The structure of string theory at finite temperature

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Menika Sharma

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Abstract

This thesis deals with string theory at finite temperature. String theory has attracted considerable attention in recent years because of its ability to unify the fundamental forces and particles in nature and provide a quantized description of gravity. However, many aspects of this theory remain mysterious, including its behavior at high temperature. One guiding principle for finite temperature string theory is the observation that a quantum theory at finite temperature can be recast as a zero-temperature theory in which a Euclidean time dimension is compactified on a circle. This temperature/radius correspondence holds in quantum mechanics as well as quantum field theory, and is normally assumed to hold in string theory as well. However it was shown recently that this correspondence fails for a class of string theories, called heterotic strings. This motivates a search for an alternate way to restore this correspondence, as well as a reevaluation of the thermodynamic behaviour of other classes of string theories, namely Type II and Type I. We find that contrary to the established wisdom, all ten dimensional string theories have a similar behaviour at finite temperature. This also leads us to the conclusion that the Heterotic and Type I theory behave in a dual way at finite temperature.

Chapter 1
INTRODUCTION

1.1 Introduction

In this thesis we will mainly be concerned with String theory at finite temperature. Our motivation for studying string theory at finite temperature is twofold. Firstly we will investigate to what extent various ten-dimensional string theories at finite temperature can be interpreted as purely geometric compactified nine-dimensional theories. This kind of interpretation is standard for quantum field theory, where it is well-established that a finite-temperature quantum theory can be recast as a zero-temperature theory in which a Euclidean time dimension is compactified on a circle. It has been assumed for a long time that this relation is true for string theories as well. However recently this assumption has been challenged for a particular ten dimensional string theory called the Heterotic String. In Chapter 2, we will explain the basis of this challenge in detail and give a proof that indeed the temperature–correspondence breaks down for Heterotic theories. In Chapter 3 we will show that there is a alternate way that this correspondence can be preserved. Of course, if this correspondence breaks down for a given string theory, it motivates an examination into its status for other string theories as well. A close relative of Heterotic String theory is Type I string theory. In Chapter 4 we study Type I string theories at finite temperature.

The second reason we are interested in studying String theory at finite temperature is because we want to find what happens to duality symmetries that hold for ten-dimensional strings at zero temperature, as the temperature is increased. Specifically we are interested in a duality symmetry called S-duality. S-duality says that there exist dual pair of theories, such that the strongly coupled limit of a string/field theory is equivalent to another weakly coupled theory. Since any calculation for a strongly coupled theory is hard to carry out, S-duality is a very useful symmetry - it says that the same calculation can be done perturbatively in a different weakly coupled theory. In string theory the Heterotic and Type I theories are known S-dual pairs. However it is not known whether this symmetry survives as the temperature is increased from zero. We take up this question in Chapter 5.

In order that Chapters 2-5 are understandable, we need to lay out some preliminaries. We shall do so in this chapter. Our goal is to present *ten*-dimensional string theories at zero temperature. Note that superstrings can only exist in 9+1 infinitely large dimensions and that to construct lower dimensional string models we have to compactify some dimensions. In section 1.2, we give a introduction to the bosonic string and its spectrum of states. Bosonic string theory is not phenomenologically viable and only functions as a toy model for introducing string theory (at least as far as our current understanding of string theory tells us). In section 1.3 we explain how superstring theory is constructed. In section 1.4 we describe the five superstring theories that exist in ten dimensions. Note that as opposed to four dimensions where there exist 10^{500} consistent string theories, there are only five consistent superstring theories in ten dimensions. One of the exciting developments of the last decade was that even these five superstring theories are all related to each other by duality symmetries. These duality symmetries are T-duality which holds in perturbative string theory and the previously mentioned S-duality which can only be seen non-perturbatively. Briefly T-duality relates a string theory with one dimension compactified at large radius with another string theory compactified at small radius. As the radii are taken to $R \to \infty$ and $R \to 0$ respectively, we get a relation between two string theories in *ten* dimensions. Although the T-duality and S-duality symmetries are unrelated to each other, T-duality provides us with extra information about String theory which is crucial for proving S-duality. In fact T-duality led to the discovery of objects called D-Branes, without which S-duality between string theories would only have remained a beautiful guess and not a proven fact. In section 1.5 therefore we introduce T-duality and show how it leads to D-branes. D-branes are actually hyperplanes in spacetime and while strings can only (by definition) be one-dimensional, D-branes come in various dimensions. The specific dimensions of D-branes that can exist will depend on the theory in question. In section 1.5, we also list the stable D-branes that can be present in the various ten dimensional string theories.

As is well known string theories generically contain in their spectrum a state that can be identified with the graviton. So in theory, string theory should be able to handle strings propagating in a non-Minkowski background. In practice this is a complex problem and we do not need to deal with it for the purpose of this thesis as we will always be in a Minkowski background. However there are massless fields other than the graviton in string theory - namely vector gauge fields, tensors and a scalar. A background for the vector fields is of interest and indeed there are papers in the literature dealing with strings in background electric and magnetic fields. As we will see in Chapters 2 and 3, string theories at finite temperature naturally admit a background gauge field. This background field has a vanishing field strength, yet it is a physical dynamical parameter at non-zero temperature. In field theory such a background field is referred to as a Wilson line. Interestingly we will see in section 1.5.3 that Wilson lines have a geometrical interpretation in terms of D-brane positions in string theory.

Since this is a review chapter it draws heavily from pre-existing literature in String theory. Specifically the material in this chapter is from the books([1, 2, 3]) and the review papers([4, 5, 6]).

1.2 The Bosonic String

A string is a one-dimensional object moving through space-time. As this object travels through spacetime it sweeps out a two-dimensional worldsheet. This worldsheet can be labeled by the coordinates,

$$(\sigma^0, \sigma^1) = (\tau, \sigma) \ . \tag{1.1}$$

Here σ is the spatial coordinate along the string, while τ describes its propagation in time. Strings come in two types – if the end points are joined together it is a closed string, otherwise it is an open string. The two cases are shown in Fig 1.1.

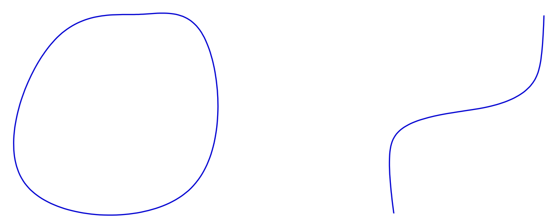

FIGURE 1.1. A closed and an open string

Classical Bosonic String

The string's embedding in space-time is given by the functions $X^\mu(\tau,\sigma)$, where μ varies from 1 to $d-1$. These functions describe the shape of the string worldsheet in the target spacetime.

The string action can be obtained by minimizing the total area of the string worldsheet in spacetime. Such an action is called the Nambu-Goto action and is given by,

$$S = -T \int \mathrm{d}\tau \; \mathrm{d}\sigma \; \sqrt{\dot{X}^2 \, X'^2 - (\dot{X} \cdot X')^2} \tag{1.2}$$

where

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} \; , \quad X'^\mu = \frac{\partial X^\mu}{\partial \sigma} \; . \tag{1.3}$$

The quantity T has dimensions of mass per unit length and is the tension of the string. It is related to the length ℓ_s of the string by

$$T = \frac{1}{2\pi\alpha'} \; , \quad \alpha' = \ell_s^2 \tag{1.4}$$

However since the Nambu-Goto action Eq. (1.2) contains square-roots of derivatives of X^μ, it is difficult to work with. An equivalent action called the Polyakov action can be defined after introducing an auxiliary worldsheet field γ^{ab} (here $a, b = 0, 1$) This action is given by,

$$S = -T \int \mathrm{d}^2\sigma \, \sqrt{-\gamma}\, \gamma^{ab}\, \partial_a X^\mu \, \partial_b X^\nu \, \eta_{\mu\nu} \tag{1.5}$$

γ is the determinant of γ_{ab}. The rank two symmetric tensor field γ_{ab} has a natural interpretation as a metric on the string worldsheet.

The equations of motion obtained by varying the Polyakov action with respect to γ^{ab} are

$$T_{ab} \equiv \partial_a X^\mu \, \partial_b X_\mu - \frac{1}{2}\, \gamma_{ab}\, \gamma^{cd}\, \partial_c X^\mu \, \partial_d X_\mu = 0 \tag{1.6}$$

This equation can be used to eliminate the worldsheet metric γ_{ab} from the action and recover the Nambu-Goto action.

Worldsheet Symmetries

The string worldsheet has the following local symmetries,

- Reparametrization invariance, also known as diff invariance

$$(\tau, \sigma) \;\rightarrow\; \Big(\tau(\tau', \sigma')\,,\, \sigma(\tau', \sigma')\Big)\;, \tag{1.7}$$

- Invariance under Weyl rescalings

$$\gamma_{ab} \;\rightarrow\; \mathrm{e}^{\,2\phi(\tau,\sigma)}\, \gamma_{ab} \tag{1.8}$$

  where $\phi(\tau, \sigma)$ is an arbitrary function on the worldsheet.

In short we refer to these two symmetry groups as diff$\times$weyl. These two symmetries of the Polyakov action allow us to fix the metric γ_{ab}. Reparametrization invariance allows us to choose two components of γ_{ab}. The remaining component can be gauged away using invariance under weyl rescalings. This lets us choose the convenient gauge,

$$\gamma_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \;. \tag{1.9}$$

With this choice of a flat worldsheet metric the Polyakov action takes the simple form,

$$S = T \int \mathrm{d}\tau \; \mathrm{d}\sigma \; \left(\dot{X}^2 - X'^2 \right) \tag{1.10}$$

Note that Eq. (1.10) defines a free field theory apart from some constraints.

Equations of Motion

Since the worldsheet theory is a free field theory the equation of motion for X^μ, is just the two-dimensional wave equation,

$$\left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) X^\mu(\tau, \sigma) = 0 \tag{1.11}$$

This has to be supplemented by the vanishing of the energy-momentum tensor,

$$\begin{aligned} T_{01} &= T_{10} = \dot{X} \cdot X' = 0 \,, \\ T_{00} &= T_{11} = \frac{1}{2} \left(\dot{X}^2 + X'^2 \right) = 0 \end{aligned} \tag{1.12}$$

The total variation of the action has to be zero, this means that the boundary terms must vanish. The specific way that these terms vanish will depend on whether we are dealing with *closed* or *open* strings.

- Closed Strings

  The field X^μ on the closed string has to be periodic. The boundary conditions are given by,

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \pi) \tag{1.13}$$

- Open Strings

  For open strings there are two choices of boundary conditions.

  - *Neumann* boundary conditions are given by

$$X'^\mu(\tau, \sigma) \Big|_{\sigma = 0, \pi} = 0 \tag{1.14}$$

    In this case the string is free to move in the ten-dimensional spacetime.

- *Dirichlet* boundary conditions are defined by

$$\dot{X}^{\mu}(\tau,\sigma)\Big|_{\sigma=0,\pi} = 0 \tag{1.15}$$

If we integrate the above condition over τ we get

$$\delta X^{\mu}(\tau,\sigma)\Big|_{\sigma=0,\pi} = 0 \tag{1.16}$$

This means that the ends of the open string are *fixed* in spacetime. The plane on which the open string endpoints are fixed corresponds to a physical object called a D-brane.

Mode Expansions

We know write down the solution for the wave equations. To do this, we define the worldsheet light-cone coordinates,

$$\sigma^{\pm} = \tau \pm \sigma \ , \quad \partial_{\pm} = \frac{\partial}{\partial \sigma^{\pm}} \ . \tag{1.17}$$

The wave equation Eq. (1.11) then becomes,

$$\partial_{+}\partial_{-}X^{\mu} = 0 \tag{1.18}$$

The '+' and '−' directions decouple from each other. They are referred to as left-moving and right-moving. The solution of (1.18) is therefore the sum of a function of σ^{+} alone and a function of σ^{-} alone, as:

$$X^{\mu}(\tau,\sigma) = X^{\mu}_{\mathrm{L}}(\sigma^{+}) + X^{\mu}_{\mathrm{R}}(\sigma^{-}) \tag{1.19}$$

For closed strings, the most general solution to the wave equation satisfying the boundary conditions, Eq. (1.13), is

$$\begin{aligned} X^{\mu}_{\mathrm{L}}(\sigma^{+}) &= \frac{1}{2}\, x^{\mu}_{0} + \alpha'\, p^{\mu}_{0}\, \sigma^{+} + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}^{\mu}_{n}}{n}\ \mathrm{e}^{-2\,\mathrm{i}\,n\sigma^{+}} \\ X^{\mu}_{\mathrm{R}}(\sigma^{-}) &= \frac{1}{2}\, x^{\mu}_{0} + \alpha'\, p^{\mu}_{0}\, \sigma^{-} + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha^{\mu}_{n}}{n}\ \mathrm{e}^{-2\,\mathrm{i}\,n\sigma^{-}} \end{aligned}$$

x_0^μ and p_0^μ represent the center of mass position and momentum of the string, respectively. The $\tilde{\alpha}_n^\mu$ and α_n^μ represent the oscillatory modes of the string.

Since the string function X^μ has to be real, both x_0^μ and p_0^μ are real, and

$$(\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu \ , \quad (\alpha_n^\mu)^* = \alpha_{-n}^\mu \ . \tag{1.20}$$

For open strings, with the Neumann boundary conditions (1.14), the general solution to the wave equation is given by,

$$X^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' \, p_0^\mu \, \tau + \mathrm{i}\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} \ \mathrm{e}^{-\mathrm{i}\, n\tau} \ \cos(n\sigma) \ . \tag{1.21}$$

The left and right moving sectors of the closed string are combined together by the Neumann boundary condition (1.14) and the open string equation of motion has a standing wave as the solution.

The total classical Hamiltonian can be derived from the Polyakov action with the flat metric and is given by,

$$H = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot \alpha_n \equiv L_0 & \text{for open strings} \\ \dfrac{1}{2} \displaystyle\sum_{n=-\infty}^{\infty} (\tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + \alpha_{-n} \cdot \alpha_n) \equiv \tilde{L}_0 + L_0 & \text{for closed strings} \end{cases} \tag{1.22}$$

Quantization

To quantize we write down commutators for the oscillator modes. The oscillator modes have the commutators

$$\begin{array}{rcl} [x_0^\mu \, , \, p_0^\nu] & = & \mathrm{i}\,\eta^{\mu\nu} \\ [x_0^\mu \, , x_0^\nu] & = & [p_0^\mu \, , p_0^\nu] \ = \ 0 \\ [\alpha_m^\mu \, , \, \alpha_n^\nu] & = & [\tilde{\alpha}_m^\mu \, , \, \tilde{\alpha}_n^\nu] \ = \ m\,\delta_{m+n,0}\,\eta^{\mu\nu} \\ [\alpha_m^\mu \, , \, \tilde{\alpha}_n^\nu] & = & 0 \end{array} \tag{1.23}$$

Writing down the quantum versions of the Hamiltonian defined in Eq. (1.22), needs some care as α_n^μ and α_{-n}^μ do not commute. The quantum Hamiltonian is generally denoted by L_0 and is given by

$$L_0 = \frac{1}{2}\,\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n - a \tag{1.24}$$

where a is the total zero-point energy of the families of infinite field oscillators. The quantity a is the Casimir energy and is present in our case because the quantum field theory we are dealing with is defined on a closed area - an infinite strip for open strings and an infinite cylinder for closed strings.

From the physical constraint that the energy-momentum tension vanish, we can an infinite set of constraints in terms of the oscillator modes. These constraints correspond to the presence of an infinite-dimensional symmetry on the worldsheet known as conformal symmetry. We saw in Sect 1.2 that the full symmetry group of the Polyakov action, with the metric unfixed, is diff×weyl. Even after we fix the metric there remains a large residual symmetry on the classical worldsheet that is this conformal symmetry. The requirement that conformal invariance on the string worldsheet hold at a quantum field theory level fixes $a = 1$ for the bosonic string and the total dimension of spacetime as, $D = 26$.

The Bosonic String Spectrum

We will start with the open string. The constraint $(L_0 - 1)|\text{phys}\rangle = 0$ gives the mass-shell condition,

$$m^2 = \frac{1}{\alpha'}\left(N - 1\right)\,, \tag{1.25}$$

here we have fixed the casimir energy value, $a = 1$. N is the level number defined as,

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \tag{1.26}$$

The closed string case can be handled similarly by taking the tensor product of two copies of the open string result. However we get a extra constraint now. The operator

L_0 has another right-moving copy now $\tilde{L}_0$. Therefore there are two mass shell conditions $(L_0 - 1)|\text{phys}\rangle = (\tilde{L}_0 - 1)|\text{phys}\rangle = 0$. Combining them gives

$$\begin{aligned} \left(L_0 + \tilde{L}_0 - 2\right)|\text{phys}\rangle &= 0 \\ \left(L_0 - \tilde{L}_0\right)|\text{phys}\rangle &= 0 \end{aligned} \tag{1.27}$$

The first constraint yields the mass-shell relation just as for open strings. It is now given by,

$$m^2 = \frac{4}{\alpha'}\left(N - 1\right) \tag{1.28}$$

Since the second constraint says that the mass-shell condition has to hold individually for the left moving and right moving modes we get an additional condition called the level-matching condition

$$N = \tilde{N} \tag{1.29}$$

Later we will see that the level matching condition is a part of a larger constraint on the string spectrum coming from higher-loop order string interactions.

We now look at the bosonic string spectrum. The ground state for the bosonic theory is $|k;0\rangle$ and it has mass-squared

$$m^2 = -\frac{4}{\alpha'} < 0\ . \tag{1.30}$$

It is therefore a spin 0 tachyon and the string vacuum is unstable.

The first excited state is

$$|k;\zeta\rangle = \zeta_{\mu\nu}\left(\alpha^{\mu}_{-1}|k;0\rangle \otimes \tilde{\alpha}^{\nu}_{-1}|k;0\rangle\right) \tag{1.31}$$

and has mass-squared

$$m^2 = 0\ . \tag{1.32}$$

We can decompose this rank 2 tensor into a symmetric, traceless tensor $g_{\mu\nu}$ corresponding to a spin 2 graviton, an antisymmetric spin 2 tensor $B_{\mu\nu}$ called the NS B-field and a scalar field Φ, which is called the dilaton. These set of fields is referred to as the gravity multiplet.

It can be shown that the spin 2 symmetric tensor obeys the Einstein equations. Therefore the tensor $g_{\mu\nu}$ can be identified with the graviton field. The vacuum expectation value of $\langle g_{\mu\nu} \rangle$ determines the spacetime metric. The vacuum expectation value of the dilaton field Φ determines the string coupling constant,

$$g = \mathrm{e}^{\langle \Phi \rangle} \tag{1.33}$$

The graviton and the dilaton are generically present in any consistent string theory. The tensor $B_{\mu\nu}$ is a generalization of the electromagnetic field. A fundamental string acts as a source for the B-field, just like a charged particle is a source for an electromagnetic vector potential A_μ.

1.3 Superstrings

Since the bosonic theory contains a tachyon and no fermions in its spectrum, it is of little practical use. We will now generalize the bosonic string to include supersymmetry and hence discuss superstrings. This can be accomplished by adding fermions to the string worldsheet. While it is not clear that such a procedure should yield spacetime supersymmetry, it turns out that this is so.

The classical theory

For constructing the superstring theory, we retain the bosonic string action with d free, massless scalar fields $X^\mu(\tau, \sigma)$ and add d free, massless Majorana spinors $\psi^\mu(\tau, \sigma)$ to it which transform as d-dimensional vectors under Lorentz transformations in spacetime. The worldsheet action becomes,

$$S = -\frac{T}{2} \int \mathrm{d}^2\sigma \, \left(\partial_a X^\mu \, \partial^a X_\mu - \mathrm{i} \, \bar{\psi}^\mu \, \rho^a \, \partial_a \psi_\mu \right) \tag{1.34}$$

with the addition of the Dirac term for the spinor fields.

The ρ^a are 2-dimensional gamma matrices given by,

$$\rho^0 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \ , \quad \rho^1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{1.35}$$

They satisfy the commutation relations,

$$\left\{\rho^a \, , \, \rho^b\right\} = -2\,\eta^{ab} \tag{1.36}$$

The fermion field

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \tag{1.37}$$

is a two-component Majorana spinor, $\psi_\pm^* = \psi_\pm$. The two-dimensional Dirac term becomes in this notation

$$\overline{\psi} \cdot \rho^a \, \partial_a \psi = \psi_- \cdot \partial_+ \psi_- + \psi_+ \cdot \partial_- \psi_+ \tag{1.38}$$

The equations of motion for the spinor fields are given by the massless Dirac equation,

$$\partial_+ \psi_-^\mu = \partial_- \psi_+^\mu = 0 \tag{1.39}$$

Therefore the fermions ψ_-^μ are right-moving while ψ_+^μ are left-moving.

Mode Expansions

To find the solutions for the Dirac equation we need to consider the boundary conditions for the spinor fields $\psi^\mu(\tau, \sigma)$. Let us first look at open strings. To ensure the vanishing of the Polyakov action (1.34) under variation we need,

$$\psi_+ \cdot \delta\psi_+ - \psi_- \cdot \delta\psi_- = 0 \quad \text{ at } \ \sigma = 0, \pi \tag{1.40}$$

There are therefore two possible boundary conditions, $\psi_+ = \pm\,\psi_-$ at $\sigma = 0, \pi$. The overall relative sign between the fields ψ_- and ψ_+ is just convention, so we can set

$$\psi_+^\mu(\tau, 0) = \psi_-^\mu(\tau, 0) \ . \tag{1.41}$$

This still leaves two possibilities at the other endpoint $\sigma = \pi$. These are referred to as the Ramond (R) and Neveu-Schwarz (NS) boundary conditions:

$$\begin{aligned} \psi_+^\mu(\tau,\pi) &= \psi_-^\mu(\tau,\pi) \quad \text{(R)} \ , \\ \psi_+^\mu(\tau,\pi) &= -\psi_-^\mu(\tau,\pi) \quad \text{(NS)} \ . \end{aligned}$$

The solutions to the Dirac Eq. (1.39) are now given by,

$$\begin{aligned} \psi_\pm(\tau,\sigma) &= \frac{1}{\sqrt{2}} \sum_r \psi_r^\mu \ \mathrm{e}^{-\mathrm{i}\, r(\tau \pm \sigma)} \ , \\ r &= \text{integer} \quad \text{(R)} \\ &= \text{half} - \text{integer} \quad \text{(NS)} \end{aligned}$$

where the condition that the field be real requires

$$\psi_{-r}^\mu = (\psi_r^\mu)^* \tag{1.42}$$

The closed string sector is again given by a tensor product of left and right moving sectors. Each component of ψ^μ can now have periodic or anti-periodic boundary conditions separately. This gives us the mode expansions

$$\begin{aligned} \psi_+^\mu(\tau,\sigma) &= \sum_r \tilde{\psi}_r^\mu \ \mathrm{e}^{-2\,\mathrm{i}\, r(\tau+\sigma)} \\ \psi_-^\mu(\tau,\sigma) &= \sum_r \psi_r^\mu \ \mathrm{e}^{-2\,\mathrm{i}\, r(\tau-\sigma)} \end{aligned} \tag{1.43}$$

We can pair the left-moving and right-moving modes in four different ways which gives us four different closed string sectors:

- NS–NS
- NS–R
- R–NS
- NS–NS

The Hamiltonian in this case is given by,

$$L_0 = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{-m} \cdot \alpha_m + \frac{1}{2} \sum_r r\, \psi_{-r} \cdot \psi_r \tag{1.44}$$

The Superstring Spectrum

We now quantize the worldsheet theory for superstrings.

The anti-commutators for the spinor fields are given by,

$$\{\psi_r^\mu \,,\, \psi_s^\nu\} = \delta_{r+s,0}\, \eta^{\mu\nu} \tag{1.45}$$

The spacetime dimension D for the superstring is fixed to $D = 10$. The normal ordering constants are found to be

$$a = \left\{ \begin{array}{ll} 0 & \text{(R)} \\ \frac{1}{2} & \text{(NS)} \end{array} \right\} \tag{1.46}$$

The $L_0 = a$ constraint gives the open string mass formula

$$m^2 = \frac{1}{\alpha'} \Big(N - a \Big) \tag{1.47}$$

where the total level number is given by

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r>0} r\, \psi_{-r} \cdot \psi_r \tag{1.48}$$

As we saw in the previous section, the open string spectrum of states has two sectors - NS and R.

The vacuum which is the NS ground state, $|k;0\rangle_{\text{NS}}$ has a mass given by $m^2 = -\frac{1}{2\alpha'}$ and is tachyonic. Therefore we will have to find a way to eliminate it from the physical spectrum.

The first excited levels in the NS sector contain the massless states $\psi^\mu_{-\frac{1}{2}}|k;0\rangle_{\text{NS}}$, $m^2 = 0$. These are spacetime vectors. They are actually states of the massless field $A_\mu(x)$ in ten spacetime dimensions. All states in the NS sector are spacetime bosons.

The NS ground sector has total worldsheet fermion number -1 and such states are labeled as the NS$-$ sector. (This is not apparent at the level of detail we are going into, but can be seen in a more careful quantization). The first excited level has total fermion number $+1$ and such states are labeled as the NS$+$ sector.

In the Ramond sector there are zero modes ψ_0^μ which satisfy the ten dimensional Dirac algebra

$$\{\psi_0^\mu\,,\,\psi_0^\nu\} = \eta^{\mu\nu} \tag{1.49}$$

The ψ_0's are therefore a representation of the Dirac matrices. Consequently, the ground state is a Dirac spinor. Since the excited states are built up by acting by spacetime vectors on the ground states, all states in the R sector are spacetime fermions. The ground state of the Ramond sector is massless and therefore the ground state can be decomposed into two chiral spinors. The spinors in each chiral sector are denoted as coming from the R$+$ and R$-$ sectors respectively. Just like in the NS sector $+$ and $-$ refer to the total worldsheet fermion number.

The spectrum of closed strings is obtained by taking tensor products of left-movers and right-movers. There are four sectors. In the NS–NS sector, the lowest lying level is again a closed string tachyon. We need to find a way to get rid of the tachyon in these theories. In the next section we discuss how this is done and also enumerate the consistent supersymmetric theories in ten dimensions.

1.4 Supersymmetric theories in ten dimensions

So far we have concentrated on the tree level formulation of superstring theory. However for consistent closed string theories in ten dimensions there is an important constraint that comes from the fact that the diff$\times$Weyl invariance hold at one loop. Since the one loop two dimensional worldsheet structure of a closed string theory is a torus - this translates to the useful fact that the string one loop amplitude be invariant under the modular group of the torus. This requirement is called *modular invariance*. The entire spectrum of spacetime states arising from the worldsheet theory, is not compatible with modular invariance for instance. A truncation of the spectrum is generally required - known as the GSO projection.

In the next section we will discuss in brief consistent string theories in ten dimensions. We will write down the one-loop vacuum amplitude for such theories. The vacuum amplitude is in general given by

$$F_{\mathrm{T}} = V_{10} \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{4\tau_2} \int \frac{\mathrm{d}^{10}k}{2\pi^{10}} \sum_{i\in\mathcal{H}^{\perp}} (-1)^F \; q^{\;\alpha'(k^2+m_i^2)/4} \bar{q}^{\;\alpha'(k^2+\tilde{m}_i^2)/4} \tag{1.50}$$

where $q \equiv \mathrm{e}^{2\pi\mathrm{i}\tau}$ and $\tau = \tau_1 + \mathrm{i}\tau_2$ parametrizes the torus. $(-1)^F$ is the space-time fermion number. The trace includes a sum over different sectors of the superstring Hilbert space and is the partition function of the theory denoted by Z_{T}, where the T stands for the torus. Note that the integration is over the fundamental domain of the torus and therefore avoids the dangerous region $\tau_2 \to 0$. This is shown in Figure 1.2.

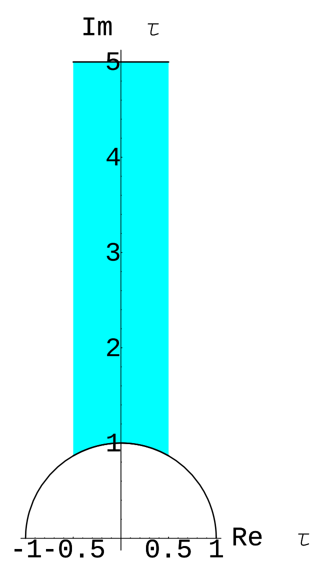

FIGURE 1.2. Fundamental domain of the torus

Type II theories

Two consistent closed string theories can be defined in ten dimensions. The Type IIA theory containing the sectors, (NS+,NS+), (R+,NS+), (NS+,R−), (R+,R−) and the Type IIB theory containing the sectors (NS+,NS+), (R+,NS+), (NS+,R+), (R+,R+).

For writing the vacuum amplitude we need to take the trace over all states in a given sector. The traces are denoted by:

$$\text{NS}+: \;\; \chi_V, \qquad \text{NS}-: \;\; \chi_I, \qquad \text{R}+: \;\; \chi_S, \qquad \text{R}-: \;\; \chi_C$$

The actual form of the traces is given in the appendix of Chapter 2. The partition function for the Type IIA and Type IIB theories can now be written as,

$$\begin{aligned} Z_{\text{IIA}} &= Z^{(8)}_{\text{boson}} \, (\overline{\chi}_V - \overline{\chi}_S) \, (\chi_V - \chi_C) \\ Z_{\text{IIB}} &= Z^{(8)}_{\text{boson}} \, (\overline{\chi}_V - \overline{\chi}_S) \, (\chi_V - \chi_S) \end{aligned} \tag{1.51}$$

Th Type II theories contain in their massless spectrum, the gravity multiplet and its supersymmetric counterparts. In addition they contain massless p-forms called Ramond-Ramond fields (refering to the sector they arise form). For some time it was a mystery whether there exist any objects in string theory that couple to these fields. Note that the Type II theories have no gauge fields in their spectrum.

Type I theory

The Type IIB theory with the same chiralties on both sides has a worldsheet parity symmetry, Ω. The action of Ω is,

$$\sigma \longrightarrow -\sigma \tag{1.52}$$

This symmetry can be gauged to obtain an unoriented closed string theory. Acting by Ω on a closed string propgating in a loop, that is essentially reversing the orientation of the string, results in the Klein bottle. The torus and the Klein bottle are shown in Figure 1.3. The double covering torus for the Klein bottle is shown in Fig 1.4. From

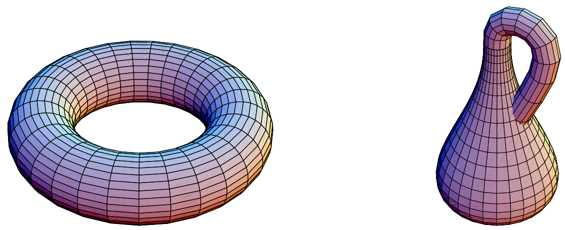

FIGURE 1.3. The closed sector contribution to Type I string one loop amplitudes corresponds to a superposition of the torus and the Klein surfaces.

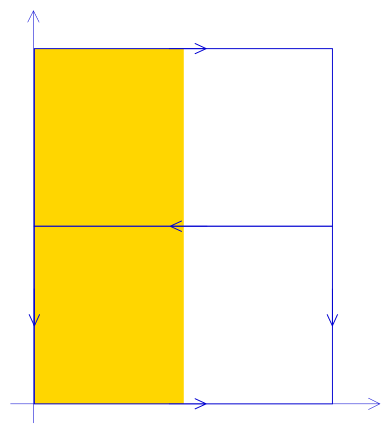

FIGURE 1.4. The Klein bottle and its double cover. The shaded area represents time flowing horizontally instead of vertically.

the mode expansions it can be seen that the action of Ω interchanges left and right moving oscillators. Therefore the unoriented theory obtained after gauging Ω will keep only those sectors that are left-right symmetric. That means it is not possible to get a consistent unoriented supersymmetric string theory from the Type IIA string. On the other hand a valid orientifold of Type IIB theory can be constructed. The torus and klein bottle partition function of such a model are given by,

$$\begin{aligned} Z_{\mathrm{T}}(\tau) &= \frac{1}{2}\, Z^{(8)}_{\text{closed}}\, \left(\overline{\chi}_V - \overline{\chi}_S\right)\left(\chi_V - \chi_S\right) \\ Z_{\mathrm{K}}(\tau_2) &= \frac{1}{2}\, Z^{(8)}_{\text{open}}\, \left(\chi_V - \chi_S\right) \end{aligned} \tag{1.53}$$

If we take time to flow horizontally rather than vertically, the klein bottle amplitude can also be viewed as a closed string propagating in a tube and ending in two crosscaps.

This closed theory by itself is not consistent because now there is a divergence coming from the Klein bottle amplitude. Unlike the torus amplitude, where we integrate over the fundamental region and avoid the dangerous region $\tau_2 \to 0$, the integration region for the Klein bottle is from 0 to ∞. Therefore massless states lead to a divergence in the Klein bottle amplitude. To cancel this divergence a something new has to be introduced. It turns out that open strings are the answer. Exactly what sectors will be

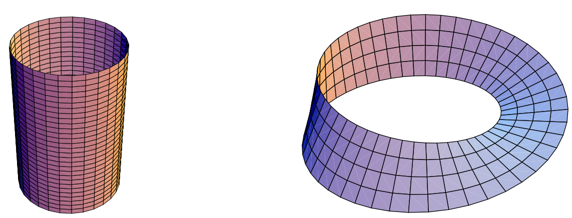

FIGURE 1.5. The open sector contribution to the Type I string one loop amplitudes corresponds to a superposition of the Cylinder and Mobius strip.

present for the open strings can be determined by looking at the sectors already present for closed strings. The one-loop diagram for an open string can be interpreted in two ways. It corresponds to a open string propagating in a loop, if time is taken to flow vertically. However if time is taken to flow horizontally it corresponds to a *closed* string moving from one point to another, with a reflection at the open ends. Therefore in this interpretation, know as the tree channel, the closed sectors present should again be the ones that are left-right symmetric. Finally the action of Ω on the open strings will result in another non-orientable surface - the Mobius strip. These surfaces are shown in Figure (1.5) and their double covering torus is shown in Figure (1.6).In the loop channel, this is in open string propogating and undergoing orientation reversal. In the tree channel this corresponds to a closed string in a tube with a boundary at one end and a crosscap at another. The tree channel amplitude for the Mobius strip will therefore be the geometric mean of the Klein and cylinder amplitudes. The cylinder and mobius

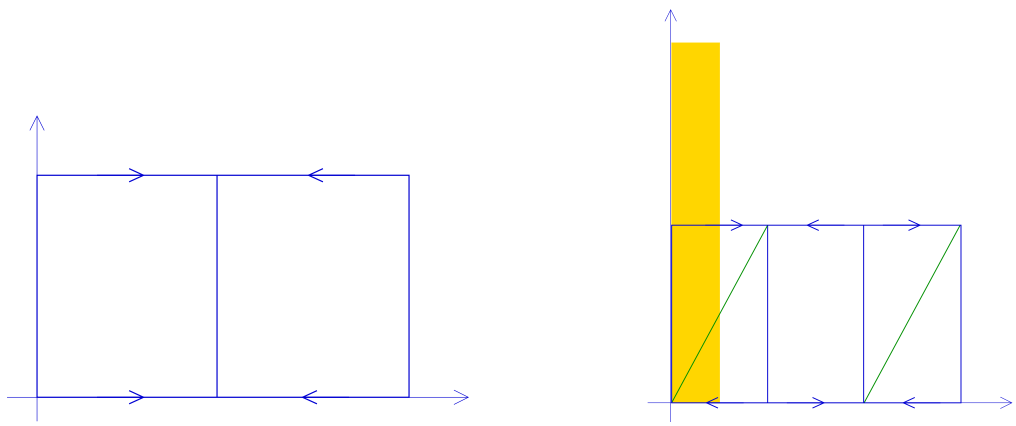

FIGURE 1.6. Double covers of the cylinder and mobius strip. The shaded area represents time flowing horizontally instead of vertically.

contributions to the partition functions are given by,

$$\begin{aligned} Z_{\mathrm{C}}(\tau_2) &= \frac{1}{2} N^2 \, Z^{(8)}_{\mathrm{open}} \, (\chi_V - \chi_S) \\ Z_{\mathrm{M}}(\tau_2) &= -\frac{1}{2} N \hat{Z}^{(8)}_{\mathrm{open}} \, (\hat{\chi}_V - \hat{\chi}_S) \end{aligned} \tag{1.54}$$

Cancelling the divergence present in the closed sector fixes the value of N to be 32.

The closed sector of Type I theory gives the massless graviton and dilaton (the B-field is projected out) as well as the p-form fields that survive the orientifold projection. The open sector of Type I gives us massless gauge fields and the corresponding gauge group $SO(32)$.

Heterotic theory

The heterotic string is constructed in a different manner than the Type II and Type I Strings. It is a theory of closed oriented strings - however it is constructed as a combination of the bosonic string and the superstring. The heterotic string construction takes advantage of the fact that the left and the right moving modes of closed strings factorize completely and therefore can be treated in disparate ways. In the right moving sector one can have the usual worldsheet field content of the superstring which is given by the bosonic and fermionic fields $\tilde{X}^\mu(\sigma^+)$ and $\tilde{\psi}^\mu(\sigma^+)$ with $\mu = 0, 1, \ldots, 9$. In the left moving sector one can have only bosonic fields $X^\nu(\sigma^-)$ with $\nu = 0, 1, \ldots, 25$. Since the dimension of uncompactified space on both the right and left moving side has to be equal, not all the X^ν's can be interpreted as spacetime coordinates. Therefore the left-moving sector of the heterotic string is partitioned into two sets. When $\nu = 0, \ldots, 9$, the left-moving fields $X^\nu(\sigma^-)$ add to the right-moving fields $\tilde{X}^\mu(\sigma^+)$ to give the uncompactified coordinates $X^\mu(\tau, \sigma)$. For $\nu = 10, \ldots, 25$, the left-moving fields $X^\nu(\sigma^-)$ are interpreted as internal degrees of freedom only. These 16 left moving internal bosonic fields can also be though of as 32 majorana worldsheet fermions by the boson-fermion correspondence in two dimensions.

Quantization for the heterotic string is the same as for the Type II Strings. There are two consistent supersymmetric field theories in two dimensions, the $SO(32)$ and the $E(8) \times E(8)$ strings. Here we concentrate on the $SO(32)$ theory. The sectors that can potentially be present for a heterotic theory and their corresponding traces are given by,

$$\text{NS}+: \quad \tilde{\chi}_I, \qquad \text{NS}-: \quad \tilde{\chi}_V, \qquad \text{R}+: \quad \tilde{\chi}_S, \qquad \text{R}-: \quad \tilde{\chi}_C$$

These are for the left moving side. The right moving traces remain the same as for the superstring. The partition function is given by,

$$Z_{SO(32)} \;=\; Z^{(8)}_{\text{boson}} \;(\overline{\chi}_V - \overline{\chi}_S)\,(\tilde{\chi}_I + \tilde{\chi}_S) \tag{1.55}$$

The massless spectrum of the heterotic theory consists of the $N = 1$ supersymmetric gravity multiplet. In addition there are massless gauge fields transforming in the adjoint of $SO(32)$ just like in Type I. There are however no RR p-forms in this case.

1.5 T-duality

A major concern of this thesis will be the duality symmetries that exist in String theory. While we will chiefly examine Strong-weak coupling duality in string theory, here we look at another famous duality - T-duality. There are two reasons for this - one it lets us examine a duality symmetry in a simple context, since this duality holds in perturbative string theory. Second it gives us a rationale for the existence of non-perturbative objects called D-branes that are vital for looking at strong-weak coupling duality. T-duality also provides a pleasing geometrical picture and is a first clue that String theory may not just be a human invention, but at the least a consistent and beautiful mathematical theory.

T-duality symmetry is a consequence of the finite length of strings and therefore it has no field theory counterpart. We will first consider T-duality for closed strings.

Recalling, the mode expansions of the string worldsheet fields:

$$\begin{aligned} X^\mu(\tau,\sigma) &= X^\mu_{\mathrm{L}}(\tau+\sigma) + X^\mu_{\mathrm{R}}(\tau-\sigma) \\ &= x_0^\mu + \tilde{x}_0^\mu + \sqrt{\frac{\alpha'}{2}}\left(\alpha_0^\mu + \tilde{\alpha}_0^\mu\right)\tau + \sqrt{\frac{\alpha'}{2}}\left(\alpha_0^\mu - \tilde{\alpha}_0^\mu\right)\sigma + (\ldots) \end{aligned} \tag{1.56}$$

The total spacetime momentum of the string is,

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}\left(\alpha_0^\mu + \tilde{\alpha}_0^\mu\right) . \tag{1.57}$$

A periodic shift in σ to $\sigma + 2\pi$ changes the function Eq. (1.56) to

$$X^\mu(\tau,\sigma+2\pi) = x^\mu(\tau,\sigma) + 2\pi\sqrt{\frac{\alpha'}{2}}\left(\alpha_0^\mu - \tilde{\alpha}_0^\mu\right) + (\ldots) \tag{1.58}$$

Since the oscillators terms, indicated by $(\ldots)$, were already periodic they are unchanged. X^μ has to be single-valued under $\sigma \to \sigma + 2\pi$. Therefore we get,

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu \tag{1.59}$$

where p^μ is real.

This is true if all space directions are uncompactified. We want to see how the mode expansion is modified if one of the directions, X^9, is compact. For this, we compactify X^9 on a circle of radius R. This spacetime coordinate is then periodic,

$$X^9 = X^9 + 2\pi R \tag{1.60}$$

The momentum p^9 becomes quantized and is given by,

$$p^9 = \frac{n}{R} \tag{1.61}$$

for some integer n. This is also true in quantum field theory. We now have the constraint,

$$\alpha_0^9 + \tilde{\alpha}_0^9 = \frac{2n}{R}\sqrt{\frac{\alpha'}{2}} \tag{1.62}$$

In addition now, under a periodic shift $\sigma \to \sigma + 2\pi$, the string can wind around the compact spacetime dimension. This means that from Eq. (1.60) the coordinate X^9 can be periodic upto $2\pi w R$, where w is an integer, known as the winding number. Therefore we can add the term $wR\sigma$ to the mode expansion Eq. (1.56) for $X^9(\tau, \sigma)$, and comparing with Eq. (1.56) this gives another constraint

$$\alpha_0^9 - \tilde{\alpha}_0^9 = wR\sqrt{\frac{2}{\alpha'}} \ . \tag{1.63}$$

Solving (1.62) and (1.63) gives

$$p_{\mathrm{L}} = \frac{n}{R} + \frac{wR}{\alpha'} \ , \quad p_{\mathrm{R}} = \frac{n}{R} - \frac{wR}{\alpha'} \tag{1.64}$$

where p_{L} and p_{R} are the left-moving and right-moving momenta.

The mass spectrum is now given by,

$$\begin{aligned} m^2 &= -p_\mu p^\mu \\ &= \frac{2}{\alpha'}\left(\alpha_0^9\right)^2 + \frac{4}{\alpha'}\left(N - 1\right) \\ &= \frac{2}{\alpha'}\left(\tilde{\alpha}_0^9\right)^2 + \frac{4}{\alpha'}\left(\tilde{N} - 1\right) \end{aligned} \tag{1.65}$$

Simplifying we get,

$$m^2 = \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right) \tag{1.66}$$

$$nw + N - \tilde{N} = 0 \tag{1.67}$$

In field theory a compactified direction leads to just a Kaluza-Klein tower of momentum states. In string theory there is a tower of winding states as well. The winding modes can of course just occur for strings because only a string can wrap around a circle.

Let us look at the large and small radius limits of the compactified string theory. In the limit $R \to \infty$, all $w \neq 0$ winding states disappear as they are very heavy. But the $w = 0$ states with all values of n become the usual continuum of momentum zero modes. This is the same thing that would happen in quantum field theory.

In the limit, $R \to 0$, all $n \neq 0$ momentum states become infinitely massive and decouple. But now the pure winding states $n = 0$, $w \neq 0$ form a continuum as it costs very little energy to wind around a small circle. So as $R \to 0$ an *extra* uncompactified dimension reappears. This is very different from what happens in quantum field theory, where all surviving fields in the limit $R \to 0$, would just become independent of the coordinate X^9.

The mass formula (1.67) for the spectrum is invariant under the simultaneous exchanges

$$n \longleftrightarrow w \ , \quad R \longleftrightarrow R' = \frac{\alpha'}{R} \ , \tag{1.68}$$

This symmetry of any compactified string theory is known as the T-duality symmetry. In fact T-duality symmetry is an *exact* quantum symmetry of perturbative closed string theory. By T-duality, very small circles are equivalent to very large ones in string theory. Thus strings see spacetime geometry very differently from point particles.

1.5.1 T-Duality for Open Strings

We now look at the more interesting case of open strings. Open strings cannot wind around the periodic direction of spacetime. Therefore they behave more like point particles. Open string theory looks like a quantum field theory in the limit $R \to 0$, in that states with non-zero Kaluza-Klein momentum become very heavy but no new continuum of states arise.

So there is an interesting dichotomy in the behaviour of open and closed strings. Now any consistent string theory can not be built using open strings alone, it needs closed strings too. However, the open strings effectively live in *nine* spacetime dimensions as

$R \to 0$, while the closed strings live in *ten* dimensions. The way out of this paradox is to realize that the interior of the open string still vibrates in ten dimensions. But the open string endpoints are now restricted to lie on a nine dimensional plane in spacetime.

The mode expansion for the open string worldsheet fields is given by

$$\begin{aligned} X^\mu(\tau,\sigma) &= X^\mu(\tau+\sigma) + X^\mu(\tau-\sigma) \\ &= x_0^\mu + \alpha' p_0^\mu \, \tau + \mathrm{i}\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} \; \mathrm{e}^{-\mathrm{i}n\tau} \cos(n\sigma) \end{aligned} \tag{1.69}$$

We put X^9 on a circle of radius R again, so that

$$X^9(\tau,\sigma) \sim X^9(\tau,\sigma) + 2\pi R \tag{1.70}$$

and the momentum is again quantized as $p^9 = \frac{n}{R}$, where n is an integer.

We can work out the mode expansions for the T-dual open string coordinate. Note that we can use the closed string procedure for getting the dual coordinate, $X'^9(\tau,\sigma) = X^9(\tau+\sigma) - X^9(\tau-\sigma)$. The spacetime mode expansions is given by

$$\begin{aligned} X'^9(\tau,\sigma) &= X^9(\tau+\sigma) - X^9(\tau-\sigma) \\ &= x_0'^9 + 2\alpha' \, \frac{n}{R} \, \sigma + \sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^9}{n} \; \mathrm{e}^{-\mathrm{i}n\tau} \sin(n\sigma) \end{aligned} \tag{1.71}$$

However now the zero mode sector of (1.71) is independent of the worldsheet time coordinate τ, and hence the new string carries no center-of-mass momentum. Thus the dual string cannot move in the X^9 direction. This corresponds to the Dirichlet boundary condition we discussed before. The open string endpoints are at a fixed location in spacetime given by,

$$X'^9(\tau,\pi) - X'^9(\tau,0) = 2\pi n R' \tag{1.72}$$

Thus the endpoints $X'^9|_{\sigma=0,\pi}$ are equivalent up to the a periodicity in R'. The quantized momentum of the open string is therefore converted into a 'winding number' in the dual theory. An open string with winding is shown in Fig. (1.7).

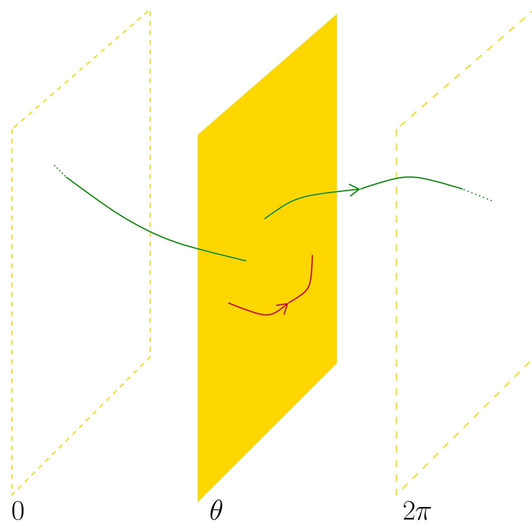

FIGURE 1.7. Open Strings end in hyperplanes called D-branes. Here we show an open string with winding number 0 and a open string that winds around the compact dimension once, and therefore has winding number 1. The dashed hyperplanes are periodically identified.

The open string can still move freely in the other spacetime directions. These dimensions taken together are a hyperplane called the "D-brane". For eight dimensions, it is a D8-brane. If we take the T-dual of q directions (intead of a single direction that we have considered so far), it gives Dirichlet boundary conditions in the q directions. This results in the open strings being confined to a plane with $p = 9 - q$ spatial dimensions. This hyperplane is called a 'Dp-brane'.Thus we see that T-duality, as a symmetry of string theory inevitably leads to D-branes.

Let us briefly discuss T-duality for the ten dimensional superstring theories. T-duality interchanges the Type IIA and Type IIB superstring theories. Type I theory (only the closed sector of Type I is invariant under T-duality) has a T-dual called Type I'. For the heterotic theory T-duality operates in a more complicated manner. It interchanges the $SO(32)$ and the $E(8) \times E(8)$ theory, if a specific background field is present. Otherwise each of these theories is self-dual under T-duality.

1.5.2 D-Branes in superstring theories

We can figure out exactly what dimension Dp-branes will be present in a given superstring theory. Dp-branes actually couple to the massless R-R forms present in superstring theory. In ten dimensions, an n-form potential, represented by $C^{(n)}$, couples electrically to a $p = n - 1$ brane and magnetically to a $p = 7 - n$ brane. Given this fact, we can easily find the D-branes that are present in the different superstring theories:

Type IIA Dp-Branes: The Ramond-Ramond potentials in Type IIA are $C^{(1)}$, $C^{(3)}$, $C^{(5)}$, $C^{(7)}$. Therefore Dp-branes exist in this theory for all even values of p,

$$p \;=\; 0,\; 2,\; 4,\; 6,\; 8\;. \tag{1.73}$$

The $p = 0$ case is called the D-particle.

Type IIB Dp-Branes: The Ramond-Ramond potentials in Type IIB are $C^{(0)}$, $C^{(2)}$, $C^{(4)}$, $C^{(6)}$, and $C^{(8)}$. Here we find branes for all odd values of p,

$$p \;=\; -1,\; 1,\; 3,\; 5,\; 7,\; 9\;. \tag{1.74}$$

The case $p = -1$ describes an object which is localized in time and is called a D-instanton. $p = 1$ is a D-string.

Type I Dp-Branes: Since Type I theory is a orientifold of Type II B, it's D-branes are a subset of those of Type IIB. Some RR forms and their corresponding D-branes are projected out by Ω. The remaining ones are,

$$p \;=\; 1,\; 5,\; 9 \tag{1.75}$$

Note that the $p = 1$ brane, the D-string survives. This is important for strong-weak coupling duality of the Type I string.

