PTEP

Prog. Theor. Exp. Phys. **2022** 113B07(16 pages)
DOI: 10.1093/ptep/ptac141

IYBSSD2022
International Year of Basic Sciences for Sustainable Development

Mass matrices with CP phase in modular flavor symmetry

Shota Kikuchi[1], Tatsuo Kobayashi[1], Morimitsu Tanimoto[2], and Hikaru Uchida[1,⋆]

[1] *Department of Physics, Hokkaido University, Sapporo 060-0810, Japan*
[2] *Department of Physics, Niigata University, Niigata 950-2181, Japan*
⋆E-mail: h-uchida@particle.sci.hokudai.ac.jp

Received September 17, 2022; Accepted October 20, 2022; Published October 29, 2022

We study the CP violation and the CP phase of quark mass matrices in modular flavor symmetric models. The CP symmetry remains at $\tau = e^{2\pi i/3}$ by a combination of the T-symmetry of the modular symmetry. However, T-symmetry breaking may lead to CP violation at the fixed point $\tau = e^{2\pi i/3}$. We study such a possibility in magnetized orbifold models as examples of modular flavor symmetric models. These models, in general, have more than one candidate for Higgs modes, while generic string compactifications also lead to several Higgs modes. These Higgs modes have different behaviors under the T-transformation. The light Higgs mode can be a linear combination of those modes so as to lead to realistic quark mass matrices. The CP phase of the mass matrix does not appear in a certain case, which is determined by the T-transformation behavior. Deviation from this is important to realize the physical CP phase. We discuss an example leading to a non-vanishing CP phase at the fixed point $\tau = e^{2\pi i/3}$.

Subject Index B41, B52

1. Introduction

The origin of the flavor structure including the CP violation is an important issue to study in particle physics. The 4D CP symmetry can be embedded into a proper Lorentz symmetry in higher-dimensional theory such as superstring theory [1–6]. In such a theory, the CP symmetry can be broken spontaneously at the compactification scale or below it within the framework of 4D effective field theory.

In addition to the CP symmetry, geometrical symmetries of compact space can be sources of flavor symmetries among quarks and leptons. For example, D_4 and $\Delta(54)$ flavor symmetries can be derived from heterotic string theory on orbifolds and magnetized/intersecting D-brane models [7–11]. These non-Abelian discrete flavor symmetries have been used in model building for the quark and lepton flavors with the bottom-up approach [12–21].

The torus T^2 and the orbifold T^2/Z_2 compactifications have modular symmetry, which corresponds to the change of basis and is generated by S and T generators. Interestingly, the modular symmetry transforms zero-modes of matter fields. That is, the modular symmetry can also be

Funded by SCOAP³. SCOAP3 and OUP support the goals of the International Year of Basic Sciences for Sustainable Development

a source of the flavor symmetry of quarks and leptons. (See Refs. [22–24] for heterotic string theory on orbifolds and Refs. [25–31] for magnetized D-brane models.[1])

Inspired by these extra-dimensional models and superstring theory, recently 4D modular flavor symmetric models have been studied in lepton and quark sectors. Indeed, the well-known finite groups S_3, A_4, S_4, and A_5 are isomorphic to the finite modular groups Γ_N for $N = 2$, 3, 4, 5, respectively [36]. The lepton mass matrices have been given successfully in terms of $\Gamma_3 \simeq A_4$ modular forms [37]. Modular invariant flavor models have also been proposed on $\Gamma_2 \simeq S_3$ [38], $\Gamma_4 \simeq S_4$ [39], and $\Gamma_5 \simeq A_5$ [40,41]. Other finite groups are also derived from magnetized D-brane models [26]. By using these modular forms, phenomenological studies of the lepton flavors have been done intensively based on A_4 [42–46] and S_4 [47–49]. The quark mass matrices have also been discussed in order to reproduce the observed Cabibbo–Kobayashi–Maskawa (CKM) matrix elements [50,51]. A lot of references to modular invariant flavor models are seen in Ref. [21].

