ. As we now explain, in this case M admits a natural contact structure which is invariant under the action of $U(1)$. As a result, our reformulation of Chern-Simons theory in (4.3.11) still respects this crucial symmetry of M.

To describe this $U(1)$ invariant contact structure on M, we simply exhibit an invariant one-form κ, defined globally on M, which satisfies the contact condition that $\kappa \wedge d\kappa$ is nowhere vanishing. To describe κ, we begin by choosing a symplectic form ω on Σ which is normalized so that

$$\int_\Sigma \omega \,=\, 1\,. \tag{4.3.13}$$

Regarding M as the total space of a principal $U(1)$-bundle, we take κ to be a connection on this bundle (and hence a real-valued one-form on M) whose curvature satisfies

$$d\kappa = n\,\pi^* \omega\,, \tag{4.3.14}$$

where we recall that $n \geq 1$ is the degree of the bundle. For a nice, explicit description of κ in this situation, see the description of the angular form in §6 of [52].

We let R (for "rotation") be the non-vanishing vector field on M which generates the $U(1)$ action and which is normalized so that its orbits have unit period. By the fundamental properties of a connection, κ is invariant under the $U(1)$ action and satisfies $\langle \kappa, R \rangle = 1$. Here we use $\langle\,\cdot\,,\,\cdot\,\rangle$ generally to denote the canonical dual pairing. Thus, κ pulls back to a non-zero one-form which generates the integral cohomology of each S^1 fiber of M, and we immediately see from (4.3.14) that $\kappa \wedge d\kappa$ is everywhere non-vanishing on M so long as the bundle is nontrivial.

For future reference, we note that the integral of $\kappa\wedge d\kappa$ over M is determined as follows. Because κ satisfies $\langle\kappa, R\rangle = 1$, where R is the generator of the $U(1)$ action whose orbits correspond to the S^1 fibers over Σ in (4.3.12), the integral of κ over any such fiber is given by

$$\int_{S^1} \kappa \,=\, 1\,. \tag{4.3.15}$$

Upon integrating over the S^1 fiber of M, we see from (4.3.13), (4.3.14), and (4.3.15) that

$$\int_M \kappa\wedge d\kappa \,=\, n\int_M \kappa\wedge\pi^*\omega \,=\, n\int_\Sigma \omega \,=\, n\,. \tag{4.3.16}$$

Orbifold Generalization

Of course, in the above construction we have assumed that M admits a free $U(1)$ action, which is a more stringent requirement than the condition that no point of M is completely fixed by the $U(1)$ action. However, an arbitrary Seifert manifold does admit an orbifold description precisely analogous to the description of M as a principal $U(1)$-bundle over a Riemann surface. This point of view is taken in a nice paper by Furuta and Steer [94] for an application somewhat related to ours, and we follow their basic exposition below.

To generalize our previous discussion to the case of an arbitrary Seifert manifold, we simply replace the Riemann surface Σ with an orbifold, and we replace the principal $U(1)$-bundle over Σ with its orbifold counterpart. Concretely, the orbifold base $\widehat{\Sigma}$ of M is now described by a Riemann surface of genus g with N marked points p_j, $j = 1, \ldots, N$, at which the coordinate neighborhoods are modeled not on $\mathbb{C}$ but on $\mathbb{C}/\mathbb{Z}_{\alpha_j}$ for some cyclic group $\mathbb{Z}_{\alpha_j}$, which acts on the local coordinate z at p_j as

$$z \mapsto \zeta\cdot z\,, \qquad \zeta = \mathrm{e}^{2\pi i/\alpha_j}\,. \tag{4.3.17}$$

The choice of the particular orbifold points p_j is topologically irrelevant, and the orbifold base $\widehat{\Sigma}$ can be completely specified by the genus g and the set of integers $\{\alpha_1, \ldots, \alpha_N\}$.

We now consider a line V-bundle over $\widehat{\Sigma}$. Such an object is precisely analogous to a complex line bundle, except that the local trivialization over each orbifold point p_j of $\widehat{\Sigma}$ is now modeled on $\mathbb{C}\times\mathbb{C}/\mathbb{Z}_{\alpha_j}$, where $\mathbb{Z}_{\alpha_j}$ acts on the local coordinates (z, s) of the base and fiber as

$$z \mapsto \zeta \cdot z\,, \qquad s \mapsto \zeta^{\beta_j} \cdot s\,, \qquad \zeta = \mathrm{e}^{2\pi i/\alpha_j}\,, \tag{4.3.18}$$

for some integers $0 \le \beta_j < \alpha_j$.

Given such a line V-bundle over $\widehat{\Sigma}$, an arbitrary Seifert manifold M can be described as the total space of the associated S^1 fibration. Of course, we require that M itself is smooth. This condition implies that each pair of integers (α_j, β_j) above must be relatively prime so that the local action (4.3.18) of the orbifold group $\mathbb{Z}_{\alpha_j}$ on $\mathbb{C} \times S^1$ is free (in particular, we require $\beta_j \neq 0$ above).

The $U(1)$ action on M again arises from rotations in the fibers over $\widehat{\Sigma}$, but this action is no longer free. Rather, the points in the S^1 fiber over each ramification point p_j of $\widehat{\Sigma}$ are fixed by the cyclic subgroup $\mathbb{Z}_{\alpha_j}$ of $U(1)$, due to the orbifold identification in (4.3.18).

Once the integers $\{\beta_1, \ldots, \beta_N\}$ are fixed, the topological isomorphism class of a line V-bundle on $\widehat{\Sigma}$ is specified by a single integer n, the degree. Thus, in total, the description of an arbitrary Seifert manifold M is given by the Seifert invariants

$$\Big[g; n; (\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)\Big]\,, \quad \gcd(\alpha_j, \beta_j) = 1\,. \tag{4.3.19}$$

Because the basic notions of bundles, connections, curvatures, and (rational) characteristic classes generalize immediately from smooth manifolds to orbifolds [95,96], our previous construction of an invariant contact form κ as a connection on a principal $U(1)$-bundle immediately generalizes to the orbifold situation here. In this case, if $\widehat{\mathcal{L}}$ denotes the line V-bundle over $\widehat{\Sigma}$ which describes M, with Seifert invariants (4.3.19), then $\widehat{\mathcal{L}}$ is nontrivial so long as its Chern class is non-zero (and positive by convention),

$$c_1(\widehat{\mathcal{L}}) \,=\, n + \sum_{j=1}^{N} \frac{\beta_j}{\alpha_j} \,>\, 0\,, \tag{4.3.20}$$

which generalizes our previous condition that $n \geq 1$. In particular, n can now be any integer such that the expression in (4.3.20) is positive.

In the Chern-Weil description of the Chern class, $c_1(\widehat{\mathcal{L}})$ is represented by smooth curvature in the bulk of the orbifold $\widehat{\Sigma}$. In contrast, the degree n receives contributions from both the bulk curvature in $\widehat{\Sigma}$ and from local, delta-function curvatures at the orbifold points of $\widehat{\Sigma}$. That is why n is an integer but the orbifold first Chern class $c_1(\widehat{\mathcal{L}})$ is not. The delta-function contributions to n are cancelled by the rational numbers β_j/α_j appearing explicitly in the formula (4.3.20) for $c_1(\widehat{\mathcal{L}})$.

From (4.3.20), to define a contact structure on M we choose the connection κ so that its curvature is given by

$$d\kappa \,=\, \left(n + \sum_{j=1}^{N} \frac{\beta_j}{\alpha_j}\right) \pi^* \widehat{\omega}\,, \tag{4.3.21}$$

where $\widehat{\omega}$ is a symplectic form on $\widehat{\Sigma}$ of unit volume, as in (4.3.13). Then, exactly as in (4.3.16), the integral of $\kappa \wedge d\kappa$ over M is determined by the Chern class of $\widehat{\mathcal{L}}$,

$$\int_M \kappa \wedge d\kappa \,=\, n + \sum_{j=1}^{N} \frac{\beta_j}{\alpha_j}\,. \tag{4.3.22}$$

For future reference, we also note that the Riemann-Roch formula for a line bundle on a Riemann surface has a direct generalization to the case of a line V-bundle on an orbifold [97], so that

$$\chi(\widehat{\mathcal{L}}) = \dim_{\mathbb{C}} H^0(\widehat{\Sigma}, \widehat{\mathcal{L}}) - \dim_{\mathbb{C}} H^1(\widehat{\Sigma}, \widehat{\mathcal{L}}) \,=\, n + 1 - g\,, \tag{4.3.23}$$

which justifies calling n the degree of $\widehat{\mathcal{L}}$.

In this discussion, we have used the notation $\widehat{\Sigma}$ and $\widehat{\mathcal{L}}$ to distinguish these orbifold quantities from their smooth counterparts Σ and $\mathcal{L}$. In the future, we will not make this artificial distinction, and in our discussion of Chern-Simons theory we will use Σ and $\mathcal{L}$ to denote general orbifold quantities.

4.3.3 A Symplectic Structure For Chern-Simons Theory

We now specialize to the case of Chern-Simons theory on a Seifert manifold M, which carries a distinguished $U(1)$ action and an invariant contact form κ. Initially, the path integral of Chern-Simons theory on M is an integral over the affine space $\mathcal{A}$ of all connections on M. Unlike the case of two-dimensional Yang-Mills theory, $\mathcal{A}$ is not naturally symplectic and cannot play the role of the symplectic manifold X that appears in the canonical symplectic integral (4.1.4).

However, we now reap the reward of our reformulation of Chern-Simons theory to decouple one component of A. Specifically, we consider the following two-form Ω on $\mathcal{A}$. If η and ξ are any two tangent vectors to $\mathcal{A}$, and hence are represented by sections of the bundle $\Omega^1_M \otimes \mathfrak{g}$ on M, then we define Ω by

$$\Omega(\eta, \xi) = -\int_M \kappa \wedge \mathrm{Tr}\,(\eta \wedge \xi)\ . \tag{4.3.24}$$

Because κ is a globally-defined one-form on M, this expression is well-defined. Further, Ω is closed and invariant under all the symmetries. In particular, Ω is invariant under the group $\mathcal{S}$ of shift symmetries, and by virture of this shift invariance Ω is degenerate along tangent vectors to $\mathcal{A}$ of the form $\sigma\kappa$, where σ is an arbitrary section of $\Omega^0_M \otimes \mathfrak{g}$. However, unlike the gauge symmetry $\mathcal{G}$, which acts nonlinearly on $\mathcal{A}$, the shift symmetry $\mathcal{S}$ acts in a simple, linear fashion on $\mathcal{A}$. Thus, we can trivially take the quotient of $\mathcal{A}$ by the action of $\mathcal{S}$, which we denote as $\overline{\mathcal{A}}$,

$$\overline{\mathcal{A}} \,=\, \mathcal{A}/\mathcal{S}\,. \tag{4.3.25}$$

Under this quotient, the presymplectic form Ω on $\mathcal{A}$ descends immediately to a symplectic form on $\overline{\mathcal{A}}$, which becomes a symplectic space naturally associated to Chern-Simons theory on M. In the following, $\overline{\mathcal{A}}$ plays the role of the abstract symplectic manifold X in (4.1.4).

More About the Path Integral Measure

Our reformulation of the Chern-Simons action $S(A)$ in (4.3.9) is invariant under the shift symmetry $\mathcal{S}$, so $S(A)$ descends to the quotient $\overline{\mathcal{A}}$ of $\mathcal{A}$ by $\mathcal{S}$. But we should also think (at least formally) about the path integral measure $\mathcal{D}A$. As in Yang-Mills theory, we define $\mathcal{D}A$ up to norm as a translation-invariant measure on $\mathcal{A}$, and a convenient way both to describe $\mathcal{D}A$ and to fix its normalization is to consider this measure as induced from a Riemannian metric on $\mathcal{A}$. In turn, we describe this metric on $\mathcal{A}$ as induced from a corresponding metric on M, so that a tangent vector η to $\mathcal{A}$ has norm

$$(\eta, \eta) \,=\, -\int_M \mathrm{Tr}(\eta \wedge \star \eta)\,. \tag{4.3.26}$$

We normalize the volume of $\mathcal{G}$ in (4.3.1) using the similarly induced, invariant metric on $\mathcal{G}$.

We assume that $U(1)$ acts on M by isometries, so that the metric on M associated to the operator $\star$ in (4.3.26) takes the form

$$ds^2_M \,=\, \pi^* ds^2_\Sigma \,+\, \kappa \otimes \kappa\,. \tag{4.3.27}$$

Here ds^2_Σ represents any Kahler metric on Σ which is normalized so that the corresponding Kahler form pulls back to $d\kappa$. As a result of this normalization convention, the duality operator $\star$ defined by the metric (4.3.27) satisfies $\star 1 \,=\, \kappa\wedge d\kappa$.

Tangent vectors to the orbits of the shift symmetry $\mathcal{S}$ are described by sections of $\Omega^1_M \otimes \mathfrak{g}$ which take the form $\sigma\kappa$, where σ is any function taking values in $\mathfrak{g}$ on M. Similarly, tangent vectors to the quotient $\overline{\mathcal{A}}$ are naturally represented by sections of $\Omega^1_M \otimes \mathfrak{g}$ which are annihilated by the interior product ι_R with the vector field R, the generator of the $U(1)$ action on M. When the metric on M takes the form in (4.3.27), the one-forms annihilated by ι_R are orthogonal to the one-forms proportional to κ. Thus, the tangent space to $\mathcal{S}$ is orthogonal to the tangent space to $\overline{\mathcal{A}}$ in the corresponding metric (4.3.26) on $\mathcal{A}$.

We can exhibit the orthogonal decomposition of the metric in (4.3.26) explicitly as

$$(\eta,\eta) = -\int_M \kappa\wedge d\kappa\,\mathrm{Tr}\left[(\iota_R\eta)^2\right] - \int_M \kappa\wedge\mathrm{Tr}\Big[\Pi(\eta)\wedge\star_2\Pi(\eta)\Big]\,. \tag{4.3.28}$$

The first term in (4.3.28) describes the metric on $\mathcal{S}$ which we have already introduced in (4.3.8), and the second term describes the induced metric on $\overline{\mathcal{A}}$. The form of the first term follows immediately from the fact that $\star\kappa = d\kappa$.

In the second term of (4.3.28), we have introduced two natural operators. First, we introduce the the operator Π which projects from the tangent space of $\mathcal{A}$ to the tangent space of $\overline{\mathcal{A}}$, so that Π is given by

$$\Pi(\eta) = \eta - (\iota_R\eta)\,\kappa\,. \tag{4.3.29}$$

Trivially, $\iota_R \circ \Pi = 0$.

Second, we introduce an effective “two-dimensional” duality operator $\star_2$ on M which induces a corresponding complex structure on $\overline{\mathcal{A}}$. This operator is defined globally on M by

$$\star_2 \,=\, -\iota_R \circ \star\,. \tag{4.3.30}$$

Using that $\star\kappa = d\kappa$ and $\star 1 = \kappa\wedge d\kappa$, we see immediately that $\star_2\,\kappa = \star_2\,(\kappa\wedge d\kappa) = 0$ and that $\star_2\,1 = -d\kappa$. Also, one can easily check (for instance by considering local coordinates) that $\star_2$ satisfies $(\star_2)^2 = -1$ when acting on one-forms in the image of Π, representing tangent vectors to $\overline{\mathcal{A}}$. This latter property is important, since it implies that $\star_2$ defines a complex structure on $\overline{\mathcal{A}}$ exactly as in two-dimensional Yang-Mills theory.

With this notation in place, the form of the second term in (4.3.28) follows immediately from the simple computation below,

$$\begin{aligned} \Pi(\eta)\wedge\star\Pi(\eta) &= \iota_R\Big(\kappa\wedge\Pi(\eta)\Big)\wedge\star\Pi(\eta)\,, \\ &= -\kappa\wedge\Pi(\eta)\wedge\,\iota_R\Big(\star\Pi(\eta)\Big)\,, \\ &= \kappa\wedge\Pi(\eta)\wedge\star_2\Pi(\eta)\,. \end{aligned} \tag{4.3.31}$$

In passing from the first to the second line of (4.3.31), we have "integrated by parts" with respect to the operator ι_R, as $\iota_R\,(\kappa\wedge\Pi(\eta)\wedge\star\Pi(\eta))$ is trivially zero on the three-manifold M by dimensional reasons.

We thus see from the second term in (4.3.28) that the induced metric on $\overline{\mathcal{A}}$ is Kahler with respect to the symplectic form Ω in (4.3.24) and the complex structure $\star_2$. Hence the Riemannian measure induced on $\overline{\mathcal{A}}$ from (4.3.28) is identical to the symplectic measure induced by Ω.

Finally, because the measure along the orbits of $\mathcal{S}$ in $\mathcal{A}$ is the same as the invariant measure (4.3.8) which we defined on $\mathcal{S}$ itself, we can trivially integrate over these orbits, which simply contribute a factor of the volume $\mathrm{Vol}(\mathcal{S})$ to the path integral. Consequently, the Chern-Simons path integral in (4.3.11) reduces to an integral over $\overline{\mathcal{A}}$ with its symplectic measure,

$$\begin{aligned} Z(\epsilon) &= \frac{1}{\mathrm{Vol}(\mathcal{G})}\left(\frac{-i}{2\pi\epsilon}\right)^{\Delta_{\mathcal{G}}/2}\int_{\overline{\mathcal{A}}}\exp\left[\Omega + \frac{i}{2\epsilon}\,S\,(A)\right], \\ S(A) &= \int_M \mathrm{Tr}\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\Big) - \int_M \frac{1}{\kappa\wedge d\kappa}\mathrm{Tr}\Big[(\kappa\wedge F_A)^2\Big]\,. \end{aligned} \tag{4.3.32}$$

4.3.4 Hamiltonian Symmetries of Chern-Simons Theory

To complete our symplectic description of the Chern-Simons path integral on M, we must show that the action $S(A)$ in (4.3.32) is the square of a moment map μ for the Hamiltonian action of some symmetry group $\mathcal{H}$ on the symplectic space $\overline{\mathcal{A}}$.

By analogy to the case of Yang-Mills theory on Σ, one might naively guess that the relevant symmetry group for Chern-Simons theory would also be the group $\mathcal{G}$ of gauge transformations. One can easily check that the action of $\mathcal{G}$ on $\mathcal{A}$ descends under the quotient to a well-defined action on $\overline{\mathcal{A}}$, and clearly the symplectic form Ω on $\overline{\mathcal{A}}$ is invariant under $\mathcal{G}$. However, one interesting aspect of non-abelian localization for Chern-Simons theory is the fact that the group $\mathcal{H}$ which we use for localization must be somewhat more complicated than $\mathcal{G}$ itself.

A trivial objection to using $\mathcal{G}$ for localization is that, by construction, the square of the moment map μ for any Hamiltonian action on $\overline{\mathcal{A}}$ defines an invariant function on $\overline{\mathcal{A}}$, but the action $S(A)$ is not invariant under the group $\mathcal{G}$. Instead, the action $S(A)$ is the sum of a manifestly gauge invariant term and the usual Chern-Simons action, and the Chern-Simons action shifts by integer multiples of 2π under "large" gauge transformations, those not continuously connected to the identity in $\mathcal{G}$.

This trivial objection is easily overcome. We consider not the disconnected group $\mathcal{G}$ of all gauge transformations but only the identity component $\mathcal{G}_0$ of this group, under which $S(A)$ is invariant.

We now consider the action of $\mathcal{G}_0$ on $\overline{\mathcal{A}}$, and our first task is to determine the corresponding moment map μ. If ϕ is an element of the Lie algebra of $\mathcal{G}_0$, described by a section of the bundle $\Omega^0_M \otimes \mathfrak{g}$ on M, then the corresponding vector field $V(\phi)$ generated by ϕ on $\mathcal{A}$ is given by $V(\phi) = d_A\phi$. Thus, from our expression for the symplectic form Ω in (4.3.24) we see that

$$\iota_{V(\phi)}\Omega \,=\, -\int_M \kappa\wedge\mathrm{Tr}\left(d_A\phi\wedge\delta A\right)\,. \tag{4.3.33}$$

Integrating by parts with respect to d_A, we can rewrite (4.3.33) in the form $\delta\langle\mu,\phi\rangle$, where

$$\langle\mu,\phi\rangle = \int_M \kappa\wedge\mathrm{Tr}\Big(\phi F_A\Big) - \int_M d\kappa\wedge\mathrm{Tr}\Big(\phi(A-A_0)\Big)\,. \tag{4.3.34}$$

Here A_0 is an arbitrary connection, corresponding to a basepoint in $\mathcal{A}$, which we must choose so that the second term in (4.3.34) can be honestly interpreted as the integral of a differential form on M. Geometrically, the choice of A_0 corresponds to the choice of a trivialization for the principal G-bundle over M. We will say more about this choice momentarily, but we first observe that the expression for μ in (4.3.34) is invariant under the shift symmetry and immediately descends to a moment map for the action of $\mathcal{G}$ on $\overline{\mathcal{A}}$.

The fact that we must choose a basepoint A_0 in $\mathcal{A}$ to define the moment map is very important in the following, and it is fundamentally a reflection of the affine structure of $\mathcal{A}$. In general, an affine space is a space which can be identified with a vector space only after some basepoint is chosen to represent the origin. In the case at hand, once A_0 is chosen, we can identify $\mathcal{A}$ with the vector space of sections η of the bundle $\Omega^1_M \otimes \mathfrak{g}$ on M, via $A = A_0 + \eta$, as we used in (4.3.34). However, $\mathcal{A}$ is not naturally itself a vector space, since $\mathcal{A}$ does not intrinsically possess a distinguished origin. This statement corresponds to the geometric statement that, though our principal G-bundle on M is trivial, it does not possess a canonical trivialization.

In terms of the moment map μ, the choice of A_0 simply represents the possibility of adding an arbitrary constant to μ. In general, our ability to add a constant to μ means that μ need *not* determine a Hamiltonian action of $\mathcal{G}_0$ on $\overline{\mathcal{A}}$. Indeed, as we show below, the action of $\mathcal{G}_0$ on $\overline{\mathcal{A}}$ is not Hamiltonian and we cannot simply use $\mathcal{G}_0$ to perform localization.

In order not to clutter the expressions below, we assume henceforth that we have fixed a trivialization of the G-bundle on M and we simply set $A_0 = 0$.

To determine whether the action of $\mathcal{G}_0$ on $\overline{\mathcal{A}}$ is Hamiltonian, we must check the condition (4.2.5) that μ determine a homomorphism from the Lie algebra of $\mathcal{G}_0$ to the algebra of functions on $\overline{\mathcal{A}}$ under the Poisson bracket. So we directly compute

$$\begin{aligned}\Big\{\langle\mu,\phi\rangle,\langle\mu,\psi\rangle\Big\} &= \Omega\Big(d_A\phi, d_A\psi\Big) = -\int_M \kappa\wedge\mathrm{Tr}\left(d_A\phi\wedge d_A\psi\right)\,,\\ &= \int_M \kappa\wedge\mathrm{Tr}\Big([\phi,\psi]F_A\Big) - \int_M d\kappa\wedge\mathrm{Tr}\Big(\phi\, d_A\psi\Big)\,,\\ &= \langle\mu,[\phi,\psi]\rangle - \int_M d\kappa\,\mathrm{Tr}\Big(\phi\, d\psi\Big)\,.\end{aligned} \tag{4.3.35}$$

Thus, the failure of μ to determine an algebra homomorphism is measured by the cohomology class of the Lie algebra cocycle

$$\begin{aligned} c(\phi, \psi) \;&=\; \Big\{\langle\mu, \phi\rangle, \langle\mu, \psi\rangle\Big\} - \langle\mu, [\phi, \psi]\rangle\,, \\ &=\; -\int_M d\kappa\wedge \mathrm{Tr}\Big(\phi\, d\psi\Big) \;=\; -\int_M \kappa\wedge d\kappa \mathrm{Tr}\Big(\phi \pounds_R \psi\Big)\,. \end{aligned} \tag{4.3.36}$$

In the second line of (4.3.36), we have rewritten the cocycle more suggestively by using the Lie derivative $\pounds_R$ along the vector field R on M which generates the $U(1)$ action. The class of this cocycle is not zero, and no Hamiltonian action on $\overline{\mathcal{A}}$ exists for the group $\mathcal{G}_0$.

Some Facts About Loop Groups

The cocycle appearing in (4.3.36) has a very close relationship to a similar cocycle that arises in the theory of loop groups, and some well-known loop group constructions feature heavily in our study of Chern-Simons theory. We briefly review these ideas, for which a general reference is [98].

When G is a finite-dimensional Lie group, we recall that the loop group LG is defined as the group of smooth maps $\mathrm{Map}(S^1, G)$ from S^1 to G. Similarly, the Lie algebra $L\mathfrak{g}$ of LG is the algebra $\mathrm{Map}(S^1, \mathfrak{g})$ of smooth maps from S^1 to $\mathfrak{g}$. When $\mathfrak{g}$ is simple, then the Lie algebra $L\mathfrak{g}$ admits a unique, G-invariant cocycle up to scale, and this cocycle is directly analogous to the cocycle we discovered in (4.3.36). If ϕ and ψ are elements in the Lie algebra $L\mathfrak{g}$, then this cocycle is defined by

$$c(\phi, \psi) = -\int_{S^1} \mathrm{Tr}\Big(\phi\, d\psi\Big) \;=\; -\int_{S^1} dt\, \mathrm{Tr}\Big(\phi \pounds_R \psi\Big)\,. \tag{4.3.37}$$

In passing to the last expression, we have by analogy to (4.3.36) introduced a unit-length parameter t on S^1, so that $\int_{S^1} dt = 1$, and we have introduced the dual vector field $R = \partial/\partial t$ which generates rotations of S^1.

In general, if $\mathfrak{g}$ is any Lie algebra and c is a nontrivial cocycle, then c determines a corresponding central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$,

$$\mathbb{R} \longrightarrow \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g}\,. \tag{4.3.38}$$

As a vector space, $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$, and the Lie algebra of $\widetilde{\mathfrak{g}}$ is given by the bracket

$$\Big[(\phi, a), (\psi, b)\Big] = \Big([\phi, \psi], c(\phi, \psi)\Big)\,, \tag{4.3.39}$$

where ϕ and ψ are elements of $\mathfrak{g}$, and a and b are elements of $\mathbb{R}$.

In the case of the Lie algebra $L\mathfrak{g}$, the cocycle c appearing in (4.3.37) consequently determines a central extension $\widetilde{L\mathfrak{g}}$ of $L\mathfrak{g}$. When G is simply connected, the extension determined by c or any integral multiple of c lifts to a corresponding extension of LG by $U(1)$,

$$U(1) \longrightarrow \widetilde{LG} \longrightarrow LG\,. \tag{4.3.40}$$

Topologically, the extension $\widetilde{LG}$ is the total space of the S^1 bundle over LG whose Euler class is represented by the cocyle of the extension, interpreted as an invariant two-form on LG. The fact that the Euler class must be integral is responsible for the corresponding quantization condition on the cocycle of the extension.

When $\mathfrak{g}$ is simple, the algebra $L\mathfrak{g}$ has a non-degenerate, invariant inner product which is unique up to scale and is given by

$$(\phi, \psi) \;=\; -\int_{S^1} dt\, \mathrm{Tr}\,(\phi\psi)\;. \tag{4.3.41}$$

On the other hand, the corresponding extension $\widetilde{L\mathfrak{g}}$ does *not* possess a non-degenerate, invariant inner product, since any element of $\widetilde{L\mathfrak{g}}$ can be expressed as a commutator, so that $[\widetilde{L\mathfrak{g}}, \widetilde{L\mathfrak{g}}] = \widetilde{L\mathfrak{g}}$, and the center of $\widetilde{L\mathfrak{g}}$ is necessarily orthogonal to every commutator under an invariant inner product.

However, we can also consider the semidirect product $U(1) \ltimes \widetilde{LG}$. Here, the rigid $U(1)$ action on S^1 induces a natural $U(1)$ action on $\widetilde{LG}$ by which we define the product, and the important observation about this group $U(1) \ltimes \widetilde{LG}$ is that it does admit an invariant, non-degenerate inner product on its Lie algebra.

Explicitly, the Lie algebra of $S^1 \ltimes \widetilde{LG}$ is identified with $\mathbb{R} \oplus \widetilde{L\mathfrak{g}} = \mathbb{R} \oplus L\mathfrak{g} \oplus \mathbb{R}$ as a vector space, and the Lie algebra is given by the bracket

$$\Big[(p, \phi, a), (q, \psi, b)\Big] \;=\; \Big(0, [\phi, \psi] + p\pounds_R\psi - q\pounds_R\phi, c(\phi, \psi)\Big)\,, \tag{4.3.42}$$

where $\pounds_R$ is the Lie derivative with respect to the vector field R generating rotations of S^1. We then consider the manifestly non-degenerate inner product on $\mathbb{R} \oplus \widetilde{L\mathfrak{g}}$ which is given by

$$\Big((p, \phi, a), (q, \psi, b)\Big) \;=\; -\int_M dt\, \mathrm{Tr}(\phi\psi) - pb - qa\,. \tag{4.3.43}$$

One can directly check that this inner product is invariant under the adjoint action determined by (4.3.42). We note that although this inner product is non-degenerate, it is not positive-definite because of the last two terms in (4.3.43).

Extension To Chern-Simons Theory

We now return to our original problem, which is to find a Hamiltonian action of a group $\mathcal{H}$ on $\overline{\mathcal{A}}$ to use for localization. The natural guess to consider the identity component $\mathcal{G}_0$ of the gauge group does not work, because the cocycle c in (4.3.36) obstructs the action of $\mathcal{G}_0$ on $\overline{\mathcal{A}}$ from being Hamiltonian.

However, motivated by the loop group constructions, we consider now the central extension $\widetilde{\mathcal{G}_0}$ of $\mathcal{G}_0$ by $U(1)$ which is determined by the cocycle c in (4.3.36),

$$U(1) \longrightarrow \widetilde{\mathcal{G}_0} \longrightarrow \mathcal{G}_0 \,. \tag{4.3.44}$$

We assume that the central $U(1)$ subgroup of $\widetilde{\mathcal{G}_0}$ acts trivially on $\overline{\mathcal{A}}$, so that the moment map for the central generator $(0, a)$ of the Lie algebra is constant. Then, by construction, we see from (4.3.36) and (4.3.39) that the new moment map for the action of $\widetilde{\mathcal{G}_0}$ on $\overline{\mathcal{A}}$, which is given by

$$\langle \mu, (\phi, a) \rangle \;=\; \int_M \kappa \wedge \mathrm{Tr}\,(\phi F_A) - \int_M d\kappa \wedge \mathrm{Tr}\,(\phi A) + a \,, \tag{4.3.45}$$

satisfies the Hamiltonian condition

$$\Big\{ \langle \mu, (\phi, a) \rangle, \langle \mu, (\psi, b) \rangle \Big\} = \Big\langle \mu, [(\phi, a), (\psi, b)] \Big\rangle \,. \tag{4.3.46}$$

The action of the extended group $\widetilde{\mathcal{G}}_0$ on $\overline{\mathcal{A}}$ is thus Hamiltonian with moment map in (4.3.45).

But $\widetilde{\mathcal{G}_0}$ is still not the group $\mathcal{H}$ which we must use to perform non-abelian localization in Chern-Simons theory! In order to realize the action $S(A)$ as the square of the moment map μ for some Hamiltonian group action on $\overline{\mathcal{A}}$, the Lie algebra of the group must first possess a non-degenerate, invariant inner product. Just as for the loop group extension $\widetilde{LG}$, the group $\widetilde{\mathcal{G}_0}$ does not possess such an inner product.

However, we can elegantly remedy this problem, just as it was remedied for the loop group, by also considering the action of $U(1)$ on M. The $U(1)$ action on M induces an action of $U(1)$ on $\widetilde{\mathcal{G}_0}$, so we consider the associated semidirect product $U(1) \ltimes \widetilde{\mathcal{G}}_0$. Then a non-degenerate, invariant inner product on the Lie algebra of $U(1) \ltimes \widetilde{\mathcal{G}}_0$ is given by

$$\Big((p, \phi, a), (q, \psi, b)\Big) = -\int_M \kappa \wedge d\kappa\, \mathrm{Tr}(\phi\psi) - pb - qa\,, \tag{4.3.47}$$

in direct correspondence with (4.3.43). As for the loop group, this quadratic form is of indefinite signature, due to the hyperbolic form of the last two terms in (4.3.47).

Finally, the $U(1)$ action on M immediately induces a corresponding $U(1)$ action on $\mathcal{A}$. Since the contact form κ is invariant under this action, the induced $U(1)$ action on $\mathcal{A}$ descends to a corresponding action on the quotient $\overline{\mathcal{A}}$. In general, the vector field upstairs on $\mathcal{A}$ which is generated by an arbitrary element (p, ϕ, a) of the Lie algebra of $U(1) \ltimes \widetilde{\mathcal{G}}_0$ is then given by

$$\delta A = d_A\phi + p\, \pounds_R A\,, \tag{4.3.48}$$

where R is the vector field on M generating the action of $U(1)$. Clearly the moment for the new generator $(p, 0, 0)$ is given by

$$\Big\langle \mu, (p, 0, 0) \Big\rangle \;=\; -\frac{1}{2}\, p \int_M \kappa \wedge \mathrm{Tr}\left(\pounds_R A \wedge A\right)\,. \tag{4.3.49}$$

This moment is manifestly invariant under the shift symmetry and descends to $\overline{\mathcal{A}}$.

In fact, the action of $U(1) \ltimes \widetilde{\mathcal{G}}_0$ on $\overline{\mathcal{A}}$ is Hamiltonian, with moment map

$$\Big\langle \mu, (p, \phi, a) \Big\rangle \;=\; -\frac{1}{2} p \int_M \kappa \wedge \mathrm{Tr}\left(\pounds_R A \wedge A\right) + \int_M \kappa \wedge \mathrm{Tr}\left(\phi F_A\right) - \int_M d\kappa \wedge \mathrm{Tr}\left(\phi A\right) + a\,. \tag{4.3.50}$$

To check this statement, it suffices to compute $\Big\{\langle \mu, (p, 0, 0)\rangle, \langle \mu, (0, \psi, 0)\rangle\Big\}$, which is the only nontrivial Poisson bracket that we have not already computed. Thus,

$$\begin{aligned} \Big\{\langle \mu, (p, 0, 0)\rangle, \langle \mu, (0, \psi, 0)\rangle\Big\} \;&=\; \Omega\Big(p\, \pounds_R A, d_A\psi\Big) \;=\; -p \int_M \kappa \wedge \mathrm{Tr}\left(\pounds_R A \wedge d_A\psi\right)\,, \\ &= p \int_M \kappa \wedge \mathrm{Tr}\left(\pounds_R \psi\, F_A\right) - p \int_M d\kappa \wedge \mathrm{Tr}\left(\pounds_R \psi\, A\right)\,, \\ &= \Big\langle \mu, (0, p\, \pounds_R \psi, 0) \Big\rangle\,, \end{aligned} \tag{4.3.51}$$

as required by the Lie bracket (4.3.42).

Thus, we identify $\mathcal{H} = U(1) \ltimes \widetilde{\mathcal{G}}_0$ as the relevant group of Hamiltonian symmetries which we use for localization in Chern-Simons theory.

4.3.5 The Action $S(A)$ as the Square of the Moment Map

By construction, the square (μ,μ) of the moment map μ in (4.3.50) for the Hamiltonian action of $\mathcal{H}$ on $\overline{\mathcal{A}}$ is a function on $\overline{\mathcal{A}}$ invariant under $\mathcal{H}$. The new Chern-Simons action $S(A)$ in (4.3.9) is also a function on $\overline{\mathcal{A}}$ invariant under $\mathcal{H}$. Given the high degree of symmetry, we certainly expect that (μ,μ) and $S(A)$ agree up to normalization. We now check this fact directly and fix the relative normalization.

We first observe that, in terms of the invariant form $(\cdot,\cdot)$ in (4.3.47) on the Lie algebra of $\mathcal{H}$, we can express the moment map dually as determined by the inner product with the vector $\Big(-1,\, -\left(\kappa\wedge F_A - d\kappa\wedge A\right)/\kappa\wedge d\kappa,\, \frac{1}{2}\int_M \kappa\wedge\mathrm{Tr}(\pounds_R A\wedge A)\Big)$ in the Lie algebra of $\mathcal{H}$, so that

$$\Big\langle\mu,(p,\phi,a)\Big\rangle = \left(\left(-1,-\left(\frac{\kappa\wedge F_A - d\kappa\wedge A}{\kappa\wedge d\kappa}\right),\frac{1}{2}\int_M \kappa\wedge\mathrm{Tr}(\pounds_R A\wedge A)\right),(p,\phi,a)\right). \tag{4.3.52}$$

Thus, by duality, the square of μ is determined to be

$$\begin{aligned}(\mu,\mu) &= \left\langle\mu,\left(-1,-\left(\frac{\kappa\wedge F_A - d\kappa\wedge A}{\kappa\wedge d\kappa}\right),\frac{1}{2}\int_M \kappa\wedge\mathrm{Tr}(\pounds_R A\wedge A)\right)\right\rangle,\\ &= \int_M \kappa\wedge\mathrm{Tr}\Big(\pounds_R A\wedge A\Big) - \int_M \kappa\wedge d\kappa\,\mathrm{Tr}\Big(\big(\frac{\kappa\wedge F_A - d\kappa\wedge A}{\kappa\wedge d\kappa}\big)^2\Big).\end{aligned} \tag{4.3.53}$$

To simplify the first term of (4.3.53), we use the fact that the Lie derivative $\pounds_R$ can be expressed as an anti-commutator $\pounds_R = \{\iota_R, d\}$, so that

$$\int_M \kappa\wedge\mathrm{Tr}\Big(\pounds_R A\wedge A\Big) \;=\; \int_M \kappa\wedge\mathrm{Tr}\Big(\{\iota_R,d\}A\wedge A\Big). \tag{4.3.54}$$

We now observe that $\iota_R A$ can be expressed as

$$\iota_R A = \frac{A\wedge d\kappa}{\kappa\wedge d\kappa}. \tag{4.3.55}$$

Using this fact and integrating by parts with respect to the outermost operator d or ι_R in both of the two terms from the anti-commutator (4.3.54), we find that

$$\begin{aligned}\int_M \kappa\wedge\mathrm{Tr}\Big(\pounds_R A\wedge A\Big) \;&=\; \int_M \Big[\iota_R\kappa\wedge\mathrm{Tr}\left(dA\wedge A\right) \;-\; \kappa\wedge\mathrm{Tr}\left(dA\;\iota_R A\right) +\\ &\qquad + \;d\kappa\wedge\mathrm{Tr}\left(\iota_R A\;A\right) - \kappa\wedge\mathrm{Tr}\left(\iota_R A\;dA\right)\Big],\\ &= \int_M\Bigg[\mathrm{Tr}\left(A\wedge dA\right) \;-\; 2\kappa\wedge\mathrm{Tr}\left(\frac{d\kappa\wedge A}{\kappa\wedge d\kappa}\,dA\right) +\\ &\qquad + \;d\kappa\wedge\mathrm{Tr}\left(\frac{d\kappa\wedge A}{\kappa\wedge d\kappa}\,A\right)\Bigg].\end{aligned} \tag{4.3.56}$$

(We observe that trivially $\iota_R\,(\kappa\wedge\mathrm{Tr}\,(dA\wedge A)) = 0$.)

Consequently, after some algebra, we find that (4.3.53) becomes

$$(\mu,\mu) = -\int_M \frac{1}{\kappa\wedge d\kappa}\mathrm{Tr}\Big((\kappa\wedge F_A)^2\Big) + \int_M \mathrm{Tr}\Big(A\wedge dA\Big) + 2\int_M \kappa\wedge\mathrm{Tr}\Big((\iota_R A)A\wedge A\Big)\,. \quad (4.3.57)$$

In arriving at (4.3.57), we have observed that the terms involving κ in (4.3.56) are cancelled by corresponding terms from the second term in (4.3.53), arising from the perfect square $((\kappa\wedge F_A - d\kappa\wedge A)\,/\kappa\wedge d\kappa)^2$, after expanding $F_A = dA + A\wedge A$. The last term in (4.3.57), cubic in A, arises from the cross-term in this perfect square when we express $F_A = dA + A\wedge A$ and we apply the identity (4.3.55).

To simplify the last term of (4.3.57), we observe that

$$0 = \iota_R\Big(\kappa\wedge\mathrm{Tr}(A\wedge A\wedge A)\Big) = -3\kappa\wedge\mathrm{Tr}\Big((\iota_R A)A\wedge A\Big) + \mathrm{Tr}\Big(A\wedge A\wedge A\Big)\,, \quad (4.3.58)$$

so that

$$(\mu,\mu) = -\int_M \frac{1}{\kappa\wedge d\kappa}\mathrm{Tr}\Big((\kappa\wedge F_A)^2\Big) + \int_M \mathrm{Tr}\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\Big)\,. \quad (4.3.59)$$

We thus find the beautiful result,

$$S(A) = (\mu,\mu)\,. \quad (4.3.60)$$

We finally write the Chern-Simons path integral as a symplectic integral over $\overline{\mathcal{A}}$ of the canonical form,

$$Z(\epsilon) \;=\; \frac{1}{\mathrm{Vol}(\mathcal{G})}\left(\frac{-i}{2\pi\epsilon}\right)^{\Delta_{\mathcal{G}}/2}\int_{\overline{\mathcal{A}}}\exp\left[\Omega + \frac{i}{2\epsilon}\,(\mu,\mu)\right]. \quad (4.3.61)$$

4.4 Non-Abelian Localization and Two-Dimensional Yang-Mills Theory

In this section, we recall following [18] how the technique of non-abelian localization can be generally applied to study a symplectic integral of the canonical form

$$Z(\epsilon) \;=\; \frac{1}{\mathrm{Vol}(H)}\left(\frac{1}{2\pi\epsilon}\right)^{\Delta_H/2}\int_X \exp\left[\Omega - \frac{1}{2\epsilon}\,(\mu,\mu)\right], \qquad \Delta_H = \dim H\,. \quad (4.4.1)$$

Here X is a symplectic manifold with symplectic form Ω, and H is a Lie group which acts on X in a Hamiltonian fashion with moment map μ. Finally, $(\,\cdot\,,\,\cdot\,)$ is an invariant, positive-definite quadratic form on the Lie algebra $\mathfrak{h}$ of H and dually on $\mathfrak{h}^*$ which we use to define the "action" $S = \frac{1}{2}(\mu, \mu)$ and the volume $\mathrm{Vol}(H)$ of H that appear in (4.4.1).

In the case of Chern-Simons theory, the corresponding quadratic form (4.3.47) on $\mathfrak{h}$ has indefinite signature, due to the hyperbolic summand associated to the two extra $U(1)$ generators of $\mathcal{H}$ relative to the group of gauge transformations $\mathcal{G}_0$. Also, invariance under large gauge transformations requires the Chern-Simons symplectic integral (4.3.61) to be oscillatory, instead of exponentially damped. These features do not essentially change our discussion of localization below, and we reserve further comment until Section 5.

Later in this section, we also review and extend the ideas of [18] to apply non-abelian localization to Yang-Mills theory on a Riemann surface.

4.4.1 General Aspects of Non-Abelian Localization

To apply non-abelian localization to an integral of the form (4.4.1), we first observe that $Z(\epsilon)$ can be rewritten as

$$Z(\epsilon) \;=\; \frac{1}{\mathrm{Vol}(H)} \int_{\mathfrak{h}\times X} \left[\frac{d\phi}{2\pi}\right] \exp\left[\Omega - i\,\langle\mu, \phi\rangle - \frac{\epsilon}{2}(\phi, \phi)\right]. \tag{4.4.2}$$

Here ϕ is an element of the Lie algebra $\mathfrak{h}$ of H, and $[d\phi]$ is the Euclidean measure on $\mathfrak{h}$ that is determined by the same invariant form $(\cdot, \cdot)$ which we use to define the volume $\mathrm{Vol}(H)$ of H. The Gaussian integral over ϕ in (4.4.2) leads immediately to the expression in (4.4.1). The measure $[d\phi/2\pi]$ includes a factor of $1/2\pi$ for each real component of ϕ.