1.5.3 Wilson Lines

The D-brane picture admits a generalization based on switching on a background gauge field. To understand this we need a simple result from quantum mechanics. Let us say a particle of mass m and charge q is propagating in a background electromagnetic potential A_μ. Then the action is given by,

$$S = \int \mathrm{d}\tau \, \left(\frac{m}{2} \dot{X}^\mu \dot{X}_\mu - \mathrm{i}\, q\, \dot{X}^\mu A_\mu\right) \tag{1.76}$$

If one direction, X^9 is compactified on a circle of radius R, then a constant gauge field $A_9 = -\frac{\Phi}{2\pi R}$ shifts the momentum of the particle according to,

$$p_9 = \frac{n}{R} + \frac{q\,\Phi}{2\pi R} \tag{1.77}$$

We want to see the effect such a background gauge field has on the string spectrum. Closed Type II strings are not charged under gauge fields, but open string states can carry charges at the end-points - these are called Chan-Paton factors. An oriented open string can carry a charge transforming in the N representation of $U(N)$ on one end and a charge transforming in the $\bar{N}$ representation of $U(N)$ on the other. There will be a total of N^2 states in all. Note that in the D-brane picture the Chan-Paton factors correspond to labels for open strings ending on D-branes. The gauge group $U(N)$ then corresponds to N D-branes sitting on top of each other. Let us investigate the behaviour of an open string under a background abelian gauge field given by,

$$A_9 = \begin{pmatrix} \frac{\Phi_1}{2\pi R} & & 0 \\ & \ddots & \\ 0 & & \frac{\Phi_N}{2\pi R} \end{pmatrix} \tag{1.78}$$

here Φ_i, $i = 1, \ldots, N$ are constants. The presence of such a background clearly breaks the $U(N)$ gauge symmetry to $U(1)^N$. Just like the point particle case, this background field is pure gauge and can be gauged away locally by a gauge transformation. However this is not true globally. All charged open string states pick up a phase factor W under the periodic translation $X^9 \to X^9 + 2\pi R$ and therefore the gauge field (1.78) is an actual physical parameter. It will result in a phase shift as the string ends wind around the compactified spacetime direction.

This phase factor W is in fact just the 'Wilson line' for the given gauge field, given by,

$$W = \exp\left(\mathrm{i}\int \mathrm{d}t\ \dot{X}^\mu(t)\, A_\mu\right) = \exp\left(\mathrm{i}\int_0^{2\pi R} \mathrm{d}X^9\ A_9\right) \tag{1.79}$$

This effect is quite similar to the Aharonov-Bohm phenomenon in quantum mechanics.

We have seen that the Wilson line breaks the $U(N)$ symmetry. In the T-dual theory this has a geometrical interpretation in terms of D-branes. Since the string momenta along the X^9 direction is fractional for a general wilson line, and momentum gets mapped to winding in the T-dual theory, it implies that the open strings in the T-dual description will have fractional winding numbers. We now explain what this means.

Let us consider a Chan-Paton state $|k;ij\rangle$. The state i attached to an end of the open string will acquire a factor $\mathrm{e}^{-\mathrm{i}\,\Phi_i X^9/2\pi R}$ due to the gauge field , while state j will acquire a phase $\mathrm{e}^{\mathrm{i}\,\Phi_j X^9/2\pi R}$. The total open string wavefunction will therefore be shifted by, $|k;ij\rangle \cdot \mathrm{e}^{-\mathrm{i}\,(\Phi_i-\Phi_j)X^9/2\pi R}$. Under the periodic translation $X^9 \to X^9 + 2\pi R$ this shift becomes,

$$|k;ij\rangle \ \rightarrow\ \mathrm{e}^{\mathrm{i}\,(\Phi_j-\Phi_i)}\,|k;ij\rangle \tag{1.80}$$

This means that the momentum of the state will shift to

$$p^9_{ij} = \frac{n}{R} + \frac{\Phi_j - \Phi_i}{2\pi R}\ , \tag{1.81}$$

This means that the the Dirichlet boundary condition corresponding to this Chan-Paton state is now altered to

$$X'^9(\tau,\pi) - X'^9(\tau,0) = (2\pi n + \Phi_j - \Phi_i)\ R' \tag{1.82}$$

where R' is the T-dual compactification radius. Therefore the open string endpoint in Chan-Paton state i can be taken to be at the spacetime position,

$$X'^9_i = \Phi_i\, R' \quad i = 1,\dots,N \tag{1.83}$$

The stack of overlaying D-branes has now separated into D-branes at different positions (fig. 1.8). T-duality maps gauge fields in open string theory to the positions of D-branes in the dual theory. We can think of the original ten-dimensional open strings as ending

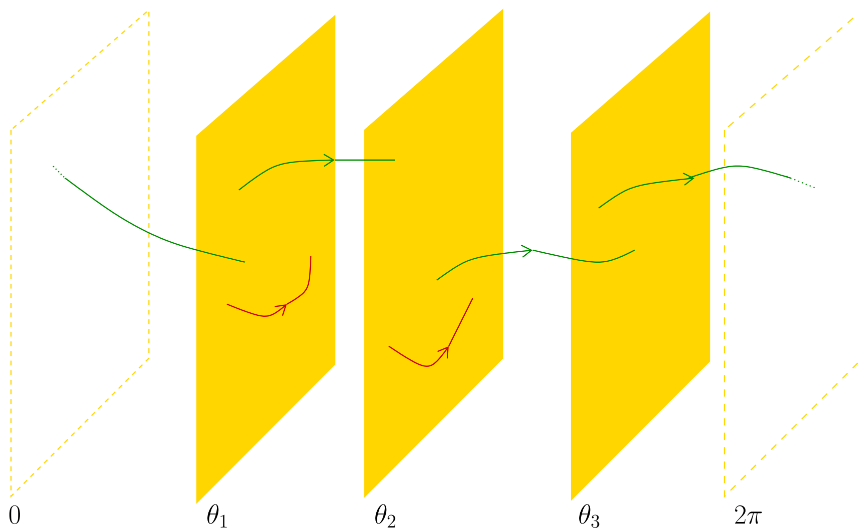

FIGURE 1.8. N separated parallel D-branes with open strings attached. The dashed hyperplanes are periodically identified. The angle θ is equal to the flux Φ used in the text.

on N spacetime filling D9-branes. In this picture the string endpoints can sit anywhere in spacetime. Compactifying a single coordinate, confines the open string endpoints to N D8-branes.

Let us now look at the massless states associated with strings ending on the same and different D-branes. The mass-shell condition is

$$\begin{aligned} m^2 &= \left(p^9\right)^2 + \frac{1}{\alpha'}\left(N-1\right) \\ &= \frac{\delta X}{(2\pi\alpha')^2} + \frac{1}{\alpha'}\left(N-1\right) , \end{aligned} \tag{1.84}$$

where

$$\delta X = \Big|2\pi n + (\Phi_i - \Phi_j)\Big| R' \tag{1.85}$$

is the minimum length of an open string which winds n times between the D-branes i

and j. From this equation we can see that massless states can only occur when $\delta X = 0$. This can only happen for zero winding and $\Phi_i = \Phi_j$ - that is the open string end-points lie on the same D-brane, $i = j$.

If none of the D-branes coincide, then there is only a single massless vector state associated with each of the N D-brane. These states give rise to a gauge theory with gauge group $U(1)^N$. However if k D-branes coincide,

$$\Phi_1 = \Phi_2 = \cdots = \Phi_k = \Phi \tag{1.86}$$

new massless states appear in the spectrum of the open string theory, because strings which have end-points on these overlapping branes can have zero length. This gives rise to the gauge group $U(k)$. If all of the N D-branes coincide we recover the original $U(N)$ gauge symmetry.

Chapter 2

THE BREAKDOWN OF THE TEMPERATURE/RADIUS CORRESPONDENCE FOR HETEROTIC STRINGS

Summary

It is an old observation that a quantum theory at finite temperature T can be recast as a zero-temperature theory in which a Euclidean time dimension is compactified on a circle of radius $R = (2\pi T)^{-1}$. The traditional thermodynamic Boltzmann sum is then achieved by taking bosonic (fermionic) states to be periodic (anti-periodic) around the thermal circle. This temperature/radius correspondence is a deep one, holding in quantum mechanics as well as quantum field theory, and it is normally assumed to hold in string theory as well. In this chapter, however, we demonstrate that while this correspondence holds for bosonic strings as well as Type II strings, it actually fails for heterotic strings. Specifically, we demonstrate that the traditional Boltzmann sum for heterotic strings at finite temperature cannot be recast as the partition function corresponding to any self-consistent heterotic compactification. This chapter mainly serves as a review chapter.

2.1 Introduction

The connection between temperature and geometry is a deep one, stretching from quantum mechanics and quantum field theory all the way into string theory. The fundamental idea is that a theory at finite temperature T can be reformulated as zero-temperature theory in which a Euclidean time dimension is compactified on a circle of radius $R = (2\pi T)^{-1}$. However, the process of compactification in string theory is very different from the analogous process in field theory, since many new features and complications can arise due to the finite spatial extent of the string. As a result, the extent to which the temperature/radius correspondence holds in string theory is not immediately clear.

In this chapter, we shall develop a rigorous test of this so-called “temperature/radius correspondence”, and then proceed to apply it to perturbative bosonic strings, Type II superstrings, and heterotic strings in their critical dimensions. We shall find, as expected, that bosonic strings and Type II superstrings pass this test. We shall also find, however, that the heterotic string does not. As a result, we shall find that the heterotic string actually violates the temperature/radius correspondence.

This is clearly an important and unexpected observation, so we shall proceed in developing our argument as slowly and carefully as possible. In Sect. 2.2, we shall begin with a short review of the temperature/radius correspondence, touching on only the major points which will be necessary for our subsequent discussion. We shall discuss this correspondence from the point of view of both standard quantum field theory as well as string theory. In Sect. 2.3, we shall then develop an explicit and rigorous test which will enable us to determine whether the standard Boltzmann sum associated with a given string theory is consistent with a potential geometric compactification of that string. The virtue of this test is that it is relatively straightforward to understand and apply. In Sects. 2.4 and 2.5, we shall then apply this test to the bosonic and Type II strings, respectively, and find that both of these strings pass the test. However, in Sect. 2.6, we shall demonstrate that the heterotic string actually fails this test, and we shall provide several different ways of understanding and demonstrating this result. Finally, in Sect. 2.7, we shall provide a short discussion in which we relate this result to prior results in the literature. Appendix A contains a listing of most of the background mathematical results which we shall be using in this chapter, and serves to define our notation and conventions. Appendix B sketches the derivation of a result quoted in the body of the chapter.

2.2 Preliminaries: The geometry of temperature

The connection between temperature and geometry, *i.e.*, the so-called "temperature/radius correspondence", is the central focus of this thesis. We shall therefore begin by reviewing the basic points of this correspondence, providing a brief sketch of the primary observation that links the physics of finite temperatures with the physics of compactified dimensions. Our goal is not to provide a complete technical derivation of these rather standard results, but merely to recall the basic points that enter into the mathematics of this correspondence.

2.2.1 The temperature/radius correspondence

The story begins with two mathematical identities which express hyperbolic trigonometric functions as infinite products:

$$\begin{aligned} \sinh(x) &= x \prod_{n=1}^{\infty} \left(\frac{x^2}{\pi^2 n^2} + 1 \right) \\ \cosh(x) &= \prod_{n=0}^{\infty} \left(\frac{x^2}{\pi^2 (n+1/2)^2} + 1 \right) . \end{aligned} \tag{2.1}$$

Substituting $x = E/(2T)$ and taking the logarithm of both sides then yields

$$\begin{aligned} \log(1 - e^{-E/T}) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \log\left[E^2 + 4\pi^2 n^2 T^2\right] \ + \ ... \\ \log(1 + e^{-E/T}) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \log\left[E^2 + 4\pi^2 (n+1/2)^2 T^2\right] \ + \ ... \ , \end{aligned} \tag{2.2}$$

where we have dropped terms beyond the infinite products as well as terms which compensate for the dimensionalities of the arguments of the logarithms.

The results in Eq. (2.2) can be viewed as mere mathematical identities involving two quantities E and T. However, their physical implications are readily apparent upon considering the thermodynamics of free bosons and fermions. Let us begin by considering a free real boson in four spacetime dimensions:

$$\Phi(\mathbf{x}) ~\sim~ \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, a^\dagger_{\mathbf{p}}\, e^{i\mathbf{p}\cdot\mathbf{x}} ~. \tag{2.3}$$

Since the creation operators $a^\dagger_{\mathbf{p}}$ obey Bose-Einstein statistics, the available states in the Fock space of the free boson are $|0\rangle$, $a^\dagger_{\mathbf{p}}|0\rangle$, $(a^\dagger_{\mathbf{p}})^2|0\rangle$, ..., for each $\mathbf{p}$. The (grand-canonical) partition function of this theory is therefore

$$Z_b(T) ~=~ \prod_{\mathbf{p}} \left(1 + e^{-E_{\mathbf{p}}/T} + e^{-2E_{\mathbf{p}}/T} + ...\right) ~=~ \prod_{\mathbf{p}} \frac{1}{1-e^{-E_p/T}} \tag{2.4}$$

with $E^2_{\mathbf{p}} \equiv \mathbf{p}\cdot\mathbf{p} + m^2$, whereupon the free energy of this theory (strictly, the four-dimensional free-energy density) is given by

$$F_b(T) ~\equiv~ -T\log Z_b(T) ~=~ +T\int \frac{d^3\mathbf{p}}{(2\pi)^3} \log(1-e^{-E_{\mathbf{p}}/T}) ~. \tag{2.5}$$

For a free fermion, by contrast, the associated Fermi-Dirac statistics restrict the corresponding Fock space to only $|0\rangle$ and $b^\dagger_{\mathbf{p}}|0\rangle$ for each $\mathbf{p}$, where $b^\dagger_{\mathbf{p}}$ is the fermionic creation operator. The fermionic partition function is therefore $Z_f(T) = \prod_{\mathbf{p}}(1+e^{-E_{\mathbf{p}}/T})$, whereupon the fermionic free energy is given by

$$F_f(T) ~\equiv~ -T\log Z_f(T) ~=~ -T\int \frac{d^3\mathbf{p}}{(2\pi)^3} \log(1+e^{-E_{\mathbf{p}}/T}) ~. \tag{2.6}$$

Using the identities in Eq. (2.2), the free-energy densities in Eqs. (2.5) and (2.6) can thus be rewritten in the forms

$$\begin{aligned} F_b(T) ~&=~ +\frac{T}{2}\int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \log\left[E^2_{\mathbf{p}} + 4\pi^2 n^2 T^2\right] ~+~ ... \\ F_f(T) ~&=~ -\frac{T}{2}\int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \log\left[E^2_{\mathbf{p}} + 4\pi^2 (n+1/2)^2 T^2\right] ~+~ ... \end{aligned} \tag{2.7}$$

Separately, let us also recall that the zero-point one-loop vacuum-energy density of a zero-temperature theory consisting of a single real quantum field of mass m in four uncompactified dimensions is given by

$$\Lambda ~\equiv~ \tfrac{1}{2}\,(-1)^F \int \frac{d^4p}{(2\pi)^4}~\log(p^2+m^2) \tag{2.8}$$

where $(-1)^F$ indicates the spacetime statistics of the quantum field ($= 1$ for a bosonic field, $= -1$ for a fermionic field). Moreover, if we imagine that the time dimension is compactified on a circle of radius R (so that the integral over p^0 can be replaced by a discrete sum), and if the quantum field in question is taken to be periodic around this compactification circle, then Eq. (2.8) takes the form

$$\Lambda \;=\; \tfrac{1}{2}\,\frac{1}{2\pi R}\,(-1)^F \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \log[\mathbf{p}\cdot\mathbf{p} + m^2 + n^2/R^2]\ . \tag{2.9}$$

By contrast, if the quantum field is taken to be anti-periodic around this circle, then Eq. (2.8) takes the alternate form

$$\Lambda \;=\; \tfrac{1}{2}\,\frac{1}{2\pi R}\,(-1)^F \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \log[\mathbf{p}\cdot\mathbf{p} + m^2 + (n+1/2)^2/R^2]\ . \tag{2.10}$$

Given these results, it is now possible to make the "temperature/radius" correspondence. Comparing Eq. (2.7) with Eqs. (3.13) and (3.14), we see that we can identify the *free-energy density* $F_{b,f}$ of a boson (fermion) in four spacetime dimensions at temperature T with the zero-temperature *vacuum-energy density* Λ of a boson (fermion) in four spacetime dimensions, where a (Euclidean) timelike dimension is compactified on a circle of radius $R \equiv 1/(2\pi T)$ about which the boson (fermion) is taken to be periodic (anti-periodic). This is the essence of the temperature/radius correspondence, connecting uncompactified theories at finite temperatures with compactified theories at zero temperature. In this correspondence, the Matsubara modes associated with finite temperatures are identified as the Kaluza-Klein modes associated with spacetime compactification. Moreover, these relations hold not only for theories in four dimensions, but for theories with arbitrary dimensionalities. These relations are summarized in Table 2.1.

These results, of course, are completely standard (see, *e.g.*, Ref. [9]), and there are many important theoretical details which we have omitted in this discussion. However, our purpose in providing this short review has been to emphasize three key points which will be crucial in the following:

Temperature	Geometry
temperature T	compactification radius $R \equiv (2\pi T)^{-1}$
$T = 0$	uncompactified theory
$T > 0$	compactified theory
D uncompactified dimensions, $T > 0$	$D-1$ uncompactified dimensions, $T = 0$
Free energy $F(T)$	Vacuum energy $\Lambda(R)$
Matsubara modes, masses $\sim nT$	KK modes, masses $\sim n/R$
bosons: $n \in \mathbb{Z}$	periodic compactification, $n \in \mathbb{Z}$
fermions: $n \in \mathbb{Z} + 1/2$	anti-periodic compactification, $n \in \mathbb{Z} + 1/2$

Table 2.1. The temperature/radius correspondence, wherein the free energy of a boson (fermion) in D uncompactified spacetime dimensions at temperature T can be identified as the the zero-temperature vacuum energy of a boson (fermion) in D spacetime dimensions with the (Euclidean) timelike dimension compactified on a circle of radius $R \equiv 1/(2\pi T)$ about which the bosonic (fermionic) field is taken to be periodic (anti-periodic).

- First, we observe that bosons contribute with modings $\in \mathbb{Z}$, while fermions contribute with modings $\in \mathbb{Z} + 1/2$. Likewise, bosons contribute to the free energy or vacuum energy with an overall + sign, while fermions contribute with an overall minus sign. This overall sign choice is therefore *correlated* with the above modings.

- Second, we see that the identification of bosons having integer modings and fermions having half-integer modings is the inescapable consequence of the different assumed quantization statistics (Bose-Einstein versus Fermi-Dirac) of our original underlying quantum field. Thus, our definition of "boson" and "fermion" in the discussion of the temperature/radius correspondence rests on quantization statistics alone (having nothing whatsoever to do with the Lorentz spin of the field in question), and indeed this connection between the assumed quantization statistics and the resulting modings and overall sign follows from a *mathematical identity*. As a result, this connection is also rigorous and inescapable.

- Finally, we point out that there are only two additional physics assumptions which have implicitly entered this derivation. On the thermal side, we have taken our partition functions as $Z \equiv \sum g_n e^{-E_n/T}$. This of course implicitly assumes a vanishing chemical potential; otherwise, we would use the grand canonical partition function instead. Likewise, on the geometric side, our expressions in Eqs. (2.8), (3.13), and (3.14) implicitly assume that there is no non-trivial gauge potential present. Otherwise, in the presence of a gauge potential, the momentum p^μ in these expressions would be replaced by the gauge-invariant kinematic momentum $\Pi^\mu \equiv p^\mu - \vec{q} \cdot \vec{A}^\mu$ where the vector signs indicate vectors in the root space of the gauge group.

We also emphasize that this temperature/radius correspondence has nothing whatsoever do with the spin-statistics theorem. The spin-statistics theorem is a relation between the quantization statistics of the field and its spacetime Lorentz spin. However, the above discussion is independent of the Lorentz spin, and only serves to correlate the quantization statistics of a given field with its thermal modings and with the sign of its overall contribution to the vacuum energy.

2.2.2 Extension to string theory

While this connection between temperature and geometry is well established in quantum field theory, at first glance it may seem surprising that it should hold in string theory. The reason is that strings (particularly closed strings) behave in very non-field-theoretic ways when a spacetime dimension is compactified. For example, upon spacetime compactification, closed strings accrue not only infinite towers of Kaluza-Klein "momentum" states, but also infinite towers of winding states. *A priori*, it is not clear what might be the thermal analogues of these winding states. Likewise, as a more general (but not unrelated) issue, string one-loop vacuum energies generally exhibit additional symmetries (*e.g.*, modular invariance) which transcend field-theoretic expectations. While the emergence of modular invariance is clearly understood for zero-temperature geometric compactifications, the need for modular invariance is perhaps less obvious from the thermal perspective in which one would simply write down a Boltzmann sum corresponding to each string state which survives the GSO projections.

Indeed, both of these issues tended to dominate the earliest discussions of string thermodynamics in the mid-1980's. Historically, they were first flashpoints which seemed to show apparent conflicts between the thermal and geometric approaches which had otherwise been consistent in quantum field theory. However, as is now understood through explicit studies of the bosonic string and the Type II superstring, there are ultimately no conflicts between these two approaches [2, 10]. Modular invariance emerges naturally upon relating the integral of the Boltzmann sum over the "strip" in the complex τ-plane to the integral of the partition function over the fundamental domain of the modular group [11], and likewise thermal windings emerge naturally as a consequence of modular invariance and can be viewed as artifacts arising from this mapping between the strip and the modular-group fundamental domain.

The net result, then, is that the temperature/radius correspondence can potentially survive intact, leading to the expectation that a string theory at temperature T in D uncompactified dimensions can be reformulated as a zero-temperature string theory of the same type in a spacetime with $D-1$ uncompactified dimensions. Moreover, it is also commonly assumed that this relation holds exactly as sketched in Sect. 2.2.1, namely that bosonic states accrue modings $\in \mathbb{Z}$ around the additional circle of compactification (commonly called the "thermal circle"), while fermionic states accrue modings $\in \mathbb{Z}+1/2$. Strictly speaking, this last requirement is commonly assumed to hold only for string states with zero windings. However, these are the assumptions that underlie string thermodynamics as it is currently practiced in the literature.

Given these observations, let us now review how the temperature/radius correspondence is generalized to the case of string theory. Because of its central role in determining the thermodynamic properties of the corresponding string theory, we shall focus on the calculation of the string thermal partition function $Z_{\text{string}}(\tau, T)$.

We begin by reviewing the case of a partition function for a string at zero temperature. For simplicity, in this chapter we shall restrict our attention to the case of closed strings: this includes not only bosonic strings, but also Type II superstrings as well as heterotic strings. In such cases, we then have

$$Z_{\rm model}(\tau) ~\equiv~ {\rm Tr}~(-1)^F ~\overline{q}^{H_R}\, q^{H_L} \tag{2.11}$$

where the trace is over the complete Fock space of states in the theory, weighted by a spacetime statistics factor $(-1)^F$. Here $q \equiv \exp(2\pi i\tau)$, and (H_R, H_L) denote the worldsheet energies for the right- and left-moving worldsheet degrees of freedom, respectively. For example, in the case of the bosonic string compactified to D spacetime dimensions, $Z_{\rm model}$ takes the general form

$$Z_{\rm model}(\tau) ~=~ \tau_2^{1-D/2}\, \frac{\overline{\Theta}^{26-D}\Theta^{26-D}}{\overline{\eta}^{24}\eta^{24}} \tag{2.12}$$

where the numerator $\overline{\Theta}^{26-D}\Theta^{26-D}$ schematically represents a sum over the $2(26-D)$-dimensional compactification lattice for left- and right-movers and where η represents the Dedekind η-function defined in Eq. (2.59).

Note that in general, $Z_{\rm model}$ is the quantity which appears in the calculation of the one-loop cosmological constant (vacuum energy density) of the model:

$$\Lambda^{(D)} ~\equiv~ -\,\tfrac{1}{2}\,\mathcal{M}^D \int_{\mathcal{F}} \frac{d^2\tau}{({\rm Im}\,\tau)^2} Z_{\rm model}(\tau) \tag{2.13}$$

where $\mathcal{M} \equiv M_{\rm string}/(2\pi)$ is the reduced string scale and where

$$\mathcal{F} ~\equiv~ \{\tau:~ |{\rm Re}\,\tau| \leq \tfrac{1}{2},~ {\rm Im}\,\tau > 0,~ |\tau| \geq 1\} \tag{2.14}$$

is the fundamental domain of the modular group. Of course, the quantity in Eq. (2.13) is divergent for the compactified bosonic string as a result of the physical bosonic-string tachyon.

Given this general form for the zero-temperature string partition function in Eq. (2.11), it is straightforward to develop its generalization to finite temperature. As dictated by the temperature/radius correspondence, finite-temperature effects can be incorporated [2, 10] by compactifying an extra (Euclidean) time dimension on a circle of radius $R = (2\pi T)^{-1}$. However, as also discussed above, for closed strings we must include not only "momentum" Matsubara states but also "winding" Matsubara states, as both types of states are necessary for the modular invariance of the underlying theory at finite temperature. As a result, a given zero-temperature string state will accrue not a single sum of Matsubara/Kaluza-Klein modes at finite temperature, but actually a *double sum* consisting of not only the Matsubara/Kaluza-Klein momentum modes, but also a corresponding set of winding modes.

The final expressions for our finite-temperature string partition functions $Z(\tau, T)$ must also be modular invariant, satisfying the constraint $Z(\tau, T) = Z(\tau + 1, T) = Z(-1/\tau, T)$. Since the temperature/radius correspondence requires that we consider momentum quantum numbers $n \in \mathbb{Z}$ as well as $n \in \mathbb{Z} + 1/2$, it turns out that modular invariance requires that we consider winding quantum numbers $w \in \mathbb{Z}$, both even and odd. As a result, the most general thermal string-theoretic partition function must take the form [12, 15, 13, 14]

$$\begin{aligned} Z_{\text{string}}(\tau, T) \;&=\; Z^{(1)}(\tau)\; \mathcal{E}_0(\tau, T) \;+\; Z^{(2)}(\tau)\; \mathcal{E}_{1/2}(\tau, T) \\ &\quad +\; Z^{(3)}(\tau)\; \mathcal{O}_0(\tau, T) \;+\; Z^{(4)}(\tau)\; \mathcal{O}_{1/2}(\tau, T)\;. \end{aligned} \tag{2.15}$$

Here $\mathcal{E}_{0,1/2}$ and $\mathcal{O}_{0,1/2}$ represent the thermal portions of the partition function, namely the double sums over appropriate combinations of thermal momentum and winding modes [12]. Specifically, the $\mathcal{E}_{0,1/2}$ functions include the contributions from even winding numbers w along with either integer or half-integer momenta n, while the $\mathcal{O}_{0,1/2}$ functions include the contributions from odd winding numbers w with either integer or half-integer momenta n. These functions are defined explicitly in Appendix A, which also serves to fix our precise notation and conventions. Likewise, the terms $Z^{(i)}$ $(i = 1, ..., 4)$ represent the traces over those subsets of the zero-temperature string states in Eq. (2.11) which accrue the corresponding thermal modings at finite temperature. For example, $Z^{(1)}$ represents a trace over those string states in Eq. (2.11) which accrue even thermal windings $w \in 2\mathbb{Z}$ and integer thermal momenta $n \in \mathbb{Z}$, and so forth. Modular invariance for Z_{string} as a whole is then achieved by demanding that each $Z^{(i)}$ transform exactly as does its corresponding $\mathcal{E}/\mathcal{O}$ function. These modular transformation properties are also listed explicitly in Appendix A.

In the $T \to 0$ limit, it is easy to verify that $\mathcal{O}_0$ and $\mathcal{O}_{1/2}$ each vanish while $\mathcal{E}_0, \mathcal{E}_{1/2} \to \mathcal{M}/T$. As a result, we find that

$$Z_{\text{string}}(T) \;\to\; \frac{\mathcal{M}}{T}\,\left[Z^{(1)} + Z^{(2)}\right] \qquad \text{as}\;\; T \to 0\;. \tag{2.16}$$

The divergent prefactor proportional to $1/T$ in Eq. (2.16) is a mere rescaling factor which reflects the effective change of the dimensionality of the theory in the $T \to 0$ limit. Specifically, this is an expected dimensionless volume factor which emerges as the spectrum of surviving Matsubara momentum states becomes continuous. However, we already know that Z_{model} in Eq. (2.11) is the partition function of the zero-temperature theory. Consequently, we see that we can relate Eq. (2.11) and (2.15) by identifying

$$Z_{\text{model}} \;=\; Z^{(1)} + Z^{(2)}\;. \tag{2.17}$$

For completeness, we observe that a similar simplification holds in the $T \to \infty$ limit: $\mathcal{E}_{1/2}$ and $\mathcal{O}_{1/2}$ each vanish while $\mathcal{E}_0, \mathcal{O}_0 \to T/(2\mathcal{M})$, whereupon the partition function of our thermal model reduces to

$$Z_{\text{string}}(\tau, T) \;\to\; \frac{T}{2\mathcal{M}}\,\left[Z^{(1)} + Z^{(3)}\right] \qquad \text{as}\;\; T \to \infty\;. \tag{2.18}$$

Once again, the divergent prefactor reflects a T-dualized (and $\mathbb{Z}_2$-orbifolded) volume factor which emerges as the spectrum of surviving Matsubara winding states becomes continuous.

We see, then, that the procedure for extrapolating a given zero-temperature string model to finite temperature is relatively straightforward. Any zero-temperature string model is described by a partition function Z_{model}, the trace over its Fock space. The remaining task is then simply to determine which states within Z_{model} are to accrue integer modings around the thermal circle, and which are to accrue half-integer modings. Those that are to accrue integer modings become part of $Z^{(1)}$, while those that are to accrue half-integer modings become part of $Z^{(2)}$. In this way, we are essentially decomposing Z_{model} in Eq.(2.17) into separate components $Z^{(1)}$ and $Z^{(2)}$. Once this is done, modular invariance alone determines the unique resulting forms for $Z^{(3)}$ and $Z^{(4)}$:

$$\begin{aligned} Z^{(3)} &= \tfrac{1}{2}\left[Z^{(1)}(-1/\tau) - Z^{(2)}(-1/\tau) + Z^{(1)}(-1/\tau+1) - Z^{(2)}(-1/\tau+1)\right] \\ Z^{(4)} &= \tfrac{1}{2}\left[Z^{(1)}(-1/\tau) - Z^{(2)}(-1/\tau) - Z^{(1)}(-1/\tau+1) + Z^{(2)}(-1/\tau+1)\right] \;. \end{aligned} \tag{2.19}$$

The final thermal partition function $Z_{\text{string}}(\tau,T)$ is then given in Eq. (2.15). In complete analogy to Eq. (2.13), we can then proceed to define the $(D-1)$-dimensional vacuum energy density

$$\Lambda^{(D-1)} \;\equiv\; -\tfrac{1}{2}\,\mathcal{M}^{D-1}\int_{\mathcal{F}} \frac{d^2\tau}{(\text{Im}\,\tau)^2} Z_{\text{string}}(\tau,T)\;, \tag{2.20}$$

whereupon the corresponding D-dimensional free-energy density $F(T)$ is given by

$$F(T) \;=\; T\,\Lambda^{(D-1)}\;. \tag{2.21}$$

A derivation of Eq. (2.21), starting from Eqs. (3.13) and (3.14), is sketched in Appendix B. Given this result, Eq. (2.16) implies that $F(T)\to\Lambda^{(D)}$ as $T\to 0$, where $\Lambda^{(D)}$ is defined in Eq. (2.13).

Once again, we emphasize that all of the results in this section are completely standard, and correspond to string thermodynamics as it is currently practiced in the string literature.

2.3 Temperature versus Geometry: A General Test of the Temperature/Radius Correspondence

Zero-temperature string theories are relatively well understood, and there are no unresolved issues concerning the zero-temperature partition functions Z_{model} corresponding to a given string model at zero temperature. Moreover, as we have sketched above, the extension to finite temperature is relatively straightforward, and relies on the temperature/radius correspondence as far as possible. *Indeed, the only remaining question is how to determine which states within $Z_{\rm model}$ are to accrue integer thermal modings (and thereby be included within $Z^{(1)}$), and which are to accrue half-integer modings (and thereby be included within $Z^{(2)}$).*

It is important to realize that if the temperature/radius correspondence is to hold, then we do not have much freedom in making this choice. Clearly, according to Eq. (2.17), we must have $Z^{(1)}+Z^{(2)} = Z_{\rm model}$. Of course, once $Z^{(1)}$ and $Z^{(2)}$ are determined, we find that $Z^{(3)}$ and $Z^{(4)}$ are also determined through Eq. (2.19). This then completely fixes the finite-temperature string partition function $Z_{\rm string}(\tau, T)$ in Eq. (2.15). *However, the temperature/radius correspondence then additionally demands that the resulting thermal partition function in Eq. (2.15) correspond to a bona-fide geometric compactification of the original zero-temperature model $Z_{\rm model}$ on a circle (or more technically speaking, a $\mathbb{Z}_2$ orbifold of a circle) of radius $R \equiv (2\pi T)^{-1}$.* Thus, if we assume the validity of the temperature/radius correspondence, this last constraint becomes a highly non-trivial requirement that can be exploited in order to accept or reject different possible choices for $Z^{(1)}$ and $Z^{(2)}$.

2.3.1 The standard Boltzmann approach

In general, a given string model will give rise to states which are spacetime bosons as well as states which are spacetime fermions. In making this statement, we are identifying "bosons" and "fermions" on the basis of their spacetime Lorentz spins. (By the spin-statistics theorem, this is equivalent to identifying these states on the basis of their Bose-Einstein or Fermi-Dirac quantizations. For a proof of the spin-statistics theorem in string theory, see Appendix 2.E.) As a result, we can always decompose $Z_{\rm model}$ into separate contributions from spacetime bosons and spacetime fermions:

$$Z_{\text{model}} = Z_{\text{boson}} + Z_{\text{fermion}} . \tag{2.22}$$

Given this decomposition, the standard approach which is taken in the string literature is to follow our field-theoretic expectations, identifying

$$\begin{aligned} Z^{(1)} &= Z_{\text{boson}} \\ Z^{(2)} &= Z_{\text{fermion}} . \end{aligned} \tag{2.23}$$

This makes sense, since $Z^{(1)}$ corresponds to the $\mathcal{E}_0$ sector which accrues integer thermal Matsubara modes, while $Z^{(2)}$ corresponds to the $\mathcal{E}_{1/2}$ sector which accrues half-integer thermal Matsubara modes. Indeed, since both $Z^{(1)}$ and $Z^{(2)}$ correspond to thermal sectors which include states with zero thermal winding number, this identification seems almost unavoidable.

Note that the choice in Eq. (2.23) is the one which reproduces the standard Boltzmann sum for the states in the string spectrum. Although this is not obvious from the modular-invariant partition function in Eq. (2.15), this becomes clear upon transforming to the so-called "strip" representation [11, 16]. We shall therefore refer to Eq. (2.23) as the Boltzmann choice.

The question, then, is whether the Boltzmann choice is consistent with a *geometric* interpretation. Specifically, if we make the choices for $Z^{(1)}$ and $Z^{(2)}$ given in Eq. (2.23), we can ask whether the resulting thermal partition function given Eqs. (2.15) and (2.19) corresponds to a self-consistent geometric compactification of the original zero-temperature model specified in $Z_{\text{model}} = Z^{(1)} + Z^{(2)}$.

2.3.2 The geometric approach

At first glance, it may seem that *any* function of the form given in Eq. (2.15) will correspond to a self-consistent compactification of a closed string on a thermal circle.

Certainly the general form of Eq. (2.15) resembles what one would expect from such a compactification. However, *not every modular-invariant function $Z(\tau, T)$ is the trace of the Fock space of a self-consistent string model.* In other words, the space of modular-invariant functions is larger than the space of actual string partition functions.[1] As a result, the issue of whether Eq. (2.15) corresponds to an actual self-consistent geometric compactification has teeth, and we must develop a *test* which will help us determine when this is the case.

We shall develop such a test as follows. The process by which a string model can be compactified is well understood, for it is well known how to compactify a string model while preserving its self-consistency. Therefore, we shall attempt to derive a partition function of the form in Eq. (2.15) solely by following standard geometric compactification techniques and taking only those allowed steps which preserve the worldsheet self-consistency of the underlying theory. We will allow $Z^{(1)}$ and $Z^{(2)}$ to remain arbitrary in this discussion, so that we ultimately obtain a general understanding of which choices for $Z^{(1)}$ and $Z^{(2)}$ can consistently arise from geometric compactifications.

The following discussion follows the mathematical treatment in Refs. [12, 20, 19], suitably T-dualized in order to apply to temperature rather than geometric radius. Let us suppose that we begin with a D-dimensional zero-temperature closed string model whose one-loop partition function is given by $Z(\tau)$. The first step in the thermal construction is to compactify this theory on circle of radius R. At this stage, we then have a thermal string partition function $Z_{\text{therm}}(\tau, T)$ of the form

$$Z_{\text{therm}}(\tau, T) \;\equiv\; Z(\tau)\, Z_{\text{circ}}(\tau, T) \tag{2.24}$$

where the extra factor Z_{circ} represents the double summation over integer Matsubara

[1] A reader who doubts this assertion can consult, for example, some of the early string literature on the cosmological-constant problem. Various modular-invariant functions of the correct form were proposed which contained a so-called “Atkin-Lehner symmetry” [17] and which therefore led to a vanishing one-loop vacuum energy without exhibiting spacetime supersymmetry. Ultimately, however, it was proven [18] that these functions could not be the partition functions of self-consistent string models. Many other examples of this phenomenon exist as well.

momentum and winding modes given in Eq. (2.55). However, at this stage in the construction, we see that each of the states within $Z(\tau)$ is multiplied by the same thermal spectrum of integer momentum and winding modes within Z_{circ}. The next step, therefore, is to break this degeneracy, allowing some states within $Z(\tau)$ to continue to have integer Matsubara modes (so that they are periodic around the thermal circle) while other states within $Z(\tau)$ have *half-integer* modings (so that they are anti-periodic around the thermal circle).

In string theory, the only way to accomplish this in a self-consistent geometric manner is by twisting or *orbifolding* the compactified theory in Eq. (2.24). But what orbifold should we choose? Clearly, one piece of this must be a $\mathbb{Z}_2$ operator that distinguishes between the states which are to accrue periodic (integer) modings and those that are to accrue anti-periodic (half-integer) modings. We shall generally let Q denote such an operator. For example, we would take $Q = (-1)^F$ if we wished to reproduce the Boltzmann choice in Eq. (2.23), but we shall keep our discussion general for now except to remark that Q should at least *contain* a factor of $(-1)^F$ in order to break whatever supersymmetry might have existed at zero temperature. However, regardless of the precise form of Q, we will also need to couple Q with an operator that can distinguish between between integer and half-integer thermal momenta. As we shall see, such an operator is given by $\mathcal{T} : y \to y + \pi R$, where y is the (T-dual) coordinate along the compactified dimension. This is nothing but a shift around half the circumference of the (dualized) thermal circle, so that the states which are invariant under $\mathcal{T}$ are those with even winding numbers. This will then necessarily re-introduce states with odd winding numbers in the twisted sectors, along with states having half-integer momentum numbers.

Given these operators, the final step in our procedure is to orbifold the circle-compactified theory in Eq. (2.24) by the $\mathbb{Z}_2$ product operator $\mathcal{T}Q$. What does this do to our partition function? While Q acts on the original non-thermal component $Z(\tau)$, the operator $\mathcal{T}$ acts on the thermal sum $Z_{\text{circ}}(\tau, T)$. Since states contributing to

$Z_{\rm circ}$ with even (odd) values of thermal momentum numbers are even (odd) under $\mathcal{T}$, we distinguish the specific values of momentum and winding numbers by introducing the four thermal functions $\mathcal{E}_{0,1/2}$ and $\mathcal{O}_{0,1/2}$ defined in Eq. (2.56). Since $Z_{\rm circ} = \mathcal{E}_0 + \mathcal{O}_0$, our original (untwisted) thermal partition function in Eq. (2.24) can be rewritten as

$$Z^{+}_{\rm therm,+} \;=\; Z^{+}_{+}\,(\mathcal{E}_0 + \mathcal{O}_0) \tag{2.25}$$

where $Z^{+}_{+}(\tau) \equiv Z(\tau)$. Therefore, in order to project onto the states invariant under $\mathcal{T}Q$, we add to Eq. (2.25) the contributions from the projection sector

$$Z^{-}_{\rm therm,+} \;=\; Z^{-}_{+}\,(\mathcal{E}_0 - \mathcal{O}_0) \tag{2.26}$$

where Z^{-}_{+} is the Q-projection sector for the non-thermal contribution Z^{+}_{+}. In the usual fashion, modular invariance then requires us to add the contribution from the twisted sector

$$Z^{+}_{\rm therm,-} \;=\; Z^{+}_{-}\,(\mathcal{E}_{1/2} + \mathcal{O}_{1/2}) \tag{2.27}$$

as well as its corresponding projection sector

$$Z^{-}_{\rm therm,-} \;=\; Z^{-}_{-}\,(\mathcal{E}_{1/2} - \mathcal{O}_{1/2})~. \tag{2.28}$$

The net result of the orbifold procedure, then, is a $(D-1)$-dimensional thermal string model with total partition function

$$\begin{aligned} & Z_{\rm string}(\tau,T) \\ =\; & \frac{1}{2}\,\left(Z^{+}_{\rm therm,+} + Z^{-}_{\rm therm,+} + Z^{+}_{\rm therm,-} + Z^{-}_{\rm therm,-}\right) \\ =\; & \frac{1}{2}\left\{ (Z^{+}_{+} + Z^{-}_{+})\,\mathcal{E}_0 \;+\; (Z^{+}_{-} + Z^{-}_{-})\,\mathcal{E}_{1/2} \;+\; (Z^{+}_{+} - Z^{-}_{+})\,\mathcal{O}_0 \;+\; (Z^{+}_{-} - Z^{-}_{-})\,\mathcal{O}_{1/2} \right\}~. \end{aligned} \tag{2.29}$$

In deriving the result in Eq. (2.29), we have followed only standard steps (specifically, compactification and orbifolding) which ensure the geometric self-consistency of our underlying string model. What we have learned, however, is that the individual $Z^{(i)}$

factors in Eq. (2.15) are not completely arbitrary. Indeed, comparing Eqs. (2.15) and (2.29), we see that it must be possible to express these factors in the form

$$\begin{aligned} Z^{(1)} &= \tfrac{1}{2}\left(Z_+^+ + Z_+^-\right) \\ Z^{(2)} &= \tfrac{1}{2}\left(Z_-^+ + Z_-^-\right) \\ Z^{(3)} &= \tfrac{1}{2}\left(Z_+^+ - Z_+^-\right) \\ Z^{(4)} &= \tfrac{1}{2}\left(Z_-^+ - Z_-^-\right) \end{aligned} \tag{2.30}$$

where Z_+^+ is the partition function of a self-consistent string model in D dimensions, and where the $\pm$ projections correspond to a legitimate $\mathbb{Z}_2$ orbifold symmetry of that model. Any other relation between the different $Z^{(i)}$ factors will render Eq. (2.15) inconsistent from a geometric perspective.

We can sharpen this result even further by examining the $T \to 0$ and $T \to \infty$ limits of Eq. (2.29). For closed strings, each of these endpoints should reproduce a D-dimensional [rather than $(D-1)$-dimensional] theory. Given the results in Eq. (2.30), we can exploit the limits in Eqs. (2.17) and (2.18) in order to derive these "endpoint" models:

$$\begin{aligned} T \to 0: \qquad & Z_{\text{model}} = Z^{(1)} + Z^{(2)} = \tfrac{1}{2}(Z_+^+ + Z_+^- + Z_-^+ + Z_-^-) \\ T \to \infty: \qquad & Z'_{\text{model}} = Z^{(1)} + Z^{(3)} = Z_+^+ \ . \end{aligned} \tag{2.31}$$

However, we see from Eq. (2.31) that the $T \to 0$ endpoint model is nothing but the Q-orbifold of the $T \to \infty$ endpoint model. Moreover, since Q is a $\mathbb{Z}_2$ action satisfying $Q^2 = \mathbf{1}$, the reverse is also true: the $T \to \infty$ endpoint model is Q-orbifold of the $T \to 0$ endpoint model. Thus, we see that the $T \to 0$ and $T \to \infty$ endpoint models are both D-dimensional, and must be $\mathbb{Z}_2$ orbifolds of each other with respect to the $\mathbb{Z}_2$ action Q. As a result, assuming that such a geometric compactification exists, we see that the thermal partition function in Eq. (2.15) can be viewed as mathematically *interpolating* between the one zero-temperature string model at $T = 0$ and a *different* zero-temperature string model as $T \to \infty$. These two models can then be related directly in D dimensions through the action of the $\mathbb{Z}_2$ orbifold operator Q.

This is a general result, so it bears repeating: A self-consistent geometric realization requires that *all D-dimensional thermal models be* $(D-1)$*-dimensional interpolating models, with the temperature* T *serving as an interpolating parameter. As* $T \to 0$*, we obtain a D-dimensional string model* M_1*; this is identified as the zero-temperature string model whose thermal extrapolation we have constructed. By contrast, as* $T \to \infty$*, we obtain a different D-dimensional string model* M_2 *which must be a* $\mathbb{Z}_2$ *orbifold of* M_1*.* This situation is illustrated in Fig. 2.1.

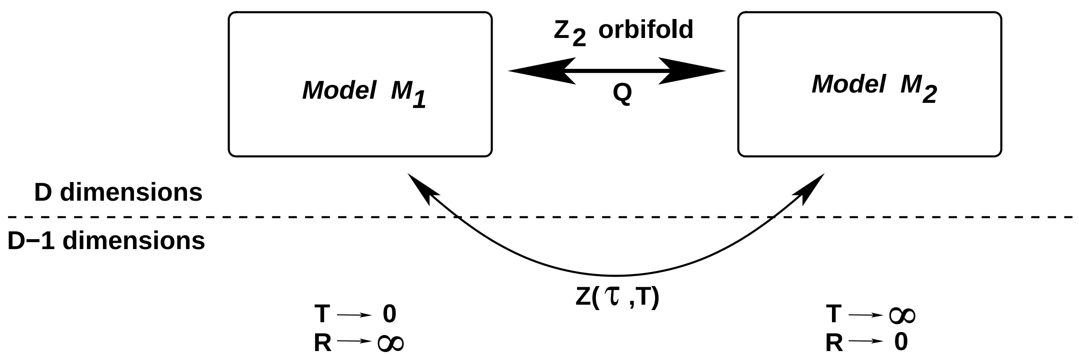

FIGURE 2.1. A self-consistent geometric realization for the $(D-1)$-dimensional thermal partition function in Eq. (2.15) requires that it *interpolate* between the partition functions of two self-consistent D-dimensional string models which are orbifolds of each other with respect to an appropriate $\mathbb{Z}_2$ action Q.

It is also easy to understand these results physically. The requirement that our thermal model be a bona-fide *interpolating* model is nothing but the requirement that the temperature T — like the radius R — be a free, adjustable *modulus* of the theory. As long as this holds, our theory will remain consistent at the string (worldsheet) level for any value of temperature T. Indeed, modular invariance, in and of itself, is not enough.

A comment on semantics is in order here. Strictly speaking, in the $T \to \infty$ limit we obtain a $(D-1)$-dimensional degenerate (*i.e.*, zero-radius) model M_2 which is actually only T-dual to a D-dimensional model. Thus, if M_2 is the $T \to \infty$ limit of our $(D-1)$-dimensional thermal interpolating model, then we should more correctly state that our $(D-1)$-dimensional thermal model interpolates between the D-dimensional models M_1

and $\tilde{M}_2$, where $\tilde{M}_2$ is the T-dual of M_2. In some sense, this distinction is only a matter of semantics, having to do with the naming of the $T \to \infty$ endpoint of the interpolation; moreover, for closed strings we should properly regard both M_2 and $\tilde{M}_2$ as being D-dimensional since they each have a continuous spectrum of states associated with the formerly compactified dimension. For simplicity, therefore, we shall continue to refer to such an interpolating model as connecting M_1 and M_2 in the remainder of this chapter. However, it is important to note that it is M_2 (and not $\tilde{M}_2$) which must be the Q-orbifold of M_1.

Similarly, we observe that the $\mathcal{E}/\mathcal{O}$ functions satisfy the identities given in Eq. (2.58). As a result, for every partition function of the form in Eq. (2.15), there is another in which we replace $a \to 2/a$ and exchange $Z^{(2)}$ and $Z^{(3)}$. This has the net effect of preserving the interpolation, but exchanging the $T \to 0$ and $T \to \infty$ limits. However, we emphasize that these are not the same thermal model: while the first may be regarded as the finite-temperature extension of model M_1, the second can at best only be regarded as a possible finite-temperature extension of M_2.

Finally, we emphasize that our interpolating functions must take the form discussed above even though tachyons might appear as the temperature is dialed from $T = 0$ to $T \to \infty$. This is in fact guaranteed to happen if the endpoint model M_2 is itself non-supersymmetric and tachyonic, and we shall see an explicit example of this below. The appearance of such a tachyon at a critical temperature T_H is precisely what triggers the Hagedorn transition [21, 13], and the theory beyond this temperature is expected to enter a new phase whose physics need not be captured by these thermal extrapolations and interpolations. However, our interest here is on the physics of the system prior to the Hagedorn transition, and our claim is that this physics must be described by functions which mathematically interpolate all the way from $T = 0$ to $T \to \infty$ in the manner described above. In other words, even though we might only be interested in describing the thermodynamic properties of strings within the temperature range $0 \le T \le T_H$, the corresponding thermal partition functions will be consistent with the temperature/radius correspondence only if they exhibit certain mathematical properties in the $T = 0$ and $T \to \infty$ limits.

Given these results, we see that we are now able to apply a sharp test to determine whether a particular choice for $Z^{(1)}$ and $Z^{(2)}$ in Eqs. (2.15) is consistent with a geometric realization, as required by the temperature/radius correspondence: this choice must lead to a thermal partition function which interpolates between two D-dimensional zero-temperature models, M_1 and M_2, which are directly related to each other through a self-consistent $\mathbb{Z}_2$ orbifold action Q. If so, then our choice for $Z^{(1)}$ and $Z^{(2)}$ is consistent with a geometric realization; otherwise, it is not. Moreover, we expect that this orbifold action Q should break supersymmetry. Otherwise, the entire interpolation will be supersymmetric for all values of the temperature T.

This is the essence of the test which we shall now apply. Specifically, in order to test whether the traditional Boltzmann sum is consistent with the temperature/radius correspondence, we shall assume the choice in Eq. (2.23) and test whether

$$\begin{aligned} M_1 : \qquad & Z_{\text{model}} = Z_{\text{boson}} + Z_{\text{fermion}} \\ M_2 : \qquad & Z'_{\text{model}} = Z_{\text{boson}} + \\ & \tfrac{1}{2}\Big[Z_{\text{boson}}(-1/\tau) - Z_{\text{fermion}}(-1/\tau) \\ & \quad + Z_{\text{boson}}(-1/\tau+1) - Z_{\text{fermion}}(-1/\tau+1)\Big] \end{aligned} \tag{2.32}$$

correspond to models which are *bona-fide* Z_2 orbifolds of each other.

2.4 Applying the test: The bosonic string

For completeness, we shall begin by applying this test to the trivial (and somewhat null) case of the bosonic string. Compactified to D dimensions, the bosonic string at zero temperature has a partition function of the form

$$Z = Z^{(24)}_{\text{boson}} \, \overline{\Theta}^{26-D} \Theta^{26-D} \tag{2.33}$$

where $Z^{(24)}_{\rm boson}$ denotes the contribution from 24 transverse coordinate bosons, as defined in Eq. (2.60), and where $\overline{\Theta}^{26-D}\Theta^{26-D}$ schematically denotes a summation over a $(26-D, 26-D)$-dimensional self-dual Lorentzian compactification lattice, as in Eq. (2.12). Note that $Z^{(24)}_{\rm boson}$ and $\overline{\Theta}^{26-D}\Theta^{26-D}$ are each individually modular invariant.