We denote the complex structure modulus τ on T^2 as well as its orbifolds. The modulus τ transforms under the CP symmetry as

$$\tau \to -\tau^*, \tag{1}$$

because the 4D CP symmetry can be embedded into a proper Lorentz transformation of higher dimensions. Thus, the CP symmetry remains along $\mathrm{Re}\tau = 0$. On the other hand, the modulus value $\mathrm{Re}\tau = -1/2$ transforms to $\mathrm{Re}\tau = 1/2$ under the above CP transformation. However, these modulus values $\mathrm{Re}\tau = \pm 1/2$ are equivalent in modular symmetric models by the T-transformation. The CP symmetry remains along $\mathrm{Re}\tau = \pm 1/2$. This issue was studied explicitly in Ref. [52]. Hence, the CP symmetry and modular flavor symmetries are combined so as to construct a larger symmetry [35,53–57].[2] The CP can be violated at a generic value of the modulus τ. For example, realization of the Kobayashi–Maskawa CP phase as well as quark masses and mixing angles in magnetized orbifold models was studied in Ref. [59].

Indeed, how to fix the modulus value is an important issue, i.e., the moduli stabilization problem, although the modulus value is used as a free parameter in many modular flavor symmetric models. Spontaneous CP violation has been studied through moduli stabilization. (For early works, see Refs. [60–63].) Recently the CP violation was studied through the moduli stabilization due to the three-form fluxes [34,64]. For example, the modulus stabilization analysis in Ref. [65] shows that the modulus can be stabilized at the fixed point $\tau = \omega$ with the highest provability, where $\omega = e^{2\pi i/3}$ (see also Refs. [66–68]). At the fixed point $\tau = \omega$ the residual symmetry Z_3 of the modular symmetry remains, while the residual Z_2 and Z_4 symmetries remain at the fixed point $\tau = i$ for $PSL(2, \mathbb{Z})$ and $SL(2, \mathbb{Z})$, respectively. These fixed points are also attractive from the viewpoint of model building in the bottom-up approach. The large flavor mixing angle is simply realized at $\tau = i$ [69] due to the residual Z_2 symmetry (see also Ref. [70]). Interestingly, the hierarchy of charged lepton masses is successfully obtained at nearby $\tau = \omega$ without tuning parameters [71] (see also Ref. [72]). The challenge to the quark sector is promising. The residual symmetries at the fixed points are also useful to stabilize dark matter candidates [73]. Thus, the modular flavor symmetric model presents a phenomenologically spe-

[1]Calabi–Yau compactifications include more moduli and they have larger symplectic modular symmetries [32–35].

[2]See also Ref. [58] for CP in Calabi–Yau compactification.

cial feature at the fixed points. More studies at the fixed points are required to solve the flavor problem such as the CP violation as well as the mass hierarchy.

The CP is not violated at the fixed point $\tau = \omega$ through the above discussion. That is, the CP symmetry is preserved at $\tau = \omega$ if the T-symmetry remains. On the other hand, if the T-symmetry is broken, CP violation may occur at the fixed point $\tau = \omega$. In generic string compactification, there is more than one candidate for Higgs modes, which have the same $SU(2)_L \times U(1)_Y$ quantum numbers and can couple with quarks and leptons. The torus and orbifold compactifications with magnetic fluxes are interesting compactifications. They can lead to 4D chiral theory, where the generation number is determined by magnetic fluxes [74–77]. These magnetic fluxes determine the number of Higgs modes, which can couple with three generations of quarks and leptons [78,79]. Yukawa couplings are written as theta functions. Realistic quark and lepton mass matrices have been studied [80–83]. In this paper, we show that such magnetized orbifold models with multi-Higgs modes can break the T-symmetry, and they can lead to CP violation even at the fixed point $\tau = \omega$.