A BRST Symmetry

The advantage of writing Z in the form (4.4.2) is that, once we introduce ϕ, then Z becomes invariant under a BRST symmetry, and this BRST symmetry leads directly to a localization formula for (4.4.2).

To describe this BRST symmetry, we recall that the moment map satisfies

$$d\langle\mu, \phi\rangle \;=\; \iota_{V(\phi)}\Omega\,, \tag{4.4.3}$$

where $V(\phi)$ is the vector field on X associated to the infinitesimal action of ϕ. Because of the relation (4.4.3), the argument of the exponential in (4.4.2) is immediately annihilated by the BRST operator D defined by

$$D \,=\, d + i\,\iota_{V(\phi)}\,. \tag{4.4.4}$$

To exhibit the action of D locally, we choose a basis ϕ^a for $\mathfrak{h}$, and we introduce local coordinates x^m on X. We also introduce the notation $\chi^m \equiv dx^m$ for the corresponding basis of local one-forms on X, and we expand the vector field $V(\phi)$ into components as $V(\phi) = \phi^a\, V_a^m\, \partial/\partial x^m$. Then the action of D in (4.4.4) is described in terms of these local coordinates by

$$\begin{aligned} Dx^m \,&=\, \chi^m\,,\\ D\chi^m \,&=\, i\,\phi^a\, V_a^m\,,\\ D\phi^a \,&=\, 0\,. \end{aligned} \tag{4.4.5}$$

From this local description (4.4.5), we see that the action of D preserves a ghost number, or grading, under which x carries charge 0, χ carries charge +1, ϕ carries charge +2, and D itself carries charge +1.

The most important property of a BRST operator is that it squares to zero. In this case, either from (4.4.4) or from (4.4.5), we see that D squares to the Lie derivative along the vector field $V(\phi)$,

$$D^2 = i\,\{d, \iota_{V(\phi)}\} = i\,\pounds_{V(\phi)}\,. \tag{4.4.6}$$

Thus, $D^2 = 0$ exactly when D acts on the subspace of H-invariant functions $\mathcal{O}(x, \chi, \phi)$ of x, χ, and ϕ.

For simplicity, we restrict attention to functions $\mathcal{O}(x, \chi, \phi)$ which are polynomial in ϕ. Then an arbitrary function of this form can be expanded as a sum of terms

$$\mathcal{O}(x)_{m_1 \ldots m_p\, a_1 \ldots a_q}\, \chi^{m_1} \cdots \chi^{m_p}\, \phi^{a_1} \cdots \phi^{a_q}\,, \tag{4.4.7}$$

for some $0 \le p \le \dim X$ and $q \ge 0$. (The restriction on p arises from the fact that χ satisfies Fermi statistics, whereas ϕ satisfies Bose statistics.)

More globally, each term of the form (4.4.7) is specified by a section of the bundle $\Omega^p_X \otimes \mathrm{Sym}^q(\mathfrak{h}^*)$ of p-forms on X which take values in the q-th symmetric tensor product of the dual $\mathfrak{h}^*$ of the Lie algebra of H. Thus, if we consider the complex $\left(\Omega^*_X \otimes \mathrm{Sym}^*\left(\mathfrak{h}^*\right)\right)^H$ of all H-invariant differential forms on X which take values in the ring of polynomial functions on $\mathfrak{h}$, then we see that D defines a cohomology theory associated to the action of H on X. This cohomology theory is known as the Cartan model of the H-equivariant cohomology of X. With the exception of the last computation in Section 5.3, our applications will not require a greater familiarity with equivariant cohomology than what we have described here. However, in Section 5.3 we will need to use a few additional properties of equivariant cohomology that we discuss in Appendix C, and we recommend [20,99] as basic references.

Localization for Z

Because the argument of the exponential in (4.4.2) is annihilated by D and because this argument is manifestly invariant under H, the integrand of the symplectic integral Z determines an equivariant cohomology class on X. Furthermore, by the usual arguments, Z is formally unchanged by the addition of any D-exact invariant form to its integrand. This formal statement can fail if X is not compact and Z suffers from divergences, as we analyze in great detail in Appendix A, but for the moment we ignore this issue and assume X is compact. Thus, Z depends only on the equivariant cohomology class of its integrand.

We now explain how this fact leads immediately to a localization formula for Z. We first observe that we can add to the argument of the exponential in (4.4.2) an arbitrary term of the form $t\,D\lambda$, where λ is any H-invariant one-form on X and t is a real parameter, so that

$$Z(\epsilon) = \frac{1}{\mathrm{Vol}(H)} \int_{\mathfrak{h}\times X} \left[\frac{d\phi}{2\pi}\right] \exp\left[\Omega - i\,\langle\mu,\phi\rangle - \frac{\epsilon}{2}(\phi,\phi) + t\,D\lambda\right]. \qquad (4.4.8)$$

This deformation of the integrand of (4.4.2) is D-exact and does not change Z. In particular, Z does not depend on t.

By definition, $D\lambda$ is given explicitly by

$$D\lambda \,=\, d\lambda + i\,\langle\lambda, V(\phi)\rangle\,. \qquad (4.4.9)$$

As before, $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the canonical dual pairing, so that in components the last term of (4.4.9) is given by $\lambda_m V_a^m \phi^a$.

Thus, apart from a polynomial in t that arises from expanding the term $\exp(t\,d\lambda)$, all of the dependence on t in the integrand of Z arises from the factor $\exp\left[i\,t\,\langle\lambda, V(\phi)\rangle\right]$ that now appears in (4.4.8). So if we consider the limit $t \to \infty$, then the stationary phase approximation to the integral is valid, and all contributions to Z localize around the critical points of the function $\langle\lambda, V(\phi)\rangle$.

We expand this function in the basis ϕ^a for $\mathfrak{h}$ which we introduced previously,

$$\langle\lambda, V(\phi)\rangle \,=\, \phi^a\,\langle\lambda, V_a\rangle\,. \tag{4.4.10}$$

Thus, the critical points of $\langle\lambda, V(\phi)\rangle$ arise from the simultaneous solutions in $\mathfrak{h}\times X$ of the equations

$$\begin{aligned}\langle\lambda, V_a\rangle \,&=\, 0\,,\\ \phi^a\,d\langle\lambda, V_a\rangle \,&=\, 0\,.\end{aligned} \tag{4.4.11}$$

The first equation in (4.4.11) implies that Z necessarily localizes on points in $\mathfrak{h}\times X$ for which $\langle\lambda, V_a\rangle$ vanishes. As for the second equation in (4.4.11), we see that it is invariant under an overall scaling of ϕ in the vector space $\mathfrak{h}$. Consequently, upon integrating over ϕ in (4.4.8), we see that the critical locus of the function $\langle\lambda, V(\phi)\rangle$ in $\mathfrak{h}\times X$ projects onto the vanishing locus of $\langle\lambda, V_a\rangle$ in X. So Z localizes on the subset of X where $\langle\lambda, V_a\rangle = 0$.

By making a specific choice of the one-form λ, we can describe the localization of Z more precisely. In particular, we now show that Z localizes on the set of critical points of the function $S = \frac{1}{2}(\mu,\mu)$ on X.

We begin by choosing an almost complex structure J on X. That is, $J : TX \to TX$ is a linear map from TX to itself such that $J^2 = -1$. We assume that J is compatible with the symplectic form Ω in the sense that Ω is of type $(1,1)$ with respect to J and the associated metric $G(\cdot,\cdot) = \Omega(\cdot, J\cdot)$ on X is positive-definite. Such an almost complex structure always exists.

Using J and S, we now introduce the invariant one-form

$$\lambda \,=\, J\,dS \,=\, (\mu, J\,d\mu)\,. \tag{4.4.12}$$

In components, $\lambda \,=\, dx^m J_m^n \partial_n S \,=\, dx^m \mu^a J_m^n \partial_n \mu_a$.

The integral Z now localizes on the subset of X where $\langle \lambda, V_a \rangle = 0$. Comparing to (4.4.12), we see that this subset certainly includes all critical points of S, since by definition $dS = 0$ at these points.

Conversely, we now show that if $\langle \lambda, V_a \rangle = 0$ at some point on X, then this point is a critical point of S. To prove this assertion, we use the inverse Ω^{-1} to Ω, which arises by considering the symplectic form as an isomorphism $\Omega : TM \rightarrow T^*M$ with inverse $\Omega^{-1} : T^*M \rightarrow TM$. In components, Ω^{-1} is defined by $(\Omega^{-1})^{lm}\,\Omega_{mn} \,=\, \delta^l_n$.

In terms of Ω^{-1}, the moment map equation (4.4.3) is equivalent to the relation

$$V \,=\, \Omega^{-1}\, d\mu\,, \tag{4.4.13}$$

or $V_a^m \,=\, (\Omega^{-1})^{mn}\,\partial_n \mu_a$. Thus,

$$\Omega^{-1}\, dS \,=\, \Big(\mu,\, \Omega^{-1} d\mu\Big) \,=\, (\mu, V)\,, \tag{4.4.14}$$

or $(\Omega^{-1})^{mn}\,\partial_n S \,=\, \mu^a V_a^m$.

In particular, the condition that $\langle \lambda, V_a \rangle = 0$ implies that

$$0 \,=\, (\mu,\, \langle \lambda, V \rangle) \,=\, \langle \lambda,\, \Omega^{-1} dS \rangle \,=\, \langle J\, dS,\, \Omega^{-1} dS \rangle\,, \tag{4.4.15}$$

or more explicitly, $0 \,=\, \mu^a \lambda_m V_a^m \,=\, \lambda_m\,(\Omega^{-1})^{mn}\,\partial_n S \,=\, (\Omega^{-1})^{mn} J_m^l\,\partial_l S\,\partial_n S$. We recognize the last expression in (4.4.15) as the norm of the one-form dS with respect to the metric G on X. As G is positive-definite, we conclude that the condition $\langle \lambda, V_a \rangle = 0$ implies the vanishing of dS. Thus, the symplectic integral Z localizes precisely on the critical set of S.

4.4.2 Non-Abelian Localization For Yang-Mills Theory, Part I

In the rest of this section, we apply non-abelian localization to perform path integral computations in two-dimensional Yang-Mills theory on a smooth Riemann surface Σ. These computations are an essential warmup for our later computations in Chern-Simons theory.

As we discussed in Section 2, the Yang-Mills path integral is naturally a symplectic integral of the canonical form (4.4.1), where the abstract symplectic manifold X is now the affine space $\mathcal{A}(P)$ of connections on a fixed principal G-bundle P over Σ, and where the abstract group H is now the group $\mathcal{G}(P)$ of gauge transformations. Also, the moment map for the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$ is simply the curvature of the connection, $\mu = F_A$.

As a result of our general discussion above, the Yang-Mills path integral localizes on critical points of the Yang-Mills action. These critical points fall into two qualitatively different sorts. Because the action $S = \frac{1}{2}(\mu, \mu)$ is quadratic in the moment map μ, so that $dS = (\mu, d\mu)$, we see that the critical locus of S includes all points where μ vanishes, as well as other points where μ is generally non-zero. The points at which $\mu = 0$ are clearly stable minima of S, and any other critical points at which $\mu \neq 0$ are higher extrema of S, which in our applications are unstable. In the case of Yang-Mills theory, the stable minima of the action are the flat connections on Σ, and the higher extrema are connections with non-zero curvature which represent classical solutions of Yang-Mills theory, so that $d_A{\star}F_A = 0$ with $F_A \neq 0$.

For our application to Chern-Simons theory, we must understand localization at both the flat and the non-flat solutions of classical Yang-Mills theory. So in the rest of Section 4.2, we review following [18] how non-abelian localization works for flat connections, and then in Section 4.3 we discuss the generalization for solutions of Yang-Mills theory with curvature.

Localization on a Smooth Component of the Moduli Space of Flat Connections

We assume that $\mathcal{M}_0$ is a smooth component of the moduli space of flat connections on Σ. For ease of future notation, we make the identifications

$$\begin{aligned} X &= \mathcal{A}(P)\,, \\ H &= \mathcal{G}(P)\,, \\ \mu &= F_A\,. \end{aligned} \tag{4.4.16}$$

We now identify $\mathcal{M}_0$ abstractly as a symplectic quotient of the zero locus $\mu^{-1}(0) \subset X$ by the free action of the group H, so that $\mathcal{M}_0 = \mu^{-1}(0)/H$.

The fundamental result of [18], whose derivation we now recall, is that the local contribution $Z(\epsilon)|_{\mathcal{M}_0}$ to the path integral from $\mathcal{M}_0$ is given by the topological expression

$$Z(\epsilon)|_{\mathcal{M}_0} \;=\; \int_{\mathcal{M}_0} \exp\left(\Omega + \epsilon\,\Theta\right). \tag{4.4.17}$$

Here Ω is the symplectic form on $\mathcal{M}_0$ induced from the corresponding symplectic form on X (also denoted previously by Ω), and Θ is a characteristic class of degree four on $\mathcal{M}_0$ which appears explicitly as part of the derivation of (4.4.17). In particular, when the coupling ϵ is zero, then $Z(0)|_{\mathcal{M}_0}$ is the symplectic volume of $\mathcal{M}_0$.

To derive (4.4.17) by localization, we start by considering the local geometry of the zero set $\mu^{-1}(0)$ in X. Thus, we let N be a small open neighborhood of $\mu^{-1}(0)$ in X, so that $\mu^{-1}(0) \subset N \subset X$. We assume that this neighborhood is chosen so that N is preserved by the action of H and so that N retracts equivariantly onto $\mu^{-1}(0)$. By composing this retraction with the quotient by the action of H, we define a projection $pr : N \to \mathcal{M}_0$. Thus, denoting the fiber of pr by F, we have the following equivariant bundle

$$F \longrightarrow N \xrightarrow{pr} \mathcal{M}_0\,. \tag{4.4.18}$$

The symplectic integral which describes the local contribution of $\mathcal{M}_0$ to Z is now given by

$$Z(\epsilon)|_{\mathcal{M}_0} \;=\; \frac{1}{\mathrm{Vol}(H)} \int_{\mathfrak{h}\times N} \left[\frac{d\phi}{2\pi}\right] \exp\left[\Omega - i\,\langle\mu,\phi\rangle - \frac{\epsilon}{2}(\phi,\phi) + t\,D\lambda\right], \tag{4.4.19}$$

where λ is the invariant one-form that we introduced in (4.4.12) to localize Z. Because N is noncompact, this integral in (4.4.19) is only defined by localization, so that we require $t \neq 0$.

As explained in detail in [18], because N retracts equivariantly onto $\mathcal{M}_0$ and because the action of H is free near $\mu^{-1}(0)$, the equivariant cohomology class of degree two represented by the expression $\Omega - i\,\langle\mu,\phi\rangle$ in (4.4.19) is simply the pullback by pr of the induced symplectic form on $\mathcal{M}_0$. (We recall that ϕ carries degree $+2$ with respect to equivariant cohomology.) Similarly, the equivariant cohomology class of degree four represented by $-\frac{1}{2}(\phi,\phi)$ in (4.4.19) is the pullback by pr of an ordinary cohomology class Θ of degree four on $\mathcal{M}_0$. Since H acts freely on $\mu^{-1}(0)$, Θ represents a degree four characteristic class of $\mu^{-1}(0)$ regarded as a principal H-bundle over $\mathcal{M}_0$.

Thus, as the only term appearing in the argument of the exponential in (4.4.19) which does not pull back from $\mathcal{M}_0$ is $t\,D\lambda$ itself, to derive (4.4.17) from (4.4.19) we must only show that the integral of $\exp(tD\lambda)$ over the fiber F of (4.4.18) produces a trivial factor of 1,

$$\frac{1}{\mathrm{Vol}(H)}\int_{\mathfrak{h}\times F}\left[\frac{d\phi}{2\pi}\right]\exp\left[t\,D\lambda\right]\,=\,1\,. \tag{4.4.20}$$

This computation is what we must essentially generalize to discuss localization at non-flat Yang-mills solutions, so we review it in detail.

A Local Model For F From Hodge Theory

In order to perform the direct computation of the integral in (4.4.20), we first identify the correct local model for the geometry of F. By assumption, the group H acts freely on F, so F must contain a copy of H. Since F must also be symplectic, the simplest local model for F is just the cotangent bundle T^*H of H, with its canonical symplectic structure.

In fact, the simple guess that $F = T^*H$ is precisely correct, and it has an important infinite-dimensional interpretation in the context of Yang-Mills theory. To explain this interpretation, we consider the tangent space to $\mathcal{A}(P)$ at a point corresponding to a flat connection A. As we have discussed, the tangent space to $\mathcal{A}(P)$ at A can be identified with the space of smooth sections $\Gamma(\Sigma, \Omega^1_\Sigma \otimes \mathrm{ad}(P))$ of the bundle of one-forms on Σ taking values in the adjoint bundle $\mathrm{ad}(P)$.

By definition, the flatness of A implies that the covariant derivative d_A satisfies $d_A^2 = 0$. Because of this fact, d_A has many of the same properties as the de Rham exterior derivative d, and the usual Hodge decomposition for d has an immediate analogue for d_A.

In the case of the covariant derivative d_A, the Hodge decomposition implies that the vector space $\Gamma(\Sigma, \Omega^1_\Sigma \otimes \mathrm{ad}(P))$ decomposes into three subspaces, orthogonal with respect to the metric induced by $\star$ on $\mathcal{A}(P)$, of the form

$$\Gamma(\Sigma, \Omega^1_\Sigma \otimes \mathrm{ad}(P))\,=\,\mathcal{H}_1 \oplus \mathrm{Im}(d_A) \oplus \mathrm{Im}(d_A^\dagger)\,. \tag{4.4.21}$$

Here $d_A^\dagger = -\star d_A \star$ is the standard adjoint to d_A with respect to the metric on $\mathcal{A}(P)$. Also, $\mathcal{H}_1$ denotes the finite-dimensional subspace of harmonic one-forms taking values in

$\mathrm{ad}(P)$, so that elements of $\mathcal{H}_1$ are annihilated by the Laplacian $\Delta_A = d_A d_A^\dagger + d_A^\dagger d_A$. Finally, $\mathrm{Im}(d_A)$ and $\mathrm{Im}(d_A^\dagger)$ denote the images of d_A and $d_A^\dagger$ when these operators act respectively on sections of the bundles $\mathrm{ad}(P)$ and $\Omega^2_\Sigma \otimes \mathrm{ad}(P)$ on Σ.

Concretely, the Hodge decomposition implies that, if η is any section of $\Omega^1_\Sigma \otimes \mathrm{ad}(P)$, then η can be uniquely written as a sum of three terms, all orthogonal,

$$\eta \,=\, \xi \,+\, d_A\phi \,+\, d_A^\dagger\Psi\,, \tag{4.4.22}$$

where ξ satisfies $\Delta_A\xi = 0$ and where ϕ and Ψ are respectively sections of the bundles $\mathrm{ad}(P)$ and $\Omega^2_\Sigma \otimes \mathrm{ad}(P)$.

To interpret the Hodge decomposition (4.4.21) as a geometric statement, we note that the finite-dimensional vector space $\mathcal{H}_1$ of harmonic one-forms can be identified with the tangent space to the moduli space $\mathcal{M}_0$ of flat connections at A. For instance, since $d_A^2 = 0$, we can consider the cohomology of d_A. As usual, we identify the harmonic forms in $\mathcal{H}_1$ as representatives of cohomology classes in $H^1(\Sigma, \mathrm{ad}\,(P))$. These cohomology classes describe infinitesimal deformations of the flat connection A.

On the other hand, since we assume that the gauge group $\mathcal{G}(P)$ acts freely at A, d_A has no kernel when acting on sections of $\mathrm{ad}(P)$. Otherwise, if a section ϕ of $\mathrm{ad}(P)$ did satisfy $d_A\phi = 0$, then the gauge transformation generated by ϕ would fix A. Equivalently, we have that $H^0(\Sigma, \mathrm{ad}\,(P)) = 0$.

Because d_A has no kernel when acting on sections of $\mathrm{ad}(P)$, d_A can be formally inverted and the image of d_A in $\Gamma(\Sigma, \Omega^1_\Sigma \otimes \mathrm{ad}(P))$ identified with the space of sections of $\mathrm{ad}(P)$ itself. Of course, a section ϕ of $\mathrm{ad}(P)$, as appears in (4.4.22), is interpreted geometrically as a tangent vector to the gauge group $\mathcal{G}(P)$.

Similarly, we can also identify the image of the adjoint $d_A^\dagger$ with the space of sections of the bundle $\Omega^2_\Sigma \otimes \mathrm{ad}(P)$. Such a section Ψ, as in (4.4.22), is interpreted geometrically as a cotangent vector to the gauge group $\mathcal{G}(P)$.

Furthermore, if we recall the natural symplectic form Ω on $\mathcal{A}(P)$ in (4.2.3), we see that $\mathrm{Im}(d_A)$ is isotropic with respect to Ω. For if ϕ and ψ are any two sections of the bundle ad(P) on Σ, then

$$\Omega(d_A\phi, d_A\psi) \,=\, -\int_\Sigma \mathrm{Tr}(d_A\phi\wedge d_A\psi) \,=\, \int_\Sigma \mathrm{Tr}(\phi\, d_A^2\psi) \,=\, 0\,. \tag{4.4.23}$$

This fact crucially relies on the flatness of A, since we use that $d_A^2 = 0$ in deducing the last equality of (4.4.23). Of course, the fact that $\mathrm{Im}(d_A)$ is isotropic with respect to Ω is mirrored by the fact that H is a Lagrangian submanifold of T^*H.

Thus, the Hodge decomposition (4.4.21) applied to $\Gamma(\Sigma, \Omega^1_\Sigma \otimes \mathrm{ad}(P))$ locally reflects the geometric statement that F is modeled on the cotangent bundle T^*H. In this example, it may seem perverse to translate the simple statement that $F = T^*H$ into the infinite-dimensional statement of the Hodge decomposition. However, when we consider the corresponding local geometry for higher critical points, this infinite-dimensional perspective allows us to deduce directly how the simple symplectic model based on T^*H must be modified to describe higher critical points of Yang-Mills theory.

Computing a Symplectic Integral on T^*H

Having identified the symplectic model for F as the cotangent bundle T^*H, we compute in the remainder of this subsection the symplectic integral

$$\frac{1}{\mathrm{Vol}(H)} \int_{\mathfrak{h}\times T^*H} \left[\frac{d\phi}{2\pi}\right] \exp\left[t\, D\lambda\right]. \tag{4.4.24}$$

We review this short computation from [18] simply because we must generalize it to discuss localization at non-flat Yang-Mills connections.

Thus, we consider the symplectic manifold T^*H with its canonical symplectic structure. By convention, the action of H on T^*H is induced from the right action of H on itself. By passing to a basis of right-invariant one-forms and using the invariant metric $(\cdot,\cdot)$ on H, we identify $T^*H \cong H\times\mathfrak{h}$. Under this identification, we introduce coordinates (g,γ) on $H\times\mathfrak{h}$.

In these coordinates, the canonical right-invariant one-form on H which takes values in $\mathfrak{h}$ is given by

$$\theta \,=\, dg g^{-1}\,. \tag{4.4.25}$$

In terms of θ, the canonical symplectic structure on T^*H is given by the invariant two-form

$$\begin{aligned} \Omega \,&=\, d(\gamma,\theta) \,=\, (d\gamma,\theta) + (\gamma, d\theta)\,,\\ &= \left(d\gamma + \frac{1}{2}[\gamma,\theta],\theta\right)\,, \end{aligned} \tag{4.4.26}$$

where in passing to the second line of (4.4.26) we recall that $d\theta = \theta\wedge\theta = \frac{1}{2}[\theta,\theta]$. Also, if ϕ is an element of $\mathfrak{h}$, then the corresponding vector field $V(\phi)$ on T^*H which is generated by the infinitesimal right-action of ϕ is given by

$$\delta g \,=\, -g\phi\,, \qquad \delta\gamma \,=\, 0\,. \tag{4.4.27}$$

To proceed, we require an explicit formula for the invariant one-form λ appearing in (4.4.24). Abstractly, $\lambda = (\mu, J\,d\mu)$ is determined by the moment map μ for the H-action on T^*H and an almost complex structure J compatible with Ω in (4.4.26), both of which are easy to determine. A convenient formula for λ was obtained in [18]. In brief, one has $\langle\mu,\phi\rangle = -(\gamma, g\phi g^{-1})$, and one defines a G-invariant almost complex structure compatible with Ω by

$$J(\theta) = -\left(d\gamma + \frac{1}{2}[\gamma,\theta]\right)\,, \qquad J\left(d\gamma + \frac{1}{2}[\gamma,\theta]\right) = \theta\,. \tag{4.4.28}$$

One then finds that $(\mu, J\,d\mu) = (\gamma,\theta)$ after using the fact that $[\gamma,\gamma]=0$. So finally

$$\lambda \,=\, (\mu, J\,d\mu) \,=\, (\gamma,\theta)\,. \tag{4.4.29}$$

Thus, from (4.4.27), (4.4.29), and the definition of D in (4.4.4), we see that

$$D\lambda \,=\, \Omega - i\left(\gamma, g\phi g^{-1}\right)\,. \tag{4.4.30}$$

Without loss, we set $t=1$ in (4.4.24) and we change variables from ϕ to $g\phi g^{-1}$, under which the measure $[d\phi]$ on $\mathfrak{h}$ is invariant. Then the symplectic integral takes the simple form

$$\frac{1}{\mathrm{Vol}(H)}\int_{\mathfrak{h}\times T^*H}\left[\frac{d\phi}{2\pi}\right]\exp\Big[\Omega - i\,(\gamma,\phi)\Big]\,. \tag{4.4.31}$$

The integral over γ can be done using the fact that

$$\int_{-\infty}^{+\infty} dy\,\exp\left(-ixy\right) \,=\, 2\pi\,\delta(x)\,, \tag{4.4.32}$$

and the resulting multi-dimensional delta function can be used to perform the integral over ϕ. We note that the factors of 2π from (4.4.32) nicely cancel the factors of 2π in the measure for ϕ. Finally, the remaining integral over g in H produces a factor of the volume $\mathrm{Vol}(H)$ which cancels the prefactor in (4.4.31). Thus, assuming T^*H is suitably oriented, the symplectic integral over T^*H is indeed 1, as claimed in (4.4.20).

4.4.3 Non-Abelian Localization For Yang-Mills Theory, Part II

We now study localization at the higher, unstable critical points of the Yang-Mills action, which correspond to non-flat connections which solve the Yang-Mills equation on Σ. We begin with some generalities about these connections.

We first introduce the notation f for the section of $\mathrm{ad}(P)$ dual to the curvature F_A,

$$f = \star F_A \,. \tag{4.4.33}$$

Then, by definition, any Yang-Mills solution on Σ satisfies the classical equation of motion

$$d_A f = 0 \,. \tag{4.4.34}$$

This equation simply expresses the geometric condition that f be a covariantly constant section of $\mathrm{ad}(P)$, and we can consequently regard f as an element of the Lie algebra $\mathfrak{g}$ of G.

Because f is constant, f yields a reduction of the structure group G of the bundle to the subgroup $G_f \subset G$ which commutes with f. In physical terms, the background curvature breaks the gauge group from G to G_f.

As a result of the reduction from G to G_f, any non-flat Yang-Mills solution for gauge group G can be succinctly described as a flat connection for gauge group G_f which is twisted by a constant curvature line bundle associated to the $U(1)$ subgroup of G generated by f.

In general, we denote by $\mathcal{M}_f$ the moduli space of Yang-Mills connections whose curvature lies in the conjugacy class of f. We have already discussed localization on the moduli space $\mathcal{M}_0$ of flat connections, for which $G_0 = G$. At the opposite extreme, f breaks G to a maximal torus G_f commuting with f. We refer to such a Yang-Mills solution as "maximally reducible," and one basic goal in this section is to obtain an explicit formula, as in

(4.4.17), for the contribution to the path integral from the corresponding moduli space $\mathcal{M}_f$ of maximally reducible Yang-Mills solutions. Of course, we could also consider the local contributions from Yang-Mills solutions between the extremes of the flat and maximally reducible connections, but this further generalization is not necessary for our discussion of Chern-Simons theory.

Because f is constant, the adjoint action of f determines a bundle map from $\mathrm{ad}(P)$ to itself, and a good idea is to decompose $\mathrm{ad}(P)$ under this action. With our conventions, f is anti-hermitian, so following [79] we introduce a hermitian operator Λ,

$$\Lambda \,=\, i\,[f,\,\cdot\,]\,, \tag{4.4.35}$$

which acts on a section ϕ of $\mathrm{ad}(P)$ as $\Lambda\,\phi \,=\, i\,[f,\phi]$.

When we consider the action of Λ, it is natural to work with complex, as opposed to real, quantities. So we now consider in place of the real bundle $\mathrm{ad}(P)$ the complex bundle $\mathrm{ad}_{\mathbb{C}}(P) = \mathrm{ad}(P)\otimes\mathbb{C}$. When we complexify $\mathrm{ad}(P)$, the $(1,0)$ and $(0,1)$ parts of an $\mathrm{ad}(P)$-valued connection become independent complex variables. After picking a local complex coordinate z on Σ, these can be written locally as A_z and $A_{\overline{z}}$.

Under the action of Λ, the bundle $\mathrm{ad}_{\mathbb{C}}(P)$ decomposes into a direct sum of subbundles, each associated to a distinct eigenvalue of Λ. For our purposes, we need only consider the decomposition of $\mathrm{ad}_{\mathbb{C}}(P)$ into the positive, zero, and negative eigenspaces of Λ,

$$\mathrm{ad}_{\mathbb{C}}(P) \,=\, \mathrm{ad}_{+}(P)\oplus\mathrm{ad}_{0}(P)\oplus\mathrm{ad}_{-}(P)\,, \tag{4.4.36}$$

where $\mathrm{ad}_{\pm}(P)$ and $\mathrm{ad}_{0}(P)$ denote respectively the subbundles of $\mathrm{ad}_{\mathbb{C}}(P)$ associated to these eigenspaces. The eigenspace decomposition of $\mathrm{ad}_{\mathbb{C}}(P)$ in (4.4.36) will play an important role shortly.

Example: $G = SU(2)$

As a simple example of these ideas, we consider the higher Yang-Mills critical points when the gauge group G is $SU(2)$. In this case, all non-flat Yang-Mills solutions are maximally reducible, since any $f\neq 0$ reduces the structure group to a maximal torus $U(1)\subset SU(2)$.

The rank-one case $G = SU(2)$ of Yang-Mills theory is also the essential case to understand for our application to Chern-Simons gauge theory, with gauge group of arbitrary rank. As we explain in Section 5, near a flat Chern-Simons connection on the three-manifold M, the local geometry in the symplectic manifold $\overline{\mathcal{A}}$ of (4.3.25) can be modeled on the geometry of infinitely-many copies of the geometry near a higher $SU(2)$ Yang-Mills critical point. This correspondence arises heuristically by identifying the background Yang-Mills curvature f, which generates the torus $U(1) \subset SU(2)$, with the geometric curvature of M regarded as a principal $U(1)$-bundle over the surface Σ.

In the case of Yang-Mills theory, since f reduces the structure group of the $SU(2)$ bundle to $U(1)$, the $SU(2)$ bundle on Σ splits as a direct sum of line bundles. As f itself is associated to a constant curvature line bundle on Σ, up to conjugacy f takes the form

$$f = 2\pi i \begin{pmatrix} n & 0 \\ 0 & -n \end{pmatrix}, \tag{4.4.37}$$

for some integer $n \neq 0$. Because the Weyl group of $SU(2)$ acts on f by sending $n \to -n$, without loss we can assume that $n > 0$.

Introducing the standard generators of $\mathfrak{su}(2)$ regarded as a complex Lie algebra,

$$\sigma_z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4.4.38}$$

we see that Λ acts on $\mathfrak{su}(2)$, and hence on $\mathrm{ad}_{\mathbb{C}}(P)$, with eigenvalues 0 and $\pm 4\pi n$. Thus, in this case the general decomposition of $\mathrm{ad}_{\mathbb{C}}(P)$ in (4.4.36) takes the simple form

$$\mathrm{ad}_{\mathbb{C}}(P) = \mathcal{L}^{-1}(-2n) \oplus \mathcal{O} \oplus \mathcal{L}(2n). \tag{4.4.39}$$

Here $\mathcal{O}$ is the trivial line bundle on Σ, $\mathcal{L}$ is an arbitrary flat line bundle on Σ, and we use the standard notation $\mathcal{L}(2n) = \mathcal{L} \otimes \mathcal{O}(2n)$, where $\mathcal{O}(2n)$ is the $2n$-th tensor power of a fixed line bundle $\mathcal{O}(1)$ of degree one on Σ.

Thus, for each $n > 0$, the choice of a non-flat connection solving the Yang-Mills equation on Σ is determined by the choice of the flat line bundle $\mathcal{L}$ on Σ. Such a line bundle is specified by the $U(1)$ holonomy of its connection, and hence the moduli space of flat line bundles on Σ is parametrized by a complex torus, the Jacobian of Σ. If Σ has genus g, with $2g$

periods, then the Jacobian has complex dimension g. Thus, for fixed $f \neq 0$, the moduli space $\mathcal{M}_f$ of higher critical points of $SU(2)$ Yang-Mills theory on Σ is simply a complex torus of dimension g.

More generally, if we consider an arbitrary gauge group G of rank r such that f breaks G to a maximal torus, then the corresponding moduli space $\mathcal{M}_f$ is again a complex torus of dimension $g\,r$ which describes the holonomy in $U(1)^r$.

The Partition Function of $SU(2)$ Yang-Mills Theory

One of our basic goals in the rest of this section is to compute directly the contributions from higher critical points to the partition function Z of $SU(2)$ Yang-Mills theory. Of course, Z can be computed exactly [100], and we can readily extract from the known expression for Z a formula for the local contributions from the higher critical points. So before we delve into our path integral computation, we present now the answer which we expect to reproduce and we preview its most interesting features.

In general, if the gauge group G is simply-connected, then the partition function of Yang-Mills theory on a unit area Riemann surface of genus g is given by a sum over representations $\mathcal{R}$ of G of the form

$$Z(\epsilon) \;=\; (\mathrm{Vol}(G))^{2g-2} \sum_{\mathcal{R}} \frac{1}{\dim(\mathcal{R})^{2g-2}} \, \exp\left(-\frac{1}{2}\epsilon\, \widetilde{C}_2\,(\mathcal{R})\right). \tag{4.4.40}$$

Here $\widetilde{C}_2(\mathcal{R})$ is a renormalized version of the quadratic Casimir associated to the representation $\mathcal{R}$, and the volume $\mathrm{Vol}(G)$ of G is determined in our conventions by the invariant form $-\mathrm{Tr}$ on the Lie algebra $\mathfrak{g}$. We recall that for $G = SU(r+1)$, "Tr" denotes the trace in the fundamental representation, and the renormalized quadratic Casimir $\widetilde{C}_2(\mathcal{R})$ differs from the usual quadratic Casimir solely by an additive constant.

Finally, because of the possibility of weighting the Yang-Mills path integral on Σ by a purely topological factor $\exp\left(c\,(2g-2)\right)$ for an arbitrary constant c, we have fixed the prefactor in (4.4.40) so that $Z(0)$ agrees, at least up to a sign which we will not try to fix, with the symplectic volume of the moduli space $\mathcal{M}_0$ of flat connections on Σ as computed in [101] from the theory of Reidemeister-Ray-Singer torsion. Our choice of c differs from the choice in [101] simply because the symplectic form Ω in (4.2.3) which we use here is related to the integral symplectic form Ω' used in [101] by $\Omega = 4\pi^2\,\Omega'$.

We now evaluate (4.4.40) in the case $G = SU(2)$. In this case, each representation is labelled by its dimension, so we denote by $\mathcal{R}_n$ the $SU(2)$ representation of dimension n. The renormalized quadratic Casimir of $\mathcal{R}_n$, which is just the usual quadratic Casimir with an additive constant, is then

$$\widetilde{C}_2(\mathcal{R}_n) \;=\; \frac{1}{2}\,n^2\,. \tag{4.4.41}$$

Finally, using the metric on $SU(2)$ determined by $-\mathrm{Tr}$, the volume of $SU(2)$ is given by $\mathrm{Vol}(SU(2)) = 2^{5/2}\pi^2$. This fact follows immediately if we recall that the volume of an S^3 of unit radius is $2\pi^2$. However, in our metric on $SU(2)$, the $U(1)$ subgroup associated to the normalized generator $T_z = \frac{1}{\sqrt{2}}\,\sigma_z$, as in (4.4.38), has length $2\pi\sqrt{2}$, so $SU(2)$ has radius $r = \sqrt{2}$ in our metric. Thus, the partition function (4.4.40) of $SU(2)$ Yang-Mills theory on Σ becomes

$$Z(\epsilon) \;=\; \left(32\pi^4\right)^{g-1} \sum_{n=1}^{\infty} \frac{1}{n^{2g-2}}\,\exp\left(-\frac{\epsilon\,n^2}{4}\right). \tag{4.4.42}$$

In order to extract the contributions of the higher critical points from (4.4.42), we first differentiate $Z(\epsilon)$ with respect to ϵ to cancel the prefactor $n^{-2(g-1)}$ in the summand of (4.4.42),

$$\frac{\partial^{g-1} Z(\epsilon)}{\partial\epsilon^{g-1}} \;=\; \left(-8\pi^4\right)^{g-1} \sum_{n=1}^{\infty} \exp\left(-\frac{\epsilon\,n^2}{4}\right) \;=\; \frac{1}{2}\,\left(-8\pi^4\right)^{g-1}\left(-1 \,+\, \sum_{n\in\mathbb{Z}} \exp\left(-\frac{\epsilon\,n^2}{4}\right)\right). \tag{4.4.43}$$

To obtain a manifestly convergent expression in the weak coupling regime of small ϵ, we apply Poisson summation to the last term of (4.4.43) to obtain

$$\frac{\partial^{g-1} Z(\epsilon)}{\partial\epsilon^{g-1}} \;=\; \frac{1}{2}\,\left(-8\pi^4\right)^{g-1}\left(-1 \,+\, \sqrt{\frac{4\pi}{\epsilon}} \sum_{n\in\mathbb{Z}} \exp\left(-\frac{(2\pi n)^2}{\epsilon}\right)\right). \tag{4.4.44}$$

Finally, to identify the contribution in (4.4.44) from higher Yang-Mills critical points, we observe that at a higher critical point of degree n, the classical Yang-Mills action S_n determined by f in (4.4.37) is given by $S_n = (2\pi n)^2/\epsilon$ (assuming Σ has unit area). The semiclassical contribution to Z from such a critical point is weighted by the usual exponential factor $\exp(-S_n)$, which we see directly in the last term of (4.4.44). Thus, the locus $\mathcal{M}_n$ of higher critical points of degree n contributes to the sum in (4.4.44) as

$$\left.\frac{\partial^{g-1} Z(\epsilon)}{\partial \epsilon^{g-1}}\right|_{\mathcal{M}_n} = \left(-8\pi^4\right)^{g-1} \sqrt{\frac{4\pi}{\epsilon}} \exp\left(-\frac{(2\pi n)^2}{\epsilon}\right). \tag{4.4.45}$$

We note that a trivial factor of two in (4.4.45) arises from the action of the Weyl group, since the two terms in (4.4.44) for both $\pm n$ arise from the higher critical points of degree n.

This expression (4.4.45) is what we compute using localization, and it has a number of interesting features. Most fundamentally, we see that the natural quantity to consider is not Z but its derivative $\partial^{g-1} Z(\epsilon)/\partial\epsilon^{g-1}$. In discussing the higher critical points, we lose nothing by considering this derivative, since any terms in Z that are polynomial in ϵ, and hence are annihilated by the derivative, arise as contributions from the moduli space $\mathcal{M}_0$ of flat connections. Moreover, although the formula in (4.4.45) is expressed in terms of elementary functions, its integral with respect to ϵ cannot be expressed so simply.

We also see from (4.4.45) that the local contributions from the higher critical points to $\partial^{g-1} Z(\epsilon)/\partial\epsilon^{g-1}$ are essentially independent of g and n, apart from a numerical prefactor and the usual exponential dependence on the classical action S_n.

Finally, we see that the only dependence on ϵ in (4.4.45) besides the classical dependence on S_n is through the prefactor proportional to $\epsilon^{-1/2}$. As we will see, this prefactor reflects the geometric fact that the gauge group does not act freely on the locus of non-flat Yang-Mills solutions. To explain this fact, we note that for any Yang-Mills solution the section f of $\mathrm{ad}(P)$ satisfies $d_A f = 0$, so that $f \neq 0$ generates a $U(1)$ subgroup of the gauge group $\mathcal{G}(P)$ that fixes the corresponding point of $\mathcal{A}(P)$.

This geometric observation about higher critical points of Yang-Mills theory is actually a general property of any higher critical points of the abstract symplectic model with quadratic action $S = \frac{1}{2}(\mu, \mu)$. Namely, the abstract Hamiltonian group H can never act freely at a higher critical point of S.

By definition, such a higher critical point x_0 in the symplectic manifold X is described by the conditions $dS = (\mu, d\mu) = 0$ with $\mu \neq 0$ at x_0. To show that H does not act freely at x_0, we now exhibit a Hamiltonian vector field which vanishes at x_0. We first recall the quantity

$V = \Omega^{-1} d\mu$ which we introduced in Section 4.1. Geometrically V, or $V_a^m = (\Omega^{-1})^{mn} \partial_n \mu_a$ in components, is a linear map from the Lie algebra $\mathfrak{h}$ of H to the space of Hamiltonian vector fields on X. From (4.4.13) and (4.4.14), we see that V trivially satisfies $(\mu, V) = \mu^a V_a^m = 0$ at x_0. But since $\mu(x_0)$ is non-zero, we can consider on X the Hamiltonian vector field generated by $\mu(x_0)$ itself. This vector field is given by $(\mu(x_0)\,, V) = \mu(x_0)^a\, V_a^m$, and by our observations above it vanishes at x_0.

The Hodge Decomposition at a Higher Yang-Mills Critical Point

In many respects, localization at an irreducible, flat Yang-Mills solution is precisely opposite to localization at a maximally reducible, non-flat Yang-Mills solution. In both cases, the local geometry in $\mathcal{A}(P)$ near these critical points can be described as the total space N of an equivariant bundle with infinite-dimensional fiber F over a finite-dimensional moduli space $\mathcal{M}_f$,

$$F \longrightarrow N \stackrel{pr}{\longrightarrow} \mathcal{M}_f\,. \tag{4.4.46}$$

However, in the case of a flat connection the interesting contributions to the integral over N arise from the moduli space $\mathcal{M}_0$ itself, and the integral over the infinite-dimensional fiber $F = T^*H$ contributes a trivial factor of 1. In contrast, for a maximally reducible Yang-Mills solution, the integral over $\mathcal{M}_f$ is essentially trivial, and the interesting contributions arise from the fiber F. Therefore, the most important aspect of our discussion of non-abelian localization at higher critical points in Yang-Mills theory is to identify the correct symplectic model for F, analogous to the identification $F = T^*H$ used previously.