If we follow the Boltzmann prescription in Eq. (2.23), the fact that all states of the bosonic string are spacetime bosons leads us to make the identifications

$$\begin{aligned} Z^{(1)} &= Z^{(24)}_{\rm boson}\,\overline{\Theta}^{26-D}\Theta^{26-D} \\ Z^{(2)} &= 0~. \end{aligned} \tag{2.34}$$

Using Eq. (2.19), we then find that

$$\begin{aligned} Z^{(3)} &= Z^{(24)}_{\rm boson}\,\overline{\Theta}^{26-D}\Theta^{26-D} \\ Z^{(4)} &= 0~, \end{aligned} \tag{2.35}$$

whereupon we obtain the thermal interpolating function

$$Z_{\rm string}(\tau,T) ~=~ Z^{(24)}_{\rm boson}\,\overline{\Theta}^{26-D}\Theta^{26-D}\,(\mathcal{E}_0+\mathcal{O}_0)~. \tag{2.36}$$

Note that $\mathcal{E}_0+\mathcal{O}_0$ is nothing but $Z_{\rm circ}$, as defined in Eq. (2.55).

Of course, Eq. (2.36) is the correct thermal partition function for the bosonic string. Due to the absence of spacetime fermions, the only thermal momentum and winding modes that are needed have integer quantum numbers. In other words, for special case of the bosonic string, we are essentially compactifying on a true thermal circle rather than on a $\mathbb{Z}_2$ orbifold of this circle.

The presence of the factor $\mathcal{E}_0+\mathcal{O}_0$ in this special case already assures us that this partition function corresponds to a true geometric compactification. However, it is easy to apply the test we have developed above in order to verify this. Since $Z^{(1)}=Z^{(3)}$, we find from Eq. (2.31) that the two endpoints of our interpolation are actually identical and have the same partition function.[1] This corresponds to the orbifold choice $Q=\mathbf{1}$, which is certainly a self-consistent (though trivial) orbifold choice for the bosonic string.

[1]The reader might be disturbed by the fact that the formalism outlined in Sect. 2.3 appears to give

We thus conclude that the temperature/radius correspondence is valid for the case of the bosonic string, in agreement with standard results. In other words, for the bosonic string, the traditional Boltzmann sum can be consistently reformulated as a geometric compactification.

2.5 Applying the test: Type II superstrings

We now proceed to apply this test to the more complex case of the ten-dimensional Type II superstrings. For concreteness, we shall focus on the (chiral) Type IIB string; the case of the (non-chiral) Type IIA string proceeds in exactly the same manner.

As indicated in Appendix A, the Type IIB string at zero temperature has the partition function

$$Z_{\rm IIB} ~=~ Z^{(8)}_{\rm boson}\, (\overline{\chi}_V - \overline{\chi}_S)\, (\chi_V - \chi_S) ~. \tag{2.37}$$

where $Z^{(8)}_{\rm boson}$ denotes the contribution from the eight worldsheet bosons, and where the contributions from the worldsheet fermions are written in terms of the characters of the transverse $SO(8)$ Lorentz group, Since χ_V and χ_S (and their anti-holomorphic counterparts) are respectively the vectorial and spinorial chiral characters of the transverse $SO(8)$ Lorentz group, states contributing to the terms $\chi_V\chi_V$ and $\chi_S\chi_S$ are spacetime bosons, while states contributing to the terms $\chi_V\chi_S$ and $\chi_S\chi_V$ are spacetime fermions. If we apply the standard Boltzmann prescription in Eq. (2.23), we are therefore led to

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an extra factor of two for the partition function of the bosonic string in the $T\to\infty$ limit, as compared with the $T=0$ limit. However, this arises because the formalism in Sect. 2.3 implicitly assumes that we are dealing with $\mathbb{Z}_2$ orbifolds of the thermal circle, and not the thermal circle itself. Indeed, it was for this reason that that the extra factor of two appearing in Eq. (2.18) was not retained into the definition of $Z'_{\rm model}$ appearing in Eq. (2.31). However, for the special case of the bosonic string, our "orbifold" action is nothing but the identity. Consequently, in this case the factor of two in Eq. (2.17) should properly be retained in Eq. (2.31), thereby cancelling the extra factor of two that would have appeared in $Z'_{\rm model}$ due to the equality of $Z^{(1)}$ and $Z^{(3)}$. Note that this issue only arises in theories without spacetime fermions, such as the bosonic string; in all other cases, Q must correspond to actual $\mathbb{Z}_2$ orbifolds, and all factors of two in Sect. 2.3 are correct as written.

identify

$$\begin{aligned} Z^{(1)} &= Z^{(8)}_{\rm boson}\,(\;\;\overline{\chi}_V\chi_V + \overline{\chi}_S\chi_S) \\ Z^{(2)} &= Z^{(8)}_{\rm boson}\,(-\overline{\chi}_V\chi_S - \overline{\chi}_S\chi_V)~. \end{aligned} \tag{2.38}$$

As required by the Boltzmann prescription, this decomposition places the bosonic states within $Z^{(1)}$ and the fermionic states within $Z^{(2)}$. Given Eq. (2.38), we then find that

$$\begin{aligned} Z^{(3)} &= Z^{(8)}_{\rm boson}\,(\;\;\overline{\chi}_I\chi_I + \overline{\chi}_C\chi_C) \\ Z^{(4)} &= Z^{(8)}_{\rm boson}\,(-\overline{\chi}_I\chi_C - \overline{\chi}_C\chi_I)~, \end{aligned} \tag{2.39}$$

leading to the final thermal partition function

$$\begin{aligned} Z_{\rm string}(\tau,T) ~=~ Z^{(8)}_{\rm boson} \times \bigl\{ \quad & [\overline{\chi}_V\chi_V + \overline{\chi}_S\chi_S] ~~ \mathcal{E}_0 \\ - ~& [\overline{\chi}_V\chi_S + \overline{\chi}_S\chi_V] ~~ \mathcal{E}_{1/2} \\ + ~& [\overline{\chi}_I\chi_I + \overline{\chi}_C\chi_C] ~~ \mathcal{O}_0 \\ - ~& [\overline{\chi}_I\chi_C + \overline{\chi}_C\chi_I] ~~ \mathcal{O}_{1/2} ~~\bigr\}~. \end{aligned} \tag{2.40}$$

This is the result of the Boltzmann prescription. Let us now apply our test from Sect. 2.3 to see if this result is also consistent with a geometric interpolation. Given the results in Eqs. (2.38) and (2.39), we find that the $T\to\infty$ limit of the interpolation in Eq. (3.42) is given by

$$Z ~=~ Z^{(8)}_{\rm boson}\,(\overline{\chi}_I\chi_I + \overline{\chi}_V\chi_V + \overline{\chi}_S\chi_S + \overline{\chi}_C\chi_C)~. \tag{2.41}$$

However, this is nothing but the partition function of the non-supersymmetric Type 0B string! Moreover, as we know, the Type 0B string is indeed $\mathbb{Z}_2$ orbifold of the Type IIB string; in this case, the relevant orbifold action is nothing but $Q = (-1)^F$ where F denotes the spacetime fermion number.

A similar situation emerges for the Type IIA string: we simply replace $\chi_S \leftrightarrow \chi_C$ for the left-moving characters throughout the above expressions. The Type IIA thermal extrapolation is therefore one which interpolates between the Type IIA string at $T = 0$ and the Type 0A string as $T \to \infty$.

We therefore conclude that the temperature/radius correspondence holds for the ten-dimensional Type II strings, as expected. Moreover, we make three critical observations:

- First, we observe that the $T \to \infty$ endpoint of the interpolation, namely the Type 0B string, is non-supersymmetric. As a result, even though the Type IIB theory is supersymmetric at $T = 0$, we see that the theory becomes non-supersymmetric for all $T > 0$. This is entirely expected, since thermal effects treat bosons and fermions differently and thereby break whatever supersymmetry might have existed at zero temperature. This is also reflected in the fact that the $Q = (-1)^F$ orbifold breaks supersymmetry explicitly.

- Second, we observe that the Type 0B string is *tachyonic*. Indeed, this tachyon is the lowest-energy state contributing to the $\overline{\chi}_I \chi_I$ term within $Z^{(3)}$, and corresponds to the $(H_R, H_L) = (-1/2, -1/2)$ ground state of the Type II superstring. Since there is no such tachyon in the $T = 0$ theory, this implies that our interpolating thermal theory in Eq. (3.42) must develop a tachyon beyond some critical temperature T_H. In other words, there is a thermal state (in this case, a thermal winding mode) in the interpolating theory which is massive for relatively small temperatures, but which becomes massless at a critical temperature T_H before becoming the Type 0B tachyon as $T \to \infty$. Of course, this is nothing but the Hagedorn phenomenon, with the appearance of a new massless state in the theory triggering a phase transition at the critical temperature T_H.

- Finally, we observe that the interpolation in Eq. (3.42) has the expected *structure*. All spacetime bosonic (fermionic) states contribute positively (negatively) to the partition function. Likewise, bosonic (fermionic) states only appear in sectors with integer (half-integer) thermal momentum modings. Indeed, this holds true not only in the $\mathcal{E}$ sectors (which contain states without non-trivial thermal windings), but also in the $\mathcal{O}$ sectors (where non-trivial thermal windings are present).

2.6 Applying the Test: Heterotic Strings

Let us now turn to the case of the ten-dimensional supersymmetric $SO(32)$ heterotic string. As indicated in Appendix A, this string has the zero-temperature partition function

$$Z_{SO(32)} \;=\; Z^{(8)}_{\text{boson}}\,(\overline{\chi}_V - \overline{\chi}_S)\,(\chi_I + \chi_S) \tag{2.42}$$

where the contributions from the right-moving worldsheet fermions are written in terms of the barred characters $\overline{\chi}_i$ of the transverse $SO(8)$ Lorentz group while the contributions from the left-moving (internal) worldsheet degrees of freedom are now written in terms of the unbarred characters χ_i of the internal $SO(32)$ gauge group. States which are spacetime bosons or fermions thus give contributions proportional to $\overline{\chi}_V$ or $\overline{\chi}_S$, respectively, whereupon the usual Boltzmann prescription instructs us to identify

$$\begin{aligned} Z^{(1)} &= \;\;\, Z^{(8)}_{\text{boson}}\,\overline{\chi}_V\,(\chi_I + \chi_S) \\ Z^{(2)} &= -Z^{(8)}_{\text{boson}}\,\overline{\chi}_S\,(\chi_I + \chi_S)\;. \end{aligned} \tag{2.43}$$

Modular invariance then dictates

$$\begin{aligned} Z^{(3)} &= -Z^{(8)}_{\text{boson}}\,\overline{\chi}_C\,(\chi_I + \chi_S) \\ Z^{(4)} &= \;\;\, Z^{(8)}_{\text{boson}}\,\overline{\chi}_I\,(\chi_I + \chi_S)\;. \end{aligned} \tag{2.44}$$

whereupon we see that the Boltzmann prescription leads to the $SO(32)$ heterotic thermal partition function

$$\begin{aligned} Z(\tau,T) \;=\; Z^{(8)}_{\text{boson}} \times \Big\{ \quad & \overline{\chi}_V\;(\chi_I + \chi_S)\;\;\mathcal{E}_0 \\ - \;& \overline{\chi}_S\;(\chi_I + \chi_S)\;\;\mathcal{E}_{1/2} \\ - \;& \overline{\chi}_C\;(\chi_I + \chi_S)\;\;\mathcal{O}_0 \\ + \;& \overline{\chi}_I\;(\chi_I + \chi_S)\;\;\mathcal{O}_{1/2}\;\Big\}\;. \end{aligned} \tag{2.45}$$

This is indeed the standard result in the string literature [13], and the analysis of the thermodynamics of heterotic strings traditionally flows from this result. Indeed, this result is modular invariant, and bosonic (fermionic) states contribute to the partition function with proper overall signs and with the correct modings in the $\mathcal{E}$ sectors.

Despite these successes, we shall now demonstrate that the temperature/radius correspondence actually fails for this expression. In and of itself, this does not necessarily imply that anything is wrong with the expression in Eq. (3.34); this is, indeed, the correct Boltzmann thermal sum which would follow from the $SO(32)$ string in ten dimensions. It is simply our claim that result is inconsistent with the temperature/radius correspondence, *i.e.*, that the temperature/radius correspondence fails to hold in this case.

To demonstrate this fact, we shall apply the test we developed in Sect. 2.3. Specifically, while the $T = 0$ limit of Eq. (3.34) reproduces the partition function of the original $SO(32)$ heterotic theory in Eq. (3.5), the $T \to \infty$ limit of Eq. (3.34) yields the expression

$$Z'_{SO(32)} \;=\; Z^{(8)}_{\rm boson}\,(\overline{\chi}_V - \overline{\chi}_C)\,(\chi_I + \chi_S)\ . \tag{2.46}$$

This expression is the partition function of the same model with which we started at zero temperature, only now involving spacetime spinors of opposite chirality. Given this, we are then led to ask: *Is there a self-consistent* $\mathbb{Z}_2$ *orbifold* Q *in ten dimensions which relates the supersymmetric* $SO(32)$ *heterotic theory to a chirality-flipped version of itself?*

It is our claim that no such orbifold exists. In fact, we shall demonstrate that no suitable orbifold exists that can accomplish this feat *even if we remove the requirement that the resulting model be chirality-flipped*. In other words, the flipping of the chirality is not the stumbling block, and can be safely omitted if needed.

At first glance, it might seem that the required orbifold continues to be $Q = (-1)^F$ where F represents spacetime fermion number. However, as we shall implicitly demonstrate, this does not represent a consistent orbifold choice for the ten-dimensional supersymmetric $SO(32)$ heterotic string.

In a nutshell, our argument is that no self-consistent orbifold in ten dimensions can simultaneously break supersymmetry (as required for compatibility with our thermal extrapolation) and yet also preserve it (as required in order to reproduce a supersymmetric ten-dimensional theory in the $T \to \infty$ limit). However, this argument is rather subtle, since (as we shall see) such a phenomenon can indeed occur in dimensions less than ten. Indeed, the reason for the subtlety is that the orbifold which acts to produce the nine-dimensional interpolation is actually $\mathcal{T}Q$, as discussed in Sect. 2.3.1, while the orbifold which directly relates the ten-dimensional endpoints is merely Q. We shall therefore develop our argument in several steps.

Let us begin by understanding what the orbifolding action of Q would need to do in order to relate the supersymmetric $SO(32)$ heterotic theory to (a chirality-flipped version of) itself.

In general, the act of orbifolding projects certain states out of the spectrum from untwisted sectors. However, the act of orbifolding also simultaneously introduce new twisted sectors from which additional states may emerge. It will be sufficient for our argument to focus on one particular state in the $SO(32)$ heterotic string: the gravitino. In order to relate the supersymmetric $SO(32)$ heterotic theory to (a chirality-flipped version of) itself, our Q orbifold must project out the gravitino that appears in the original $SO(32)$ heterotic string. However, our orbifolding procedure must also somehow restore the (opposite-chirality) gravitino from a twisted sector. This is necessary so that the net result of the orbifolding procedure can be the (chirality-flipped) supersymmetric $SO(32)$ model.

At first glance, one might question whether it is truly necessary that our orbifold Q project out the gravitino from the $T = 0$ theory: after all, both endpoints of the heterotic interpolation are supersymmetric. However, it is easy to see that any orbifold Q which aspires to reproduce Eq. (3.34) must ultimately project out whatever gravitino state might exist in the zero-temperature theory, placing this state within $Z^{(2)}$ rather than $Z^{(1)}$. The reason is simple: If the gravitino were invariant (preserved) by the Q orbifold, then it would contribute within $Z^{(1)}$ rather than $Z^{(2)}$. But among the thermal excitations for states within $Z^{(1)}$ are those with zero thermal momentum and winding modes. The masses of such modes are not affected by the increase in temperature above zero, and therefore any massless state which resides within $Z^{(1)}$ at $T = 0$ will continue to remain massless for all non-zero temperatures. Thus, if the gravitino were invariant under Q, then this gravitino would exist in the spectrum and remain massless not only for $T = 0$, but for the entire interpolation. In other words, our entire interpolation would necessarily be supersymmetric for all temperatures! But we can easily verify that Eq. (3.34) is non-supersymmetric for all intermediate temperatures $0 < T < \infty$. This is nothing but a reflection of the fact that the Boltzmann prescription — and indeed thermal effects more generally — treat bosons and fermions differently. Thus, if Q is to reproduce Eq. (3.34), then the $T = 0$ gravitino must indeed be projected out by the action of Q.

Note that we have *not* argued that any orbifold Q which seeks to reproduce Eq. (3.34) must (in and of itself) break supersymmetry. Such a claim would be false. Rather, we have merely argued that Q must project out whatever gravitino exists at zero temperature. In particular, at this stage of the argument, we have still left open the possibility that a new gravitino could re-emerge from a twisted sector, thereby allowing Q to preserve supersymmetry overall. In other words, it is *a priori* possible that the ten-dimensional orbifold Q preserves supersymmetry even though the nine-dimensional orbifold $\mathcal{T}Q$ does not. Indeed, this would enable the $T = 0$ and $T \to \infty$ endpoints to be supersymmetric, while allowing a non-supersymmetric interpolation for all intermediate temperatures $0 < T < \infty$.

We shall now argue that this cannot occur. Specifically, we shall argue that any self-consistent orbifold Q that projects out the gravitino from the ten-dimensional supersymmetric $SO(32)$ heterotic string cannot reintroduce a gravitino (chirality-flipped or not) from a twisted sector. The reason is simple. Let us consider the worldsheet sector which gives rise to the gravitino of the original supersymmetric $SO(32)$ model. In any ten-dimensional heterotic string model, a gravitino state can emerge only from the Ramond sector as the spin-3/2 component within the tensor product

$$\text{gravitino:} \qquad \tilde{g}^{\alpha\nu} \ \subset \ \{\tilde{b}_0\}^\alpha|0\rangle_R \ \otimes \ \alpha^\nu_{-1}|0\rangle_L \ . \tag{2.47}$$

Here α^ν_{-1} denotes the lowest excitation of the left-moving worldsheet coordinate boson X^ν, with its Lorentz vector index ν, while $\{\tilde{b}_0\}^\alpha$ schematically indicates the Ramond zero-mode combinations which collectively give rise to the spacetime Lorentz spinor index α, as required for the spin-3/2 gravitino state. Note that in order for such a state to be massless and level-matched, it must emerge from a sector in which the left-moving (conformal) side of the heterotic string is in the completely Neveu-Schwarz ground state, while the right-moving (superconformal) side of the heterotic string is in the completely Ramond ground state. Indeed, in ten dimensions, this is the *only* sector which can ever give rise to spacetime gravitinos, and as such this sector is unique.

But this uniqueness is precisely the problem. This is, quite simply, the only sector which can provide gravitinos of either chirality. It has a unique worldsheet construction. Thus, if this is the original untwisted sector which produced the original gravitino prior to the orbifolding procedure, it cannot simultaneously also be the twisted sector which yields the new gravitino that survives after the orbifolding procedure.

Note that we are *not* saying that an orbifold cannot project out certain states from an untwisted sector, only to have these states re-emerge (even with chirality flips) from a twisted sector. This indeed happens quite often. Rather, we are claiming that the heterotic gravitino sector in ten dimensions is special because it is unique. As such, this sector cannot simultaneously be both untwisted and twisted with respect to the same orbifold. Since this is the only sector which can possibly produce a gravitino (of any chirality) for the ten-dimensional heterotic string, there cannot exist an orbifold which both projects out a gravitino (with any chirality) from the untwisted sector and then restores it (with any chirality) from a twisted sector.

It is critical for this argument that we are discussing the ten-dimensional theory, for the same assertion would not hold upon compactification. To see this with most generality, it is useful to consider the heterotic string from a conformal field theory (CFT) perspective. In ten dimensions, the worldsheet theory of the heterotic string is constructed from ten right- and left-moving coordinate bosons, along with their right-moving superpartners. Together this is a (super)conformal field theory with right- and left-moving central charges $(c_R, c_L) = (15, 10)$. Thus, in order to achieve the total central charges $(c_R, c_L) = (15, 26)$ required for anomaly cancellation, the ten-dimensional heterotic string also contains an additional purely internal left-moving CFT of central charge $c_L = 16$. As always, any on-shell massless state such as the gravitino must have conformal dimensions $(h_R, h_L) = (1/2, 1)$. In particular, the gravitino emerges from the Ramond sector of the right-moving theory [which has conformal dimension $h_R = 1/2$, corresponding to the conformal dimension of a spinor of the transverse Lorentz group $SO(8)$], tensored with the lowest excitation of the left-moving coordinate boson (providing $h_L = 1$). Thus, the left-moving internal CFT with $c_L = 16$ remains its ground state with $h_L = 0$, and *this is a unique configuration for the internal CFT*.

However, this picture changes upon compactification. In four dimensions, for example, the worldsheet theory of the heterotic string consists of four left- and right-moving coordinate bosons, along with their right-moving superpartners. This provides the central charges $(c_R, c_L) = (6, 4)$, requiring the introduction of additional purely internal right- and left-moving (super)CFT's with central charges $(c_R, c_L) = (9, 22)$ in order to achieve conformal anomaly cancellation. Of course, an on-shell massless state such as the gravitino continues to require worldsheet excitations totalling $(h_R, h_L) = (1/2, 1)$, just as in ten dimensions. For the left-movers, the gravitino is realized in precisely the same way as in ten dimensions: we excite the lowest mode of the coordinate boson, thereby providing the $h_L = 1$ excitation that yields the vector index. Thus, the purely internal left-moving CFT with $c_L = 22$ remains in its (unique) $h_L = 0$ ground state. However, the right-moving configuration of the gravitino is greatly altered by compactification.

The Ramond sector in four dimensions now contributes only $h_R = 1/8$, in keeping with the general result (see Appendix A) that the spinor representation of the $SO(2n)$ group has conformal dimension $h = n/8$. As a result, the gravitino state in four dimensions involves further excitations within the $c_R = 9$ purely internal right-moving super-CFT in order to contribute the required additional $h_R = 3/8$. However, in principle there are many ways in which these purely internal excitations can be accomplished. This implies that a gravitino in four dimensions (and more generally, in any dimension $D < 10$) can potentially emerge from a variety of different worldsheet sectors with different underlying CFT constructions, some of which might indeed be the twisted versions of others with respect to an appropriate $\mathbb{Z}_2$ orbifold action Q.

Although the above arguments are completely general and rely on only the most general features of the CFT's underlying these string theories, it is also possible to provide specific mathematical proofs of our assertions using particular CFT representations such as the free-fermionic construction [22, 23]. While this construction is not completely general, it is capable of yielding all closed string theories in ten dimensions — including all of the theories we have been discussing here [24] — and as such is sufficient for our purposes. Using the precise notation of Ref. [23], it is clear that any gravitino in a ten-dimensional heterotic string must emerge from a sector which is defined through the fermionic boundary-condition basis vector typically called $\mathbf{V_1}$:[1]

$$\mathbf{V_1} \;=\; \left[(\tfrac{1}{2})^4 \,|\, (0)^{16}\right] \;. \tag{2.48}$$

It is well known to any practitioner of heterotic string model-building using the free-fermionic construction that this is the only possible sector which can give rise to gravitinos in ten dimensions. A $\mathbb{Z}_2$ orbifold twist then corresponds to the introduction of a new

[1] In the free-fermionic construction, all worldsheet CFT's other than those associated with the spacetime coordinate bosons are represented in terms of free complex worldsheet fermions. In D dimensions, this requires $14 - D$ right-moving complex fermions and $26 - D$ left-moving complex fermions. A given boundary-condition basis vector then describes a specific spin-structure, *i.e.*, a specific set of boundary conditions that such fermions experience as they traverse the space-like direction of the worldsheet torus. In such vectors, a component 0 (1/2) indicates Neveu-Schwarz (Ramond) boundary conditions for the corresponding worldsheet fermion.

vector $\mathbf{V}_{\rm orb}$ which introduces an additional GSO constraint in each untwisted sector. It is easy to arrange $\mathbf{V}_{\rm orb}$ to project out the gravitino emerging from Eq. (2.48). However, it is also easy to see that there cannot be a twisted sector which can possibly restore the gravitino (of any chirality). In general, a twisted sector must take the general form $\mathbf{V}_{\rm orb} + \alpha\mathbf{V}$ where $\alpha\mathbf{V}$ represents any untwisted sector that previously existed in the model prior to the introduction of the orbifold vector $\mathbf{V}_{\rm orb}$. Since the gravitino sector in Eq. (2.48) is the unique sector which can possibly produce gravitinos, such a twisted sector can produce a new gravitino only if $\mathbf{V_1} = \mathbf{V}_{\rm orb} + \alpha\mathbf{V}$. However, this violates the required linear independence of the basis vectors in the free-fermionic construction. Indeed, the linear independence of the basis vectors is one of the primordial ingredients of the free-fermionic construction, one which actually transcends the fact that this construction represents the worldsheet CFT's in terms of free fermions. As a result, we once again see that there does not exist any $\mathbb{Z}_2$ orbifold Q in ten dimensions which can simultaneously project out the gravitino from an untwisted sector and yet re-introduce the gravitino from a twisted sector.

We stress, again, that this argument holds only in ten dimensions. In four dimensions, for example, there are a variety of gravitino sectors which are possible:

$$\begin{aligned}
\mathbf{V_1} &= \left[(\tfrac{1}{2})(\tfrac{1}{2}00)^3\,|\,(0)^{22}\right] ~, \\
\mathbf{V_1'} &= \left[(\tfrac{1}{2})(0\tfrac{1}{2}0)^3\,|\,(0)^{22}\right] ~, \\
\mathbf{V_1''} &= \left[(\tfrac{1}{2})(00\tfrac{1}{2})^3\,|\,(0)^{22}\right] ~, \qquad \textit{etc.}
\end{aligned} \tag{2.49}$$

Thus it is possible, in principle, for one of these sectors to be untwisted and another to be twisted with respect to a given $\mathbb{Z}_2$ orbifold.

We can also use the free-fermionic construction to analyze Eq. (3.34) directly, without worrying about specific ten-dimensional orbifolds. At the special radius corresponding to $T = \sqrt{2}\mathcal{M}$, the identities in Eq. (2.62) indicate that Eq. (3.34) should have a specific free-fermionic realization, and indeed it does. However, the model to which it corresponds fails to have a modulus associated with variations of the radial degree of freedom, as would be required in a true geometric compactification for which the radius (or temperature) is a free parameter. Indeed, all self-consistent nine-dimensional geometric interpolations with variable radii have been constructed in Ref. [19], and Eq. (3.34) is not among them.

It may seem strange that $(-1)^F$ is a legitimate orbifold for the Type II string, yet not for the heterotic string. However, the fermion number F for the heterotic string is nothing but F_R (the contribution from the right-movers), while for the Type II string we have $F = F_R + F_L$. Consequently, the true comparison for $(-1)^F = (-1)^{F_R}$ in the heterotic theory is therefore *not* with $(-1)^F = (-1)^{F_R+F_L}$ in the Type II theory, but merely with $(-1)^{F_R}$ — and this too represents an inconsistent orbifold for the Type II string. Indeed, if we were to attempt to use $Q = (-1)^{F_R}$ as our thermal orbifold choice for the Type IIB string, we would find similar problems as we find in the heterotic case, obtaining a model which interpolates from the Type IIB string at zero temperature to the Type IIA string as $T \to \infty$ while passing through a region for all finite non-zero T in which the $\mathcal{N} = 2$ supersymmetry is broken to $\mathcal{N} = 1$. This is inconsistent for all of the same reasons that applied in the heterotic case.

We conclude, then, that Eq. (3.34) — although properly representing the expected Boltzmann thermal sum of the supersymmetric $SO(32)$ heterotic string — fails to have a simultaneous interpretation as resulting from a *bona-fide* geometric compactification of this string. Specifically, there is no self-consistent orbifold Q which reproduces Eq. (3.34) upon geometric compactification to nine dimensions. As such, we conclude that the temperature/radius correspondence fails to hold for this string. Note that identical arguments apply as well to the $E_8 \times E_8$ heterotic string, since the only difference between the supersymmetric $SO(32)$ and $E_8 \times E_8$ theories is the nature of the purely internal left-moving CFT.

Our arguments in this section have been completely general. However, a little detective work can also uncover additional difficulties if we attempt to interpret Eq. (3.34) as resulting from a *bona-fide* geometric compactification, along with an associated CFT interpretation. Perhaps the most problematic feature is the term $\overline{\chi_I}\,\chi_I$ which appears within $Z^{(4)}$ in Eq. (2.44), yielding the overall contribution

$$\overline{\chi_I}\,\chi_I\,\mathcal{O}_{1/2} \tag{2.50}$$

which appears within Eq. (3.34). A similar term also appears within the analogous Boltzmann sum for the $E_8 \times E_8$ heterotic string. This term represents the identity sectors (*i.e.*, the tachyonic ground states) of the right- and left-moving worldsheet CFT's of the heterotic string. Indeed, since the ground state of the heterotic string has worldsheet energies $(H_R, H_L) = (-1/2, -1)$, we see from level-matching constraints that such a state can indeed only appear within the term $Z^{(4)}$, which multiplies the similarly non-level-matched thermal function $\mathcal{O}_{1/2}$. In other words, modular invariance requires that a term such as $\overline{\chi_I}\chi_I$ — if it appears at all — can only appear within $Z^{(4)}$, multiplying $\mathcal{O}_{1/2}$.

The problem, however, is that $Z^{(4)}$ represents a twisted sector of the $\mathbb{Z}_2$ thermal orbifold, and this is true regardless of whether we choose to run our interpolations from $T = 0$ to $T \to \infty$ or backwards from $T \to \infty$ to $T = 0$. Indeed, we see from Eq. (2.31) that the $Z^{(4)}$ sector is the only sector for which this is true: $Z^{(1)}$ and $Z^{(2)}$ can be regarded as the untwisted sectors if run our interpolations in the direction of increasing temperature, while $Z^{(1)}$ and $Z^{(3)}$ can be regarded as the untwisted sectors if run our interpolations in the direction of decreasing temperature. However, we do not expect to see the ground state of a worldsheet conformal field theory emerging from a twisted sector such as $Z^{(4)}$ — the ground state of a given theory is one which inherently has no twists at all in its worldsheet formulation. Thus the appearance of a term such as Eq. (2.50) signals an inconsistency in providing a consistent CFT interpretation to the Boltzmann sum in Eq. (3.34), as would be required for its interpretation as corresponding to a geometric compactification.

We note, in passing, that no such problem appears for the Type II string: while the ground state $\overline{\chi_I}\chi_I$ of the Type II string does appear in Eq. (3.42), it does so only within

$Z^{(3)}$ (*i.e.*, within the $\mathcal{O}_0$ sector). While this sector is twisted from the perspective of the supersymmetric $T = 0$ theory, it is completely untwisted from the perspective of the non-supersymmetric $T \to \infty$ theory [and in fact represents the physical tachyon whose presence in Eq. (3.42) ultimately triggers the Hagedorn transition in the Type II case]. As a result, there is no difficulty in providing a consistent geometric interpretation to the expression in Eq. (3.42). Indeed, from this perspective, we see that the root of the difficulty in the heterotic case is the fact that the ground state of the heterotic string (unlike that of the Type II string) fails to be level-matched.

This argument is particularly useful because it also extends to dimensions $D < 10$. In $D < 10$ dimensions, any heterotic string model which exhibits spacetime supersymmetry must have a zero-temperature partition function which factorizes into the form

$$Z \;=\; Z_{\text{boson}}^{(D-2)}\, \overline{J} \sum_{ij} \overline{\chi_i'} N_{ij} \chi_j' \;. \tag{2.51}$$

Here $J \equiv (\vartheta_3^4 - \vartheta_2^4 - \vartheta_4^4)/\eta^4 \;=\; 0$ is the spacetime Jacobi factor whose appearance reflects the assumed spacetime supersymmetry of the model, and $(\overline{\chi'}, \chi')$ represent the characters of the remaining right- and left-moving CFT's of central charges $(c_R, c_L) = (10 - D, 26 - D)$. Moreover, as a purely algebraic matter, we can write $J = \chi_V - \chi_S$ where these are the characters of the underlying transverse $SO(8)$ Lorentz group (now broken by compactification). Likewise, although many different coefficients N_{ij} and CFT's in Eq. (2.51) are possible (thereby yielding an incredible richness of possible heterotic strings in lower dimensions), the fact that our model presumably contains the gravity multiplet in its spectrum implies that $N_{II} = 1$, where $\overline{\chi_I'}$ and χ_I' represent the vacuum (identity) sectors of these CFT's. As a result, we can write

$$\sum_{ij} \overline{\chi_i'} N_{ij} \chi_j' \;=\; \overline{\chi_I'} \chi_I' + \ldots \;, \tag{2.52}$$

whereupon the standard Boltzmann prescription requires that the different thermal contributions corresponding to any such model take the form

$$\begin{aligned} Z^{(1)} &= Z^{(D-2)}_{\text{boson}}\ \overline{\chi_V}\ \left(\overline{\chi'_I}\chi'_I + ...\right) \\ Z^{(2)} &= -Z^{(D-2)}_{\text{boson}}\ \overline{\chi_S}\ \left(\overline{\chi'_I}\chi'_I + ...\right) \\ Z^{(3)} &= -Z^{(D-2)}_{\text{boson}}\ \overline{\chi_C}\ \left(\overline{\chi'_I}\chi'_I + ...\right) \\ Z^{(4)} &= +Z^{(D-2)}_{\text{boson}}\ \overline{\chi_I}\ \left(\overline{\chi'_I}\chi'_I + ...\right)\ . \end{aligned} \tag{2.53}$$

We thus see that the $Z^{(4)}$ sector is once again forced to contain the heterotic ground state $\overline{\chi_I}\,\overline{\chi'_I}\chi'_I$, and by the same arguments as discussed above, this precludes a self-consistent geometric interpretation. As a result, our conclusions for ten-dimensional heterotic strings actually extend to *all* supersymmetric heterotic strings, regardless of their spacetime dimensionalities.

2.7 Discussion

In this chapter, we have demonstrated that the standard thermal Boltzmann sum for the heterotic string is inconsistent with the temperature/radius correspondence according to which this sum should have a geometric interpretation corresponding to a spacetime compactification. Without a doubt, this result runs counter to much of what is currently believed in the string thermodynamics literature. Of course, as we have already emphasized, this does not necessarily imply that anything is wrong with the expression in Eq. (3.34); this is, indeed, the correct Boltzmann thermal sum which would follow from the $SO(32)$ string in ten dimensions. It is simply our observation that this result is inconsistent with the temperature/radius correspondence.

The results in Eq. (3.42) and (3.34) first implicitly appeared in the classic work of Atick and Witten [13] more than two decades ago, and since then they have formed the backbone of most work in string thermodynamics as applied to Type II and heterotic strings. In the case of the Type II string, the authors of Ref. [13] accepted the result in Eq. (3.42) and used it to successfully discuss the high-temperature behavior of Type II string theories, including the Hagedorn transition. Moreover, as we have seen, the result in Eq. (3.42) is perfectly consistent with the temperature/radius correspondence.

However, it is perhaps less widely appreciated that the authors of Ref. [13] themselves seriously questioned the self-consistency of the heterotic result in Eq. (3.34) as remaining valid for all temperatures. They too noticed the uncomfortable fact that the $T \to \infty$ limit of this result yields another supersymmetric theory, and flatly rejected this possibility as being unphysical, representing a violation of the expectation that the free energy of any self-consistent theory must evolve monotonically as a function of temperature. They then speculated that "there must be some other effect" which alters this conclusion, and proceeded to discuss the possibility of turning on various additional Wilson lines in the underlying theory in such a way as to ensure that the $T \to \infty$ theory is not only non-supersymmetric, but also tachyonic. As we shall explicitly demonstrate in chapter 3, the turning on of such additional Wilson lines is precisely what naturally occurs in geometric compactifications, and can be viewed as the fundamental feature that prevents the Boltzmann sum (which has no such Wilson lines intrinsically embedded within it) from corresponding to any self-consistent geometric compactification. Indeed, turning on a non-trivial Wilson line is also precisely what is needed in order to project the ground state $\overline{\chi_I}\chi_I$ in Eq. (3.34) out of the thermal spectrum — just as occurs in bona-fide geometric compactifications.

Thus, our complaint against Eq. (3.34) is not entirely new. Rather, we view the primary result of the present chapter as the realization and demonstration that the fundamental problem with Eq. (3.34) is that it ultimately represents a violation of the temperature/radius correspondence, a fact which has not hitherto been realized. We believe that our placement of the problem in this context enables us to clearly see the underlying origin of the difficulties associated with Eq. (3.34). Moreover, as we shall discuss in chapter 3, this realization will also enable us to make a set of concrete proposals towards rectifying these difficulties.

2.A Useful Trace Formulae

In this Appendix, we collect the mathematical expressions which are used in this chapter for the traces over relevant string Fock spaces. These results also serve to define our notations and conventions.

Thermal Sums

For any temperature T, we define the corresponding dimensionless temperature $a \equiv 2\pi T/M_{\text{string}} \equiv T/\mathcal{M}$ where $\mathcal{M} \equiv M_{\text{string}}/(2\pi) = (2\pi\sqrt{\alpha'})^{-1}$. We also define the associated thermal radius $R \equiv (2\pi T)^{-1}$. A field compactified on a circle with this radius then accrues integer Matsubara momentum and winding modes around this thermal circle, resulting in left- and right-moving spacetime momenta of the forms

$$p_R \;=\; \frac{1}{\sqrt{2\alpha'}}(ma - n/a)\;, \qquad p_L \;=\; \frac{1}{\sqrt{2\alpha'}}(ma + n/a)\;. \tag{2.54}$$

Here m and n respectively represent the momentum and winding quantum numbers of the field in question. The contribution to the partition function from such thermal modes then takes the form of the double summation

$$Z_{\text{circ}}(\tau, T) \;=\; \sqrt{\tau_2}\sum_{m,n\in\mathbb{Z}} \overline{q}^{\alpha' p_R^2/2} q^{\alpha' p_L^2/2} \;=\; \sqrt{\tau_2}\sum_{m,n\in\mathbb{Z}} \overline{q}^{(ma-n/a)^2/4}\, q^{(ma+n/a)^2/4} \tag{2.55}$$

where $q \equiv \exp(2\pi i\tau)$ and where $\tau_{1,2}$ respectively denote $\text{Re}\,\tau$ and $\text{Im}\,\tau$. Note that $Z_{\text{circ}} \to 1/a$ as $a \to 0$, while $Z_{\text{circ}} \to a$ as $a \to \infty$.

The trace Z_{circ} is sufficient for compactifications on a thermal circle. However, in this chapter we are interested in compactifications on $\mathbb{Z}_2$ orbifolds of the thermal circle. Towards this end, we introduce [12] four new functions $\mathcal{E}_{0,1/2}$ and $\mathcal{O}_{0,1/2}$ which are the same as the summation in Z_{circ} in Eq. (2.55) except for the following restrictions on their summation variables:

$$\begin{aligned} \mathcal{E}_0 &= \{m \in \mathbb{Z},\ n \text{ even}\} \\ \mathcal{E}_{1/2} &= \{m \in \mathbb{Z} + \tfrac{1}{2},\ n \text{ even}\} \\ \mathcal{O}_0 &= \{m \in \mathbb{Z},\ n \text{ odd}\} \\ \mathcal{O}_{1/2} &= \{m \in \mathbb{Z} + \tfrac{1}{2},\ n \text{ odd}\}\ . \end{aligned} \tag{2.56}$$

Note that these functions are to be distinguished from a related (and also often used) set of functions with the same names in which the roles of m and n are exchanged. Under the modular transformation $T : \tau \to \tau + 1$, the first three functions are invariant while $\mathcal{O}_{1/2}$ picks up a minus sign; likewise, under $S : \tau \to -1/\tau$, these functions mix according to

$$\begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_{1/2} \\ \mathcal{O}_0 \\ \mathcal{O}_{1/2} \end{pmatrix} (-1/\tau) \;=\; \tfrac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_{1/2} \\ \mathcal{O}_0 \\ \mathcal{O}_{1/2} \end{pmatrix} (\tau)\ . \tag{2.57}$$

In the $a \to 0$ limit, $\mathcal{O}_0$ and $\mathcal{O}_{1/2}$ each vanish while $\mathcal{E}_0, \mathcal{E}_{1/2} \to 1/a$; by contrast, as $a \to \infty$, $\mathcal{E}_{1/2}$ and $\mathcal{O}_{1/2}$ each vanish while $\mathcal{E}_0, \mathcal{O}_0 \to a/2$. Clearly, $\mathcal{E}_0 + \mathcal{O}_0 = Z_{\text{circ}}$. Note that the $\mathcal{E}/\mathcal{O}$ functions also satisfy the identities

$$\begin{aligned} \mathcal{E}_0(1/a) &= \mathcal{E}_0(2a)\ , & \mathcal{E}_{1/2}(1/a) &= \mathcal{O}_0(2a) \\ \mathcal{O}_0(1/a) &= \mathcal{E}_{1/2}(2a)\ , & \mathcal{O}_{1/2}(1/a) &= \mathcal{O}_{1/2}(2a)\ . \end{aligned} \tag{2.58}$$

2.B $SO(2n)$ characters

We begin by recalling the standard definitions of the Dedekind η and Jacobi ϑ_i functions:

$$\begin{aligned}
\eta(q) &\equiv q^{1/24} \prod_{n=1}^{\infty} (1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n \, q^{3(n-1/6)^2/2} \\
\vartheta_1(q) &\equiv 2\sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2/2} \\
\vartheta_2(q) &\equiv 2q^{1/8}\prod_{n=1}^{\infty}(1+q^n)^2(1-q^n) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2/2} \\
\vartheta_3(q) &\equiv \prod_{n=1}^{\infty}(1+q^{n-1/2})^2(1-q^n) = 1+2\sum_{n=1}^{\infty} q^{n^2/2} \\
\vartheta_4(q) &\equiv \prod_{n=1}^{\infty}(1-q^{n-1/2})^2(1-q^n) = 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2/2} \; .
\end{aligned} \tag{2.59}$$

These functions satisfy the identities ${\vartheta_3}^4 = {\vartheta_2}^4 + {\vartheta_4}^4$ and $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3$. Note that $\vartheta_1(q)$ has a vanishing q-expansion and is modular invariant; its infinite-product representation has a vanishing coefficient and is thus not shown. This function is nevertheless included here because it plays a role within string partition functions as the indicator of the chirality of fermionic states, as discussed below.

The partition function of n free bosons is given by

$$Z^{(n)}_{\text{boson}} \equiv {\tau_2}^{-n/2} \, (\overline{\eta}\eta)^{-n} \; . \tag{2.60}$$

By contrast, the characters of the level-one $SO(2n)$ affine Lie algebras are defined in terms of both the η- and the ϑ-functions. Recall that at level one, the $SO(2n)$ algebra for each $n \in \mathbb{Z}$ has four distinct representations: the identity (I), the vector (V), the spinor (S), and the conjugate spinor (C). In general, these representations have conformal dimensions $\{h_I, h_V, h_S, h_C\} = \{0, 1/2, n/8, n/8\}$, and their characters are given by

$$\begin{aligned}
\chi_I &= \tfrac{1}{2}\,({\vartheta_3}^n + {\vartheta_4}^n)/\eta^n \;=\; q^{h_I - c/24}\,(1 + n(2n-1)\,q + ...)\\
\chi_V &= \tfrac{1}{2}\,({\vartheta_3}^n - {\vartheta_4}^n)/\eta^n \;=\; q^{h_V - c/24}\,(2n + ...)\\
\chi_S &= \tfrac{1}{2}\,({\vartheta_2}^n + i^{-n}{\vartheta_1}^n)/\eta^n \;=\; q^{h_S - c/24}\,(2^{n-1} + ...)\\
\chi_C &= \tfrac{1}{2}\,({\vartheta_2}^n - i^{-n}{\vartheta_1}^n)/\eta^n \;=\; q^{h_C - c/24}\,(2^{n-1} + ...)
\end{aligned} \tag{2.61}$$

where the central charge is $c = n$ at affine level one. The vanishing of ϑ_1 implies that χ_S and χ_C have identical q-expansions; this is a reflection of the conjugation symmetry between the spinor and conjugate spinor representations. When $SO(2n)$ represents a transverse spacetime Lorentz group, the distinction between S and C is equivalent to relative spacetime chirality; the choice of which spacetime chirality is to be associated with S or C is a matter of convention. Note that the special case $SO(8)$ has a further triality symmetry under which the vector and spinor representations are indistinguishable. Thus, for $SO(8)$, we find that $\chi_V = \chi_S$, an identity already given above in terms of ϑ-functions.

Finally, we observe that there is also a connection between the $\mathcal{E}/\mathcal{O}$ functions in Eq. (2.56) and the ϑ_i-functions in Eq. (2.59). At the specific thermal radius $a = \sqrt{2}$, the $\mathcal{E}/\mathcal{O}$ functions reduce to the forms:

$$\begin{aligned}
\mathcal{E}_0 &\longrightarrow \tfrac{1}{2}\sqrt{\tau_2}\,\left(\overline{\vartheta_3}\vartheta_3 + \overline{\vartheta_4}\vartheta_4\right)\\
\mathcal{E}_{1/2} &\longrightarrow \tfrac{1}{2}\sqrt{\tau_2}\,\left(\overline{\vartheta_2}\vartheta_2 + \overline{\vartheta}_1\vartheta_1\right)\\
\mathcal{O}_0 &\longrightarrow \tfrac{1}{2}\sqrt{\tau_2}\,\left(\overline{\vartheta_2}\vartheta_2 - \overline{\vartheta}_1\vartheta_1\right)\\
\mathcal{O}_{1/2} &\longrightarrow \tfrac{1}{2}\sqrt{\tau_2}\,\left(\overline{\vartheta_3}\vartheta_3 - \overline{\vartheta_4}\vartheta_4\right)\ .
\end{aligned} \tag{2.62}$$

These identities are nothing but a reflection of the equivalence between a compactified boson and a Dirac fermion at the so-called "free-fermionic" radius $a = \sqrt{2}$.

2.C Partition functions of ten-dimensional strings

We now collect together the partition functions of the ten-dimensional Type II and heterotic strings which are relevant to the discussions in this chapter.

In ten dimensions, the contributions from the left-moving (holomorphic) and right-moving (anti-holomorphic) degrees of freedom of the Type II strings can be written in terms of the characters $\chi_{V,S,C}$ of the $SO(8)$ transverse Lorentz group. In terms of these characters, the partition functions of the two supersymmetric Type II strings in ten dimensions are:

$$\begin{aligned} Z_{\text{IIA}} &= Z^{(8)}_{\text{boson}}\,(\overline{\chi}_V - \overline{\chi}_S)\,(\chi_V - \chi_C) \\ Z_{\text{IIB}} &= Z^{(8)}_{\text{boson}}\,(\overline{\chi}_V - \overline{\chi}_S)\,(\chi_V - \chi_S)\,, \end{aligned} \tag{2.63}$$

where $Z^{(8)}_{\text{boson}}$ denotes the contribution from the eight worldsheet bosons, as defined in Eq. (2.60). Note that the difference between χ_S and χ_C is equivalent to relative spacetime chirality, and that the triality symmetry of $SO(8)$ guarantees that $\chi_V = \chi_S = \chi_C$ and $\overline{\chi}_V = \overline{\chi}_S = \overline{\chi}_C$. This in turn indicates that the partition functions in Eq. (2.63) vanishes, consistent with the spacetime supersymmetry of these theories; in fact, the presence of two such factors within each partition function in Eq. (3.1) reflects the $\mathcal{N} = 2$ supersymmetry of these theories at zero temperature.

There are also two *non*-supersymmetric Type II theories in ten dimensions. These are the Type 0A and 0B strings, with partition functions

$$\begin{aligned} Z_{0A} &= Z^{(8)}_{\text{boson}}\,(\overline{\chi}_I\chi_I + \overline{\chi}_V\chi_V + \overline{\chi}_S\chi_C + \overline{\chi}_C\chi_S) \\ Z_{0B} &= Z^{(8)}_{\text{boson}}\,(\overline{\chi}_I\chi_I + \overline{\chi}_V\chi_V + \overline{\chi}_S\chi_S + \overline{\chi}_C\chi_C)\,. \end{aligned} \tag{2.64}$$

For heterotic strings in ten dimensions, the contributions from the right-moving (anti-holomorphic) degrees of freedom continue to be written in terms of the characters $\chi_{V,S,C}$ of the $SO(8)$ transverse Lorentz group, just as for the Type II strings, while the contributions from the left-moving (holomorphic) degrees of freedom are written in terms of the characters appropriate for the internal gauge symmetry of the particular heterotic string under consideration.

For the ten-dimensional supersymmetric $SO(32)$ heterotic string, the partition function can be written as

$$Z_{SO(32)} \;=\; Z^{(8)}_{\text{boson}}\,(\overline{\chi}_V - \overline{\chi}_S)\,(\chi_I + \chi_S) \tag{2.65}$$

where the holomorphic characters are those of $SO(32)$. As before, the appearance of the anti-holomorphic factor $\overline{\chi}_V - \overline{\chi}_S$ signals the $\mathcal{N} = 1$ spacetime supersymmetry of this theory. However, Eq. (2.65) may equivalently be written in terms of the (product) characters of an $SO(16) \times SO(16)$ subgroup:

$$Z_{SO(32)} \;=\; Z^{(8)}_{\text{boson}}\,(\overline{\chi}_V - \overline{\chi}_S)\,(\chi_I\chi_I + \chi_V\chi_V + \chi_S\chi_S + \chi_C\chi_C)\;. \tag{2.66}$$

Likewise, the partition function of the $E_8 \times E_8$ heterotic string can also be written in terms of $SO(16) \times SO(16)$ characters:

$$Z_{E_8\times E_8} \;=\; Z^{(8)}_{\text{boson}}\,(\overline{\chi}_V - \overline{\chi}_S)\,(\chi_I + \chi_S)^2\;. \tag{2.67}$$

There are also a number of *non*-supersymmetric heterotic strings in ten dimensions. Those which will be the most relevant for the work in this thesis are the *non*-supersymmetric $SO(32)$ string and the $SO(16) \times E_8$ heterotic string [25, 26]. Both of these strings are tachyonic. The non-supersymmetric $SO(32)$ string has partition function

$$Z^{(\mathcal{N}=0)}_{SO(32)} \;=\; Z^{(8)}_{\text{boson}}\,(\overline{\chi}_I\chi_V + \overline{\chi}_V\chi_I - \overline{\chi}_S\chi_S - \overline{\chi}_C\chi_C) \tag{2.68}$$

when expressed in terms of $S0(32)$ characters; this takes the alternate form

$$\begin{aligned} Z^{(\mathcal{N}=0)}_{SO(32)} \;=\; Z^{(8)}_{\text{boson}}\,\Big[&\overline{\chi}_I(\chi_I\chi_V + \chi_V\chi_I) + \overline{\chi}_V(\chi_I^2 + \chi_V^2) \\ &- \overline{\chi}_C(\chi_S\chi_C + \chi_c\chi_S) - \overline{\chi}_S(\chi_S^2 + \chi_C^2)\Big] \end{aligned} \tag{2.69}$$

when expressed in terms of $SO(16)\times SO(16)$ characters. Likewise, the $SO(16)\times E_8$ heterotic string has the partition function

$$Z_{SO(16)\times E_8} ~=~ Z^{(8)}_{\rm boson}\,(\overline{\chi}_I\chi_V+\overline{\chi}_V\chi_I-\overline{\chi}_S\chi_S-\overline{\chi}_C\chi_C)\,(\chi_I+\chi_S) \tag{2.70}$$

when expressed in terms of $SO(16)\times SO(16)$ characters.