This paper is organized as follows. In Sect. 2, we give a brief review of the CP violation in modular flavor symmetric models, and then study the importance of the T-symmetry at the fixed point $\tau = \omega$. In Sect. 3, we study the CP violation in the quark sector of the magnetized orbifold models, which were studied in Ref. [83]. In particular, the T-symmetry is broken and that leads to CP violation at the fixed point $\tau = \omega$. Section 4 is our conclusion.

2. CP

Here, we first give a review of the CP in modular flavor symmetric models, and then study one of key points in the CP violation within the framework of the modular flavor symmetry. We focus on 6D theory, i.e., two extra dimensions. Similarly, we can study 10D theory with six extra dimensions.

2.1 *CP symmetry*

Here, we briefly review the CP in modular flavor symmetric models. We use the complex coordinate $z = y_1 + \tau y_2$ on two extra dimensions, where y_1 and y_2 are real coordinates and τ is the complex structure modulus. On T^2, we identify $z \sim z + m + n\tau$, where m and n are integers. We transform z as $z \to z^*$ or $z \to -z^*$ at the same time as the 4D CP transformation. Such a transformation corresponds to a 6D proper Lorentz transformation. Here, we use the latter transformation $z \to -z^*$, because it maps the upper half plane $\mathrm{Im}\tau > 0$ to the same half plane. Then, the modulus τ transforms as Eq. (1) under this CP symmetry. Obviously the line $\mathrm{Re}\tau = 0$ is symmetric under Eq. (1). However, the CP symmetry seems to be violated at other points. For example, the line $\mathrm{Re}\tau = -1/2$ transforms as

$$\tau = -1/2 + i\mathrm{Im}\tau \quad \to \quad 1/2 + i\mathrm{Im}\tau, \tag{2}$$

and it is not invariant.

The modular symmetry transforms the modulus τ as

$$\tau \;\to\; \gamma\tau = \frac{a\tau + b}{c\tau + d}, \tag{3}$$

by the $SL(2, \mathbb{Z})$ element γ,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4}$$

where a, b, c, d are integers satisfying $ad - bc = 1$. The modular symmetry is generated by two elements, S and T:

$$S : \tau \rightarrow -\frac{1}{\tau}, \qquad T : \tau \rightarrow \tau + 1. \tag{5}$$

As mentioned above, the line Re$\tau = -1/2$ transforms to the line Re$\tau = 1/2$ under Eq. (1), and is not invariant. However, these lines transform each other by the T-transformation. Thus, the line Re$\tau = -1/2$ is also CP-symmetric by combining the T-symmetry.

Here, we consider 4D supersymmetric effective theory derived from higher-dimensional theory.[3] For example, we study the superpotential terms including quark Yukawa couplings,

$$W(\tau) = Y^{(u)}_{ij\ell}(\tau) Q_i u_j H^u_\ell + Y^{(d)}_{ij\ell}(\tau) Q_i d_j H^d_\ell, \tag{6}$$

where Q_i, u_j, d_j denote superfields corresponding to three generations of left-handed quarks, right-handed up-sector quarks, and right-handed down-sector quarks, respectively, and $H^{u,d}_\ell$ are up-sector and down-sector Higgs superfields. Here we have added indices for Higgs fields $H^{u,d}_\ell$, because string compactifications, in general, lead to more than one pair of Higgs fields. These quark and Higgs fields transform under the modular symmetry

$$\Phi_i \rightarrow \frac{1}{(c\tau + d)^{k_i}} \rho(\gamma)_{ij} \Phi_j, \tag{7}$$

where $-k_i$denotes the modular weights of 4D fields and $\rho(\gamma)_{ij}$ is a unitary matrix to represent the modular group. The Yukawa couplings $Y^{(u,d)}_{ij}(\tau)$ depend on the modulus τ, and they are modular forms. Similarly, we can study other terms in the superpotential.

The typical Kähler potential of the modulus field τ is written as

$$\hat{K} = -\ln(2\text{Im}\tau), \tag{8}$$

and the tree-level Kähler potential of the matter field with the modular weight $-k_i$is written as

$$K_m = \frac{|\Phi_i|^2}{(2\text{Im}\tau)^{k_i}}. \tag{9}$$

The modular symmetry within the framework of supergravity theory requires $e^{\hat{K}}|W|^2$ to be invariant.