At this point, we can immediately see that a local symplectic model for F based on T^*H does not correctly describe the geometry near $\mathcal{M}_f$ if $f \neq 0$. First, as we have already observed, the gauge group does not act freely at points on $\mathcal{M}_f$, as we used in identifying F with T^*H when we considered the geometry near $\mathcal{M}_0$. Second, if ϕ and ψ are any two sections of $\mathrm{ad}(P)$ representing tangent vectors to $\mathcal{G}(P)$, then the computation in (4.4.23) shows that the symplectic form Ω at a point on $\mathcal{M}_f$ satisfies

$$\Omega(d_A\phi, d_A\psi) \,=\, -\int_\Sigma \mathrm{Tr}(d_A\phi \wedge d_A\psi) \,=\, \int_\Sigma \mathrm{Tr}(\phi\, d_A^2 \psi) \,=\, \int_\Sigma \mathrm{Tr}\left(\phi\; [F_A, \psi]\right)\,. \tag{4.4.47}$$

Here we just use the fact that $d_A^2 = F_A$ is nonzero, and we observe that the last expression in (4.4.47) need not vanish for suitable ϕ and ψ. Thus, the orbit of $\mathcal{G}(P)$ through any point on $\mathcal{M}_f$ is no longer an isotropic submanifold of $\mathcal{A}(P)$, as would be required to model this orbit on H embedded in the cotangent bundle T^*H with its canonical symplectic structure.

Now, the fact that F is not modelled on T^*H at a higher critical point of Yang-Mills theory must be reflected in a breakdown of the naive Hodge decomposition for the corresponding covariant derivative d_A, so that

$$\Gamma(\Sigma, \Omega^1_\Sigma \otimes \mathrm{ad}(P)) \neq \mathcal{H}_1 \oplus \mathrm{Im}(d_A) \oplus \mathrm{Im}(d_A^\dagger)\,. \tag{4.4.48}$$

Thus, a natural strategy to determine the correct symplectic model for F is just to consider how the Hodge decomposition is modified when A is a non-flat solution of the Yang-Mills equation.

In expanding around a flat connection, the tangent space to the moduli space $\mathcal{M}_0$ of flat connections is given by $H^1_{d_A}(\Sigma, \mathrm{ad}(P))$. For a non-flat Yang-Mills connection, d_A only squares to zero when restricted to $\mathrm{ad}_0(P)$, the subspace of $\mathrm{ad}(P)$ that commutes with f. However, any infinitesimal deformation of a Yang-Mills solution must preserve f up to a gauge transformation, since the eigenvalues of f are quantized to take integral values. As a result, tangent vectors to $\mathcal{M}_f$ arise from one-forms valued in the bundle $\mathrm{ad}_0(P)$. More globally, these sections of $\Omega^1_\Sigma \otimes \mathrm{ad}_0(P)$ represent deformations of the Yang-Mills solution by flat connections valued in the subgroup of G that commutes with f. So the tangent space to $\mathcal{M}_f$ is $\mathcal{H}_1 = H^1_{d_A}(\Sigma, \mathrm{ad}_0(P))$. By standard Hodge theory, this can also be defined as

$$\mathcal{H}_1 = H^1_{\bar\partial}(\Sigma, \mathrm{ad}_0(P)). \tag{4.4.49}$$

Similarly, the Lie algebra of the unbroken subgroup G_f, which leaves fixed the given Yang-Mills connection, is

$$\mathcal{H}_0 = H^0_{d_A}(\Sigma, \mathrm{ad}_0(P)) = H^0_{\bar\partial}(\Sigma, \mathrm{ad}_0(P)). \tag{4.4.50}$$

What we have said so far is a fairly direct generalization of the usual statements in the flat case. However, if A is a non-flat Yang-Mills solution, then the usual Hodge theory needs to be modified from the flat case in two essential ways. First, once we get out of $\mathrm{ad}_0(P)$, the image of d_A and the image of $d_A^\dagger$ are no longer transverse. They have a nonzero, finite-dimensional intersection that we will call $\mathcal{E}_0$:

$$\mathrm{Im}(d_A) \cap \mathrm{Im}(d_A^\dagger) = \mathcal{E}_0. \tag{4.4.51}$$

Second, the image of d_A plus the image of $d_A^\dagger$ plus the tangent space $\mathcal{H}_1$ to the moduli space no longer generates $T_P = \Gamma(\Sigma, \Omega^1_\Sigma \otimes \mathrm{ad}(P))$. The quotient $T_P/(\mathrm{Im}(d_A) \oplus \mathrm{Im}(d_A^\dagger))$ is another finite-dimensional vector space $\mathcal{E}_1$. The bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ both have natural complex structures. They will turn out to be

$$\begin{aligned} \mathcal{E}_0 &= H^0_{\overline{\partial}}(\Sigma, \mathrm{ad}_+(P))\,, \\ \mathcal{E}_1 &= H^1_{\overline{\partial}}(\Sigma, \mathrm{ad}_+(P)) \oplus H^1_{\overline{\partial}}(\Sigma, \mathrm{ad}_-(P)). \end{aligned} \tag{4.4.52}$$

We will often regard these complex vector spaces as real vector spaces of twice the dimension.

Thus, the correct generalization of (4.4.48) is informally

$$\Gamma(\Sigma, \Omega^1_\Sigma \otimes \mathrm{ad}(P)) \;=\; \mathcal{H}_1 \oplus \mathrm{Im}(d_A) \oplus \mathrm{Im}(d_A^\dagger) \ominus \mathcal{E}_0 \oplus \mathcal{E}_1\,. \tag{4.4.53}$$

As indicated by our use of "$\ominus$", the expression in (4.4.53) is to be interpreted somewhat in the sense of K-theory. Since $\mathrm{Im}(d_A)$ and $\mathrm{Im}(d_A^\dagger)$ have a non-trivial intersection $\mathcal{E}_0$, this extra copy of $\mathcal{E}_0$ must be removed to get the right description of $\Gamma(\Sigma, \Omega^1_\Sigma \otimes \mathrm{ad}(P))$.

The definition of the Dolbeault cohomology groups in (4.4.52) requires a complex structure on Σ. Abstractly, this complex structure is induced from the duality operator $\star$ on Σ. Because $\star^2 = -1$ when $\star$ acts on any one-form on Σ, we can define the bundles $\Omega^{0,1}$ and $\Omega^{1,0}$ of complex one-forms of either type on Σ by the respective $+i$ and $-i$ eigenspaces of $\star$. This decomposition by type determines the complex structure and hence the Dolbeault $\overline{\partial}$ operator appearing in (4.4.52).

However, for the following we find it useful to give an explicit formula for the operator $\overline{\partial}$, acting on the bundle $\mathrm{ad}_{\mathbb{C}}(P)$, in terms of $\star$ and the covariant derivative d_A. We define the operators $\overline{\partial}^{(p)}$ acting on complex p-forms on Σ taking values in $\mathrm{ad}_{\mathbb{C}}(P)$ by

$$\begin{aligned} \overline{\partial}^{(0)} &= d_A - i\star d_A\,, \\ \overline{\partial}^{(1)} &= -i\,d_A + d_A\star\,, \\ \overline{\partial}^{(2)} &= 0\,. \end{aligned} \tag{4.4.54}$$

Again because $\star^2 = -1$ when acting on one-forms on Σ, one can easily check the essential requirement that $\overline{\partial}^{(1)} \circ \overline{\partial}^{(0)} = 0$. From the expression for $\overline{\partial}^{(1)}$ in (4.4.54), we also see that $\overline{\partial}^{(1)}$ annihilates all one-forms in the $+i$ eigenspace of $\star$, which we have identified with the space of one-forms of type $(0,1)$.

The subbundle $\mathrm{ad}_0(P)$ has a de Rham cohomology (with respect to d_A) that we have already encountered. The subbundles $\mathrm{ad}_+(P)$ and $\mathrm{ad}_-(P)$ do not have de Rham cohomology, but they have Dolbeault cohomology groups

$$H^0_{\overline{\partial}}(\Sigma, \mathrm{ad}_+(P))\,, \quad H^0_{\overline{\partial}}(\Sigma, \mathrm{ad}_-(P))\,, \quad H^1_{\overline{\partial}}(\Sigma, \mathrm{ad}_+(P))\,, \quad H^1_{\overline{\partial}}(\Sigma, \mathrm{ad}_-(P)) \tag{4.4.55}$$

that we should expect will enter somehow. Of these cohomology groups, $H^0_{\overline{\partial}}(\Sigma, \mathrm{ad}_-(P))$ is zero by the Kodaira vanishing theorem [79], which is the reason that $\mathcal{E}_0$ in (4.4.52) only involves $\mathrm{ad}_+(P)$. (We also note parenthetically that $H^1_{\overline{\partial}}(\Sigma, \mathrm{ad}_+(P))$ is similarly zero for critical points associated to line bundles of sufficiently high degree.) So we are left to show that $\mathcal{E}_0$ corresponds to the finite-dimensional intersection of $\mathrm{Im}(d_A)$ and $\mathrm{Im}(d_A^\dagger)$ and $\mathcal{E}_1$ describes the tangent vectors to $\mathcal{A}(P)$ not contained in $\mathrm{Im}(d_A) \oplus \mathrm{Im}(d_A^\dagger) \oplus \mathcal{H}_1$.

We identify $\mathcal{E}_0$ as described in (4.4.51) immediately from our formula for $\overline{\partial}^{(0)}$ in (4.4.54). It is convenient to write $\mathrm{ad}(P) = \mathrm{ad}_0(P) \oplus \mathrm{ad}_\perp(P)$, with $\mathrm{ad}_\perp(P)$ (whose complexification is $\mathrm{ad}_+(P) \oplus \mathrm{ad}_-(P)$) the orthocomplement of $\mathrm{ad}_0(P)$. By standard Hodge theory, if we restrict to $\mathrm{ad}_0(P)$, $\mathrm{Im}(d_A) \cap \mathrm{Im}(d_A^\dagger) = 0$. So the nontrivial intersection of $\mathrm{Im}(d_A)$ and $\mathrm{Im}(d_A^\dagger)$ occurs in $\mathrm{ad}_\perp(P)$. Such an intersection arises if there is $\phi \in \Gamma(\Sigma, \mathrm{ad}_\perp(P))$ and $\Psi \in \Omega^2(\Sigma, \mathrm{ad}_\perp(P))$ such that $d_A\phi = d_A^\dagger\Psi$. If so, let $\psi = \star\Psi$, whereupon, since $d_A^\dagger = -\star d_A \star$ and $\star^2 = -1$, we have $d_A\phi = -\star d_A\psi$. So if $\varphi = \phi + i\psi$, we have $\overline{\partial}^{(0)}\varphi = (d_A - i\star d_A)\varphi = 0$. Hence $\varphi \in H^0_{\overline{\partial}}(\Sigma, \mathrm{ad}_+(P) \oplus \mathrm{ad}_-(P))$. But by Kodaira vanishing, $\mathrm{ad}_-(P)$ does not contribute, and $\varphi \in H^0_{\overline{\partial}}(\Sigma, \mathrm{ad}_+(P))$. This argument can also be run backwards, to map $H^0_{\overline{\partial}}(\Sigma, \mathrm{ad}_+(P))$ to

$\mathcal{E}_0$. This explains the claim that $\mathcal{E}_0 = H^0_{\overline{\partial}}(\Sigma, \mathrm{ad}_+(P))$.

Finally, we can identify $\mathcal{E}_1$, the subspace of $\Gamma(\Sigma, \mathrm{ad}_\perp(P))$ that is orthogonal to the image of d_A and the image of $d_A^\dagger$. We begin with the tautological observation that the orthocomplement of the image of d_A is precisely the kernel of $d_A^\dagger$, and similarly the orthocomplement of the image of $d_A^\dagger$ is precisely the the kernel of d_A. Thus, $\mathcal{E}_1$, the orthocomplement to the image of d_A and $d_A^\dagger$, consists of forms annihilated by both $d_A^\dagger$ and d_A. (We note that although d_A^2 and ${d_A^\dagger}^2$ are nonzero, they annihilate $\Omega^1(\Sigma, \mathrm{ad}_\perp(P))$ for dimensional reasons, so d_A and $d_A^\dagger$ can have a kernel.) Given the formula $\overline{\partial}^{(1)} = -i d_A + d_A \star$, it follows that $\overline{\partial}^{(1)}$ annihilates $\mathcal{E}_1$. Moreover, $\overline{\partial}^{\dagger(1)}$, the $\overline{\partial}^\dagger$ operator acting on one-forms, is $\overline{\partial}^{\dagger(1)} = d_A^\dagger - i d_A^\dagger \star$, and so annihilates $\mathcal{E}_1$. This reasoning can also be read backwards to show that a form annihilated by $\overline{\partial}^{(1)}$ and its adjoint $\overline{\partial}^{\dagger(1)}$ is annihilated by d_A and $d_A^\dagger$ and hence is contained in $\mathcal{E}_1$. By Hodge theory, the joint kernel of $\overline{\partial}$ and $\overline{\partial}^\dagger$ is the same as the cohomology of $\overline{\partial}$. So finally, $\mathcal{E}_1 = H^1_{\overline{\partial}}(\Sigma, \mathrm{ad}_+(P) \oplus \mathrm{ad}_-(P))$, as we have claimed.

A New Symplectic Model For Localization

The Hodge decomposition (4.4.53) implicitly describes the local symplectic model to use at a higher Yang-Mills critical point. We now present this model and compute via localization the canonical symplectic integral in this case.

Abstractly, our local model for F now differs in two ways from the model based on the cotangent bundle T^*H. First, H no longer acts freely at the given critical point. We let $H_0 \subset H$ denote the subgroup of H which fixes the critical point. Thus, the orbit of H through the critical point can be identified with H/H_0. In the case of Yang-Mills theory, the vector space $\mathcal{H}_0$ of harmonic sections of $\mathrm{ad}_0(P)$ is abstractly identified with the Lie algebra $\mathfrak{h}_0$ of H_0.

Second, because of the appearance of $\mathcal{E}_0$ and $\mathcal{E}_1$ in the Hodge decomposition in (4.4.53), the naive model based on the cotangent bundle of the orbit H/H_0 must be modified in the following way. If we simply wanted to discuss the cotangent bundle of the orbit H/H_0, then we could again pass to a basis of right-invariant forms and use the invariant metric $(\cdot, \cdot)$ on

$\mathfrak{h}$ to present $T^*(H/H_0)$ as a homogeneous bundle

$$T^*(H/H_0) \cong H \times_{H_0} (\mathfrak{h} \ominus \mathfrak{h}_0) \,. \tag{4.4.56}$$

Here $\mathfrak{h} \ominus \mathfrak{h}_0$ denotes the orthogonal complement to $\mathfrak{h}_0$ in $\mathfrak{h}$, and "$\times_{H_0}$" indicates that we identify points (g, γ) in the product $H \times (\mathfrak{h} \ominus \mathfrak{h}_0)$ under the following action of H_0,

$$h \cdot (g, \gamma) = \Big(hg \,, h\gamma h^{-1}\Big) \,, \qquad h \in H_0 \,. \tag{4.4.57}$$

To incorporate the appearance of $\mathcal{E}_0$ and $\mathcal{E}_1$ in (4.4.53), we now introduce abstractly a subspace E_0 of the Lie algebra $\mathfrak{h}$ which has a trivial intersection with $\mathfrak{h}_0$ and is preserved under the adjoint action of H_0, so that infinitesimally $[\mathfrak{h}_0, E_0] \subseteq E_0$. This condition certainly holds in Yang-Mills theory for the vector space $\mathcal{E}_0$. Similarly, we introduce another vector space E_1 on which H_0 acts in some representation. We assume that, like the subspace E_0, the representation E_1 admits a metric invariant under the action of H_0.

We now describe our model for F as a homogeneous bundle over the orbit H/H_0 which generalizes (4.4.56). To describe this bundle, we need only specify the fiber of F over the identity coset of H/H_0 and the action of H_0 on the fiber. Thus, as in the modified Hodge decomposition (4.4.53), we subtract E_0 from the cotangent fiber of H/H_0 in (4.4.56), meaning that we take the orthogonal complement to E_0 in $\mathfrak{h} \ominus \mathfrak{h}_0$, and we also add E_1 to the cotangent fiber of H/H_0. So the resulting fiber of F over the identity is given by $\mathfrak{h} \ominus \mathfrak{h}_0 \ominus E_0 \oplus E_1$. By our assumptions on E_0 and E_1, this vector space transforms as a representation of H_0.

In summary, the local model for F is given abstractly by the following homogeneous bundle over H/H_0,

$$F = H \times_{H_0} (\mathfrak{h} \ominus \mathfrak{h}_0 \ominus E_0 \oplus E_1) \,. \tag{4.4.58}$$

We now use γ to denote an element of the orthogonal complement $\mathfrak{h}^\perp$ to $\mathfrak{h}_0 \oplus E_0$ in $\mathfrak{h}$,

$$\gamma \in \mathfrak{h}^\perp = \mathfrak{h} \ominus \mathfrak{h}_0 \ominus E_0 \,, \tag{4.4.59}$$

and we use v to denote a vector in E_1. So in (4.4.58), we identify points (g, γ, v) in the product $H \times (\mathfrak{h}^\perp \oplus E_1)$ under the following action of H_0,

$$h \cdot (g, \gamma, v) \,=\, \Big(hg\,, h\gamma h^{-1}\,, h \cdot v\Big)\,, \qquad h \in H_0\,. \tag{4.4.60}$$

To specify completely our local model, we must also discuss the symplectic structure and the Hamiltonian H-action on F. We will be somewhat brief, since we are just applying standard techniques to construct symplectic bundles, as explained for instance in Ch. 35–41 of [102].

In order to construct a symplectic structure on F, we must make some additional assumptions about the representations E_0 and E_1 of H_0. We first introduce an element γ_0 of $\mathfrak{h}_0$. Abstractly, γ_0 corresponds to the value of the moment map at the given critical point, and in the Yang-Mills context γ_0 is identified with f.

As in Yang-Mills theory, we assume that the hermitian operator Λ,

$$\Lambda \,=\, i\,[\gamma_0, \,\cdot\,]\,, \tag{4.4.61}$$

annihilates $\mathfrak{h}_0$ and acts on the vector spaces E_0 and E_1 with strictly non-zero eigenvalues. The first assumption implies that γ_0 is central in $\mathfrak{h}_0$ and is invariant under the adjoint action of H_0,

$$H_0\,\gamma_0\,H_0^{-1} \,=\, \gamma_0\,. \tag{4.4.62}$$

Because the action of γ_0 preserves the invariant metrics on E_0 and E_1, the action of γ_0 is represented by a real, anti-symmetric matrix. By our second assumption above, this matrix is non-degenerate. Consequently, the decomposition of E_0, and similarly E_1, into the positive and negative eigenspaces of Λ defines a complex structure which is invariant under the action of H_0 and for which the invariant metric $(\cdot,\cdot)$ is hermitian.

Having introduced γ_0, we now describe the symplectic structure on F. As in Section 4.2, we let θ be the canonical right-invariant one-form on H taking values in $\mathfrak{h}$,

$$\theta \,=\, dg g^{-1}\,. \tag{4.4.63}$$

We recall that in the case of the cotangent bundle T^*H or $T^*(H/H_0)$, we can immediately describe the sympletic structure with the manifestly closed and non-degenerate two-form

Ω_0,

$$\Omega_0 \,=\, d(\gamma, \theta)\,, \tag{4.4.64}$$

which reduces on the orbit H/H_0, where $\gamma = 0$, to the canonical form $(d\gamma, \theta)$.

Similarly, when we consider the homogeneous bundle F in (4.4.58), Ω_0 in (4.4.64) still descends to a closed two-form on F. However, because γ now takes values in $\mathfrak{h}^\perp$ as in (4.4.59), the restriction of Ω_0 to the orbit H/H_0 is degenerate on the subspace E_0 of the tangent space to the orbit. Thus, if we ignore the vector space E_1 for the moment, then to construct a symplectic structure on the homogeneous bundle with fiber $\mathfrak{h}^\perp$ over H/H_0 we must supplement the canonical two-form Ω_0 with an additional two-form which is non-degenerate on E_0.

What other two-form should we consider? For motivation, while keeping $E_1 = 0$, let us consider the opposite case from the cotangent bundle. As the cotangent bundle has $E_0 = 0$, the other extreme is for E_0 to be all of $\mathfrak{h} \ominus \mathfrak{h}_0$, so that $\mathfrak{h} \ominus \mathfrak{h}_0 \ominus E_0 = 0$ and $F = H/H_0$. Since we have postulated that γ_0 acts non-degenerately on E_0, while commuting with $\mathfrak{h}_0$, it follows in this case that $\mathfrak{h}_0$ is precisely the subalgebra of $\mathfrak{h}$ that commutes with γ_0. Therefore, H/H_0 is precisely the orbit of γ_0 in the Lie algebra of H. Such an orbit is called a coadjoint orbit (for compact Lie groups the difference between the adjoint representation and its dual is not important here) and has a natural symplectic structure, namely

$$\Omega_1 \,=\, d(\gamma_0, \theta) \,=\, \frac{1}{2}\left(\theta, [\gamma_0, \theta]\right)\,, \tag{4.4.65}$$

where we observe that $d\theta \,=\, \theta \wedge \theta \,=\, \frac{1}{2}[\theta, \theta]$ in deducing the second equality of (4.4.65). Because γ_0 is invariant under the adjoint action of H_0 in (4.4.62), Ω_1 is also invariant under the action of H_0 in (4.4.60) and descends to a manifestly closed and nondegenerate two-form on H/H_0. Indeed, coadjoint orbits are the basic examples of homogeneous symplectic manifolds.

In fact, we have already seen the coadjoint form Ω_1 arise in the context of Yang-Mills theory. We recall from (4.4.47) that the restriction of the Yang-Mills symplectic form Ω on the affine space $\mathcal{A}(P)$ to the orbit of $\mathcal{G}(P)$ through a non-flat Yang-Mills solution is given by

$$\Omega(d_A\phi, d_A\psi) \,=\, \int_\Sigma \mathrm{Tr}\left(\phi\;[F_A, \psi]\right)\,. \tag{4.4.66}$$

Upon identifying the abstract element γ_0 with f, we see that Ω_1 in (4.4.65) precisely represents (4.4.66).

The general case, still with $E_1 = 0$, is a mixture of the cotangent bundle and the coadjoint orbit. We thus naturally add the two two-forms that arise in those two cases and consider the sum

$$\Omega_0 + \Omega_1 \,=\, d(\gamma + \gamma_0, \theta)\,, \tag{4.4.67}$$

which restricts on the orbit H/H_0, where $\gamma = 0$, to the simple expression

$$\left(\Omega_0 + \Omega_1\right)|_{H/H_0} \,=\, (d\gamma, \theta) + \frac{1}{2}\,(\theta, [\gamma_0, \theta])\,. \tag{4.4.68}$$

We see immediately from (4.4.68) that $\Omega_0 + \Omega_1$ defines a symplectic form on a neighborhood of H/H_0 in the homogeneous bundle with fiber $\mathfrak{h}^\perp$. For instance, since the expression in (4.4.67) is manifestly invariant under the right action of H on H/H_0, we need only consider (4.4.68) as restricted to the tangent space $(\mathfrak{h} \ominus \mathfrak{h}_0) \oplus \mathfrak{h}^\perp$ of the bundle at the identity coset on H/H_0. The top power of (4.4.68) on this tangent space is then manifestly non-zero, since all tangent vectors in $\mathfrak{h}^\perp$ are paired by Ω_0 and the remaining tangent vectors to the orbit in E_0 are paired by Ω_1.

Finally, we need to include E_1. By assumption, E_1 has a metric and a complex structure invariant under the action of H_0, so that E_1 has an associated symplectic form $\widetilde{\Omega}$ invariant under H_0.

In order to pass from the symplectic form $\widetilde{\Omega}$ on E_1 to a closed two-form on F which is non-degenerate on the E_1 fiber at the identity coset of H/H_0 and compatible with the bundle structure of F, we must further suppose that H_0 acts on E_1 in a Hamiltonian fashion with moment map $\widetilde{\mu}$. We can always choose $\widetilde{\mu}$ to vanish at the origin of E_1. We also observe that since the action of H_0 on E_1 is linear, of the form $\delta v = \psi \cdot v$ for v in E_1 and ψ in $\mathfrak{h}_0$, the moment map $\widetilde{\mu}$ depends quadratically on v and satisfies $d\widetilde{\mu} = 0$ at the origin of E_1.

With these observations in hand, we consider the two-form Ω_2 defined below,

$$\Omega_2 \,=\, \widetilde{\Omega} + d\langle\widetilde{\mu}\,,\theta\rangle\,. \tag{4.4.69}$$

This two-form is manifestly closed, as $\widetilde{\Omega}$ is closed. It also is clearly invariant under the action of H_0 in (4.4.60).

Finally, to explain the appearance of the second term in (4.4.69), we note that the action of $\mathfrak{h}_0$ on F can be described as follows. For $\psi \in \mathfrak{h}_0$, the corresponding vector field $V(\psi)$ on F acts by

$$\delta g \,=\, \psi g\,, \qquad \delta\gamma \,=\, [\psi, \gamma]\;, \qquad \delta v \,=\, \psi \cdot v. \tag{4.4.70}$$

In order that Ω_2 descend under the quotient by H_0 which defines the bundle, we require that Ω_2 be invariant under H_0 (as we have already seen) and that Ω_2 be annihilated by contraction with $V(\psi)$. By the defining moment map relation, the contraction of $V(\psi)$ with $\widetilde{\Omega}$ is $\iota_{V(\psi)}\widetilde{\Omega} = d\langle\widetilde{\mu}, \psi\rangle$. As for the second term in (4.4.69), the one-form $\langle\widetilde{\mu}, \theta\rangle$ is invariant under the action of H_0 and hence annihilated by the Lie derivative $\mathcal{L}_{V(\psi)} = \{d, \iota_{V(\psi)}\}$. Thus we see that $\iota_{V(\psi)}\, d\langle\widetilde{\mu}, \theta\rangle = -d\,\iota_{V(\psi)}\langle\widetilde{\mu}, \theta\rangle = -d\langle\widetilde{\mu}, \psi\rangle$, which cancels the contraction of $\iota_{V(\psi)}$ with $\widetilde{\Omega}$.

Because $\widetilde{\mu} = d\widetilde{\mu} = 0$ at the origin of E_1, the restriction of Ω_2 to the orbit H/H_0 in F is simply the symplectic form $\widetilde{\Omega}$ on E_1. Thus, the sum of Ω_0, Ω_1, and Ω_2 defines a symplectic form Ω on a neighborhood of the orbit H/H_0 in F,

$$\begin{aligned} \Omega \,&=\, \Omega_0 + \Omega_1 + \Omega_2\,, \\ &=\, d\,(\gamma + \gamma_0\,, \theta) \,+\, d\langle\widetilde{\mu}\,, \theta\rangle + \widetilde{\Omega}\,. \end{aligned} \tag{4.4.71}$$

Having placed a symplectic structure on F, we are left to consider the action of H on F. As in the model based on the cotangent bundle, we assume that H acts from the right on the orbit H/H_0 in F, so that

$$h \cdot (g, \gamma, v) \,=\, (gh^{-1}, \gamma, v)\,, \qquad h \in H\,. \tag{4.4.72}$$

The corresponding element ϕ in $\mathfrak{h}$ generates the vector field

$$\delta g \,=\, -g\phi\,, \qquad \delta\gamma \,=\, 0\,, \qquad \delta v \,=\, 0\,. \tag{4.4.73}$$

Since the one-form θ appearing in Ω is right-invariant, the symplectic form Ω is manifestly invariant under H.

Finally, using (4.4.71) and (4.4.73), one can easily check that the action of H on F is Hamiltonian with moment map μ given by

$$\langle \mu, \phi \rangle \,=\, \left(\gamma + \gamma_0, g\phi g^{-1}\right) + \left\langle \widetilde{\mu}, g\phi g^{-1} \right\rangle . \tag{4.4.74}$$

In particular, we see that the value of μ at the point corresponding to the identity coset on the orbit H/H_0 is just the dual of γ_0 in $\mathfrak{h}^*$, as we have claimed.

Computing the Symplectic Integral over F

For our applications to both Yang-Mills theory and Chern-Simons theory, we now compute the canonical symplectic integral over F,

$$Z(\epsilon) \,=\, \frac{1}{\mathrm{Vol}(H)} \int_{\mathfrak{h}\times F} \left[\frac{d\phi}{2\pi}\right] \exp\left[\Omega - i\,\langle \mu, \phi \rangle - \frac{\epsilon}{2}(\phi,\phi) + t\, D\lambda\right] . \tag{4.4.75}$$

In this expression, λ is the canonical one-form defined as in (4.4.12) by $\lambda = J\,dS$, where $S = \frac{1}{2}(\mu,\mu)$ and J is a compatible almost-complex structure, and t is a non-zero parameter.

Before we delve into computations, let us make a few remarks about how this symplectic integral over F is to be interpreted. We start by considering the canonical symplectic integral (4.4.8) of the same form as (4.4.75) but defined as an integral over a compact symplectic manifold X instead of F. Because X is compact, this integral is convergent for arbitrary t, including $t = 0$, and does not depend on either t or λ.

By our general analysis of Section 4.1, in the limit $t \to \infty$ and for λ of the canonical form, the integral over X localizes on the critical set of S and reduces to a finite sum of contributions from the components of this set. Although the global integral over X is perfectly defined, independent of t and λ, the contributions from the critical locus of S are only defined via localization, with $t \neq 0$ and λ of the canonical form. For instance, at a higher critical point of S, for which we model the normal symplectic geometry on F, the unstable modes of S make the integral over the non-compact fibers of F ill-defined when $t = 0$. Thus, the symplectic integral $Z(\epsilon)$ over F as in (4.4.75) represents a *definition* of the local contribution from an unstable critical point of S in X.

Although we use the canonical one-form $\lambda = J\,dS$ to define via localization the integral over F in (4.4.75), we are free to compute $Z(\epsilon)$ using any other invariant form λ' which is homotopic to λ on F. In particular, though λ is defined globally on X, λ' need only be defined locally on F.

The reason that we might want to compute $Z(\epsilon)$ using some alternative form λ' instead of the canonical one-form λ is just that generically the integral over F defined by λ is not Gaussian even in the limit $t \to \infty$ and cannot be easily evaluated in closed form. See the appendix of [18] for a simple example of this behavior. However, by making a convenient choice for λ', we can greatly simplify our computation and essentially reduce it to the evaluation of Gaussian integrals.

So in order to compute $Z(\epsilon)$ in (4.4.75), we first make a convenient choice for λ'. Since the motivation for our choice is fundamentally to simplify the evaluation of $Z(\epsilon)$, we next evaluate (4.4.75) using λ' in place of λ. Finally, in Appendix A, we perform the analysis required to show that $Z(\epsilon)$ as defined using the canonical one-form λ can be equivalently evaluated using λ'.

To describe our choice for λ', we introduce a projection $\Pi_{\mathfrak{h}_0}$ onto $\mathfrak{h}_0$ and a projection Π_{E_0} onto E_0 in the Lie algebra $\mathfrak{h}$ of H. We define these projections using the invariant metric on $\mathfrak{h}$, so that they are invariant under the adjoint action of H_0 on $\mathfrak{h}$. We then introduce the quantities

$$\begin{aligned} \theta_{\mathfrak{h}_0} &= \Pi_{\mathfrak{h}_0}(\theta)\,, \qquad (g\phi g^{-1})_{\mathfrak{h}_0} = \Pi_{\mathfrak{h}_0}(g\phi g^{-1})\,,\\ \theta_{E_0} &= \Pi_{E_0}(\theta)\,, \qquad (g\phi g^{-1})_{E_0} = \Pi_{E_0}(g\phi g^{-1})\,. \end{aligned} \tag{4.4.76}$$

We now define λ' as

$$\lambda' = (\gamma\,,\theta) - i\left(\theta_{E_0}, g\phi g^{-1}\right) + i\left(\left(g\phi g^{-1}\right)_{\mathfrak{h}_0}\cdot v\,,dv\right) - i\left(\left(g\phi g^{-1}\right)_{\mathfrak{h}_0}\cdot v\,,\theta_{\mathfrak{h}_0}\cdot v\right)\,. \tag{4.4.77}$$

The first term in (4.4.77) has the same form as the canonical one-form which we used for localization on T^*H. However, we recall that now γ takes values not in $\mathfrak{h}$ but in $\mathfrak{h}^\perp = \mathfrak{h} \ominus \mathfrak{h}_0 \ominus E_0$. As before, this first term has degree one under the grading on equivariant cohomology. The other three terms are associated to the new vector spaces E_0 and E_1 that appear at a higher critical point. Since ϕ carries charge $+2$ under the grading on equivariant cohomology, these terms are all of degree three.

The most basic requirement that λ' must satisfy is that it descends to an invariant form on F under the quotient by H_0 which defines the homogeneous bundle. So we first observe that λ' is manifestly invariant under the action of H_0 in (4.4.60). Furthermore, if $V(\psi)$ denotes the vector field on the product $H \times (\mathfrak{h}^\perp \oplus E_1)$ generated by ψ in $\mathfrak{h}_0$ as in (4.4.70), then the first two terms in λ' are trivially annihilated upon contraction with $V(\psi)$ since both γ and θ_{E_0} take values in the orthocomplement to $\mathfrak{h}_0$. Because of the identity

$$\iota_{V(\psi)}\, dv \;=\; \psi \cdot v \;=\; \Big(\iota_{V(\psi)} \theta_{\mathfrak{h}_0}\Big) \cdot v\,, \tag{4.4.78}$$

the last two terms in λ' are also annihilated upon contraction with $V(\psi)$. So λ' descends to a well-defined form on F.

Finally, to check that λ' is invariant under the action of H on F in (4.4.72), we simply note that ϕ transforms under the adjoint action of H so that the quantity $g\phi g^{-1}$ is invariant. Since θ is also invariant under the action of H, λ' is manifestly invariant.

To motivate our definition (4.4.77), we now use λ' to compute the symplectic integral over F. We first compute $D\lambda'$. As we saw when we considered localization on T^*H, the final expression for $D\lambda'$ will only involve ϕ in the invariant combination $g\phi g^{-1}$. Thus, even before presenting our formula for $D\lambda'$, we make the change of variables from ϕ to $g\phi g^{-1}$ in the symplectic integral in order to simplify slightly our result. If we recall that $D = d + i\,\iota_{V(\phi)}$ and we use the formula in (4.4.73) for $V(\phi)$, we find by a straightforward computation that

$$\begin{aligned} D\lambda' \;=\; & (d\gamma, \theta) \,-\, i\,(\gamma, \phi) \,-\, i\,(\theta_{E_0}, [\phi_{\mathfrak{h}_0}, \theta_{E_0}]) \,-\, (\phi_{E_0}, \phi_{E_0}) \,+ \\ & +\, i\,(\phi_{\mathfrak{h}_0} \cdot dv, dv) \,-\, (\phi_{\mathfrak{h}_0} \cdot v, \phi_{\mathfrak{h}_0} \cdot v) \,+\, \mathcal{X}\,. \end{aligned} \tag{4.4.79}$$

Here $\mathcal{X}$ consists of extra terms in $D\lambda'$ that will not actually contribute to the symplectic integral in the limit $t \to \infty$. Explicitly,

$$\begin{aligned} \mathcal{X} \;=\; & \left(\gamma, \frac{1}{2}[\theta,\theta]\right) \,-\, i\left(\frac{1}{2}\left[\theta^\perp, \theta^\perp\right], \phi_{E_0}\right) \,-\, i\left(\left[\theta^\perp, \theta_{E_0}\right], \phi^\perp\right) \,-\, i\left(\frac{1}{2}\left[\theta_{E_0}, \theta_{E_0}\right], \phi^\perp\right) \,- \\ & -\, i\left(\phi_{\mathfrak{h}_0} \cdot v, \frac{1}{2}\left[\theta,\theta\right]_{\mathfrak{h}_0} \cdot v\right) \quad \mathrm{mod}\ \theta_{\mathfrak{h}_0}\,. \end{aligned} \tag{4.4.80}$$

(Terms involving $\theta_{\mathfrak{h}_0}$ in $D\lambda'$, some of which are omitted here, actually cancel since $D\lambda'$ is a pullback from F.) We use the fact that $d\theta = \frac{1}{2}[\theta, \theta]$ to simplify somewhat the form of $\mathcal{X}$, and we use the natural notation $\theta^\perp$ and $\phi^\perp$ to denote the projections of θ and ϕ onto $\mathfrak{h}^\perp$.

In (4.4.79), the first two terms arise from the action of D on the first term in λ', the next two arise from the action of D on the second term in λ', and the final two terms arise from the action of D on the last two terms in λ'. We remark that our choice of the i's that appear in the definition (4.4.77) of λ' was made to ensure that the quadratic terms in (4.4.79) involving ϕ_{E_0} and $\phi_{\mathfrak{h}_0} \cdot v$ are both negative-definite.

We now consider the canonical symplectic integral in (4.4.75) with λ' in place of λ and in the limit $t \to \infty$. This symplectic integral is an integral over the product $\mathfrak{h} \times F$. We can perform this integral over $\mathfrak{h} \times F$ in two steps. First, we hold the projection $\phi_{\mathfrak{h}_0}$ of the variable ϕ in $\mathfrak{h}_0 \subset \mathfrak{h}$ fixed, and we perform the integral over the remaining variables in $\widetilde{F} = (\mathfrak{h} \ominus \mathfrak{h}_0) \times F$. This integral produces a measure on $\mathfrak{h}_0$, which we then use to perform the remaining integral over $\mathfrak{h}_0$. The utility of this way of performing the symplectic integral is that, with our ansatz for λ', we will see that the first integral over $(\mathfrak{h} \ominus \mathfrak{h}_0) \times F$ can be performed directly as a Gaussian integral in the limit $t \to \infty$ and under the assumption that $\phi_{\mathfrak{h}_0}$ acts in a non-degenerate fashion on E_0 and E_1.

To prove this fact, we first consider the symplectic integral over $\widetilde{F} = (\mathfrak{h} \ominus \mathfrak{h}_0) \times F$ which arises if $\mathcal{X}$ is omitted from $D\lambda'$. So we consider the integral

$$I(\phi_{\mathfrak{h}_0}) = \frac{1}{\mathrm{Vol}(H)} \int_{\widetilde{F}} \left[\frac{d\phi}{2\pi}\right] \exp\left[t\,(d\gamma, \theta) - it\,(\gamma, \phi) - it\,(\theta_{E_0}, [\phi_{\mathfrak{h}_0}, \theta_{E_0}]) - t\,(\phi_{E_0}, \phi_{E_0})\right] \times \\ \times \exp\left[it\,(\phi_{\mathfrak{h}_0} \cdot dv, dv) - t\,(\phi_{\mathfrak{h}_0} \cdot v, \phi_{\mathfrak{h}_0} \cdot v)\right]. \tag{4.4.81}$$

For fixed $\phi_{\mathfrak{h}_0}$ acting non-degenerately on E_0 and E_1, this integral (4.4.81) is a Gaussian integral, which we now evaluate. In performing this integral, we recall that the vector spaces E_0 and E_1 carry a complex structure, invariant under the action of $\phi_{\mathfrak{h}_0}$, for which the metric $(\cdot, \cdot)$ is hermitian.

Assuming E_1 is suitably oriented, the Gaussian integral over v in E_1 first produces a factor

$$\det\left(\frac{\phi_{\mathfrak{h}_0}}{2\pi}\bigg|_{E_1}\right)^{-1}\,. \tag{4.4.82}$$

This expression does not depend on t, due to a cancellation between the factors of t that arise from the Gaussian integral over v and the factors of t that appear in the measure on E_1.

The Gaussian integral over ϕ_{E_0} in (4.4.81) next produces a factor proportional to t^{-d_0}, where $d_0 = \dim_{\mathbb{C}} E_0$, which we will absorb momentarily into another determinantal factor arising from E_0.

As in Section 4.2, the integral over γ in $\mathfrak{h}^\perp$ then produces a delta function of $t\phi$ that can be used to perform the integral over the remaining values of ϕ in $\mathfrak{h}^\perp$. This delta function contributes a factor $t^{-\dim \mathfrak{h}^\perp}$.

We are left with an integral over the orbit H/H_0 itself. The measure on H/H_0 now arises from the both the terms $t\,(d\gamma, \theta)$ and $t\,(\theta_{E_0}, [\phi_{\mathfrak{h}_0}, \theta_{E_0}])$ that appear in the exponential in (4.4.81). The term involving $d\gamma$ only receives contributions from directions tangent to $\mathfrak{h}^\perp$ at the identity coset of H/H_0, and the factors of t that arise from expanding the exponential $\exp\left[t\,(d\gamma, \theta)\right]$ cancel the factor $t^{-\dim \mathfrak{h}^\perp}$ from the delta function.

Of course, the remaining term $t\,(\theta_{E_0}, [\phi_{\mathfrak{h}_0}, \theta_{E_0}])$ only receives contributions from directions on H/H_0 tangent to E_0. Upon expanding the exponential $\exp\left[-it\,(\theta_{E_0}, [\phi_{\mathfrak{h}_0}, \theta_{E_0}])\right]$ and absorbing the factor proportional to t^{-d_0} that arises from the corresponding integral over ϕ_{E_0}, we see that the integral over H/H_0 produces an overall factor

$$\frac{\mathrm{Vol}(H)}{\mathrm{Vol}(H_0)}\,\det\left(\frac{\phi_{\mathfrak{h}_0}}{2\pi}\bigg|_{E_0}\right)\,. \tag{4.4.83}$$

Again, the explicit factors of t that arise from the measure on E_0 cancel the factor t^{-d_0} that arises from the Gaussian integral over ϕ_{E_0}. In writing the determinant of $\phi_{\mathfrak{h}_0}$ in (4.4.83), we regard $\phi_{\mathfrak{h}_0}$ as a linear operator acting via the adjoint representation on the complex vector space E_0.

So finally, simplifying the notation by setting $\psi = \phi_{\mathfrak{h}_0}$, the result arising from the

Gaussian integration is

$$I(\psi) \;=\; \frac{1}{\mathrm{Vol}(H_0)}\,\det\left(\frac{\psi}{2\pi}\Big|_{E_0}\right)\,\det\left(\frac{\psi}{2\pi}\Big|_{E_1}\right)^{-1}\,,\qquad \psi\in\mathfrak{h}_0\,. \tag{4.4.84}$$

The result (4.4.84) for the integral (4.4.81) is independent of t. We now observe that the terms in $\mathcal{X}$ which we omitted from $D\lambda'$ when computing (4.4.84) are all of at least third order in the integration variables on $\widetilde{F} = (\mathfrak{h}\ominus\mathfrak{h}_0)\times F$ (which do *not* include the constant $\phi_{\mathfrak{h}_0}$). Thus, upon rescaling all the integration variables by $t^{-\frac{1}{2}}$ so that the quadratic terms in (4.4.81) become independent of t, we see that any contributions from terms in $\mathcal{X}$ to the symplectic integral fall off at least as fast as $t^{-\frac{1}{2}}$ for large t. Thus, our Gaussian evaluation of the symplectic integral over $\widetilde{F}$ is exact as $t\to\infty$.

So we are left to consider the remaining integral over $\mathfrak{h}_0$, which is now given formally by

$$Z'(\epsilon) \;=\; \frac{1}{\mathrm{Vol}(H_0)}\int_{\mathfrak{h}_0}\left[\frac{d\psi}{2\pi}\right]\,\det\left(\frac{\psi}{2\pi}\Big|_{E_0}\right)\,\det\left(\frac{\psi}{2\pi}\Big|_{E_1}\right)^{-1}\,\exp\left[-i\,(\gamma_0,\psi)-\frac{\epsilon}{2}\,(\psi,\psi)\right]. \tag{4.4.85}$$

In obtaining this expression, we recall from (4.4.74) that the value of the moment map μ at the identity coset on the orbit H/H_0 is γ_0. Also, we denote this quantity as $Z'(\epsilon)$, instead of $Z(\epsilon)$, to emphasize that we compute it with λ' instead of the canonical form λ that defines the local contributions to $Z(\epsilon)$.