2.D Derivation of Eq. (2.21)

In this Appendix, we discuss the origin of the result in Eq. (2.21). Specifically, we shall sketch the steps involved in passing from the field-theoretic results in Eqs. (3.13) and (3.14) to the string-theoretic result in Eq. (2.21).

We begin with the expressions for the vacuum energy Λ of the compactified field theory given in Eqs. (3.13) and (3.14), as it is this quantity which must be identified with the free energy $F(T)$ according to the temperature/radius correspondence. The first step is to rewrite the logarithms in Eqs. (3.13) and (3.14) in terms of a Schwinger proper-time parameter t using the identity

$$\log x ~=~ \int_1^x \frac{dy}{y} ~=~ \int_1^x dy \int_0^\infty dt\; e^{-yt} ~=~ -\int_0^\infty \frac{dt}{t}\; e^{-xt} ~+~ ... \tag{2.71}$$

where we have dropped an x-dependent term. We thus find that Eqs. (3.13) and (3.14) take the form

$$\Lambda ~=~ -\tfrac{1}{2}\,\frac{1}{2\pi R}\int \frac{d^{D-1}\mathbf{p}}{(2\pi)^{D-1}}\,(-1)^F \sum_n \exp\left[-\left(\mathbf{p}\cdot\mathbf{p}+m^2+\frac{n^2}{R^2}\right)\,t\right] \tag{2.72}$$

where $\mathbf{p}$ denotes the space-vector associated with the D-vector p and where $\sum_n$ denotes a summation over either integer or half-integer values of n depending on whether we are considering the periodic case in Eq. (3.13) or the anti-periodic case in Eq. (3.14). Performing the $\mathbf{p}$-integration then yields

$$\Lambda ~=~ -\tfrac{1}{2}\,\frac{1}{2\pi R}\,\frac{1}{(4\pi)^{(D-1)/2}}\int_0^\infty \frac{dt}{t^{(D+1)/2}}\,(-1)^F\, e^{-m^2 t}\sum_n e^{-n^2 t/R^2}~. \tag{2.73}$$

Our next step is to define the dimensionless real parameter $\tau_2 = 4\pi\mathcal{M}^2 t$ where $\mathcal{M} \equiv M_{\rm string}/(2\pi)$ is the reduced string scale. Similarly, we introduce an additional dimensionless real variable τ_1 by inserting

$$1 = \int_{-1/2}^{1/2} d\tau_1 \tag{2.74}$$

into Eq. (2.73). Defining the complex variable $\tau \equiv \tau_1 + i\tau_2$, we then find that Eq. (2.73) takes the form

$$\Lambda = -\tfrac{1}{2}\,\frac{1}{2\pi R}\,\mathcal{M}^{D-1}\int_{\mathcal{S}}\frac{d^2\tau}{\tau_2^2}\left[\frac{1}{\tau_2^{D/2-1}}\,(-1)^F\,(\overline{q}q)^{\alpha' m^2/4}\right]\left[\sqrt{\tau_2}\,\sum_n\,(\overline{q}q)^{\alpha' n^2/4R^2}\right] \tag{2.75}$$

where $q \equiv e^{2\pi i\tau}$, where $\alpha' \equiv 1/M_{\rm string}^2$, and where $\mathcal{S}$ denotes the strip in the complex τ-plane defined by $\mathcal{S} \equiv \{\tau :\ \tau_2 > 0,\ |\tau_1| < 1/2\}$.

Eq. (2.75) represents the contribution to the vacuum energy from a single level-matched string state with spacetime mass m and fermion number F. Summing over the complete string spectrum including the off-shell (non-matched) string states, we then find that the first factor in square brackets in Eq. (2.75) yields the $Z^{(i)}$ terms in Eq. (2.15). Likewise, the second square-bracketed factors in Eq. (2.75) become the $\mathcal{E}/\mathcal{O}$-functions in Eq. (2.15) once we include the appropriate thermal winding modes and consider the different possible moding combinations. Thus, with these pieces properly stitched together, the resulting integrand in Eq. (2.75) becomes nothing but the thermal partition function $Z_{\rm string}(\tau, T)$, and the modular invariance of this quantity then allows us to truncate the strip $\mathcal{S}$ to the fundamental domain $\mathcal{F}$ defined in Eq. (3.33). Likewise, the temperature/radius correspondence allows us to identify the leading factor $(2\pi R)^{-1}$ in Eq. (2.75) as the temperature T. We then obtain the result given in Eq. (2.21).

2.E Spin statistics for string theory

String theories generally seem to obey the spin-statistics relation. In this appendix we show that if a string theory satisfies certain conditions it will also give rise to normal spin-statistics relation.

In quantum field theory, the spin-statistics connection results from Lorentz invariance, causality and positivity of energy. In string theory also we can ask whether these conditions (along with others) are enough to ensure normal spin-statistics. The proper arena for addressing this question may be string field theory, however it is possible to attempt to answer this within the S-matrix formulation of string theory. In this context, a string theory obeys normal spin-statistics if all scattering amplitudes are symmetric/antisymmetric under corresponding exchange of bosons/fermions.

The exchange symmetry of scattering amplitudes actually follows from exchange symmetry of vertex operators on the string worldsheet ([2, 27, 28]). The string S-Matrix in superstring theory can be written as:

$$A_{j_1,j_2,\ldots,j_N}(k_1,k_2\ldots.k_N) = \sum_{topologies}\int\frac{[dXd\Psi dg]}{V_{diff\times Weyl}}\exp(-S-\lambda_\chi)\prod_{i=1}^{n}\int d^2\sigma_i g(\sigma_i))^{1/2}V_{j_i}(k_i,\sigma_i) \quad (2.76)$$

What happens when we exchange particle 1 having quantum numbers k_1, j_1 with particle 2 having quantum numbers k_2, j_2 in the amplitude on the left hand side? This corresponds to an exchange of the corresponding vertex operators of the particles on the right hand side. So for all string amplitudes to obey normal space-time spin-statistics under particle exchange, it is necessary that the corresponding vertex operators on the worldsheet also obey the same relations.

In any string theory if all the vertex operators corresponding to the spectrum, behave as expected under exchange, then that theory satisfies the spin-statistics relation. Taking this approach, our aim in this paper is to show that:

Proposition. *Any String theory satisfies normal spin-statistics in space-time if the following conditions hold:*

1. *The theory is conformally invariant, which leads to lorentz invariance in space-time.*

2. *The Vertex Operator Product Expansion (O.P.E) for the theory closes.*

Since any consistent string theory obeys the above two conditions, it follows that it also satisfies proper spin-statistics. In the rest of this appendix we demonstrate how this is so, for ten dimensional string theories and also for string theories compactified on orbifolds.

Proof

The Vertex operator corresponding to a closed string state is a linear combination of operators of the form,

$$V_{closed}(k,j) = V^L_{creation} V^L_{ghost} V^L_{momentum} V^R_{creation} V^R_{ghost} V^R_{momentum} \tag{2.77}$$

Where the superscripts L and R denote the left and right vertex operators. The left-hand side creation operator $V^L_{creation}$ contains gauge group creation operators for heterotic strings and lorentz group creation operators for other strings.

In bosonized form the operators $V_{creation}, V_{ghost}, V_{momentum}$, are written in terms of bosonic fields H, ϕ, X which are functions of $z/\bar{z}$. The general form of the creation operator (whether lorentz or gauge group) is:

$$V_{creation} = \prod_i (\partial^{m_i} X^\mu_i) \exp{[\pm i H^a]} ... \exp{[\pm i H^b]} \exp{[i \sum_a s_a H^a]} \tag{2.78}$$

The last term in the creation operator is the spin field. $\mathbf{s}$ is the spinor weight vector which is $\mathbf{s} = (\pm\frac{1}{2}, \pm\frac{1}{2}, ..., \pm\frac{1}{2})$ for Ramond sector operators and 0 in case of NS sector. If the different exponentials in the creation operator are combined, the creation operator can be written as derivatives of free bosonic fields times a single exponential:

$$V_{creation} = \prod_i (\partial^{m_i} X^\mu_i) \prod_{a,p} (\partial^{p_a} H^a) \exp{[i \mathbf{H} \cdot \mathbf{n}]} \tag{2.79}$$

The conformal dimension of this operator is $m + \mathbf{n}^2/2$, where $\mathbf{n}$ is a weight vector of the weight lattice of the corresponding group (SO(10) for lorentz, SO(32) etc for gauge) and m is the (integral) weight of the derivative part. The ghost vertex operator is given by,

$$V_{ghost} = \exp{[-l\phi]} \tag{2.80}$$

The conformal dimension of the ghost vertex is $-l^2/2 - l$, due to the presence of the background charge. All these operators are to be understood with the L or R subscript. The momentum vertex is

$$V_{momentum} = \exp\left[i\mathbf{k}\cdot\mathbf{X}\right] = \exp\left[i\mathbf{k_L}\cdot\mathbf{X_L} + i\mathbf{k_R}\cdot\mathbf{X_R}\right] \tag{2.81}$$

The form of the momentum vertex remains the same, if the string theory is compactified on a orbifold. Only the values taken by $k_L \pm k_R$, corresponding to compactified dimensions, are restriced to integer and half-integer values(when multiplied suitably by α and R).

For open strings, vertex operators are on the boundary so they are a function of y(real variable). Also, open strings can carry chan-paton factors, so the general form of an open string vertex operator is:

$$V_{open}(k, j) = \lambda^a V_{creation} V_{ghost} V_{momentum} \tag{2.82}$$

Where, λ^a is the group matrix.

To get the phase under exchange of operators (corresponding to states) we write down the O.P.E of the two operators. Conservation of charge implies that the quantum numbers $\mathbf{k}, \mathbf{n}$ and l will be conserved by the O.P.E. Ensuring that the scaling dimension of the L.H.S and R.H.S are equal we get:

$$\begin{aligned} &: V_{k_1,m_2,n_1,l_1}(z_1,\bar{z_1}) :: V_{k_2,m_2,n_2,l_2}(z_2,\bar{z_2}) :\sim \\ &\qquad \frac{: e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{X}} e^{i(\mathbf{n}_1+\mathbf{n}_2)\cdot\mathbf{H}} e^{i(\mathbf{l}_1+\mathbf{l}_2)\cdot\phi} :}{z_{12}^{\mathbf{k}_{1L}\cdot\mathbf{k}_{2L}} z_{12}^{\mathbf{n}_{1L}\cdot\mathbf{n}_{2L}} z_{12}^{l_{1L}\cdot l_{2L}} \bar{z}_{12}^{\mathbf{k}_{1R}\cdot\mathbf{k}_{2R}} \bar{z}_{12}^{\mathbf{n}_{1R}\cdot\mathbf{n}_{2R}} \bar{z}_{12}^{l_{1R}\cdot l_{2R}}} \end{aligned} \tag{2.83}$$

where, $\mathbf{k}\cdot\mathbf{X} = \mathbf{k}_L\cdot\mathbf{X}_L + \mathbf{k}_R\cdot\mathbf{X}_R$. From the above equation the total phase for exchanging particle 1 with 2 is, for closed strings:

$$\exp[i\pi(\frac{\alpha}{2}\mathbf{k}_{1L}\cdot\mathbf{k}_{2L} - \frac{\alpha}{2}\mathbf{k}_{1R}\cdot\mathbf{k}_{2R} + \mathbf{n}_{1L}\cdot\mathbf{n}_{2L} - \mathbf{n}_{1R}\cdot\mathbf{n}_{2R} - l_{1L}l_{2L} + l_{1R}l_{2R})] \tag{2.84}$$

For open strings the Chan-Paton factor does not make a difference, therefore the phase is given by:

$$\exp[i\pi(2\alpha\mathbf{k}_1\cdot\mathbf{k}_2 + \mathbf{n}_1\cdot\mathbf{n}_2 - l_1 l_2)] \tag{2.85}$$

Note that the above phases are only valid when the O.P.E closes(the relevant charges are conserved). This therefore gives us condition (2) of our theorem.

For String states with identical quantum numbers the total phase reduces to $\exp[i\pi\theta]$, where θ is

$$\theta = \theta_L - \theta_R = (\frac{\alpha}{2}\mathbf{k}_L{}^2 - \frac{\alpha}{2}\mathbf{k}_R{}^2 + \mathbf{n}_L{}^2 - \mathbf{n}_R{}^2 - l_L{}^2 + l_R{}^2) \tag{2.86}$$

Now the total dimension of a closed string vertex operator, (h_L, h_R), is equal to:

$$\begin{aligned} h_L &= \frac{\alpha}{4}\mathbf{k}_L{}^2 + m_L + \frac{1}{2}\mathbf{n}_L{}^2 - \frac{1}{2}l_L{}^2 - l_L \\ h_R &= \frac{\alpha}{4}\mathbf{k}_R{}^2 + m_R + \frac{1}{2}\mathbf{n}_R{}^2 - \frac{1}{2}l_R{}^2 - l_R \end{aligned} \tag{2.87}$$

and that of a open string vertex operator is:

$$h = \alpha\mathbf{k}^2 + m + \frac{1}{2}\mathbf{n}^2 - \frac{1}{2}l^2 - l \tag{2.88}$$

Using eqs.2.87, eq.2.86 can be reduced to

$$\theta = \theta_L - \theta_R = 2h_L - 2h_R + 2l_L - 2l_R \mod 2 \tag{2.89}$$

Note that the calculation of this phase, does not depend on details of the exact form of the vertex operator. Rather (θ_L, θ_R), is always related to the dimension (h_L, h_R) of the vertex operator in this way.

If the string theory obeys the mass-shell condition, $(h_L, h_R) = (1, 1)$, the phase becomes:

$$\begin{aligned} \exp[i2\pi(l_L - l_R)] & \quad \text{for closed strings} \\ \exp[i2\pi l] & \quad \text{for open strings} \end{aligned} \tag{2.90}$$

The values of l for different string theories are:

$$\begin{aligned} l_{lorentz\ SO(10)\ group} &= \begin{cases} 1 & \text{for NS states} \\ \frac{1}{2} & \text{for R states} \end{cases} \\ l_{internal\ gauge\ groups} &= 0 \end{aligned} \tag{2.91}$$

From equations 2.90 and 2.91 it is evident that the only string state which anticommutes with itself is a state having a single Lorentz-group Ramond field. This coincides with the definition of a space-time fermion. Note also that the statistics for a string state is determined solely by the ghost sector.

We can see from equation 2.89, that even when the mass-shell condition,$(h_L, h_R) = (1, 1)$, does not hold, string theories can still satisfy the spin-statistics relation (for instance when h_L and h_R are integers). However the mass-shell condition(though not necessary) is sufficient for proper spin-statistics to hold at least at the tree-level formulation of string theory. In this setting we have not looked at the constraints coming from higher loops. Since the mass-shell condition is part of conformal invariance in string theory, it follows that any conformally invariant string theory will obey the spin-statistics condition.

Chapter 3

A GEOMETRIC APPROACH TO FINITE-TEMPERATURE STRING THEORY AND ITS IMPLICATIONS FOR THE HAGEDORN TRANSITION

Summary

The temperature/radius correspondence states that a quantum theory at finite temperature T can be recast as a zero-temperature theory in which a Euclidean time dimension is compactified on a circle of radius $R = (2\pi T)^{-1}$. In chapter 2, however, it was demonstrated that this correspondence is actually broken for heterotic strings at finite temperature — *i.e.*, the traditional Boltzmann sum for heterotic strings cannot be recast as the partition function corresponding to any self-consistent heterotic compactification. The question then arises as to whether one should follow the traditional Boltzmann approach or a geometric approach when extrapolating a zero-temperature heterotic string model to finite temperature. The Boltzmann approach is the one typically followed in the string literature. In this chapter, however, we investigate the consequences of pursuing a geometric approach to finite-temperature string theory, and show that this corresponds to turning on a non-trivial Wilson line (or equivalently, an imaginary temperature-dependent chemical potential) in the standard Boltzmann thermal extrapolation. This in turn leads to many surprising results which differ from standard expectations. For example, we demonstrate that the geometric approach actually leads to a universal Hagedorn temperature for all tachyon-free closed string theories in ten dimensions — both Type II and heterotic. As we show, these results are not in conflict with the well-known exponential growth in the degeneracies of string states in such models.

3.1 Introduction

In the last chapter, we demonstrated that while the "temperature/radius correspondence" holds for bosonic strings as well as Type II strings, it is actually broken for heterotic strings at finite temperature. Specifically, we showed that the traditional Boltzmann sum for heterotic strings cannot be recast as the partition function corresponding to any self-consistent heterotic compactification.

This result places us at a cross-roads when it comes to extrapolating heterotic strings to finite temperature. On the one hand, we could take the attitude that the traditional Boltzmann sum actually *defines* what we mean by thermodynamics at finite temperatures, and proceed to follow this approach for heterotic strings regardless of the absence of possible interpretations of such results as corresponding to legitimate geometric compactifications of the heterotic string. If there is no corresponding geometric underpinning to this approach, so be it. This is essentially the path taken in most if not all of the current literature on finite-temperature heterotic strings.

In this chapter, however, we shall take a more speculative path and investigate the consequences of following the opposite philosophy. While the thermodynamics of string theory remains, in many ways, a mysterious subject, the physics of compactifying a string on a circle is well understood. Thus, in this chapter we shall follow the geometric approach as a guide for extrapolating zero-temperature string theories to finite temperatures. Needless to say, following this path will produce results which differ from many of the standard results in the string literature. However, as discussed in chapter 2, maintaining a geometric underpinning to our thermal extrapolations has a number of advantages. For example, perhaps the most important of these advantages is that any finite-temperature string models derived in this manner will have self-consistent *worldsheet* interpretations for all values of the temperature T. In other words, T will be a bona-fide *modulus* in such theories. As a result, these theories will continue to retain the self-consistent conformal field theory (CFT) underpinnings at finite temperature that they had at zero temperature.

As we shall show, following a geometric approach to finite-temperature heterotic string theory corresponds to making only one small modification to the standard Boltzmann approach: one must turn on a non-trivial Wilson line, or equivalently introduce a temperature-dependent imaginary chemical potential. While such features occasionally play a role in finite-temperature field theory, they have not historically played a significant role in finite-temperature string theory. Nevertheless, as we shall demonstrate, turning on such a chemical potential actually *restores* the temperature/radius correspondence for heterotic strings, and in fact leads to a number of exciting results which introduce a certain theoretical "unity" to finite-temperature string theory. For example, we shall find that the geometric approach actually leads to a universal Hagedorn temperature for all tachyon-free closed string theories in ten dimensions — both Type II as well as heterotic! Moreover, as we shall show, these strange results are not in conflict with the well-known exponential growth in the degeneracies of string states in such models.

This chapter is organized as follows. In Sect. 3.2, we discuss the appearance of this new Wilson line, and show that it corresponds to the introduction of a chemical potential in the standard Boltzmann approach. Using this, we then construct in Sect. 3.3 what we believe to be the correct "geometric" extrapolations of the $SO(32)$ and $E_8 \times E_8$ heterotic strings to finite temperature — *i.e.*, finite-temperature versions of these theories which actually correspond to their geometric compactifications. Finally, in Sect. 3.4, we discuss the implications of these new results for the Hagedorn transition, and briefly mention some other related thermal transitions which can occur.

3.2 Wilson lines and imaginary chemical potentials

The overall question we face concerns the manner in which a given zero-temperature string model can be extrapolated to finite temperature. The standard way to do this is to follow the standard Boltzmann approach, and construct an appropriate Boltzmann sum for each state in the zero-temperature string theory according to whether it is a spacetime boson or spacetime fermion. However, as discussed in chapter 2, there is a simple criterion that can be used in order to test whether a given finite-temperature extrapolation of a D-dimensional string model is consistent with the temperature/radius correspondence (*i.e.*, whether it corresponds to a geometric compactification of the zero-temperature theory): such an extrapolation must be a $(D-1)$-dimensional *interpolating model*, with the temperature T serving as an interpolating parameter. As $T \to 0$, we obtain a D-dimensional string model M_1; this is identified as the zero-temperature string model whose thermal extrapolation we have constructed. By contrast, as $T \to \infty$, we obtain a different D-dimensional string model M_2 which must be a $\mathbb{Z}_2$ orbifold of M_1. The corresponding $\mathbb{Z}_2$ orbifold action is denoted Q. In general, we expect that M_2 must be non-supersymmetric, even if M_1 was originally supersymmetric. This is because thermal effects should break whatever supersymmetry might have existed at zero temperature.

3.2.1 The need for a non-trivial thermal orbifold Q for heterotic strings

Clearly, if the Boltzmann approach is to be consistent with the temperature/radius correspondence, it must correspond to choosing the trivial $\mathbb{Z}_2$ orbifold $Q = (-1)^F$ where F is the spacetime fermion number. This is because the Boltzmann approach depends only on whether a state is bosonic or fermionic, and is insensitive to all other features (such as the gauge charges which such a state may carry).

As shown in chapter 2, the Boltzmann approach turns out to be fully consistent with the geometric temperature/radius correspondence for the Type II superstrings. Indeed, $Q = (-1)^F$ is a fully consistent orbifold choice for Type II strings; moreover, it operates as expected, yielding a non-supersymmetric string theory as the $T \to \infty$ endpoint of the corresponding thermal extrapolation. As an example, let us consider the case where

M_1 is the Type IIB string. As discussed in Appendix A of chapter 2, this string has the partition function

$$Z_{\rm IIB} \;=\; Z^{(8)}_{\rm boson}\,(\overline{\chi}_V - \overline{\chi}_S)\,(\chi_V - \chi_S) \tag{3.1}$$

where $Z^{(8)}_{\rm boson}$ denotes the contribution from the eight worldsheet bosons in light-cone gauge and where the contributions from the worldsheet fermions are written in terms of the characters χ of the $SO(8)$ transverse Lorentz group. Specifically, χ_V and χ_S are respectively the vectorial and spinorial chiral characters of this group, and satisfy the identity $\chi_V = \chi_S$. As a result, Eq. (3.1) vanishes, which is a reflection of the unbroken spacetime supersymmetry of the Type IIB string. Orbifolding the theory in Eq. (3.1) with the Boltzmann choice $Q = (-1)^F$ then requires that we identify Eq. (3.1) as our unprojected sector Z^+_+ and introduce a contribution from the corresponding projection sector

$$Z^-_+ \;=\; Z^{(8)}_{\rm boson}\,(\overline{\chi}_V + \overline{\chi}_S)\,(\chi_V + \chi_S)~. \tag{3.2}$$

This in turn requires that we introduce a corresponding twisted sector along with its projection sector:

$$\begin{aligned} Z^+_- &= Z^{(8)}_{\rm boson}\,(\overline{\chi}_I - \overline{\chi}_C)\,(\chi_I - \chi_C) \\ Z^+_- &= Z^{(8)}_{\rm boson}\,(\overline{\chi}_I + \overline{\chi}_C)\,(\chi_I + \chi_C)~, \end{aligned} \tag{3.3}$$

and taken together, the net result of this orbifolding procedure is the partition function of model M_2:

$$Z \;=\; \tfrac{1}{2}\left(Z^+_+ + Z^-_+ + Z^+_- + Z^-_-\right) \;=\; Z^{(8)}_{\rm boson}\,\left(\overline{\chi}_I\chi_I + \overline{\chi}_V\chi_V + \overline{\chi}_S\chi_S + \overline{\chi}_C\chi_C\right)~. \tag{3.4}$$

However, as seen in Appendix A of chapter 2, this is nothing but the partition function of the Type 0B string, and this string is indeed non-supersymmetric. A similar result holds for Type IIA strings: orbifolding by $Q = (-1)^F$ produces the non-supersymmetric Type 0A string. Thus, for Type II strings, we see that the Boltzmann choice $Q = (-1)^F$ succeeds in breaking the spacetime supersymmetry of the original Type II theory, leaving us with its non-supersymmetric (and tachyonic) counterpart as the model M_2.

This situation changes dramatically for the heterotic string, and this difference is directly related to the breakdown of the temperature/radius correspondence in this case. For concreteness, let us consider the case where M_1 is the ten-dimensional $\mathcal{N} = 1$ supersymmetric $SO(32)$ heterotic string. This string has the partition function

$$Z_{SO(32)}^{(\mathcal{N}=1)} = Z_{\rm boson}^{(8)} \, (\overline{\chi}_V - \overline{\chi}_S)\,(\chi_I + \chi_S) \, , \tag{3.5}$$

where the contributions from the left-moving (internal) worldsheet fermions are now written in terms of the unbarred characters χ_i of an internal $SO(32)$ gauge group. According to the temperature/radius correspondence, there must be a $\mathbb{Z}_2$ orbifold Q which describes the finite-temperature behavior of this theory. Moreover, our expectations from the Type II case lead us to suspect that we should choose an orbifold Q such that the corresponding model M_2 turns out to be the non-supersymmetric version of this string, namely the ten-dimensional *non*-supersymmetric (tachyonic) $SO(32)$ heterotic string with partition function

$$Z_{SO(32)}^{(\mathcal{N}=0)} = Z_{\rm boson}^{(8)} \, \left(\overline{\chi}_I \chi_V + \overline{\chi}_V \chi_I - \overline{\chi}_S \chi_S - \overline{\chi}_C \chi_C\right) \, . \tag{3.6}$$

Indeed, identifying this string as our model M_2 would mirror the Type II situation as closely as possible: this string has exactly the same gauge symmetry as its supersymmetric counterpart, but the supersymmetry is broken and tachyons appear. Both features are exactly as expected.

However, it is easy to demonstrate that the Boltzmann choice $Q = (-1)^F$ fails to accomplish the transition from Eq. (3.5) to Eq. (3.6). If $Q = (-1)^F$ were a self-consistent orbifold of the $SO(32)$ heterotic string (and we have already shown in chapter 2 that it is not), we would proceed in exactly the same manner as in the Type II case to find

$$\begin{aligned}
Z_+^+ &= Z_{\rm boson}^{(8)} \, (\overline{\chi}_V - \overline{\chi}_S)\,(\chi_I + \chi_S) \\
Z_+^- &= Z_{\rm boson}^{(8)} \, (\overline{\chi}_V + \overline{\chi}_S)\,(\chi_I + \chi_S) \\
Z_-^+ &= Z_{\rm boson}^{(8)} \, (\overline{\chi}_I - \overline{\chi}_C)\,(\chi_I + \chi_S) \\
Z_-^- &= Z_{\rm boson}^{(8)} \, (-\overline{\chi}_I - \overline{\chi}_C)\,(\chi_I + \chi_S) \, .
\end{aligned} \tag{3.7}$$

However, the net result of this "orbifolding" procedure is *not* the non-supersymmetric tachyonic $SO(32)$ theory, but rather the supersymmetric tachyon-free $SO(32)$ theory with a spacetime chirality flip:

$$Z'_{SO(32)} \;=\; Z^{(8)}_{\text{boson}}\,(\overline{\chi}_V - \overline{\chi}_C)\,(\chi_I + \chi_S)\;. \tag{3.8}$$

Thus, we see that even if $Q = (-1)^F$ were a valid orbifold choice for the supersymmetric $SO(32)$ heterotic string, the supersymmetry of this string would remain intact even after the orbifolding. In other words, the Boltzmann choice $Q = (-1)^F$ does not break supersymmetry for the heterotic string! A similar problem emerges for the $E_8 \times E_8$ heterotic string.

How then can one produce the non-supersymmetric $SO(32)$ theory from the supersymmetric $SO(32)$ theory? It turns out that one must choose a different $\mathbb{Z}_2$ orbifold which not only contains a $(-1)^F$ factor but which *also contains a non-trivial extra factor (ultimately corresponding to a non-trivial Wilson line) which acts on the gauge degrees of freedom.* Indeed, we must do this *even though we do not wish to break the $SO(32)$ gauge symmetry in passing from our original theory to our final theory.* This the key difference between heterotic strings and Type II strings.

It turns out that the Wilson line we need in this case is one in which states in vectorial (respectively spinorial) representations of the gauge group pick up positive (respectively negative) signs. This choice then implies the projection/twisted sectors

$$\begin{aligned}
Z^+_+ &= Z^{(8)}_{\text{boson}}\,(\overline{\chi}_V - \overline{\chi}_S)\,(\chi_I + \chi_S)\\
Z^-_+ &= Z^{(8)}_{\text{boson}}\,(\overline{\chi}_V + \overline{\chi}_S)\,(\chi_I - \chi_S)\\
Z^+_- &= Z^{(8)}_{\text{boson}}\,(\overline{\chi}_I - \overline{\chi}_C)\,(\chi_V + \chi_C)\\
Z^-_- &= Z^{(8)}_{\text{boson}}\,(\overline{\chi}_I + \overline{\chi}_C)\,(\chi_V - \chi_C)\;,
\end{aligned} \tag{3.9}$$

and together these correctly combine to produce the partition function of the *non*-supersymmetric $SO(32)$ string in in Eq. (3.6). As we shall see, a non-trivial orbifold Q

is required for the $E_8 \times E_8$ string as well; indeed, such non-trivial orbifolds are required for heterotic strings in general.

These observations do not constitute a proof of the failure of the temperature/radius correspondence for heterotic strings. Indeed, such a proof is given in chapter 2, and centers on the demonstration that the Boltzmann choice $Q = (-1)^F$ — regardless of its would-be effects — is not a self-consistent orbifold choice for the supersymmeric heterotic strings in ten dimensions. However, the observations given here clearly show that in order to break spacetime supersymmetry in passing from our original model M_1 to our final model M_2, something unique is required for heterotic strings which is not required for Type II superstrings: the presence of an additional Wilson-line action in the orbifold Q. *Indeed, such an extra Wilson line acting on the gauge degrees of freedom is unavoidable in any geometric compactification of the heterotic string — even if we do not wish to alter the gauge symmetry.* Thus, any potential geometric approach towards understanding heterotic string theories at finite temperatures will inevitably involve the introduction of such non-trivial Wilson lines.

3.2.2 Interpreting the Wilson line

Thus far, we have shown that geometric compactifications require non-trivial $\mathbb{Z}_2$ orbifolds Q that transcend the simple Boltzmann choice $Q = (-1)^F$. However, although we have referred to the extra contributions to the orbifold Q as coming from a Wilson line, we have not yet demonstrated this fact. Moreover, we have not yet interpreted this Wilson line *thermodynamically*; our discussion has thus far has focused on the purely geometric side of the temperature/radius correspondence. Our goal is now to address both of these points.

We begin by observing that according to the results in Eq. (3.9), the contributions to the individual $\mathcal{E}_0$ and $\mathcal{E}_{1/2}$ sectors in the corresponding thermal extrapolation of the supersymmetric $SO(32)$ heterotic string must take the forms[1]

$$\begin{aligned} Z^{(1)} &= \overline{\chi}_V \chi_I - \overline{\chi}_S \chi_S \\ Z^{(2)} &= -\overline{\chi}_S \chi_I + \overline{\chi}_V \chi_S \ . \end{aligned} \tag{3.10}$$

While these contributions continue to have the correct overall signs [with bosonic (fermionic) states contributing with overall plus (minus) signs], we nevertheless observe that these results imply non-standard thermal modings for some of the states in the $SO(32)$ heterotic string. In particular, $Z^{(1)}$ is generally expected to receive the contributions from states with integer thermal modings $n \in \mathbb{Z}$, yet the second term within $Z^{(1)}$ corresponds to spacetime fermions, not spacetime bosons. Likewise, $Z^{(2)}$ is generally expected to receive the contributions from states with thermal modings $n \in \mathbb{Z} + 1/2$, yet the second term within $Z^{(1)}$ corresponds to spacetime bosons, not spacetime fermions. Indeed, in each case, it is the states which transform in spinorial representations of the $SO(32)$ gauge group which appear to have the "wrong" modings.

It turns out that a non-trivial Wilson line is precisely what produces such "wrong" modings. Recall that in Sect. 2.2 of Chapter 2, we demonstrated that spacetime bosons (fermions) are expected to have thermal modings $n \in \mathbb{Z}$ ($n \in \mathbb{Z} + 1/2$). We did this by comparing thermal free energies with the vacuum energies of zero-temperature theories compactified on a circle. However, when we calculate these vacuum energies in the presence of a non-trivial gauge field A^μ, we must use the kinematic momenta $\Pi^\mu \equiv p^\mu - \vec{q} \cdot \vec{A}^\mu$ where $\vec{q}$ is the charge (expressed as a vector in root space) of the field in question. If the field A^μ is pure-gauge (*i.e.*, with vanishing corresponding field strength) and our spacetime geometry is trivial, then this change in momenta from p^μ to Π^μ will have no physical effect. However, if we are compactifying on a circle, there is always the possibility that our compactification encloses a gauge-field flux. As in the Aharonov-Bohm effect, this then has the potential to introduce a non-trivial change in modings

[1]Here we are using the definitions and notation in chapter 2, to which we refer the reader for a complete discussion.

for fields around this circle, even if the gauge field A^μ is pure-gauge at all points along the compactification circle. Such a flat (pure-gauge) background for the gauge field A^μ is nothing but a Wilson line.

To be specific, let us first consider the situation in which our compactification circle of radius R completely encloses a $U(1)$ magnetic flux of magnitude Φ which is entirely contained within a radius $\rho < R$. At all points along the compactification circle, this then corresponds to a $U(1)$ gauge field A^μ whose only non-zero component is the component $A^i = -\Phi/(2\pi R)$ along the compactified dimension. Because of the non-trivial topology of the circle, we then find that the shift from p^μ to Π^μ induces a corresponding shift in the corresponding modings:[2]

$$\frac{n}{R} ~\to~ \frac{n}{R} ~+~ \frac{1}{2\pi R}\, q\Phi ~. \tag{3.11}$$

While this result holds for $U(1)$ gauge fields, it is easy to generalize this to the gauge fields of any gauge group G. For any gauge group G, we can describe a corresponding gauge flux in terms of the parameters Φ_i for each $i = 1, ..., r$, where r is the rank of G. Collectively, we can write $\vec{\Phi}$ as a vector in root space. Likewise, the gauge charge of any given state can be described in terms of its Cartan components q_i for $i = 1, ..., r$; collectively, $\vec{q}$ is nothing but the weight of the state in root space. We then find that the modings are shifted according to

$$\frac{n}{R} ~\to~ \frac{n}{R} ~+~ \frac{1}{2\pi R}\, \vec{q}\cdot\vec{\Phi} ~. \tag{3.12}$$

As a result, complex fields which are chosen to be periodic around the compactification circle will have vacuum energies given by

$$\Lambda ~=~ \frac{1}{2\pi R}\,(-1)^F \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \log\left[E_{\mathbf{p}}^2 + \frac{1}{R^2}\left(n + \frac{1}{2\pi}\,\vec{q}\cdot\vec{\Phi}\right)^2\right] \tag{3.13}$$

[2]This discussion of the effects of Wilson lines is mostly field-theoretic. For closed strings, however, there will also be an additional shift due to the possible appearance of a non-trivial winding number. We shall disregard this in the following, since it will play no essential role in our discussion.

where $E_{\mathbf{p}}^2 \equiv \mathbf{p}\cdot\mathbf{p} + m^2$, while complex fields chosen to be anti-periodic around the compactification circle will have vacuum energies given by

$$\Lambda \;=\; \frac{1}{2\pi R}\,(-1)^F \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \log\left[E_{\mathbf{p}}^2 + \frac{1}{R^2}\left(n+\frac{1}{2}+\frac{1}{2\pi}\,\vec q\cdot\vec\Phi\right)^2\right]~. \tag{3.14}$$

Note that in each case, the underlying periodicity properties of the field are unaffected; rather, it is the *manifestations of these periodicities in terms of the modings* which are affected by the appearance of the Wilson line. Thus, we see that effects such as those in Eq. (3.10) — in which certain bosonic and fermionic fields appear to have the wrong modings — can be easily understood as the effects of a non-trivial Wilson line. In particular, the results in Eq. (3.10) for the $SO(32)$ string can be obtained directly if our Wilson line $\vec\Phi$ is chosen such that $\vec q\cdot\vec\Phi = \pi$ (mod 2π) for states in spinorial representations of $SO(32)$, while $\vec q\cdot\vec\Phi = 0$ (mod 2π) for states in vectorial representations of $SO(32)$. Given that $q^i \in \mathbb{Z}$ for vectorial representations of $SO(32)$ and $q^i\in\mathbb{Z}+1/2$ for vectorial representations of $SO(32)$, we see that a simple choice such as $\vec\Phi = (2\pi, 0, ..., 0)$ can easily accomplish this.

This, then, explains how a non-trivial Wilson line can produce unexpected modings due to the non-trivial compactification geometry. However, we still wish to understand the appearance of such a Wilson line thermally. What is the thermal analogue of the non-trivial Wilson line? Specifically, what effect on the thermal side can restore the temperature/radius correspondence if a non-trivial Wilson line has been introduced on the geometric side?

It turns that introducing a non-trivial Wilson line on the geometric side corresponds to introducing a non-zero chemical potential on the thermal side. In fact, this chemical potential will be imaginary. To see this, let us reconsider the partition functions of complex bosons and fermions in the presence of a non-zero chemical potential $\mu \equiv i\tilde\mu$ where $\tilde\mu \in \mathbb{R}$. In general, a complex bosonic field

$$\Phi(\mathbf{x}) \;\sim\; \int \frac{d^3\mathbf{p}}{(2\pi)^3}\left(a_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{3.15}$$

has a grand-canonical partition function given by

$$Z_b(T) \;=\; \prod_{\mathbf{p}} \left[1 + e^{-(E_{\mathbf{p}}-\mu)/T} + e^{-2(E_{\mathbf{p}}-\mu)/T} + ...\right] \left[1 + e^{-(E_{\mathbf{p}}+\mu)/T} + e^{-2(E_{\mathbf{p}}+\mu)/T} + ...\right] \tag{3.16}$$

where the two factors in Eq. (3.16) correspond to particle and anti-particle excitations respectively. The corresponding free energy $F_b(T) \equiv -T \log Z_b$ then takes the form

$$\begin{aligned}
F_b(T) \;&=\; T \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left\{ \log[1 - e^{-(E_{\mathbf{p}}-\mu)/T}] + \log[1 - e^{-(E_{\mathbf{p}}+\mu)/T}] \right\} \\
&=\; \frac{T}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \left\{ \log[(E_{\mathbf{p}} - \mu)^2 + 4\pi^2 n^2 T^2] + \log[(E_{\mathbf{p}} + \mu)^2 + 4\pi^2 n^2 T^2] \right\} \\
&=\; \frac{T}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \log[(E_{\mathbf{p}}^2 - \tilde{\mu}^2 + 4\pi^2 n^2 T^2)^2 + 4\tilde{\mu}^2 E_{\mathbf{p}}^2] \\
&=\; \frac{T}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \log[E_{\mathbf{p}}^4 + 2E_{\mathbf{p}}^2(4\pi^2 n^2 T^2 + \tilde{\mu}^2) + (4\pi^2 n^2 T^2 - \tilde{\mu}^2)^2] \\
&=\; \frac{T}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \log[E_{\mathbf{p}}^4 + 2E_{\mathbf{p}}^2(2\pi n T + \tilde{\mu})^2 + 2E_{\mathbf{p}}^2(-2\pi n T + \tilde{\mu})^2 \\
&\qquad\qquad + (2\pi n T + \tilde{\mu})^2(-2\pi n T + \tilde{\mu})^2] \\
&=\; \frac{T}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \left\{ \log[E_{\mathbf{p}}^2 + (2\pi n T + \tilde{\mu})^2] + \log[E_{\mathbf{p}}^2 + (-2\pi n T + \tilde{\mu})^2] \right\} \\
&=\; T \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \log[E_{\mathbf{p}}^2 + (2\pi n T + \tilde{\mu})^2] \; .
\end{aligned} \tag{3.17}$$

In Eq. (3.17), the second equality follows from the algebraic identities in Sect. 2.2 of Chapter 2, while the final equality results upon exchanging $n \to -n$ in the second term. Thus, comparing the result in Eq. (3.17) with the result in Eq. (3.13), we see that the free energy of a bosonic field at temperature T is equal to the vacuum energy of a periodically-moded field on a circle of radius R, where $R \equiv 1/(2\pi T)$ and where

$$\tilde{\mu} \;=\; (\vec{q}\cdot\vec{\Phi})\, T \qquad \Longrightarrow \qquad \mu \;=\; i\,(\vec{q}\cdot\vec{\Phi})\, T \; . \tag{3.18}$$

A similar result holds for complex fermions and anti-periodic fields, with the same chemical potential. We thus conclude that the introduction of a non-trivial Wilson line on the geometric side corresponds to the introduction of an imaginary, temperature-dependent chemical potential on the thermal side.

We stress, again, that these results do not alter the conclusion of chapter 2, namely that the temperature/radius correspondence is broken for heterotic strings. Specifically, the Boltzmann sum intrinsically assumes that no chemical potential is present, yet we have seen that a non-trivial Wilson line must be introduced for all geometric compactifications of the heterotic string. As a result, the temperature/radius correspondence fails to hold for heterotic strings. However, we now see that this correspondence can be restored if we are willing to introduce an imaginary chemical potential into the heterotic Boltzmann sum. Specifically, once we have determined the appropriate Wilson line for a given heterotic string compactification, we can use the mapping in Eq. (3.18) in order to determine the thermal chemical potential to which it corresponds.

3.3 Choosing the correct Wilson line

Given that the temperature/radius correspondence requires us to turn on a non-trivial Wilson line when constructing the finite-temperature extrapolation of a given zero-temperature string model, we must now tackle the fundamental question: which Wilson line do we choose? In other words, given a specific zero-temperature heterotic string model, which of its many self-consistent $\mathbb{Z}_2$ orbifolds Q leads to the correct finite-temperature theory?

We are forced to address this issue because a given heterotic string model can often be orbifolded in a number of self-consistent ways. Thus, *a priori*, there are many potential choices for the $\mathbb{Z}_2$ orbifold Q. Of course, we have already shown(chapter 2) that the required orbifold Q must transform our original zero-temperature string model M_1 into a different zero-temperature string model M_2 in which spacetime supersymmetry is broken. However, given a particular model M_1, there can be many suitable choices for the corresponding model M_2, and we currently have no guidance as to which model M_2

should be chosen for a given model M_1.

In Sect. 3.2.1, we examined one particular $\mathbb{Z}_2$ orbifold of the ten-dimensional supersymmetric $SO(32)$ heterotic string, namely the one that produces the ten-dimensional *non-supersymmetric* $SO(32)$ heterotic string. However, we chose this orbifold for our discussion merely in order to illustrate the fact that for heterotic strings, any self-consistent $\mathbb{Z}_2$ orbifold must incorporate the effects of a non-trivial Wilson line acting on the gauge sector of the heterotic string — even if we wish to preserve our overall gauge symmetry. However, even though this orbifold is a natural choice, there was no *a priori* reason why the orbifold that describes the finite-temperature version of the supersymmetric $SO(32)$ heterotic string must be the one that preserves its gauge symmetry completely.

This issue is perhaps illustrated even more dramatically for the ten-dimensional $E_8 \times E_8$ heterotic string. *A priori*, following the same logic, we might attempt to construct an orbifold of the ten-dimensional $E_8 \times E_8$ heterotic string which produces a ten-dimensional *non-supersymmetric* $E_8 \times E_8$ heterotic string. Yet, as is well known, no non-supersymmetric $E_8 \times E_8$ heterotic string exists in ten dimensions. Thus, for the $E_8 \times E_8$ heterotic string, it will not even be possible to choose an orbifold Q which preserves the gauge symmetry completely. In this case, the Wilson-line contribution will necessarily break the gauge symmetry to some extent.

Let us therefore systematically survey the possible self-consistent choices for the orbifold Q, for both the $SO(32)$ and $E_8 \times E_8$ heterotic strings. It is easiest to do this by surveying the possible self-consistent non-supersymmetric heterotic string models in ten dimensions, each of which might potentially serve as a suitable endpoint model M_2. These have been classified in Ref. [55], and it turns out that there are only seven such models. These are the tachyon-free $SO(16) \times SO(16)$ string model as well as six tachyonic string models with gauge groups $SO(32)$, $SO(8)\times SO(24)$, $U(16)$, $SO(16)\times E_8$, $(E_7)^2 \times SU(2)^2$, and E_8. The tachyons in the latter six models all have worldsheet energies $(H_R, H_L) = (-1/2, -1/2)$.

However, not all of these models can be realized as $\mathbb{Z}_2$ orbifolds of the original supersymmetric $SO(32)$ or $E_8 \times E_8$ models. Indeed, of the seven non-supersymmetric models listed above, only four are $\mathbb{Z}_2$ orbifolds of the supersymmeric $SO(32)$ string; likewise, only four are $\mathbb{Z}_2$ orbifolds of the $E_8 \times E_8$ string. These $\mathbb{Z}_2$ orbifold relations are shown in Fig. 3.1.

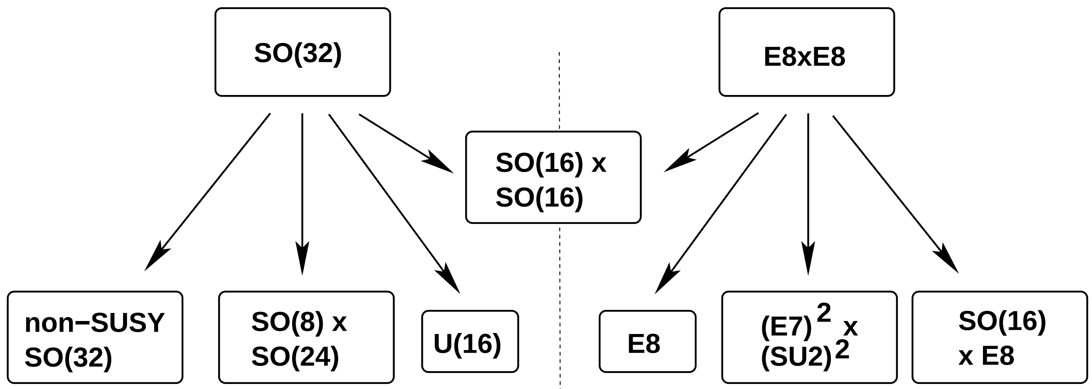

FIGURE 3.1. Possible Wilson-line choices for the supersymmetric $SO(32)$ and $E_8 \times E_8$ heterotic strings, each corresponding to a $\mathbb{Z}_2$ orbifold which breaks spacetime supersymmetry. Note that the $SO(16) \times SO(16)$ string is unique in that it can be realized as a $\mathbb{Z}_2$ orbifold of either the $SO(32)$ or $E_8 \times E_8$ heterotic strings; it is also the only non-supersymmetric heterotic string in ten dimensions which is tachyon-free. By contrast, each of the remaining six non-supersymmetric strings in ten dimensions has a physical tachyon with worldsheet energies $H_L = H_R = -1/2$.

Given these results, we see that there are only four candidate Wilson-line choices for the finite-temperature $SO(32)$ heterotic string. Likewise, there are only four candidate Wilson-line choices for the finite-temperature $E_8 \times E_8$ heterotic string. For each of these Wilson-line choices, we can then construct the corresponding finite-temperature interpolating model following the procedures outlined in chapter 2. Each of these interpolating models is thus a potential candidate for the finite-temperature version of the corresponding zero-temperature supersymmetric theory.

It is straightforward to write down the partition functions of these models. In each case, we shall follow the exact notations and conventions established in Appendix A of chapter 2. However, for convenience, we shall also establish one further convention.

Although the anti-holomorphic (right-moving) parts of these partition functions will always be expressed in terms of the (barred) characters $\overline{\chi}_i$ of the transverse $SO(8)$ Lorentz group, it turns out that we can express the holomorphic (left-moving) parts of each of these partition functions in terms of the (unbarred) characters $\chi_i\chi_j$ associated with the group $SO(16)\times SO(16)$. Indeed, it turns out that such a rewriting is possible in each case regardless of the actual gauge group G of the endpoint model as $T\to\infty$. Of course, if $SO(16)\times SO(16)$ is a subgroup of G, then such a rewriting is meaningful and the characters which appear in the resulting partition function correspond to the actual gauge-group representations which appear in spectrum of the model. By contrast, if $SO(16)\times SO(16)$ is not a subgroup of G, then such a rewriting is merely an algebraic exercise; the $SO(16)\times SO(16)$ characters then have no meaning beyond their q-expansions, and can appear with non-integer coefficients. In all cases, however, these expressions represent the true partition functions of these interpolating models as far as their q-expansions are concerned. We shall therefore follow these conventions in what follows.

Let us begin by considering the supersymmetric $SO(32)$ heterotic string, which has partition function

$$Z^{(8)}_{\rm boson}\,(\overline{\chi}_V-\overline{\chi}_S)\,(\chi_I^2+\chi_V^2+\chi_S^2+\chi_C^2) \tag{3.19}$$

when expressed in terms of the characters of $SO(16)\times SO(16)$. For this string, our four candidate finite-temperature extrapolations are then as follows. The partition function of the interpolating model associated with the non-supersymmetric $SO(32)$ endpoint is given by

$$\begin{aligned}
Z_{SO(32)} \;=\; Z^{(8)}_{\rm boson} \;\times \Big\{ & \\
& [\overline{\chi}_V\,(\chi_I^2+\chi_V^2) \;-\; \overline{\chi}_S\,(\chi_S^2+\chi_C^2)]\;\mathcal{E}_0 \\
+\; & [\overline{\chi}_V\,(\chi_S^2+\chi_C^2) \;-\; \overline{\chi}_S\,(\chi_I^2+\chi_V^2)]\;\mathcal{E}_{1/2} \\
+\; & [\overline{\chi}_I\,(\chi_I\chi_V+\chi_V\chi_I) \;-\; \overline{\chi}_C\,(\chi_S\chi_C+\chi_C\chi_S)]\;\mathcal{O}_0 \\
+\; & [\overline{\chi}_I\,(\chi_S\chi_C+\chi_C\chi_S) \;-\; \overline{\chi}_C\,(\chi_I\chi_V+\chi_V\chi_I)]\;\mathcal{O}_{1/2}\;\Big\}\,,
\end{aligned} \tag{3.20}$$

while the partition functions of the interpolating models associated with the $SO(8) \times SO(24)$, U_{16}, and $SO(16) \times SO(16)$ endpoints are respectively given by

$$\begin{aligned} Z_{SO(8)\times SO(24)} \;=\;\; & Z^{(8)}_{\rm boson} \;\; \times \Big\{ \\ & [\overline{\chi}_V \, (\chi_I^2 + \frac{1}{4}\chi_V^2 + \frac{3}{4}\chi_S^2) \; - \; \overline{\chi}_S \, (\frac{1}{4}\chi_S^2 + \frac{3}{4}\chi_V^2 + \chi_C^2)] \; \mathcal{E}_0 \\ + \;\; & [\overline{\chi}_V \, (\frac{1}{4}\chi_S^2 + \frac{3}{4}\chi_V^2 + \chi_C^2) \; - \; \overline{\chi}_S \, (\chi_I^2 + \frac{1}{4}\chi_V^2 + \frac{3}{4}\chi_S^2)] \; \mathcal{E}_{1/2} \\ + \;\; & [\overline{\chi}_I \, (\frac{1}{2}\chi_I\chi_V + \frac{3}{2}\chi_V\chi_C) \; - \; \overline{\chi}_C \, (\frac{1}{2}\chi_S\chi_C + \frac{3}{2}\chi_I\chi_S)] \; \mathcal{O}_0 \\ + \;\; & [\overline{\chi}_I \, (\frac{1}{2}\chi_S\chi_C + \frac{3}{2}\chi_I\chi_S) \; - \; \overline{\chi}_C \, (\frac{1}{2}\chi_I\chi_V + \frac{3}{2}\chi_V\chi_C)] \; \mathcal{O}_{1/2} \;\; \Big\} \; , \end{aligned} \tag{3.21}$$

$$\begin{aligned} Z_{U(16)} \;=\;\; & Z^{(8)}_{\rm boson} \;\; \times \Big\{ \\ & [\overline{\chi}_V \, (\chi_I^2 + \frac{1}{16}\chi_V^2 + \frac{15}{16}\chi_S^2) \; - \; \overline{\chi}_S \, (\frac{1}{16}\chi_S^2 + \frac{15}{16}\chi_V^2 + \chi_C^2)] \; \mathcal{E}_0 \\ + \;\; & [\overline{\chi}_V \, (\frac{1}{16}\chi_S^2 + \frac{15}{16}\chi_V^2 + \chi_C^2) \; - \; \overline{\chi}_S \, (\chi_I^2 + \frac{1}{16}\chi_V^2 + \frac{15}{16}\chi_S^2)] \; \mathcal{E}_{1/2} \\ + \;\; & [\overline{\chi}_I \, (\frac{1}{8}\chi_I\chi_V + \frac{15}{8}\chi_V\chi_C) \; - \; \overline{\chi}_C \, (\frac{1}{8}\chi_S\chi_C + \frac{15}{8}\chi_I\chi_S)] \; \mathcal{O}_0 \\ + \;\; & [\overline{\chi}_I \, (\frac{1}{8}\chi_S\chi_C + \frac{15}{8}\chi_I\chi_S) \; - \; \overline{\chi}_C \, (\frac{1}{8}\chi_I\chi_V + \frac{15}{8}\chi_V\chi_C)] \; \mathcal{O}_{1/2} \;\; \Big\} \; , \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} Z_{SO(16)\times SO(16)} \;=\;\; & Z^{(8)}_{\rm boson} \;\; \times \Big\{ \\ & [\overline{\chi}_V \, (\chi_I^2 + \chi_S^2) \; - \; \overline{\chi}_S \, (\chi_V^2 + \chi_C^2)] \; \mathcal{E}_0 \\ + \;\; & [\overline{\chi}_V \, (\chi_V^2 + \chi_C^2) \; - \; \overline{\chi}_S \, (\chi_I^2 + \chi_S^2)] \; \mathcal{E}_{1/2} \\ + \;\; & [\overline{\chi}_I \, (\chi_V\chi_C + \chi_C\chi_V) \; - \; \overline{\chi}_C \, (\chi_I\chi_S + \chi_S\chi_I)] \; \mathcal{O}_0 \\ + \;\; & [\overline{\chi}_I \, (\chi_I\chi_S + \chi_S\chi_I) \; - \; \overline{\chi}_C \, (\chi_V\chi_C + \chi_C\chi_V)] \; \mathcal{O}_{1/2} \;\; \Big\} \; . \end{aligned} \tag{3.23}$$

Using the identities listed in the Appendix of chapter 2, it is straightforward to verify that as $T \to 0$, each of these expressions reduces to Eq. (3.19), as required.