The supersymmetric models are CP-symmetric if $|W|^2$ is invariant, i.e.,

$$W(\tau) \rightarrow e^{i\chi}\overline{W(\tau)}, \tag{10}$$

under the CP transformation with $\tau \rightarrow -\tau^*$ including the CP transformation of chiral matter fields.

2.2 *CP and T-symmetry*

Here, we study the implication of T-symmetry from the viewpoint of CP violation. Suppose that the T-transformation is represented by $\rho(T)$ in our 4D effective field theory, and $\rho(T)$ satisfies

$$\rho(T^N) = \mathbb{I}, \tag{11}$$

i.e., the $Z^{(T)}_N$ symmetry. We consider the field basis such that the T-transformation is represented by diagonal matrices,

$$T : Q_i \rightarrow e^{2\pi i P[Q_i]/N} Q_i, \;\; u_i \rightarrow e^{2\pi i P[u_i]/N} u_i, \;\; d_i \rightarrow e^{2\pi i P[d_i]/N} d_i, \;\; H^{u,d}_i \rightarrow e^{2\pi i P[H^{u,d}_i]/N} H^{u,d}_i, \tag{12}$$

[3] See, e.g., Ref. [84] and references therein.

where the $P[\Phi_i]$ denote $Z_N^{(T)}$ charges of fields Φ_i. The T-invariance of the superpotential requires the following T-transformation of Yukawa couplings:

$$T : Y_{ij\ell}^{(u)}(\tau) \rightarrow e^{2\pi i P[Y^u_{(ij\ell)}]/N} Y_{ij\ell}^{(u)}(\tau), \qquad Y_{ij\ell}^{(d)}(\tau) \rightarrow e^{2\pi i P[Y^d_{(ij\ell)}]/N} Y_{ij\ell}^{(d)}(\tau), \tag{13}$$

where

$$P\left[Y^u_{(ij\ell)}\right] = -\left(P[Q_i] + P[u_j] + P\left[H^u_\ell\right]\right), \qquad P\left[Y^d_{(ij\ell)}\right] = -\left(P[Q_i] + P[d_j] + P\left[H^d_\ell\right]\right). \tag{14}$$

We use these $Z_N^{(T)}$ charges satisfying $0 \leq P[Y^d_{(ij\ell)}] < N$ and $0 \leq P[Y^u_{(ij\ell)}] < N$.

The Yukawa couplings, which are modular forms, can be expanded in terms of $q = e^{2\pi i\tau}$. Since they satisfy the above T-transformation behavior, they can be written as

$$\begin{aligned} Y_{ij\ell}^{(u)}(\tau) &= a_0 q^{P\left[Y^u_{(ij\ell)}\right]/N} + a_1 q q^{P\left[Y^u_{(ij\ell)}\right]/N} + a_2 q^2 q^{P\left[Y^u_{(ij\ell)}\right]/N} + \cdots = \tilde{Y}_{ij\ell}^{(u)}(q) q^{P\left[Y^u_{(ij\ell)}\right]/N}, \\ Y_{ij\ell}^{(d)}(\tau) &= b_0 q^{P\left[Y^d_{(ij\ell)}\right]} + b_1 q q^{P\left[Y^d_{(ij\ell)}\right]/N} + b_2 q^2 q^{P\left[Y^d_{(ij\ell)}\right]/N} + \cdots = \tilde{Y}_{ij\ell}^{(d)}(q) q^{P\left[Y^u_{(ij\ell)}\right]/N}, \end{aligned} \tag{15}$$

where the functions $\tilde{Y}_{ij\ell}^{(u)}(q)$ and $\tilde{Y}_{ij\ell}^{(d)}(q)$ include only integer powers of q, i.e., q^n.