Now, this formal integral over $\mathfrak{h}_0$ in (4.4.85) might or might not actually be defined. Due to the exponential factor in the integrand of (4.4.85), the integral is certainly convergent at large ψ. However, on the locus in $\mathfrak{h}_0$ where the determinant of ψ acting on E_1 vanishes (for instance at the origin of $\mathfrak{h}_0$), the measure $I(\psi)$ in (4.4.84) might be singular if there is no compensating zero from the determinant of ψ acting on E_0. If $I(\psi)$ is singular, then the integral in (4.4.85) could fail to be convergent at the singularity. Since $Z(\epsilon)$ as defined using the canonical one-form λ is always finite, our computation using λ' cannot generally be valid.

On the other hand, because E_0 and E_1 are both finite-dimensional vector spaces, with

$$\dim_{\mathbb{C}} E_0 \;=\; d_0\,,\qquad \dim_{\mathbb{C}} E_1 \;=\; d_1\,, \tag{4.4.86}$$

the determinants appearing in $I(\psi)$ in (4.4.84) are just invariant polynomials, homogeneous of degrees d_0 and d_1, of ψ in $\mathfrak{h}_0$. For our application to $SU(2)$ Yang-Mills theory, for which $H_0 = U(1)$, we need only consider the simplest case that $\mathfrak{h}_0 = \mathbb{R}$ is one-dimensional. In this case, the invariant polynomials are just monomials

$$\det\left(\frac{\psi}{2\pi}\Big|_{E_0}\right) = c_0\,\psi^{d_0}\,, \qquad \det\left(\frac{\psi}{2\pi}\Big|_{E_1}\right) = c_1\,\psi^{d_1}\,, \tag{4.4.87}$$

for some constants c_0 and c_1.

Assuming (4.4.87), we see that (4.4.85) becomes

$$Z'(\epsilon) = \frac{1}{\mathrm{Vol}(H_0)} \int_{\mathfrak{h}_0} \left[\frac{d\psi}{2\pi}\right] \left(\frac{c_0}{c_1}\right) \psi^{d_0-d_1} \exp\left[-i\,(\gamma_0,\psi) - \frac{\epsilon}{2}\,(\psi,\psi)\right]. \tag{4.4.88}$$

Although this expression in (4.4.88) is ill-defined if $d_1 > d_0$, we can still apply our previous work to compute using λ' a completely well-defined integral. Namely, instead of considering the symplectic integral $Z'(\epsilon)$, we introduce the differential operator Q,

$$Q = \left(-2\frac{\partial}{\partial\epsilon}\right)^{\frac{1}{2}(d_1-d_0)}\,, \tag{4.4.89}$$

and we consider instead the quantity

$$Q\cdot Z'(\epsilon) = \frac{1}{\mathrm{Vol}(H)} \int_{\mathfrak{h}\times F} \left[\frac{d\phi}{2\pi}\right] (\phi,\phi)^{\frac{1}{2}(d_1-d_0)} \exp\left[\Omega - i\,\langle\mu,\phi\rangle - \frac{\epsilon}{2}(\phi,\phi) + t\,D\lambda'\right]. \tag{4.4.90}$$

Using the same definition for λ' and proceeding exactly as before, we compute

$$\begin{aligned} Q\cdot Z'(\epsilon) &= \frac{1}{\mathrm{Vol}(H_0)} \int_{\mathfrak{h}_0} \left[\frac{d\psi}{2\pi}\right] \left(\frac{c_0}{c_1}\right) \exp\left[-i\,(\gamma_0,\psi) - \frac{\epsilon}{2}\,(\psi,\psi)\right], \\ &= \frac{1}{\mathrm{Vol}(H_0)} \left(\frac{c_0}{c_1}\right) \frac{1}{\sqrt{2\pi\epsilon}} \exp\left[-\frac{(\gamma_0,\gamma_0)}{2\epsilon}\right]. \end{aligned} \tag{4.4.91}$$

The fact that the differential operator Q in (4.4.89) can be used to cancel the determinants of ψ in (4.4.87) that arise from localization is a special consequence of our assumption that $\dim\mathfrak{h}_0 = 1$. For an arbitrary Lie algebra $\mathfrak{h}_0$, we cannot generally express these determinants as functions of only the quadratic invariant (ψ,ψ) that appears in the canonical symplectic integral. As a result, in the general case we cannot cancel such determinants simply by differentiating $Z(\epsilon)$ with respect to the coupling ϵ. Though we will not require the generalization for this chapter, we explain in Appendix B how to extend the discussion above to the case of general $\mathfrak{h}_0$.

We see from (4.4.91) that, although our computation using λ' does not always give a sensible answer for $Z'(\epsilon)$, it does give a sensible answer for the derivative $Q{\cdot}Z'(\epsilon)$. Knowledge of this derivative implicitly determines the contribution of a higher critical point to $Z'(\epsilon)$, as the only ambiguity in integrating (4.4.91) is a polynomial in ϵ which cannot arise from a higher critical point. Finally, as we show in Appendix A, the quantity $Q \cdot Z'(\epsilon)$ in (4.4.91) defined using λ' agrees with the corresponding quantity $Q \cdot Z(\epsilon)$ defined using the canonical one-form λ. Hence, provided we take derivatives when necessary, we can use λ' for localization computations on F.

Our computation also shows that it may be easier to consider the contributions of higher critical points not to $Z(\epsilon)$ but to the derivative $Q \cdot Z(\epsilon)$. We have already seen an example of this phenomenon in our discussion of $SU(2)$ Yang-Mills theory. In that case, we found it more natural to compute the contributions of higher Yang-Mills critical points to the derivative $\partial^{g-1} Z(\epsilon)/\partial\epsilon^{g-1}$ in (4.4.45) as opposed to $Z(\epsilon)$ itself.

Application to Higher Critical Points of Yang-Mills Theory

To finish this section, we apply our abstract study of localization on F to compute the path integral contributions from maximally reducible Yang-Mills solutions. We focus on the specific case of $SU(2)$ Yang-Mills theory, for which we reproduce the explicit expression in (4.4.45) for the contributions from the locus $\mathcal{M}_n$ of degree n critical points.

As we have discussed, if $f = \star F_A$ is the curvature of a maximally reducible Yang-Mills solution for gauge group G of rank r, then f breaks the gauge group to a maximal torus $G_f = U(1)^r$. In terms of our abstract model, we thus identify the stabilizer group H_0 with the subgroup $U(1)^r \subset \mathcal{G}(P)$ of constant gauge transformations in this maximal torus. As we have also discussed, this fact implies that the corresponding moduli space $\mathcal{M}_f$ of maximally reducible Yang-Mills solutions is just a complex torus of dimension gr.

Now, our description of the local symplectic model F for the normal geometry over a higher Yang-Mills critical point is completely general, since in deriving the model for F

we did not make any assumptions about the reducibility of the connection. However, if we wish to use this local model to compute contributions from arbitrary higher Yang-Mills critical points, we will generally find that both the integral over F and the integral over the associated moduli space $\mathcal{M}_f$ make nontrivial contributions to $Z(\epsilon)$ which depend on ϵ.

In contrast, if we restrict to the special case that $\mathcal{M}_f$ describes maximally reducible Yang-Mills solutions, then only the integral over F is nontrivial, and the integral over the torus $\mathcal{M}_f$ contributes a multiplicative factor $\mathrm{Vol}(\mathcal{M}_f)$ independent of ϵ, where

$$\mathrm{Vol}\,(\mathcal{M}_f) \,=\, \int_{\mathcal{M}_f} \exp\,(\Omega)\,. \tag{4.4.92}$$

From a physical perspective, the contribution from $\mathcal{M}_f$ to $Z(\epsilon)$ does not involve the coupling ϵ because abelian gauge theory is free. From a mathematical perspective, the Donaldson theory of $U(1)$ bundles is simple, as the corresponding universal bundle is a line bundle having only a first Chern class, which is proportional to Ω.

In the case of $SU(2)$ Yang-Mills theory, the stabilizer group H_0 is just $U(1)$, and $\mathfrak{h}_0$ has dimension one. Thus, we can apply our computation of the integral over F in (4.4.91) to conclude that the local contribution from the moduli space $\mathcal{M}_n$ of higher critical points of degree n is described by

$$\left(-2\frac{\partial}{\partial\epsilon}\right)^{\frac{1}{2}(d_1-d_0)} \cdot Z(\epsilon)\Big|_{\mathcal{M}_n} \,=\, \frac{\mathrm{Vol}(\mathcal{M}_n)}{\mathrm{Vol}(H_0)}\,\left(\frac{c_0}{c_1}\right)\,\frac{1}{\sqrt{2\pi\epsilon}}\,\exp\left[-\frac{(2\pi n)^2}{\epsilon}\right]. \tag{4.4.93}$$

We immediately see that this expression has the same form as the expression that appeared earlier in (4.4.45).

To make a precise comparison of our formula (4.4.93) to (4.4.45), we must compute the various constants appearing in (4.4.93). To start, we introduce the normalized generator T_0 of H_0,

$$T_0 \,=\, \frac{1}{\sqrt{2}}\,\sigma_z \,=\, \frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{4.4.94}$$

which satisfies $\mathrm{Tr}(T_0^2) = -1$. From (4.4.94), we immediately see that the volume of H_0 in our metric on $\mathfrak{h}_0$ is

$$\mathrm{Vol}(H_0) \,=\, 2\pi\sqrt{2}\,. \tag{4.4.95}$$

In the case of $SU(2)$ Yang-Mills theory, we have already identified in (4.4.39) the bundles $\mathrm{ad}_\pm(P)$ with the line bundles $\mathcal{L}(+2n)$ and $\mathcal{L}^{-1}(-2n)$. Thus, from (4.4.52), the complex vector spaces $\mathcal{E}_0$ and $\mathcal{E}_1$, abstractly identified with E_0 and E_1, are now given by the following Dolbeault cohomology groups,

$$\begin{aligned} E_0 &= H^0_{\overline{\partial}}(\Sigma, \mathcal{L}(2n))\,, \\ E_1 &= H^1_{\overline{\partial}}(\Sigma, \mathcal{L}(2n)) \oplus H^1_{\overline{\partial}}(\Sigma, \mathcal{L}^{-1}(-2n))\,. \end{aligned} \tag{4.4.96}$$

The index theorem, in combinating with the vanishing of $H^0_{\overline{\partial}}(\Sigma, \mathcal{L}^{-1}(-2n))$, implies that

$$\begin{aligned} \chi(\mathcal{L}(2n)) &= \dim_{\mathbb{C}} H^0_{\overline{\partial}}(\Sigma, \mathcal{L}(2n)) - \dim_{\mathbb{C}} H^1_{\overline{\partial}}(\Sigma, \mathcal{L}(2n)) = 2n + 1 - g\,, \\ \chi(\mathcal{L}^{-1}(-2n)) &= \dim_{\mathbb{C}} H^1_{\overline{\partial}}(\Sigma, \mathcal{L}^{-1}(-2n)) = 2n - 1 + g\,. \end{aligned} \tag{4.4.97}$$

Thus, from (4.4.97) we determine the exponent $\frac{1}{2}(d_1 - d_0)$ appearing in (4.4.93) to be

$$\frac{1}{2}(d_1 - d_0) = \frac{1}{2}\Big[\chi\left(\mathcal{L}^{-1}(-2n)\right) - \chi\left(\mathcal{L}(2n)\right)\Big] = g - 1\,. \tag{4.4.98}$$

To fix the ratio c_0/c_1 appearing in (4.4.93), which is determined by the determinant of $\psi/2\pi$ acting on E_0 and E_1 as in (4.4.87), we recall that $\mathcal{L}(2n)$ and $\mathcal{L}^{-1}(-2n)$ arise from the standard generators $\sigma_\pm$ of the complex Lie algebra of $SU(2)$, as in (4.4.38). Since σ_z in (4.4.94) acts with eigenvalues $\pm 2i$ on $\sigma_\pm$, we see that $\psi \equiv \psi \cdot T_0$ acts on sections of $\mathcal{L}(2n)$ and $\mathcal{L}^{-1}(-2n)$ with eigenvalues $\pm i\sqrt{2}\,\psi$. Thus, in this case,

$$\begin{aligned} \det\left(\frac{\psi}{2\pi}\Big|_{E_0}\right) \det\left(\frac{\psi}{2\pi}\Big|_{E_1}\right)^{-1} &= \left(\frac{i\sqrt{2}\,\psi}{2\pi}\right)^{2n+1-g} \left(\frac{-i\sqrt{2}\,\psi}{2\pi}\right)^{-2n+1-g}, \\ &= \left(\frac{\psi^2}{2\pi^2}\right)^{1-g}. \end{aligned} \tag{4.4.99}$$

So

$$\left(\frac{c_0}{c_1}\right) = (2\pi^2)^{g-1}\,. \tag{4.4.100}$$

Finally, we must compute the symplectic volume $\mathrm{Vol}(\mathcal{M}_n)$. This is equivalent to the moduli space of flat connections for the group $U(1)$, and appears with the same symplectic structure as if we were doing $U(1)$ gauge theory. The symplectic form is hence equivalent to $\Omega = \sum_{i=1}^g dx_i \wedge dy_i$, where our normalization is such that each of dx_i and dy_i have period

$2\pi\sqrt{2}$ on the appropriate one-cycle. (This is the same factor that appeared in (4.4.95).) Thus,

$$\mathrm{Vol}(\mathcal{M}_n) \,=\, \left(8\pi^2\right)^g \,. \tag{4.4.101}$$

So from (4.4.95), (4.4.98), (4.4.100), and (4.4.101), we evaluate (4.4.93) as

$$\frac{\partial^{g-1} Z(\epsilon)}{\partial \epsilon^{g-1}}\Big|_{\mathcal{M}_n} \,=\, \left(-8\pi^4\right)^{g-1} \sqrt{\frac{4\pi}{\epsilon}}\, \exp\left(-\frac{(2\pi n)^2}{\epsilon}\right), \tag{4.4.102}$$

which agrees with (4.4.45).

4.5 Non-Abelian Localization For Chern-Simons Theory

We now discuss non-abelian localization for Chern-Simons theory on a Seifert manifold M. As we recall from Section 3, the Chern-Simons path integral then takes the symplectic form

$$Z(\epsilon) \,=\, \frac{1}{\mathrm{Vol}(\mathcal{G})} \left(\frac{1}{2\pi i \epsilon}\right)^{\Delta_{\mathcal{G}}/2} \int_{\overline{\mathcal{A}}} \exp\left[\Omega - \frac{1}{2 i\epsilon}\,(\mu, \mu)\right]. \tag{4.5.1}$$

Our general discussion in Section 4 implies that $Z(\epsilon)$ localizes on critical points of the action $S = \frac{1}{2}(\mu, \mu)$. Explicitly,

$$S \,=\, \int_M \mathrm{Tr}\left(A{\wedge}dA + \frac{2}{3} A{\wedge}A{\wedge}A\right) \,-\, \int_M \frac{1}{\kappa{\wedge}d\kappa} \mathrm{Tr}\Big[(\kappa{\wedge}F_A)^2\Big]. \tag{4.5.2}$$

Our first task is thus to classify the critical points of S. We claim that, up to the action of the shift symmetry, the critical points of S correspond precisely to the flat connections on M. To prove this statement, we simply observe that the critical points of S satisfy the equation of motion

$$F_A - \left(\frac{\kappa{\wedge}F_A}{\kappa{\wedge}d\kappa}\right) d\kappa - \kappa{\wedge}d_A \left(\frac{\kappa{\wedge}F_A}{\kappa{\wedge}d\kappa}\right) \,=\, 0\,, \tag{4.5.3}$$

where the first term of (4.5.3) arises from the variation of the Chern-Simons functional and the last two terms arise from the variation of the last term in (4.5.2). To classify solutions of (4.5.3), we recall that S is invariant under the shift symmetry $\delta A \,=\, \sigma\kappa$, where σ is an arbitrary function on M taking values in the Lie algebra $\mathfrak{g}$ of the gauge group G. Under the shift symmetry, the quantity $\kappa{\wedge}F_A$ transforms as

$$\kappa{\wedge}F_A \,\longrightarrow\, \kappa{\wedge}F_A + \sigma\, \kappa{\wedge}d\kappa\,. \tag{4.5.4}$$

Thus, since $\kappa\wedge d\kappa$ is everywhere non-zero on M, we can unambiguously fix a gauge for the shift symmetry by the condition

$$\kappa\wedge F_A \,=\, 0\,. \tag{4.5.5}$$

In this gauge, any solution of the equation of motion (4.5.3) is precisely a flat connection on M. So, as we certainly expect, the Chern-Simons path integral localizes around points of $\overline{\mathcal{A}}$ which represent flat connections on M.

It is interesting to contrast this situation to the case of Yang-Mills theory on a Riemann surface Σ. In that case, the path integral receives contributions from two qualitatively different kinds of critical points, for which the moment map $\mu = F_A$ satisfies either $\mu = 0$ or $\mu \neq 0$, and the critical point is respectively stable or unstable. Since the critical points of Chern-Simons theory are described by flat connections on M, one might naively suppose that these critical points are analogous to the stable critical points of Yang-Mills theory, which are also described by flat connections. However, let us recall our expression from Section 3 for the Chern-Simons moment map,

$$\Big\langle \mu, (p, \phi, a) \Big\rangle \,=\, -\frac{1}{2} p \int_M \kappa\wedge\mathrm{Tr}\,(\pounds_R A\wedge A) + \int_M \kappa\wedge\mathrm{Tr}\,(\phi F_A) - \int_M d\kappa\wedge\mathrm{Tr}\,(\phi A) + a\,. \tag{4.5.6}$$

The last term of (4.5.6) is simply a constant piece of μ dual to the generator a of the central extension of the group $\mathcal{G}_0$, and this generator acts trivially on $\overline{\mathcal{A}}$. As a result of this term, the Chern-Simons moment map is everywhere non-zero, and the critical points of Chern-Simons theory are actually of the same kind as the higher, unstable critical points of Yang-Mills theory.

Our goal in the rest of the chapter is now to compute the local contributions to $Z(\epsilon)$ from two especially simple sorts of flat connections on M. First, we compute the contribution to $Z(\epsilon)$ from the trivial connection when M is a Seifert homology sphere. Second, we compute the contribution to $Z(\epsilon)$ from a smooth component in the moduli space of irreducible flat connections when M is a principal $U(1)$-bundle over a Riemann surface. As we will see, these local computations in Chern-Simons theory are direct generalizations of the local computation at a higher critical point of two-dimensional Yang-Mills theory. The two cases we consider are the extreme cases in which the connection is either trivial or irreducible. Other cases are intermediate between these.

The Normalization of $Z(\epsilon)$

Before we perform any detailed computations, we must make a few general remarks about the normalization of $Z(\epsilon)$. As we see from (4.5.1), we have normalized the Chern-Simons path integral with the formal prefactor

$$\frac{1}{\mathrm{Vol}(\mathcal{G})}\left(\frac{1}{2\pi i\epsilon}\right)^{\Delta_{\mathcal{G}}/2}\,, \qquad \Delta_{\mathcal{G}} = \dim\mathcal{G}\,, \tag{4.5.7}$$

which is defined in terms of the group $\mathcal{G}$ of gauge transformations.

On the other hand, as we discussed in Section 3, the Hamiltonian group which we use for localization in Chern-Simons theory is *not* $\mathcal{G}$ but rather the group $\mathcal{H} = U(1) \ltimes \widetilde{\mathcal{G}}_0$, where $\widetilde{\mathcal{G}}_0$ is a central extension by $U(1)$ of the identity component $\mathcal{G}_0$ of $\mathcal{G}$. We also introduce the group $\mathcal{H}' = U(1) \ltimes \widetilde{\mathcal{G}}$, which arises from the corresponding central extension $\widetilde{\mathcal{G}}$ of the full group $\mathcal{G}$ of all gauge transformations.

When we apply non-abelian localization to Chern-Simons theory, the path integral which we compute most directly is not given by (4.5.1) but by the canonically normalized symplectic integral

$$Z_0(\epsilon) = \frac{1}{\mathrm{Vol}(\mathcal{H}')}\int_{\mathfrak{h}\times\overline{\mathcal{A}}}\left[\frac{d\phi}{2\pi}\right]\exp\left[\Omega - i\,\langle\mu,\phi\rangle - \frac{i\epsilon}{2}(\phi,\phi)\right], \tag{4.5.8}$$

as we computed abstractly in Section 4. The appearance of the volume of the disconnected group $\mathcal{H}'$ in (4.5.8), as opposed to the connected group $\mathcal{H}$, accounts for the action of gauge transformations in the disconnected components of $\mathcal{G}$ on critical points in $\overline{\mathcal{A}}$. Also, because the Chern-Simons path integral is oscillatory, an imaginary coupling $i\epsilon$ now appears in (4.5.8).

If we perform the Gaussian integral over ϕ in (4.5.8), then $Z_0(\epsilon)$ becomes

$$Z_0(\epsilon) = \frac{i}{\mathrm{Vol}(\mathcal{H}')}\left(\frac{1}{2\pi i\epsilon}\right)^{\Delta_{\mathcal{H}}/2}\int_{\overline{\mathcal{A}}}\exp\left[\Omega - \frac{1}{2i\epsilon}\,(\mu,\mu)\right], \qquad \Delta_{\mathcal{H}} = \dim\mathcal{H}\,. \tag{4.5.9}$$

In computing this integral over ϕ, we must be careful to remember that the quadratic form $(\,\cdot\,,\,\cdot\,)$ on the Lie algebra $\mathfrak{h}$ of $\mathcal{H}$ is the direct sum of a positive-definite form on the

Lie algebra of the gauge group $\mathcal{G}$ and a hyperbolic form (with signature $(+,-)$) on the two additional generators in $\mathcal{H}$ relative to $\mathcal{G}$. Had the form on $\mathfrak{h}$ been positive-definite, the Gaussian integral over each generator in $\mathfrak{h}$ would have contributed an identical factor $(2\pi i\epsilon)^{-\frac{1}{2}}$ to the prefactor in front of (4.5.9). However, due to the hyperbolic summand in $(\,\cdot\,,\,\cdot\,)$, the phases that result from the Gaussian integral over the two generators in the hyperbolic subspace of $\mathfrak{h}$ actually cancel. To account for this cancellation, we include the extra factor of 'i' appearing in (4.5.9).

Although $Z_0(\epsilon)$ in (4.5.9) takes the same form as the physical Chern-Simons path integral $Z(\epsilon)$ in (4.5.1), evidently the prefactor (4.5.7) which fixes the normalization of $Z(\epsilon)$ differs from the corresponding prefactor in $Z_0(\epsilon)$ by the ratio

$$\frac{\mathrm{Vol}(\mathcal{H}')}{i\,\mathrm{Vol}(\mathcal{G})}\cdot\left(\frac{1}{2\pi i\epsilon}\right)^{\frac{1}{2}(\Delta_{\mathcal{G}}-\Delta_{\mathcal{H}})} = \mathrm{Vol}(U(1)^2)\cdot 2\pi\epsilon\,. \tag{4.5.10}$$

The finite factors $\mathrm{Vol}(U(1)^2)$ and $2\pi\epsilon$ arise in the obvious way from the two extra generators in $\mathcal{H}$ relative to $\mathcal{G}$.

When we perform localization computations in Chern-Simons theory, we apply our abstract localization computations in Section 4 to compute $Z_0(\epsilon)$. By our observation above, for the purpose of computing the physical Chern-Simons path integral $Z(\epsilon)$, we must multiply the results from our abstract local computations by the finite factor in (4.5.10). As we will see, this expression turns out to cancel nicely against corresponding factors from the local computation.

4.5.1 A Two-Dimensional Interpretation of Chern-Simons Theory on M

Our symplectic interpretation of Chern-Simons theory on M fundamentally relies on the fact that the shift symmetry decouples one component of the gauge field A. As a result, we can essentially perform Kaluza-Klein reduction over the S^1 fiber of M to the base Σ to express Chern-Simons theory as a two-dimensional topological theory on Σ. From this two-dimensional perspective, we can immediately apply our localization computations in Section 4 to Chern-Simons theory.

In fact, the two-dimensional topological theory on Σ arising from Chern-Simons theory on M is closely related to Yang-Mills theory on Σ, a point also recently emphasized in [78]. At the level of the classical moduli spaces, the relationship between Chern-Simons theory on M and Yang-Mills theory on Σ was noted long ago by Furuta and Steer in [94]. These authors identify a correspondence between the moduli space of flat connections on M and certain components of the moduli space of Yang-Mills solutions on Σ. Since the relationship between flat connections on M and Yang-Mills solutions on Σ underlies our study of Chern-Simons theory, we now explain the fundamental aspects of this correspondence.

Flat Connections on M From Yang-Mills Solutions on Σ

We start by considering the moduli space of flat connections on M. As before, we suppose that the gauge group G is compact, connected, simply-connected, and simple.

A flat connection on M is determined by its holonomies, and the moduli space of flat connections on M, up to gauge equivalence, can be concretely described as the space of group homomorphisms from the fundamental group $\pi_1(M)$ to G, up to conjugacy. Hence the structure of the moduli space of flat connections on M is determined by $\pi_1(M)$.

On the other hand, because M is a Seifert manifold, and hence generally a $U(1)$ V-bundle over an orbifold Σ, the structure of $\pi_1(M)$ is closely tied to the structure of the orbifold fundamental group $\pi_1(\Sigma)$. This topological fact underlies the close relationship between flat connections on M and Yang-Mills solutions on Σ, and to explain it we now present the group $\pi_1(M)$.

As in Section 3, we describe M using the Seifert invariants

$$\Big[g; n; (\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)\Big], \quad \gcd(\alpha_j, \beta_j) = 1 \,. \tag{4.5.11}$$

We recall that g is the genus of Σ, n is the degree of the $U(1)$ V-bundle over Σ, and the relatively prime integers (α_j, β_j) for $j = 1, \ldots, N$ specify the local geometry of M near the N orbifold points on Σ.

To present $\pi_1(M)$, we introduce elements

$$\begin{aligned} & a_p\,, b_p\,, \quad p = 1\,, \dots\,, g\,, \\ & c_j\,, \quad j = 1\,, \dots\,, N\,, \\ & h\,. \end{aligned} \tag{4.5.12}$$

Then $\pi_1(M)$ is generated by these elements in (4.5.12) subject to the following relations,

$$\begin{aligned} & [a_p, h] \,=\, [b_p, h] \,=\, [c_j, h] \,=\, 1\,, \\ & c_j^{\alpha_j} h^{\beta_j} \,=\, 1\,, \\ & \prod_{p=1}^{g} \, [a_p, b_p] \, \prod_{j=1}^{N} c_j \,=\, h^n\,. \end{aligned} \tag{4.5.13}$$

We will not give a formal proof of this presentation of $\pi_1(M)$, which follows from the standard surgery construction of M and which can be found in [93], but we will describe the geometric interpretation of the generators in (4.5.12). The generator h, which is a central element of $\pi_1(M)$ by the first line of (4.5.13), arises from the generic S^1 fiber over Σ. Since Σ has genus g, the generators a_p and b_p for $p = 1, \dots, g$ arise from the $2g$ non-contractible cycles on Σ. Finally, the generators c_p for $p = 1, \dots, N$ arise from small one-cycles in Σ about each of the orbifold points. We note that from the presentation of $\pi_1(M)$ in (4.5.12) and (4.5.13) one can immediately compute the corresponding homology group $H_1(M, \mathbb{Z})$ as the abelianization of $\pi_1(M)$.

For example, with a view to our application below, let us determine the condition to have $H_1(M) = 0$. This requires $g = 0$, else the homology of Σ will appear in $H_1(M)$. So $\pi_1(M)$ has generators c_j, $j = 1, \dots, N$, and $c_0 = h$. There are $N + 1$ relations, namely $c_j^{\alpha_j} c_0{}^{\beta_j} = 1$, $j = 1 \dots, N$, and $\prod_{j=1}^N c_j \cdot c_0{}^{-n} = 1$. These relations can be written $\prod_{j=0}^N c_j^{K_{j,l}} = 1$ in terms of an $N + 1 \times N + 1$ matrix K. A general element of $H_1(M)$ of the form $\prod_{j=0}^N c_j^{v_j}$ is trivial if and only if one can write $v_j = \sum_{j'} K_{jj'} w_{j'}$ for some integer-valued vector w; that is, $\prod_{j=0}^N c_j^{v_j}$ is trivial if and only if the vector v_j lies in the integral lattice generated by the matrix $K_{jj'}$. Consequently $H_1(M)$ is trivial if and only if $\det(K) = \pm 1$. With the actual form of K, one can work out this determinant and find that the condition is that

$$n + \sum_{j=1}^{N} \frac{\beta_j}{\alpha_j} = \pm \prod_{j=1}^{n} \frac{1}{\alpha_j}. \tag{4.5.14}$$

The left hand side is also equal to the orbifold first Chern class $c_1(\mathcal{L})$ of the line V-bundle $\mathcal{L}$ discussed in Section 3.2.

With the presentation of $\pi_1(M)$ in (4.5.12) and (4.5.13), we can immediately present $\pi_1(\Sigma)$ as well. Thus, $\pi_1(\Sigma)$ is generated by the elements a_p, b_p, and c_j in (4.5.12), omitting the generator h which arises from the S^1 fiber, and the relations in $\pi_1(\Sigma)$ are given by the relations in (4.5.13) upon setting $h = 1$. A very succinct description of this relation between $\pi_1(M)$ and $\pi_1(\Sigma)$ is to recognize $\pi_1(M)$ as a central extension of $\pi_1(\Sigma)$,

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1(\Sigma) \longrightarrow 1\,, \tag{4.5.15}$$

where h is the generator of $\mathbb{Z}$ above.

Given the close relationship between the groups $\pi_1(M)$ and $\pi_1(\Sigma)$ expressed in (4.5.15), we can immediately deduce a relationship between flat connections on M and Yang-Mills solutions on Σ. To describe this relationship, we consider a homomorphism ρ,

$$\rho : \pi_1(M) \longrightarrow G\,, \tag{4.5.16}$$

which describes the holonomies of a given flat connection on M.

Because h is central in $\pi_1(M)$, the image of ρ must lie in the centralizer $G_{\rho(h)}$ of the element $\rho(h)$ in G. To simplify the following discussion, we suppose that $\rho(h)$ actually lies in the center Γ of G, implying that $G_{\rho(h)} = G$. This condition is necessary whenever the connection described by ρ is irreducible, and it certainly holds also when the connection is trivial, which are the two main cases we consider when we perform computations in Chern-Simons theory. We refer to [94] for a discussion of the general case.

Clearly if $\rho(h) = 1$, so that the corresponding flat connection on M has trivial holonomy around the S^1 fiber over Σ, then ρ factors through the extension (4.5.15) to induce a homomorphism from $\pi_1(\Sigma)$ to G. Hence ρ describes a flat connection on M that pulls back from a flat Yang-Mills connection on Σ.

More generally, when $\rho(h)$ is non-trivial in Γ, then the corresponding flat connection on M has non-trivial holonomy around the S^1 fiber of M and is not the pull back of a flat G-connection on Σ. However, if we pass from G to the quotient group $\overline{G} = G/\Gamma$, so that we consider the connection on M as a flat connection on the trivial $\overline{G}$-bundle, then the holonomy of this connection around the S^1 fiber of M becomes trivial.

As a result, the homomorphism ρ can be interpreted as describing a flat connection on M which arises from the pull back of a flat Yang-Mills connection on a generally non-trivial V-bundle over Σ whose structure group is now $\overline{G}$, as opposed to G. In general, a flat connection on a non-trivial $\overline{G}$-bundle over Σ can be described as a flat connection on the trivial G-bundle over Σ such that the connection has non-trivial monodromies in Γ around the orbifold points as well as around one additional, arbitrarily chosen smooth point of Σ. These monodromies represent the obstruction to smoothly extending the given flat connection to the trivial G-bundle over all of Σ, and hence they describe the non-trivial $\overline{G}$-structure on the bundle.

In the case at hand, we see from the relations (4.5.13) which describe $\pi_1(M)$ as an extension of $\pi_1(\Sigma)$ that the relevant monodromies are determined by the holonomies of the connection on M associated to the elements h^{β_j} and h^n, so that these holonomies determine the topology of the corresponding $\overline{G}$-bundle on Σ. For instance, if we consider the simplest case that the gauge group G is $SU(2)$ and M arises from a principal $U(1)$-bundle over a smooth Riemann surface Σ such that the degree n is odd, then flat connections on M whose holonomies satisfy $\rho(h) = \rho(h)^n = -1$ correspond bijectively to flat $SU(2)$ connections on Σ which have monodromy -1 around a specified puncture. Such flat $SU(2)$ connections can then be identified with flat connections on the topologically non-trivial principal $SO(3)$-bundle over Σ.

On the other hand, if the degree n of the principal $U(1)$-bundle is even, then $\rho(h)^n = 1$ for both $\rho(h) = \pm 1$, so points in both of these components of the moduli space of flat connections on M are identified with flat $SU(2)$ connections on Σ.

The Local Symplectic Geometry Near a Critical Point of Chern-Simons Theory

The discussion above shows that irreducible flat connections on M can be identified with corresponding flat Yang-Mills connections on Σ. We now extend this observation to give a "two-dimensional" description of the local symplectic geometry in $\overline{\mathcal{A}}$ around such a critical point of Chern-Simons theory.

Because $\overline{\mathcal{A}}$ is the quotient of the affine space $\mathcal{A}$ by the shift symmetry $\mathcal{S}$, we are free to work in any convenient gauge for $\mathcal{S}$. For instance, in order to identify the critical points of the new Chern-Simons action S in (4.5.2), we found it convenient to impose the gauge condition (4.5.5).

However, in order to describe the local geometry in $\overline{\mathcal{A}}$ in terms of geometric quantities on Σ, we make a new gauge choice for $\mathcal{S}$, corresponding to the gauge condition

$$\iota_R A \,=\, 0\,. \tag{4.5.17}$$

Because A transforms under the shift symmetry as $\delta A = \sigma\,\kappa$, the quantity $\iota_R A$ transforms as $\iota_R A \to \iota_R A + \sigma$, and the gauge condition in (4.5.17) is unambiguous.

To describe a critical point of the action S in the gauge (4.5.17), we consider as above a flat Yang-Mills connection B_0 on a generally non-trivial V-bundle with structure group $\overline{G}$ over Σ. Then, in the gauge (4.5.17), the full tangent space to the symplectic manifold $\overline{\mathcal{A}}$ at B_0 is described by the space of sections ξ of the bundle $\Omega^1_M \otimes \mathfrak{g}$ which satisfy the gauge condition

$$\iota_R \xi \,=\, 0\,. \tag{4.5.18}$$

Because our symplectic description of Chern-Simons theory respects the geometric $U(1)$ action on M, we naturally consider the decomposition of the tangent space to $\overline{\mathcal{A}}$ under the action of this $U(1)$. In terms of the section ξ, this statement simply means that we consider the Fourier decomposition of ξ into eigenmodes of the operator $\pounds_R$. Thus we write

$$\xi \,=\, \sum_{t=-\infty}^{+\infty} \xi_t\,, \tag{4.5.19}$$

where, in addition to the gauge condition (4.5.18), each eigenmode ξ_t satisfies

$$\pounds_R \xi_t \,=\, -2\pi i t \cdot \xi_t\,. \tag{4.5.20}$$

We can similarly perform this Fourier decomposition on the tangent space to the group of gauge transformations $\mathcal{G}$. Thus, if ϕ is a section of $\Omega^0_M \otimes \mathfrak{g}$, we write

$$\phi \;=\; \sum_{t=-\infty}^{+\infty} \phi_t\,, \tag{4.5.21}$$

where

$$\pounds_R \phi_t \;=\; -2\pi i t \cdot \phi_t\,. \tag{4.5.22}$$

To describe these eigenmodes ξ_t and ϕ_t geometrically on Σ, we recall that $\mathcal{L}$ denotes the line V-bundle over Σ associated to the Seifert manifold M. Since non-trivial representations of the $U(1)$ action on M are associated to non-zero powers of $\mathcal{L}$ on Σ, we can describe the modes ξ_t and ϕ_t geometrically on Σ as being respectively sections of the bundles $\Omega^1_\Sigma \otimes \mathrm{ad}(P) \otimes \mathcal{L}^t$ and $\Omega^0_\Sigma \otimes \mathrm{ad}(P) \otimes \mathcal{L}^t$. Here we have also replaced the trivial bundle $\mathfrak{g}$ on M by the possibly nontrivial $\overline{G}$-bundle $\mathrm{ad}(P)$ on Σ.

So, at least formally, the tangent space to $\overline{\mathcal{A}}$ at B_0 decomposes into the following sum of spaces of sections on Σ,

$$T\overline{\mathcal{A}} = \bigoplus_{t=-\infty}^{+\infty} \Gamma\left(\Sigma, \Omega^1_\Sigma \otimes \mathrm{ad}(P) \otimes \mathcal{L}^t\right)\,, \tag{4.5.23}$$

and similarly for the Lie algebra of $\mathcal{G}$,

$$T\mathcal{G} = \bigoplus_{t=-\infty}^{+\infty} \Gamma\left(\Sigma, \Omega^0_\Sigma \otimes \mathrm{ad}(P) \otimes \mathcal{L}^t\right)\,. \tag{4.5.24}$$

By assumption, the covariant derivative d_{B_0} commutes with the Lie derivative $\pounds_R$,

$$[d_{B_0}\,, \pounds_R] \;=\; 0\,, \tag{4.5.25}$$

so these decompositions are compatible with the action of d_{B_0}.

As in Section 4.2, the local structure of the space of fields over which we integrate near a given component $\mathcal{M}$ of the moduli space of critical points is a fibration

$$F \longrightarrow N \xrightarrow{pr} \mathcal{M}. \tag{4.5.26}$$

As before, F is given by a symplectic bundle

$$F \;=\; \mathcal{H} \times_{H_0} \left(\mathfrak{h} \ominus \mathfrak{h}_0 \ominus \mathcal{E}_0 \oplus \mathcal{E}_1\right), \tag{4.5.27}$$

where the invariance group H_0 and the exceptional bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ must be identified. As we observed at the start of this section, because the Chern-Simons moment map is non-vanishing, the local model is analogous to the geometry near a higher critical point of Yang-Mills theory, with some $\mathcal{E}_0$ and $\mathcal{E}_1$.

In the model (4.5.27) for F, $\mathcal{H} = U(1) \ltimes \widetilde{\mathcal{G}}_0$ is the Hamiltonian group which we use for localization, and H_0 is the subgroup of $\mathcal{H}$ which fixes B_0. In general, H_0 is a finite-dimensional group of the form

$$H_0 \;=\; U(1)^2 \times K_0\,. \tag{4.5.28}$$

One $U(1)$ factor in H_0 arises from the action of $\pounds_R$ on $\overline{\mathcal{A}}$, which fixes B_0 by assumption, and the other $U(1)$ factor arises from the central $U(1)$ in $\widetilde{\mathcal{G}}_0$. This $U(1)$ acts trivially on all of $\overline{\mathcal{A}}$. Finally, K_0 denotes the group of gauge transformations acting on $\mathrm{ad}(P)$ which fix B_0. These gauge transformations are generated by covariantly constant sections ϕ of $\mathrm{ad}(P) \otimes \mathcal{L}^0$, so that ϕ is annihilated by $\pounds_R$, and consequently K_0 commutes with both $U(1)$ factors in H_0.

To identify $\mathcal{E}_0$ and $\mathcal{E}_1$, we must look at the images of d_{B_0} and of $\star_2 d_{B_0}$ mapping $T\mathcal{G}$ to $T\overline{\mathcal{A}}$. The bundle $\mathrm{ad}(P) \otimes \mathcal{L}^t$ has connection $C = B_0 + t\kappa$ (κ is the constant curvature connection on $\mathcal{L}$ introduced in Section 3.2). For fixed t, the three-dimensional operators d_{B_0} and $\star_2 d_{B_0}$ reduce to two-dimensional operators d_C and $\star d_C$. As B_0 is flat, the connection C has curvature equal to t times a positive two-form. So the analysis of the intersection and unions of the images of d_C and $\star d_C$ precisely follows Section 4.3, with the following dictionary between quantities in the two-dimensional analysis of that section and quantities in the present three-dimensional problem:

$$\begin{aligned} \mathrm{ad}_0(P) &\longleftrightarrow \mathrm{ad}(P) \\ \mathrm{ad}_+(P) &\longleftrightarrow \bigoplus_{t>0} \mathrm{ad}(P) \otimes \mathcal{L}^t \\ \mathrm{ad}_-(P) &\longleftrightarrow \bigoplus_{t<0} \mathrm{ad}(P) \otimes \mathcal{L}^t. \end{aligned} \tag{4.5.29}$$

In two dimensions, we decomposed $\mathrm{ad}(P)$ into $\mathrm{ad}_0(P)$, $\mathrm{ad}_+(P)$, and $\mathrm{ad}_-(P)$ according to the sign of the curvature. Here, curvature comes only from $\mathcal{L}$. So finally, we get

$$\begin{aligned} \mathcal{E}_0 \,&=\, \bigoplus_{t\neq 0} H^0_{\bar\partial}(\Sigma, \mathrm{ad}(P)\otimes\mathcal{L}^t) \,=\, \bigoplus_{t\geq 1} H^0_{\bar\partial}(\Sigma, \mathrm{ad}(P)\otimes(\mathcal{L}^t\oplus\mathcal{L}^{-t}))\,,\\ \mathcal{E}_1 \,&=\, \bigoplus_{t\neq 0} H^1_{\bar\partial}(\Sigma, \mathrm{ad}(P)\otimes\mathcal{L}^t) \,=\, \bigoplus_{t\geq 1} H^1_{\bar\partial}(\Sigma, \mathrm{ad}(P)\otimes(\mathcal{L}^t\oplus\mathcal{L}^{-t}))\,. \end{aligned} \tag{4.5.30}$$

Unlike in the case of Yang-Mills theory, these exceptional bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ now have infinite dimension, since the cohomology groups in (4.5.30) are non-zero for infinitely many t's.

4.5.2 Localization at the Trivial Connection on a Seifert Homology Sphere

We are finally prepared to carry out a computation in Chern-Simons theory using non-abelian localization. We consider localization at the trivial connection when M is a Seifert manifold that also is a homology sphere, that is, it has $H_1 = 0$. We start by stating some necessary facts about the topology of M in this case.

Seifert Homology Spheres and a Slight Generalization

We recall that we generally characterize M with the Seifert invariants

$$\Big[g; n; (\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)\Big]\,, \quad \gcd(\alpha_i, \beta_i) = 1\,. \tag{4.5.31}$$

As we have explained above, M is a homology sphere, with $H_1(M, \mathbb{Z}) = 0$, if and only if the invariants in (4.5.31) satisfy

$$g \,=\, 0\,, \qquad\qquad c_1(\mathcal{L}_0) \,=\, n + \sum_{j=1}^N \frac{\beta_j}{\alpha_j} \,=\, \pm \prod_{j=1}^N \frac{1}{\alpha_j}\,. \tag{4.5.32}$$

Here $\mathcal{L}_0$ denotes the line V-bundle over the orbifold Σ which describes M.

To interpret geometrically the condition on $\mathcal{L}_0$ in (4.5.32), we note that this condition implies the arithmetic condition that the numbers α_j be pairwise relatively prime, so that

$$\gcd(\alpha_j, \alpha_{j'}) = 1\,, \qquad j \neq j'\,. \tag{4.5.33}$$

In turn, as explained in Section 1 of [94], this arithmetic condition on the orders of the orbifold points of Σ implies that the Picard group of line V-bundles on Σ is isomorphic to

$\mathbb{Z}$, just as for $\mathbb{CP}^1$. In analogy to the case of S^3, which arises from a generator of the Picard group of $\mathbb{CP}^1$, the condition on $c_1(\mathcal{L}_0)$ in (4.5.32) is then precisely the condition that $\mathcal{L}_0$ generate the Picard group of Σ.