A similar situation exists for the $E_8 \times E_8$ heterotic string, which has partition function

$$Z^{(8)}_{\text{boson}}\,(\overline{\chi}_V - \overline{\chi}_S)\,(\chi_I + \chi_S)^2 \tag{3.24}$$

when expressed in terms of the characters of $SO(16)\times SO(16)$. The partition function of the interpolating model associated with the non-supersymmetric $SO(16)\times E_8$ endpoint is given by

$$\begin{aligned} Z_{SO(16)\times E_8} \;=\;\; & Z^{(8)}_{\text{boson}} \;\times \Big\{ \\ & \quad [\overline{\chi}_V\,\chi_I \;-\; \overline{\chi}_S\,\chi_S]\;\mathcal{E}_0 \\ & + \; [\overline{\chi}_V\,\chi_S \;-\; \overline{\chi}_S\,\chi_I]\;\mathcal{E}_{1/2} \\ & + \; [\overline{\chi}_I\,\chi_V \;-\; \overline{\chi}_C\,\chi_C]\;\mathcal{O}_0 \\ & + \; [\overline{\chi}_I\,\chi_C \;-\; \overline{\chi}_C\,\chi_V]\;\mathcal{O}_{1/2} \;\Big\} \;\times\; (\chi_I + \chi_S)\;, \end{aligned} \tag{3.25}$$

while the partition functions of the interpolating models associated with the $(E_7)^2 \times SU(2)^2$, E_8, and $SO(16)\times SO(16)$ endpoints are respectively given by

$$\begin{aligned} Z_{E_7^2\times SU(2)^2} \;=\;\; & Z^{(8)}_{\text{boson}} \;\times \Big\{ \\ & \quad [\overline{\chi}_V\,(\chi_I^2 + \frac{1}{4}\chi_I\chi_S + \frac{3}{4}\chi_S^2) \;-\; \overline{\chi}_S\,(\frac{1}{4}\chi_S^2 + \frac{7}{4}\chi_I\chi_S)]\;\mathcal{E}_0 \\ & + \; [\overline{\chi}_V\,(\frac{1}{4}\chi_S^2 + \frac{7}{4}\chi_I\chi_S) \;-\; \overline{\chi}_S\,(\chi_I^2 + \frac{1}{4}\chi_I\chi_S + \frac{3}{4}\chi_S^2)]\;\mathcal{E}_{1/2} \\ & + \; [\overline{\chi}_I\,(\frac{1}{4}\chi_I\chi_V + \frac{7}{4}\chi_V\chi_S) \;-\; \overline{\chi}_C\,(\frac{1}{4}\chi_S\chi_S + \frac{7}{4}\chi_I\chi_S)]\;\mathcal{O}_0 \\ & + \; [\overline{\chi}_I\,(\frac{1}{4}\chi_S\chi_S + \frac{7}{4}\chi_I\chi_S) \;-\; \overline{\chi}_C\,(\frac{1}{4}\chi_I\chi_V + \frac{7}{4}\chi_V\chi_S)]\;\mathcal{O}_{1/2} \;\Big\}\;, \end{aligned} \tag{3.26}$$

$$
\begin{aligned}
Z_{E_8} \;=\; & Z^{(8)}_{\rm boson} \times \Big\{ \\
& [\overline{\chi}_V\,(\chi_I^2 + \frac{1}{16}\chi_I\chi_S + \frac{15}{16}\chi_S^2) \;-\; \overline{\chi}_S\,(\frac{1}{16}\chi_S^2 + \frac{31}{16}\chi_I\chi_S)]\;\mathcal{E}_0 \\
+\; & [\overline{\chi}_V\,(\frac{1}{16}\chi_S^2 + \frac{31}{16}\chi_I\chi_S) \;-\; \overline{\chi}_S\,(\chi_I^2 + \frac{1}{16}\chi_I\chi_S + \frac{15}{16}\chi_S^2)]\;\mathcal{E}_{1/2} \\
+\; & [\overline{\chi}_I\,(\frac{1}{16}\chi_I\chi_V + \frac{31}{16}\chi_V\chi_S) \;-\; \overline{\chi}_C\,(\frac{1}{16}\chi_S\chi_S + \frac{31}{16}\chi_I\chi_S)]\;\mathcal{O}_0 \\
+\; & [\overline{\chi}_I\,(\frac{1}{16}\chi_S\chi_S + \frac{31}{16}\chi_I\chi_S) \;-\; \overline{\chi}_C\,(\frac{1}{16}\chi_I\chi_V + \frac{31}{16}\chi_V\chi_S)]\;\mathcal{O}_{1/2} \;\;\Big\}~,
\end{aligned}
\tag{3.27}
$$

and

$$
\begin{aligned}
Z_{SO(16)\times SO(16)} \;=\; & Z^{(8)}_{\rm boson} \times \Big\{ \\
& [\overline{\chi}_V\,(\chi_I^2 + \chi_S^2) \;-\; \overline{\chi}_S\,(\chi_I\chi_S + \chi_S\chi_I)]\;\mathcal{E}_0 \\
+\; & [\overline{\chi}_V\,(\chi_I\chi_S + \chi_S\chi_I) \;-\; \overline{\chi}_S\,(\chi_I^2 + \chi_S^2)]\;\mathcal{E}_{1/2} \\
+\; & [\overline{\chi}_I\,(\chi_V\chi_C + \chi_C\chi_V) \;-\; \overline{\chi}_C\,(\chi_V^2 + \chi_C^2)]\;\mathcal{O}_0 \\
+\; & [\overline{\chi}_I\,(\chi_V^2 + \chi_C^2) \;-\; \overline{\chi}_C\,(\chi_V\chi_C + \chi_C\chi_V)]\;\mathcal{O}_{1/2} \;\;\Big\}~.
\end{aligned}
\tag{3.28}
$$

Once again, using the identities listed in the Appendix of chapter 2, it is straightforward to verify that each of these expressions reduces to Eq. (3.24) as $T \to 0$. Moreover, the expressions in Eqs. (3.23) and (3.28) are actually equal as the result of the further identity on $SO(16)$ characters given by

$$
\chi_I\chi_S + \chi_S\chi_I \;=\; \chi_V^2 + \chi_C^2~. \tag{3.29}
$$

This is ultimately the identity which is responsible for the fact that Eqs. (3.19) and (3.24) are equal at the level of their q-expansions, *i.e.*, that the ten-dimensional supersymmetric $SO(32)$ and $E_8 \times E_8$ heterotic strings have the same bosonic and fermionic state degeneracies at each mass level.

As an aside, it is interesting to note that all of these interpolating functions can be written in a common form parametrized by a single integer ℓ:

$$
\begin{aligned}
Z_{\text{boson}}^{(8)} \times \Bigg\{ \quad & \left[\overline{\chi}_V \left(\chi_I^2 + \frac{1}{\ell}\chi_V^2 + \frac{\ell-1}{\ell}\chi_S^2\right) - \overline{\chi}_S \left(\chi_C^2 + \frac{1}{\ell}\chi_S^2 + \frac{\ell-1}{\ell}\chi_V^2\right)\right] \mathcal{E}_0 \\
+ & \left[\overline{\chi}_V \left(\chi_C^2 + \frac{1}{\ell}\chi_S^2 + \frac{\ell-1}{\ell}\chi_V^2\right) - \overline{\chi}_S \left(\chi_I^2 + \frac{1}{\ell}\chi_V^2 + \frac{\ell-1}{\ell}\chi_S^2\right)\right] \mathcal{E}_{1/2} \\
+ & \left[\overline{\chi}_I \left(\frac{1}{\ell}\chi_I\chi_V + \frac{1}{\ell}\chi_V\chi_I + \frac{\ell-1}{\ell}\chi_V\chi_C + \frac{\ell-1}{\ell}\chi_C\chi_V\right)\right. \\
& \left. - \overline{\chi}_C \left(\frac{1}{\ell}\chi_S\chi_C + \frac{1}{\ell}\chi_C\chi_S + \frac{\ell-1}{\ell}\chi_V^2 + \frac{\ell-1}{\ell}\chi_C^2\right)\right] \mathcal{O}_0 \\
+ & \left[\overline{\chi}_I \left(\frac{1}{\ell}\chi_S\chi_C + \frac{1}{\ell}\chi_C\chi_S + \frac{\ell-1}{\ell}\chi_V^2 + \frac{\ell-1}{\ell}\chi_C^2\right)\right. \\
& \left. - \overline{\chi}_C \left(\frac{1}{\ell}\chi_I\chi_V + \frac{1}{\ell}\chi_V\chi_I + \frac{\ell-1}{\ell}\chi_V\chi_C + \frac{\ell-1}{\ell}\chi_C\chi_V\right)\right] \mathcal{O}_{1/2} \Bigg\} .
\end{aligned}
\tag{3.30}
$$

In particular, the values $\ell = \{1, 2, 4, 8, 16, 32, \infty\}$ correspond to the partition functions in Eqs. (3.20), (3.25), (3.21), (3.26), (3.22), (3.27), and (3.23) [or (3.28)] respectively, where Eq. (3.29) has been used wherever needed.

Note that each of these finite-temperature partition functions corresponds to a *bona-fide* geometric compactification of the appropriate ten-dimensional heterotic string [either $SO(32)$ or $E_8 \times E_8$], and thereby satisfies the temperature/radius correspondence. Moreover, we see that none of these partition functions contains a ground-state term of the form

$$\overline{\chi_I}\, \chi_I^2\, \mathcal{O}_{1/2} \tag{3.31}$$

which was shown in Sect. 2.4 of Chapter 2 to signal an inconsistent orbifold. Thus, all of these partition functions pass the tests developed in chapter 2.

Our final task, then, is to determine which of these partition functions describe the appropriate finite-temperature extrapolations for the supersymmetric $SO(32)$ and $E_8 \times E_8$ heterotic strings. However, there is no mystery in making this selection: the orbifold choice which is preferred dynamically for each string is the one which leads to a minimization of the ten-dimensional free-energy density $F(T)$. Moreover, given the thermal partition functions $Z(\tau, T)$ listed above, calculating $F(T)$ in each case is relatively straightforward: we simply evaluate

$$F(T) \;=\; -\tfrac{1}{2}\, T\, \mathcal{M}^9 \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\, Z(\tau, T) \tag{3.32}$$

where $\mathcal{M} \equiv M_{\mathrm{string}}/(2\pi)$ and where $\mathcal{F}$ is the fundamental domain of the modular group:

$$\mathcal{F} \;\equiv\; \{\tau :\; |\mathrm{Re}\,\tau| \le \tfrac{1}{2}, \mathrm{Im}\,\tau > 0, |\tau| \ge 1\}\;. \tag{3.33}$$

We then obtain the results shown in Fig. 3.2.

We observe from Fig. 3.2 that *the Wilson line which minimizes the free energy in each case is the one which breaks the gauge group minimally. For the $SO(32)$ string, this is the Wilson line leading to the non-supersymmetric $SO(32)$ string as $T \to \infty$, while for the $E \times E_8$ heterotic string, this is the Wilson line leading to the non-supersymmetric $SO(16) \times E_8$ string.* As discussed above, it is perhaps not surprising that the Wilson lines which break the gauge group minimally are those which lead to the most negative free energies, for this result is the closest way in which one can mimic mimics what happens in the Type II situation (in which there are no internal gauge groups to be modified at all by thermal effects). The critical difference in the heterotic case, however, is that we cannot preserve our gauge groups, even in the $SO(32)$ case, without turning on a non-trivial Wilson line. Such non-trivial Wilson lines (or equivalently, the non-zero chemical potentials to which they correspond) are then the means by which the temperature/radius correspondence is restored.

We can therefore summarize our results as follows. If we were to follow the standard Boltzmann approach to finite-temperature string theory, the zero-temperature ten-dimensional supersymmetric $SO(32)$ heterotic string [with partition function given in

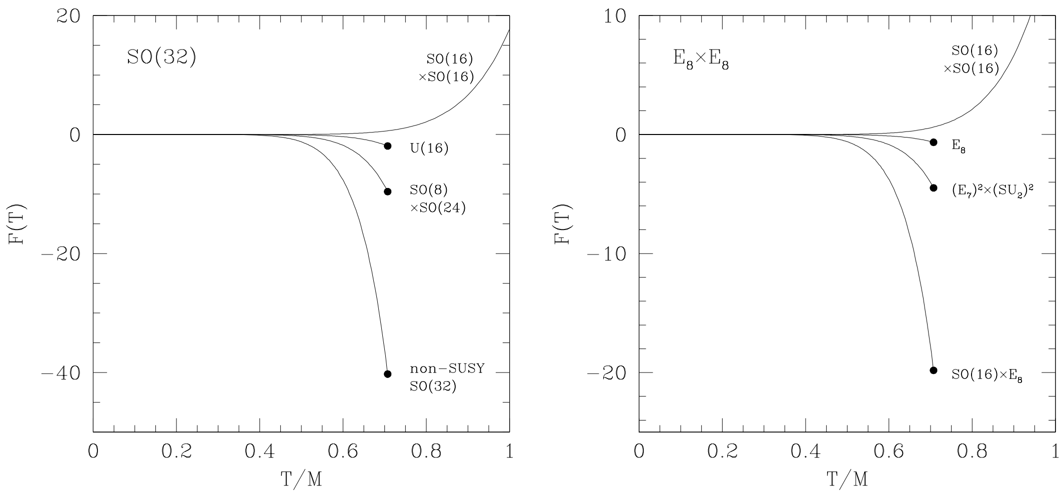

FIGURE 3.2. Free energies $F(T)$ in units of $\frac{1}{2}\mathcal{M}^{10} = \frac{1}{2}(M_{\rm string}/2\pi)^{10}$, plotted as functions of the normalized temperature $T/\mathcal{M}$ for the $SO(32)$ heterotic string (left plot) and $E_8 \times E_8$ heterotic string (right plot). In each case, the free energies are shown for the four corresponding choices of allowed Wilson lines. We see that in general $F(T) \to 0$ as $T \to 0$, in accordance with the spacetime supersymmetry which exists at zero temperature. At non-zero temperatures, however, we see that the Wilson line which minimizes the free energy in each case is the one which breaks the gauge group minimally: for the $SO(32)$ string, this is the Wilson line leading to the non-supersymmetric $SO(32)$ string as $T \to \infty$, while for the $E \times E_8$ heterotic string, this is the Wilson line leading to the non-supersymmetric $SO(16) \times E_8$ string. With the sole exception of the Wilson line leading to the $SO(16) \times SO(16)$ heterotic string, each of the Wilson-line choices in each case leads to a free energy which is negative for all $T > 0$ and diverges discontinuously at the critical temperature $T_H \equiv \mathcal{M}/\sqrt{2}$ (indicated in each case with a solid black dot). As will be discussed in Sect. 3.4, these divergences arise in each case due to the existence of a thermal winding state which is massive for all $T < T_H$, massless at $T = T_H$, and tachyonic for all $T > T_H$.

Eq. (3.19)] would have a finite-temperature extrapolation given by

$$\begin{aligned} Z(\tau,T) \;=\; Z^{(8)}_{\rm boson} \times \Big\{ \quad & \overline{\chi}_V \; (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \;\; \mathcal{E}_0 \\ - \;\; & \overline{\chi}_S \; (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \;\; \mathcal{E}_{1/2} \\ - \;\; & \overline{\chi}_C \; (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \;\; \mathcal{O}_0 \\ + \;\; & \overline{\chi}_I \; (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \;\; \mathcal{O}_{1/2} \; \Big\} \end{aligned} \tag{3.34}$$

where we are continuing to express our internal (holomorphic) degrees of freedom in terms of $SO(16) \times SO(16)$ characters. This is indeed the standard result [13] which corresponds to string thermodynamics as it is currently practiced in the string literature. However, as we have shown in chapter 2, this result is at odds with the temperature/radius correspondence: in particular, it does not correspond to any self-consistent nine-dimensional geometric compactification of the supersymmetric $SO(32)$ string. What we have now shown is that there *does* exist a finite-temperature extrapolation of the supersymmetric $SO(32)$ heterotic string which is fully consistent with the temperature/radius correspondence and which minimizes the associated free energy: this is the alternative result given by

$$\begin{aligned} Z(\tau,T) \;=\; Z^{(8)}_{\text{boson}} \times \Big\{ \quad & [\overline{\chi}_V\,(\chi_I^2+\chi_V^2) \;-\; \overline{\chi}_S\,(\chi_S^2+\chi_C^2)]\;\mathcal{E}_0 \\ +\; & [\overline{\chi}_V\,(\chi_S^2+\chi_C^2) \;-\; \overline{\chi}_S\,(\chi_I^2+\chi_V^2)]\;\mathcal{E}_{1/2} \\ +\; & [\overline{\chi}_I\,(\chi_I\chi_V+\chi_V\chi_I) \;-\; \overline{\chi}_C\,(\chi_S\chi_C+\chi_C\chi_S)]\;\mathcal{O}_0 \\ +\; & [\overline{\chi}_I\,(\chi_S\chi_C+\chi_C\chi_S) \;-\; \overline{\chi}_C\,(\chi_I\chi_V+\chi_V\chi_I)]\;\mathcal{O}_{1/2} \;\; \Big\}\;. \end{aligned} \tag{3.35}$$

A similar conclusion exists for the ten-dimensional $E_8 \times E_8$ heterotic string [whose partition function is given in Eq. (3.24)]: the traditional Boltzmann approach would yield the finite-temperature extrapolation

$$\begin{aligned} Z(\tau,T) \;=\; Z^{(8)}_{\text{boson}} \times \Big\{ \quad & \overline{\chi}_V\;(\chi_I+\chi_S)^2 \;\;\mathcal{E}_0 \\ -\;\; & \overline{\chi}_S\;(\chi_I+\chi_S)^2 \;\;\mathcal{E}_{1/2} \\ -\;\; & \overline{\chi}_C\;(\chi_I+\chi_S)^2 \;\;\mathcal{O}_0 \\ +\;\; & \overline{\chi}_I\;(\chi_I+\chi_S)^2 \;\;\mathcal{O}_{1/2}\;\Big\}\;, \end{aligned} \tag{3.36}$$

yet the result which is consistent with the temperature/radius correspondence and which minimizes the corresponding free energy is actually given by

$$\begin{aligned} Z(\tau,T) \;=\; Z^{(8)}_{\rm boson} \times \Big\{ & \quad [\overline{\chi}_V\,\chi_I \;-\; \overline{\chi}_S\,\chi_S]\;\mathcal{E}_0 \\ & + \; [\overline{\chi}_V\,\chi_S \;-\; \overline{\chi}_S\,\chi_I]\;\mathcal{E}_{1/2} \\ & + \; [\overline{\chi}_I\,\chi_V \;-\; \overline{\chi}_C\,\chi_C]\;\mathcal{O}_0 \\ & + \; [\overline{\chi}_I\,\chi_C \;-\; \overline{\chi}_C\,\chi_V]\;\mathcal{O}_{1/2} \;\Big\} \;\times\; (\chi_I + \chi_S)\;. \end{aligned} \tag{3.37}$$

All of these partition functions reduce to the appropriate ten-dimensional zero-temperature partition functions as $T \to 0$, and are modular invariant for all T. However, in each case, it is easy to see how our proposed "geometric" result differs from the traditional Boltzmann result: certain zero-temperature states which accrue periodic modings around the thermal circle in the traditional Boltzmann formulation now accrue anti-periodic modings around the thermal circle in our "geometric" formulation, and *vice versa*. *This is not a violation of any spin-statistics relations.* Instead, as shown in Sect. 3.2, this is nothing but the direct effect of turning on a non-trivial Wilson line upon compactification, or equivalently introducing a temperature-dependent chemical potential into the traditional Boltzmann sum. Such non-trivial Wilson lines are needed in order to ensure that heterotic strings are compatible with the temperature/radius correspondence, and in particular ensure that the problematic $\overline{\chi_I}\chi_I^2\mathcal{O}_{1/2}$ terms which appear in Eqs. (3.34) and (3.36) are actually eliminated from Eqs. (3.35) and (3.37) without breaking modular invariance.

3.4 Implications for the Hagedorn transition

The Hagedorn transition is one of the central hallmarks of string thermodynamics. Originally discovered in the 1960's through studies of hadronic resonances and the so-called "statistical bootstrap" [21, 29, 30], the Hagedorn transition is now understood to be a generic feature of any theory exhibiting a density of states which rises exponentially as a function of mass. In string theory, the number of states of a given total mass depends on the number of ways in which that mass can be partitioned amongst individual quantized mode contributions, leading to an exponentially rising density of states [2]. Thus, string theories should exhibit a Hagedorn transition [31, 32, 11, 13, 33]. Originally, it was assumed that the Hagedorn temperature is a limiting temperature at which the internal energy of the system diverges. However, later studies demonstrated that the internal energy actually remains finite at this temperature. This then suggests that the Hagedorn temperature is merely the critical temperature corresponding to a first- or second-order phase transition.

There have been many speculations concerning possible interpretations of this phase transition, including a breakdown of the string worldsheet into vortices [32] or a transition to a single long-string phase [33]. It has also been speculated that there is a dramatic loss of degrees of freedom at high temperatures [13]. Over the past two decades, studies of the Hagedorn transition have reached across the entire spectrum of modern string-theory research, including open strings and D-branes, strings with non-trivial spacetime geometries (including AdS backgrounds and *pp*-waves), strings in magnetic fields, $\mathcal{N}=4$ strings, tensionless strings, non-critical strings, two-dimensional strings, little strings, matrix models, non-commutative theories, as well as possible cosmological implications and implications for the brane world. A brief selection of papers in many of these areas appears in Refs. [14, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46].

Our goal in this section will be to understand the implications of our "geometric" results for the Hagedorn transition. In particular, we shall examine the consequences of replacing Eq. (3.34) with Eq. (3.35), and replacing Eq. (3.36) with Eq. (3.37). Our focus here will be on the tachyons and temperature associated with the Hagedorn transition, since it turns out that both of these features will be changed if we adopt the "geometric" rather than the traditional approach.

Surprisingly, we shall find that the traditional Hagedorn temperature for heterotic strings is shifted to a new value which happens to coincide with the traditional Type II value. As a result, our "geometric" approach to finite-temperature string theory results in a universal Hagedorn temperature for all tachyon-free closed string models in ten dimensions. Moreover, in chapter 4, we shall demonstrate that a similar result also holds for Type I (open) strings. Thus, we see that our "geometric" approach restores a certain unity to all finite-temperature string theories, endowing string theory with a universal Hagedorn temperature regardless of the particular class of model under consideration.

3.4.1 The Hagedorn transition: UV versus IR

We begin with several preliminary remarks concerning the UV/IR nature of the Hagedorn transition.

In general, once we have determined the correct finite-temperature partition function $Z_{\text{string}}(\tau, T)$ that describes the thermodynamics associated with a given zero-temperature string model, the one-loop thermal vacuum amplitude $\mathcal{V}(T)$ (the analogue of the logarithm of the statistical-mechanical partition function) is given by the modular integral [10]

$$\mathcal{V}(T) \;\equiv\; -\,\tfrac{1}{2}\,\mathcal{M}^{D-1} \int_{\mathcal{F}} \frac{d^2\tau}{{\tau_2}^2}\, Z_{\text{string}}(\tau, T) \qquad (3.38)$$

where $\mathcal{F}$ is the fundamental domain of the modular group given in Eq. (3.33) and where $\tau_2 \equiv \text{Im}\,\tau$. Given this definition for $\mathcal{V}$, the full panoply of thermodynamic quantities such as the free energy F, internal energy U, entropy S, and specific heat c_V then follow from the standard definitions $F \equiv T\mathcal{V}$, $U \equiv -T^2 d\mathcal{V}/dT$, $S \equiv -dF/dT$, and $c_V \equiv dU/dT$. In string theory, the Hagedorn transition is usually associated with a divergence or other discontinuity in the vacuum amplitude $\mathcal{V}(T)$ as a function of temperature. It turns out that are only two ways in which such a divergence may arise within the expression in Eq. (3.38).

First, of course, is the possibility of a divergence or discontinuity due to the well-known exponential rise in the degeneracy of string states which contribute to $Z_{\text{string}}(\tau, T)$. This may be considered an ultraviolet (UV) divergence because it is triggered by the behavior of the extremely massive end of the string spectrum. However, it turns out that this rise in the state degeneracies ultimately does not cause $\mathcal{V}(T)$ to diverge. To understand why, we may expand $Z(\tau, T)$ in the form $\sum_{MN} a_{MN}\overline{q}^M q^N$ where (M, N) describe the right- and left-moving worldsheet energies (with thermal contributions included), where a_{MN} describe the corresponding degeneracies of bosonic minus fermionic states, and where $q \equiv e^{2\pi i\tau}$. Although the degeneracies a_{MN} indeed experience exponential growth of the generic form $a_{MN} \sim \exp\left(C_R\sqrt{M} + C_L\sqrt{N}\right)$ where $C_{L,R}$ are positive coefficients, the contribution of each such state to the modular integrand is suppressed according to $|\overline{q}^M q^N| \sim \exp[-2\pi\tau_2(M+N)]$. For all $\tau_2 > 0$ and sufficiently large (M, N), this exponential suppression easily overwhelms the exponential rise in the degeneracy of states. As a result, the integrand in Eq. (3.38) remains convergent everywhere except as $\tau_2 \to 0$. However, this dangerous UV region is explicitly excised from the fundamental domain $\mathcal{F}$ in Eq. (3.33). Thus, we conclude that the expression in Eq. (3.38) does not suffer from any UV divergences resulting from the exponential growth in the asymptotic degeneracies of states.

On the other hand, the expression in Eq. (3.38) may experience a divergence due to on-shell states within $Z_{\text{string}}(\tau, T)$ which may become massless or tachyonic at specific critical temperatures. This can therefore be considered an infrared (IR) divergence. Indeed, the geometric approach we have been following in this chapter almost guarantees that such a divergence will arise. As required by the temperature/radius correspondence, $Z_{\text{string}}(\tau, T)$ will generically interpolate from a supersymmetric (and hence tachyon-free) theory at $T = 0$ to a non-supersymmetric (and indeed tachyonic) theory as $T \to \infty$. As a result, there necessarily exists a critical temperature T_H at which certain states which were massive for $T < T_H$ become massless at $T = T_H$ and ultimately tachyonic for $T > T_H$. Since such on-shell tachyons correspond to states with worldsheet energies $M = N < 0$, their contributions to the modular integral in Eq. (3.38) *grow* as $(\overline{q}q)^N \sim \exp(+4\pi\tau_2|N|)$. The contributions from the (infrared) $\tau_2 \to \infty$ region of the fundamental domain then lead to a divergence for $\mathcal{V}(T)$.

Thus, a study of the Hagedorn transition in string theory essentially reduces to a study of the *tachyonic* structure of $Z_{\rm string}(\tau,T)$ as a function of temperature. Before proceeding further, however, we caution that we have reached this conclusion only because we have chosen to work in the so-called $\mathcal{F}$-representation for $\mathcal{V}(T)$ given in Eq. (3.38). By contrast, utilizing Poisson resummations and modular transformations [11], we can always rewrite $\mathcal{V}(T)$ as the integration of a different integrand $Z'_{\rm string}(\tau,T)$ over the strip

$$\mathcal{S} \equiv \{\tau : \, |{\rm Re}\,\tau| \leq \tfrac{1}{2}, {\rm Im}\,\tau > 0\} \, . \tag{3.39}$$

In such an $\mathcal{S}$-representation, the IR divergence as $\tau_2 \to \infty$ is transformed into a UV divergence as $\tau_2 \to 0$. This formulation thus has the advantage of relating the Hagedorn transformation directly to a UV phenomenon such as the exponential rise in the degeneracy of states. However, both formulations are mathematically equivalent; indeed, modular invariance provides a tight relation between the tachyonic structure of a given partition function and the rate of exponential growth in its asymptotic degeneracy of states [48, 57, 49, 54]. In the following, therefore, we shall utilize the $\mathcal{F}$-representation for $\mathcal{V}(T)$ and focus on only the tachyonic structure of $Z_{\rm string}(\tau,T)$, but we shall comment on the connection to the asymptotic degeneracy of states in Sect. 3.4 C.

3.4.2 A new Hagedorn temperature for heterotic strings

So what then are the potential tachyonic states within $Z_{\rm string}(\tau,T)$, and at what temperature T_H do they first arise? Note that we are concerned with states whose masses are temperature-dependent: positive at temperatures below a certain critical temperature, zero at the critical temperature, and tachyonic at temperatures immediately above the critical temperature. The sudden appearance of such new "thermally massless" states at a critical temperature T_H is the signal of the appearance of the long-range order normally associated with a phase transition, and the fact that such states generally become tachyonic immediately above T_H reflects the instabilities which are also normally associated with a phase transition.

As a result, in order to derive the Hagedorn temperature of a given theory, it is sufficient to search for states within the thermal partition function Z_{string} whose masses decrease as a function of temperature, reaching (and perhaps even crossing) zero at a certain critical temperature. We shall refer to such states as "thermally massless" at the critical temperature. Since thermal effects always provide a positive contribution to the squared masses of any states, such states must intrinsically be tachyonic at zero temperature. In other words, for such thermally massless states, masslessness is achieved at the critical temperature T_H as the result of a balance between a tachyonic non-thermal mass contribution (arising from the characters $\overline{\chi}_i \chi_j \chi_k$ within Z_{string}) and an additional positive temperature-dependent thermal mass contribution (arising from the thermal $\mathcal{E}, \mathcal{O}$ functions).

We can quantify this mathematically as follows. A given state with worldsheet energies (H_R, H_L) will contribute a term of the form $\overline{q}^{H_R} q^{H_L}$ to the characters $\overline{\chi}_i \chi_j \chi_k$ within Z_{string}. Likewise, as evident from their definitions in Appendix A of chapter 2, the thermal $\mathcal{E}, \mathcal{O}$ functions will make an additional, thermal contribution to these energies which is given by

$$[\Delta H_R, \Delta H_L] \;=\; [\tfrac{1}{4}(ma - n/a)^2,\; \tfrac{1}{4}(ma + n/a)^2] \tag{3.40}$$

where (m, n) are respectively the momentum and winding quantum numbers around the thermal circle and where $a \equiv T/\mathcal{M} = T/(2\pi M_{\text{string}})$. The conditions for thermal masslessness then become

$$H_R + \tfrac{1}{4}(ma - n/a)^2 \;=\; 0\;, \qquad H_L + \tfrac{1}{4}(ma + n/a)^2 \;=\; 0\;, \tag{3.41}$$

which together imply the useful relation $mn = H_R - H_L$. Since the thermal contributions in Eq. (3.40) are strictly non-negative (and are not zero, according to our assumption of *thermal* masslessness), we see that the possibility of obtaining a thermally massless state requires that either H_L or H_R (or both) must be negative, and neither can be positive. In other words, the zero-temperature state contributing within the characters

$\overline{\chi}_i\chi_j\chi_k$ within Z_{string} must be a tachyon which is either on-shell (if $H_R = H_L$) or off-shell (if $H_R \neq H_L$); this tachyonic mode is then "dressed" with specific thermal contributions in order to become massless at the critical temperature a_H. Moreover, if our solution to Eq. (3.41) has non-zero n, then such a state will be massive for all temperatures below this critical temperature, as desired. It will also usually be tachyonic for temperatures immediately above this critical temperature.

Given these observations, our procedure for determining the Hagedorn temperature implied by a given thermal partition function $Z_{\text{string}}(\tau, T)$ is then fairly straightforward. First, we identify any zero-temperature states which are tachyonic (either on- or off-shell) contributing to the characters appearing within $Z_{\text{string}}(\tau, T)$. For each such state, we then attempt to solve the conditions in Eq. (3.41), subject to the constraints that (m, n) are restricted to the values appropriate for the corresponding thermal function (*i.e.*, $m \in \mathbb{Z}$ or $\mathbb{Z} + 1/2$ and $n \in 2\mathbb{Z}$ or $2\mathbb{Z} + 1$). If such a solution exists and has non-zero n, then we have succeeded in identifying a massive state in the full thermal theory which will become massless at the corresponding critical temperature a_H. This then signals a Hagedorn transition. In situations where multiple thermally massless states exist, the Hagedorn temperature is identified as the lowest of the corresponding critical temperatures, since the presumed existence of a phase transition at that temperature invalidates any analysis based on Z_{string} at temperatures above it.

Let us now calculate the Hagedorn temperatures corresponding to the partition functions $Z_{\text{string}}(\tau, T)$ in Sect. 3.3. We begin by first considering the case of the Type IIB string, for which the appropriate thermal function is given by

$$\begin{aligned} Z_{\text{string}}(\tau, T) \;=\; Z^{(8)}_{\text{boson}} \times \Big\{ \quad & [\overline{\chi}_V\chi_V + \overline{\chi}_S\chi_S] \;\; \mathcal{E}_0 \\ - \; & [\overline{\chi}_V\chi_S + \overline{\chi}_S\chi_V] \;\; \mathcal{E}_{1/2} \\ + \; & [\overline{\chi}_I\chi_I + \overline{\chi}_C\chi_C] \;\; \mathcal{O}_0 \\ - \; & [\overline{\chi}_I\chi_C + \overline{\chi}_C\chi_I] \;\; \mathcal{O}_{1/2} \;\; \Big\} \,. \end{aligned} \tag{3.42}$$

Note that all of the characters in Eq. (3.42), both holomorphic and anti-holomorphic, correspond to the transverse $SO(8)$ Lorentz group, as appropriate for ten-dimensional Type II strings. Recalling the conventions of Appendix A of chapter 2, we see that the only potentially tachyonic contributions in this expression arise from the term $\overline{\chi}_I \chi_I \mathcal{O}_0$. Indeed, only this sector has the potential to give rise to thermally massless level-matched states: these are the $(H_R, H_L) = (-1/2, -1/2)$ tachyons within $\overline{\chi}_I \chi_I$, "dressed" with the $(m, n) = (0, \pm 1)$ thermal excitations within $\mathcal{O}_0$. Solving for masslessness, we find that these $(H_R, H_L) = (-1/2, -1/2)$ states will indeed become thermally massless at the temperature $T_H = \mathcal{M}/\sqrt{2}$; they are massive for $T < T_H$, and tachyonic for $T > T_H$. We thus identify $T_H = \mathcal{M}/\sqrt{2}$ as the Hagedorn temperature for the Type IIB string. Note that this analysis is nothing but the standard derivation [2, 13] of the Hagedorn temperature for the Type IIB string, and indeed an identical result holds for the Type IIA string and for Type II strings in general.

However, this situation changes dramatically in the case of the heterotic string. Let us focus first on the $SO(32)$ heterotic string. If we were to utilize the standard Boltzmann result given in Eq. (3.34), we would find that the sector $\overline{\chi}_I \chi_I^2 \mathcal{O}_{1/2}$ is the only sector which is capable of providing thermally massless states. Indeed, solving the conditions for masslessness in Eq. (3.41), we see that the $(H_R, H_L) = (-1/2, -1)$ off-shell tachyon within $\overline{\chi}_I \chi_I^2$ — dressed with the thermal excitations $(m, n) = \pm(1/2, 1)$ within $\mathcal{O}_{1/2}$ — becomes thermally massless at the critical temperature $T_H = 2\mathcal{M}/(2 + \sqrt{2}) = (2 - \sqrt{2})\mathcal{M}$. This, of course, is nothing but the traditional Hagedorn temperature associated with the $SO(32)$ heterotic string, and indeed a similar result would emerge for the $E_8 \times E_8$ string upon using Eq. (3.36).

However, the main point of this chapter is that we should not be using Eqs. (3.34) or (3.36) to describe the finite-temperature behavior of these ten-dimensional strings; consistency with the temperature/radius correspondence requires that we instead use Eqs. (3.35) and (3.37). However, performing exactly the same analysis for Eq. (3.35), we now find that only the term $\overline{\chi}_I (\chi_I \chi_V + \chi_V \chi_I) \mathcal{O}_0$ is capable of giving rise to thermally massless level-matched states. Indeed, the $SO(16) \times SO(16)$ character $(\chi_I \chi_V + \chi_V \chi_I)$

gives rise to 32 on-shell $(H_R, H_L) = (-1/2, -1/2)$ tachyons, and these are nothing but the 32 tachyons of the non-supersymmetric $SO(32)$ heterotic string which serves as the $T \to \infty$ endpoint of the interpolation. Moreover, we find that the $(m, n) = (0, \pm 1)$ thermal excitations of these states are massless at $T_H = \mathcal{M}/\sqrt{2}$, massive below this temperature, and tachyonic above it. Thus the Hagedorn temperature associated with Eq. (3.35) is actually given by $T_H = \mathcal{M}/\sqrt{2}$, not $T_H = 2\mathcal{M}/(2+\sqrt{2})$. Remarkably, this new temperature is exactly the same as the Hagedorn temperature of the Type II string, and there are no other tachyonic sectors within Eq. (3.35) which could give rise to other phase transitions at lower temperatures.

A similar situation exists for the $E_8 \times E_8$ string. Examining Eq. (3.37), we see that only the sector $\overline{\chi}_I \chi_V \chi_I \mathcal{O}_0$ is capable of giving rise to thermally massless level-matched states; once again, these are the tachyons with energies $(H_R, H_L) = (-1/2, -1/2)$ within $\overline{\chi}_I \chi_V \chi_I$, dressed with the $(m, n) = (0, \pm 1)$ thermal excitations within $\mathcal{O}_0$. These states are massless at $T_H = \mathcal{M}/\sqrt{2}$, massive below this temperature, and tachyonic above it. Thus, we see that $T_H = \mathcal{M}/\sqrt{2}$ emerges as the Hagedorn temperature following from Eq. (3.37) as well.

We conclude, then, that our "geometric" approach to string thermodynamics leads to a remarkable property: both of the supersymmetric heterotic strings in ten dimensions have a new Hagedorn temperature given by $T_H = \mathcal{M}/\sqrt{2}$, which is exactly the same as the Hagedorn temperature for the Type II string! Following this approach, we thus find that

$$T_H = \frac{\mathcal{M}}{\sqrt{2}} = \frac{M_{\text{string}}}{2\sqrt{2}\pi} \qquad \text{for } \textit{all} \text{ supersymmetric closed strings in } D = 10 \;, \tag{3.43}$$

both Type II and heterotic! In other words, by carefully maintaining the temperature/radius correspondence through the introduction of a suitable non-zero Wilson line, we have uncovered a universal Hagedorn temperature for all closed supersymmetric strings in ten dimensions.

Clearly, the major difference between the traditional Boltzmann results in Eqs. (3.34) and (3.36) and the "geometric" results in Eqs. (3.35) and (3.37) is the fact that the ground-state tachyonic sector $\overline{\chi}_I \chi_I^2 \mathcal{O}_{1/2}$ appears in the former but not in the latter. As we have seen above, it is this term which is responsible for yielding the traditional Hagedorn temperature for heterotic strings. However, as discussed in chapter 2, it is the appearance of precisely this term within Eqs. (3.34) and (3.36) which signals the inconsistency of these partition functions as far as the temperature/radius correspondence is concerned. Thus, when we introduce the non-trivial Wilson line in order to achieve consistency with the temperature/radius correspondence, it is no surprise that this term no longer appears in Eqs. (3.35) and (3.37). It is this fact which is responsible for the corresponding shift in the heterotic Hagedorn temperature.

It is also easy to understand why the new heterotic Hagedorn temperature precisely matches the Type II Hagedorn temperature. The lowest mode contributing within $\overline{\chi}_I \chi_I^2$ is the (tachyonic) ground state of the heterotic theory, with non-level-matched vacuum energies $(H_R, H_L) = (-1/2, -1)$. However, as we have seen, turning on the Wilson line effectively projects this non-level-matched state out of the finite-temperature theory and leaves behind only the "next-deepest" tachyon with $(H_R, H_L) = (-1/2, -1/2)$ within $\overline{\chi}_I \chi_I \chi_V$. Thus, with the Wilson line turned on, this new tachyon becomes the effective ground state of the theory. However, this "next-deepest" tachyon has exactly the same worldsheet energies $(H_R, H_L) = (-1/2, -1/2)$ as the ground state of the Type II string. Thus it is not surprising that the presence of the non-trivial Wilson line shifts the heterotic Hagedorn temperature in such a way that it now matches the Type II value.

It is important to emphasize that our conclusions concerning the new Hagedorn temperature for heterotic strings depend almost exclusively on our observation that a thermal Wilson line must be introduced in order to render heterotic strings consistent with the temperature/radius correspondence. In particular, the specific *choice* of Wilson line is largely irrelevant to this conclusion, and we can verify that all of the proposed Wilson lines discussed in Sect. 3.3 which lead to tachyonic $T \to \infty$ endpoints would have generated the same result.

3.4.3 Reconciling the new Hagedorn temperature with the asymptotic degeneracy of states

As discussed in Sect. 3.4.1, our analysis of the Hagedorn temperature has thus far been based on an analysis of the tachyonic structure of our thermal interpolating models. Yet we know that there is a tight relation between the Hagedorn temperature of a given theory and the exponential rate of growth of its asymptotic degeneracies of bosonic and fermionic states. Specifically, if g_M denotes the number of string states with mass M, then the thermal partition function is given by $Z(T) = \sum g_M e^{-M/T}$. However, if $g_M \sim e^{\alpha M}$ as $M \to \infty$, then $Z(T)$ diverges for $T \geq T_H \equiv 1/\alpha$. This appears to provide a firm link between the Hagedorn temperature and the asymptotic degeneracy of states. Of course, $\sum g_M e^{-M/T}$ is not a proper string-theoretic partition function. However, even when we utilize a proper string-theoretic partition function $Z_{\text{string}}(\tau, T)$ and calculate a proper string-theoretic amplitude as in Eq. (3.38) in the $\mathcal{S}$-representation, the same basic argument continues to apply.

We are thus left with the critical question: *How can we justify a new Hagedorn temperature $T_H = \mathcal{M}/\sqrt{2}$ for heterotic strings, given that the zero-temperature bosonic and fermionic densities of heterotic states are apparently unchanged?* Specifically, an increase in the Hagedorn temperature of the heterotic string from the traditional value $T_H = 2\mathcal{M}/(2 + \sqrt{2})$ to a new, higher value $T_H = \mathcal{M}/\sqrt{2}$ would seem to require a corresponding decrease in the exponential rate of growth of the asymptotic density of heterotic string states. In what sense can we understand such a decrease?

To answer this question, let us look again at the original partition function of the zero-temperature ten-dimensional $SO(32)$ heterotic string model in Eq. (3.19). Recall that $\overline{\chi}_V$ and $\overline{\chi}_S$ indicate the transverse $SO(8)$ Lorentz spins of the different states which contribute in this theory. As a result of spacetime supersymmetry, this partition function vanishes identically — *i.e.*, all of its level-degeneracy coefficients are identically zero. There is no exponential growth here at all. But of course one does not examine the *whole* partition function in order to derive a Hagedorn temperature; one instead looks at its separate bosonic and fermionic contributions. Ordinarily, these contributions would be identified on the basis of the Lorentz spins of these states as

$$\begin{aligned} Z^{(\text{bosonic})}_{SO(32)} &= Z^{(8)}_{\text{boson}}\, \overline{\chi}_V \,(\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) ~, \\ Z^{(\text{fermionic})}_{SO(32)} &= -Z^{(8)}_{\text{boson}}\, \overline{\chi}_S \,(\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) ~, \end{aligned} \tag{3.44}$$

and indeed each of these expressions separately exhibits an exponential rise in the degeneracy of states which is consistent with the traditional heterotic Hagedorn temperature.

But what do we really mean by "bosonic" and "fermionic" in this context? For most purposes, we would identify states as "bosonic" or "fermionic" based on their Lorentz spins, as above. Moreover, by the spin-statistics theorem, this is equivalent to identifying states as bosonic or fermionic based on their zero-temperature quantization statistics. However, *for the purposes of computing a Hagedorn temperature*, we should really be focused on a *thermodynamic* definition of "bosonic" and "fermionic" wherein we identify states as bosons or fermions on the basis of their Matsubara frequencies, *i.e.*, on the basis of their modings around the thermal circle. Of course, under normal circumstances, all three of these identifications are equivalent. However, we have already seen in Sect. 3.2 that this chain of equivalences is modified in the presence of a non-trivial Wilson line: certain states which are "bosonic" in terms of their Lorentz spins and zero-temperature quantization statistics can nevertheless have anti-periodic modings around the thermal circle, while other states which are "fermionic" in terms of their Lorentz spins and zero-temperature quantization statistics can nevertheless have periodic modings around the thermal circle. Thus, *in the presence of a non-trivial Wilson line,* certain bosonic states can behave as fermions from a thermodynamic standpoint, and certain fermionic states can behave as bosons.

We emphasize that this is *not* a violation of the spin-statistics theorem. Indeed, the spin-statistics theorem is believed to hold without alteration in string theory, providing a connection between the Lorentz spin of a state and its zero-temperature quantization statistics [2]. Rather, as discussed in Sect. 2.2, the effect of the Wilson line is to modify the *thermodynamic manifestation* of these properties as far as their Matsubara modings are concerned. For issues pertaining to zero-temperature physics, these thermodynamic manifestations may be of little consequence. However, when we seek to understand the thermal properties of a theory, these modifications are critical.

Therefore, if we seek to understand the spectra of bosonic and fermionic states in the supersymmetric $SO(32)$ heterotic string *for thermodynamic purposes*, we should return to the partition function in Eq. (3.19) and separate this expression into bosonic and fermionic contributions *on the basis of their Matsubara modings around the thermal circle*. However, given the interpolating function in Eq. (3.35), this is easy to do: the "bosonic" contributions to Eq. (3.19) are those which multiply the thermal sum $\mathcal{E}_0$ in Eq. (3.35), while the "fermionic" contributions to Eq. (3.19) are those which multiply the thermal sum $\mathcal{E}_{1/2}$ in Eq. (3.35). In other words, we replace the identifications in Eq. (3.44) with

$$\begin{aligned} \widetilde{Z}_{SO(32)}^{(\text{bosonic})} &= \quad Z_{\text{boson}}^{(8)} \left[\overline{\chi}_V \, (\chi_I^2 + \chi_V^2) \; - \; \overline{\chi}_S \, (\chi_S^2 + \chi_C^2) \right] \; , \\ \widetilde{Z}_{SO(32)}^{(\text{fermionic})} &= -Z_{\text{boson}}^{(8)} \left[\overline{\chi}_S \, (\chi_I^2 + \chi_V^2) \; - \; \overline{\chi}_V \, (\chi_S^2 + \chi_C^2) \right] \; . \end{aligned} \tag{3.45}$$

In the presence of the non-trivial Wilson line, it is therefore *these* expressions which serve to define our separate bosonic and fermionic contributions to Eq. (3.19), and indeed their sum

$$\widetilde{Z}_{SO(32)}^{(\text{bosonic})} \; + \; \widetilde{Z}_{SO(32)}^{(\text{fermionic})} \tag{3.46}$$

correctly reproduces the expression in Eq. (3.19).

Given these results, we can now calculate the exponential rates of growth in the degeneracies of the states contributing to $\widetilde{Z}_{SO(32)}^{(\text{bosonic})}$ and $\widetilde{Z}_{SO(32)}^{(\text{fermionic})}$. *We find that while each term within these expressions continues to exhibit the traditional rate of growth associated with the traditional Hagedorn temperature for heterotic strings, the minus signs within* $\widetilde{Z}_{SO(32)}^{(\text{bosonic})}$ *and* $\widetilde{Z}_{SO(32)}^{(\text{fermionic})}$ *have the net effect of cancelling this dominant exponential behavior, leaving behind only a smaller exponential rate of growth for the state* *degeneracies*. Moreover, as expected, this smaller exponential rate of growth precisely matches the rate of growth that corresponds to the new (increased) heterotic Hagedorn temperature $T_H = \mathcal{M}/\sqrt{2}$.

It may seem strange that two terms, each exhibiting a dominant exponential growth rate, can be subtracted and leave behind a sub-dominant exponential growth rate. Yet this phenomenon is well known in modular functions such as these, and has been shown to operate in other string-theoretic contexts [48, 57, 49, 50]. We emphasize that this subtraction is relevant only in the sense that it deforms the exponential growth rate when we count $\widetilde{Z}^{(\text{bosonic})}_{SO(32)}$ and $\widetilde{Z}^{(\text{fermionic})}_{SO(32)}$ separately. Each string state still continues to appear with positive unit weight in the string Fock space, as always, and still contributes to the overall partition function with an appropriate sign (positive for spacetime bosons, negative for spacetime fermions).