Here, let us consider the model with one pair of H^u and H^d, which are T-invariant. In addition, we set $\mathrm{Re}\tau = -1/2$. In this model, the Yukawa couplings are written as

$$\begin{aligned} Y_{ij\ell}^{(u)}(\tau) &= \hat{Y}_{ij}^{(u)}(q) e^{-\pi i P\left[Y^u_{(ij)}\right]/N}, \\ Y_{ij\ell}^{(d)}(\tau) &= \hat{Y}_{ij}^{(d)}(q) e^{-\pi i P\left[Y^d_{(ij)}\right]/N}, \end{aligned} \tag{16}$$

where we have omitted the indices for Higgs fields, and

$$P\left[Y^u_{(ij)}\right] = -(P[Q_i] + P[u_j]), \qquad P\left[Y^d_{(ij)}\right] = -(P[Q_i] + P[d_j]), \tag{17}$$

$$\hat{Y}_{ij}^{(u)}(q) = \tilde{Y}_{ij}^{(u)}(q) e^{-2\pi P\left[Y^u_{(ij)}\right]\mathrm{Im}\tau/N}, \qquad \hat{Y}_{ij}^{(d)}(q) = \tilde{Y}_{ij}^{(d)}(q) e^{-2\pi P\left[Y^d_{(ij)}\right]\mathrm{Im}\tau/N}. \tag{18}$$

The Yukawa couplings have phases $e^{-\pi i P[Y^{u,d}_{(ij)}]/N}$, although $\hat{Y}_{ij}^{(u)}(q)$ and $\hat{Y}_{ij}^{(d)}(q)$ are real. However, these phases can be removed by the following rephasing of fields:

$$Q_i' = e^{-\pi i P[Q_i]/N} Q_i, \quad u_i' = e^{-\pi i P[u_i]/N} u_i, \quad d_i' = e^{-\pi i P[d_i]/N} d_i. \tag{19}$$

Then, this model is CP-invariant. In this discussion, the T-symmetry is important.

Similarly, we can study the model with one pair of H^u and H^d, which transform non-trivially under the T-transformation:

$$T : H^{u,d} \rightarrow e^{2\pi i P[H^{u,d}]/N} H^{u,d}. \tag{20}$$

When these Higgs fields develop their vacuum expectation values (VEVs), mass matrices have phases, but those are overall phases, and not physical. The Higgs VEVs break the T-symmetry and the T-symmetry is broken through the moduli stabilization at $\tau = \omega$, but these are not enough to realize the physical CP phase.

Unless the above structure is violated by any effects such as non-perturbative effects, the above discussion suggests that we need two or more Higgs VEV directions. We denote them by

$$v^u_\ell = \left|v^u_\ell\right| e^{2\pi i P\left[v^u_\ell\right]/N} = \left\langle H^u_\ell \right\rangle, \qquad v^d_\ell = \left|v^d_\ell\right| e^{2\pi i P\left[v^d_\ell\right]/N} = \left\langle H^d_\ell \right\rangle, \tag{21}$$

where $P[v_\ell^u]$ or $P[v_\ell^d]$ is not integer for a generic VEV. At any rate, the phase structure of mass matrices at $\tau = \omega$ is controlled by the T-symmetry. If they satisfy

$$-\frac{1}{2}\left(P\left[Y^u_{(ij\ell)}\right] + P[Q_i] + P\left[u_j\right]\right) + P\left[v^u_\ell\right] = \text{constant independent of } \ell,$$
$$-\frac{1}{2}\left(P\left[Y^d_{(ij\ell)}\right] + P[Q_i] + P\left[d_j\right]\right) + P\left[v^d_\ell\right] = \text{constant independent of } \ell, \quad (22)$$

in all of the allowed Yukawa couplings with i, j fixed, one can cancel the phase of mass matrix elements up to an overall phase by the $Z_N^{(T)}$ rotation. We can compare this condition with the relations (14), where the factor $-1/2$ originates from $\mathrm{Re}\tau = -1/2$. We study this point in the next section by using magnetized orbifold models as an example.