As previously, we orient M so that $c_1(\mathcal{L}_0)$ is positive, and we introduce the notation β_j^0 to distinguish the orbifold invariants of this fundamental line V-bundle $\mathcal{L}_0$ on Σ,

$$c_1(\mathcal{L}_0) \,=\, n + \sum_{j=1} \frac{\beta_j^0}{\alpha_j} \,=\, \prod_{j=1}^{N} \frac{1}{\alpha_j}\,. \tag{4.5.34}$$

The reason that we distinguish the invariants β_j^0 of $\mathcal{L}_0$ is that, more generally, we will also consider the case that M arises not from the fundamental line V-bundle $\mathcal{L}_0$ on Σ but from some multiple $\mathcal{L}_0^d$ for $d \geq 1$. In this case, we simply require that $g = 0$ in (4.5.32) and that the invariants α_j be relatively prime to each β_j and also pairwise relatively prime, as in (4.5.33). The Seifert manifold arising from $\mathcal{L}_0^d$ is a quotient by the cyclic group $\mathbb{Z}_d$ of the Seifert manifold associated to $\mathcal{L}_0$, and in this case $H_1(M, \mathbb{Z}) = \mathbb{Z}_d$. So the integer d can be characterized topologically as the order of $H_1(M, \mathbb{Z})$,

$$d \,=\, |H_1(M, \mathbb{Z})|\,. \tag{4.5.35}$$

These Seifert manifolds are still rational homology spheres, with $H_1(M, \mathbb{R}) = 0$, and the trivial connection on M is an isolated flat connection.

We note that when the Seifert manifold M is described by a smooth, degree n line-bundle over $\mathbb{CP}^1$, then M is a lens space, and the Seifert invariant n coincides with d in (4.5.35).

The Result of Lawrence and Rozansky

Our basic results on localization for Chern-Simons theory imply that the Chern-Simons partition function Z can be expressed as a sum of local contributions from the flat connections on M. In the case $G = SU(2)$ and with M as above, Lawrence and Rozansky [17] have already made this simple structure of Z explicit by working backwards from the previously known formula for Z. Our goal here is to compute directly one term in their formula, the local contribution from the trivial connection. However, because the general result in [17] is both very elegant and very suggestive, we now pause to present it.

To express Z as in [17], we find it useful to introduce the numerical quantities

$$\begin{aligned} \epsilon_r &= \frac{2\pi}{k+2}\,, \\ P &= \prod_{j=1}^{N} \alpha_j \quad \text{if } N \geq 1\,, \qquad P = 1 \quad \text{otherwise}\,, \\ \theta_0 &= 3 - \frac{d}{P} + 12 \sum_{j=1}^{N} s(\beta_j, \alpha_j)\,. \end{aligned} \tag{4.5.36}$$

Here ϵ_r is the renormalized coupling incorporating the famous shift $k \to k+2$ in the level in the case $G = SU(2)$, and $s(\beta, \alpha)$ is the Dedekind sum,

$$s(\beta, \alpha) = \frac{1}{4\alpha} \sum_{l=1}^{\alpha-1} \cot\left(\frac{\pi l}{\alpha}\right) \cot\left(\frac{\pi l \beta}{\alpha}\right)\,. \tag{4.5.37}$$

For brevity, we also introduce the analytic functions

$$\begin{aligned} F(z) &= \left(2\sinh\left(\frac{z}{2}\right)\right)^{2-N} \cdot \prod_{j=1}^{N} \left(2\sinh\left(\frac{z}{2\alpha_j}\right)\right)\,, \\ G^{(l)}(z) &= \frac{i}{4\epsilon_r}\left(\frac{d}{P}\right) z^2 - \frac{2\pi\, l}{\epsilon_r}\, z\,. \end{aligned} \tag{4.5.38}$$

Then, from the results of [17], the partition function $Z(\epsilon)$ of Chern-Simons theory on M can be written as

$$\begin{aligned} Z(\epsilon) = (-1)\, \frac{\exp\left(\frac{3\pi i}{4} - \frac{i}{4}\theta_0\epsilon_r\right)}{4\sqrt{P}} \Bigg\{ &\sum_{l=0}^{d-1} \frac{1}{2\pi i} \int_{\mathcal{C}^{(l)}} dz\, F(z)\, \exp\left[G^{(l)}(z)\right] - \\ - \sum_{m=1}^{2P-1} \operatorname{Res}\left(\frac{F(z)\exp\left[G^{(0)}(z)\right]}{1 - \exp\left(-\frac{2\pi}{\epsilon_r}\, z\right)} \right)\Bigg|_{z=2\pi i\, m} &- \sum_{l=1}^{d-1} \sum_{m=1}^{\left[\frac{2Pl}{d}\right]} \operatorname{Res}\left(F(z)\, \exp\left[G^{(l)}(z)\right]\right)\Bigg|_{z=-2\pi i\, m} \Bigg\}\,. \end{aligned} \tag{4.5.39}$$

Our notation differs somewhat from [17], and we have normalized $Z(\epsilon)$ so that the partition function on $S^2 \times S^1$ is 1, whereas the authors of [17] normalize the partition function on S^3 to be 1.

Here $\mathcal{C}^{(l)}$ for $l = 0, \ldots, d-1$ denote a set of contours in the complex plane over which we evalute the integrals in the first line of (4.5.39). In particular, $\mathcal{C}^{(0)}$ is the diagonal line contour through the origin,

$$\mathcal{C}^{(0)} = \mathrm{e}^{\frac{i\pi}{4}} \times \mathbb{R}\,, \tag{4.5.40}$$

and the other contours $\mathcal{C}^{(l)}$ for $l > 0$ are diagonal line contours parallel to $\mathcal{C}^{(0)}$ running through the stationary phase point of the integrand, given by $z = -4\pi i\, l\,(P/d)$. Also, "Res" denotes the residue of the given analytic function evaluated at the given point.

We now wish to point out a few general features of this result (4.5.39) from the perspective of non-abelian localization.

First, the d contour integrals in the first term of (4.5.39) are identified in [17] with the local contributions from the d reducible flat connections on M. In particular, the integral arising from $l = 0$ above is the local contribution from the trivial connection, which takes the form

$$\begin{aligned} Z(\epsilon)\Big|_{\{0\}} &= (-1)\,\frac{\exp\left(\frac{3\pi i}{4} - \frac{i}{4}\theta_0\epsilon_r\right)}{4\sqrt{P}} \times \\ &\times \frac{1}{2\pi i}\int_{\mathcal{C}^{(0)}} dz\, \exp\left[\frac{i}{4\epsilon_r}\left(\frac{d}{P}\right)z^2\right]\left(2\sinh\left(\frac{z}{2}\right)\right)^{2-N}\cdot\prod_{j=1}^{N}\left(2\sinh\left(\frac{z}{2\alpha_j}\right)\right). \end{aligned} \tag{4.5.41}$$

For instance, one can directly check that, in the case $M = S^3$, the integral in (4.5.41) reduces to our much simpler expression for $Z(\epsilon)$ in (4.1.5).

Similarly, the integrals for $l > 0$ arise from reducible flat connections whose holonomies lie in a maximal torus of $SU(2)$, and hence these connections are fixed by a $U(1)$ subgroup of the gauge group. As we generally saw in Section 4 when we considered higher critical points of Yang-Mills theory, non-abelian localization at a reducible connection leads to an integral over the Lie algebra $\mathfrak{h}_0$ of the stablizer group H_0. This integral over $\mathfrak{h}_0$ is represented by the contour integrals above.

In contrast, the residues in the remaining terms of (4.5.39) are identified in [17] with the local contributions from the irreducible flat connections on M. As we show later, at least in the non-orbifold case $N = 0$ and $g > 0$, the local path integral contribution from a smooth component $\mathcal{M}$ in the moduli space of irreducible flat connections on M is given by a computation in the cohomology ring of $\mathcal{M}$. In the context of two-dimensional Yang-Mills theory, cohomology computations on $\mathcal{M}$ are often expressed in the form of residues, and we expect the residues in (4.5.39) to arise in this fashion.

Finally, the phase of $Z(\epsilon)$ in (4.5.39) is quite subtle. As explained in [103], this phase can be defined given the choice of a 2-framing on M, meaning a trivialization of $TM \oplus TM$, and for each three-manifold M a canonical choice of 2-framing exists. The partition function can thus be presented with a canonical phase, as originally computed in [104,105] and as given in (4.5.39). The phase of $Z(\epsilon)$ which arises naturally when we define Chern-Simons theory via localization differs from this canonical phase, and we discuss this fact at the end of the section.

Localization at the Trivial Connection

We now compute using localization the contribution from the trivial connection to $Z(\epsilon)$ when M is a Seifert homology sphere. Although the results of Lawrence and Rozansky in (4.5.39) hold for gauge group $G = SU(2)$, Mariño has presented in [81] an expression for the contribution from the trivial connection for an arbitrary simply-laced gauge group G. With our methods, the generalization from $G = SU(2)$ to arbitrary simply-laced G is immediate, so we also consider the general case.

At the trivial connection, the moduli space $\mathcal{M}$ is trivial, so the local geometry in $\overline{\mathcal{A}}$ is entirely described by the normal symplectic fiber F in (4.5.27), with the appropriate $\mathfrak{h}_0$, E_0, and E_1. So we need only evaluate the canonical symplectic integral over F for this case.

We first observe that the stabilizer subgroup $H_0 \subset \mathcal{H}$ for the trivial connnection is given as in (4.5.28) by

$$H_0 \,=\, U(1)^2 \times G\,, \tag{4.5.42}$$

where the factor G arises from the constant gauge transformations on M. Since H_0 decomposes as a product, we decompose an arbitrary element ψ of its Lie algebra $\mathfrak{h}_0 = \mathbb{R} \oplus \mathfrak{g} \oplus \mathbb{R}$ as

$$\psi \,=\, p + \phi + a\,, \tag{4.5.43}$$

where p and a generate the $U(1)$ factors of H_0 and ϕ is an element of $\mathfrak{g}$, according to the notation of Section 3.

As in (4.5.30), the exceptional bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ at the trivial connection are now given by

$$\mathcal{E}_0 \,=\, \bigoplus_{t\ge 1} H^0_{\bar\partial}(\Sigma, \mathfrak{g}\otimes(\mathcal{L}^t\oplus\mathcal{L}^{-t}))\,,$$
$$\mathcal{E}_1 \,=\, \bigoplus_{t\ge 1} H^1_{\bar\partial}(\Sigma, \mathfrak{g}\otimes(\mathcal{L}^t\oplus\mathcal{L}^{-t}))\,. \tag{4.5.44}$$

Here $\mathcal{L} = \mathcal{L}_0^d$ is the line V-bundle on Σ which describes M.

From our localization formula (4.4.85) in Section 4, the contribution of the trivial connection to $Z(\epsilon)$ is now given formally by the following integral over $\mathfrak{h}_0$,

$$Z(\epsilon)\Big|_{\{0\}} \,=\, \frac{(2\pi\epsilon)}{\mathrm{Vol}(G)}\int_{\mathfrak{h}_0}\left[\frac{d\psi}{2\pi}\right] e(\psi)\,\exp\left[-i\,(\gamma_0,\psi) - \frac{i\epsilon}{2}\,(\psi,\psi)\right], \tag{4.5.45}$$

where $e(\psi)$ is an infinite-dimensional determinant,

$$e(\psi) \,=\, \det\left(\frac{\psi}{2\pi}\Big|_{\mathcal{E}_0}\right)\,\det\left(\frac{\psi}{2\pi}\Big|_{\mathcal{E}_1}\right)^{-1}. \tag{4.5.46}$$

In normalizing (4.5.45), we have cancelled the factor $\mathrm{Vol}(U(1)^2)$ that appears in the relative normalization (4.5.10) against a corresponding factor in $1/\,\mathrm{Vol}(H_0)$ from the localization formula (4.4.85), leaving the factor $1/\,\mathrm{Vol}(G)$. We have also included the factor $(2\pi\epsilon)$ from (4.5.10).

Evaluating $e(\psi)$

We first evaluate $e(\psi)$, which turns out to be the only non-trivial piece of our computation. From (4.5.46), we see that $e(\psi)$ is described formally by the determinant of the operator ψ acting on the infinite-dimensional vector spaces $\mathcal{E}_0$ and $\mathcal{E}_1$. So to evaluate $e(\psi)$, we will have to decide how to define such a determinant.

Here we employ the standard analytic technique of zeta/eta-function regularization to define the various infinite products that represent the determinant $e(\psi)$. This choice is somewhat *ad hoc*, and our best justification for it is the fact that it eventually leads to agreement with the results of Lawrence and Rozansky. However, this method of regularization does feature in the usual perturbative approach to Chern-Simons gauge theory, for instance in the one-loop computation in [15]. So, optimistically, one might be able to better justify the use of zeta/eta-function regularization here by comparing the localization computation with conventional perturbation theory. We make a few further remarks in Section 5.3.

Since the general element of $\mathcal{H}$ acts on $\overline{\mathcal{A}}$ as

$$\delta A = d_A\phi + p\,\pounds_R A\,, \tag{4.5.47}$$

we see that the determinants in $e(\psi)$ can be written concretely in terms of p and ϕ in (4.5.43) as

$$e(\psi) \,=\, e(p,\phi) \,=\, \det\left[\frac{1}{2\pi}\left(p\pounds_R - [\phi,\,\cdot\,]\right)\Big|_{\mathcal{E}_0}\right] \det\left[\frac{1}{2\pi}\left(p\pounds_R - [\phi,\,\cdot\,]\right)\Big|_{\mathcal{E}_1}\right]^{-1}. \tag{4.5.48}$$

In particular, $e(p,\phi)$ does not depend on a in $\mathfrak{h}_0$, since this generator acts trivially. This fact is important later.

As $\pounds_R$ acts on sections of $\mathcal{L}^t$ with eigenvalue $-2\pi i t$, we rewrite $e(p,\phi)$ as a product over the non-zero eigenvalues of $\pounds_R$ as

$$e(p,\phi) \,=\, \prod_{t\neq 0} \det\left[\left(-itp - \frac{[\phi,\,\cdot\,]}{2\pi}\right)\Big|_{\mathfrak{g}}\right]^{\chi(\mathcal{L}^t)}. \tag{4.5.49}$$

Here $\chi(\mathcal{L}^t)$ is the Euler character of $\mathcal{L}^t$, so that we incorporate the cancellation between the action of ψ on elements of $\mathcal{E}_0$ and $\mathcal{E}_1$, and the determinant in (4.5.49) indicates the determinant with respect to the action on $\mathfrak{g}$.

We now evaluate this finite-dimensional determinant on $\mathfrak{g}$. This determinant is invariant under the adjoint action on $\mathfrak{g}$, and without loss we assume that ϕ lies in the Lie algebra $\mathfrak{t}$ of a maximal torus T of G. In this case, if β denotes a root of $\mathfrak{g}$ and g_β the corresponding generator of $\mathfrak{g}$, then the adjoint action of ϕ on g_β is given by $[\phi, g_\beta] = i\,\langle\beta,\phi\rangle\, g_\beta$. Thus diagonalizing the adjoint action of ϕ, we see that

$$\begin{aligned}\det\left(-itp - \frac{[\phi,\,\cdot\,]}{2\pi}\right)\Big|_{\mathfrak{g}} \,&=\, (-itp)^{\Delta_G}\prod_{\beta}\left(1 + \frac{\langle\beta,\phi\rangle}{2\pi tp}\right),\\ &=\, (-itp)^{\Delta_G}\prod_{\beta>0}\left(1 - \left(\frac{\langle\beta,\phi\rangle}{2\pi tp}\right)^2\right).\end{aligned} \tag{4.5.50}$$

Here Δ_G denotes the dimension of G. In the first line of (4.5.50), the product runs over all the roots β of $\mathfrak{g}$, whereas in the second line of (4.5.50), we have grouped together the two terms arising from the roots $\pm\beta$ and rewritten the product over a set of positive roots.

Now from (4.5.49) and (4.5.50), we rewrite $e(p,\phi)$ as

$$e(p,\phi) \;=\; \exp\left(-\frac{i\pi}{2}\eta\right)\cdot\prod_{t\ge1}\left|(tp)^{\Delta_G}\prod_{\beta>0}\left(1-\left(\frac{\langle\beta,\phi\rangle}{2\pi tp}\right)^2\right)\right|^{\chi(\mathcal{L}^t)+\chi(\mathcal{L}^{-t})}\,. \tag{4.5.51}$$

Here $\exp\left(-\frac{i\pi}{2}\eta\right)$ represents the phase of $e(p,\phi)$, which involves an infinite product of factors $\pm i$, and the product written explicity in (4.5.51) represents the norm. We first evaluate this norm, as the quantity η is much more delicate to determine.

To start, we evaluate the exponent that appears in (4.5.51). By the Riemann-Roch theorem in (4.3.23),

$$\chi(\mathcal{L}^t)+\chi(\mathcal{L}^{-t}) \;=\; \deg(\mathcal{L}^t)+\deg(\mathcal{L}^{-t})+2\,. \tag{4.5.52}$$

In general, the degree of a line V-bundle is *not* multiplicative, so that $\deg(\mathcal{L}^t)\neq t\deg(\mathcal{L})$, and the first two terms on the right of (4.5.52) do not necessarily cancel as they do for ordinary line bundles.

So we must work a little bit to simplify (4.5.52). As we now show, this exponent can be simplified as

$$\chi(\mathcal{L}^t)+\chi(\mathcal{L}^{-t}) \;=\; 2-N+\sum_{j=1}^{N}\varphi_{\alpha_j}(t)\,, \tag{4.5.53}$$

where $\varphi_{\alpha_j}(t)$ is an arithmetic function which takes the value 1 if α_j divides t and is 0 otherwise,

$$\begin{aligned}\varphi_{\alpha_j}(t) \;&=\; 1 \quad\text{if}\quad \alpha_j\,|\,t\,,\\ &=\; 0 \quad\text{otherwise}\,.\end{aligned} \tag{4.5.54}$$

To deduce (4.5.53), we suppose that the line V-bundle $\mathcal{L}^t$ is characterized on Σ by isotropy invariants γ_j, where

$$\gamma_j\equiv t\,\beta_j \mod \alpha_j\,,\quad 0\le\gamma_j<\alpha_j\,, \tag{4.5.55}$$

and, as before, the isotropy invariants β_j characterize the line V-bundle $\mathcal{L}$ itself. From (4.5.14), the degree of $\mathcal{L}^t$ is given in terms of the first Chern class, which is multiplicative, and γ_j as

$$\deg(\mathcal{L}^t) \;=\; t\,c_1(\mathcal{L})-\sum_{j=1}^{N}\frac{\gamma_j}{\alpha_j}\,. \tag{4.5.56}$$

On the other hand, the isotropy invariants $\overline{\gamma}_j$ for the inverse line V-bundle $\mathcal{L}^{-t}$ are given by

$$\overline{\gamma}_j \equiv -t\,\beta_j \mod \alpha_j\,, \quad 0 \le \overline{\gamma}_j < \alpha_j\,, \tag{4.5.57}$$

so that in terms of γ_j,

$$\begin{aligned} \overline{\gamma}_j \;&=\; \alpha_j - \gamma_j \quad \text{if } \gamma_j \neq 0\,, \\ &=\; \gamma_j \;=\; 0 \quad \text{otherwise}\,. \end{aligned} \tag{4.5.58}$$

We note from (4.5.55) that γ_j vanishes whenever $t\beta_j \equiv 0 \mod \alpha_j$. Because β_j is relatively prime to α_j by assumption, the vanishing of γ_j is then equivalent to the condition that α_j divide t, so that

$$\gamma_j = 0 \quad \Longleftrightarrow \quad \alpha_j \,|\, t\,. \tag{4.5.59}$$

Thus, using the arithemetic function $\varphi_{\alpha_j}(t)$ defined in (4.5.54) in conjunction with (4.5.58) and (4.5.59), we see that the degree of $\mathcal{L}^{-t}$ can be written as

$$\begin{aligned} \deg(\mathcal{L}^{-t}) \;&=\; -t\,c_1(\mathcal{L}) - \sum_{j=1}^N \frac{\overline{\gamma}_j}{\alpha_j}\,, \\ &=\; -t\,c_1(\mathcal{L}) - \sum_{j=1}^N \left(1 - \frac{\gamma_j}{\alpha_j} - \varphi_{\alpha_j}(t)\right)\,. \end{aligned} \tag{4.5.60}$$

From (4.5.52), (4.5.56), and (4.5.60), we immediately deduce (4.5.53).

Consequently, $e(p,\phi)$ now becomes

$$\begin{aligned} e(p,\phi) \;&=\; \exp\left(-\frac{i\pi}{2}\eta\right) \cdot \prod_{t\ge1} \left| (tp)^{\Delta_G} \prod_{\beta>0} \left(1 - \left(\frac{\langle\beta,\phi\rangle}{2\pi tp}\right)^2\right) \right|^{2-N+\sum_{j=1}^N \varphi_{\alpha_j}(t)}\,, \\ &=\; \exp\left(-\frac{i\pi}{2}\eta\right) \cdot f_0(p,\phi)^2 \cdot \prod_{j=1}^N \left|\frac{f_{\alpha_j}(p,\phi)}{f_0(p,\phi)}\right|\,, \end{aligned} \tag{4.5.61}$$

where

$$f_0(p,\phi) \;=\; \prod_{t\ge1} \left[(tp)^{\Delta_G} \prod_{\beta>0} \left(1 - \left(\frac{\langle\beta,\phi\rangle}{2\pi tp}\right)^2\right)\right]\,, \tag{4.5.62}$$

and f_{α_j} is related to f_0 by

$$f_{\alpha_j}(p,\phi) \;=\; f_0(\alpha_j \cdot p, \phi)\,. \tag{4.5.63}$$

In deducing (4.5.61) from (4.5.62) and (4.5.63), we apply the following arithmetic identity, which holds for an arbitrary function $f(t)$,

$$\prod_{t\geq 1} f(t)^{\varphi_{\alpha_j}(t)} \,=\, \prod_{t\geq 1} f(\alpha_j \cdot t)\,. \tag{4.5.64}$$

We finally evaluate the infinite product which defines $f_0(p,\phi)$. We use the well known identity below,

$$\frac{\sin(x)}{x} \,=\, \prod_{t\geq 1}\left(1 - \frac{x^2}{\pi^2 t^2}\right)\,, \tag{4.5.65}$$

and we use the Riemann zeta-function ζ to define trivial, but infinite, products

$$\begin{aligned} \prod_{t\geq 1} p^{\Delta_G} \,&=\, \exp\left(\Delta_G \ln p \cdot \zeta\,(0)\right) \,=\, p^{-\Delta_G/2}\,,\\ \prod_{t\geq 1} t^{\Delta_G} \,&=\, \exp\left(-\Delta_G \cdot \zeta'(0)\right) \,=\, (2\pi)^{\Delta_G/2}\,. \end{aligned} \tag{4.5.66}$$

So from (4.5.65) and (4.5.66), we evaluate $f_0(p,\phi)$ to be

$$\begin{aligned} f_0(p,\phi) \,&=\, \left(\frac{p}{2\pi}\right)^{-\Delta_G/2} \prod_{\beta>0}\left[\frac{2p}{\langle\beta,\phi\rangle}\,\sin\left(\frac{\langle\beta,\phi\rangle}{2p}\right)\right]\,,\\ &=\, (2\pi)^{\Delta_G/2}\; p^{-\Delta_T/2} \prod_{\beta>0}\left[\frac{2}{\langle\beta,\phi\rangle}\,\sin\left(\frac{\langle\beta,\phi\rangle}{2p}\right)\right]\,. \end{aligned} \tag{4.5.67}$$

Here Δ_T denotes the dimension of the maximal torus T of G (hence the rank of G), and in passing to the second line of (4.5.67) we just pull the factors of p outside the product over the positive roots of G.

From (4.5.61), (4.5.63), and (4.5.67), we finally evaluate $e(p,\phi)$ to be

$$\begin{aligned} e(p,\phi) \,&=\, \exp\left(-\frac{i\pi}{2}\eta\right)\cdot\frac{(2\pi)^{\Delta_G}}{(p\sqrt{P})^{\Delta_T}}\,\times\\ &\times \prod_{\beta>0}\langle\beta,\phi\rangle^{-2}\left|2\sin\left(\frac{\langle\beta,\phi\rangle}{2p}\right)\right|^{2-N}\prod_{j=1}^{N}\left|2\sin\left(\frac{\langle\beta,\phi\rangle}{2\alpha_j p}\right)\right|\,, \end{aligned} \tag{4.5.68}$$

where P is defined in (4.5.36) as the product of all the α_j.

Evaluating η and the Quantum Shift in the Chern-Simons Level

We now evaluate the phase factor $\exp\left(-\frac{i\pi}{2}\eta\right)$, from which we will find the famous quantum shift in the Chern-Simons level $k \to k + \check{c}_{\mathfrak{g}}$, where $\check{c}_{\mathfrak{g}}$ is the dual Coxeter number of $\mathfrak{g}$. For instance, we recall that in the case $G = SU(r+1)$, $\check{c}_{\mathfrak{g}} = r+1$.

To start, we consider the operator

$$\frac{i}{2\pi}\left(p\pounds_R - [\phi,\,\cdot\,]\right)\,, \tag{4.5.69}$$

acting on the vector spaces $\mathcal{E}_0$ and $\mathcal{E}_1$ in (4.5.44). The spectrum of this operator is real, so at least formally, we see from the definition of $e(p,\phi)$ in (4.5.48) that the phase η is given by

$$\eta \approx \sum_{\lambda_{(0)}\neq 0} \mathrm{sign}(\lambda_{(0)}) - \sum_{\lambda_{(1)}\neq 0} \mathrm{sign}(\lambda_{(1)})\,, \tag{4.5.70}$$

where $\lambda_{(0)}$ and $\lambda_{(1)}$ range, respectively, over the eigenvalues of the operator in (4.5.69) acting on $\mathcal{E}_0$ and $\mathcal{E}_1$.

We have not written (4.5.70) with an equality because the sums on the right of (4.5.70) are ill-defined without a regulator. To regulate these sums, we follow the philosophy of [106] and introduce the eta-function

$$\eta_{(p,\phi)}(s) \;=\; \sum_{\lambda_{(0)}\neq 0} \mathrm{sign}(\lambda_{(0)})\,|\lambda_{(0)}|^{-s} - \sum_{\lambda_{(1)}\neq 0} \mathrm{sign}(\lambda_{(1)})\,|\lambda_{(1)}|^{-s}\,. \tag{4.5.71}$$

Here s is a complex variable. When the real part of s is sufficiently large, the sums in (4.5.71) are absolutely convergent so that $\eta_{(p,\phi)}(s)$ is defined in this case. Otherwise, $\eta_{(p,\phi)}(s)$ is defined by analytic continuation in the s-plane. Assuming that the limit $s \to 0$ exists, we then set

$$\eta = \eta_{(p,\phi)}(0)\,. \tag{4.5.72}$$

Thus, η is basically the classic eta-invariant of [106] which is here associated to the operator in (4.5.69) acting on the virtual vector space $\mathcal{E}_0 \ominus \mathcal{E}_1$, where the "$\ominus$" simply indicates the relative sign in (4.5.71).

In our problem, because we explicitly know the spectrum of the operator in (4.5.69), we can directly evaluate $\eta_{(p,\phi)}(0)$ without too much work. One advantage of this direct approach is that it very concretely displays the origin of the finite shift in the Chern-Simons level k, a very subtle quantum effect to understand otherwise.

Ultimately this shift in k arises because, despite what might be one's naive expectation from (4.5.70), η depends nontrivially on p and ϕ. To isolate this interesting functional dependence of $\eta_{(p,\phi)}(0)$ on p and ϕ, we observe that, for $s = 0$, the sum in (4.5.71) is invariant under an overall scaling of the eigenvalues $\lambda_{(0)}$ and $\lambda_{(1)}$, so that $\eta_{(p,\phi)}(0)$ is invariant under an overall scaling of the operator itself in (4.5.69). In particular, so long as $p > 0$ (as holds when we later set $p = 1/\epsilon$), we are free to rescale the operator in (4.5.69) by $1/p$ without changing η.

As a technical convenience, we thus introduce another eta-function $\eta'_{(p,\phi)}(s)$ which is defined as in (4.5.71) but is associated to the rescaled operator

$$\frac{i}{2\pi}\left(\pounds_R - \left[\frac{\phi}{p}, \cdot\right]\right). \tag{4.5.73}$$

Because $\eta = \eta_{(p,\phi)}(0) = \eta'_{(p,\phi)}(0)$, we see from (4.5.73) that η can only depend on p and ϕ in the combination ϕ/p.

We also introduce the eta-function $\eta_0(s)$ which is associated to the constant operator $i\pounds_R/2\pi$, and to isolate the functional dependence of η on p and ϕ we define

$$\delta\eta(p,\phi) \;=\; \eta'_{(p,\phi)}(0) - \eta_0(0)\,. \tag{4.5.74}$$

As we now compute directly,

$$\delta\eta(p,\phi) \;=\; -\frac{\check{c}_{\mathfrak{g}}}{2(\pi p)^2}\left(\frac{d}{P}\right)\mathrm{Tr}(\phi^2) \quad \mathrm{mod}\ 2. \tag{4.5.75}$$

The role of the mod 2 terms is to remove the absolute value bars $|\cdot|$ that appear in (4.5.68), so that $e(p,\phi)$ depends analytically on p and ϕ as its definition suggests.

Of course, η itself is given by $\eta = \delta\eta(p,\phi) + \eta_0(0)$. We also discuss $\eta_0(0)$, though this constant is much less interesting than $\delta\eta(p,\phi)$.

A Warmup Computation on S^1

Before we directly evaluate $\delta\eta$, $\eta_0(0)$, and η for the case at hand, we find it useful to warm up with a simpler example, originally presented in [106,II]. Thus, we consider the eta-function $\eta_\nu(s)$ which is associated to the operator D_ν acting on functions on S^1,

$$D_\nu \;=\; \frac{i}{2\pi}\frac{d}{dx} + \nu\,. \tag{4.5.76}$$

Here ν is a real parameter in the interval $0 < \nu < 1$, and x is a coordinate on S^1 with period 2π. If we wish, we can equivalently consider D_ν as the covariant derivative acting on sections of a flat $U(1)$ bundle over S^1 whose connection has holonomy parametrized by ν.

Clearly the eigenvalues λ of D_ν are given by $\lambda = t + \nu$ as t runs over all integers. So we compute

$$\begin{aligned} \eta_\nu(s) &= \sum_\lambda \text{sign}(\lambda)\,|\lambda|^{-s}\,, \\ &= \sum_{t\geq 0} \frac{1}{(t+\nu)^s} - \sum_{t\geq 1} \frac{1}{(t-\nu)^s}\,, \\ &= \frac{1}{\nu^s} - \sum_{t\geq 1} \frac{2\nu s}{t^{s+1}} + \sum_{t\geq 1} s\cdot\mathcal{O}\left(\frac{1}{t^{s+2}}\right)\,. \end{aligned} \tag{4.5.77}$$

In passing from the second to the third lines of (4.5.77), we apply the binomial expansion, and we collect into $\mathcal{O}(1/t^{s+2})$ the terms in this expansion for which the sum over t is absolutely convergent near $s = 0$. Thus, when we evaluate $\eta_\nu(s)$ at $s = 0$, the last term of (4.5.77) vanishes.

On the other hand, for the term involving the sum over $1/t^{s+1}$, we have

$$\sum_{t\geq 1} \frac{2\nu s}{t^{s+1}} \,=\, 2\nu s\,\zeta(1+s)\,. \tag{4.5.78}$$

Because $\zeta(1+s)$ has a simple pole with residue 1 at $s = 0$, we see that (4.5.78) makes a non-zero contribution to $\eta_\nu(0)$, and

$$\eta_\nu(0) \,=\, 1 - 2\nu\,. \tag{4.5.79}$$

Physically the term involving ν arises as a finite renormalization effect, due to the divergence in the sum over eigenvalues in (4.5.78).

The Computation of η on M

Given the formal similarity of the operators in (4.5.73) and (4.5.76), we now evaluate $\eta_{(p,\phi)}(0)$ just as in our warmup computation on S^1. In the case at hand, we must consider the eigenvalue multiplicities which are associated to the dimensions of the Dolbeault cohomology groups $H^0_{\bar\partial}(\Sigma, \mathcal{L}^t)$ and $H^1_{\bar\partial}(\Sigma, \mathcal{L}^t)$, and as in our earlier computation we must also consider the eigenvalues of the adjoint action of ϕ on $\mathfrak{g}$. Taking these considerations into account, we find the following compact expression for $\eta'_{(p,\phi)}(s)$,

$$\begin{aligned} \eta'_{(p,\phi)}(s) \,&=\, \sum_{t=-\infty}^{+\infty} \sum_{\beta} \chi(\mathcal{L}^t)\, \mathrm{sign}\left(\lambda\left(t,\beta\right)\right)\, |\lambda(t,\beta)|^{-s}\,, \\ \lambda(t,\beta) \,&=\, t + \frac{\langle\beta,\phi\rangle}{2\pi p}\,. \end{aligned} \tag{4.5.80}$$

Here the sum over β is again a sum over the roots of $\mathfrak{g}$, including the roots $\beta = 0$ from the Cartan subalgebra. We note that the appearance of the Euler character $\chi(\mathcal{L}^t)$ in (4.5.80) accounts both for the multiplicities and the relative signs of the eigenvalue contributions from $\mathcal{E}_0$ and $\mathcal{E}_1$ in (4.5.71).

We can give a similar, simpler expression for $\eta_0(s)$,

$$\begin{aligned} \eta_0(s) \,&=\, \sum_{t\neq 0} \sum_{\beta} \chi(\mathcal{L}^t)\, \mathrm{sign}(t)\, |t|^{-s}\,, \\ &=\, \sum_{t\geq 1} \sum_{\beta} \frac{\chi(\mathcal{L}^t) - \chi(\mathcal{L}^{-t})}{t^s}\,. \end{aligned} \tag{4.5.81}$$

In the general orbifold case, the index difference $\chi(\mathcal{L}^t) - \chi(\mathcal{L}^{-t})$ that arises in (4.5.81) appears to be a somewhat complicated arithmetic function of t, in contrast to our simple expression for the index sum in (4.5.53), and we will not evaluate $\eta_0(0)$ in complete generality here.

However, if we consider the special case of a degree d line-bundle $\mathcal{L}$ over a smooth Riemann surface Σ, then the Riemann-Roch theorem immediately implies that

$$\chi(\mathcal{L}^t) - \chi(\mathcal{L}^{-t}) \,=\, 2dt\,, \tag{4.5.82}$$

independent of the genus of Σ. So in this special case, we have from (4.5.81) that

$$\begin{aligned} \eta_0(s) \,&=\, \Delta_G \sum_{t\geq 1} \frac{2dt}{t^s}\,, \\ &=\, 2d\Delta_G\, \zeta(s-1)\,. \end{aligned} \tag{4.5.83}$$

Thus,

$$\eta_0(0) \,=\, 2d\Delta_G\, \zeta(-1) \,=\, -\frac{d\Delta_G}{6}\,. \tag{4.5.84}$$

Having discussed $\eta_0(0)$, we now compute the more interesting quantity $\delta\eta(p,\phi)$ in (4.5.74). Upon expressing (4.5.80) as in (4.5.81) and collecting terms, we find that

$$\begin{aligned}\eta'_{(p,\phi)}(s) - \eta_0(s) \,=\, &\sum_{t\geq 0}\sum_{\beta>0} \Big(\chi(\mathcal{L}^t) - \chi(\mathcal{L}^{-t})\Big)\cdot\left[\frac{1}{\left(t+\frac{\langle\beta,\phi\rangle}{2\pi p}\right)^s} - \frac{1}{t^s}\right] + \\ &+ \sum_{t\geq 1}\sum_{\beta>0} \Big(\chi(\mathcal{L}^t) - \chi(\mathcal{L}^{-t})\Big)\cdot\left[\frac{1}{\left(t-\frac{\langle\beta,\phi\rangle}{2\pi p}\right)^s} - \frac{1}{t^s}\right].\end{aligned} \tag{4.5.85}$$

In writing this expression, we assume without loss that the condition below holds for each positive root β,

$$0 < \frac{\langle\beta,\phi\rangle}{2\pi p} < 1\,. \tag{4.5.86}$$

Otherwise, when the quantity in (4.5.86) undergoes an integral shift, then the overall phase $\exp\left(-i\pi\eta/2\right)$ of $e(p,\phi)$ simply picks up a sign so as to effectively remove the absolute value bars $|\cdot|$ appearing in (4.5.68). Hence $e(p,\phi)$ depends analytically on p and ϕ.

We now observe from our general expressions (4.5.56) and (4.5.60) for $\deg(\mathcal{L}^t)$ and $\deg(\mathcal{L}^{-t})$ that the index difference in (4.5.85) depends generally on t as

$$\chi(\mathcal{L}^t) - \chi(\mathcal{L}^{-t}) \,=\, 2t\left(\frac{d}{P}\right) + \mathcal{O}(t^0). \tag{4.5.87}$$

We have used the fact that $c_1(\mathcal{L}) = d/P$, since $\mathcal{L} = \mathcal{L}_0^d$, and $c_1(\mathcal{L}_0) = \prod_j 1/\alpha_j = 1/P$.

If we now consider the binomial expansion of the denominators in (4.5.85), we see immediately that no contribution at $s=0$ can arise from the terms of order t^0 in (4.5.87). The leading terms in the expansion which arise from these $\mathcal{O}(t^0)$ terms are proportional to $\pm\langle\beta,\phi\rangle/(2\pi p)\cdot t^{-(s+1)}$, and such terms linear in ϕ cancel between the two sums in (4.5.85). The same cancellation occurs between the leading expansion terms which arise from the term linear in t in (4.5.87), and fundamentally these cancellations reflect the fact that no invariant linear function of ϕ exists.

Thus, expanding the denominators in (4.5.85) to second order, we find

$$\eta'_{(p,\phi)}(s) - \eta_0(s) \,=\, 2\left(\frac{d}{P}\right)\sum_{t\geq 1}\sum_{\beta>0}\left(\frac{\langle\beta,\phi\rangle}{2\pi p}\right)^2\cdot\frac{s(s+1)}{t^{s+1}} + \sum_{t\geq 1}\sum_{\beta>0} s\cdot\mathcal{O}\left(\frac{1}{t^{s+2}}\right). \tag{4.5.88}$$

We evaluate (4.5.88) at $s=0$ to determine $\delta\eta(p,\phi)$, which is thus given by

$$\delta\eta(p,\phi) \,=\, 2\left(\frac{d}{P}\right)\sum_{\beta>0}\left(\frac{\langle\beta,\phi\rangle}{2\pi p}\right)^2. \tag{4.5.89}$$

To simplify the sum over roots on the right side of (4.5.89), we note that this sum defines an invariant quadratic polynomial of ϕ and hence must be proportional to $\mathrm{Tr}(\phi^2)$. When $\mathfrak{g}$ is simply-laced, we have the following identity, as shown for instance in [107,VI],

$$\sum_{\beta>0} \langle\beta,\phi\rangle^2 \;=\; -\check{c}_{\mathfrak{g}}\,\mathrm{Tr}(\phi^2)\,. \tag{4.5.90}$$

Together, (4.5.89) and (4.5.90) imply the main result in (4.5.75).

Thus the full determinant $e(p,\phi)$ is now given by

$$\begin{aligned} e(p,\phi) \;=\; &\exp\left(-\frac{i\pi}{2}\,\eta_0(0)\right)\cdot\frac{(2\pi)^{\Delta_G}}{(p\sqrt{P})^{\Delta_T}}\times \\ &\times\,\exp\left[\frac{i\,\check{c}_{\mathfrak{g}}}{4\pi p^2}\,\left(\frac{d}{P}\right)\,\mathrm{Tr}(\phi^2)\right]\prod_{\beta>0}\langle\beta,\phi\rangle^{-2}\,\left[2\sin\left(\frac{\langle\beta,\phi\rangle}{2p}\right)\right]^{2-N}\prod_{j=1}^{N}\left[2\sin\left(\frac{\langle\beta,\phi\rangle}{2\alpha_j p}\right)\right]. \end{aligned} \tag{4.5.91}$$

As we will see directly, the exponential term involving $\mathrm{Tr}(\phi^2)$ in $e(p,\phi)$ describes the quantum shift in the Chern-Simons level k.

Evaluating the Integral over $\mathfrak{h}_0$

We are finally left to consider the integral over $\mathfrak{h}_0$ in (4.5.45). We first observe that the norm (ψ,ψ) appearing in the exponent of the integrand there is given explicity by

$$\begin{aligned} (\psi,\psi) \;&=\; -\int_M \kappa\wedge d\kappa\,\mathrm{Tr}(\phi^2) - 2pa\,, \\ &=\; -\left(\frac{d}{P}\right)\mathrm{Tr}(\phi^2) - 2pa\,. \end{aligned} \tag{4.5.92}$$

In passing to the second line of (4.5.92), we use the fact that ϕ is constant so that the integral over M simply evaluates to $c_1(\mathcal{L}) = d/P$. Second, we recall from Section 3 that the moment map at the trivial connection satisfies

$$\langle\mu,\psi\rangle \;=\; (\gamma_0,\psi) \;=\; a\,. \tag{4.5.93}$$

Hence the integral over $\mathfrak{h}_0$ takes the explicit form

$$Z(\epsilon)\Big|_{\{0\}} \;=\; \frac{(2\pi\epsilon)}{\mathrm{Vol}(G)}\int_{\mathfrak{h}_0}\left[\frac{dp}{2\pi}\right]\left[\frac{da}{2\pi}\right]\left[\frac{d\phi}{2\pi}\right]\,e(p,\phi)\,\exp\left[-ia + i\epsilon pa + \frac{i\epsilon}{2}\left(\frac{d}{P}\right)\mathrm{Tr}(\phi^2)\right]. \tag{4.5.94}$$

We now evaluate the integral over a, which is easy since a only appears in the exponent of the integrand in (4.5.94). From a previous identity (4.4.32), this integral produces the delta function $2\pi\,\delta(1-\epsilon p)$.

In turn, we use the delta function to perform the integral over p, setting $p = 1/\epsilon$. In the process, we cancel the explicit factor of $2\pi\epsilon$ which appears in the normalization of (4.5.94), and the integral over $\mathfrak{h}_0$ simplifies to an integral over $\mathfrak{g}$,

$$Z(\epsilon)\Big|_{\{0\}} = \frac{1}{\mathrm{Vol}(G)} \int_{\mathfrak{g}} \left[\frac{d\phi}{2\pi}\right] e(\epsilon^{-1},\phi)\, \exp\left[\frac{i\epsilon}{2}\left(\frac{d}{P}\right)\mathrm{Tr}(\phi^2)\right]. \tag{4.5.95}$$

Because the integrand of (4.5.95) is invariant under the adjoint action on $\mathfrak{g}$, we can apply the classical Weyl integral formula to reduce the integral over $\mathfrak{g}$ to an integral over the Cartan subalgebra $\mathfrak{t}$, in which form we make contact with the results in [17,81]. In its infinitesimal version, the Weyl integral formula states that, if f is a function on $\mathfrak{g}$ invariant under the adjoint action, then

$$\int_{\mathfrak{g}} [d\phi]\; f(\phi) = \frac{1}{|W|}\,\frac{\mathrm{Vol}(G)}{\mathrm{Vol}(T)} \int_{\mathfrak{t}} [d\phi] \prod_{\beta>0} \langle\beta,\phi\rangle^2\, f(\phi)\,. \tag{4.5.96}$$

Here $|W|$ is the order of the Weyl group of G, and the product over the positive roots β of G appearing on the right of (4.5.96) is a Jacobian factor.