Similar results hold for the $E_8 \times E_8$ heterotic string. Without a Wilson line, the usual identification of bosonic and fermionic states is nothing but

$$Z^{(\text{bosonic})}_{E_8\times E_8} = Z^{(8)}_{\text{boson}}\, \overline{\chi}_V\, (\chi_I + \chi_S)^2 ~, \qquad Z^{(\text{fermionic})}_{E_8\times E_8} = -Z^{(8)}_{\text{boson}}\, \overline{\chi}_S\, (\chi_I + \chi_S)^2 ~. \tag{3.47}$$

However, with the Wilson line turned on, we find from Eq. (3.37) that our new thermal identification of bosonic and fermionic states is given by

$$\begin{aligned} \widetilde{Z}^{(\text{bosonic})}_{E_8\times E_8} &= Z^{(8)}_{\text{boson}}\, (\overline{\chi}_V\, \chi_I ~-~ \overline{\chi}_S\, \chi_S)\, (\chi_I + \chi_S) \\ \widetilde{Z}^{(\text{fermionic})}_{E_8\times E_8} &= -Z^{(8)}_{\text{boson}}\, (\overline{\chi}_S\, \chi_I ~-~ \overline{\chi}_V\, \chi_S)\, (\chi_I + \chi_S) ~. \end{aligned} \tag{3.48}$$

Just as with the $SO(32)$ string, the minus signs within Eq. (3.48) lead to state degeneracies which show a reduced expoential growth — one which is precisely in accordance with the new, increased heterotic Hagedorn temperature.

This, then, is the essence of the manner in which the asymptotic density of states is reconciled with the modified Hagedorn temperature for heterotic strings. The presence of the non-trivial Wilson line "deforms" the thermal identification of bosonic and fermionic states, trading states between the separate sets of bosonic and fermionic states in such a way that the net exponential rate of growth for the asymptotic state degeneracy of each set is reduced.

There are also other ways to understand this result. For example, one might argue on general conformal-field-theory (CFT) grounds that such a change in the Hagedorn temperature should not be possible. After all, there exists a general result which relates the Hagedorn temperature of a given closed-string theory to the central charges (c_R, c_L) of its underlying worldsheet CFT's:

$$T_H = \left(\sqrt{\frac{c_L}{24}} + \sqrt{\frac{c_R}{24}}\right)^{-1} \mathcal{M} \ . \tag{3.49}$$

For Type II strings, we have $(c_R, c_L) = (12, 12)$, while for heterotic strings, we have $(c_R, c_L) = (12, 24)$. However, in deriving Eq. (3.49), there is only place in which the central charges enter: this is in setting the ground-state energies $(H_R, H_L) = (-c_R/24, -c_L/24)$. Moreover, as we have already seen in Sect. 3.4.2, the Wilson line has effectively projected the true heterotic ground state with $(H_R, H_L) = (-1/2, -1)$ out of the spectrum, leaving behind only the "next-deepest" tachyon with $(H_R, H_L) = (-1/2, -1/2)$ to serve as the effective ground state of the theory. Thus, in the presence of the Wilson line, the effective central charges of the theory become $(c_R, c_L) = (12, 12)$, just as for Type II strings.

In general, the shift in the vacuum energy of the effective ground state and the shift in the asymptotic rates of growth for the state degeneracies are flip sides of the same coin, deeply related to each other through modular transformations. Indeed, these are nothing but equivalent UV/IR descriptions of the same phenomenon, all arising due to the existence of the non-trivial Wilson line. Under Poisson resummation, a half-integer shift in the moding of a given set of string states around the thermal circle translates into an overall $\mathbb{Z}_2$ phase (*i.e.*, a minus sign) in front of the corresponding character in the partition function. Thus the Wilson line, which shifts the apparent thermal modings of certain states in the theory, necessarily induces a corresponding change in the asymptotic state degeneracies and a corresponding shift in the Hagedorn temperature of the theory.

We thus conclude that all tachyon-free closed strings in ten dimensions share a universal Hagedorn temperature. Although the heterotic string would naïvely appear to have a slightly lower Hagedorn temperature than the Type II string due to its non-level-matched ground state, consistency with the temperature/radius correspondence in the heterotic case requires the introduction a non-trivial thermal Wilson line. This Wilson line then deforms the effective worldsheet central charge of the heterotic theory as far as thermal properties are concerned, and leads to a new effective ground state for the theory as well as a new rate of exponential growth for the corresponding density of states. Both effects then alter Hagedorn temperature of the heterotic string, and bring it into agreement with the Type II value.

3.4.4 Beyond ten dimensions: Additional general observations

Thus far, we have shown that consistency with the temperature/radius correspondence requires that all supersymmetric closed strings in ten dimensions have a common Hagedorn temperature $T_H \equiv \mathcal{M}/\sqrt{2}$. This applies to Type II strings as well as to heterotic strings. Yet it is natural to wonder how general this result might be. In particular, we would like to determine whether this result might hold regardless of the spacetime dimension.

In the standard Boltzmann approach, it is a straightforward matter to demonstate that the Hagedorn temperatures found for ten-dimensional strings are unchanged by the process of compactification. This observation follows from the fact that compactification does not change the underlying worldsheet central charges of these strings, and no Wilson lines arise in the Boltzmann approach to provide complications to this standard argument. However, this is less obvious in our "geometric" approach because the space of possible distinct heterotic string theories grows significantly as the spacetime dimension is reduced, and we do not know, *a priori*, whether or not a Wilson line must be introduced in each case, and if so, which Wilson line is appropriate. Note that for dimensions below ten, this worry is no longer unique to the heterotic string; the Type II string also develops gauge symmetries upon compactification, and thus non-trivial Wilson lines are in principle possible (although unlikely) for these strings as well.

However, it is possible to make some general observations for the case of the hererotic string. First, we claim that a non-trivial Wilson line must indeed be introduced for *all* heterotic strings, regardless of spacetime dimensionality. Second, we claim that the introduction of this Wilson line will necessarily increase the Hagedorn temperature above its traditional heterotic value.

Our arguments here are relatively simple. Without any Wilson lines, we know that the Hagedorn temperatures for all heterotic strings are given by the traditional value $T_H = 2\mathcal{M}/(2+\sqrt{2}) = (2-\sqrt{2})\mathcal{M}$ regardless of dimensionality. This is indeed nothing but the standard situation. Moreover, as we have seen above, $T_H = (2-\sqrt{2})\mathcal{M}$ can emerge as the Hagedorn temperature of a given heterotic string model only if its thermal partition function $Z_{\text{string}}(\tau, T)$ contains a term of the form

$$\overline{\chi}_I \chi_I \mathcal{O}_{1/2} \tag{3.50}$$

where χ_I and $\overline{\chi}_I$ represent the characters corresponding to the identity sectors of the complete left- and right-moving worldsheet CFT's of the string. Indeed, only such a term is capable of providing the $(H_R, H_L) = (-1/2, -1)$ tachyon, dressed with the thermal excitations $(m, n) = \pm(1/2, 1)$ within $\mathcal{O}_{1/2}$, which becomes massless at $T_H = (2-\sqrt{2})\mathcal{M}$. However, as we have argued in chapter 2, the appearance of such a term is inconsistent with the temperature/radius correspondence. Specifically, if $Z_{\text{string}}(\tau, T)$ contains such a term, then it cannot correspond to an interpolating model which relates two endpoints which are themselves directly connected through a *bona-fide* $\mathbb{Z}_2$ orbifold Q. Consequently, we know that any expression $Z_{\text{string}}(\tau, T)$ which is consistent with the temperature/radius correspondence cannot possibly contain a term of the form (3.50); a non-trivial Wilson line is needed in order to remove it. It then follows from the absence of this term that the Hagedorn temperature of the resulting theory can no longer be given by its usual heterotic value, but must be higher.

While this argument demonstrates that a shift in the heterotic Hagedorn temperature is inevitable, it does not demonstrate that the new, shifted temperature is always the same as the Type II value $T_H \equiv \mathcal{M}/\sqrt{2}$. Indeed, given the analysis in Sect. 3.4.2, we see that this will only happen if the interpolating model in question has, as its formal $T \to \infty$ endpoint, a non-supersymmetric heterotic string model which contains on-shell tachyons with worldsheet energies $(H_R, H_L) = (-1/2, -1/2)$. Of course, due to level-matching constraints, this is indeed the "deepest" (*i.e.*, most tachyonic) state that such an endpoint string model can contain. Moreover, it seems likely that the corresponding free-energy density $F(T)$ would indeed be minimized only when a model with such a tachyonic structure serves as the formal $T \to \infty$ endpoint of the thermal interpolation. Thus, on this basis, it seems that $T_H \equiv \mathcal{M}/\sqrt{2}$ should indeed be the *generic* result for the new, shifted Hagedorn temperature for all heterotic strings, regardless of the spacetime dimension.

However, this alone does not prove that there cannot be special cases in which even these worldsheet tachyons $(H_R, H_L) = (-1/2, -1/2)$ might also fail to appear within $Z_{\text{string}}(\tau, T)$. In such cases, there would be no Hagedorn transition at $T_H = \mathcal{M}/\sqrt{2}$. Yet, even in such cases, there might be other terms within $Z_{\text{string}}(\tau, T)$ which — although normally subdominant relative to the tachyonic $(H_R, H_L) = (-1/2, -1/2)$ states — might also trigger a Hagedorn transition. Such a Hagedorn transition would then take place at an even higher temperature.

It is possible to classify all of these possibilities. For concreteness, let us restrict our attention to theories built from only $\mathbb{Z}_2$ orbifolds, so that $H_{L,R}$ are quantized in half-integer values. Given the heterotic constraints $H_L \geq -1$ and $H_R \geq -1$ (which also subsume the Type II constraints $H_{L,R} \geq -1/2$), we then find that there are only eight different terms which could possibly appear in $Z_{\text{string}}(\tau, T)$ and trigger a Hagedorn transition. These are listed in Table 3.1, along with their corresponding thermal excitations and Hagedorn temperatures [obtained by solving Eq. (3.41)]. It is interesting to note that these terms come in "dual" pairs under which $T_H/\mathcal{M} \to 2\mathcal{M}/T_H$ and $(\mathcal{E}_0, \mathcal{E}_{1/2}, \mathcal{O}_0, \mathcal{O}_{1/2}) \to (\mathcal{E}_0, \mathcal{O}_0, \mathcal{E}_{1/2}, O_{1/2})$. Roughly speaking, this duality corresponds to exchanging the direction of the interpolation: for every solution that might exist for a finite-temperature model which interpolates from M_1 to M_2, there is another solution which would arise in the model in which the direction of the interpolation is reversed from M_2 back to M_1. Of course, although the "forward" model is presumed to represent the finite-temperature behavior of M_1, it is not guaranteed that the "reverse" interpolation in any way represents the finite-temperature behavior of M_2. As a result, this "thermal duality" phenomenon shall not concern us further (although it has played a role in other work [51]), and we can view the emergence of this duality as a mere mathematical curiosity.

As we have already seen, Case A is responsible for the traditional heterotic Hagedorn transition. Although it leads to the lowest possible Hagdorn temperature, we have shown that this case cannot arise in any $Z_{\text{string}}(\tau, T)$ which is consistent with the temperature/radius correspondence. Likewise, as we have discussed above, Case C with $n = 1$ is responsible for the traditional Type II Hagedorn transition as well as our new "geometric" heterotic Hagedorn transition. Indeed, this case leads to the "next-lowest" Hagedorn temperature, and as such it dominates [when present within $Z_{\text{string}}(\tau, T)$] over any other terms which may also simultaneously appear within $Z_{\text{string}}(\tau, T)$. Moreover, in ten dimensions, our complete enumeration of all possible Wilson lines in the heterotic case has demonstrated that Case C with $n = 1$ *must* appear, which renders the existence of the other cases moot in ten dimensions. Yet, in dimensions below ten, it is logically possible that this case might also fail to appear for a given Wilson-line choice. In that case, the existence of these other possibilities becomes relevant, leading to the possibility of a Hagedorn transition with an even higher temperature.

Ultimately, the question of which of these terms ends up dominating for a given $D < 10$ string model is likely to be addressable only on a case-by-case basis. Nevertheless, it is easy to see that Cases B and D can only arise for string models which are already

	H_R	H_L	T F	T M (m,n)	$T_H/\mathcal{M}$
A	$-1/2$	-1	$\mathcal{O}_{1/2}$	$\pm(1/2,1)$	$2-\sqrt{2}$ (also $2+\sqrt{2}$)
B	$-1/2$	$-1/2$	$\mathcal{E}_0$	$(0,n),\ \ n\in 2\mathbb{Z}$ $(m,0),\ \ m\in\mathbb{Z}$	$\lvert n\rvert/\sqrt{2}$ $\sqrt{2}/\lvert m\rvert$
C D	$-1/2$ $-1/2$	$-1/2$ $-1/2$	$\mathcal{O}_0$ $\mathcal{E}_{1/2}$	$(0,n),\ \ n\in 2\mathbb{Z}+1$ $(m,0),\ \ m\in\mathbb{Z}+1/2$	$\lvert n\rvert/\sqrt{2}$ $\sqrt{2}/\lvert m\rvert$
E	0	$-1/2$	$\mathcal{O}_{1/2}$	$\pm(1/2,1)$	$\sqrt{2}$
F	$-1/2$	0	$\mathcal{O}_{1/2}$	$\pm(1/2,-1)$	$\sqrt{2}$
G H	0 0	-1 -1	$\mathcal{O}_0$ $\mathcal{E}_{1/2}$	$\pm(1,1)$ $\pm(1/2,2)$	1 2

TABLE 3.1. Eight possible terms (labelled A through H) which can potentially trigger a Hagedorn transition. T F means Thermal Function and T M means Thermal Modes. As discussed in the text, Case A is responsible for the traditional heterotic Hagedorn transition, while Case C with $n=1$ is responsible for the traditional Type II Hagedorn transition as well as our new "geometric" heterotic Hagedorn transition. Cases B and D can only arise in theories which are already tachyonic (and hence unstable) at zero temperature, while Case H is guaranteed to arise for all heterotic strings which are supersymmetric at zero temperature. Observe that all of these possibilities come in "dual" pairs under which $T_H/\mathcal{M}\to 2\mathcal{M}/T_H$ and $(\mathcal{E}_0,\mathcal{E}_{1/2},\mathcal{O}_0,\mathcal{O}_{1/2})\to(\mathcal{E}_0,\mathcal{O}_0,\mathcal{E}_{1/2},O_{1/2})$. Thus the two possibilities within Cases A and B are dual to each other, while Cases C and G are dual to Cases D and H respectively (and *vice versa*). By contrast, Cases E and F are each self-dual. Note that Cases A, G, and H are unique to heterotic strings, while all other cases can in principle arise in both heterotic and Type II strings.

tachyonic (and hence unstable) at zero temperature; this follows from the fact that the solutions for their corresponding Hagedorn temperatures, as shown in Table 3.1, always include the cases with $n=0$ or $m\to\infty$. This can also be seen by taking the direct $T\to 0$ limit of the terms in each of these cases. Thus, Cases B and D need not concern us further.

Given this situation, it is natural to wonder whether there are any string models in which the Hagedorn transition is eliminated *completely* — *i.e.*, models in which no thermally massless states appear at *any* temperature, and in which *none* of the remaining cases listed in Table 3.1 arise. Following the analysis in Sect. 3.4.2, this would have to be a zero-temperature heterotic string model for which the thermal extrapolation $Z_{\rm string}(\tau, T)$ that minimizes the free energy involves a $T \to \infty$ endpoint model which is non-supersymmetric but tachyon-free. While this did not occur in ten dimensions, this remains a logical possibility in lower dimensions where many such non-supersymmetric tachyon-free string models exist.

We shall now prove that this cannot happen for any heterotic string which is supersymmetric at zero temperature, regardless of its spacetime dimension. Specifically, we shall now demonstrate that Case H will always arise for such strings, leading a Hagedorn transition at $T_H = 2\mathcal{M}$ if no earlier Hagedorn transition has occurred at lower temperature.

Our argument is completely general since it is based on considerations of the most generic massless states in the perturbative heterotic string: those associated with the gravity multiplet. Recall that in the heterotic string, the graviton is realized in the Neveu-Schwarz sector as

$$\text{graviton:} \qquad g^{\mu\nu} \subset \tilde{b}^{\mu}_{-1/2}|0\rangle_R \otimes \alpha^{\nu}_{-1}|0\rangle_L \tag{3.51}$$

where $\tilde{b}^{\mu}_{-1/2}$ and α^{ν}_{-1} are respectively the excitations of the right-moving worldsheet Neveu-Schwarz fermion $\tilde{\psi}^{\mu}$ and left-moving worldsheet coordinate boson X^{ν}. Since the Neveu-Schwarz heterotic string ground state has vacuum energies $(H_R, H_L) = (-1/2, -1)$, the states in Eq. (3.51) are both level-matched and massless, with $(H_R, H_L) = (0, 0)$. These states include the spin-two graviton, the spin-one antisymmetric tensor field, and the spin-zero dilaton.

In a similar vein, any model exhibiting spacetime supersymmetry must also contain the gravitino state, realized in the Ramond sector of the heterotic string as

$$\text{gravitino:} \qquad \tilde{g}^{\alpha\nu} \subset \{\tilde{b}_0\}^{\alpha}|0\rangle_R \otimes \alpha^{\nu}_{-1}|0\rangle_L \ . \tag{3.52}$$

Here $\{\tilde{b}_0\}^{\alpha}$ schematically indicates the Ramond zero-mode combinations which collectively give rise to the spacetime Lorentz spinor index α, as required for the spin-3/2 gravitino state.

Regardless of the particular GSO projections inherent in the particular string model under consideration, we know that the graviton state in Eq. (3.51) must always appear in the string spectrum. Likewise, if the model has spacetime supersymmetry, we know that the gravitino state in Eq. (3.52) must exist as well. However, it is then straightforward to show that this implies that certain additional off-shell tachyons must also exist in the string spectrum. Specifically, regardless of the particular GSO projections, the off-shell spectrum will always contain a spin-one "proto-graviton" state ϕ^μ in the Neveu-Schwarz sector:

$$\text{proto-graviton:} \qquad \phi^\mu \;\equiv\; \tilde{b}^\mu_{-1/2}|0\rangle_R \;\otimes\; |0\rangle_L \;; \tag{3.53}$$

likewise, if the model is spacetime supersymmetric, the off-shell spectrum will always contain a spin-1/2 "proto-gravitino" state ψ^α in the Ramond sector:

$$\text{proto-gravitino:} \qquad \tilde{\phi}^\alpha \;\equiv\; \{\tilde{b}_0\}^\alpha|0\rangle_R \;\otimes\; |0\rangle_L \;. \tag{3.54}$$

Note that these are the same states as the graviton/gravitino, except that in each case the left-moving bosonic excitation is lacking. However, it is important to realize that *GSO projections are completely insensitive to the presence or absence of excitations of the worldsheet coordinate bosonic fields*. Thus, since the graviton is always present in the on-shell spectrum, it then follows that the proto-graviton must also always be present in the off-shell spectrum; likewise, if the model is supersymmetric and the gravitino is present in the on-shell spectrum, then the proto-gravitino must also always be present in the off-shell spectrum. Thus, we conclude that the proto-graviton and proto-gravitino are two off-shell tachyons with worldsheet energies $(H_R, H_L) = (0, -1)$ which generically appear in all supersymmetric heterotic string models.

This does not, in and of itself, guarantee that these states will contribute to the thermal partition function $Z_{\text{string}}(\tau, T)$ in the specific $\mathcal{O}_{1/2}$ or $\mathcal{E}_{1/2}$ sectors that Cases G or H would require. Fortunately, it is not too difficult to determine which sectors will contain these states. Like the graviton and gravitino states from which they are derived, these proto-graviton and proto-gravitino states must exist in the zero-temperature theory and thus must survive the zero-temperature limit. This implies that these states must appear in the $\mathcal{E}$ sectors, not the $\mathcal{O}$ sectors. Moreover, since neither of these states carries any gauge charges, neither can be affected by the presence of a Wilson line. As a result, we know that the (bosonic) proto-graviton state must appear in the $\mathcal{E}_0$ sector (which has integer modings around the thermal circle), while the (fermionic) proto-gravitino state must appear in the $\mathcal{E}_{1/2}$ sector (which has half-integer modings).

Given these results, we conclude that while the proto-graviton state will never lead to any of the cases in Table 3.1, the proto-gravitino state leads directly to Case H. Moreover, as we have argued on general grounds, this state is always present in any heterotic model which is supersymmetric at zero temperature. As a result, we conclude that the proto-gravitino state — dressed with $(m,n) = \pm(1/2,2)$ thermal excitations — will always exist and trigger a Hagedorn-like transition at temperature $T_H = 2\mathcal{M}$ (provided no other phase transition has occurred at any lower temperature).

This transition is somewhat different from the typical Hagedorn transition, however. In general, the total spacetime mass M_{tot} of a given (H_R, H_L) state dressed with (m,n) thermal excitations varies with the temperature T according to

$$\alpha' M_{\text{tot}}^2 = 2\left[H_L + \tfrac{1}{4}(ma - n/a)^2 + H_L + \tfrac{1}{4}(ma + n/a)^2\right] \tag{3.55}$$

where $a \equiv T/\mathcal{M}$. However, for the proto-gravitino (Case H), this becomes

$$\alpha' M_{\text{tot}}^2 \;=\; \frac{a^2}{4} \;+\; \frac{4}{a^2} \;-\; 2\;, \tag{3.56}$$

whereupon we see that the thermal excitation of the proto-gravitino state *never becomes tachyonic*! Indeed, this state is massive for all $a < 2$, and merely hits masslessness at $a = 2$ before becoming massive again at higher temperatures. Of course, this result is completely consistent with the fact that the proto-gravitino state is fermionic, since the existence of a physical fermionic tachyon at any temperature would violate Lorentz invariance.

However, given that this state never becomes tachyonic, it is natural to wonder whether this state can ever give rise to a Hagedorn transition. Indeed, since no tachyon ever develops, the thermal vacuum amplitude $\mathcal{V}(T)$ will never diverge. To see this, we observe that the $(m,n)=(1/2,2)$ thermal excitation of the proto-gravitino state makes a contribution to $\mathcal{V}(T)$ given by

$$\begin{aligned}\mathcal{V}(T) &= -\tfrac{1}{2}\mathcal{M}^{D-1}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\,\tau_2^{1-D/2}\sqrt{\tau_2}\;\;\frac{1}{q}\;\left[\overline{q}^{(a/2-2/a)^2/4}q^{(a/2+2/a)^2/4}\right]\;+\;...\\ &= -\tfrac{1}{2}\mathcal{M}^{D-1}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\,\tau_2^{1-D/2}\sqrt{\tau_2}\;\;e^{2\pi\tau_2}\;e^{-\pi\tau_2(a^2/4+4/a^2)}\;+\;...\end{aligned} \tag{3.57}$$

where we have left the temperature $a\equiv T/\mathcal{M}$ arbitrary. Note that the leading $1/q$ factor in the first line of Eq. (3.57) represents the zero-temperature contribution from the proto-gravitino, with $(H_R,H_L)=(0,-1)$, while the remaining factor in brackets represents the thermal contribution with $(m,n)=(1/2,2)$. Likewise, we have carefully recorded all factors of $\tau_2\equiv \mathrm{Im}\,\tau$: two factors of τ_2 arise in the denominator from the modular-invariant measure of integration, $(1-D/2)$ factors arise in the numerator from the zero-temperature partition function, and an additional factor $\sqrt{\tau_2}$ arises in the numerator from the definitions of the $\mathcal{E},\mathcal{O}$ thermal sums. However, at $a=2$, this expression reduces to

$$\mathcal{V}(T)\Big|_{a=2} \;=\; -\,\tfrac{1}{2}\mathcal{M}^{D-1}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\,\tau_2^{1-D/2}\sqrt{\tau_2}\;+\;... \tag{3.58}$$

and as $\tau_2\to\infty$, this contribution scales like

$$\int^{\infty}\frac{d\tau_2}{\tau_2^{(1+D)/2}}\;. \tag{3.59}$$

This contribution is therefore finite for all $D\geq 2$. This, of course, agrees with our usual expectation that a massless state does not lead to a divergent vacuum amplitude in two or more spacetime dimensions.

It is important to realize that even though $\mathcal{V}(T)$ remains finite for all temperatures, a phase transition still occurs; indeed the sudden appearance of a new massless state at a critical temperature signals the appearance of a new long-range order that was not present previously. Therefore, in order to uncover the effects of this massless state, let us now investigate temperature derivatives of $\mathcal{V}(T)$. As evident from the second line of

Eq. (3.57), each temperature derivative $d/dT \sim d/da$ brings down an extra factor of τ_2. In general, this thereby increases the tendency towards divergence of our thermodynamic quantities.

Our results are as follows. The contribution of this thermally excited proto-gravitino state to the first derivative $d\mathcal{V}/da$ is given by

$$\frac{d\mathcal{V}}{da} ~=~ \pi \mathcal{M}^{D-1} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, \tau_2^{1-D/2} \sqrt{\tau_2}\; \tau_2 \left(\frac{a}{4} - \frac{4}{a^3}\right)~ e^{2\pi\tau_2}~ e^{-\pi\tau_2(a^2/4+4/a^2)} ~+~ ..., \qquad (3.60)$$

but at the temperature $a = 2$ we see that the factor in parenthesis within Eq. (3.60) actually vanishes:

$$\left. \frac{d\mathcal{V}}{da}\right|_{a=2} ~=~ 0 ~. \qquad (3.61)$$

It turns out that this is a general property, reflecting nothing more than the fact that the slope of the mass function in Eq. (3.56) vanishes at its minimum, as it must. However, taking subsequent derivatives and evaluating at $a = 2$, we find the general pattern

$$\left. \frac{d^p\mathcal{V}}{da^p}\right|_{a=2} ~=~ \mathcal{M}^{D-1} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\, \tau_2^{1-D/2} \sqrt{\tau_2}~ f_p(\tau_2) ~+~ ... \qquad (3.62)$$

where $f_p(\tau_2)$ for $p \geq 2$ is a rank-r polynomial in τ_2 of the form

$$f_p(\tau_2) ~=~ A_p\, \tau_2^r ~+~ B_p\, \tau_2^{r-1} ~+~ C_p \tau_2^{r-2} ~... ~, \qquad (3.63)$$

where

$$r ~=~ \begin{cases} p/2 & \text{for } p \text{ even} \\ (p-1)/2 & \text{for } p \text{ odd,} \end{cases} \qquad (3.64)$$

and where the leading coefficients A_p are positive for $p = 1, 2 \pmod 4$ and negative for $p = 0, 3 \pmod 4$, with alternating signs for the lower-order coefficients B_p, C_p, *etc.* Given these extra leading powers of τ_2, we thus find that as a result of the proto-gravitino state,

$$\frac{d^p\mathcal{V}}{dT^p} \qquad \text{diverges for} \qquad \begin{cases} D \leq p & \text{for } p \text{ odd} \\ D \leq p+1 & \text{for } p \text{ even.} \end{cases} \qquad (3.65)$$

Equivalently, in $D \geq 2$ spacetime dimensions, the proto-gravitino state results in a divergence that first occurs for $d^p\mathcal{V}/dT^p$, where

$$p \;=\; \begin{cases} D & \text{for } D \text{ even} \\ D-1 & \text{for } D \text{ odd.} \end{cases} \tag{3.66}$$

This divergence then corresponds to a very weak, p^{th}-order phase transition. In particular, for $D = 4$, this would be a fourth-order phase transition in which d^2c_V/dT^2 diverges, causing dc_V/dT to experience a discontinuity, the specific heat c_V itself to experience a kink, and the internal energy function to have a discontinuous change in curvature.

We stress that it is not merely the masslessness of this thermally-enhanced proto-gravitino state that results in this phase transition. It is the fact that this masslessness is achieved *thermally*, with non-trivial thermal momentum and winding quanta, that induces this phase transition. By contrast, a regular massless state such as the usual graviton or gravitino does not contribute to any temperature derivatives of $\mathcal{V}$.

Thus, we conclude that for supersymmetric heterotic strings, it is never possible to completely evade a Hagedorn-like phase transition. However, the phase transition associated with the proto-gravitino state appears only at the relatively high temperature $T_H \equiv 2\mathcal{M}$, and thus will be completely irrelevant if tachyon-induced Hagedorn transitions appear at lower temperatures.

3.5 Conclusions

In this chapter, we investigated the manner in which a given zero-temperature string model may be extrapolated to finite temperature. Following relatively conservative conditions for self-consistency, we nevertheless found that the traditional Hagedorn transition does not exist for heterotic strings but is instead replaced by a new, “re-identified” Hagedorn transition which emerges at the somewhat higher temperature normally associated with Type II strings. This allowed us to uncover a universal Hagedorn temperature for all tachyon-free closed string theories in ten dimensions. We also showed that these results are not in conflict with the exponential rise in the degeneracy of string states in these models.

Clearly, many outstanding questions remain. Perhaps the two most critical are the issues of the *existence* and *uniqueness* of thermal extrapolations satisfying the general criteria we put forth in Sect. 3.3. In other words, it is important to demonstrate that, for any given D-dimensional zero-temperature string model, there always exists one and only one suitable corresponding $T \to \infty$ endpoint D-dimensional string model such that the corresponding $(D-1)$-dimensional interpolation is thermally consistent according to our general criteria, including proper spin-statistics relations. In ten dimensions, we have already seen that such extrapolations exist and are unique. However, neither property has been proven in lower dimensions. This is clearly an important issue that requires further study.

Another interesting question concerns the thermal fate of string models which are non-supersymmetric but tachyon-free at zero temperature: is it ever possible that such a non-supersymmetric model will have a thermal extrapolation whose $T \to \infty$ limit is *supersymmetric*? If so, this would be an example of a situation in which the zero-temperature theory is non-supersymmetric, but in which thermal effects compensate for this inequity between bosons and fermions and thereby *introduce* (rather than break) supersymmetry as $T \to \infty$. In other words, such thermal effects would be "SUSY-making" rather than SUSY-breaking, with SUSY-breaking occurring at *lower* temperatures. This phenomenon would be intrinsically string-theoretic, since only for closed strings does the $T \to \infty$ limit yield a theory of the same dimensionality as the original zero-temperature theory. No examples exhibiting this phenomenon exist in ten dimensions, but it would be interesting to explore whether such examples might exist in lower dimensions.

Chapter 4

FINITE-TEMPERATURE TYPE I STRING THEORY AND THE HAGEDORN TRANSITION

Summary

The temperature/radius correspondence states that a quantum theory at finite temperature T can be recast as a zero-temperature theory in which a Euclidean time dimension is compactified on a circle of radius $R = (2\pi T)^{-1}$. In chapter 2, however, it was demonstrated that this correspondence is actually broken for heterotic strings at finite temperature — *i.e.*, the traditional Boltzmann sum for heterotic strings cannot be recast as the partition function corresponding to any self-consistent heterotic compactification. It was further shown in a chapter 3 that enforcing this correspondence for heterotic string theory results in the same Hagedorn temperature for Type II and Heterotic strings. In this chapter we look at finite temperature Type I string theory in detail and show that all ten-dimensional string theories have a universal Hagedorn temperature. We also comment on the stability of the Type I finite temperature state.

4.1 Introduction

It is generally believed that all strings at finite temperature can be understood in terms of the "temperature/radius correspondence" - the observation that a finite-temperature string theory is equivalent to a zero-temperature string theory in which a Euclidean time dimension is compactified on a circle. However, in the second chapter, it was shown that while the "temperature/radius correspondence" correspondence holds for bosonic strings as well as Type II strings, it is broken for heterotic strings at finite temperature. In the third chapter, we showed that this correspondence can be restored by choosing an alternate finite temperature model for Heterotic strings which differs from the original model in having a non-trivial Wilson Line (or imaginary chemical potential). As a bonus, the Hagedorn temperature for this new finite temperature model matches that of Type II Strings.

In this chapter we analyse in detail Type I strings at finite temperature. Unlike the Heterotic string case the purpose of this chapter is not to overturn the conventional finite temperature formulation of Type I strings. Indeed as far as the Hagedorn temperature is concerned, it has been known for a long time that Type I strings have the same Hagedorn temperature as Type II strings since the early work of [60] (For a review see [1, 2]). So based on the results of chapters 2 and 3, we can already claim that all string theories have a universal Hagedorn Temperature.

The accepted way to identify the finite temperature formulation for any string theory is to start with the traditional Boltzmann sum. However it was shown in chapter 2 that this approach cannot be trusted as far as String Thermodynamics is concerned. In chapter 3 we developed a methodology for identifying the correct finite temperature string theory without reliance on the Boltzmann sum as a starting point. The purpose of this chapter is to use this method for understanding the finite temperature behaviour of Type I strings. Since Type I and Heterotic string theories are closely related by S-duality, it is of interest to look for similarities and differences in their finite temperature behaviour. Using the exact same methodology of chapter 3 lets us analyse the extent to which the finite temperature behaviour of Type I strings mimics that of Heterotic Strings.

In this chapter we will focus on the *perturbative* behaviour of Type I strings. We will not discuss the thermodynamics of Dp-branes or the thermodynamics of open strings on Dp-branes where $p < 9$. There has been considerable activity in these areas, some selected papers are listed in [61, 62], as well as in cosmological applications of finite T D-branes [63]. Because of the "temperature/radius correspondence" most of the results we present in this paper are already known in the context of Type I string compactified on a Scherk-Schwarz orbifold, S^1/Z_2 [19, 67]. We are therefore not writing down new partition functions but rather interpreting compactified 9-dimensional Type I partition functions as finite temperature partition functions.

This chapter is organized as follows. In Sect. 4.2 we write down all potential Type I finite temperature partition functions. We do this by orientifolding [65] the Type IIB thermal partition function which results in a number of descendants. (Note that since we are not starting with the traditional Boltzmann sum, we are not going to pick a partition function from these descendants which is equal to the Boltzmann sum and assume it to be the finite T string partition). In choosing to identify the descendants of thermal Type IIB as potential finite T Type I candidates we are deviating somewhat from the way we identified similar candidates for the Heterotic String theory. For the Heterotic theory we constructed the potential finite temperature functions by starting with the supersymmetric heterotic theory at $T = 0$, identifying the possible non-supersymmetric theories at the $T = \infty$ limit and then constructing the corresponding interpolations. This approach does not work well for Type I since there are various subtleties associated with the $T = \infty$ limit of Type I. Therefore in this chapter we shall use the standard orientifolding procedure while writing down potential thermal partition functions. As we have already stated such a procedure will give as a number of partition functions with different gauge groups. This variety represents nothing else but a degree of freedom corresponding to a choice of a Wilson Line. It is well-known that turning on a background gauge field for a compactified string theory leads to a family of theories. As we showed in Chapter 3 this Wilson line can be interpreted under the "temperature/radius correspondence" as an imaginary chemical potential on the thermal side. In Sect. 4.3 we dynamically select the correct thermal partition function by looking at the free energy associated with all our potential finite T candidates. In Sect. 4.4 we take this thermal partition function and turn on continuous background gauge fields, thereby reproducing all the partition functions found in Sect. 4.2. We then use this partition function with general Wilson lines to show how the Type I thermal partition function is stable under small changes in the background gauge fields and to look at the stability of other critical points. In Sect. 4.5 we comment on the correspondence to Heterotic strings and talk briefly about the implications of this for S-duality at finite temperature.

4.2 Finite temperature extrapolation of Type I $SO(32)$ string

We first establish our notation and write down the partition function for the zero temperature Type I $SO(32)$ string.

4.2.1 The Type I $SO(32)$ string

The Type I $SO(32)$ open string is obtained as an orientifold projection of Type IIB. The Type IIB partition function is:

$$Z_{\rm IIB} \;=\; Z^{(8)}_{\rm closed}\,(\overline{\chi}_V - \overline{\chi}_S)\,(\chi_V - \chi_S) \tag{4.1}$$

where $Z^{(8)}_{\rm closed}$ denotes the contribution from the eight worldsheet bosons in light-cone gauge given by,

$$Z^{(n)}_{\rm closed} = {\tau_2}^{-n/2}(\bar{\eta}\eta)^{-n} \tag{4.2}$$

and where the contributions from the worldsheet fermions are written in terms of the characters χ of the $SO(8)$ transverse Lorentz group. All $SO(8)$ characters are defined in the appendix of Chapter 2. The Type I string is invariant under orientation-reversal. This symmetry is implemented by the orientation-reversal operator, Ω, that exchanges left and right sectors for the closed string. The closed sector of Type I is thus given by:

$$\text{Tr}\ \tfrac{1}{2}(1+\Omega)Z_{\rm IIB} \tag{4.3}$$

The first term in Eq. 4.3 corresponds to the Torus contribution and the second term corresponds to the Klein bottle, which is an unoriented surface of genus one. In addition the Type I partition function includes an open sector which has contributions from genus-one surfaces with boundaries, the Cylinder and the Mobius strip.

We will write down all these partition functions in terms of bosonic and $SO(8)$ characters which will be functions of q. We always define q as $q \equiv \exp[2\pi i\tau]$. The variable τ will depend upon the surface under consideration. For the torus it is simply the modulus, while for the Klein, Cylinder and Mobius partition functions it is the modulus of their double covering torus. For each case τ is defined as,

$$\tau = \begin{cases} \tau_1 + i\tau_2 & \text{for Torus} \\ 2i\tau_2 & \text{for Klein} \\ \frac{1}{2}i\tau_2 & \text{for Cylinder} \\ \frac{1}{2}i\tau_2 + \frac{1}{2} & \text{for Mobius} \end{cases} \tag{4.4}$$

Since the Klein, Cylinder and Mobius partition functions ultimately depend only on τ_2, we shall always refer to them as functions of τ_2.

The bosonic contribution for the Klein, Cylinder and Mobius partitions is given by,

$$Z^{(n)}_{\text{open}} = {\tau_2}^{-n/2}\eta^{-n} \tag{4.5}$$

where η is a function of q as defined in Eq. 4.4.

We now write down the partition functions for the Type I SO(32) string. The Torus contribution is given by,

$$Z_{\text{T}}(\tau) \;=\; \frac{1}{2}\, Z^{(8)}_{\text{closed}}\; (\overline{\chi}_V - \overline{\chi}_S)\,(\chi_V - \chi_S) \tag{4.6}$$

The Klein partition is given by,

$$Z_{\text{K}}(\tau_2) \;=\; \frac{1}{2}\, Z^{(8)}_{\text{open}}\; (\chi_V - \chi_S) \tag{4.7}$$

The cylinder partition function is given by,

$$Z_{\text{C}}(\tau_2) \;=\; \frac{1}{2}\, N^2\, Z^{(8)}_{\text{open}}\, (\chi_V - \chi_S) \tag{4.8}$$

The Mobius partition is,

$$Z_{\text{M}}(\tau_2) \;=\; -\;\frac{1}{2}\, N \hat{Z}^{(8)}_{\text{open}}\, (\hat{\chi}_V - \hat{\chi}_S) \tag{4.9}$$

The hatted characters ensure that the Mobius partition is real and are defined as

$$\hat{\chi}_i = e^{-i\pi h_i}\chi_i \tag{4.10}$$

where h_i are the conformal weights of the corresponding primary fields. The factors of N^2 and N in the Cylinder and Mobius contributions arise from the presence of internal Chan-Paton symmetry that associates a multiplicity N to each end of the open string. These amplitudes are written in the direct channel - one can go to the corresponding transverse channels and write the tadpole cancellation condition for the NS-NS and R-R sectors. These conditions uniquely select $N = 32$, corresponding to the gauge group $SO(32)$ for the Type I string.

4.2.2 Possible Type I Partitions

The partition function for the thermal extrapolation of Type IIB is

$$\begin{aligned} Z_{\text{IIB}}(\tau,T) \;=\; Z^{(8)}_{\text{closed}} \times \Big\{ \quad & \mathcal{E}_0 \quad [\overline{\chi}_V\chi_V + \overline{\chi}_S\chi_S] \\ + \quad & \mathcal{O}_0 \quad [\overline{\chi}_I\chi_I + \overline{\chi}_C\chi_C] \\ - \quad & \mathcal{E}_{1/2} \quad [\overline{\chi}_V\chi_S + \overline{\chi}_S\chi_V] \\ - \quad & \mathcal{O}_{1/2} \quad [\overline{\chi}_I\chi_C + \overline{\chi}_C\chi_I] \; \Big\} \end{aligned} \tag{4.11}$$

where $\mathcal{E}_0, \mathcal{O}_0, \mathcal{E}_{1/2}$ and $\mathcal{O}_{1/2}$, represent the bosonic contribution for the compactified Euclidean time direction, equivalent to T under the "temperature/radius correspondence", and are defined in the appendix of Chapter 2. This partition function develops a tachyon above the Hagedorn temperature $T = \mathcal{M}/\sqrt{2}$ and the $T \to \infty$ limit of the model is the Type 0B string. To identify the correct thermal extrapolation of Type I we write down all descendants of Eq. (4.11).

The torus partition function for the Type I descendants is uniquely given by,

$$Z_{\text{T}}(\tau,T) \;=\; \frac{1}{2}\, Z_{\text{IIB}}(\tau,T) \tag{4.12}$$

To write down the bosonic contribution to the partition function from the compactified direction for the Klein, Cylinder and Mobius contributions, we define:

$$P_m = \exp[-\pi m^2 a^2 \tau_2] \tag{4.13}$$

$$\begin{aligned} \mathcal{E} &= \sum_{m\in \mathbb{Z}} P_m \\ \mathcal{O} &= \sum_{m\in \mathbb{Z}} P_{m+\frac{1}{2}} \end{aligned} \tag{4.14}$$

In Eq. 5.26, a is the normalized temperature defined as $a = T/\mathcal{M}$. The Klein partition function is also uniquely given by,

$$Z_{\rm K}(\tau_2, T) = \frac{1}{2} Z^{(8)}_{\rm open} \times \left\{ \quad \mathcal{E} \quad [\chi_V - \chi_S] \ \right\} \tag{4.15}$$

The Cylinder partition function is,

$$\begin{aligned} Z_{\rm C}(\tau_2, T) = \frac{1}{2} Z^{(8)}_{\rm open} \times \Big\{ & \quad \mathcal{E} \quad [({n_1}^2 + {n_2}^2)\chi_V - 2n_1 n_2 \chi_S] \\ + & \quad \mathcal{O} \quad [2n_1 n_2 \chi_V - ({n_1}^2 + {n_2}^2)\chi_S]\Big\} \end{aligned} \tag{4.16}$$

And for the Mobius strip,

$$Z_{\rm M}(\tau_2, T) = -\frac{1}{2} \hat{Z}^{(8)}_{\rm open} \times \Big\{ (n_1 + n_2) \, [\mathcal{E} \, \hat{\chi}_V - \mathcal{O} \, \hat{\chi}_S]\Big\} \tag{4.17}$$

These partition functions result in an open string spectrum with gauge group $SO(n_1) \times SO(n_2)$, where tadpole cancellations from the transverse channel fix $n_1 + n_2 = 32$. This is a single family of partition functions. The sum $n_1 + n_2$ also counts the total number of D9-branes in the theory.

There is one more partition function with a unitary gauge group whose cylinder partition function is given by,

$$\begin{aligned} Z_{\rm C}(\tau_2, T) = \frac{1}{2} Z^{(8)}_{\rm open} \times \Big\{ & \quad \mathcal{E} \quad [(2n\bar{n})\chi_V - (n^2 + \bar{n}^2)\chi_S] \\ + & \quad \mathcal{O} \quad [(n^2 + \bar{n}^2)\chi_V - (2n\bar{n})\chi_S]\Big\} \end{aligned} \tag{4.18}$$

And for the Mobius strip

$$\begin{aligned} Z_{\rm M}(\tau_2, T) \;=\; -\frac{1}{2}\,\hat{Z}^{(8)}_{\rm open} \times \Big\{ \quad & \mathcal{O} \quad [(n+\bar{n})\hat{\chi}_V] \\ - \quad & \mathcal{E} \quad [(n+\bar{n})\hat{\chi}_S]\Big\} \end{aligned} \tag{4.19}$$

With $n = \bar{n} = 16$ and the gauge group is therefore $U(16)$.

4.3 Selection of thermal partition function for Type I

To select the correct Type I partition at finite T we use the idea outlined in chapter 3 - the thermal partition function is the one that has the least free energy. To this end we compute the free energies for the partition functions written down in the previous section. For any general partition the free energy contributed by the torus is given by,

$$F_{\rm T}(T) \;=\; -\tfrac{1}{2}\,T\,\mathcal{M}^9 \int_{\mathcal{F}} \frac{d^2\tau}{(\tau_2)^2}\,Z_{\rm T}(\tau, T) \tag{4.20}$$

where $\mathcal{F}$ is the fundamental domain of the modular group. The free energy contributed by the other genus one surfaces is:

$$F_{\rm X}(T) = -\tfrac{1}{2}\,T\,\mathcal{M}^9 \int_0^\infty \frac{d\tau_2}{(\tau_2)^2} Z_{\rm X}(\tau_2, T) \tag{4.21}$$

where X is K, C or M. The total free energy is the sum,

$$F(T) \;=\; F_{\rm T}(T) + F_{\rm K}(T) + F_{\rm C}(T) + F_{\rm M}(T) \tag{4.22}$$

We explicitly write out the free energy contributions of the Klein, Cylinder and Mobius partitions with gauge group $SO(n_1) \times SO(n_2)$ (opening out all bosonic contributions). Changing variables for Klein $t = 2\tau_2$, for cylinder $t = \tau_2/2$ and for Mobius $t = \tau_2$, the free energy integrals are,

$$
\begin{aligned}
F_{\mathrm{K}} &= -\tfrac{1}{2}\, T\, \mathcal{M}^9\, 2^{\frac{9}{2}} \int_0^\infty \frac{dt}{t^{\frac{11}{2}}}\, \eta(it)^{-8} \times \\
&\qquad \frac{1}{2}\left\{ [\chi_V(it) - \chi_S(it)] \sum_{m\in\ \mathbb{Z}} e^{-\pi m^2 a^2 t/2} \right\} \\
F_{\mathrm{C}} &= -\tfrac{1}{2}\, T\, \mathcal{M}^9\, 2^{-\frac{9}{2}} \int_0^\infty \frac{dt}{t^{\frac{11}{2}}}\, \eta(it)^{-8} \times \\
&\qquad \frac{1}{2}\Big\{ [({n_1}^2 + {n_2}^2)\chi_V(it) - (2n_1 n_2)\chi_S(it)] \sum_{m\in\ \mathbb{Z}} e^{-2\pi m^2 a^2 t} \\
&\qquad + [(2n_1 n_2)\chi_V(it) - ({n_1}^2 + {n_2}^2)\chi_S(it)] \sum_{m\in\ \mathbb{Z}+\frac{1}{2}} e^{-2\pi m^2 a^2 t} \Big\} \\
F_{\mathrm{M}} &= \tfrac{1}{2}\, T \mathcal{M}^9 \int_0^\infty \frac{dt}{t^{\frac{11}{2}}}\, \hat{\eta}(\tfrac{it}{2} + \tfrac{1}{2})^{-8} \times \\
&\qquad \frac{1}{2}\Big\{ (n_1 + n_2)[\hat{\chi}_V(\tfrac{it}{2} + \tfrac{1}{2}) \sum_{m\in\ \mathbb{Z}} e^{-\pi m^2 a^2 t} \\
&\qquad -\ \hat{\chi}_S(\tfrac{it}{2} + \tfrac{1}{2}) \sum_{m\in\ \mathbb{Z}+\frac{1}{2}} e^{-\pi m^2 a^2 t}] \Big\}
\end{aligned} \tag{4.23}
$$

Using the methods of [19] we can evaluate these free energies. Note that irrespective of the gauge group, the contribution of the Torus, the Klein bottle and the Mobius strip is always the same for the $SO(n_1) \times SO(n_2)$ family. The Klein bottle contribution is zero in all cases due to the abstruse identity. Only the contribution of the cylinder depends upon the gauge group. Simplifying the cylinder partition function we see that this dependence is given by $(n_1 - n_2)^2 = (2n_1 - 32)^2$. The contribution to the free energy by the cylinder is a monotonic function of n_1 and hence the total free energy also depends on a monotonic way on n_1 as it varies from 32 to 16 by integers. Therefore even without calculating the free energy it is apparent that the partition function with the $SO(32)$ gauge group will have the lowest free energy. Nevertheless to see the exact behaviour at finite T of the choices available to us, we plot the free energy for specific values of n_1 in Fig. 4.1. For comparison with the Heterotic string we restrict n_1 to be a multiple of 8, with corresponding gauge groups given by,

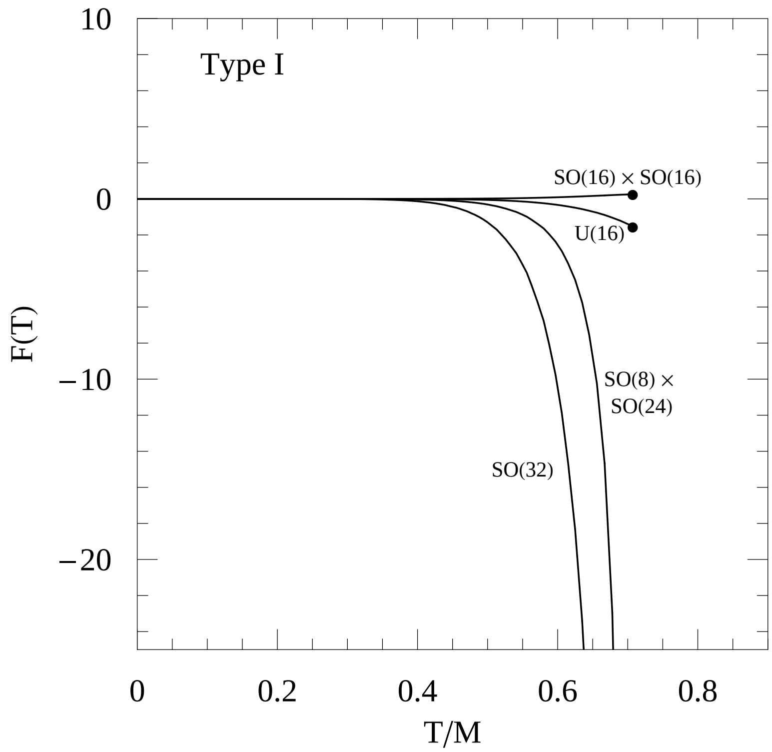

FIGURE 4.1. Free energies $F(T)$ in units of $\frac{1}{2}\mathcal{M}^{10} = \frac{1}{2}(\mathcal{M}_{\text{string}}/2\pi)^{10}$, plotted as functions of the normalized temperature $T/\mathcal{M}$ for the Type I string. The free energies are shown for the same gauge groups as the Heterotic string. Just as in the case of the Heterotic String, we see that the choice that leads to minimum free energy - the non-supersymmetric $SO(32)$ theory - also breaks the gauge group minimally. One important difference from the Heterotic theory is that the free energies for the non-supersymmetric $SO(32)$ and $SO(8) \times SO(24)$ theories grow without bound as the critical temperature $T = \mathcal{M}/\sqrt{2}$ is approached.

- $SO(32)$
- $SO(24) \times SO(8)$
- $SO(16) \times SO(16)$

and the

- $U(16)$

gauge group.

From the plot it is clear that the theory with $SO(32)$ gauge group has the lowest free energy at all temperatures. All four theories also become tachyonic at $T = \mathcal{M}/\sqrt{2}$, which we refer to as the critical temperature. The $SO(32)$ theory and the $SO(8) \times SO(24)$ theory develop tachyons in both the closed and open sector beyond the critical temperature. Because of symmetry the open sector tachyon is absent for the $SO(16) \times SO(16)$ and $U(16)$ theories. Correspondingly there is a NS-NS tadpole divergence at the critical temperature for the $SO(32)$ and $SO(8) \times SO(24)$ theories while there is no such divergence for the $SO(16) \times SO(16)$ and $U(16)$ theories. This is reflected in the way the free energy varies with T for these theories, with the free energy of the $SO(16) \times SO(16)$ and $U(16)$ theories staying finite till the critical temperature similar to the behaviour of closed string theories, while the free energy for $SO(32)$ theory and the $SO(8) \times SO(24)$ theory diverges as the critical temperature is approached. Since we have chosen the $SO(32)$ theory as the finite temperature extrapolation for Type I, the Hagedorn temperature, equal to the critical temperature for this theory, is given by $T = \mathcal{M}/\sqrt{2}$ which is the same as the Hagedorn temperature of Type II and Heterotic strings. However because the free energy for this theory diverges asymptotically at the Hagedorn temperature, it is widely assumed that the Hagedorn temperature is a limiting temperature for Type I strings unlike for the Heterotic theory.