This condition is also applied for the fixed point $\tau = i$. The Yukawa couplings are real in our basis because of $\mathrm{Re}\tau = 0$. Then, the coefficients of $P[Y^u_{(ij\ell)}] + P[Q_i] + P[u_j]$ and $P[Y^d_{(ij\ell)}] + P[Q_i] + P[d_j]$ are zero. That is, if we choose non-trivial phases of Higgs VEVs, which cannot be removed by rephasing, the CP violation occurs. For example, if all of the Higgs VEVs are real in our basis, the CP symmetry remains at the fixed point $\tau = i$. However, the CP can be violated at the fixed point $\tau = \omega$, even if all of the Higgs VEVs are real in our basis. Differences of Higgs VEV phases from their $Z_N^{(T)}$ charges are important. The $Z_N^{(T)}$ charge pattern is the reference to judge whether the non-trivial CP phase appears or not.

3. CP phase in magnetized orbifold models

Here we study the CP phase derived from magnetized orbifold models as an example of modular flavor symmetric models.

3.1 *Magnetized orbifold models*

First, we give a brief review of zero-mode wave functions on magnetized T^2 [74]. Higher-dimensional fields, e.g., the spinor field $\Psi(x, z)$, can be decomposed by

$$\Psi(x, y) = \sum \chi_i(x)\psi_i(z) + \cdots , \quad (23)$$

where x denotes the 4D coordinate, the first term $\chi_i(x)\psi_i(z)$ corresponds to zero-modes, and the ellipsis denotes massive modes. For simplicity, we explain them by use of $U(1)$ theory. We introduce the $U(1)$ background magnetic flux,

$$F = dA = \frac{\pi i M}{\mathrm{Im}\tau} dz \wedge d\bar{z}, \qquad A = \frac{\pi M}{\mathrm{Im}\tau}\mathrm{Im}(\bar{z}dz), \quad (24)$$

where M must be integer because of the Dirac quantization condition. We consider the Dirac equation for a spinor with $U(1)$ charge $q = 1$. On T^2, the spinor ψ has two components, $\psi = (\psi_+, \psi_-)^T$. For $M > 0$, ψ_+ has M zero-modes, but ψ_- has no zero-modes. On the other hand, for $M < 0$, ψ_- has $|M|$ zero-modes but ψ_+ has no zero-modes. Thus, we can realize a chiral theory. When $M > 0$, the jth zero-mode can be written as

$$\psi_+^{j,|M|}(z, \tau) = \left(\frac{|M|}{\mathcal{A}}\right)^{1/4} e^{i\pi|M|z\frac{\mathrm{Im}z}{\mathrm{Im}\tau}} \vartheta\begin{bmatrix} \frac{j}{|M|} \\ 0 \end{bmatrix}(|M|z, |M|\tau), \quad (25)$$

where $\mathcal{A}$ denotes the area of T^2 and ϑ denotes the Jacobi theta function defined by

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu, \tau) = \sum_{\ell\in\mathbb{Z}} e^{\pi i(a+\ell)^2\tau} e^{2\pi i(a+\ell)(\nu+b)}. \quad (26)$$

Similarly, we can write zero-mode wave functions of ψ_- for $M < 0$. Hereafter, we omit the chirality sign index $\pm$, but we denote the wave function by ψ_{T^2}. Here we use the normalization

$$\int d^2z \psi^{i,|M|}_{T^2}(z,\tau)\left(\psi^{j,|M|}_{T^2}(z,\tau)\right)^* = (2\mathrm{Im}\tau)^{-1/2}\delta_{i,j}. \tag{27}$$