Applying (4.5.96) and recalling the form of E in (4.5.91), we rewrite (4.5.95) explicitly as

$$\begin{aligned} Z(\epsilon)\Big|_{\{0\}} &= \mathrm{e}^{\left(-\frac{i\pi}{2}\eta_0(0)\right)} \frac{1}{|W|}\,\frac{1}{\mathrm{Vol}(T)} \left(\frac{\epsilon}{\sqrt{P}}\right)^{\Delta_T} \int_{\mathfrak{t}} [d\phi] \exp\left[\frac{i\epsilon}{2}\left(\frac{d}{P}\right)\left(1+\frac{\epsilon\,\check{c}_{\mathfrak{g}}}{2\pi}\right)\mathrm{Tr}(\phi^2)\right] \times \\ &\quad\times \prod_{\beta>0}\left[2\sin\left(\frac{\epsilon\,\langle\beta,\phi\rangle}{2}\right)\right]^{2-N} \prod_{j=1}^{N}\left[2\sin\left(\frac{\epsilon\,\langle\beta,\phi\rangle}{2\alpha_j}\right)\right]. \end{aligned} \tag{4.5.97}$$

We finally make the change of variables $\phi \to \epsilon\phi$ to remove some of the extraneous factors of ϵ in front of (4.5.97), so that

$$\begin{aligned} Z(\epsilon)\Big|_{\{0\}} &= \exp\left(-\frac{i\pi}{2}\,\eta_0(0)\right) \frac{1}{|W|}\,\frac{1}{\mathrm{Vol}(T)} \left(\frac{1}{\sqrt{P}}\right)^{\Delta_T} \times \\ &\times \int_{\mathfrak{t}} [d\phi]\, \exp\left[\frac{i}{2\epsilon_r}\left(\frac{d}{P}\right)\mathrm{Tr}(\phi^2)\right] \prod_{\beta>0}\left[2\sin\left(\frac{\langle\beta,\phi\rangle}{2}\right)\right]^{2-N} \prod_{j=1}^{N}\left[2\sin\left(\frac{\langle\beta,\phi\rangle}{2\alpha_j}\right)\right]. \end{aligned} \tag{4.5.98}$$

Here we introduce the usual renormalized coupling ϵ_r,

$$\epsilon_r \;=\; \frac{2\pi}{k + \check{c}_{\mathfrak{g}}}\,, \tag{4.5.99}$$

to absorb the explicit shift in the coefficient of $\mathrm{Tr}(\phi^2)$ that arises from the phase $\delta\eta$ and that appears in (4.5.97).

As it stands, the integral over $\mathfrak{t}$ in (4.5.98) has oscillatory, as opposed to exponentially damped, behavior at infinity due to purely imaginary Gaussian factor involving $\mathrm{Tr}(\phi^2)$. Such oscillatory Gaussian integrals typically arise in quantum field theory. For instance, we saw an earlier example in our path integral manipulations at the end of Section 3.1, when we integrated out the auxiliary scalar field Φ that appeared there.

Exactly as in Section 3.1, the standard analytic prescription to define such an oscillatory integral is to shift the integration contour slightly off the real axis. That is, in the context of (4.5.98) we consider the complexification $\mathfrak{t}\otimes\mathbb{C}$ of the real Lie algebra $\mathfrak{t}$, and we define (4.5.98) by integrating over $\mathfrak{t}\times(1-i\varepsilon)$ for a small real parameter ε. This $i\varepsilon$ prescription has the added virtue that the new contour avoids any poles of the integrand on the real axis that generally occur for $N > 2$.

Once we define (4.5.98) with the $i\varepsilon$ prescription, we are free to analytically continue the contour to lie along the diagonal $\mathfrak{t}\times \mathrm{e}^{-i\pi/4}$, so that the Gaussian factor in (4.5.98) becomes purely real and negative-definite. (We recall that Tr is a negative-definite form.) To make contact with the result of Lawrence and Rozansky in (4.5.39), we finally make another change of variables $\phi \;\rightarrow\; i\phi$, so that

$$\begin{aligned} Z(\epsilon)\Big|_{\{0\}} \;=\; &\exp\left(-\frac{i\pi}{2}\,\eta_0(0)\right)\frac{1}{|W|}\,\frac{(-1)^{(\Delta_G-\Delta_T)/2}}{\mathrm{Vol}(T)}\,\left(\frac{1}{i\sqrt{P}}\right)^{\Delta_T}\times\\ &\times\int_{\mathcal{C}\times\mathfrak{t}}[d\phi]\,\exp\left[-\frac{i}{2\epsilon_r}\left(\frac{d}{P}\right)\mathrm{Tr}(\phi^2)\right]\prod_{\beta>0}\left[2\sinh\left(\frac{\langle\beta,\phi\rangle}{2}\right)\right]^{2-N}\prod_{j=1}^{N}\left[2\sinh\left(\frac{\langle\beta,\phi\rangle}{2\alpha_j}\right)\right]\,, \end{aligned} \tag{4.5.100}$$

where $\mathcal{C}$ is the diagonal contour $\mathbb{R}\times\mathrm{e}^{\frac{i\pi}{4}}$, as in (4.5.40).

We immediately see that (4.5.100) has the same form as our earlier expression in (4.5.41) for the contribution from the trivial connection in the case $G = SU(2)$, and with a suitable choice of generator for $\mathfrak{t}$ one can see that (4.5.100) agrees, up to the overall phase, with the result of Lawrence and Rozansky. For general G, our expression takes the same form as that found by Mariño in [81].

The Phase of $Z(\epsilon)$

We now discuss the phase of our result (4.5.100) for the contribution of the trivial connection to the Chern-Simons path integral. In the simplest case that M is described by a smooth line-bundle of degree $d = n$ over $\mathbb{CP}^1$, we have computed this phase explicitly, as determined by the constant

$$\eta_0(0) \,=\, -\frac{d\Delta_G}{6}\,. \tag{4.5.101}$$

Since we have not performed a careful analysis of the path integral phases that arise from the η invariant when M is an orbifold, we restrict attention to the smooth case in the following.

If we compare our result to the result (4.5.41) of Lawrence and Rozansky for gauge group $SU(2)$, we see that the overall phase of $Z(\epsilon)$ which arises naturally from localization does not agree with the canonical phase. To be more precise, the result of Mariño [81] in the case of a general gauge group G shows that the ratio $\exp(i\,\delta\Psi)$ between the canonical phase of $Z(\epsilon)$ and the phase we determine via (4.5.101) is given by

$$\begin{aligned}\exp(i\,\delta\Psi) \,&=\, \exp\left(\frac{i\pi\Delta_G}{4} - \frac{i\pi\Delta_G\check{c}_{\mathfrak{g}}}{12(k+\check{c}_{\mathfrak{g}})}\,\theta_0 + \frac{i\pi}{2}\eta_0(0)\right),\\ &=\, \exp\left(\frac{i\pi\Delta_G}{12}(3-d) - \frac{i\pi\Delta_G\check{c}_{\mathfrak{g}}}{12(k+\check{c}_{\mathfrak{g}})}\,\theta_0\right).\end{aligned} \tag{4.5.102}$$

Here k is the Chern-Simons level. The quantity θ_0 is defined in general in (4.5.36), and in the smooth case we see that θ_0 is given by

$$\theta_0 \,=\, 3 - d\,. \tag{4.5.103}$$

Hence the expression in (4.5.102) simplifies greatly to

$$\exp(i\,\delta\Psi) \,=\, \exp\left(\frac{i\pi k\Delta_G}{12(k+\check{c}_{\mathfrak{g}})}\,(3-d)\right). \tag{4.5.104}$$

As we now explain, the phase discrepancy in (4.5.104) is not really a discrepancy at all, and it merely reflects the fact that our path integral computation is effectively performed in a framing of M which differs from the canonical two-framing of Atiyah [103], which has been used by Lawrence and Rozansky. We first recall from [15] that the partition function of Chern-Simons theory generally transforms under a change in the framing of M by

$$Z \longrightarrow \exp\left(\frac{i\pi c}{12}\, s\right) Z\,, \qquad c = \frac{k\Delta_G}{k + \check{c}_{\mathfrak{g}}}\,, \qquad s \in \mathbb{Z}\,. \tag{4.5.105}$$

Here c arises as the central charge of the two-dimensional WZW model associated to the group G, and s is an integer that labels the shift in the frame. As a result, we see immediately from (4.5.105) that the phase discrepancy (4.5.104) can be eliminated by a shift in $s = (3-d)$ units from the canonical framing of M.

Of course, in evaluating the Chern-Simons path integral by localization, we did not explicitly specify any framing of M. Given the framing ambiguity (4.5.105) in Z, one might naturally wonder how we managed to obtain a definite answer for the phase of Z in the first place.

To answer this question, we observe generally that if M is an integral homology sphere, then the choice of a locally-free $U(1)$ action on M implies a canonical choice, up to homotopy, of a framing of M. Concretely, a framing of M amounts to the choice of three linearly independent, non-vanishing vector fields on M, and the $U(1)$ action on M immediately supplies us with one such vector field, the generating vector field R of $U(1)$. We decompose the tangent bundle to M as $TM = L \oplus W$, where L is a one-dimensional bundle generated by R and W is the complement. We are left to make a choice for the other two vector fields, which must span the rank two sub-bundle W of TM which lies in the kernel of the contact form κ. The choice of these two vector fields amounts to a trivialization of W, so if the Euler class of W is non-zero, W is non-trivial and our construction fails. However, since the Euler class of W lies in the cohomology group $H^2(M, \mathbb{Z})$, which vanishes for an integral homology sphere, W is automatically trivial in this case. Finally, because W has rank two, possible changes of trivialization of W are classified by homotopy classes of maps of M to $SO(2)$. But for a homology sphere M (or even a rational homology sphere), the space of maps to $SO(2)$ is connected (as we recall below). So, given the choice of the original $U(1)$ action, we produce a unique framing of M up to homotopy.

Parenthetically, to show that the space of maps $M \to SO(2)$ is connected, we let $w = d\theta$ be an angular form on $SO(2) \cong S^1$ and we let $m : M \to SO(2)$ be any map. As $H^1(M;\mathbb{R})$ vanishes by assumption, $m^*(w)$ vanishes in de Rham cohomology; hence $m^*(w) = df$ for some real-valued function $f : M \to \mathbb{R}$. We can now define a trivial homotopy from f to a constant map from M to $\mathbb{R}$ by simply setting $f_t = tf$, $0 \leq t \leq 1$. Now let $\pi : \mathbb{R} \to S^1 \cong \mathbb{R}/2\pi$ be the projection. Then setting $m_t = \pi \circ f_t$, we get the desired homotopy from m to a constant map from M to S^1.

More generally, if M is not assumed to be a homology sphere, then W might be nontrivial. To define the Chern-Simons invariant of a three-manifold M, however, it is not quite necessary to have a framing of TM. It is enough to have a two-framing, a trivialization of $TM \oplus TM$. We claim that every Seifert fibration $\pi : M \to \Sigma$ determines a natural two-framing on M (which might depend on the choice of π, as a given M may admit more than one Seifert fibration). As $TM \oplus TM = L \oplus L \oplus W \oplus W$, and L has rank one, it suffices to trivialize $W \oplus W$.

The trivialization of $W \oplus W$ which we need is not arbitrary but must satisfy two conditions. For the first condition, we observe that $W \oplus W$ has a natural spin structure, the spin bundle being the sum of exterior powers of W. On the other hand, any trivial bundle associated to a vector space V also has a natural spin bundle associated to the Clifford module $C(V)$, which is unique up to isomorphism. So a given trivialization of $W \oplus W$ also determines a spin structure, and we require that this spin structure coincide with the natural spin structure.

Second, since $U(1)$ acts on the Seifert manifold M, we require that the trivialization of $W \oplus W$ be invariant under this action.

To show that $W \oplus W$ is trivial in the first place, we note that by definition W is a pullback from some $SO(2)$ bundle W_0 over Σ. Hence $W \oplus W$ is the pullback of $U = W_0 \oplus W_0$. The rank four real bundle U has vanishing Stiefel-Whitney classes w_1 and w_2 (being valued in $\mathbb{Z}_2$, they are killed by taking two copies of W_0), so it is trivial and hence $W \oplus W$ is also trivial.

A trivialization of $W \oplus W$ compatible with the two conditions above exists, and it is unique up to homotopy. To prove the uniqueness, we note that compatibility with a given spin structure of a rank k real bundle U — in our application $k = 4$ — means that changes of trivialization really come from maps to $Spin(k)$ rather than $SO(k)$. As $\pi_i(Spin(k)) = 0$ for $i \leq 2$, $k \geq 3$, a trivial $SO(k)$ bundle U over Σ of rank $k \geq 3$ has up to homotopy only one trivialization compatible with a given spin structure. So finally the Seifert fibration $\pi : M \to \Sigma$ endows M with a natural two-framing (which may differ from its canonical two-framing [103], which is determined by a different construction).

In sum, then, a Seifert fibration of a homology sphere M determines a natural trivialization of the tangent bundle TM, which we will call the Seifert framing, and any Seifert fibration $\pi : M \to \Sigma$ (even if M is not a homology sphere) determines a natural trivialization of $TM \oplus TM$, which we will call the Seifert two-framing. If M is a Seifert homology sphere, the Seifert two-framing just arises by applying the Seifert framing to each copy of TM.

Now we consider in detail the illustrative example $M = S^3$. S^3 has no one natural framing. However, if we identify it with the Lie group $SU(2)$, then it does have two equally natural framings, one which is left-invariant and one which is right-invariant. They are exchanged by an orientation-reversing reflection of S^3, so neither one is preferred. In regarding S^3 as a Seifert fibration over $\mathbb{CP}^1$, we write $\mathbb{CP}^1 = S^3/U(1)$, where $U(1)$ is either part of the left action of $SU(2)$ on itself or part of the right action. For either choice of $U(1)$, our construction produces a framing that is canonically determined by the choice of $U(1)$ generator and so is invariant under any symmetry that commutes with $U(1)$. If the $U(1)$ is part of the left $SU(2)$, then it commutes with the right $SU(2)$ and so we get the right-invariant framing; and likewise if the $U(1)$ is part of the right $SU(2)$, we get the left-invariant framing.

We naturally expect that the phase of Z in our computation of the Chern-Simons path integral is based on the Seifert framing. In view of our direct computation of the phase of Z, the Seifert two-framing of M must differ from the canonical two-framing of [103] by

$s = (3-d)$ units. We now give a simple proof of this fact in the case $M = S^3$ and $d = 1$ (though we will not be careful about the sign of the shift).

When $M = S^3$, the canonical two-framing of [103] can be described as follows. It is the trivialization of $TM \oplus TM$ that comes from the left-invariant framing on, say, the first copy of TM and the right-invariant framing on the second. (This is the unique reflection-invariant two-framing of S^3, so it must be the canonical two-framing.) On the other hand, the Seifert framing of M is (for a suitable choice of fibration $\pi : S^3 \to \mathbb{CP}^1$) the left-invariant framing of TM, so the Seifert two-framing comes by applying the left-invariant framing to each of the two copies of TM. Hence the comparison between the Seifert two-framing and the canonical one is the same as the comparison between the left-invariant two-framing and the right-invariant two-framing for a single copy of TM.

The right-invariant framing of S^3 is determined by the basis of right-invariant one-forms $\theta = dg\, g^{-1}$, while the left-invariant framing is determined by the basis of left-invariant one-forms $\widehat{\theta} = g^{-1}dg$. To compare the framings, we write $\theta = T\widehat{\theta}T^{-1}$, where T is a map from M to $SO(3)$. Such a map has a "degree," and this integer measures by how many units the two framings differ. Clearly, in this case, $T = g$, so T is the "identity" map from $S^3 \cong SU(2)$ to itself. This map is of degree 1 as a map to $SU(2)$. However, because the structure group of the tangent bundle of M is $SO(3) = SU(2)/\mathbb{Z}_2$, we must actually count the degree for maps to $SO(3)$. The identity map to $SU(2)$ descends to a map of degree 2 to $SO(3)$, and this shows, as expected, that the Seifert two-framing of S^3 differs from the canonical two-framing by $3 - d = 2$ units.

To illustrate the role of the structure group $SO(3)$, let us consider one more simple example, which is $M = SO(3) = S^3/\mathbb{Z}_2$. This is the case $d = 2$ of the lens space considered above, so we expect the Seifert two-framing and the canonical two-framing to differ by $3 - d = 1$ unit. The comparison again reduces to comparing the right-invariant framing of TM with the left-invariant one. So again we have to compare $\theta = dg\, g^{-1}$ with $\widehat{\theta} = g^{-1}dg$. We have $\theta = g\widehat{\theta}g^{-1}$, where now g is the identity map from $SO(3)$ to itself, which is of degree 1, showing that the two two-framings differ by one unit.

For any d, the general analysis of framings by Freed and Gompf in [104] can be used to check that the canonical two-framing and the Seifert two-framing on M differ by $s = (3 - d)$ units.

4.5.3 Localization on a Smooth Component of the Moduli Space of Irreducible Flat Connections

We now extend our work in the previous section to describe the local contribution to the Chern-Simons path integral from a smooth component $\mathcal{M}$ of the moduli space of irreducible flat connections on a Seifert manifold M. We assume here for simplicity that M is described by a line bundle $\mathcal{L}$ of degree n over a smooth Riemann surface Σ of genus $g \geq 1$. The orbifold case is also discussed by Rozansky in [80] but is somewhat more involved.

As we recall from Section 5.1, $\mathcal{M}$ is literally the moduli space of flat connections on the trivial G-bundle over M such that the holonomy $\rho(h)$ around the S^1 fiber of M is a fixed element of the center Γ of G. This moduli space is not smooth for arbitrary $\rho(h)$ in Γ, but it is smooth in certain cases. The main such case, and the case we consider here, arises when the gauge group G is $SU(r+1)$, $\rho(h)$ is a generator of $\Gamma = \mathbb{Z}_{r+1}$, and n and $r+1$ are relatively prime. Under these conditions, $\rho(h)^n$ also generates Γ, and $\mathcal{M}$ is smooth and can be identified with an unramified $(r+1)^{2g}$-fold cover of the moduli space $\mathcal{M}_0$ of flat Yang-Mills connections on an associated principal bundle P over Σ with structure group $\overline{G} = G/\Gamma$. ($\overline{G}$ enters because when we project to $\overline{G}$, $\rho(h)$ projects to 1 and the representation ρ becomes a pullback from Σ. But as the three-dimensional gauge group is really G, the holonomies of ρ around one-cycles in Σ are defined as elements of G, not $\overline{G}$; this leads to the unramified cover.)

Our general discussion of non-abelian localization in Section 4 implies that the path integral contribution from $\mathcal{M}$ can be expressed entirely in terms of the cohomology ring of $\mathcal{M}$, or equivalently $\mathcal{M}_0$. One of the reasons that localization on $\mathcal{M}$ is interesting is that we find in Chern-Simons theory a natural generalization of the cohomological formula (4.4.17) for the path integral contribution from $\mathcal{M}_0$ in two-dimensional Yang-Mills theory.

We recall from our discussion in Section 5.1 that a local symplectic neighborhood N near $\mathcal{M}$ in $\overline{\mathcal{A}}$ is described by an equivariant bundle

$$F \longrightarrow N \xrightarrow{pr} \mathcal{M}\,, \tag{4.5.106}$$

where the normal fiber F takes the (by now familiar) form $F = \mathcal{H} \times_{H_0} (\mathfrak{h} \ominus \mathfrak{h}_0 \ominus \mathcal{E}_0 \oplus \mathcal{E}_1)$.

By assumption, the only gauge transformations which fix the irreducible flat connections associated to points in $\mathcal{M}$ are constant gauge transformations by elements in the center Γ of G, since the center of G always acts trivially in the adjoint representation. So the stabilizer subgroup H_0 in $\mathcal{H}$ is now given by

$$H_0 = U(1)^2 \times \Gamma\,, \tag{4.5.107}$$

where we recall that the torus $U(1)^2$ arises from the two extra generators in $\mathcal{H}$ relative to $\mathcal{G}$.

Also, we recall that the vector spaces $\mathcal{E}_0$ and $\mathcal{E}_1$ are now given over a point of $\mathcal{M}$ by

$$\begin{aligned} \mathcal{E}_0 &= \bigoplus_{t\neq 0} H^0_{\bar\partial}(\Sigma, \mathrm{ad}(P)\otimes\mathcal{L}^t) = \bigoplus_{t\geq 1} H^0_{\bar\partial}(\Sigma, \mathrm{ad}(P)\otimes(\mathcal{L}^t\oplus\mathcal{L}^{-t}))\,,\\ \mathcal{E}_1 &= \bigoplus_{t\neq 0} H^1_{\bar\partial}(\Sigma, \mathrm{ad}(P)\otimes\mathcal{L}^t) = \bigoplus_{t\geq 1} H^1_{\bar\partial}(\Sigma, \mathrm{ad}(P)\otimes(\mathcal{L}^t\oplus\mathcal{L}^{-t}))\,. \end{aligned} \tag{4.5.108}$$

The Canonical Symplectic Integral Over N

Having described the local geometry near $\mathcal{M}$ in $\overline{\mathcal{A}}$, we next consider the canonical symplectic integral over N. This integral takes the form

$$Z(\epsilon)\Big|_{\mathcal{M}} = \frac{2\pi\epsilon\cdot \mathrm{Vol}(U(1)^2)}{\mathrm{Vol}(\mathcal{H})} \int_{\mathfrak{h}\times N} \left[\frac{d\phi}{2\pi}\right] \exp\left[\Omega - i\,\langle\mu,\phi\rangle - \frac{i\epsilon}{2}(\phi,\phi) + tD\lambda\right], \tag{4.5.109}$$

where we include in the normalization of (4.5.109) the prefactor from (4.5.10). To define the integral over the non-compact directions in N, we also include in (4.5.109) the localization form $tD\lambda$.

Our goal now is to reduce the integral over $\mathfrak{h}\times N$ in (4.5.109) to an integral over the moduli space $\mathcal{M}$ itself. We have already discussed a problem of this sort in Section 4.2, when we considered the path integral contribution from irreducible flat connections in two-dimensional Yang-Mills theory. As we briefly recall, in the case of Yang-Mills theory the fiber F in (4.5.106) is modelled on the cotangent bundle T^*H (with H being the group of gauge transformations in that case), so that N retracts equivariantly onto a principal H-bundle P_H over the the moduli space $\mathcal{M}_0$. Because H acts freely on P_H, the H-equivariant cohomology of the total space P_H can be identified with the ordinary cohomology of the quotient $P_H/H = \mathcal{M}_0$, so $H^*_H(P_H) \cong H^*(\mathcal{M}_0)$. In particular, the H-equivariant cohomology classes of $[\Omega - i \langle \mu, \phi \rangle]$ and $[-\frac{1}{2}(\phi, \phi)]$ on P_H pull back from ordinary cohomology classes Ω and Θ of degrees two and four on $\mathcal{M}_0$, and we apply this fundamental fact to reduce the symplectic integral in Yang-Mills theory to an integral over $\mathcal{M}_0$.

In the case of Chern-Simons theory, the group $H \equiv \mathcal{H}$ no longer acts freely on N, but we can still apply much the same logic as for the case of Yang-Mills theory. Here a subgroup H_0 of H acts with fixed points on N, so N equivariantly retracts onto a bundle with fiber H/H_0 over $\mathcal{M}$. We denote the total space of this bundle by N_0, so that $H/H_0 \longrightarrow N_0 \longrightarrow \mathcal{M}$.

Because N_0 is an equivariant retraction from N, the H-equivariant cohomology ring of N is the same as that of N_0. As we explain in Appendix C, the formal properties of equivariant cohomology further imply that the H-equivariant cohomology ring of N_0 is identified under pullback with the H_0-equivariant cohomology ring of $\mathcal{M}$ itself. So in total, we have the relation $H^*_H(N) \cong H^*_{H_0}(\mathcal{M})$.

As a result, in precise analogy to the case of two-dimensional Yang-Mills theory, the H-equivariant cohomology classes of $[\Omega - i \langle \mu, \phi \rangle]$ and $[-\frac{1}{2}(\phi, \phi)]$ which appear in the symplectic integral over N can be identified as the pullbacks from $\mathcal{M}$ of elements in the ring $H^*_{H_0}(\mathcal{M})$.

To identify the elements of $H^*_{H_0}(\mathcal{M})$ which pull back to these classes appearing in the symplectic integral over N, we note that $H^*_{H_0}(\mathcal{M})$ has a very simple structure. As we also explain in Appendix C, because H_0 acts trivially on $\mathcal{M}$, $H^*_{H_0}(\mathcal{M})$ is given by the tensor product of the ordinary cohomology ring $H^*(\mathcal{M})$ of $\mathcal{M}$ with the H_0-equivariant cohomology ring $H^*_{H_0}(pt)$ of a point. Thus, $H^*_{H_0}(\mathcal{M}) = H^*(\mathcal{M}) \otimes H^*_{H_0}(pt)$.

Finally, our previous discussion of the Cartan model of equivariant cohomology explicitly identifies the H_0-equivariant cohomology ring of a point with the ring of invariant functions on the Lie algebra $\mathfrak{h}_0$. Thus, all elements of $H^*_{H_0}(\mathcal{M})$ can be written as sums of terms having the form $x \cdot f(\psi)$, where x is an ordinary cohomology class on $\mathcal{M}$ and $f(\psi)$ is an invariant function of ψ in $\mathfrak{h}_0$.

With our concrete description of $H^*_{H_0}(\mathcal{M})$, we can immediately identify the elements of this ring which pull back to the H-equivariant classes $[\Omega - i\,\langle\mu,\phi\rangle]$ and $[-\frac{1}{2}(\phi,\phi)]$ on N. Let us decompose the Lie algebra $\mathfrak{h}$ of $\mathcal{H}$ as a sum $\mathfrak{h} = (\mathfrak{h}\ominus\mathfrak{h}_0)\oplus\mathfrak{h}_0$. As a result, we write $\phi = \varphi + p + a$, where φ is an element of $\mathfrak{h}\ominus\mathfrak{h}_0$, which can be identified as the Lie algebra of $\mathcal{G}$, and, in the same notation from Section 3.4, p and a are elements of the Lie algebra $\mathfrak{h}_0$ of H_0.

We then identify the H-equivariant classes on N appearing in (4.5.109) with corresponding H_0-equivariant classes on $\mathcal{M}$ via

$$\begin{aligned} \Omega - i\,\langle\mu,\phi\rangle \;&\longleftrightarrow\; \Omega\,-\,i\,a\,,\\ -\frac{1}{2}(\phi,\phi)\;&\longleftrightarrow\; n\,\Theta\,+\,pa\,. \end{aligned} \tag{4.5.110}$$

We abuse notation slightly in the first line of (4.5.110). On the left, Ω is the symplectic form on $\overline{\mathcal{A}}$ restricted to N, and on the right Ω is the induced symplectic form on $\mathcal{M}$ (or equivalently $\mathcal{M}_0$), exactly as in our discussion of two-dimensional Yang-Mills theory. In identifying the dependence of this degree two class in $H^*_{H_0}(\mathcal{M})$ on p and a, we use the fact, evident from the formula for μ in (4.3.50), that the value of the moment map $\langle\mu,\phi\rangle$ evaluated at a flat connection which pulls back from Σ is just the constant a appearing on the right of the first line in (4.5.110).

Similarly, in the second line of (4.5.110), the degree four class Θ on $\mathcal{M}$ is the same degree four class that appeared in our discussion of Yang-Mills theory. The identification in (4.5.110) arises by writing the degree four invariant $-\frac{1}{2}(\phi,\phi)$ in terms of φ, p, and a as

$$-\frac{1}{2}(\phi,\phi)\;=\;\frac{1}{2}\int_M \kappa\wedge d\kappa\,\mathrm{Tr}(\varphi^2)\,+\,pa\;=\;\frac{n}{2}\int_\Sigma \omega\,\mathrm{Tr}(\varphi^2)\,+\,pa\,, \tag{4.5.111}$$

where we recall that n is the degree of the line-bundle $\mathcal{L}$ over Σ which defines M and ω is a unit-volume symplectic form on Σ. As in the case of two-dimensional Yang-Mills theory, the term quadratic in the generators φ of the gauge symmetry is associated by the Chern-Weil homomorphism to the degree four class Θ.

With the identifications in (4.5.110), we can rewrite the symplectic integral over N as

$$Z(\epsilon)\Big|_{\mathcal{M}} \;=\; \frac{2\pi\epsilon\cdot\mathrm{Vol}(U(1)^2)}{\mathrm{Vol}(\mathcal{H})}\int_{\mathfrak{h}\times N}\left[\frac{d\phi}{2\pi}\right]\exp\left[(pr^*\Omega)\;-\;ia\,(1-\epsilon p)\;+\;i\epsilon n\,(pr^*\Theta)\;+\;tD\lambda\right]. \tag{4.5.112}$$

As in the case of localization at the trivial connection, the generator a acts trivially on all of N and so does not appear in the localization form $tD\lambda$. So we can perform the integrals over a and p exactly as before, and the integral over a produces a delta-function that sets $p = 1/\epsilon$. As a result, the symplectic integral reduces to the form

$$Z(\epsilon)\Big|_{\mathcal{M}} \;=\; \frac{\mathrm{Vol}(U(1)^2)}{\mathrm{Vol}(\mathcal{H})}\int_{(\mathfrak{h}\ominus\mathfrak{h}_0)\times N}\left[\frac{d\phi}{2\pi}\right]\exp\left[(pr^*\Omega)\;+\;i\epsilon n\,(pr^*\Theta)\;+\;tD\lambda\Big|_{p=1/\epsilon}\right]. \tag{4.5.113}$$

The only term in (4.5.113) which does not pull back from $\mathcal{M}$ is the localization term $tD\lambda$, so we are left to integrate $tD\lambda$ over the fiber F of N. In the case of two-dimensional Yang-Mills theory, with $F = T^*H$, this integral gave a trivial factor of unity. In Chern-Simons theory, the result is much more interesting.

An Equivariant Euler Class From F

To evaluate (4.5.113), we consider the following integral,

$$I(\psi) \;=\; \frac{1}{\mathrm{Vol}(\mathcal{H})}\int_{\widetilde{F}}\left[\frac{d\phi}{2\pi}\right]\exp\left[tD\lambda\right], \qquad \widetilde{F} \;=\; (\mathfrak{h}\ominus\mathfrak{h}_0)\times F\,, \qquad \psi\in\mathfrak{h}_0\,. \tag{4.5.114}$$

Here we let $\psi = p + a$ be an arbitrary element of $\mathfrak{h}_0$, though in general the generator a will not appear in (4.5.114) since a acts trivally on N, and we set $p = 1/\epsilon$ at the end of the discussion, as in (4.5.113).

Of course, in Section 4.3 we computed this integral over the abstract model for F. There we assumed $\mathcal{M}$ to be a point, and we found the result

$$I(\psi) \;=\; \frac{1}{\mathrm{Vol}(H_0)}\,\det\left(\frac{\psi}{2\pi}\Big|_{E_0}\right)\det\left(\frac{\psi}{2\pi}\Big|_{E_1}\right)^{-1}, \qquad \psi\in\mathfrak{h}_0\,. \tag{4.5.115}$$

Unfortunately, we cannot apply this result directly to the case at hand. When F is fibered over a non-trivial moduli space $\mathcal{M}$, then $I(\psi)$ will generally involve cohomology classes on $\mathcal{M}$ which are associated to the twisting of the bundle and which our previous computation did not detect.

To compute $I(\psi)$ in (4.5.114), one approach is simply to generalize the abstract localization computation in Section 4.3 to allow for a non-trivial moduli space $\mathcal{M}$. We perform this computation in Appendix D. However, we can also make an immediate guess, on the basis of mathematical naturality, for what the generalization of the formula (4.5.115) must be when $\mathcal{M}$ is non-trivial. This guess relies on a more intrinsic topological interpretation of the result (4.5.115) even in the case that $\mathcal{M}$ is a point. For this reason, it turns out to be much more illuminating to "guess" the generalization of (4.5.115) rather than simply to compute, so we pursue this approach now.

Let us think about what our result for $I(\psi)$ really *means* in the case that $\mathcal{M} = pt$. Abstractly, the data which enter the formula (4.5.115) are the group H_0, which acts trivially on $\mathcal{M}$, and the finite-dimensional unitary representations E_0 and E_1 of H_0. In general, to say that E is a representation of H_0 is the same thing as to say that E is an H_0-equivariant bundle over a point, so if we like, we can consider E_0 and E_1 as H_0-equivariant bundles over $\mathcal{M} = pt$.

This language is useful, since whenever we have a vector bundle (even a vector bundle over a point!) an extremely natural set of topological invariants to consider are the characteristic classes of the bundle. In our context, we naturally consider the H_0-equivariant characteristic classes of E_0 and E_1 as H_0-equivariant bundles over $\mathcal{M} = pt$. (Although we will not require the generalization here, we refer the reader to Chapter 8.5 of [99] for a general discussion of equivariant characteristic classes.) These characteristic classes are valued in the H_0-equivariant cohomology ring of $\mathcal{M}$ — since $\mathcal{M}$ is a point, this ring is the ring of invariant functions on the Lie algebra $\mathfrak{h}_0$ of H_0.

If E is a unitary representation of H_0 and we consider E as an H_0-equivariant bundle over a point, then the H_0-equivariant characteristic classes of E have a simple description. We let $U(E)$ be the unitary group acting on E. Since H_0 acts in a unitary fashion on E, the relevant characteristic classes of E to consider are the equivariant Chern classes. As is well known, the ordinary Chern classes of a vector bundle are associated via the Chern-Weil homomorphism to the generators c_i of the ring of invariant polynomials on the Lie algebra of the unitary group. To describe the corresponding H_0-equivariant Chern classes of E, we observe that, since E is a unitary representation of H_0, we have an induced map $H_0 \longrightarrow U(E)$. Consequently, any invariant polynomial on the Lie algebra of $U(E)$ pulls back to an invariant polynomial on the Lie algebra $\mathfrak{h}_0$ of H_0. The pullbacks of the generators c_i to invariant polynomials on $\mathfrak{h}_0$ are then the H_0-equivariant Chern classes of E. In particular, if the action of H_0 on E is non-trivial, then the equivariant Chern classes of E can also be non-trivial, despite the fact that E is a bundle over only a point.

The invariant polynomials appearing in $I(\psi)$, namely

$$e_{H_0}(pt, E_0) \equiv \det\left(\frac{\psi}{2\pi}\Big|_{E_0}\right), \qquad e_{H_0}(pt, E_1) \equiv \det\left(\frac{\psi}{2\pi}\Big|_{E_1}\right), \tag{4.5.116}$$

arise from determinants. The Chern-Weil homomorphism associates the determinant to the top Chern class, so by our discussion above the invariant polynomials in (4.5.116) can be characterized intrinsically as the H_0-equivariant top Chern classes, or equivalently Euler classes, of E_0 and E_1 as equivariant bundles over a point. Thus, when $\mathcal{M}$ is a point, we write $I(\psi)$ in (4.5.115) intrinsically as

$$I(\psi) = \frac{1}{\mathrm{Vol}(H_0)} \frac{e_{H_0}(pt, E_0)}{e_{H_0}(pt, E_1)}. \tag{4.5.117}$$

More generally, if E is an H_0-equivariant vector bundle over a complex manifold $\mathcal{M}$, then we can still consider the H_0-equivariant Euler class $e_{H_0}(\mathcal{M}, E)$ of E, which takes values in the H_0-equivariant cohomology ring of $\mathcal{M}$. If H_0 acts trivially on $\mathcal{M}$ (but not necessarily trivially on E), we have already identified this cohomology ring as a product $H^*_{H_0}(\mathcal{M}) \cong H^*(\mathcal{M}) \otimes H^*_{H_0}(pt)$. We describe $e_{H_0}(\mathcal{M}, E)$ in this case explicitly below.

In our application to Chern-Simons theory, the infinite-dimensional vector spaces $\mathcal{E}_0$ and $\mathcal{E}_1$ in (4.5.108) determine associated H_0-equivariant bundles over the moduli space $\mathcal{M}$, on which H_0 in (4.5.107) acts trivially. Given our intrinsic interpretation of $I(\psi)$ when $\mathcal{M}$ is a point, we certainly expect that the integral over F in (4.5.114) produces the natural generalization of (4.5.117), involving the H_0-equivariant Euler classes of the bundles associated to $\mathcal{E}_0$ and $\mathcal{E}_1$ over $\mathcal{M}$. That is,

$$I(\psi) = \frac{1}{\mathrm{Vol}(H_0)} \frac{e_{H_0}(\mathcal{M}, \mathcal{E}_0)}{e_{H_0}(\mathcal{M}, \mathcal{E}_1)}. \tag{4.5.118}$$

As our direct computation in Appendix D shows, this formula is correct.

We remark that the appearance of the equivariant Euler class of the bundle $\mathcal{E}_1$ in the denominator of (4.5.118) is quite standard. This class appears in precisely the same way in the classic Duistermaat-Heckman formula [19] for abelian localization, as was explained in [20]. The essentially new feature of the formula (4.5.118) is the appearance of a corresponding Euler class from $\mathcal{E}_0$ in the numerator.

We set

$$e(\psi) \;=\; \frac{e_{H_0}(\mathcal{M}, \mathcal{E}_0)}{e_{H_0}(\mathcal{M}, \mathcal{E}_1)}\,. \tag{4.5.119}$$

Then from (4.5.113), (4.5.114), and (4.5.118), the local contribution from $\mathcal{M}$ in Chern-Simons theory is given abstractly by

$$Z(\epsilon)\Big|_{\mathcal{M}} \;=\; \frac{1}{|\Gamma|}\int_{\mathcal{M}} e(p)\Big|_{p=1/\epsilon}\, \exp\left(\Omega \,+\, i\epsilon n\Theta\right). \tag{4.5.120}$$

In arriving at (4.5.120), we note that the prefactor $\mathrm{Vol}(U(1)^2)$ in (4.5.113) cancels against a corresponding factor in $\mathrm{Vol}(H_0)$ from $I(\psi)$. This cancellation leaves the factor $1/|\Gamma|$ in (4.5.114), where $|\Gamma|$ is the order of the center Γ of G.

As we recall in writing (4.5.120), since the generator a in $\mathfrak{h}_0$ acts trivially on N, $e(\psi) \equiv e(p)$ depends only on p in $\mathfrak{h}_0$. Once we set $p = 1/\epsilon$ in (4.5.120), $e(\epsilon^{-1})$ will become an ordinary cohomology class on $\mathcal{M}$. As in the case of localization at the trivial connection, our computation now reduces to determining explicitly this class.

More About the Equivariant Euler Class

Before we evaluate the equivariant Euler classes of the infinite-dimensional bundles corresponding to $\mathcal{E}_0$ and $\mathcal{E}_1$, we first give a more explicit description of the equivariant Euler class in a simpler, finite-dimensional situation. To make contact with Chern-Simons theory, we assume abstractly that H_0 is a torus which acts trivially on a complex manifold $\mathcal{M}$, and we assume that E is a complex representation of H_0 which is fibered over $\mathcal{M}$ to determine an associated H_0-equivariant bundle. Our goal is now to give a concrete topological formula for $e_{H_0}(\mathcal{M}, E)$, which we will then apply to evaluate $e(\psi)$ in (4.5.119) for Chern-Simons theory.

In general, $e_{H_0}(\mathcal{M}, E)$ incorporates both the algebraic data associated to the action of H_0 on E as well as the topological data that describes the twisting of E over $\mathcal{M}$. To encode the data related to the action of H_0 on E, we decompose E under the action of H_0 into a sum of one-dimensional complex eigenspaces

$$E \;=\; \bigoplus_{j=1}^{\dim E} E_{\beta_j}\,, \tag{4.5.121}$$

where each β_j is a weight in $\mathfrak{h}_0^*$ which describes the action of H_0 on the eigenspace E_{β_j}.

To encode the topological data associated to the vector bundle determined by E over $\mathcal{M}$, we apply the splitting principle in topology, as explained for instance in Chapter 21 of [52]. By this principle, we can assume that the vector bundle determined by E over $\mathcal{M}$ splits equivariantly into a sum of line-bundles associated to each of the eigenspaces E_{β_j} for the action of H_0. Under this assumption, we let $x_j = c_1(E_{\beta_j})$ be the first Chern class of the corresponding line-bundle. These virtual Chern roots x_j determine the total Chern class of E as

$$c(E) \;=\; \prod_{j=1}^{\dim E} (1 + x_j)\,. \tag{4.5.122}$$

In particular, the ordinary Euler class of E over $\mathcal{M}$ is then given by

$$e(\mathcal{M}, E) \;=\; \prod_{j=1}^{\dim E} x_j\,. \tag{4.5.123}$$

The equivariant Euler class $e_{H_0}(\mathcal{M}, E)$ is now determined in terms of the weights β_j and the Chern roots x_j. Since H_0 acts trivially on $\mathcal{M}$, we recall that $e_{H_0}(\mathcal{M}, E)$ is defined as an element of $H^*_{H_0}(\mathcal{M}, E) = H^*(\mathcal{M}) \otimes H^*_{H_0}(pt)$. Thus, $e_{H_0}(\mathcal{M}, E)$ will be a function of $\psi \in \mathfrak{h}_0$ with values in the cohomology of $\mathcal{M}$. Explicitly, the H_0-equivariant Euler class of E over $\mathcal{M}$ is given by

$$e_{H_0}(\mathcal{M}, E) \;=\; \prod_{j=1}^{\dim E} \left(\frac{i\,\langle \beta_j, \psi \rangle}{2\pi} + x_j \right)\,. \tag{4.5.124}$$

We see that this expression is a natural generalization of the ordinary Euler class (4.5.123) of E. Also, when $\mathcal{M}$ is only a point, the Chern roots x_j do not appear in (4.5.124) for dimensional reasons, and the product over the weights β_j in (4.5.124) reproduces the determinant of ψ acting on E as in (4.5.116).

Evaluating $e(p)$

We now evaluate $e(p)$ for Chern-Simons theory. We set $p = 1/\epsilon$ only at the very end of the computation. First we recall that the complex vector spaces $\mathcal{E}_0$ and $\mathcal{E}_1$ appearing in (4.5.119) arise from the Dolbeault cohomology groups of the bundles $\mathrm{ad}(P) \otimes \mathcal{L}^t$ over Σ, with

$$\begin{aligned} \mathcal{E}_0 &= \bigoplus_{t \neq 0} H^0_{\bar\partial}(\Sigma, \mathrm{ad}(P) \otimes \mathcal{L}^t) = \bigoplus_{t \geq 1} H^0_{\bar\partial}(\Sigma, \mathrm{ad}(P) \otimes (\mathcal{L}^t \oplus \mathcal{L}^{-t}))\,, \\ \mathcal{E}_1 &= \bigoplus_{t \neq 0} H^1_{\bar\partial}(\Sigma, \mathrm{ad}(P) \otimes \mathcal{L}^t) = \bigoplus_{t \geq 1} H^1_{\bar\partial}(\Sigma, \mathrm{ad}(P) \otimes (\mathcal{L}^t \oplus \mathcal{L}^{-t}))\,. \end{aligned} \tag{4.5.125}$$

We also recall that the action of H_0 on $\mathcal{E}_0$ and $\mathcal{E}_1$ is determined by the operator $p\pounds_R$, whose action in turn only depends on the grading t in (4.5.125). We naturally decompose $\mathcal{E}_0$ and $\mathcal{E}_1$ under the action of H_0, and we consider the finite-dimensional eigenspaces

$$\mathcal{E}_0^{(t)} = H^0_{\bar\partial}(\Sigma, \mathrm{ad}(P) \otimes \mathcal{L}^t)\,, \qquad \mathcal{E}_1^{(t)} = H^1_{\bar\partial}(\Sigma, \mathrm{ad}(P) \otimes \mathcal{L}^t)\,. \tag{4.5.126}$$

The abelian group H_0 acts canonically on both $\mathcal{E}_0^{(t)}$ and $\mathcal{E}_1^{(t)}$ with eigenvalue $-2\pi i t$.