Finally for future reference we explicitly write down the partition function for the thermal Type I string. It consists of the Torus and Klein contributions given by Equations 4.12 and 4.15, while the Cylinder and Mobius contributions are given by,

$$\begin{aligned} Z_{\rm C}(\tau_2, T) &= \frac{1}{2}\, Z^{(8)}_{\rm open} \times \left\{\; N^2\, [\mathcal{E}\, \chi_V - \mathcal{O}\, \chi_S]\right\} \\ Z_{\rm M}(\tau_2, T) &= \frac{1}{2}\, \hat{Z}^{(8)}_{\rm open} \times \left\{\; -N\, [\mathcal{E}\, \hat{\chi}_V - \mathcal{O}\, \hat{\chi}_S]\right\} \end{aligned} \tag{4.24}$$

where $N = 32$. These are the same as the partitions of Equations 4.16 and 4.17 with $n_1 = 32$ and $n_2 = 0$.

As we mentioned in the introduction while talking about finite temperature string theory one usually starts with the Boltzmann sum which is an integral over the strip region of the torus, which in turn is equal to a modular invariant partition function integrated over the fundamental region of the torus([64]). For Type II strings this modular invariant partition function is obtained from Type II theory on a circle by modding out by the orbifold $\mathcal{T}(-1)^F$, where F is the spacetime fermion number and $\mathcal{T}$ acts on the compactified Euclidean time direction, X^0 as $\mathcal{T}:\ X^0 \to X^0 + \pi R$, here R is the radius of the thermal circle. For Type I strings the Boltzmann sum similarly equals the sum of the Torus and the Klein amplitudes for the closed sector while it trivially equals the sum of cylinder and mobius contributions of Eq. 4.24 in the open sector. Note that Equations 4.12, 4.15 and 4.24 are also the result of modding out/projecting the Type I partition function on a circle by $\mathcal{T}(-1)^F$.

4.4 Stable Thermodynamic states

4.4.1 Stable states for the Type I string

In the previous section we assumed that there is only one state the Type I theory can occupy at finite temperature. However at a fixed temperature the thermal partition function (and hence the free energy) can be modified by turning on continuous background gauge fields. We will only be interested in flat background gauge fields with vanishing field strengths also referred to as Wilson lines[1]. The $T = 0$ limit of any finite T theory with a general Wilson lines is the Type I theory. Therefore we can treat such background gauge fields as free parameters of the partition function which appear at finite temperature and determine the thermodynamically stable states by finding the extrema of the free energy with respect to the background gauge fields. Taking this approach means that at finite temperature the string theory may have possible metastable states. However we find that Type I theory only has a single stable state at any temperature.

[1] For some representative papers for strings in general background fields at finite temperature see [69].

It can be analytically shown that the partition function with all background gauge fields set to zero will always be a global minimum of the free energy. We verify this and also show that there are no other local minima present for the Type I $SO(32)$ finite Temperature theory.

First we review how the thermal partition function of Type I gets modified when a flat background gauge field is turned on (Similar analysis is done in [67, 68] for the Scherk-Schwarz compactification of Type I theory). We saw in Chapter 3 that a background gauge field given by $\vec{A}^0 = -\vec{\Phi}/(2\pi R)$, shifts the Matsubara momentum of a state in the following way:

$$\frac{m}{R} \rightarrow \frac{m}{R} + \frac{1}{2\pi R}\vec{\lambda}\cdot\vec{\Phi} \ . \tag{4.25}$$

Here $\vec{\lambda}$ is the weight vector of the state and R is the radius of the thermal circle related to the temperature T by $R = \frac{1}{2\pi T}$.
Defining $\vec{\Phi} = 2\pi\vec{\ell}$, so that the gauge field is given by

$$\vec{A} = -\vec{\Phi}/(2\pi R) = -T\,2\pi\vec{\ell} \tag{4.26}$$

we see from Eq. 4.25 that each momentum mode, m will shift as,

$$m \rightarrow \quad m + \vec{\lambda}\cdot\vec{\ell} \tag{4.27}$$

for a state with weight vector $\vec{\lambda}$ in the partition function. A background gauge field will not affect the closed sector consisting of the Torus and the Klein bottle partition functions since the closed string does not carry any Chan-Paton charges. The open sector partition functions get modified. Since the associated gauge group of the thermal partition function is $SO(32)$ the most general background gauge field can be written in the form Eq. 4.26, where the vectors $\vec{A}$ and $\vec{\ell}$ are sixteen dimensional corresponding to the dimension of the Cartan subgroup of $SO(32)$. The open sector partition functions will therefore be functions of the parameters, ℓ^I, where I varies from 1 to 16.

The total number of gauge group states in the Cylinder partition is given by the factor $N^2 = 32^2$, of Eq. 4.24. This decomposes as a total of $N(N-1)/2$ antisymmetric states, in the adjoint representation and $N(N+1)/2$ symmetric states comprising of the symmetric tensor and a singlet state. For future use we define the set of weights of states in the Cylinder partition function as Λ_C, which is the union of the set of weights of the symmetric states Λ_S and the set of weights of antisymmetric states Λ_A.

As is well known the background gauge fields can also be interpreted in the T-dual theory as the positions of 16 D8-branes (and 16 image D8-branes under an orientifold identification) on the thermal circle. The general background field of Eq. 4.26 corresponds to D8-brane positions given by

$$2\pi\{\ell_i\}_{i=1 \text{ to } 32} = 2\pi\{\ell^1, -\ell^1, \ell^2, -\ell^2, ..., \ell^{16}, -\ell^{16}\} \tag{4.28}$$

We now write down the partition function with a general Wilson line. The partition function for the cylinder is:

$$\begin{aligned} Z_{\rm C}(\tau_2, \vec{\ell}, T) &= \frac{1}{2} Z^{(8)}_{\rm open} \times \Big\{ \sum_{\substack{\lambda\in\Lambda_C \\ m\in \mathbb{Z}}} [\chi_V P_{m+\vec{\lambda}\cdot\vec{\ell}} \;-\; \chi_S P_{m+\vec{\lambda}\cdot\vec{\ell}+\frac{1}{2}}] \Big\} \\ &= \frac{1}{2} Z^{(8)}_{\rm open} \times \Big\{ \sum_{\substack{i,j=1 \\ m\in \mathbb{Z}}}^{32} [\chi_V P_{m+\ell_i+\ell_j} - \chi_S P_{m+\ell_i+\ell_j+\frac{1}{2}}] \Big\} \end{aligned} \tag{4.29}$$

For the mobius strip, the factor $-N = -32$ of the Mobius amplitude decomposes as the difference between the antisymmetric states and the sum of the symmetric states. Therefore the partition function for the Mobius strip becomes:

$$\begin{aligned} Z_{\rm M}(\tau_2, \vec{\ell}, T) &= \frac{1}{2} \hat{Z}^{(8)}_{\rm open} \times \Big\{ \sum_{\substack{\lambda\in\Lambda_A \\ m\in \mathbb{Z}}} [\hat{\chi}_V P_{m+\vec{\lambda}\cdot\vec{\ell}} \;-\; \hat{\chi}_S P_{m+\vec{\lambda}\cdot\vec{\ell}+\frac{1}{2}}] \\ &\qquad - \sum_{\substack{\lambda\in\Lambda_S \\ m\in \mathbb{Z}}} [\hat{\chi}_V P_{m+\vec{\lambda}\cdot\vec{\ell}} \;-\; \hat{\chi}_S P_{m+\vec{\lambda}\cdot\vec{\ell}+\frac{1}{2}}] \Big\} \\ &= \frac{1}{2} \hat{Z}^{(8)}_{\rm open} \times \Big\{ \sum_{\substack{i=1 \\ m\in \mathbb{Z}}}^{32} [\hat{\chi}_V P_{m+2\ell_i} - \hat{\chi}_S P_{m+2\ell_i+\frac{1}{2}}] \Big\} \end{aligned} \tag{4.30}$$

Setting the first derivatives of these partition functions with respect to the 16 parameters ℓ^I (proportional to the background gauge fields) to zero will give us the critical points of this theory. Note that consistency of the string vacuum demands that all one loop one-point functions vanish which is equivalent to the condition that the first derivatives of the partition function with respect to the background gauge fields vanish. To find maxima/minima of the partition function we can calculate the Hessain Matrix or numerically determine whether a particular point is an extrema by evaluating the free energy of these partition functions in the neighbourhood of the point. Choosing the second approach we can confirm that the partition function with all background gauge fields set to zero is indeed a global minimum.

To check whether there are other potential minima in the theory, we simplify the partition functions, equations 4.29 and 4.30, by reducing them to equations of two variables. We choose a special background gauge field given by

$$\vec{A} = -T\,2\pi\{\ell, \ell, ..., \ell, 0, 0, ..., 0\} \tag{4.31}$$

where ℓ occurs $n \equiv n_1/2$ times and zero occurs $n_2/2$ times with $n_1 + n_2 = 32$. In the T-dual theory this corresponds to n D8-branes coincident at the point, $2\pi\ell$, on the thermal circle and $16 - n$ D8-branes coincident with an orientifold fixed plane. Any point that is a minimum in the theory with the most general background field, Eq. 4.26, should also appear as a minimum under variation of the single parameter ℓ in Eq. 4.31. For notational convenience we define:

$$\begin{aligned} \mathcal{E}(s) &= \sum_{m\in\mathbb{Z}} P_{m+s} \\ \mathcal{O}(s) &= \sum_{m\in\mathbb{Z}} P_{m+\frac{1}{2}+s} \end{aligned} \tag{4.32}$$

Note that, $\mathcal{E}(n) = \mathcal{E}, \mathcal{E}(n+\frac{1}{2}) = \mathcal{O}$ and $\mathcal{O}(n) = \mathcal{O}, \mathcal{O}(n+\frac{1}{2}) = \mathcal{E}$, where n is any integer. The cylinder partition function becomes

$$\begin{aligned} Z_{\rm C}(\tau_2, \ell, T) \;=\; & \frac{1}{2}\, Z^{(8)}_{\rm open} \times \{ \\ & \chi_V[\tfrac{{n_1}^2}{4}\{\mathcal{E}(2\ell) + \mathcal{E}(-2\ell)\} + {n_1 n_2}\{\mathcal{E}(\ell) + \mathcal{E}(-\ell)\} + (\tfrac{{n_1}^2 + 2{n_2}^2}{2})\mathcal{E}] \\ - \; & \chi_S[\tfrac{{n_1}^2}{4}\{\mathcal{O}(2\ell) + \mathcal{O}(-2\ell)\} + {n_1 n_2}\{\mathcal{O}(\ell) + \mathcal{O}(-\ell)\} + \tfrac{{n_1}^2 + 2{n_2}^2}{2}\mathcal{O}]\} \end{aligned} \tag{4.33}$$

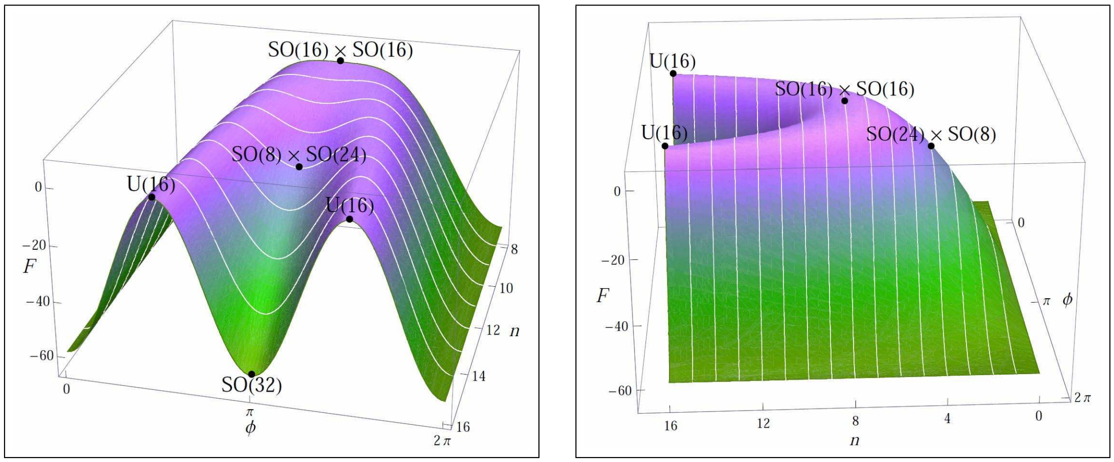

FIGURE 4.2. Variation in Free energy, F plotted in units of $\frac{1}{2}\mathcal{M}^{10}$, with n D-branes fixed at an angle ϕ and $16-n$ at an angle 0 on the thermal circle. In the T-dual theory this corresponds to a gauge field background given by $\frac{1}{T}\vec{A} = \{\phi, \phi, ..., \phi, 0, 0, ..., 0\}$, where ϕ occurs n times. The temperature here is fixed at $T/\mathcal{M} = \frac{2}{3}$, although the plot qualitatively stays the same at any value of T lower than the Hagedorn Temperature. In the first plot n varies from 8 to 16. The second plot is another view of the same data, plotted over the full range of n from 0 to 16. Four critical points are shown. The non-susy $SO(32)$ theory is a global minimum. Note that in both plots the entire lower perimeter is the $SO(32)$ theory. The $U(16)$ and $SO(24) \times SO(8)$ theories are saddle points. The $SO(16) \times SO(16)$ is expected to be a global maximum, and here in the plot it does appear as a maximum, although that is hard to discern.

And the Mobius partition

$$\begin{aligned} Z_{\mathrm{M}}(\tau_2, \ell, T) \;\; = \;\; -\frac{1}{2}\,\hat{Z}^{(8)}_{\text{open}} \times \Big\{ \; & \hat{\chi}_V[\tfrac{n_1}{2}\{\mathcal{E}(2\ell) + \mathcal{E}(-2\ell)\} + n_2 \mathcal{E}] \\ & - \hat{\chi}_S[\tfrac{n_1}{2}\{\mathcal{O}(2\ell) + \mathcal{O}(-2\ell)\} + n_2 \mathcal{O}]\Big\} \end{aligned} \tag{4.34}$$

The thermal free energy can now be treated as a function of the two variables $n \equiv n_1/2$ and ℓ. (Note that $n_2/2$ is not an independent variable here but is equal to $16-n$.) The variable $\phi \equiv 2\pi\ell$ can be varied smoothly from 0 to 2π. In contrast the above partition functions and therefore the free energy is defined only for integer n varying from 0 to 16. Nevertheless a fractional value of n physically makes sense in the T-dual theory as capturing the dynamics of a configuration with a total of 16 branes (and 16 image branes), some of which may be located at points other than 0 and ϕ.

In Fig. 4.2 we plot the total free energy as a function of n and $\phi \equiv 2\pi\ell$. Note that the equations 4.33 and 4.34 reduce to the equations 4.16 and 4.17 for special values of ℓ. We write down the gauge groups obtained for particular values of the gauge field, $\vec{A}$:

- $\frac{1}{T}\vec{A} = 2\pi(0^{16})$ – results in Non-SUSY $SO(32)$
- $\frac{1}{T}\vec{A} = 2\pi(\frac{1}{2}^4, 0^{12})$ – results in $SO(8) \times SO(24)$
- $\frac{1}{T}\vec{A} = 2\pi(\frac{1}{2}^8, 0^8)$ – results in $SO(16) \times SO(16)$
- $\frac{1}{T}\vec{A} = 2\pi(\frac{1}{4}^{16})$ – results in $U(16)$ (4.35)

where the LHS is defined Mod 2π.

In Fig. 4.2 all the above gauge groups show up as critical points of the theory as expected. Also the Non-SUSY $SO(32)$ theory shows up as a minimum as it should. However it is clear from the plot that none of the other critical points can be a minima. The points corresponding to the $SO(16) \times SO(16)$ gauge group appears as a maximum on this plot, $U(16)$ as a saddle point, whereas the $SO(8) \times SO(24)$ theory shows up as a minima at one place and as a maxima at another. Therefore we conclude from Fig. 4.2 that there is only one minimum in the theory.

4.4.2 Stable states for the Heterotic $SO(32)$ String

In the case of a Heterotic string compactified on the thermal circle we can also turn on the same background gauge fields given by Eq.4.26. The shift in momentum mode, m due to these fields will be of a form similar to Eq. 4.25 but will also depend on the string winding number, n. (For a derivation see [71]). In addition the gauge group weight vector, $\vec{\lambda}$, will also be shifted. The shifts in the momentum, winding numbers and the weight vector are given by:

$$\begin{aligned} m &\rightarrow m + \vec{\lambda}\cdot\vec{\ell} - n\,\vec{\ell}\cdot\vec{\ell}/2 \\ n &\rightarrow n \\ \vec{\lambda} &\rightarrow \vec{\lambda} - n\,\vec{\ell} \end{aligned} \tag{4.36}$$

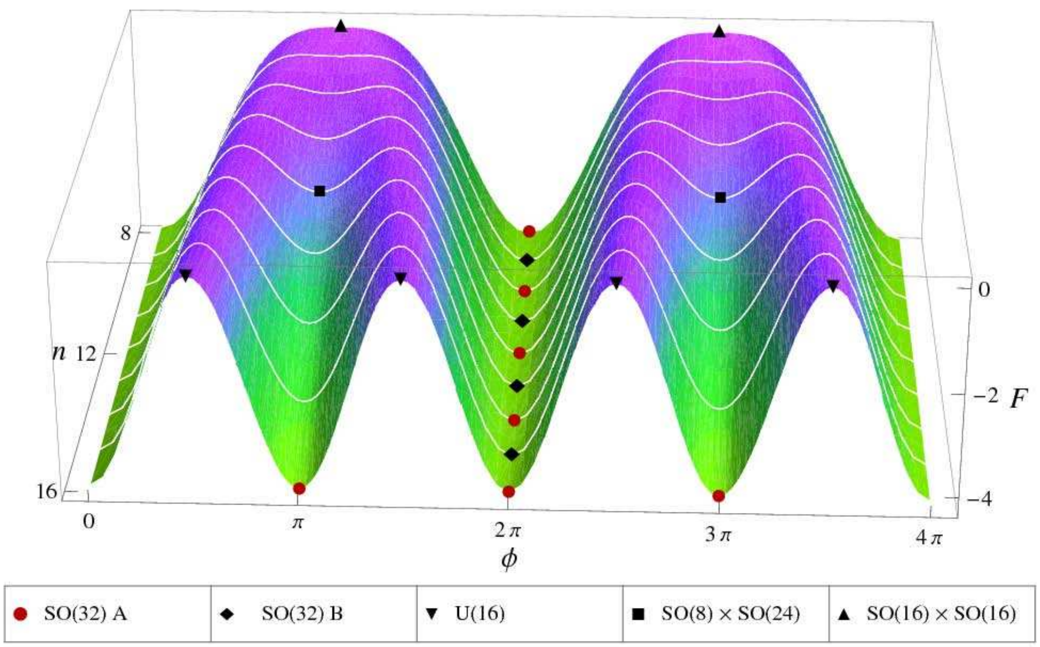

FIGURE 4.3. For comparison purposes we show the heterotic free energy, F plotted in units of $\frac{1}{2}\mathcal{M}^{10}$, as a function of the gauge field specified by the two parameters ϕ and n. Five critical points are shown on the plot. These points match the critical points of Type I theory except that the single $SO(32)$ point of Type I is now split into two different $SO(32)$ theories with very close free energies. Note that for odd values of n the plot repeats after an interval 4π, while for even values of n it repeats after an interval 2π.

The heterotic partition function on the thermal circle with no Wilson line is given by,

$$\begin{aligned} Z(\tau,T) \;=\; Z^{(8)}_{\text{boson}} \times \Big\{ \quad & \mathcal{E}_0 \quad \overline{\chi}_V \; (\tilde{\chi}_I + \tilde{\chi}_S) \\ - \quad & \mathcal{E}_{1/2} \quad \overline{\chi}_S \; (\tilde{\chi}_I + \tilde{\chi}_S) \\ - \quad & \mathcal{O}_0 \quad \overline{\chi}_C \; (\tilde{\chi}_I + \tilde{\chi}_S) \\ + \quad & \mathcal{O}_{1/2} \quad \overline{\chi}_I \; (\tilde{\chi}_I + \tilde{\chi}_S) \; \Big\} \end{aligned} \tag{4.37}$$

To see the effect of turning on a general Wilson line given by Eq. 4.26 on this partition function, we use equations (4.36), the lattices corresponding to the characters

$\mathcal{E}_0, \mathcal{O}_0, \mathcal{E}_{1/2}, \mathcal{O}_{1/2}$, which are given by

$$\begin{aligned}
\Lambda_{0,0} &= \{m \in \mathbb{Z},\ n \text{ even}\} \\
\Lambda_{0,1} &= \{m \in \mathbb{Z},\ n \text{ odd}\} \\
\Lambda_{1/2,0} &= \{m \in \mathbb{Z} + \tfrac{1}{2},\ n \text{ even}\} \\
\Lambda_{1/2,1} &= \{m \in \mathbb{Z} + \tfrac{1}{2},\ n \text{ odd}\}\ .
\end{aligned} \tag{4.38}$$

and the weight lattice corresponding to the sum of the $SO(32)$ characters, $\tilde{\chi}_I + \tilde{\chi}_S$, given by:

$$\Lambda_{SO(32)} = \left\{ \lambda^I \in \mathbb{Z}, \sum_{I=1 \text{ to } 16} \lambda^I = \text{even} \right\} + \left\{ \lambda^I \in \mathbb{Z} + \tfrac{1}{2}, \sum_{I=1 \text{ to } 16} \lambda^I = \text{even} \right\} \tag{4.39}$$

We now write down exactly the way the partition function Eq. (4.37) will deform for a general Wilson line. Using Eq. 4.36 for the shift in momentum, winding and gauge quantum numbers we can define:

$$\Xi[\vec{\ell}, r, s] = \sum_{\substack{\vec{\lambda} \in \Lambda_{SO(32)} \\ \{m,n\} \in \Lambda_{\{r,s\}}}} q^{(\vec{\lambda} - n\,\vec{\ell})^2/2} q^{[(m-k)a^2 + n^2/a^2]/4} \bar{q}^{[(m-k)a^2 - n^2/a^2]/4} \tag{4.40}$$

where $\Lambda_{\{r,s\}}$ is defined in Eq. (4.38), $k = -\vec{\lambda} \cdot \vec{\ell} + n\ \vec{\ell} \cdot \vec{\ell}/2$ and $q = \exp[2\pi i \tau]$. Then the partition function Eq. (4.37) will deform to

$$\begin{aligned}
Z[\tau, \vec{\ell}, T] = Z^{(8)}_{\text{boson}} \times \{ \quad & \overline{\chi}_V \quad \Xi[\vec{\ell}, 0, 0] \\
- \quad & \overline{\chi}_S \quad \Xi[\vec{\ell}, 1/2, 0] \\
- \quad & \overline{\chi}_C \quad \Xi[\vec{\ell}, 0, 1] \\
+ \quad & \overline{\chi}_I \quad \Xi[\vec{\ell}, 1/2, 1]\ \}
\end{aligned} \tag{4.41}$$

Note that when the background fields vanish, that is $\vec{\ell} = \vec{0}$, the $\Xi[\vec{\ell}, r, s]$ of Eq. 4.40 reduce to,

$$\begin{aligned}
\Xi[\vec{0},0,0] &= \mathcal{E}_0 \; (\tilde{\chi}_I + \tilde{\chi}_S) \\
\Xi[\vec{0},0,1] &= \mathcal{O}_0 \; (\tilde{\chi}_I + \tilde{\chi}_S) \\
\Xi[\vec{0},1/2,0] &= \mathcal{E}_{1/2} \; (\tilde{\chi}_I + \tilde{\chi}_S) \\
\Xi[\vec{0},1/2,1] &= \mathcal{O}_{1/2} \; (\tilde{\chi}_I + \tilde{\chi}_S)
\end{aligned} \tag{4.42}$$

and Eq. 4.41 reduces to Eq. 4.37.

The constant background gauge fields now act as sixteen free parameters, similar to cases of Heterotic string theories compactified on a circle ([71, 72]). This partition function is modular-invariant for all values of $\vec{\ell}$.

At the special values of the gauge field $\vec{A}$ given by,

- $\frac{1}{T}\vec{A} = 2\pi(0^{16})$ – results in Non-SUSY $SO(32)$ A
- $\frac{1}{T}\vec{A} = 2\pi(1, 0^{15})$ – results in Non-SUSY $SO(32)$ B
- $\frac{1}{T}\vec{A} = 2\pi(\frac{1}{2}^4, 0^{12})$ – results in $SO(8) \times SO(24)$
- $\frac{1}{T}\vec{A} = 2\pi(\frac{1}{2}^8, 0^8)$ – results in $SO(16) \times SO(16)$
- $\frac{1}{T}\vec{A} = 2\pi(\frac{1}{4}^{16})$ – results in $U(16)$ (4.43)

we recover the partition functions interpolating between the ten dimensional supersymmetric $SO(32)$ theory and the four ten dimensional non-supersymmetric theories with gauge groups $SO(32)$, $SO(8)\times SO(24)$, $SO(16)\times SO(16)$ and $U(16)$. For the corresponding partition functions see Chapter 3. Numerically we have determined that the points corresponding to the $SO(16)\times SO(16)$ and the $U(16)$ gauge groups are either maxima or saddle points under a change of background fields. The point $SO(8) \times SO(24)$ is a saddle point. The theory has two separate minima corresponding to the gauge background $\frac{1}{T}\vec{A} = 2\pi(0^{16})$ and $\frac{1}{T}\vec{A} = 2\pi(1, 0^{15})$. Both these theories have gauge group $SO(32)$ and we refer to them as $SO(32)$ A and $SO(32)$ B theories. Expectedly, the Heterotic string at finite temperature has critical points similar to that listed for Type I in Eq. 4.35, and at the same values of the background gauge field.

Similar to the Type I case we can reduce Eq.4.41 to a function of two variables by choosing the special background gauge field given by Eq. 4.31. In Fig. 4.3 we plot the Free energy as a function of the two variables n and ϕ. We see that the Heterotic behaviour is exactly the same as the Type I behaviour for even values of n. For odd values of n however, the free energy no longer repeats Mod 2π, but Mod 4π. This has implications for the finite temperature Heterotic String. Not only does the Heterotic theory have a global minimum at $(1/T)\vec{A} = 2\pi(0^{16})$ corresponding to the $SO(32)$ A theory with Hagedorn temperature, $T = 2\mathcal{M}/(2+\sqrt{2})$, but also a local minimum at $(1/T)\vec{A} = 2\pi(1, 0^{15})$ corresponding to the $SO(32)$ B theory with Hagedorn temperature, $T = \mathcal{M}/\sqrt{2}$. As can be seen from the Fig. 4.3, the free energies of these two theories are very close. This difference in the thermal behaviour of Type I and Heterotic theory can be traced to the presence of massive $SO(32)$ gauge group spinors in the Heterotic theory, which are absent in perturbative Type I theory. Non-perturbatively however Type I does contain $SO(32)$ spinors [70] so that non-perturbatively the Type I minimum is also expected to split into two states, bringing its thermal behaviour closer to the Heterotic theory. Of course not all points of the moduli space for background gauge fields are consistent vacua for either the Heterotic theory or Type I. In Chapter 2, we showed that the $SO(32)$ A theory is an inconsistent vacuum for the Heterotic theory. This then left us with the next available minima, the $SO(32)$ B vacuum, as the valid thermal Heterotic theory - fixing the Hagedorn Temperature to $T = \mathcal{M}/\sqrt{2}$, in agreement with perturbative Type I.

It is also interesting to note that the critical temperature of the general partition function, Eq. 4.41 is a function of the gauge field parameter, $\vec{\ell}$. We can trace the fate of the massless mode, about to become tachyonic, as we deform the partition function of Eq.(4.37) by a Wilson line. Using equation (4.36), the mass-shell and the level matching conditions it can be shown that the temperatures at which this mode will appear and disappear as a tacyhon are,

$$\{\frac{\sqrt{2}}{\sqrt{3-\vec{\ell}\cdot\vec{\ell}+\sqrt{8-4\vec{\ell}\cdot\vec{\ell}}}},\frac{\sqrt{2}}{\sqrt{3-\vec{\ell}\cdot\vec{\ell}-\sqrt{8-4\vec{\ell}\cdot\vec{\ell}}}}\} \tag{4.44}$$

where, $\vec{\ell}\cdot\vec{\ell}$ is defined Mod 2. While there maybe other massless modes (and subsequent tachyons) kicking in after the first temperature noted above the initial divergence in the free energy is always due to the mode above. This fixes the Hagedorn temperature to,

$$\frac{\sqrt{2}}{\sqrt{3-\vec{\ell}\cdot\vec{\ell}+\sqrt{8-4\vec{\ell}\cdot\vec{\ell}}}} \tag{4.45}$$

in units of $1/\mathcal{M}$. Note that this gives the right temperature for all interpolations we know.

4.5 Conclusions

Our aim in writing this chapter was to show the reader the similarities in the thermal behaviour of Heterotic and Type I strings.

Treating the background gauge field as a dynamical parameter at finite T, we saw that both Heterotic and Type I strings give rise to similar critical points with the same gauge groups. In both theories a non-supersymmetric vacuum with gauge group $SO(32)$ is a minimum of the theory, while vacua with all other gauge groups are unstable. While there is a difference in the number of stable minima in Type I and Heterotic theories, this difference is expected to vanish once one takes the non-pertubative regime of Type I into account.

Furthermore the evidence of chapters 2 and 3 suggests that both Heterotic and Type I theories have the same Hagedorn temperature.

As is well known a major difference is that in Type I the Hagedorn temperature acts as a limiting temperature unlike the Heterotic case.

The way Type I at finite T almost mirrors the Heterotic behaviour at finite T constitute a powerful argument in favour of S-duality at finite temperature. In the next chapter we show that indeed S-duality holds at finite temperature.

Chapter 5

S-DUALITY FOR FINITE TEMPERATURE STRING THEORY

Summary

The Heterotic and the Type I theories in ten dimensions are related to each other by S-duality. In this chapter we investigate whether this correspondence holds at finite temperature. The approach that we take in this chapter is a bit different than the previous ones. Here we investigate whether the Heterotic theory behaviour of having two stable states at finite temperature is reflected in Type I theory. We find that this is so and this mirroring of Heterotic behaviour on the Type I side is a powerful argument for S-duality at finite temperature.

5.1 Introduction

5.1.1 S-duality between Heterotic and Type I theories

As we saw in the introduction, the $SO(32)$ heterotic string and the Type I theory are very different from each other both in the way they are constructed and in their spectrum of states. Apriori there does not seem to be any relation between them. They do have the same massless spectrum and this was the initial motivation for suspecting that these two theories could be related - still it was not clear whether the same low energy spectrum could simply be attributed to supersymmetry, since supersymmetry completely determines the form of the action. Furthermore at the first excited level the Heterotic and the Type I spectrum are completely different. In the Type I theory all perturbative states transform in the adjoint representation of $SO(32)$, while in the heterotic theory there are scalars, spinors, symmetric and antisymmetric tensors of $SO(32)$.

However after years of investigation and a mound of evidence we now know that the $SO(32)$ heterotic string and the Type I theory are related to each other. Specifically the $SO(32)$ heterotic theory and the Type I theory are non-perturbative S-duals of each other. The strong coupling limit of the heterotic theory is weakly coupled Type I theory and vice-versa. In fact this duality symmetry is one example of numerous duality symmetries that relate ten dimensional string theories to each other.

The fact that the two theories are related by a change in the coupling constant also explains why the massive spectra of these two theories are so different. For example as the coupling constant value is increased, a generic heterotic state becomes unstable and decays and will therefore not be present in the Type I theory. The same is true for some Type I states. As an example the fundamental Type I string is a unstable state and breaks as the coupling is increased.

However as we will see there are some string states that are prevented from decaying and are therefore stable. These states should be present in both theories at all values of the coupling constant. Looking for these states and checking their multiplicities in the string spectrum is therefore a way to verify the duality conjecture, beyond the initial clue that the low energy effective action of Type I and $SO(32)$ are the same. BPS states which are present in supersymmetric theories are exactly such stable states. We will discuss these states in detail in the next section and calculate their masses and multiplicities. Such states are protected from decay because they carry a charge under one of the gauge fields present in the theory and their mass is completely determined by their charge. This connection also protects the mass of BPS states from radiative corrections and therefore one can easily trace their fate as the coupling constant is changed. For example, some states of a heterotic string wrapped around a circle break half of the original supersymmetries and are charged under the antisymmetric two-form field - the charge is the winding number. These BPS states should be identifiable in the Type I theory as a supersymmetric soliton. There is a non-perturbative object (d-brane) present in Type I, which is one-dimensional like the heterotic string and also BPS, therefore it can be expected that this is the needed soliton. In the next few sections we will show the various ways that the D1-brane of Type I, often referred to as the D-String, can be identified with the fundamental heterotic string, referred to as the F-String.

There is another way to test S-duality which does not rely on BPS states. It is in fact dangerous to just rely on BPS states to prove S-duality, as the matching of BPS states in the heterotic and Type I theories, maybe just due to the tight constraints that supersymmetry imposes in ten dimensions and may have nothing to do with S-duality. Now string theories often contain in their spectrum states that are *stable* but non-BPS, and thus such states can provide duality tests that are less dependent on supersymmetry. In fact a state in any theory that carries a conserved quantum number and is the lightest state of its type is protected against decay at all values of the coupling. Its mass may not be protected and hence may get renormalized as the coupling is changed, but nevertheless we should be able to identify these states and check their multiplicities at both strong and weak coupling. The first excited level of the perturbative heterotic string has exactly such states. The massive states in the representation $\mathbf{2^{15}}$ of $SO(32)$ are the lightest ones transforming as spinors of the gauge group. And indeed in the Type I theory we do find a non-perturbative object - the D0-brane that behaves as a massive particle in the spinor representation of $SO(32)$. In this chapter we will not directly be concerned with the exact properties of the D0-brane - we mention this fact to stress that the proof of S-duality goes beyond supersymmetry.

This chapter is organized as follows. In Section 5.1 we review the basic evidence for Type I, Heterotic duality in ten dimensions. Our ultimate goal is to prove S-duality for thermal Heterotic and Type I theories. As a preparation for that we first review how to write down general amplitudes for D-brane interactions in Sec 5.2. The spectrum of states for D-branes in Type I theory can always be read off from the D-brane amplitudes. In Sec 5.3 we review S-duality for the nine dimensional Type I and Heterotic theories in detail. The thermal theory is intimately related to the simple circle compactification case. As we have seen previously string theory at finite temperature can be obtained as a orbifold of the circle compactified theory. It is therefore crucial to understand how S-duality operates for the circle compactified theories. Finally in Sec 5.4 we make our case for S-duality at finite temperature.

For writing the review Sections 5.1, 5.2 and 5.3, we have consulted the papers [75, 73, 74, 6, 77].

5.1.2 Evidence for ten-dimensional S-duality

Supersymmetric S-duality in ten dimensions - Low energy effective action

In SO(32) heterotic string theory, the massless states(bosonic) come from the NS sector of the closed heterotic string, and contain the metric $G^H_{\mu\nu}$, the dilaton Φ^H, the rank two anti-symmetric tensor field $B^H_{\mu\nu}$, and gauge fields $A^{H\,a}_\mu$ in the adjoint representation of SO(32). The low energy field theory involving these massless bosonic fields is described by N=1 supergravity coupled to SO(32) gauge theory in ten dimensions. We now write down the action involving these fields. In the equations below we have absorbed the inverse string tension α'_H and coupling constant g_H into a redefinition of the fields. The action is given by:

$$\begin{aligned} S^H \sim \int d^{10}x\sqrt{-G^H}\Big[& R^H - \frac{1}{8}g^{H\,\mu\nu}\partial_\mu\Phi^H\partial_\nu\Phi^H \\ & - \frac{1}{4}G^{H\,\mu\mu'}G^{H\,\nu\nu'}e^{-\frac{\Phi^H}{4}}\ \mathrm{Tr}(F^H_{\mu\nu}F^H_{\mu'\nu'}) \\ & - \frac{1}{12}G^{H\,\mu\mu'}G^{H\,\nu\nu'}G^{H\,\rho\rho'}e^{-\frac{\Phi^H}{2}}H^H_{\mu\nu\rho}H^H_{\mu'\nu'\rho'}\Big] \end{aligned} \tag{5.1}$$

where R^H is the Ricci scalar, $F^H_{\mu\nu}$ denotes the non-abelian gauge field strength,

$$F^H_{\mu\nu} = \partial_\mu A^H_\nu - \partial_\nu A^H_\mu + \sqrt{2}\,[A^H_\mu, A^H_\nu] \tag{5.2}$$

Tr denotes the trace in the vector representation of SO(32), and $H^H_{\mu\nu\rho}$ is the field strength associated with the $B^H_{\mu\nu}$ field:

$$\begin{aligned} H^H_{\mu\nu\rho} = &\quad \partial_\mu B^H_{\nu\rho} - \frac{1}{2}Tr(A^H_\mu F^H_{\nu\rho} - \sqrt{\frac{2}{3}}\,A^H_\mu[A^H_\nu, A^H_\rho]) \\ + &\quad \text{cyclic permutations of } \mu,\nu,\rho \end{aligned} \tag{5.3}$$

This action is written in the Einstein frame which differs from the string frame in the metric field redefinition,

$$G_{\mu\nu} = \exp(-\phi/2) G_{S\,\mu\nu} \tag{5.4}$$

where the subscript S refers to the string frame. Type I string theory has the same massless spectrum as the Heterotic string theory. The massless bosonic states in type I theory come from three different sectors. The closed string NS-NS sector gives the metric $G^I_{\mu\nu}$ and the dilaton ϕ^I. The closed string Ramond-Ramond sector gives an anti-symmetric tensor field $B^I_{\mu\nu}$. Besides these, there are bosonic fields coming from the NS sector of the open string. This sector gives rise to gauge fields $A^{I\,a}_\mu$ in the adjoint representation of the group SO(32) - the same fields present in the heterotic theory. The low energy field theory is again described by the N=1 supergravity theory coupled to the SO(32) gauge theory. However fields coming from the open sector have a different dilaton dependence (coupling) given by $\exp(-\phi)$, rather than $\exp(-2\phi)$, as for fields coming from the closed sector. The effective action in terms of Type I variables is therefore slightly different. It is given by

$$\begin{aligned} S^I \sim \int d^{10}x \sqrt{-G^I} \Big[& R^I - \frac{1}{8} G^{I\,\mu\nu} \partial_\mu \Phi^I \partial_\nu \Phi^I \\ & - \frac{1}{4} G^{I\,\mu\mu'} g^{I\,\nu\nu'} e^{\frac{\Phi^I}{4}} \mathrm{Tr}(F^I_{\mu\nu} F^I_{\mu'\nu'}) \\ & - \frac{1}{12} G^{I\,\mu\mu'} g^{I\,\nu\nu'} G^{I\,\rho\rho'} e^{\frac{\Phi^I}{2}} H^I_{\mu\nu\rho} H^I_{\mu'\nu'\rho'} \Big] \end{aligned} \tag{5.5}$$

where R^I is the Ricci scalar, $F^I_{\mu\nu}$ denotes the non-abelian gauge field strength,

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + \sqrt{2}[A^I_\mu, A^I_\nu]\,, \tag{5.6}$$

and $H^I_{\mu\nu\rho}$ is the field strength associated with the $B^I_{\mu\nu}$ field:

$$\begin{aligned} H^I_{\mu\nu\rho} = \quad & \partial_\mu B^I_{\nu\rho} - \frac{1}{2} \mathrm{Tr}\Big(A^I_\mu F^I_{\nu\rho} - \frac{\sqrt{2}}{3} A^I_\mu [A^I_\nu, A^I_\rho] \Big) \\ + \quad & \text{cyclic permutations of } \mu, \nu, \rho \end{aligned} \tag{5.7}$$

Although the Type I and the Heterotic low energy actions are at the string tree level - the form of the effective action is determined completely by the requirement of supersymmetry for a given gauge group. Thus neither action can receive any quantum corrections.

The actions of Eq. 5.1 and Eq. 5.5 are equivalent under the identification:

$$\begin{aligned} \Phi^H &= -\Phi^I \\ G^H_{\mu\nu} &= G^I_{\mu\nu} \\ B^H_{\mu\nu} &= B^I_{\mu\nu} \\ A^{Ha}_{\mu} &= A^{I\,a}_{\mu} \end{aligned} \tag{5.8}$$

It was for this reason that it was initially proposed that the type I and the SO(32) heterotic string theories in ten dimensions are related.

Note the '$-$' sign in the relation between Φ^H and Φ^I in Eq.5.8. Since $e^{\langle\Phi\rangle/2}$ is the string coupling, we see that the strong coupling limit of one theory is related to the weak coupling limit of the other theory and vice versa. Also note that in the string frame the metrics in the Heterotic and Type I theory are related by,

$$G^H_{\mu\nu} = e^{-\phi} G^I_{\mu\nu} \tag{5.9}$$

Here we have dropped the subscript S used in Eq. 5.4. In the rest of the chapter we will use this metric.

Match between Heterotic F-string and Type I D-String

The Heterotic Fundamental string is a stable BPS state. Therefore it must be possible to identify a corresponding object in Type I that behaves as the Heterotic String. Moreover the mass per unit length or tension of such a object should exactly match the tension of the Heterotic String, since it is protected by the BPS property.

The tension of the fundamental heterotic string is given by,

$$\tau_H = \frac{1}{2\pi\alpha'_H} \tag{5.10}$$

Since the D1-brane of Type I has the same spacetime dimension as the heterotic String and is a stable BPS state, it is a prime candidate to be identified with Heterotic String. Calculating the tension of the D1-Brane, we find

$$\tau_{D1} = \frac{1}{2\pi\alpha'_I g_I} \tag{5.11}$$

Since the α' are dimensionful parameters with units of length squared we need to use Eq. (5.9) to set the same scale in both theories. Combined with the coupling constant mapping $g_I \to 1/g_H$, this shows that the two tensions are indeed equal.

5.2 General D-brane Amplitudes

In general, a Type I theory can be specified through its one-loop amplitude. This has four contributions, two from the closed-string sector (whose one-loop geometries have the topologies of a torus and Klein bottle respectively), and two from the open-string sector (with the topologies of a cylinder and Möbius strip). In each case, the contribution can be obtained by evaluating a trace over relevant string states and then integrating over all corresponding conformally inequivalent geometries. For the ten dimensional theory, these traces are defined as:

$$\begin{aligned} T(\tau) &\equiv \text{Tr } \tfrac{1}{2} \cdot (-1)^F \cdot \frac{1+(-1)^G}{2} \frac{1+(-1)^{\tilde{G}}}{2} \cdot e^{2\pi i\tau L_0}\, e^{-2\pi i\bar{\tau}\overline{L}_0} \\ K(t) &\equiv \text{Tr } \tfrac{\Omega}{2} \cdot (-1)^F \cdot \frac{1+(-1)^G}{2} \frac{1+(-1)^{\tilde{G}}}{2} \cdot e^{-2\pi t(L_0+\overline{L}_0)} \\ C(t) &\equiv \text{Tr } \tfrac{1}{2} \cdot (-1)^F \cdot \frac{1+(-1)^G}{2} \cdot e^{-2\pi t L_0} \\ M(t) &\equiv \text{Tr } \tfrac{\Omega}{2} \cdot (-1)^F \cdot \frac{1+(-1)^G}{2} \cdot e^{-2\pi t L_0} \ . \end{aligned} \tag{5.12}$$

Thus, with these normalizations, $T+K$ gives the trace over the closed-string states while $C+M$ gives the trace over the open-string states. In these traces, F is the spacetime fermion number and G is the worldsheet fermion number.

Evaluating these traces one gets the familiar amplitudes

$$\mathcal{K} = \frac{1}{2}(V_8 - S_8)\ , \quad \mathcal{A}_{99} = \frac{n^2}{2}(V_8 - S_8)\ , \quad \mathcal{M}_9 = -\frac{n}{2}(\hat{V}_8 - \hat{S}_8)\ , \tag{5.13}$$

where we have adopted the notation of paper [6] for the characters. We will use this notation in the rest of chapter. We have also omitted the bosonic contribution. We will continue to omit the bosonic contribution in D-brane amplitudes unless specified otherwise. Here n equals 32 on account of tadpole cancellation and gives rise to the gauge group $SO(32)$.

In addition to D9 branes the Type I SO(32) theory can also contain D1 and D5 branes. We can write down the Dp-Dp amplitude supplemented with amplitudes for the propagation of the bulk spectrum between the probe Dp and the background D9 and O9. The trace over states continues to be given by Eq. 5.12, but now the form of the amplitudes will be different.

While writing the D9-D9 amplitude the open strings had Neumann-Neumann (NN) boundary conditions in all directions. However for a open string ending on a Dp brane the open string will have Dirchlet boundary conditions in some space directions. To be able to write down the Dp-Dp amplitudes, we need to understand the mode expansions for a open string having NN, ND, DN and DD boundary conditions.

5.2.1 The p-p and p-p' System

In this section we consider two D-branes oriented parallel to each other. An open string can have both ends on the same D-brane or one on each. The $p - p$ and $p' - p'$ amplitudes behave the same way as the $9 - 9$ amplitude, but the $p - p'$ strings have a different behaviour. There are four possible sets of boundary conditions for each spatial coordinate X^i of the open string, namely NN (Neumann at both ends), DD, ND, and DN. T-duality can switch between NN and DD boundary conditions or ND and DN. But the sum of the number of ND and DN boundary conditions remains invariant.

The respective mode expansions are

$$\text{NN:} \qquad X^\mu(z,\bar z) = x^\mu - i\alpha' p^\mu \ln(z\bar z) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m^\mu}{m}(z^{-m} + \bar z^{-m}),$$

$$\text{DD:} \qquad X^\mu(z,\bar z) = -i\frac{\delta X^\mu}{2\pi} \ln(z/\bar z) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m^\mu}{m}(z^{-m} - \bar z^{-m})$$

$$\text{DN, ND:} \qquad X^\mu(z,\bar z) = i\sqrt{\frac{\alpha'}{2}} \sum_{r\in \mathbf{Z}+1/2} \frac{\alpha_r^\mu}{r}(z^{-r} \pm \bar z^{-r}), \tag{5.14}$$

z here is defined as $z = e^{[-i\sigma^1+\sigma^2]} = e^{[i(\tau-\sigma)]}$. Note that the DN and ND coordinates have half-integer moding. The worldsheet fermions will have the same moding in the Ramond sector and opposite in the Neveu-Schwarz sector.

So for example if we are writing the annulus amplitude for the Dp-D9 system, we will encounter $p-1$ DD boundary conditions and $9-p$ DN boundary conditions. From table (5.1) below we can read off the characters corresponding to a given projection. For example the sector $NS+$ corresponds to $V_{p-1}S_{9-p}$. Since the mode expansion for

	$p-1$		$9-p$	
	X	ψ	X	ψ
NS	P	A.P	A.P	P
R	P	P	A.P	A.P

Table 5.1. The mode expansions for NS and R sectors for a open string with $p-1$ DD boundary conditions and $9-p$ DN boundary conditions. P stands for periodic and A.P for anti-periodic. The specific periodicity for ψ follows from the fact that the supercurrent $\psi\partial X$ be periodic for the NS sector and anti-periodic for the R sector.

the X^μ in the DD sector is still integral - the characters for the Dp-Dp amplitude are unchanged from the D9-D9 amplitude. Writing out the annulus amplitude we get,

$$\mathcal{A}_{pp} = \frac{d^2}{2}\ (V_{p-1}O_{9-p} + O_{p-1}V_{9-p} - S_{p-1}S_{9-p} - C_{p-1}C_{9-p}) \tag{5.15}$$

here we have decomposed the 8-dimensional characters with respect to the $(p-1)$ light-cone directions longitudinal to the branes. In space-time language, $V_{p-1}O_{9-p}$ describes gauge bosons, $O_{p-1}V_{9-p}$ describes scalars and $S_{p-1}S_{9-p}$ and $C_{p-1}C_{9-p}$ describe space-time fermions.

The Dp-D9 amplitude on the other hand can be written in a general way as,

$$\begin{aligned}\mathcal{A}_{p9} = \frac{n \times d}{2} \Big[& \quad (O_{p-1}+V_{p-1})(S_{9-p}+C_{9-p}) \\ & - (S_{p-1}+C_{p-1})(O_{9-p}+V_{9-p}) \\ & + e^{-\frac{(9-p)i\pi}{4}}(O_{p-1}-V_{p-1})(S_{9-p}-C_{9-p}) \\ & + e^{-\frac{(p-1)i\pi}{4}}(S_{p-1}-C_{p-1})(O_{9-p}-V_{9-p})\Big] \end{aligned} \tag{5.16}$$

The D9-Dp amplitudes are thus inconsistent unless $p = 1, 5, 9$ - for only these dimensions as we know there exist BPS Dp-branes in the type I string.

The string zero point energy is 0 in the R sector, and

$$(8-\nu)\left(-\frac{1}{24}-\frac{1}{48}\right)+\nu\left(\frac{1}{48}+\frac{1}{24}\right)=-\frac{1}{2}+\frac{\nu}{8} \tag{5.17}$$

in the NS sector, where ν is the total number of ND + DN coordinates. To write down the Mobius amplitude, we need to understand the action of Ω on the open string.

Action of Ω

The operator Ω reverses the world sheet parity,

$$\sigma \to \pi - \sigma \tag{5.18}$$

Then, from the Neumann mode expansion we can immediately see that the action of the world-sheet parity on the oscillators of $X^\mu_{\rm N}$ is

$$\Omega\, \alpha^\mu_m\, \Omega^{-1} = {\rm e}^{{\rm i}m\pi}\, \alpha^\mu_m \tag{5.19}$$

The fermionic oscillators of the open sector transform as,

$$\Omega\, \psi^\mu_r\, \Omega^{-1} = {\rm e}^{{\rm i}r\pi}\, \psi^\mu_r \tag{5.20}$$

We also need to specify the action of Ω on the ground (ghost) states. This is given by

$$\Omega|0\rangle_{NS} = -\mathrm{i}\,|0\rangle_{NS} \tag{5.21}$$

and

$$\Omega|S^\alpha\rangle_R = -\,|S^\alpha\rangle_R \tag{5.22}$$

Since the mode expansion for the Dirichlet strings is different from the Neumann strings, the action of Ω is also different. Using Eq. (5.18) and the Dirichlet mode expansion we find that

$$\Omega\,\alpha_n^i\,\Omega^{-1} = -\mathrm{e}^{in\pi}\,\alpha_n^i \tag{5.23}$$

World-sheet supersymmetry dictates the action of Ω on the fermionic oscillators to be

$$\Omega\,\psi_r^i\,\Omega^{-1} = -\mathrm{e}^{\mathrm{i}r\pi}\,\psi_r^i \tag{5.24}$$

for both the NS and R sectors.