The Yukawa coupling of zero-modes is written as

$$\begin{aligned} Y^{ijk} &= g\int d^2z\, \psi^{i,|M_1|}_{T^2}(z,\tau)\, \psi^{j,|M_2|}_{T^2}(z,\tau)\, \left(\psi^{k,|M_3|}_{T^2}(z,\tau)\right)^* \\ &= g\mathcal{A}^{-1/2}\left|\frac{M_1M_2}{M_1+M_2}\right|^{1/4}\sum_m \delta_{k,i+j+|M_1M_2|m}\vartheta\begin{bmatrix}\frac{|M_2|i-|M_1|j+|M_1M_2|m}{|M_1M_2(M_1+M_2)|}\\ 0\end{bmatrix}(0,|M_1M_2(M_1+M_2)|, \end{aligned} \tag{28}$$

where g is the three-point coupling in higher-dimensional theory. The gauge invariance requires

$$|M_1| + |M_2| = |M_3| \tag{29}$$

for allowed Yukawa couplings. Furthermore, the Kronecker delta $\delta_{k,i+j+|M_1M_2|m}$ implies the coupling selection rule among these modes.

The zero-mode wave functions transform under the T-symmetry as [25–31]

$$T : \psi^{j,|M|}_{T^2}(z,\tau) \rightarrow e^{i\pi j^2/M}\psi^{j,|M|}_{T^2}(z,\tau), \tag{30}$$

when M is even.[4] Thus, the T-transformation is represented by the diagonal matrix in this basis, and zero-modes have $Z^{(T)}_{2M}$ charges. The zero-mode wave functions transform under the S-symmetry as

$$S : \psi^{j,|M|}_{T^2}(z,\tau) \rightarrow (-\tau)^{1/2}\frac{e^{i\pi/4}}{\sqrt{|M|}}\sum_\ell e^{2\pi i j\ell/M}\psi^{\ell,|M|}_{T^2}(z,\tau). \tag{31}$$

Note that the transformation of the 4D fields $\chi_i(x)$ is the inverse of $\psi_i(z)$ to make $\Psi(x, y)$ invariant. For example, the 4D fields transform as

$$T : \chi^j(z) \rightarrow e^{-i\pi j^2/M}\chi^j(z) \tag{32}$$

under the T-transformation.

The T^2/Z_2 orbifold is constructed by identifying $z \sim -z$ on T^2. Wave functions on T^2/Z_2 are classified into Z_2-even and -odd modes,

$$\psi_{T^2/Z_2^m}(-z) = (-1)^m\psi_{T^2/Z_2^m}(z), \tag{33}$$

where $m = 0$ and 1 correspond to Z_2-even and -odd modes. The zero-mode wave functions in orbifold models with magnetic fluxes can be given by use of zero-mode wave functions on T^2 as [75–77]

$$\begin{aligned} \psi^{j,|M|}_{T^2/Z_2^m}(z) &= \mathcal{N}^j\left(\psi^{j,|M|}_{T^2}(z) + (-1)^m\psi^{j,|M|}_{T^2}(-z)\right) \\ &= \mathcal{N}^j\left(\psi^{j,|M|}_{T^2}(z) + (-1)^m\psi^{|M|-j,|M|}_{T^2}(z)\right), \end{aligned} \tag{34}$$

where

$$\mathcal{N}^j = \begin{cases} 1/2 & (j = 0, |M|/2) \\ 1/\sqrt{2} & (\text{otherwise}) \end{cases}. \tag{35}$$

Table 1 shows the number of zero-modes on a magnetized T^2/Z_2 orbifold [75–77]. We can realize three generations by Z_2-even modes for $|M| = 4, 5$ and by Z_2-odd modes for $|M| = 7, 8$.

[4] For the generic case see Ref. [30].

Table 1. The number of zero-modes on a magnetized T^2/Z_2 orbifold.

$\|M\|$	1	2	3	4	5	6	7	8	9	10	11	12
Z_2-even	1	2	2	3	3	4	4	5	5	6	6	7
Z_2-odd	0	0	1	1	2	2	3	3	4	4	5	5

Table 2. Zero-mode wave functions.