In terms of this decomposition, the quantity $e(p)$ is given by the following infinite product,

$$e(p) = \prod_{t \neq 0} \left[\frac{e_{H_0}(\mathcal{M}, \mathcal{E}_0^{(t)})}{e_{H_0}(\mathcal{M}, \mathcal{E}_1^{(t)})} \right] = \prod_{t \geq 1} \left[\frac{e_{H_0}(\mathcal{M}, \mathcal{E}_0^{(t)}) \cdot e_{H_0}(\mathcal{M}, \mathcal{E}_0^{(-t)})}{e_{H_0}(\mathcal{M}, \mathcal{E}_1^{(t)}) \cdot e_{H_0}(\mathcal{M}, \mathcal{E}_1^{(-t)})} \right]. \tag{4.5.127}$$

Here $e_{H_0}(\mathcal{M}, \mathcal{E}_0^{(t)})$ and $e_{H_0}(\mathcal{M}, \mathcal{E}_1^{(t)})$ denote the H_0-equivariant Euler classes of the finite-dimensional bundles determined by $\mathcal{E}_0^{(t)}$ and $\mathcal{E}_1^{(t)}$ over $\mathcal{M}$.

Our basic strategy to evaluate the product in (4.5.127) is to deduce a recursive relation between the equivariant Euler classes of $\mathcal{E}_0^{(t)}$, $\mathcal{E}_0^{(t-1)}$, $\mathcal{E}_1^{(t)}$, and $\mathcal{E}_1^{(t-1)}$. So far, we have only specified the line-bundle $\mathcal{L}$ topologically, by specifying its degree n. The holomorphic structure of $\mathcal{L}$ really was not important. Now we want to pick a convenient holomorphic structure on $\mathcal{L}$ to simplify our computation. We pick n arbitrary points $\sigma_1, \ldots, \sigma_n$ on Σ and we take $\mathcal{L}$ to be $\mathcal{O}(\sigma_1 + \ldots + \sigma_n)$.

With this choice of $\mathcal{L}$, we have the following short exact sequence of coherent sheaves on Σ,

$$0 \longrightarrow \mathrm{ad}_{\mathbb{C}}(P) \otimes \mathcal{L}^{t-1} \longrightarrow \mathrm{ad}_{\mathbb{C}}(P) \otimes \mathcal{L}^{t} \longrightarrow \bigoplus_{i=1}^{n} \mathrm{ad}_{\mathbb{C}}(P)|_{\sigma_i} \longrightarrow 0\,. \tag{4.5.128}$$

Here t is any integer, and $\mathrm{ad}_{\mathbb{C}}(P)|_{\sigma_i}$ denotes the skyscraper sheaf associated to the fiber of $\mathrm{ad}_{\mathbb{C}}(P)$ over the point σ_i. The appearance of this skyscraper sheaf explains our need to work a bit more generally with coherent sheaves, as opposed to more innocuous bundles.

Associated to this short exact sequence we have the usual long exact sequence in sheaf cohomology,

$$\begin{aligned} 0 \longrightarrow H^0(\Sigma, \mathrm{ad}_{\mathbb{C}}(P) \otimes \mathcal{L}^{t-1}) \longrightarrow H^0(\Sigma, \mathrm{ad}_{\mathbb{C}}(P) \otimes \mathcal{L}^{t}) \longrightarrow \bigoplus_{i=1}^{n} H^0(\Sigma, \mathrm{ad}_{\mathbb{C}}(P)|_{\sigma_i}) \longrightarrow \\ \longrightarrow H^1(\Sigma, \mathrm{ad}_{\mathbb{C}}(P) \otimes \mathcal{L}^{t-1}) \longrightarrow H^1(\Sigma, \mathrm{ad}_{\mathbb{C}}(P) \otimes \mathcal{L}^{t}) \longrightarrow 0\,. \end{aligned} \tag{4.5.129}$$

Since a skyscraper sheaf has no higher cohomology, we observe that $H^1(\Sigma, \mathrm{ad}_{\mathbb{C}}(P)|_{\sigma_i}) = 0$ for the last term of (4.5.129).

Each cohomology group appearing in (4.5.129) can be considered as the fiber of an associated holomorphic bundle over the moduli space $\mathcal{M}$, and the exactness of the sequence (4.5.129) implies the exactness of the corresponding sequence of bundles on $\mathcal{M}$. Except for the single term involving the skyscraper sheaf, we see that the bundles which appear in (4.5.129) are those associated to $\mathcal{E}_0^{(t-1)}$, $\mathcal{E}_0^{(t)}$, $\mathcal{E}_1^{(t-1)}$, and $\mathcal{E}_1^{(t)}$. In analogy to (4.5.126), we set

$$\mathcal{V}_{(i)} \;=\; H^0(\Sigma, \mathrm{ad}_{\mathbb{C}}(P)|_{\sigma_i})\,. \tag{4.5.130}$$

Over $\mathcal{M}$, $\mathcal{V}_{(i)}$ also fibers as a holomorphic bundle. Although the holomorphic structure of $\mathcal{V}_{(i)}$ depends on σ_i, its topology, which is all we will care about, does not (as is clear from the fact that the points σ_i can be moved continuously), so we just write $\mathcal{V}$ for any of the $\mathcal{V}_{(i)}$. Thus, the exact sequence in (4.5.129) implies the following exact sequence of associated bundles on $\mathcal{M}$,

$$0 \longrightarrow \mathcal{E}_0^{(t-1)} \longrightarrow \mathcal{E}_0^{(t)} \longrightarrow \mathcal{V}^{\oplus n} \longrightarrow \mathcal{E}_1^{(t-1)} \longrightarrow \mathcal{E}_1^{(t)} \longrightarrow 0\,. \tag{4.5.131}$$

This sequence is an exact sequence of bundles on $\mathcal{M}$, but we need an exact sequence of H_0-equivariant bundles on $\mathcal{M}$, such that the maps in the sequence are compatible with the action of H_0. Because H_0 acts with different eigenvalues on the equivariant bundles $\mathcal{E}_0^{(t-1)}$ and $\mathcal{E}_0^{(t)}$, and similarly on $\mathcal{E}_1^{(t-1)}$ and $\mathcal{E}_1^{(t)}$, the canonical action of H_0 is not compatible with the maps in (4.5.131).

To fix this problem, we note that we are free to consider actions of H_0 on $\mathcal{E}_0^{(t)}$ and $\mathcal{E}_1^{(t)}$ other than the canonical action. That is, we consider H_0-equivariant bundles over $\mathcal{M}$ whose fibers are still given by the cohomology groups $H^0_{\bar\partial}(\Sigma, \mathrm{ad}(P) \otimes \mathcal{L}^t)$ and $H^1_{\bar\partial}(\Sigma, \mathrm{ad}(P) \otimes \mathcal{L}^t)$ but where the action of H_0 is not the canonical action fixed by t. In fact, so long as H_0 acts uniformly on the fiber, we can take H_0 to act with any eigenvalue.

Thus we let $\mathcal{E}_{0,m}^{(t)}$ and $\mathcal{E}_{1,m}^{(t)}$ denote the H_0-equivariant bundles over $\mathcal{M}$ whose fibers are determined by t as before but where H_0 acts with eigenvalue $-2\pi i m$ for some integer m. In this notation, the bundles $\mathcal{E}_0^{(t)}$ and $\mathcal{E}_1^{(t)}$ with the canonical action of H_0 are $\mathcal{E}_{0,t}^{(t)}$ and $\mathcal{E}_{1,t}^{(t)}$. We similarly denote by $\mathcal{V}_m$ the H_0-equivariant bundle associated to $\mathcal{V}$ for which H_0 acts uniformly on the fiber with eigenvalue $-2\pi i m$.

The exact sequence in (4.5.131) on $\mathcal{M}$ now determines a corresponding exact sequence of H_0-equivariant bundles,

$$0 \longrightarrow \mathcal{E}_{0,m}^{(t-1)} \longrightarrow \mathcal{E}_{0,m}^{(t)} \longrightarrow (\mathcal{V}_m)^{\oplus n} \longrightarrow \mathcal{E}_{1,m}^{(t-1)} \longrightarrow \mathcal{E}_{1,m}^{(t)} \longrightarrow 0\,. \tag{4.5.132}$$

Since the action of H_0 is the same on every term in this sequence, the maps are trivially compatible with the group action.

We now recall that a fundamental property of the equivariant Euler class is that it behaves multiplicatively with respect to an exact sequence of equivariant bundles, just like the ordinary Euler class. Thus, if E_1, E_2, and E_3 are H_0-equivariant bundles on $\mathcal{M}$ which fit into an exact sequence whose maps respect the action of H_0,

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0\,, \tag{4.5.133}$$

then the H_0-equivariant Euler classes of these bundles satisfy the relation

$$e_{H_0}(\mathcal{M}, E_2) \,=\, e_{H_0}(\mathcal{M}, E_1) \cdot e_{H_0}(\mathcal{M}, E_3)\,. \tag{4.5.134}$$

More generally, given an exact sequence of arbitrary length,

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_{2N} \longrightarrow E_{2N+1} \longrightarrow 0\,, \tag{4.5.135}$$

the relation (4.5.134) generalizes in the natural way, with

$$e_{H_0}(\mathcal{M}, E_2) \cdots e_{H_0}(\mathcal{M}, E_{2N}) \;=\; e_{H_0}(\mathcal{M}, E_1) \cdots e_{H_0}(\mathcal{M}, E_{2N+1})\,. \tag{4.5.136}$$

We apply this multiplicative property of the equivariant Euler class to the exact sequence in (4.5.132). For the following, it is very natural to introduce the ratio of equivariant Euler classes,

$$\mathcal{Q}_m^{(t)} \;\equiv\; \left[\frac{e_{H_0}(\mathcal{M}, \mathcal{E}_{0,m}^{(t)})}{e_{H_0}(\mathcal{M}, \mathcal{E}_{1,m}^{(t)})}\right]\,, \tag{4.5.137}$$

so that $e(p)$ is given by

$$e(p) \;=\; \prod_{t\neq 0} \mathcal{Q}_t^{(t)}\,. \tag{4.5.138}$$

In terms of $\mathcal{Q}_m^{(t)}$, the multiplicative relation (4.5.134) applied to (4.5.132) implies that

$$\mathcal{Q}_m^{(t)} \;=\; \mathcal{Q}_m^{(t-1)} \cdot \left[e_{H_0}(\mathcal{M}, \mathcal{V}_m)\right]^n\,. \tag{4.5.139}$$

Expanding the recursive relation (4.5.139), we find

$$\mathcal{Q}_m^{(t)} \;=\; \mathcal{Q}_m^{(0)} \cdot \left[e_{H_0}(\mathcal{M}, \mathcal{V}_m)\right]^{nt}\,. \tag{4.5.140}$$

What has this work gained us? As we now explain, we can give a very concrete expression for the quantity on the right of (4.5.140). By definition, the bundles over $\mathcal{M}$ which determine the ratios $\mathcal{Q}_{\pm t}^{(0)}$ have fibers

$$\mathcal{E}_0^{(0)} \;=\; H^0_{\bar\partial}(\Sigma, \mathrm{ad}_{\mathbb{C}}(P))\,, \qquad \mathcal{E}_1^{(0)} \;=\; H^1_{\bar\partial}(\Sigma, \mathrm{ad}_{\mathbb{C}}(P))\,. \tag{4.5.141}$$

By our assumption that all points in the moduli space $\mathcal{M}$ correspond to irreducible connections, $\mathcal{E}_0^{(0)} = 0$. Further, as we mentioned in Section 4.3, $\mathcal{E}_1^{(0)}$ is naturally identified with the holomorphic tangent bundle $T\mathcal{M}$ of the moduli space itself, so $\mathcal{E}_1^{(0)} = T\mathcal{M}$. We introduce the convenient notation $\mathcal{E}_{1,t}^{(0)} \equiv T\mathcal{M}_t$ to indicate the H_0-equivariant version of $T\mathcal{M}$. Because of this observation, we can apply the relations (4.5.138) and (4.5.140) to rewrite $e(p)$ entirely in terms of the equivariant bundles $T\mathcal{M}_t$ and $\mathcal{V}_t$,

$$e(p) \;=\; \prod_{t\neq 0} \frac{1}{e_{H_0}(\mathcal{M}, T\mathcal{M}_t)} \cdot \left[e_{H_0}(\mathcal{M}, \mathcal{V}_t)\right]^{nt}\,. \tag{4.5.142}$$

Let us make the factors appearing on the right in (4.5.142) more explicit. To this end, we introduce the Chern roots ϖ_j of $T\mathcal{M}$, where $j = 1, \ldots, \dim \mathcal{M}$, and the Chern roots ν_l of $\mathcal{V}$, where $l = 1, \ldots, \mathrm{rk}\, \mathcal{V}$. Since $\mathcal{V}$ arises from the fiber of the adjoint bundle $\mathrm{ad}_{\mathbb{C}}(P)$, the rank of $\mathcal{V}$ is simply $\mathrm{rk}\, \mathcal{V} = \dim G \equiv \Delta_G$. As in our general discussion of the equivariant Euler class, the Chern roots ϖ_j and ν_l are "virtual" degree two classes in $H^*(\mathcal{M})$ which are defined in terms of the total Chern classes of $T\mathcal{M}$ and $\mathcal{V}$ as

$$c(T\mathcal{M}) \,=\, \prod_{j=1}^{\dim \mathcal{M}} (1 + \varpi_j)\,, \qquad c(\mathcal{V}) \,=\, \prod_{l=1}^{\Delta_G} (1 + \nu_l)\,. \tag{4.5.143}$$

In terms our these Chern roots, our general description of the equivariant Euler class in (4.5.124) implies that

$$e_{H_0}(\mathcal{M}, T\mathcal{M}_t) \,=\, \prod_{j=1}^{\dim \mathcal{M}} (-itp \,+\, \varpi_j)\;, \qquad e_{H_0}(\mathcal{M}, \mathcal{V}_t) \,=\, \prod_{l=1}^{\Delta_G} (-itp \,+\, \nu_l)\;. \tag{4.5.144}$$

The terms in (4.5.144) which involve p arise via the infinitesimal action of H_0 on the fibers of $T\mathcal{M}_t$ and $\mathcal{V}_t$. We recall that H_0 acts infinitesimally as $p\pounds_R = -2\pi i t p$.

Together, (4.5.142) and (4.5.144) imply the following formal expression for $e(p)$,

$$e(p) \,=\, \prod_{t \neq 0} \left[\prod_{j=1}^{\dim \mathcal{M}} \frac{1}{(-itp \,+\, \varpi_j)}\right] \left[\prod_{l=1}^{\Delta_G} (-itp \,+\, \nu_l)^{nt}\right]. \tag{4.5.145}$$

This infinite product represents the determinant of a first-order operator $\mathcal{D}$ acting on $\mathcal{E}_0 \ominus \mathcal{E}_1$, where

$$\mathcal{D} \,=\, \frac{1}{2\pi} \left(p\pounds_R \,+\, i\mathcal{R}\right)\;. \tag{4.5.146}$$

Here $\mathcal{R}$ is the curvature operator acting on $\mathcal{E}_0$ and $\mathcal{E}_1$ as bundles over $\mathcal{M}$, as appears in the computation in Appendix D, and "$\ominus$" indicates that we actually take the inverse of the determinant of $\mathcal{D}$ acting on $\mathcal{E}_1$.

The determinant in (4.5.145) is only a formal expression, and to define it we must choose some regularization procedure. For instance, we considered the determinant of a similar operator $\mathcal{D}_0$ in our computation at the trivial connection in Section 5.2,

$$\mathcal{D}_0 \,=\, \frac{1}{2\pi} \left(p\pounds_R - [\phi,\, \cdot\,]\right)\;. \tag{4.5.147}$$

In that case, we defined the determinant of $\mathcal{D}_0$ analytically, using the zeta-function to define its absolute value and the eta-function to define its phase.

We follow a similar strategy to define the determinant of $\mathcal{D}$, or more explicitly the infinite product in (4.5.145). To start, we find it useful to rewrite the product in (4.5.145) by pulling out an overall factor of p,

$$e(p) \,=\, p^{\dim\mathcal{M}} \prod_{t\neq 0} \left[\prod_{j=1}^{\dim\mathcal{M}} \left(-it \,+\, \left(\frac{\varpi_j}{p}\right)\right)^{-1} \right] \left[\prod_{l=1}^{\Delta_G} \left(-it \,+\, \left(\frac{\nu_l}{p}\right)\right)^{nt} \right]. \qquad (4.5.148)$$

In passing from (4.5.145) to (4.5.148), we use as in Section 5.2 the classical Riemann zeta-function to define the trivial, but infinite, product over p which arises from (4.5.145),

$$\prod_{t\geq 1} p^{-2\dim\mathcal{M}} \,=\, \exp\left(-2\dim\mathcal{M}\cdot\ln p\cdot\zeta\left(0\right)\right) \,=\, p^{\dim\mathcal{M}}. \qquad (4.5.149)$$

(There is no contribution from the factors in (4.5.145) which are associated to $\mathcal{V}$ due to a cancellation between the terms for $\pm t$.) Thus, we are left to consider the determinant of the rescaled operator $\mathcal{D}'$,

$$\mathcal{D}' \,=\, \frac{1}{2\pi}\left(\pounds_R \,+\, i\frac{\mathcal{R}}{p}\right), \qquad (4.5.150)$$

which represents the infinite product appearing in (4.5.148) and which depends on p and the Chern roots only in the combinations ϖ_j/p and ν_l/p.

One interesting distinction between the operator $\mathcal{D}$, or equivalently $\mathcal{D}'$, and the operator $\mathcal{D}_0$ which appeared previously is that whereas $\mathcal{D}_0$ is an anti-hermitian operator, with a purely imaginary spectrum, the operator $\mathcal{D}$ has no particular hermiticity properties and its spectrum has no particular phase. This is manifest in the product (4.5.148), since $-it$ is imaginary but both the Chern roots and p are real. In terms of (4.5.150), both $\pounds_R$ and $\mathcal{R}$ are anti-hermitian operators, but we have an explicit factor of 'i' in front of $\mathcal{R}$. Because $\mathcal{D}'$ is neither hermitian nor anti-hermitian, we will have to generalize the zeta/eta-function regularization technique which we applied to define the determinant of $\mathcal{D}_0$ in Section 5.2.

Before we supply a definition for the determinant of $\mathcal{D}'$, or equivalently for the products in (4.5.148), let us consider what general properties our definition should possess. To start, we factorize the product in (4.5.148) into the two infinite products below,

$$f_{\mathcal{M}}(z) \,=\, \prod_{t\neq 0} \prod_{j=1}^{\dim\mathcal{M}} \left(-it \,+\, z\varpi_j\right)^{-1} \,,$$
$$f_{\mathcal{V}}(z) \,=\, \prod_{t\neq 0} \prod_{l=1}^{\Delta_G} \left(-it \,+\, z\nu_l\right)^{nt} \,, \tag{4.5.151}$$

where $z = 1/p$ is now a formal parameter.

The expressions in (4.5.151) are ill-defined as they stand. However, if we formally differentiate $\log f_{\mathcal{M}}(z)$ and $\log f_{\mathcal{V}}(z)$ with respect to z a sufficient number of times, we eventually obtain well-defined, absolutely convergent sums. For instance, in the case of $f_{\mathcal{M}}(z)$, we see that

$$\frac{d^2}{dz^2} \log f_{\mathcal{M}}(z) \,=\, \sum_{j=1}^{\dim\mathcal{M}} \sum_{t\neq 0} \frac{\varpi_j^2}{\left(-it \,+\, z\varpi_j\right)^2} \,=\, \sum_{j=1}^{\dim\mathcal{M}} \frac{d^2}{dz^2} \log\left[\frac{(\pi z\varpi_j)}{\sinh\left(\pi z\varpi_j\right)}\right]. \tag{4.5.152}$$

The second equality in (4.5.152) follows from the same product identity (4.5.65) for $\sin(x)/x$ as we applied in Section 5.2.

So any reasonable definition for $f_{\mathcal{M}}(z)$ in (4.5.151) must be compatible with the relation (4.5.152). In particular, upon integrating (4.5.152), we see that $\log f_{\mathcal{M}}(z)$ is determined up to a linear function of z, and hence $f_{\mathcal{M}}(z)$ is determined up to two arbitrary real constants a_0 and a_1,

$$f_{\mathcal{M}}(z) \,=\, \exp\left[a_0 \,+\, a_1 z\, c_1(T\mathcal{M})\right] \prod_{j=1}^{\dim\mathcal{M}} \frac{(\pi z\varpi_j)}{\sinh\left(\pi z\varpi_j\right)} \,. \tag{4.5.153}$$

Here $c_1(T\mathcal{M}) = \sum_j \varpi_j$ is the first Chern class of $\mathcal{M}$. In deducing the form (4.5.153), we have applied the fact, manifest from (4.5.151), that $f_{\mathcal{M}}(z)$ can only depend on z and the Chern roots ϖ_j in the combinations $z\varpi_j$, and we have also used the fact that only symmetric combinations of the Chern roots have any real meaning — hence each Chern root ϖ_j must appear with the same coefficient a_1 in the exponential factor of (4.5.153). Comparing to the product (4.5.151), we also note that $f_{\mathcal{M}}(z)$ is formally real (for real z), so a_0 and a_1 must be real.

We can also apply this general analysis to $f_{\mathcal{V}}(z)$ in (4.5.151). Here we observe that

$\log f_{\mathcal{V}}(z)$ should satisfy

$$\begin{aligned}\frac{d^3}{dz^3}\log f_{\mathcal{V}}(z) &= \sum_{t\neq 0}\sum_{l=1}^{\Delta_G} \frac{2nt\nu_l^3}{(-it\,+\,z\nu_l)^3}\,,\\ &= \sum_{t\geq 1}\sum_{l=1}^{\Delta_G}\left[\frac{2nt\nu_l^3}{(-it\,+\,z\nu_l)^3} + \frac{2nt\nu_l^3}{(-it\,-\,z\nu_l)^3}\right]\,,\\ &= 0\,.\end{aligned} \tag{4.5.154}$$

In contrast to the case of $f_{\mathcal{M}}(z)$, we must take three derivatives of $\log f_{\mathcal{V}}(z)$ to get a convergent sum, due to the exponent nt appearing in (4.5.151). In passing to the second equality of (4.5.154), we have simply paired terms for $\pm t$. However, to deduce the cancellation in the third line of (4.5.154), we must use some topological facts about the bundle $\mathcal{V}$.

We recall that $\mathcal{V}$ is the bundle over $\mathcal{M}$ whose fibers are given by $H^0(\Sigma, \mathrm{ad}_{\mathbb{C}}(P)|_\sigma)$ for some point σ on Σ. This bundle is naturally the complexification of a real bundle over $\mathcal{M}$, namely the bundle whose fibers are $H^0(\Sigma, \mathrm{ad}(P)|_\sigma)$. Consequently, the non-zero Chern roots of $\mathcal{V}$ are paired such that for each root ν there is a corresponding root ν' with $\nu' = -\nu$. This fact implies that any odd, symmetric function of the Chern roots vanishes. In particular, all odd Chern classes of $\mathcal{V}$ vanish.

We now consider a series expansion of the denominators in the second line of (4.5.154) in terms of the nilpotent quantities $z\nu_l$. Because of the relative signs in these denominators, and because of the explicit cubic factor ν_l^3 in the numerators, all terms of even degree in the Chern roots ν_l automatically cancel. However, by our observation about $\mathcal{V}$ above, the remaining terms of odd degree in the ν_l cancel when we sum over roots.

From (4.5.154), we see that $\log f_{\mathcal{V}}(z)$ is determined up to a quadratic function of z. Hence $f_{\mathcal{V}}(z)$ is determined up to two real constants b_0 and b_2,

$$f_{\mathcal{V}}(z) \,=\, \exp\Big[ib_0 \,+\, ib_2 z^2\,\Theta\Big]\,. \tag{4.5.155}$$

A term linear in z would necessarily appear with the first Chern class $c_1(\mathcal{V})$, which vanishes by our observation above. Since $c_1(\mathcal{V}) = 0$, the only degree two class that can appear in (4.5.155) is the characteristic class Θ. We also observe from the product (4.5.151) that

$f_{\mathcal{V}}(z)$ must be simply a phase (for real z), since under complex conjugation $f_{\mathcal{V}}(z)$ goes to $f_{\mathcal{V}}^{-1}(z)$. This observation fixes the factors of 'i' in (4.5.155).

Having fixed the general forms (4.5.153) and (4.5.155) of $f_{\mathcal{M}}$ and $f_{\mathcal{V}}$, we now compute the undetermined constants. To do this, we must still decide how to define the determinant of the operator $\mathcal{D}' = (1/2\pi)\,[\pounds_R + i(\mathcal{R}/p)]$. Motivated by our work in Section 5.2, we proceed as follows. First, although p is a positive, real variable in our problem, we will define the determinant of $\mathcal{D}'$ more generally for complex p. Second, once we allow p to be complex, we impose the requirement that the determinant of $\mathcal{D}'$ depend analytically on p. In particular, if we evaluate the determinant for purely *imaginary* p, of the form $p = i/y$ for real $y > 0$ (the fact that we use $1/y$ is just for notational convenience later), then the determinant is defined for real $p > 0$ by analytic continuation. Finally, when $p = i/y$, we see that $\mathcal{D}' = (1/2\pi)\,[\pounds_R + y\mathcal{R}]$ is an anti-hermitian operator exactly like $\mathcal{D}_0$, and we can use zeta/eta-function regularization to define the determinant of $\mathcal{D}'$ for these values of p as we did in Section 5.2.

In terms of $f_{\mathcal{M}}$ and $f_{\mathcal{V}}$ in (4.5.151), this definition of the determinant of $\mathcal{D}'$ amounts to the prescription to use zeta/eta-function regularization to define the products

$$\begin{aligned} f_{\mathcal{M}}(z=-iy) \;&=\; \prod_{t\neq 0} \prod_{j=1}^{\dim\mathcal{M}} \frac{i}{(t\,+\,y\varpi_j)}\,,\\ f_{\mathcal{V}}(z=-iy) \;&=\; \prod_{t\neq 0} \prod_{l=1}^{\Delta_G} (-i)^{nt}\,(t\,+\,y\nu_l)^{nt}\,. \end{aligned} \tag{4.5.156}$$

We first ignore the factors of 'i' in (4.5.156) and we compute the absolute values of $f_{\mathcal{M}}$ and $f_{\mathcal{V}}$.

For instance,

$$|f_{\mathcal{M}}(-iy)| \;=\; \prod_{t\geq 1} \prod_{j=1}^{\dim\mathcal{M}} \left[t^2 - (y\varpi_j)^2\right]^{-1} \;=\; \left(\frac{1}{2\pi}\right)^{\dim\mathcal{M}} \cdot \prod_{j=1}^{\dim\mathcal{M}} \frac{(\pi y\varpi_j)}{\sin(\pi y\varpi_j)}\,. \tag{4.5.157}$$

Since the Chern roots ϖ_j are nilpotent, the terms in the first product in (4.5.157) are manifestly positive. In passing to the second equality, we apply the same identities (4.5.65) and (4.5.66) from Section 5.2. This form of $|f_{\mathcal{M}}(-iy)|$ is clearly compatible with our general expression (4.5.153).

On the other hand, one can easily check that zeta-function regularization defines the absolute value of $f_{\mathcal{V}}$ to be trivial, for the same topological reason that we explained following (4.5.154), so

$$|f_{\mathcal{V}}(-iy)| \,=\, \prod_{t\geq 1}\prod_{l=1}^{\Delta_G}\left[\frac{t\,+\,y\nu_l}{t\,-\,y\nu_l}\right]^{nt} \,=\, 1\,. \tag{4.5.158}$$

We are left to compute the phases of $f_{\mathcal{M}}(-iy)$ and $f_{\mathcal{V}}(-iy)$. We define these using the eta-function, as in Section 5.2. More precisely, we write

$$f_{\mathcal{M}}(-iy) \,=\, \exp\left(-\frac{i\pi}{2}\,\eta_{\mathcal{M}}\right)\cdot|f_{\mathcal{M}}|\,,\qquad f_{\mathcal{V}}(-iy) \,=\, \exp\left(-\frac{i\pi}{2}\,\eta_{\mathcal{V}}\right). \tag{4.5.159}$$

Here $\eta_{\mathcal{M}}$ and $\eta_{\mathcal{V}}$ denote the eta-invariants which arise as the values at $s = 0$ of the eta-functions $\eta_{\mathcal{M}}(s)$ and $\eta_{\mathcal{V}}(s)$ abstractly associated to the hermitian operator $i\mathcal{D}'$ as it acts on $\mathcal{E}_0 \ominus \mathcal{E}_1$,

$$i\mathcal{D}' \,=\, \frac{i}{2\pi}\left(\pounds_R \,+\, y\mathcal{R}\right)\,. \tag{4.5.160}$$

This operator should be compared to the corresponding operator which we considered when computing the phase of $e(p,\phi)$ at the trivial connection,

$$\frac{i}{2\pi}\left(\pounds_R - \left[\frac{\phi}{p},\,\cdot\,\right]\right)\,. \tag{4.5.161}$$

We recall from Section 5.2 that the eta-invariant associated to the operator in (4.5.161) acquires an anomalous dependence on (ϕ/p) which produces the finite shift in the Chern-Simons level. In the case at hand, a similar anomalous dependence of $\eta_{\mathcal{M}}$ and $\eta_{\mathcal{V}}$ on $y\mathcal{R}$ gives rise to the same shift in the level.

Concretely, the eta-functions $\eta_{\mathcal{M}}(s)$ and $\eta_{\mathcal{V}}(s)$ are given by the following regularized sums over the factors which appear in $f_{\mathcal{M}}(-iy)$ and $f_{\mathcal{V}}(-iy)$ in (4.5.156) and which represent the eigenvalues λ of $i\mathcal{D}'$,

$$\begin{aligned}\eta_{\mathcal{M}}(s) \,&=\, \sum_{t\neq 0}\sum_{j=1}^{\dim\mathcal{M}} -\operatorname{sign}(\lambda(t,\varpi_j))\cdot|\lambda(t,\varpi_j)|^{-s}\,,\quad \lambda(t,\varpi_j) \,=\, t + y\varpi_j\,,\\ \eta_{\mathcal{V}}(s) \,&=\, \sum_{t\neq 0}\sum_{l=1}^{\Delta_G} nt\cdot\operatorname{sign}(\lambda(t,\nu_l))\cdot|\lambda(t,\nu_l)|^{-s}\,,\quad \lambda(t,\nu_l) \,=\, t + y\nu_l\,.\end{aligned} \tag{4.5.162}$$

The various constants appearing in (4.5.162) are perhaps most clear if we compare to the formal expressions for $f_{\mathcal{M}}(-iy)$ and $f_{\mathcal{V}}(-iy)$ in (4.5.156). Thus, the overall minus sign in

$\eta_{\mathcal{M}}(s)$ arises because i as opposed to $-i$ appears in $f_{\mathcal{M}}(-iy)$, which is in turn associated to the fact that we consider $\mathcal{E}_0 \ominus \mathcal{E}_1$ as opposed to $\mathcal{E}_0 \oplus \mathcal{E}_1$. Similarly, the multiplicity factor nt appears in $\eta_{\mathcal{V}}(s)$ because of the factor $(-i)^{nt}$ in $f_{\mathcal{V}}(-iy)$.

Since the Chern roots are nilpotent, we note that $\text{sign}(\lambda(t,x)) = \text{sign}(t)$, where $x = \varpi_j$ or $x = \nu_l$ as the case may be. Thus, we write the regularized sums in (4.5.162) explicitly as

$$\begin{aligned} \eta_{\mathcal{M}}(s) &= \sum_{t\geq 1}\sum_{j=1}^{\dim\mathcal{M}} \frac{-1}{(t + y\varpi_j)^s} + \sum_{t\geq 1}\sum_{j=1}^{\dim\mathcal{M}} \frac{1}{(t - y\varpi_j)^s}\,, \\ \eta_{\mathcal{V}}(s) &= \sum_{t\geq 1}\sum_{l=1}^{\Delta_G} \frac{nt}{(t + y\nu_l)^s} + \sum_{t\geq 1}\sum_{l=1}^{\Delta_G} \frac{nt}{(t - y\nu_l)^s}\,. \end{aligned} \tag{4.5.163}$$

As in Section 5.2, we are left to evaluate these sums at $s = 0$.

In fact, we have already done all of the required computation. The sum which defines $\eta_{\mathcal{M}}(s)$ is the same as the sum (4.5.77) which we evaluated in the warmup computation on S^1 in Section 5.2. Thus we find

$$\eta_{\mathcal{M}}(0) = 2y \sum_{j=1}^{\dim\mathcal{M}} \varpi_j = 2y\, c_1(T\mathcal{M})\,. \tag{4.5.164}$$

In deducing the second equality, we note that the trace over all Chern roots of $T\mathcal{M}$ is the first Chern class of $T\mathcal{M}$.

To evaluate $\eta_{\mathcal{V}}(0)$, we perform a computation precisely isomorphic to our computation of $e(p,\phi)$ in Section 5.2. Applying our earlier results, we find

$$\eta_{\mathcal{V}}(0) = \eta_0 + ny^2 \sum_{l=1}^{\Delta_G} \nu_l^2\,, \qquad \eta_0 = -\frac{n\Delta_G}{6}\,. \tag{4.5.165}$$

Here η_0 is the same constant that appeared in our localization computation at the trivial connection. As for the term quadratic in ν_l, this term arises in the same way as the term quadratic in ϕ in (4.5.89).

We now recall from Section 5.2 that we applied a Lie algebra identity (4.5.90) involving $\check{c}_{\mathfrak{g}}$ to rewrite the term quadratic in ϕ in (4.5.89) in terms of the natural quadratic invariant $\frac{1}{2}\text{Tr}(\phi^2)$. Under the Chern-Weil homomorphism, by which we identify the Chern roots ν_l with the eigenvalues of the curvature operator $i\mathcal{R}/2\pi$, we can apply the same Lie algebra identity to rewrite the degree four class $\sum_l \nu_l^2$ in terms of the class Θ that already appears in the integral over $\mathcal{M}$. We find from the identity (4.5.90) that

$$\sum_{l=1}^{\Delta_G} \nu_l^2 \,=\, \frac{\check{c}_{\mathfrak{g}} \Theta}{\pi^2}\,, \tag{4.5.166}$$

and $\eta_{\mathcal{V}}(0)$ becomes

$$\eta_{\mathcal{V}}(0) \,=\, \eta_0 \,+\, \frac{n \check{c}_{\mathfrak{g}}}{\pi^2}\, y^2 \Theta\,. \tag{4.5.167}$$

With these results (4.5.164) and (4.5.167), we evaluate $f_{\mathcal{M}}(-iy)$ and $f_{\mathcal{V}}(-iy)$ to be

$$\begin{aligned} f_{\mathcal{M}}(-iy) \,&=\, \exp\left(-i\pi y\, c_1(T\mathcal{M})\right) \cdot \left(\frac{1}{2\pi}\right)^{\dim \mathcal{M}} \cdot \prod_{j=1}^{\dim \mathcal{M}} \frac{(\pi y \varpi_j)}{\sin(\pi y \varpi_j)}\,, \\ f_{\mathcal{V}}(-iy) \,&=\, \exp\left(-\frac{i\pi}{2}\eta_0 \,-\, \frac{i n \check{c}_{\mathfrak{g}}}{2\pi} y^2 \Theta\right). \end{aligned} \tag{4.5.168}$$

Upon setting $z = -iy$, these expressions assume the same form as the general expressions in (4.5.153) and (4.5.155).

We recall that p is related to y via $p = i/y$. So $e(p)$, as determined by the analytic continuation of (4.5.168), is finally given by

$$\begin{aligned} e(p) \,&=\, p^{\dim \mathcal{M}} \cdot f_{\mathcal{M}}(p) \cdot f_{\mathcal{V}}(p)\,, \\ &=\, \exp\left(-\frac{i\pi}{2}\eta_0 \,+\, \frac{\pi}{p}\, c_1(T\mathcal{M}) \,+\, \frac{i n \check{c}_{\mathfrak{g}}}{2\pi p^2}\Theta\right) \left(\frac{p}{2\pi}\right)^{\dim \mathcal{M}} \prod_{j=1}^{\dim \mathcal{M}} \frac{(\pi \varpi_j / p)}{\sinh(\pi \varpi_j / p)}\,. \end{aligned} \tag{4.5.169}$$

As we will see, this formula incorporates the famous shift in the Chern-Simons level k, and leads to agreement with the results of Rozansky.

Some Further Remarks

Our use of zeta/eta-function regularization to define $e(p)$, and especially the analytic continuation we performed in p, is somewhat *ad hoc*. The need for this analytic continuation is an unfortunate consequence of the 'i' that appears in the Cartan differential $D = d + i\iota_V$ that we introduced in Section 3. Had we used the more standard mathematical definition of D, with $D = d + \iota_V$, then the basic symplectic volume integral on a symplectic manifold $\mathcal{M}$ would turn out to be $\int_{\mathcal{M}} \exp(i\Omega)$ rather than the more usual $\int_{\mathcal{M}} \exp(\Omega)$. The mathematical version of D would also clash with some conventions of physicists about reality conditions for fermions. However, it would clarify our discussion of the determinants, since if all factors of i are omitted from the localization form λ, then the operator $i\mathcal{D}'$ would come out to be hermitian. Hence, the zeta/eta-function definition of determinants could be implemented directly, with no need for artificial analytic continuation.

The zeta/eta-function definition is really most natural for oscillatory bosonic integrals such as appear in Chern-Simons theory. For example, if we consider a bosonic integral

$$Z = \int D\Phi \exp(i(\Phi, M\Phi)), \tag{4.5.170}$$

for some indefinite real symmetric operator M, and we regularize Z by the Feynman prescription $M \to M + i\varepsilon$, for small positive ε, then the phase of Z is naturally $\exp(i\pi\eta(M)/2)$. This is really why, in Chern-Simons theory, eta-invariants appear in the one-loop corrections. If we take $D = d + \iota_V$, and take the localization form λ to be purely imaginary rather than purely real, then all integrals in Appendix D are oscillatory Gaussian integrals rather than real Gaussians. In this framework, zeta/eta-function regularization provides a natural definition of the determinants that appear in our localization computation.

Our general analysis of $d^2 \log f_{\mathcal{M}}(z)/dz^2$ and $d^3 \log f_{\mathcal{V}}(z)/dz^3$ showed that any reasonable definition of these determinants would differ from the zeta/eta-function approach by adding a constant to η_0 and changing the coefficients of $c_1(T\mathcal{M})$ and Θ in (4.5.169). We will see shortly that the coefficients as written in (4.5.169) do agree with Chern-Simons theory; in fact, they show up in Chern-Simons theory at the one-loop level. Ultimately, to justify the coefficients in (4.5.169) on an *a priori* basis requires a more rigorous comparison between the localization procedure and Chern-Simons theory.

The Contribution From $\mathcal{M}$ in Chern-Simons Theory

Having evaluated $e(p)$, we now set $p = 1/\epsilon$ and substitute (4.5.169) into our expression (4.5.120) for the contribution from $\mathcal{M}$ to the Chern-Simons path integral. Thus,

$$\begin{aligned} Z(\epsilon)\Big|_{\mathcal{M}} = & \frac{1}{|\Gamma|} \exp\left(-\frac{i\pi}{2}\eta_0\right) \left(\frac{1}{2\pi\epsilon}\right)^{\dim \mathcal{M}} \times \\ & \times \int_{\mathcal{M}} \exp\left[\Omega + \pi\epsilon\, c_1(T\mathcal{M}) + i\epsilon n \left(1 + \frac{\epsilon \check{c}_{\mathfrak{g}}}{2\pi}\right)\Theta\right] \prod_{j=1}^{\dim \mathcal{M}} \left[\frac{\pi\epsilon\varpi_j}{\sinh(\pi\epsilon\varpi_j)}\right]. \end{aligned} \tag{4.5.171}$$

Since we are dealing with an integral, by making changes of variables we can rewrite the integrand of (4.5.171) in different ways which illuminate different features of this result. In the form at hand, we note that one can define a non-trivial scaling limit of (4.5.171) such that the Chern-Simons coupling ϵ goes to zero (so that the level k goes to ∞) and the degree n of $\mathcal{L}$ goes to ∞ with ϵn held fixed. In this limit, which physically decouples all the higher Kaluza-Klein modes of the gauge field, we see directly that the contribution from $\mathcal{M}$ in Chern-Simons theory has the same form as the simple expression (4.4.17) for the corresponding contribution from $\mathcal{M}_0$ in two-dimensional Yang-Mills theory.

To express (4.5.171) more compactly, we now rescale all elements of the cohomology ring of $\mathcal{M}$ by a factor $(2\pi\epsilon)^{q/2}$, where q is the degree of the given class. So for instance, the degree two Chern roots ϖ_j scale as $\varpi_j \rightarrow 2\pi\epsilon\,\varpi_j$. This trivial change of variables cancels the prefactor involving ϵ in (4.5.171) and reduces the product over Chern roots in (4.5.171) to a well-known characteristic class, the $\widehat{A}$-genus of $\mathcal{M}$.

We recall that the $\widehat{A}$-genus of $\mathcal{M}$ is given in terms of the Chern roots of $T\mathcal{M}$ as

$$\widehat{A}(\mathcal{M}) = \prod_{j=1}^{\dim\mathcal{M}} \frac{\varpi_j/2}{\sinh(\varpi_j/2)}\,. \tag{4.5.172}$$

In a sense, the appearance of the $\widehat{A}$-genus in our problem is not so surprising, since it appears in roughly the same way as in the standard path integral derivations of the index theorem. See [108] for a derivation of the index theorem that applies abelian localization to a sigma model path integral; at least formally, that computation shares many features of our computation here.

In terms of the $\widehat{A}$-genus, our expression in (4.5.171) simplifies to

$$Z(\epsilon)\Big|_{\mathcal{M}} = \frac{1}{|\Gamma|}\,\exp\left(-\frac{i\pi}{2}\eta_0\right)\int_{\mathcal{M}} \widehat{A}(\mathcal{M})\,\exp\left[\frac{1}{2\pi\epsilon}\,\Omega + \frac{1}{2}c_1(T\mathcal{M}) + \frac{in}{4\pi^2\epsilon_r}\,\Theta\right]. \tag{4.5.173}$$

Here we have absorbed the contribution from $\eta_{\mathcal{V}}(0)$ into a renormalization of the coupling $\epsilon_r = 2\pi/(k+\check{c}_{\mathfrak{g}})$ that appears in front of Θ.

Of course, we would like to write (4.5.171) entirely in terms of the renormalized coupling ϵ_r. To do so, we apply a theorem of [109] which relates the first Chern class $c_1(T\mathcal{M})$ to the symplectic form Ω in the case of gauge group $G = SU(r+1)$. In this case,

$$c_1(T\mathcal{M}) \,=\, 2(r+1)\,\Omega'\,, \qquad (4.5.174)$$

where $\Omega' = \Omega/(2\pi)^2$ is the standard, integral symplectic form on $\mathcal{M}$. Happily, the dual Coxeter number $\check{c}_{\mathfrak{g}}$ of $G = SU(r+1)$ is also given by $\check{c}_{\mathfrak{g}} = r+1$, so we see that (4.5.173) can be expressed very simply using ϵ_r,

$$Z(\epsilon)\Big|_{\mathcal{M}} \,=\, \frac{1}{|\Gamma|}\,\exp\left(-\frac{i\pi}{2}\eta_0\right)\int_{\mathcal{M}} \widehat{A}(\mathcal{M})\,\exp\left[\frac{1}{2\pi\epsilon_r}\left(\Omega + \frac{in}{2\pi}\,\Theta\right)\right]. \qquad (4.5.175)$$

This expression is of the same form as the corresponding result of Rozansky in [80].