The general direct-channel Mobius amplitude $\mathcal{M}_p$ is then given by,

$$\begin{aligned}\mathcal{M}_p = -\frac{d}{2}\cos\frac{(p-5)\pi}{4}\Big\{ & \quad (\hat{O}_{p-1}\hat{V}_{9-p} - \hat{V}_{p-1}\hat{O}_{9-p}) \\ & - \quad (\hat{S}_{p-1}\hat{S}_{9-p} - \hat{C}_{p-1}\hat{C}_{9-p})\Big\}\ .\end{aligned} \tag{5.25}$$

Since in the possible cases, $p = 5$ and $p = 1$ the cosines are equal to ± 1, a stack of d D-branes will have $USp(d)$ and $SO(d)$ gauge groups respectively.

5.2.2 The p-p and p-p' system in nine dimensions

When we compactify on a circle we choose to wrap the D-branes on the circle. To write down the bosonic contribution to the partition function from the compactified direction for the Klein, Cylinder and Mobius contributions, we define:

$$P_m \;=\; \exp[-\pi m^2 a^2 \tau_2] \tag{5.26}$$

$$\mathcal{E} = \sum_{m\in \mathbb{Z}} P_m$$
$$\mathcal{O} = \sum_{m\in \mathbb{Z}} P_{m+\frac{1}{2}} \tag{5.27}$$

The Dp-Dp amplitude for a Dp brane wrapped on a circle will therefore be given by:

$$\mathcal{A}_{pp} = \frac{d^2}{2}\Big\{\big(V_{p-1}O_{9-p} + O_{p-1}V_{9-p} - S_{p-1}S_{9-p} - C_{p-1}C_{9-p}\big)\big(\mathcal{E}+\mathcal{O}\big)\Big\} \tag{5.28}$$

For writing the Dp-D9 cylinder amplitude and the Mobius amplitudes we evaluate the phase in equations above for $p = 1$, since that is the relevant case. We get for the cylinder,

$$\mathcal{A}_{p9} = \frac{n\ \times d}{2}\Big\{\big(V_{p-1}C_{9-p} + O_{p-1}S_{9-p} - C_{p-1}O_{9-p} - S_{p-1}V_{9-p}\big)\big(\mathcal{E}+\mathcal{O}\big)\Big\} \tag{5.29}$$

For the Mobius amplitude,

$$\mathcal{M}_p = -\frac{d}{2}\Big\{\big(\hat{V}_{p-1}\hat{O}_{9-p} - \hat{O}_{p-1}\hat{V}_{9-p} + \hat{S}_{p-1}\hat{S}_{9-p} - \hat{C}_{p-1}\hat{C}_{9-p}\big)\big(\mathcal{E}+\mathcal{O}\big)\Big\} \tag{5.30}$$

Let us identify the massless states for the D1-String from this partition function. The massless states from the A_{pp} amplitude are:

1. In the $O_{p-1}V_{9-p}$ sector, the lowest state is in the ground state for the longitudinal directions for the brane while there are 8 transverse excitations - the massless state is therefore a V_8 spacetime vector.

2. In the $C_{p-1}C_{9-p}$ sector, the massless states transform as a spacetime spinor S_8 in the transverse directions, *and* as a worldsheet spinor in the longitudinal directions. Note that the massless states from the $V_{p-1}O_{9-p}$ and $S_{p-1}S_{9-p}$ sectors are projected out under Ω as can be seen from the Mobius amplitude.

The massless states coming from the A_{p9} amplitude are solely from the Ramond sector. As can be seen from Eq. 5.17, the ground state energy of the NS sector is 1/2, therefore all states in the NS sector are massive. From the Ramond sector the lowest state coming from the $C_{p-1}O_{9-p}$ sector is a spacetime scalar and a worldsheet spinor - for $d = 1$ and $n = 32$ in Eq. 5.29, there are 32 worldsheet spinors.

5.3 Supersymmetric S-duality in nine dimensions

5.3.1 D-string construction

In this section we show that the open string structure of a D1-brane of the Type I theory exactly reproduces the worldsheet structure of the heterotic theory. This construction first appeared in paper [76]. Here we follow the analysis of the review paper [73]. Since in Sect. 5.1.2 we showed that the NS rank two anti-symmetric tensor field,$B^H_{\mu\nu}$, of the heterotic theory is mapped to the Ramond-Ramond anti-symmetric tensor field $B^I_{\mu\nu}$ of the Type I theory, it implies that the winding charge of the heterotic string is mapped to the Ramond-Ramond charge in Type I. Now D-branes are carriers of RR charges, therefore the states of a wrapped heterotic string should match the states of a single wrapped D1-brane.

We explain a bit what a D1-brane and its fields look like. For a D1 brane, spacetime can be divided into a 2-dimensional Minkowski part along the worldsheet of the D1 brane, coordinates given by x^0 and x^9, and an 8-dimensional Euclidean part perpendicular to it (coordinates $x^1, \ldots, x^8$). As we saw in Chapter 1 such an object breaks ten-dimensional Lorentz invariance. The ten-dimensional spacetime symmetry group $SO(1,9)$ is broken into $SO(1,1) \times SO(8)$. It is natural to identify the worldsheet of the D1-brane with the world-sheet of the heterotic string. While the 8-dimensional space perpendicular to the brane can be taken to be the light-cone gauge coordinates in which the heterotic string is embedded.

We now check what massless open strings live on the D1-brane worldsheet. First we concentrate on the D1-D1 strings. As we saw in the previous section in the NS sector these are given by

$$\psi^{\mu}_{-1/2}|k\rangle_{NS} \tag{5.31}$$

with $k^2 = 0$. Since the momentum k is confined to just the longitudinal directions, the corresponding worldsheet field identified with this state will depend only on x^0 and x^9.

As we saw previously under the action of Ω, the states in Eq. (5.31) in directions 0 and 9 are odd and are therefore projected out. The states in the direction $1-8$ are even and survive. The associated fields, $\phi^i(x^0, x^9)$ with $i = 1, \ldots, 8$, are scalars with respect to the world-sheet group $SO(1,1)$ but are vectors with respect to the transverse group $SO(8)$. These is exactly the same behaviour as that of the embedding coordinates $X^i(\tau, \sigma)$ of the heterotic string, which therefore can be identified with $\phi^i(x^0, x^9)$. Naturally, x^0 and x^9 map to τ and σ respectively. Since the worldsheet coordinate σ of the heterotic theory goes from 0 and π, it means that the D1 brane should also be wrapped on a circle.

In the R sector the massless open strings are given by,

$$|S^\alpha; k\rangle_R \tag{5.32}$$

with $k^2 = 0$. Here the S^α are the spin fields. As we saw from Eq. (5.28), these states decompose into a chiral worldsheet spinor, χ, and a spacetime spinor. The chiral spinor obeys the massless Dirac equation

$$\not{k}\chi = \left(k_0\Gamma^0 + k_9\Gamma^9\right)\chi = 0 \tag{5.33}$$

Again as we saw in the previous section, the total decomposition of the open string spinor states of $SO(1,9)$ is as $(+\frac{1}{2}, \mathbf{8_s}) \oplus (-\mathbf{\frac{1}{2}}, \mathbf{8_c})$. Only the second term survives the orientifold projection. Furthermore χ satisfies the condition,

$$\Gamma^0\Gamma^9\chi = -\chi \tag{5.34}$$

Combining this with the Dirac equation (5.33) tells us that $k_0 = -k_9$. So in fact χ is a right-moving worldsheet spinor transforming in the $\mathbf{8_c}$ of $SO(8)$. This is exactly how the fermionic coordinates of the heterotic string transform in the Green-Schwarz formulation.

The heterotic string also has 32 left-moving worldsheet spinors. To identify these fields on the D1-brane we now turn to the D1-D9 sector. As we saw in the previous section the NS sector of these open strings does not contain any massless excitation.

However in the Ramond sector we have massless states. From Eq. (5.29) we see that we can associate a field Λ with this massless state which transforms a spinor of $SO(1,1)$ and a scalar of $SO(8)$. Again such a spinor is chiral,

$$\Gamma^0\Gamma^9\Lambda = \Lambda \tag{5.35}$$

Combining this condition with the Dirac equation

$$\not{k}\Lambda = \left(k_0\Gamma^0 + k_9\Gamma^1\right)\Lambda = 0 \tag{5.36}$$

we find that $k_0 = k_9$. Λ is therefore a left-moving world-sheet spinor. Since x^9 is compact, the momentum k_9 in Eq. (5.32) is an integer n in units of $1/R$.

Any such open string will carry Chan-Paton factors. These are denoted by λ^I_{19} $(I = 1, \ldots, 32)$ for the 1-9 sector, and by λ^I_{91} $(I = 1, \ldots, 32)$ for the 9-1 sector. Since Type I strings have to be invariant under orientation reversal we will need a combination of the 1-9 and 9-1 sectors. The operator Ω reverses the string orientation. It converts a 1-9 string into a 9-1 string as,

$$\Omega \ : \ \ \lambda^I_{19} \ \leftrightarrow \ \lambda^I_{91} \tag{5.37}$$

Hence the open string ground state selected by the Ω projection is given by,

$$\lambda^I_{19} \, |S^+; n\rangle_R + \lambda^I_{91} \, |S^+; n\rangle_R \tag{5.38}$$

The index I denotes a worldsheet spinor transforming in the vector representation of $SO(32)$. By summing over all possible values for the momentum variable, n, we can construct the following mode expansion for this field,

$$\Lambda^I(\tau, \sigma) = \sum_n \Lambda^I_n \, \mathrm{e}^{\mathrm{i}2n(\tau+\sigma)} \tag{5.39}$$

This agrees with the expansion of the 32 heterotic worldsheet spinors with periodic boundary conditions. Upon quantization, these periodic fermions give rise to the spinorial representations of $SO(32)$. As we can see using only the zero-modes Λ^I_0, we can construct the two representations $\mathbf{2^{15}}$ and $\mathbf{2^{15'}}$ of $SO(32)$. This is a massive state for the heterotic string and we still need to construct the massless spectra. As we know from the heterotic theory, the massless states come from the quantization of the 32 worldsheet spinors with anti-periodic boundary conditions.

Where will this condition occur on the Type I side? To construct these sectors it is necessary to understand the exact fate of gauge fields under orientifolding.

We saw that the orientifold projection removes the local (longitudinal) gauge field from the D1-D1 amplitude. However it turns out that a global gauge field $A = \frac{\theta}{2\pi R}$ is perfectly compatible with Ω. As we saw in the introduction such a constant gauge field A can always be set to zero locally by a gauge transformation. But the presence of such a gauge field affects the global properties of objects charged under the gauge field. A field Φ which carries a charge q under A picks up a non-trivial phase given by

$$W = \exp\left\{-\mathrm{i}q \oint dx^9 A\right\} = \mathrm{e}^{-\mathrm{i}q\theta} \ . \tag{5.40}$$

This translates into its momentum k^9 shifting by,

$$k^9 \to k^9 + \frac{q\theta}{2\pi R} \tag{5.41}$$

Let us now determine the open strings that will be charged under this field. The field A is confined to the D1 brane. Therefore any open string with at most one end-point on the D1 brane will be charged under A. The open strings in the sectors 1-9 and 9-1 satisfy this criteria. We can take the charge to be $q = 1$ for the 1-9 strings and to be $q = -1$ for the 9-1 strings. The Ramond states in this sector always appear as,

$$\lambda^I_{19}\,|S^+; n\rangle_R \pm \lambda^I_{91}\,|S^+; m\rangle_R \ \ , \tag{5.42}$$

When the gauge field was absent only the combinations with $n = m$ were kept as in Eq. (5.38). In the presence of the gauge field A, the momenta of the charged states has to be shifted according to Eq. (5.41). Therefore the superposition becomes now,

$$\lambda^I_{19}\,|S^+; n + \theta/2\pi\rangle_R \pm \lambda^I_{91}\,|S^+; m - \theta/2\pi\rangle_R \tag{5.43}$$

n and m are integers, therefore it is apparent that this superposition can be an eigenstate of Ω only if $\theta = 0$ or π.

This information can be interpreted in the following way. On the D1 brane of Type I there exists a Z_2 symmetry which is a remnant of the $U(1)$ gauge symmetry of the D1 brane of Type IIB after performing the orientifold projection. Now if we choose $\theta = \pi$, we get the following mode expansion for the worldsheet spinor Λ^I,

$$\Lambda^I(\tau, \sigma) = \sum_n \Lambda_n^I \, \mathrm{e}^{\mathrm{i}(2n+1)(\tau+\sigma)} \tag{5.44}$$

This matches with the expansion of the 32 worldsheet heterotic spinors with antiperiodic boundary conditions. Upon quantization, these spinors give rise to the integral representations of $SO(32)$ and among the lowest lying of these is the massless spectra of the heterotic string.

We are still not done. With this construction we somehow still have to get rid of the extra states we have that are not present in the heterotic spectrum. We have to remove the representations $\mathbf{2^{15\prime}}$ and **32** which can be constructed using the Λ^I's but which are not present in the heterotic theory at the massless level. For this it is postulated that only the states invariant under the Z_2 gauge symmetry introduced earlier be kept. Since the 1-1 and 9-9 states have both ends on the D1-brane, they just pick up a factor of $+1$ under the Z_2 symmetry and are all invariant. However things are different for the 1-9 and 9-1 sectors. For example we can define the spinor representation $\mathbf{2^{15}}$ to be invariant under symmetry. However this then means that the states in the other spinor representation $\mathbf{2^{15\prime}}$ are not invariant, since they contain one more spinor zero-mode that brings a Chan-Paton factor λ_{19} or λ_{91}, and hence an extra $-$ sign. Therefore, this state $\mathbf{2^{15\prime}}$ is removed from the spectrum and only one $SO(32)$ chiral spinor is kept just as for the heterotic string. The fields with the anti-periodic mode expansion (that we got by turning on a Wilson line) can also be thought of as a twisted sector of the Z_2 gauge symmetry. Again in this sector we have to keep only the states that are invariant - this removes the**32** representation, while the **1** and **496** are kept, again reproducing the heterotic massless spectrum.

In the next section we look at the exact multiplicities of the heterotic BPS spectrum and show that it is reproduced on the Type I side.

5.3.2 Exact match of states

BPS states in $D = 9$ heterotic string theory

The concept of BPS states is very important in proving S-duality. Massive BPS states appear in theories with supersymmetries greater than one. The BPS states satisfy the BPS bound which for point-like states is given by,

$$M = \text{ maximal eigenvalue of } \quad Z \tag{5.45}$$

where Z is the central charge matrix.

Because of this relation, BPS states behave in a special way. They are absolutely stable at any point in moduli space. since their mass depends on conserved charges, charge and energy conservation prohibit their decay. The equation for mass for BPS states is exact if one uses renormalized values for the charges and moduli as the coupling is increased.

Let us look for the BPS states of the $SO(32)$ heterotic string theory compactified on a circle with radius R_H and background gauge field $\vec{A}$. The winding modes are denoted by n_H and the KK momentum modes by m_H. $\vec{\lambda}$ denotes the the weight vector - an element of the $Spin(32)/\mathbb{Z}_2$ lattice.

The left-moving and right-moving momenta are

$$\begin{aligned} \vec{p}_L &= \sqrt{\frac{1}{2\alpha'_H}}\left\{\vec{\lambda} + n_H\vec{A},\ (m_H - \vec{A}\cdot\vec{\lambda} - n_H\frac{\vec{A}\cdot\vec{A}}{2})a_H + \frac{n_H}{a_H}\right\} \\ \vec{p}_R &= \sqrt{\frac{1}{2\alpha'_H}}\left\{(m_H - \vec{A}\cdot\vec{\lambda} - n_H\frac{\vec{A}\cdot\vec{A}}{2})a_H - \frac{n_H}{a_H}\right\} \end{aligned} \tag{5.46}$$

The mass spectrum is given by

$$M_H^2 = \frac{1}{2}\vec{p}_L \cdot \vec{p}_L + \frac{2}{\alpha'_H}(N_L - 1) + \frac{1}{2}\vec{p}_R \cdot \vec{p}_R + \frac{2}{\alpha'_H}(N_R - \delta_R)\,, \tag{5.47}$$

where N_R, N_L are the right and left-moving oscillator numbers, and $\delta_R = 0$ or $1/2$ depending on whether the right-moving fermions are periodic (R) or anti-periodic (NS). Physical states satisfy the level matching condition

$$\vec{p}_L \cdot \vec{p}_L + \frac{4}{\alpha'_H}(N_L - 1) = \vec{p}_R \cdot \vec{p}_R + \frac{4}{\alpha'_H}(N_R - \delta_R) \tag{5.48}$$

The total momentum in the compactified direction,p_1, is given by the sum of the left moving and right moving momentum,

$$p_1 = \sqrt{\frac{1}{2\alpha'_H}}\left\{\big(m_H - \vec{A}\cdot\vec{\lambda} - n_H\frac{\vec{A}\cdot\vec{A}}{2}\big)a_H\right\} \tag{5.49}$$

BPS states are given by the condition that $N_R = \delta_R$. The condition for states to be physical and BPS-saturated is therefore

$$\vec{p}_L \cdot \vec{p}_L - \vec{p}_R \cdot \vec{p}_R = \frac{4}{\alpha'_H}(1 - N_L) \tag{5.50}$$

which can be simplified to

$$m_H n_H = 1 - N_L - \frac{1}{2}\vec{\lambda}\cdot\vec{\lambda} \tag{5.51}$$

In this case, the mass formula becomes

$$M_{H,BPS} = \sqrt{\frac{1}{2\alpha'_H}}\left|\,\big(m_H - \vec{A}\cdot\vec{\lambda} - n_H\frac{\vec{A}\cdot\vec{A}}{2}\big)a_H - \frac{n_H}{a_H}\right|\,. \tag{5.52}$$

Mapping between Type I states and Heterotic theory

The Type I theory is related to the heterotic by a strong-weak duality. As we saw in Sect. 1, the coupling constants and metrics are related by

$$g_I = \frac{1}{g_H}, \quad G^I_{MN} = \frac{G^H_{MN}}{g_H}. \tag{5.53}$$

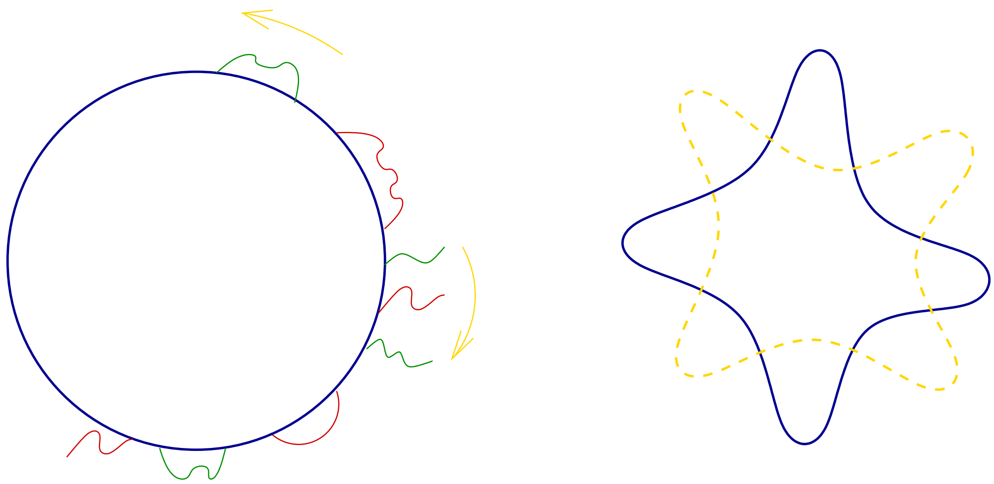

FIGURE 5.1. To prove S-duality, one attempts to show that the D-String of Type I theory (left) with a single unit of RR Charge leads to the same spectrum and degeneracies as the fundamental string of the Heterotic theory (right) with winding number 1. The D-String comes with a gas of open strings attached. Massive open string excitations decay at strong coupling, therefore only massless excitations are relevant. These massless modes are either left moving or right moving. In fact all fermionic open string excitations with both ends on the D-String are left-moving while all fermionic open string excitations with a single end on the D-String are right-moving - exactly as the oscillator modes of a single Heterotic string. So there is a one-to-one map between a oscillator excitation mode of the Heterotic string and a open string state on the D-String carrying a specific momentum. The sum of the momenta of all open strings present on the D-String is the net momentum of the D-String corresponding to the Heterotic String center of mass momentum.

The Neveu-Schwarz two-form, B_{MN}, of the heterotic string is mapped to the Ramond-Ramond antisymmetric tensor field, B^R_{MN}, of the Type I string. The states carrying n_H units of $B_{\mu 9}$ charge are the states of the fundamental heterotic string wrapped n_H times around the X^1 circle. Consequently, the winding modes of the fundamental string on the heterotic side are mapped to bound states of n_H D-strings on the Type I side. Thus, the duality between Type I string theory and the $SO(32)$ heterotic string theory requires the existence of Type I D-string bound states carrying n_H units of RR charge. In particular a single D-String ought to be able to reproduce the BPS states of a wrapped heterotic string with winding number 1.

Momentum modes of the Heterotic F-string are mapped to the total momentum of the Type I D-String.

The duality relations imply that a heterotic state labelled by n_H, m_H and $\vec{\lambda}$ is mapped to a type I state with RR charge n_I, net momentum m_I and mass $M_{I,BPS}$, given by:

$$n_I = n_H \tag{5.54}$$

$$m_I = m_H - \vec{A}\cdot\vec{\lambda} + n_H \vec{A}\cdot\vec{A}/2 \tag{5.55}$$

$$M_{I,BPS} = \sqrt{\frac{1}{2\alpha'_H}}\left|m_I a_I - \frac{n_I}{a_I}\right| \tag{5.56}$$

where a_I is the inverse type I radius along the X^9 direction.

We now match these masses and multiplicities between Type I D-String and the Heterotic F-String. We have adapted the analysis of paper [77] for the Type IIB string (which is self-dual) to the Heterotic-Type I case. Furthermore the papers [82, 83, 81], look for a match between the BPS states of the heterotic string and the BPS states of Type I and Type I'(which is T-dual to the Type I theory).

First we look at ground states of the heterotic F-string with $n_H = -1$ and the momentum, $m_H = 0$. This state has a degeneracy equal to $(\mathbf{8_v}+\mathbf{8_s})\times\{(\mathbf{8_v},\mathbf{1})+(\mathbf{1},\mathbf{496})\}$, where the 16 states in the right sector come from the 8 bosonic ground states in the NS sector and the 8 fermionic ground states in the R sector. Thus there are a total of 4032 bosonic and 4032 fermionic states of the heterotic F-string.

As we saw in the previous section for the D-string, it has 8 scalars corresponding to the transverse deformations of the D-string and 8 right-moving majorana-weyl spinors in the longitudinal directions coming from the 1-1 sector. The majorana-weyl spinors lead to a total of 16 right moving states. There are a total of 32 left moving spinors coming from the 1-9 strings. After gauging the discrete Z_2 symmetry for the 1-9 strings this gives us the same spectrum and the same degeneracies as the fundamental heterotic string.

In the next section we match the BPS states for the same winding number, $n_H = -1$.

After that we attempt to get a match for *non*-BPS states on the Type I and Heterotic side.

BPS states with $n_H = -1$

There is an infinite set of BPS states for the fundamental Heterotic string. The exact mass of these states is given by Eq. (5.52). For this section we restrict the winding number to $n_H = -1$. For simplification we will also choose the background gauge field $\vec{A} = \vec{0}$. Then equations Eq. (5.56) simplify to $m_H = m_I = m$ and $n_H = n_I = n$. One therefore has, from Eq. Eq. (5.51), $N_L = 1 + m - \frac{1}{2}\vec{\lambda} \cdot \vec{\lambda}$ $(m > 0)$. The mass becomes

$$M = 2\pi R_H T_H + \frac{m}{R_H} \tag{5.57}$$

where we have defined the Heterotic String tension as, $T_H = \frac{1}{2\pi\alpha'_H}$. The degeneracy of such states is given by the number of ways one can decompose m into levels of the 8 bosonic and 32 fermionic oscillators of the left moving side.

We identify the corresponding states of the D-string as those that have massless open strings coming from the D1-D1 and D1-D9 sectors. The total momentum of the D-String is given by,

$$p_1 = \frac{m}{R_I}, \quad m_{open} > 0, \quad \sum m_{open} = m \tag{5.58}$$

The mass of the D-String state will be the sum of the mass of the D-String, which is simply the the tension of the D-String multiplied by the length, and the mass contributed by the KK modes of the open string. Therefore we get,

$$M = 2\pi R_I T_I + \frac{m}{R_I}, \tag{5.59}$$

which agrees with the heterotic mass under the appropriate mapping. The degeneracy of such D-string states is given by the number of ways one can decompose the integer m into the positive integers m_{open}'s of the individual open string states. Note that there are 8 bosonic and 32 fermionic single string states for a given momentum. Therefore the multiplicities of states on the heterotic and Type I side ought to be equal.

Since $m = N_L - \delta_L = N_L - 1$, it might appear that there is a difference with the F-string case since for the F-String we are partitioning only N_L into 8 bosonic and 32 fermionic states. However the two ways of partitioning really are equal because for the D-String we are adding the total momentum m states over the $m = 0$ ground state discussed in the previous section. So in reality we are partitioning not just $N_L - \delta_L$, but $N_L - \delta_L + \delta_L$, where the last δ_L contribution comes from the D-String ground state. So we get the same combinatorics as for the F-String, where the oscillator levels on the left sector were positive integers that had to total upto m to generate all the BPS states.

Non-BPS states with $n_H = -1$

We now consider non-BPS states specified by some momentum $p_1 = m/R$ ($m > 0$) and $n = -1$. For F-string states it follows from previous section that the left and right oscillator levels are given by,

$$N_R = \delta_R + N, \qquad N_L + 1/2\vec{\lambda}\cdot\vec{\lambda} = 1 + N + m, \quad (N = \text{integer}) \tag{5.60}$$

The mass for such a state M_N, in the free theory, is given by

$$M_N = 2\pi R_H T_H + \frac{m}{R_H}\sqrt{\left(2\pi R_H T_H + \frac{|m|}{R_H}\right)^2 + 8\pi T_H N} \tag{5.61}$$

For large $R_H\sqrt{T}_H$ one has

$$M_N \;\approx\; 2\pi R_H T_H + \frac{|m|}{R_H} + \frac{2\pi n}{R_H} \tag{5.62}$$

where we have neglected higher order terms starting at $O(\frac{1}{T_H {R_H}^3})$. To construct the corresponding non-BPS states for the D-string we pick D-String states with open strings having both signs of $p_1 = m_{open}/R_H$, with the total momentum still being given by $\sum m_{open} = m$, but with $\sum |m_{open}| = |m| + 2N$. The mass of such a state is given by

$$M_N \;\approx\; 2\pi R_I T_I + \frac{|m|}{R_I} + \frac{2\pi N}{R_I} \tag{5.63}$$

The approximation in the above equation, comes from the fact that oppositely moving open strings on the D-String can interact and generate a change in the energy of the D-string state. However a quick calculation shows that this shift is the order of $(T_I {R_H}^3)^{-1}$. This is of the same order as the term we dropped in Eq. (5.62).

Finally we can count the non-BPS excitations of the D-string, at large coupling g_I, and see if they agree with the non-BPS excitations of the Heterotic string, at small coupling. As mentioned above, the masses of non-BPS states on the D-string will get corrections due to open string scattering to open strings. Also it is possible that two open strings travelling in opposite directions interact and decay into a closed string. To neglect the effect of these terms, we limit ourselves to the following regime. First we take the coupling to be very large, so that only the massless states of open strings on the D1-brane survive. We also take the radius, R_I to be large. In this limits it becomes hard for the open strings to find each other and interact, so that both the energy corrections to the non-BPS states and the rate of decay of these states go to zero. In this situation we can count the non-BPS states with some confidence. But since there is a one-to-one correspondence between open strings with positive and negative m_{open} on the one hand and the oscillator mode numbers in the right and left sectors of the Heterotic string on the other - it follows that the multiplicities are again identical.

5.4 Non-supersymmetric theories

5.4.1 Motivation for S-duality for Non-susy theories

The vast majority of literature on S-duality in String theory deals with supersymmetric theories. There are two reasons for this. The first is that stable non-supersymmetric vacua in String theory are notoriously hard to find. The second is that S-duality is notoriously hard to prove for non-supersymmetric theories.

Let us expand on the first reason. All supersymmetric theories are alike, but each non-supersymmetric theory breaks supersymmetry in its own messy way. Some non-supersymmetric theories come with a tachyon which implies unstability. Even for those that are tachyon-free, the cosmological constant is generically never zero. A non-zero cosmological constant generates a runaway potential for the dilaton field. Since the coupling constant g is related to the VEV of the dilaton field - such a runaway potential implies that the non-supersymmetric theory is stable only at zero coupling *or* at very strong coupling.

Even if the non-supersymmetric theory is 'almost' stable at both strong and weak coupling, it is much harder to prove that it is s-dual to another theory. Supersymmetry breaking implies that we can no longer rely on BPS states to prove S-duality. As we saw in Section 5.3.2, such states were crucial in providing a clear and direct evidence for S-duality.

However as we shall see there are still some 'almost' stable non-supersymmetric cases, for which proving S-duality is not an impossible proposition.

Our motivation for studying S-duality for non-supersymmetric theories is ostensibly because we are interested in finding whether S-duality holds at finite temperature for the supersymmetric Type I and Heterotic theories (the real reason is that this helps us develop tools for sduality for non-supersymmetric theories). As we saw previously, finite temperature effects break supersymmetry. Understanding the strong coupling regime of a finite temperature string theory is useful for calculations in black hole physics and gives String theory the much needed ability to predict something. It is therefore of practical value to understand whether and how S-duals exist at finite temperature.

Luckily, string theory at finite temperature turns out to be perfect as a candidate for proving duality in non-supersymmetric theories. Temperature can be increased continuously, so we begin with a well understood supersymmetric theory (and its S-dual pair) and then increase the temperature *slowly* (adiabatically). This gives us access to a regime which is non-supersymmetric, but where we have not strayed too far from supersymmetry and can still rely on its protective effect. For example when we start with a supersymmetric theory we start with a zero cosmological constant or free energy (in the temperature context). As we increase the temperature, the free energy stays close to zero for a substantial period and therefore the dilaton potential generated is almost vanishing, guaranteeing that the non-supersymmetric theory exists at all values of the coupling constant. This strategy of slowly moving away from the supersymmetric domain to study s-duality for non-supersymmetric theories has also been applied in papers ([19, 78, 80]).

The other reason why string theory at finite temperature is tractable is that it is easily understood as a geometric theory. As we have seen a finite T string theory can be interpreted as a string theory on a circle with $R = 1/2\pi T$ and periodic boundary conditions for spacetime bosons and anti-periodic boundary conditions for spacetime fermions.

Finally, although we no longer have BPS states, there will still exist states that are charged under some gauge field and are stable because they are the lightest states that carry that particular charge - such states are non-BPS but stable, same as in the supersymmetric case. We can make use of these states and check their existence and degeneracies in proposed dual pairs.

In this section we will aim to establish that finite T Heterotic and Type I theories are indeed S-dual upto a specific critical temperature.

5.4.2 The theories in question

In Sect. 5.3 we dealt with a supersymmetric theory on a circle. In addition to compactifying on a circle we now also project by the orbifold element,

$$\mathcal{Y} = \frac{1 + (-1)^F \mathcal{T}}{2} \tag{5.64}$$

where F is the spacetime fermion number and $\mathcal{T}$ is the shift,

$$\mathcal{T} : \quad X_1 \;\rightarrow\; X_1 + \pi R \; . \tag{5.65}$$

Here X_1 represents the coordinate of the compactified direction, and R is the radius of compactification. Note that $R = \frac{1}{2\pi T}$ defines the thermal theory. In addition to be completely general, we will switch on a background wilson line in both the Type I and the Heterotic theories. Then the heterotic theory at finite T is given by the following partition function,

$$\begin{aligned}
Z_{\rm H}(\tau, T) &= Z^{(8)}_{\rm boson} \times \Bigg\{ \\
& \left[\bar{V}_8 \left(I_{16}^2 + \frac{1}{\ell} V_{16}^2 + \frac{\ell-1}{\ell} S_{16}^2 \right) - \bar{S}_8 \left(C_{16}^2 + \frac{1}{\ell} S_{16}^2 + \frac{\ell-1}{\ell} V_{16}^2 \right) \right] \mathcal{E}_0 \\
+ & \left[\bar{V}_8 \left(C_{16}^2 + \frac{1}{\ell} S_{16}^2 + \frac{\ell-1}{\ell} V_{16}^2 \right) - \bar{S}_8 \left(I_{16}^2 + \frac{1}{\ell} V_{16}^2 + \frac{\ell-1}{\ell} S_{16}^2 \right) \right] \mathcal{E}_{1/2} \\
+ & \left[\bar{I}_8 \left(\frac{1}{\ell} I_{16} V_{16} + \frac{1}{\ell} V_{16} I_{16} + \frac{\ell-1}{\ell} V_{16} C_{16} + \frac{\ell-1}{\ell} C_{16} V_{16} \right) \right. \\
- & \left. \bar{C}_8 \left(\frac{1}{\ell} S_{16} C_{16} + \frac{1}{\ell} C_{16} S_{16} + \frac{\ell-1}{\ell} V_{16}^2 + \frac{\ell-1}{\ell} C_{16}^2 \right) \right] \mathcal{O}_0 \\
+ & \left[\bar{I}_8 \left(\frac{1}{\ell} S_{16} C_{16} + \frac{1}{\ell} C_{16} S_{16} + \frac{\ell-1}{\ell} V_{16}^2 + \frac{\ell-1}{\ell} C_{16}^2 \right) \right. \\
- & \left. \bar{C}_8 \left(\frac{1}{\ell} I_{16} V_{16} + \frac{1}{\ell} V_{16} I_{16} + \frac{\ell-1}{\ell} V_{16} C_{16} + \frac{\ell-1}{\ell} C_{16} V_{16} \right) \right] \mathcal{O}_{1/2} \Bigg\} . \qquad (5.66)
\end{aligned}$$

The values that ℓ can take are confined to $\{1, 4, 16, \infty\}$ corresponding to different wilson lines. The Type I theory partition is the sum of the torus, the klein, the cylinder and

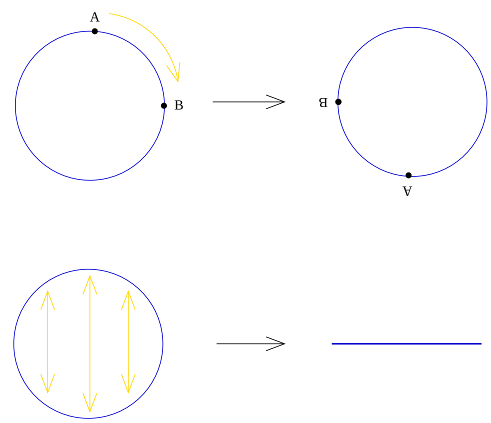

FIGURE 5.2. The figure on top shows a freely acting orbifold. The figure on the bottom shows a non-freely acting orbifold which takes a circle to a line segment.

the Mobius partitions. These are given respectively by,

$$\begin{aligned}
Z_{\rm T}(\tau,T) &= \frac{1}{2}\, Z^{(8)}_{\rm closed} \times \Big\{ \quad \mathcal{E}_0 \quad [\bar{V}_8 V_8 + \bar{S}_8 S_8] \\
&\qquad\qquad\qquad - \mathcal{E}_{1/2}\; [\bar{V}_8 S_8 + \bar{S}_8 V_8] \\
&\qquad\qquad\qquad + \mathcal{O}_0 \quad [\bar{I}_8 I_8 + \bar{C}_8 C_8] \\
&\qquad\qquad\qquad - \mathcal{O}_{1/2}\, [\bar{I}_8 C_8 + \bar{C}_8 I_8] \quad \Big\} \\
Z_{\rm K}(\tau_2,T) &= \frac{1}{2}\, Z^{(8)}_{\rm open} \times \Big\{ \quad \mathcal{E} \quad [V_8 - S_8] \quad \Big\} \\
Z_{\rm C}(\tau_2,T) &= \frac{1}{2}\, Z^{(8)}_{\rm open} \times \Big\{ \quad \mathcal{E} \quad [({n_1}^2 + {n_2}^2) V_8 - 2 n_1 n_2 S_8] \\
&\qquad\qquad\qquad + \mathcal{O} \quad [2 n_1 n_2 V_8 - ({n_1}^2 + {n_2}^2) S_8] \Big\} \\
Z_{\rm M}(\tau_2,T) &= -\frac{1}{2}\, \hat{Z}^{(8)}_{\rm open} \times \Big\{ \quad (n_1 + n_2)\, [\mathcal{E}\, \hat{V}_8 - \mathcal{O}\, \hat{S}_8] \Big\}
\end{aligned} \tag{5.67}$$

5.4.3 The Adiabatic argument

The Adiabatic argument is the statement that a freely acting orbifold that slowly deforms the supersymmetric theories that are dual pairs will lead to theories that are also dual pairs.

We began with the supersymmetric Type I and Heterotic theories in ten dimensions and compactify on a circle. Then we orbifold/project the Type I theory by $\mathcal{Y}$. This is a freely acting orbifold action since there are no fixed points in the resulting geometry. (The action of the orbifold is shown in Fig. 5.2)

Now we consider a very large R, which is equivalent to a very small T. At large R, Type I modded out by $\mathcal{Y}$ will be indistinguishable from Type I in ten dimensions, locally (for a low energy observer). So locally one can use S-duality to convert this theory to the ten-dimensional heterotic theory.

If the two theories are equivalent locally, the low energy observer can expect that the equivalence will remain valid globally as one goes around the thermal circle. Since there are no fixed points, the theory will not suffer a sudden change at any point. This does not tell us what the specific Heterotic theory is, to which out thermal Type I is dual to, but to find that all we need to do is to find the corresponding orbifold for the Heterotic theory.

Dual pairs at large radius, will remain dual pairs at smaller radius (and correspondingly higher temperatures), if we change the radius adiabatically.

5.4.4 Mapping the thermal orbifold

In the previous section we saw that to correctly identify the dual theory for thermal Type I, we need to find what the thermal orbifold $\mathcal{Y}$ maps to in the heterotic theory. Since the $\mathcal{T}$ part of $\mathcal{Y}$ is just a geometrical shift that will stay the same in the heterotic theory specifically we need to find what the orbifold $(-1)^F$ maps to in the heterotic theory.

At first sight it seems that it should simply be the same $(-1)^F$ orbifold for the Heterotic. However there is a subtlety here. The perturbative Type I theory contains only adjoint states in the $SO(32)$ gauge group. Therefore we only know the action of the $(-1)^F$ orbifold on this particular set of states. All we know is that under our particular orbifold action the spacetime vectors carrying the adjoint gauge charges get a '+' sign, while the spacetime spinors carrying the same gauge charges get a '$-$' sign. We can pick an orbifold in the Heterotic theory that has the same action for these particular set of states. However the heterotic theory also contains states transforming in the spinor representation of $SO(32)$ for example, and Type I theory tells us nothing about the way our orbifold should behave for these states. In particular the two orbifold choices $(-1)^F$ and $(-1)^{F+F_R}$, where F_R is the gauge group fermion number, have the same action for Heterotic states in the I_{32} sector (which contains the adjoint states) and only differ in their actions in the S_{32} sector - and they are both compatible with the Type I action. One can also see this from a Wilson line point of view .

Therefore we need some other way of determining which one of the two heterotic orbifolds should be selected as the correct map of the Type I thermal orbifold.

5.4.5 Matching of low energy massless fields

One of the ways to see if two theories can be dual pairs is to look at their massless fields. For supersymmetric theories the massless fields should necessarily be the same if the theories are dual pairs. For non-supersymmetric theories this is not a stringent requirement since even for a massless state the mass can get renormalized as the coupling changes and the same state can be potentially massive in the dual theory. However looking at the massless fields in the heterotic theory should still give us a clue as to which is the better orbifold out of the two we talked about in the previous section for being the map of the Type I thermal orbifold. However it turns out that both heterotic theories have the same massless fields and furthermore the thermal $SO(32)$ theory has the exact same massless fields. Therefore perturbative string theory gives us no clue as to which theory is the correct one. Both theories seemingly are equally good candidates.

We now turn to the non-perturbative sector of Type I.

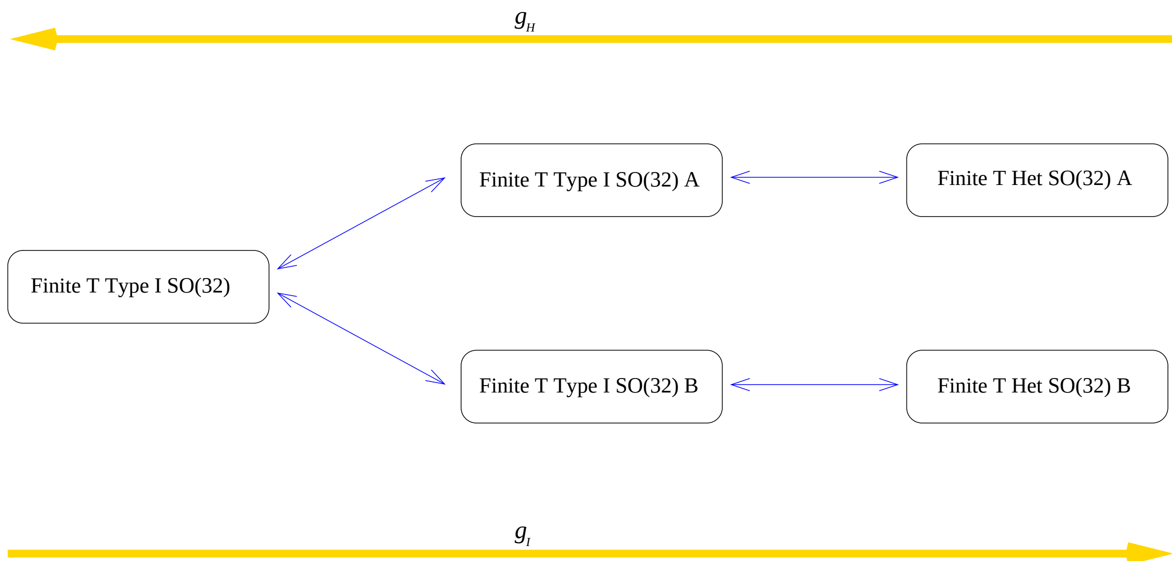

FIGURE 5.3. Relation between Finite temperature Type I and Heterotic theories.

5.4.6 Non-perturbative states of thermal Type I

Let us review the non-perturbative D-branes that are present in Type I. The stable BPS D-branes present in Type I are the D1,D5,D9-branes. In addition a non-BPS bound state of a D1 and an anti-D1 brane which acts like a D0-brane or D-particle is also present in the theory. For our present purposes the relevant objects are,

- Type I has a non-perturbative object, the D-String whose states match that of the fundamental heterotic string. In particular there is a state that has the quantum numbers of a $S0(32)$ spinor state with winding number one. The mass of this D-String as we saw earlier is given by $R_I/\alpha' g_I$
- Type I also has a non-perturbative object, the D-particle that transforms as a $S0(32)$ spinor state with winding number zero. The mass of this D-particle is given by $1/\sqrt{\alpha'} g_I$

Since the mass of these states goes as $1/g_I$, at zero coupling they are infinitely heavy and therefore not present in the free Type I theory. However as soon as the coupling is increased, these states will be present in the theory and affect its dynamics. Let us look at the effect of a non-zero wilson line on these states.

At non-zero coupling and finite T, these states will distinguish between background Wilson lines given by

$$\vec{A}_1 = -T\,2\pi\{0^{16}\} \tag{5.68}$$

$$\vec{A}_2 = -T\,2\pi\{1, 0^{15}\} \tag{5.69}$$

unlike states charged under the adjoint representation of $SO(32)$. The lowest $SO(32)$ spinor state has a weight vector given by $\vec{\lambda} = \{1/2^{16}\}$. Under the background field $\vec{A}_1$, the momentum of these states will remain unaffected while under the background field $\vec{A}_2$, the momentum of these states will shift according to

$$m \to \quad m + \vec{\lambda} \cdot \vec{\ell} \tag{5.70}$$

and therefore the momentum of the $SO(32)$ spinorial states will shift from half-integer to integer for example or vice-versa.

Note that for a adjoint state whose weight vector is given for example by $\vec{\lambda} = \{1, -1, 0^{14}\}$, the momentum remains unaffected. Only adjoint states are present at zero coupling. This means that although at zero coupling, finite T type I is unable to distinguish between the two wilson lines in Eq. 5.69, at non-zero coupling we will have two different Type I theories each corresponding to one value of the wilson line in Eq. 5.69. The single Type I theory at zero coupling therefore splits into two theories as the coupling is increased, and each of these theories can be expected to be s-dual to the corresponding heterotic theory. This relationship between the two theories is shown in Fig. 5.3.

To verify this relationship between Finite T Type I and Heterotic, it is crucial to show that we recover all stable states in the spectrum that the two finite T Heterotic theories have in the corresponding non-perturbative finite T Type I theory. We have already seen that for the simple circle compactification case we can identify the heterotic states with the D-string states. The same is true for the thermal case. We show this in the next section, confining for the moment to the simpler case $\vec{A}_1$. The $\vec{A}_2$ case we leave for later work.

5.4.7 The D-String identification

To find the spectrum of states of the type I D-String in the thermal Type I theory we project the circle amplitude Eqs. (5.28),(5.30),(5.29) by $\mathcal{Y}$. This gives the new Cylinder amplitude as,

$$\begin{aligned}
\mathcal{A}_{pp} = \frac{{d_1}^2 + {d_2}^2}{2} \Big\{ & \big(V_{p-1}O_{9-p} + O_{p-1}V_{9-p}\big)\mathcal{E} \\
- & \big(S_{p-1}S_{9-p} - C_{p-1}C_{9-p}\big)\mathcal{O}\Big\} \\
+ d_1 \times d_2 \Big\{ & \big(V_{p-1}O_{9-p} + O_{p-1}V_{9-p}\big)\mathcal{O} \\
- & \big(S_{p-1}S_{9-p} - C_{p-1}C_{9-p}\big)\mathcal{E}\Big\}
\end{aligned} \tag{5.71}$$

And Mobius amplitude as,

$$\begin{aligned}
\mathcal{M}_p = -\frac{d_1 + d_2}{2} \Big\{ & \big(\hat{V}_{p-1}\hat{O}_{9-p} - \hat{O}_{p-1}\hat{V}_{9-p}\big)\mathcal{E} \\
- & \big(\hat{S}_{p-1}\hat{S}_{9-p} - \hat{C}_{p-1}\hat{C}_{9-p}\big)\mathcal{O}\Big\}
\end{aligned} \tag{5.72}$$

While the A_{p9} amplitude is now given by,

$$\begin{aligned}
\mathcal{A}_{p9} = \frac{n_1 d_1 + n_2 d_2}{2} \Big\{ & \big(V_{p-1}C_{9-p} + O_{p-1}S_{9-p}\big)\mathcal{E} \\
- & \big(C_{p-1}O_{9-p} + S_{p-1}V_{9-p}\big)\mathcal{O}\Big\} \\
+ \frac{n_1 d_2 + n_2 d_1}{2} \Big\{ & \big(V_{p-1}C_{9-p} + O_{p-1}S_{9-p}\big)\mathcal{O} \\
- & \big(C_{p-1}O_{9-p} + S_{p-1}V_{9-p}\big)\mathcal{E}\Big\}
\end{aligned} \tag{5.73}$$

where we have written the most general amplitudes. Here n_1+n_2 count the total number of D9 branes, while d_1+d_2 is the total number of D1-branes. When there is no Wilson line present, $n_1 = 32$ while $n_2 = 0$. Similarly, when we are concerned only with D-Strings having RR charge 1, which corresponds to a winding number, $n = 1$, for the Heterotic fundamental string - $d_1 = 1$ and $d_2 = 0$.

Let us now look for the massless open string states present on the D1 string. These are the same as for the circle. The Dp-Dp amplitude gives us the $\mathbf{8_v}$ which can be identified with the spacetime vector X^μ of the Heterotic F-String. At the same time we can identify right-moving spacetime spinor S^a. The open string momentum modes associated with these states are however different now. Although the spacetime vector states continue to have integer momentum along the circle, the spacetime spinor states have half-integer momentum. Since we are identifying the momentum modes of the D-String with the raising operator level number of the Green-Schwartz heterotic string, this means that the boundary conditions for the Heterotic String are periodic for X^μ and anti-periodic for S^a. This in turn implies that the corresponding spectrum of states on the heterotic side corresponds to $I_8 + C_8$.

Now let us look at the Dp-D9 amplitude. Again like the circle the massless open string states present here are the 32 left-moving world sheet spinors λ^I. They now have half-integer momentum modes. These momentum modes after identification with the level numbers will lead to Heterotic $NS+$ and $NS-$ or $I_{32} + V_{32}$ sectors of the Heterotic string. However as we saw for the case of the circle this not the whole story. In fact since the V_{32} sector is negatively charged under the discrete Z_2 symmetry (which is a remnant of the gauge field of the D1-D1 amplitude), it will be projected out of the spectrum. Simultaneously the presence of the gauge field will lead to the presence of a Wilson line leading to integer momentum sectors for the open string states which will translate to periodic modings on the Heterotic string side. This all happens the same as for the circle case. This will then give us the sectors $R+$ and $R-$ or S_{32} and C_{32}. The C_{32} will be projected out. So the final spectrum on the Heterotic side that can be identified on the Type I side is $I_{32} + S_{32}$ - the same as in the case of the circle.

This tells us that the RR charge 1 D-string leads to the following heterotic set of states,

$$(I_8 + C_8)(I_{32} + S_{32}) = I_8 I_{32} + I_8 S_{32} + C_8 I_{32} + C_8 S_{32} \tag{5.74}$$

Let us know look at what heterotic momentum will be associated with these states. The heterotic momentum is equal to the sum of the momentum carried by open string states on the D-String as we saw in the last section. We also saw that in equating the masses on the Heterotic and the Type I side the following identification holds,

$$N_L - N_R + 1/2\vec{\lambda}\cdot\vec{\lambda} - 1 + \delta_R = m, \tag{5.75}$$

where m is the total D-String momentum (which is equal to heterotic string momentum) and N_L and $1/2\vec{\lambda}\cdot\vec{\lambda}$ are the sum of the oscillator levels on the Type I side. Let us evaluate m for the sectors we have identified Therefore the sectors together with momentum that

	N_L	N_R	$1/2\vec{\lambda}\cdot\vec{\lambda}$	δ_R	m
$I_8 I_{32}$	integer	integer	integer	half-integer	half-integer
$I_8 S_{32}$	integer	integer	integer	half-integer	half-integer
$C_8 I_{32}$	integer	integer	integer	integer	integer
$C_8 S_{32}$	integer	integer	integer	integer	integer

Table 5.2. Evaluating whether the momentum is integer or half-integer for different heterotic sectors

can be identified for the heterotic string are,

$$I_8 I_{32}\mathcal{O}_{1/2} + I_8 S_{32}\mathcal{O}_{1/2} + C_8 I_{32}\mathcal{O}_0 + C_8 S_{32}\mathcal{O}_0 \tag{5.76}$$

where the winding number of the Heterotic string is fixed at 1. These exactly match the sectors that are present on the Heterotic side as can be seen from Eq.

For other winding numbers e.g for Heterotic sector winding number 2, the corresponding configuration on the Type I side will be a bound state of two D-Strings. This can for example correspond in Eq. 5.71 to $n_1 = 32, n_2 = 0, d_1 = 1, d_2 = 1$. The identifications of the sectors and momentum are more complicated in this case. Similarly heterotic winding number 0 will correspond to a bound state of a D-String and an antiD-String on the Type I side.

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