We close with the following amusing observation. On general grounds, the $\widehat{A}$-genus of $\mathcal{M}$ is related to the Todd class $\mathrm{Td}(\mathcal{M})$ of $\mathcal{M}$ by

$$\mathrm{Td}(\mathcal{M}) \,=\, \exp\left(\frac{1}{2}c_1(T\mathcal{M})\right)\widehat{A}(\mathcal{M})\,. \qquad (4.5.176)$$

So from (4.5.173), we see that an alternative expression for the path integral contribution from $\mathcal{M}$ is

$$Z(\epsilon)\Big|_{\mathcal{M}} \,=\, \frac{1}{|\Gamma|}\,\exp\left(-\frac{i\pi}{2}\eta_0\right)\int_{\mathcal{M}} \mathrm{Td}(\mathcal{M})\,\exp\left[k\,\Omega' + \frac{in}{4\pi^2\epsilon_r}\,\Theta\right]. \qquad (4.5.177)$$

Although our derivation of (4.5.177) is not valid for the trivial case $M = S^1 \times \Sigma$, we see that, upon setting $n = 0$, our result (4.5.177) takes the same form as the index formula (4.1.1) for $Z(\epsilon)$ in the trivial case. It is satisfying to see that both the index formula (4.1.1) and the two-dimensional Yang-Mills formula (4.4.17) are reproduced as special limits of our general result.

Appendix A

Brief Analysis to Justify the Localization Computation in Chapter 4.4

In this appendix, we show that the quantity $Q \cdot Z'(\epsilon)$ computed using λ' in (4.4.91) of Chapter 4.4.3 agrees with the same quantity defined using λ, so that $Z'(\epsilon)$ as defined by integrating (4.4.91) agrees with $Z(\epsilon)$. Thus we consider the following one-parameter family of invariant forms, interpolating from λ to λ' on F,

$$\Lambda(s) \,=\, s\,\lambda + (1-s)\,\lambda'\,, \qquad s \in [0,1]\,, \tag{5.0.1}$$

and to start we consider the corresponding family $Z(\epsilon, s)$ of integrals over F,

$$Z(\epsilon, s) \,=\, \frac{1}{\mathrm{Vol}(H)} \int_{\mathfrak{h}\times F} \left[\frac{d\phi}{2\pi}\right] \exp\left[\Omega - i\,\langle\mu, \phi\rangle - \frac{\epsilon}{2}(\phi,\phi) + t\,D\Lambda(s)\right]. \tag{5.0.2}$$

If this integral is convergent for all s and also continuous as a function of s, then $Z(\epsilon, s)$ is independent of s, so that $Z(\epsilon) = Z(\epsilon, 1) = Z(\epsilon, 0) = Z'(\epsilon)$. This fact follows by differentiating the integrand of (5.0.2) with respect to s, which produces a total derivative on F.

We thus need to consider the basic convergence and continuity of $Z(\epsilon, s)$. Very broadly, divergences in the integral over F in (5.0.2) can only arise from integration over the non-compact fibers $\mathfrak{h}^\perp$ and E_1 which sit over the compact orbit H/H_0. However, the first, degree one term of λ' in (4.4.77) is precisely of the canonical form to define localization on the fiber $\mathfrak{h}^\perp$, exactly as in our computation on T^*H. Thus, no divergence arises from the integral over $\mathfrak{h}^\perp$, and we need only analyze the integral over the complex vector space E_1. As we have already seen, precisely this integral over E_1 leads to the dangerous, possibly singular factor in $I(\psi)$ in (4.4.84). Furthermore, in our application to Yang-Mills theory, the corresponding vector space $\mathcal{E}_1$ describes the set of gauge-equivalence classes of unstable modes of the Yang-Mills action, and we expect the integral over these modes to be the most delicate.

We now analyze directly the symplectic integral over E_1 that arises from (5.0.2). To set up notation, we recall that E_1 is a complex vector space, $\dim_{\mathbb{C}} E_1 = d_1$, with an invariant, hermitian metric $(\cdot,\cdot)$ and an invariant symplectic form $\widetilde{\Omega}$. In terms of holomorphic and anti-holomorphic coordinates v^n and $\overline{v}^{\overline{n}}$ on E_1, $\widetilde{\Omega}$ is given by

$$\widetilde{\Omega} \,=\, -\frac{i}{2}\,(dv, dv) \,=\, -\frac{i}{2}\,d\overline{v}_n \wedge dv^n \,. \tag{5.0.3}$$

If ψ is an element of $\mathfrak{h}_0$, then the corresponding vector field $V(\psi)$ on E_1 is described by

$$\delta v \,=\, \psi \cdot v \,, \tag{5.0.4}$$

or in coordinates, $\delta v^n = \psi^n_m v^m$, and similarly for the conjugate components of $V(\psi)$.

From (5.0.3) and (5.0.4), we see that the moment map $\widetilde{\mu}$ for the action of H_0 on E_1 is explicitly given by

$$\langle \widetilde{\mu}, \psi \rangle \,=\, \frac{i}{2}\,(v, \psi \cdot v) \,. \tag{5.0.5}$$

By our assumption that $(\cdot,\cdot)$ is invariant under (5.0.4), ψ is anti-hermitian and the expression in (5.0.5) is real.

Of course, the complex structure J acts on E_1 as $J(dv) = -i\,dv$ and $J(d\overline{v}) = +i\,d\overline{v}$. Thus, since

$$S = \frac{1}{2}(\widetilde{\mu}, \widetilde{\mu}) = \frac{1}{8}(v, v)^2 \,, \tag{5.0.6}$$

we see that the canonical one-form $\lambda = J\,dS$ is given by

$$\lambda \,=\, -\frac{i}{4}\,(v, v)\,\big((v, dv) - (dv, v)\big) \,. \tag{5.0.7}$$

On the other hand, from (4.4.77) we see that λ' on E_1 reduces to

$$\lambda' \,=\, i\,(\psi\cdot v, dv)\;. \tag{5.0.8}$$

Thus, if we restrict the integral in (5.0.2) to E_1 and keep only the terms relevant in the limit of large t (after which we set $t = 1$), we just consider the reduced integral

$$Z_{red}(\epsilon, s) \,=\, \int_{\mathfrak{h}_0\times E_1} \left[\frac{d\psi}{2\pi}\right]\,\exp\left[-i\,(\gamma_0,\psi) - \frac{\epsilon}{2}\,(\psi,\psi) + s\,D\lambda + (1-s)\;D\lambda'\right]. \tag{5.0.9}$$

Of the original integral over the full Lie algebra $\mathfrak{h}$ of H, only the integral over the subalgebra $\mathfrak{h}_0$ is relevant to the integral over E_1.

We first perform integral over ψ in $\mathfrak{h}_0$. To illustrate the essential behavior of the integral over E_1, we assume as before that $\mathfrak{h}_0 = \mathbb{R}$ has dimension one. Explicitly, $D\lambda$ and $D\lambda'$ depend on ψ as

$$D\lambda \,=\, d\lambda + \frac{1}{2}\,(v,v)\,(v,\psi\cdot v)\,, \tag{5.0.10}$$

and

$$D\lambda' \,=\, i\,(\psi\cdot dv, dv) - (\psi\cdot v, \psi\cdot v)\,, \tag{5.0.11}$$

so the integral over ψ is purely Gaussian. Upon performing this integral over ψ, we find that Z_{red} is formally given by

$$Z_{red}(\epsilon, s) \,=\, \int_{E_1} (4\pi A)^{-\frac{1}{2}}\,\exp\left[s\,d\lambda + \frac{1}{4}\left(J, A^{-1}\,J\right)\right], \tag{5.0.12}$$

where A is defined in terms of the normalized generator T_0 of $\mathfrak{h}_0$ by

$$A \,=\, \frac{\epsilon}{2} \,+\, (1-s)\,(T_0\cdot v, T_0\cdot v)\;, \tag{5.0.13}$$

and J in $\mathfrak{h}_0$ is defined by

$$J \,=\, -i\,\gamma_0 + \frac{s}{2}\,(v,v)\,(v,T_0\cdot v)\;T_0 + i\,(1-s)\,(T_0\cdot dv, dv)\;T_0\,. \tag{5.0.14}$$

We are now interested in the behavior of the integral in (5.0.12) for large $|v|$, where the non-compactness of E_1 is essential. So long as $s \neq 0$, then the integrand of (5.0.12) falls off at least as fast as $\exp\left[-(v,v)^3\right]$ for large v, due to the term quartic in v in (5.0.14) that arises from λ and the term quadratic in v in (5.0.13) that arises from λ'. Thus, the integral over E_1 is strongly convergent for $s \neq 0$ and depends smoothly on s away from 0. Of course, this integral is also non-Gaussian and cannot be simply expressed using elementary functions.

However, when $s = 0$, the integrand of (5.0.12) is no longer suppressed exponentially and decays only as a power law at infinity. This behavior arises because the bosonic term of $D\lambda'$ is quadratic in ψ, whereas the bosonic term of $D\lambda$ is linear in ψ. Because the integrand of (5.0.12) decays only as a power law for $s = 0$, the integral over E_1 does not generally converge. The prefactor proportional to $A^{-1/2}$ decays like $1/|v|$, and for $s = 0$ the measure arising from the quadratic term $(J, A^{-1} J)$ in the exponential of (5.0.12) is of the form $1/|v|^{d_1}\, d^{2d_1}v$. Consequently, the integral over E_1 behaves as $\int d^{2d_1}v \; 1/|v|^{(d_1+1)}$ for large $|v|$ and diverges.

However, we now consider the same analysis as applied to $Q \cdot Z(\epsilon, s)$. By our analysis above, we are only concerned with the potentially dangerous behavior near $s = 0$ and for large $|v|$, for which we must consider the following integral over E_1,

$$\left(-2\frac{\partial}{\partial\epsilon}\right)^{\frac{1}{2}d_1} \cdot Z_{red}(\epsilon, s) \;=\; \int_{E_1} \left(-2\frac{\partial}{\partial\epsilon}\right)^{\frac{1}{2}d_1} \left((4\pi A)^{-\frac{1}{2}} \, \exp\left[s\, d\lambda + \frac{1}{4}\left(J, A^{-1}\, J\right)\right]\right) . \tag{5.0.15}$$

To analyze (5.0.15), we first note that ϵ only appears in the quantity A in (5.0.13), and A satisfies

$$\left(-2\frac{\partial}{\partial\epsilon} + \frac{1}{(1-s)}\frac{\partial^2}{\partial\overline{v}_i\,\partial v^i}\right) A \;=\; 0\,. \tag{5.0.16}$$

Thus, we can rewrite (5.0.15) as

$$\begin{aligned}\left(-2\frac{\partial}{\partial\epsilon}\right)^{\frac{1}{2}d_1} \cdot Z_{red}(\epsilon, s) \;=\; \int_{E_1} &\left(-\frac{1}{(1-s)}\frac{\partial^2}{\partial\overline{v}_i\,\partial v^i}\right)^{\frac{1}{2}d_1} \times \\ &\times \left((4\pi A)^{-\frac{1}{2}} \, \exp\left[s\, d\lambda + \frac{1}{4}\left(J, A^{-1}\, J\right)\right]\right) .\end{aligned} \tag{5.0.17}$$

We now apply simple scaling arguments to (5.0.17) to show that this integral is convergent at $s = 0$ and behaves continuously as $s \to 0$. First, at $s = 0$, we immediately see that this integral behaves for large $|v|$ as $\int d^{2d_1}v \; 1/|v|^{(2d_1+1)}$ and hence is convergent, though just barely.

To discuss the limit $s \to 0$, we assume s is fixed at a small, non-zero value. All terms involving s which we previously dropped for $s = 0$ now appear in the argument of the exponential in (5.0.17). For large $|v|$, this argument behaves schematically as

$$s\,|v|^2\,(dv,dv) + \frac{(\gamma_0,\gamma_0)}{|v|^2} + s\,|v|^2\,(\gamma_0,T_0) + \frac{(dv,dv)}{|v|^2}\,(\gamma_0,T_0) + s^2\,|v|^6 + \frac{(dv,dv)^2}{|v|^2}\,. \tag{5.0.18}$$

Since our argument is only a scaling argument, we ignore all signs and constants in writing (5.0.18), though we do recall that the dominant term $s^2\,|v|^6$ leads to an exponential decay of the integrand at large v.

We see three terms in (5.0.18) which vanish in the limit $s \to 0$. Of these terms, we can ignore the quadratic term $s\,|v|^2(\gamma_0,T_0)$, since it is subleading compared to $s^2\,|v|^6$ for fixed s and large $|v|$.

However, we need to consider the effect of the measure $s^2\,|v|^4\,(dv,dv)^2$, which dominates the measure $(dv,dv)^2/|v|^2$ at $s = 0$ by a relative factor of $s^2\,|v|^6$. We also need to consider the terms which arise when the derivative $\partial^2/\partial\overline{v}_i\,\partial v^i$ in (5.0.17) acts on $\exp\left(-s^2\,|v|^6\right)$ to bring down the term $s^2\,|v|^4$, which dominates $1/|v|^2$ by the same relative factor $s^2|v|^6$.

These terms lead to contributions depending on s in (5.0.17) which behave for large $|v|$ as

$$\int_{E_1} d^{2d_1}v\,\frac{1}{|v|^{2d_1+1}}\,s^{2n}\,|v|^{6n}\,\exp\left(-s^2|v|^6\right), \qquad n = 1,\,\ldots\,,d_1\,. \tag{5.0.19}$$

Since these integrals only converge for $s \neq 0$, when the integrand is exponentially damped, one might have worried that these terms could cause the limit $s \to 0$ to be singular. However, we see by scaling that the expression in (5.0.19) behaves as $s^{+1/3}$ for all n and hence the asymptotic contributions to (5.0.17) from these terms still go continuously to zero as $s \to 0$.

Finally, apart from the terms in (5.0.19) with $n \geq 1$, the integrand of (5.0.17) is a smooth function $F(v,s)$ of v and s which behaves asymptotically for large $|v|$ as

$$F(v,s)\;\sim\;\frac{1}{|v|^{2d_1+1}}\,\exp\left(-s^2\,|v|^6\right). \tag{5.0.20}$$

Thus, $F(v,s)$ decays exponentially for $s \neq 0$, is integrable for all s, and is dominated by $F(v,0)$, which has a pure power law decay at infinity. On general grounds, the integral of

$F(v, s)$ over E_1 then depends continously on s, and, for the purpose of computing $Q \cdot Z(\epsilon)$, we can validly interpolate from λ to λ' on F.

Appendix B

More About Localization at Higher Critical Points: Higher Casimirs

In this appendix, we continue from Chapter 4.4.3 our general discussion of non-abelian localization at higher critical points. We recall that we obtained a formal expression for the canonical symplectic integral over F in terms of an integral over the Lie algebra $\mathfrak{h}_0$ of the stabilizer group H_0,

$$Z(\epsilon) = \frac{1}{\mathrm{Vol}(H_0)} \int_{\mathfrak{h}_0} \left[\frac{d\psi}{2\pi}\right] \det\left(\frac{\psi}{2\pi}\Big|_{E_0}\right) \det\left(\frac{\psi}{2\pi}\Big|_{E_1}\right)^{-1} \exp\left[-i\,(\gamma_0, \psi) - \frac{\epsilon}{2}\,(\psi,\psi)\right]. \tag{6.0.1}$$

As we discussed, this integral generally fails to converge when the ratio of determinants in the integrand has singularities in $\mathfrak{h}_0$. In the special case $H_0 = U(1)$, relevant for higher critical points of $SU(2)$ Yang-Mills theory, we deal with this problem by computing not $Z(\epsilon)$ itself but a higher derivative $Q \cdot Z(\epsilon)$, where $Q \equiv Q(\partial/\partial\epsilon)$ is a differential operator which we choose so that the action of Q on the integrand of (6.0.1) brings down sufficient powers of (ψ,ψ) to cancel any poles that would otherwise appear.

However, if we consider higher critical points of Yang-Mills theory with general gauge group G, then the rank of H_0 can be arbitrary, and the determinants in (6.0.1) cannot generally be expressed as a functions of only the quadratic invariant (ψ,ψ). Consequently, we cannot simply differentiate $Z(\epsilon)$ with respect to ϵ to cancel the poles in (6.0.1).

Nevertheless, by applying some simple ideas about the localization construction, we can generalize our discussion in Section 4.3 of Chapter 4 to the case that H_0 has higher rank. As in Section 4.1, we recall the form of the localization integral:

$$Z(\epsilon) \;=\; \frac{1}{\mathrm{Vol}(H)} \int_{\mathfrak{h}\times X} \left[\frac{d\phi}{2\pi}\right] \exp\left[\Omega - i\,\langle \mu, \phi\rangle - \frac{\epsilon}{2}(\phi,\phi)\right]. \tag{6.0.2}$$

In the case of Yang-Mills theory, $H = \mathcal{G}(P)$ and $X = \mathcal{A}(P)$ in the notation of Section 2.

Let us consider what natural generalizations of (6.0.2) exist. Of the terms appearing in (6.0.2), the quantity $\Omega - i\,\langle \mu, \phi\rangle$ is distinguished as an element of the equivariant cohomology ring of X, since it represents the equivariant extension of the symplectic form on X. However, nothing really distinguishes the quadratic function $-\frac{1}{2}(\phi,\phi)$ among all invariant polynomials of ϕ, and we are free to consider a general symplectic integral over $\mathfrak{h} \times X$ of the form

$$Z[V] \;=\; \frac{1}{\mathrm{Vol}(H)} \int_{\mathfrak{h}\times X} \left[\frac{d\phi}{2\pi}\right] \exp\left[\Omega - i\,\langle \mu, \phi\rangle - V(\phi)\right]. \tag{6.0.3}$$

Here $V(\phi)$ is any invariant polynomial on $\mathfrak{h}$ such that the integral over $\mathfrak{h}$ remains convergent at large ϕ. We can take

$$V(\phi) = \sum_j \epsilon_j\, C_j(\phi), \tag{6.0.4}$$

where C_j are the Casimirs of H – the homogeneous generators of the ring of invariant polynomials on $\mathfrak{h}$ – and ϵ_j are parameters. The standard localization technique can be applied to evaluate this integral. The fact that V is not quadratic in ϕ leads to no special complications.

In the case of Yang-Mills theory on a Riemann surface Σ with symplectic form ω, we would write

$$V(\phi) \;=\; \sum_{j=1}^{r} \epsilon_j \int_\Sigma \omega \cdot C_j(\phi)\,. \tag{6.0.5}$$

We assume that the gauge group G has rank r, and now $C_j(\phi)$ are the Casimirs of G. We associate to each generator a corresponding coupling ϵ_j. If we want to compare to standard methods of studying two-dimensional Yang-Mills theory by cut and paste methods, we should integrate over ϕ to express the theory in terms of the gauge field (and noninteracting fermions) alone. Of course, if $V(\phi)$ is not quadratic, we can no longer perform the integral over ϕ in (6.0.3) as a Gaussian integral. Instead, if we abstractly introduce the Fourier transform

$$\exp\left[-\widehat{V}(\phi^*)\right] \equiv \int_{\mathfrak{h}} \left[\frac{d\phi}{2\pi}\right] \exp\left[-i\langle\phi^*,\phi\rangle - V(\phi)\right], \tag{6.0.6}$$

which is an invariant function of ϕ^* in the dual algebra $\mathfrak{h}^*$, then the generalized symplectic integral over X takes the form

$$Z[V] = \frac{1}{\mathrm{Vol}(H)} \int_X \exp\left(\Omega - \widehat{V}(\mu)\right). \tag{6.0.7}$$

In the case of Yang-Mills theory, we recall that $\mu = F_A$. So in that case, (6.0.7) corresponds to a generalization of Yang-Mills theory in which the action is not the usual $\mathrm{Tr}\, f^2$ (with $f = \star F$) but $\mathrm{Tr}\, \widehat{V}(f)$, for some more general function $\widehat{V}$. The partition function of this generalized Yang-Mills theory can be computed by the usual cut and paste methods [100]. If G is simply-connected and we apply the same normalization conventions as we used in (4.4.40) for the case $G = SU(2)$, the generalized partition function is

$$Z[V] = (\mathrm{Vol}(G))^{2g-2} \sum_{\mathcal{R}} \frac{1}{\dim(\mathcal{R})^{2g-2}} \exp(-V'(\mathcal{R})), \tag{6.0.8}$$

where $V'(\mathcal{R})$ is the energy of the representation $\mathcal{R}$. (We are taking the area of Σ to be 1; for a general area α, the exponential factor would be $\exp(-\alpha V'(\mathcal{R}))$.)

To compute the energy $V'(\mathcal{R})$, we start with the action $\widehat{V}(f)$ and compute the canonical momentum $\Pi = \partial\widehat{V}/\partial f$. As usual, the energy is determined by the eigenvalue of the Hamiltonian, which is the Legendre transform of the action $\widehat{V}(f)$. Thus, the Hamiltonian is $H = f\Pi - \widehat{V}(f)$, which must be extremized with respect to f and regarded as a function of Π. After computing $H(\Pi)$, Π is interpreted as the generator of the group G and taken to act on the representation $\mathcal{R}$ to get the energy $V'(\mathcal{R})$.

Since the Legendre transform is a semiclassical approximation to the Fourier transform, the Legendre transform approximately undoes the Fourier transform in (6.0.6), and hence $H(\Pi) = V(\Pi)$+lower order terms. As discussed in [18], if the representation $\mathcal{R}$ has highest weight h, the precise formula needed to match with the localization computation is $V'(\mathcal{R}) = V(h + \delta)$, where the constant δ is half the sum of positive roots of the Lie algebra of G. This formula incorporates the difference between the Legendre transform and the Fourier transform and other possible quantum corrections.

To generalize what we said in Section 4.3, we want to find a polynomial $F(C_j)$ of the Casimirs of H which when restricted to $\mathfrak{h}_0$ is divisible by the dangerous factor in the denominator, namely $w(\psi) = \det\left(\psi/2\pi|_{E_1}\right)$. Then $Q = F(-\partial/\partial\epsilon_j)$ is a differential operator that when acting on $\exp(-V)$ will produce the factor F and cancel the denominator. Thus, Q generalizes the operator $\partial^{g-1}/\partial\epsilon^{g-1}$ that we used in Section 4.3 for two-dimensional $SU(2)$ gauge theory in genus g.

The dangerous factor w is an invariant polynomial on the Lie algebra of $\mathfrak{h}_0$ or equivalently, a polynomial on the maximal torus of H_0 that is invariant under the Weyl group of H_0. This polynomial can be extended, though not canonically, to a polynomial w' on the maximal torus of H. We can pick the extension to be invariant under the Weyl group of H_0 but not necessarily under the Weyl group of H. However, by multiplying w' by all its conjugates under the Weyl group of H, we make a polynomial $\widetilde{w}$ on the maximal torus of H that is invariant under the Weyl group of H, and whose restriction to H_0 is divisible by w. The Weyl-invariant polynomial $\widetilde{w}$ corresponds to the polynomial $F(C_j)$ of the Casimirs that was used in the last paragraph.

Finally, let us make this more explicit for Yang-Mills theory. The denominator factor in (6.0.8) that we need to cancel is $\dim(\mathcal{R})^{2g-2}$, so it suffices to know that $\dim(\mathcal{R})^2$ is a polynomial of the Casimirs. This can be proved using the Weyl character formula, discussed in §123 of [110], which provides a general formula for $\dim(\mathcal{R})$. Parametrizing the representation $\mathcal{R}_h$ by a highest weight h,

$$\dim(\mathcal{R}_h) \;=\; \prod_{\beta>0} \frac{(\beta, h+\delta)}{(\beta,\delta)}\,. \tag{6.0.9}$$

The product in (6.0.9) runs over the positive roots β, and we recall that δ is a constant, equal to half the sum of the positive roots. We regard this as a function of $h' = h + \delta$.

The formula (6.0.9) exhibits a polynomial function d on the Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G such that $\dim(\mathcal{R}_h) = d(h')$. The polynomial d is not strictly invariant under the action of the Weyl group on h', but is invariant up to sign, so d^2 is Weyl invariant.

As such, d^2 extends to an invariant polynomial on all of $\mathfrak{g}$, and thus a polyomial in the Casimirs. Finally, we observe that the shift $h \to h' = h + \delta$ is the same renormalization that we introduced for the potential $V'(\mathcal{R}_h) = V(h + \delta)$, so that by differentiating with respect to the couplings of each Casimir in V' we can cancel the denominator $\dim(\mathcal{R})^{2g-2}$.

Appendix C

A Few Additional Generalities About Equivariant Cohomology

Following the discussion in Section 5.3 of Chapter 4, we discuss in this appendix the identification of the H-equivariant cohomology of N_0 with the H_0-equivariant cohomology of $\mathcal{M}$, a fact which fundamentally leads to the correspondence (4.5.110).

To start, we find it useful to employ another topological model of equivariant cohomology, explained for instance in Chapter 1 of [99]. In this model, if X is any topological space on which a group H acts, the H-equivariant cohomology ring of X is defined as the ordinary cohomology ring of the fiber product $X_H = X \times_H EH$, where EH is any contractible space on which H acts freely. Such an EH always exists, and the choice of EH does not matter, since EH is unique up to H-equivariant homotopies. Thus, $H^*_H(X) = H^*(X_H)$.

As a simple example, if H acts freely on X, implying that X is a principal H-bundle over X/H, then X_H is equivalent to a product $X_H = (X/H) \times EH$. Since EH is contractible, we see that $H^*_H(X) = H^*(X/H)$, a fact we applied in our discussion of two-dimensional Yang-Mills theory.

At the opposite extreme, when H acts trivially on X, then X_H is also a product $X_H = X \times BH$, where $BH = EH/H$ is the classifying space associated to the group H. In this case, $H^*_H(X) = H^*(X) \otimes H^*(BH)$. However, by the definition of equivariant cohomology above, the ordinary cohomology of BH is the H-equivariant cohomology of a point, so that $H^*_H(X) = H^*(X) \otimes H^*_H(pt)$. For the latter factor, our description of the Cartan model in Section 4.1 clearly identifies $H^*_H(pt)$ with the ring of invariant functions on the Lie algebra $\mathfrak{h}$ of H.

In the case relevant for Chern-Simons theory, we suppose that X is a fiber bundle over $\mathcal{M}$ with fiber H/H_0 for some H. As a result, H acts on the fibers with isotropy subgroup H_0.

For the following, we want to realize X globally as a quotient Y/H_0. Here Y is a principal bundle over $\mathcal{M}$ with fiber H, so that $H \to Y \to \mathcal{M}$, and we suppose that Y has the following additional properties. First, we assume that $H \times H_0$ acts on Y, with H acting on the fibers on the left and H_0 on the right. As above, we also assume that $Y/H_0 = X$. In this situation, H and H_0 both act freely on Y, the quotient Y/H being $\mathcal{M}$ and the quotient Y/H_0 being X. Of course, H_0 acts trivially on X.

We can now argue as follows. First, $H^*_{H\times H_0}(Y) = H^*_H(X)$, as H_0 acts freely on Y with quotient X. On the other hand $H^*_{H\times H_0}(Y) = H^*_{H_0}(\mathcal{M})$ because H acts freely on Y with quotient $\mathcal{M}$. Finally, as H_0 acts trivially on $\mathcal{M}$, $H^*_{H_0}(\mathcal{M}) = H^*(\mathcal{M}) \otimes H^*_{H_0}(pt)$. Putting these facts together, we have our desired result that $H^*_H(X) = H^*(\mathcal{M}) \otimes H^*_{H_0}(pt)$.

In general such a Y only exists rationally (which is good enough for de Rham cohomology), but for our problem with Chern-Simons theory on a Seifert manifold, a natural Y can be constructed as follows.

First of all, over any symplectic manifold $\overline{\mathcal{A}}$, a "prequantum line bundle" $\mathcal{L}$ is a unitary line bundle with connnection whose curvature is the symplectic form. For Chern-Simons theory, $\mathcal{L}$ exists and is unique up to isomorphism as $\overline{\mathcal{A}}$ is just an affine space. We let $\mathcal{L}_0$ be the bundle of unit vectors in $\mathcal{L}$, a circle bundle over $\overline{\mathcal{A}}$.

In general, any connected Lie group of symplectomorphisms of a symplectic manifold that has an invariant moment map lifts to an action on the prequantum line bundle. For Chern-Simons theory on a Seifert manifold, the group $\mathcal{G}$ of gauge transformations does not have a moment map (due to the obstruction arising from the loop group cocycle) but its central extension $\widetilde{\mathcal{G}}$ does. We recall that $\widetilde{\mathcal{G}}$ is an extension of $\mathcal{G}$ by an abelian subgroup $U(1)_Z$ that acts trivially on $\overline{\mathcal{A}}$ but has constant moment map equal to 1. In particular, since $\widetilde{\mathcal{G}}$ has a moment map, $\widetilde{\mathcal{G}}$ acts on $\mathcal{L}$, and hence on the subbundle $\mathcal{L}_0$. Under this action, the subgroup $U(1)_Z$ acts freely by rotating the fibers of the fibration $\mathcal{L}_0 \to \overline{\mathcal{A}}$.

Finally, the Hamiltonian group $\mathcal{H}$ that we really use for our quantization is a semidirect product of $\widetilde{\mathcal{G}}$ with another abelian factor $U(1)_R$ that geometrically rotates the fibers of the Seifert fibration. The group $U(1)_R$ acts on $\mathcal{L}$ and $\mathcal{L}_0$, but not freely. To get the desired space Y on which $U(1)_R$ acts freely, we simply set $Y = U(1) \times \mathcal{L}_0$, where $U(1)_R$ acts by rotation on $U(1)$ together with its natural action on $\mathcal{L}_0$. So in fact $H_0 = U(1)_R \times U(1)_Z$ acts freely on Y.

We now want to restrict this construction from $\overline{\mathcal{A}}$, the space of all connections, to N_0, the space of flat connections on which we localize and whose quotient N_0/H is $\mathcal{M}$, the moduli space of gauge-equivalence classes of flat connections. We let Y_0 be the restriction to N_0 of the fibration $Y \to \overline{\mathcal{A}}$. So $H \times H_0$ acts on Y_0; H_0 acts freely on Y_0 with quotient N_0, and H acts freely on Y_0 with quotient $\mathcal{M}$. Finally, H_0 acts trivially on $\mathcal{M}$. With these observations, the general argument presented above shows that $H^*_H(N_0) = H^*(\mathcal{M}) \otimes H^*_{H_0}(pt)$.

Appendix D

More About Localization at Higher Critical Points: Localization Over a Nontrivial Moduli Space

In this appendix, we consider the general case that our abstract model for F in Chapter 4 is fibered over a non-trivial moduli space $\mathcal{M}$. Our goal is to compute the equivariant cohomology class on $\mathcal{M}$ which is produced by the canonical symplectic integral over F,

$$I(\psi) = \frac{1}{\mathrm{Vol}(H)} \int_{\widetilde{F}} \left[\frac{d\phi}{2\pi}\right] \exp\left[tD\lambda\right], \qquad \widetilde{F} = (\mathfrak{h} \ominus \mathfrak{h}_0) \times F, \qquad \psi \in \mathfrak{h}_0. \tag{8.0.1}$$

We begin with some geometric preliminaries. Very briefly, we recall that we model F as a vector bundle with fiber $\mathfrak{h}^{\perp} \oplus E_1$ over a homogeneous base H/H_0. Here $\mathfrak{h}^{\perp} = \mathfrak{h} \ominus \mathfrak{h}_0 \ominus E_0$, and explicitly,

$$F = H \times_{H_0} (\mathfrak{h}^{\perp} \oplus E_1). \tag{8.0.2}$$

To describe the total space N of the fiber bundle $F \longrightarrow N \longrightarrow \mathcal{M}$, we introduce a principal H-bundle P_H over $\mathcal{M}$. Besides the given action of H on P_H, we assume that P_H also admits a free action of H_0 which commutes with the action of H. As a result, we can describe the bundle N concretely in terms of P_H as

$$N = P_H \times_{H_0} (\mathfrak{h}^{\perp} \oplus E_1). \tag{8.0.3}$$

Upon setting $P_H = H$, where H acts on the right and H_0 acts on the left, this model for N reduces to the model for F itself, with $\mathcal{M}$ being a point.

Of course, the key ingredient in our localization computation is to choose a good representative of the canonical localization form λ on N. As in Section 4.3, we introduce another localization form λ' which (under the same caveats as in Section 4.3 and Appendix A) is homotopic to λ on N and takes the form

$$\lambda' = \lambda'_\perp + \lambda'_{E_0} + \lambda'_{E_1}\,, \tag{8.0.4}$$

with

$$\begin{aligned} \lambda'_\perp &= (\gamma\,,\theta)\,,\\ \lambda'_{E_0} &= -i\left(\theta_{E_0}, g\phi g^{-1} + i\mathcal{R}(\theta)\right)\,, \qquad \mathcal{R}(\theta) \,=\, d\theta - \frac{1}{2}[\theta,\theta]\,,\\ \lambda'_{E_1} &= i\left(\left(g\phi g^{-1}\right)_{\mathfrak{h}_0}\cdot v\,, dv - \theta_{\mathfrak{h}_0}\cdot v\right)\,. \end{aligned} \tag{8.0.5}$$

In these expressions, we recall that γ is an element of $\mathfrak{h}^\perp$, g is an element of H, ϕ is an element of $\mathfrak{h}$, and v is an element of the vector space E_1. Finally, θ is now a connection on the principal H-bundle P_H. In particular, θ is a globally-defined one-form on P_H. As usual, we let $\mathcal{R}(\theta)$ denote the curvature of θ.

Our choice for λ' is precisely analogous to the choice we made in Section 4.3 in the case that $P_H = H$, and in (8.0.4) we have simply grouped the terms in λ' in a natural way for the localization computation. The only term present in (8.0.5) which was not present in Section 4.3 is the term involving the curvature $\mathcal{R}(\theta)$ in λ'_{E_0}. The curvature of θ is a horizontal form on P_H, meaning that it is annihilated by contraction with the vector fields $V(\phi)$ which generate the action of H on P_H, so this curvature term could not appear when $\mathcal{M}$ was only a point. Equivalently, if the connection θ takes the global form $\theta = dg\,g^{-1}$ as in Section 4.3, then $\mathcal{R}(\theta)$ vanishes identically.

In (8.0.4) and (8.0.5) we have written λ' as an invariant form on the direct product $P_H \times (\mathfrak{h}^\perp \oplus E_1)$, but one can check exactly as in Section 4.3 that λ' descends under the quotient by H_0 to an invariant form on N.

Although λ' is globally defined on N, we have written λ' in coordinates on P_H with respect to a local trivialization of this bundle about some point m on the base $\mathcal{M}$. The integral we perform will be an integral over the fiber F_m above this point m, and since m is arbitrary, this local computation suffices to determine the cohomology class on $\mathcal{M}$ that arises after we perform the integral over all the fibers of $F \longrightarrow N \longrightarrow \mathcal{M}$. In particular, upon pulling θ back to the fiber F_m, θ takes the canonical form,

$$\theta|_{F_m} \,=\, dg\, g^{-1} \,. \tag{8.0.6}$$

However, since the curvature $\mathcal{R}(\theta)$ can be non-zero, in general $d\theta \neq \frac{1}{2}[\theta,\theta]$ at points in the fiber over m.

At this point, we repeat our earlier computation of $D\lambda'$, allowing for the presence of the curvature $\mathcal{R}(\theta)$. We find

$$\begin{aligned} D\lambda'_{\perp} \,&=\, (d\gamma,\theta) \,-\, i\,(\gamma,\phi + i\,d\theta)\;, \\ D\lambda'_{E_0} \,&=\, -i\,(d\theta_{E_0},\phi + i\,\mathcal{R}(\theta)) \,+\, i\,(\theta_{E_0},[\theta,\phi + i\,\mathcal{R}(\theta)]) \,-\, (\phi_{E_0},\phi + i\,\mathcal{R}(\theta))\;, \\ D\lambda'_{E_1} \,&=\, i\,(\phi_{\mathfrak{h}_0}\cdot dv, dv) \,-\, \Big(\phi_{\mathfrak{h}_0}\cdot v, (\phi + i\,\mathcal{R}(\theta))_{\mathfrak{h}_0}\cdot v\Big) \,+\, \mathcal{X}\,, \end{aligned} \tag{8.0.7}$$

with

$$\mathcal{X} \,=\, i\,\Big([\theta,\phi]_{\mathfrak{h}_0}\cdot v, dv\Big) \,+\, i\,\Big(\phi_{\mathfrak{h}_0}\cdot v, \frac{1}{2}\,[\theta,\theta]_{\mathfrak{h}_0}\cdot v\Big) \quad \mathrm{mod}\ \theta_{\mathfrak{h}_0}. \tag{8.0.8}$$

As before, in writing these expressions we make the change of variable from ϕ to $g\phi g^{-1}$ at the end of the calculation to simplify the result. Also, the terms appearing in $\mathcal{X}$ are at least of cubic order in the "massive" variables θ, v, and dv and so are irrelevant in the limit $t \to \infty$. Finally, we are free to work modulo terms involving $\theta_{\mathfrak{h}_0}$ since $D\lambda'$ is a pullback from the quotient $P_H \times_{H_0} (\mathfrak{h}^{\perp} \oplus E_1)$.

We now compute directly the integral below in the limit $t \to \infty$,

$$I(\phi_{\mathfrak{h}_0}) \,=\, \frac{1}{\mathrm{Vol}(H)} \int_{\widetilde{F}_m} \left[\frac{d\phi}{2\pi}\right] \exp\left[tD\lambda'_{\perp} + tD\lambda'_{E_0} + tD\lambda'_{E_1}\right], \qquad \widetilde{F}_m \,=\, (\mathfrak{h} \ominus \mathfrak{h}_0) \times F_m\,. \tag{8.0.9}$$

This integral behaves essentially the same as the integral in Section 4.3, so we will be brief.

We first consider the integral over E_1, which we perform as a Gaussian integral using the terms from $tD\lambda'_{E_1}$ in the large t limit. Explicitly, the integral over E_1 is given by

$$\int_{E_1} \exp\Big[it\,(\phi_{\mathfrak{h}_0}\cdot dv, dv) \,-\, t\,\Big(\phi_{\mathfrak{h}_0}\cdot v, (\phi + i\,\mathcal{R}(\theta))_{\mathfrak{h}_0}\cdot v\Big) \,+\, t\mathcal{X}\Big]. \tag{8.0.10}$$

Since $\mathcal{X}$ is of at least cubic order in the massive variables θ, v, and dv, this term can be dropped from the integrand when t is large. Keeping the other terms quadratic in v and dv in (8.0.10), the Gaussian integral over E_1 immediately produces

$$\det\left(\frac{1}{2\pi}\left(\phi_{\mathfrak{h}_0}+i\,\mathcal{R}(\theta)_{\mathfrak{h}_0}\right)\Big|_{E_1}\right)^{-1}. \tag{8.0.11}$$

We now integrate over both γ and ϕ in $\mathfrak{h}^\perp=\mathfrak{h}\ominus\mathfrak{h}_0\ominus E_0$. We see from (8.0.7) that γ still appears only linearly in $tD\lambda'$, so the integral over γ produces a delta-function of $\phi_\perp$, where $\phi_\perp$ denotes the component of ϕ in $\mathfrak{h}^\perp$. As is evident from the form of $tD\lambda'_\perp$, this delta-function sets $\phi_\perp=-id\theta_\perp$. (As in Section 4.3, the factors of t cancel between the integral over γ and the integral over $\phi_\perp$.)

We are left to integrate over ϕ_{E_0} and over the base H/H_0 of F_m. Of course, upon Taylor expanding the exponential $\exp\left(d\gamma,\theta\right)$ from $D\lambda'_\perp$ to produce the measure for γ, we also produce the canonical measure on the tangent directions to H/H_0 lying in $\mathfrak{h}^\perp$. So infinitesimally we have only to integrate over the remaining tangent directions to H/H_0 which lie in E_0 in addition to ϕ_{E_0}.

So we are left to integrate over E_0 using the terms in $tD\lambda'_{E_0}$. This integral takes the form

$$\begin{aligned}\int_{E_0}&\exp\left[-it\left(\theta_{E_0},[\phi_{\mathfrak{h}_0}+i\,\mathcal{R}(\theta)_{\mathfrak{h}_0},\theta_{E_0}]\right)\,+\,t\left(\mathcal{R}(\theta)_{E_0},\mathcal{R}(\theta)_{E_0}\right)\right]\times\\&\times\exp\left[-2it\left(\mathcal{R}(\theta)_{E_0},\phi_{E_0}\right)\,-\,t\left(\phi_{E_0},\phi_{E_0}\right)\right].\end{aligned} \tag{8.0.12}$$

In deducing (8.0.12), we have expanded and simplified various terms in $D\lambda'_{E_0}$ in (8.0.7). For instance, the curvature term $(\mathcal{R}(\theta)_{E_0},\mathcal{R}(\theta)_{E_0})$ arises from the linear combination of terms $(d\theta_{E_0},\mathcal{R}(\theta))-(\theta_{E_0},[\theta,\mathcal{R}(\theta)])$ in $D\lambda'_{E_0}$. To see this, we rewrite this expression as $(d\theta_{E_0}-[\theta,\theta_{E_0}],\mathcal{R}(\theta)_{E_0})\equiv(\mathcal{R}(\theta)_{E_0},\mathcal{R}(\theta)_{E_0})$, where "$\equiv$" indicates that the equality holds modulo $\theta_{\mathfrak{h}_0}$ and $\theta_\perp$, which is good enough since these forms do not contribute to the integral over E_0.

In writing (8.0.12), we also note that when we set $\phi_\perp=-i\,d\theta_\perp$ in $D\lambda'_{E_0}$, we effectively cancel similar terms in $D\lambda'_{E_0}$ which involve the components of the curvature $\mathcal{R}(\theta)$ in $\mathfrak{h}^\perp$. So $\mathcal{R}(\theta)_\perp$ does not appear in (8.0.12).

We first perform the Gaussian integral over ϕ_{E_0} in (8.0.12). The result of this integral produces a term proportional to $\exp\left[-t\left(\mathcal{R}(\theta)_{E_0}, \mathcal{R}(\theta)_{E_0}\right)\right]$ which precisely cancels the term quadratic in the curvature $\mathcal{R}(\theta)_{E_0}$ in the first line of (8.0.12). Consequently, once we collect factors of t and 2π exactly as in Section 4.3, the term quadratic in θ_{E_0} in (8.0.12) produces another determinant,

$$\det\left(\frac{1}{2\pi}\left(\phi_{\mathfrak{h}_0} + i\,\mathcal{R}(\theta)_{\mathfrak{h}_0}\right)\Big|_{E_0}\right). \tag{8.0.13}$$

Including the factor $\mathrm{Vol}(H)/\,\mathrm{Vol}(H_0)$ that arises from the integral over H/H_0 and setting $\phi_{\mathfrak{h}_0} \equiv \psi$ for notational simplicity, we find our final result for the integral in (8.0.9),

$$I(\psi) \;=\; \frac{1}{\mathrm{Vol}(H_0)}\,\det\left(\frac{1}{2\pi}\left(\psi + i\,\mathcal{R}(\theta)_{\mathfrak{h}_0}\right)\Big|_{E_0}\right)\,\det\left(\frac{1}{2\pi}\left(\psi + i\,\mathcal{R}(\theta)_{\mathfrak{h}_0}\right)\Big|_{E_1}\right)^{-1}. \tag{8.0.14}$$

Since both E_0 and E_1 are representations of H_0, the associated bundles $P_H \times_{H_0} E_0$ and $P_H \times_{H_0} E_1$ determine H_0-equivariant bundles over $\mathcal{M}$ once we divide by the action of H on P_H. The determinants appearing in (8.0.14) are then the Chern-Weil representatives of the H_0-equivariant Euler classes of these bundles.

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