i	Q_i	u_i, d_i	$H_i^{u,d}$
0	$\psi_{T^2}^{0,4}$	$\psi_{T^2}^{0,4}$	$\psi_{T^2}^{0,8}$
1	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{1,4}+\psi_{T^2}^{3,4})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{1,4}+\psi_{T^2}^{3,4})$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{1,8}+\psi_{T^2}^{7,8})$
2	$\psi_{T^2}^{2,4}$	$\psi_{T^2}^{2,4}$	$\frac{1}{\sqrt{2}}(\psi_{T^2}^{2,8}+\psi_{T^2}^{6,8})$
3			$\frac{1}{\sqrt{2}}(\psi_{T^2}^{3,8}+\psi_{T^2}^{5,8})$
4			$\psi_{T^2}^{4,8}$

Yukawa couplings in magnetized orbifold models can be calculated by

$$Y^{ijk} = g\int d^2z\,\psi^{i,|M_1|}_{T^2/Z_2^{m_1}}(z,\tau)\,\psi^{j,|M_2|}_{T^2/Z_2^{m_2}}(z,\tau)\,\left(\psi^{k,|M_3|}_{T^2/Z_2^{m_3}}(z,\tau)\right)^*. \tag{36}$$

Their explicit computations are straightforward by use of Eqs. (28) and (34). The allowed Yukawa couplings must satisfy

$$m_1+m_2=m_3 \pmod 2, \tag{37}$$

in addition to Eq. (29).

3.2 *Quark mass matrix in magnetized orbifold models*

Here, we study quark mass matrices in the magnetized orbifold model, which was studied in Ref. [83]. We consider the model in which all left-handed and right-handed quarks originate from Z_2-even zero-modes with $M=4$, leading to three zero-modes, i.e., three generations. (For details of the model building, see, e.g., Refs. [81,85].) Because of Eqs. (29) and (37), the Higgs modes correspond to Z_2-even zero-modes with $M=8$. This means five pairs of $H_\ell^{u,d}$ from Table 1. These zero-mode wave functions are summarized in Table 2.

Now we study the quark mass matrices in our model. The mass matrices in the up and down sectors of quarks can be written as

$$(M_u)_{ij} = \sum_\ell Y^{(u)}_{ij\ell}\langle H^u_\ell\rangle, \qquad (M_d)_{ij} = \sum_\ell Y^{(d)}_{ij\ell}\langle H^d_\ell\rangle, \tag{38}$$

when the Higgs fields develop their VEVs. The Yukawa couplings can be written explicitly as

$$Y^{(u),(d)}_{ij0} = c\begin{pmatrix} X_0 & & \\ & X_1 & \\ & & X_2\end{pmatrix}, \qquad Y^{(u),(d)}_{ij1} = c\begin{pmatrix} & X_3 & \\ X_3 & & X_4 \\ & X_4 & \end{pmatrix},$$
$$Y^{(u),(d)}_{ij2} = c\begin{pmatrix} & & \sqrt{2}X_1 \\ & \frac{1}{\sqrt{2}}(X_0+X_2) & \\ \sqrt{2}X_1 & & \end{pmatrix}, \qquad Y^{(u),(d)}_{ij3} = c\begin{pmatrix} & X_4 & \\ X_4 & & X_3 \\ & X_3 & \end{pmatrix}, \tag{39}$$
$$Y^{(u),(d)}_{ij4} = c\begin{pmatrix} X_2 & & \\ & X_1 & \\ & & X_0\end{pmatrix},$$

where c is an overall constant, and

$$X_0 = \eta_0 + 2\eta_{32} + \eta_{64}, \qquad X_1 = \eta_8 + \eta_{24} + \eta_{40} + \eta_{56}, \qquad X_2 = 2(\eta_{16} + \eta_{48}),$$
$$X_3 = \eta_4 + \eta_{28} + \eta_{36} + \eta_{60}, \qquad X_4 = \eta_{12} + \eta_{20} + \eta_{44} + \eta_{52}. \tag{40}$$

Here, we have used the notation

$$\eta_N = \vartheta\begin{bmatrix}\frac{N}{128}\\ 0\end{bmatrix}(0, 128\tau). \tag{41}$$

Note that each of the $Y^{(u),(d)}_{ij\ell}$ matrices with $\ell = 0, 1, 2, 3, 4$ is not a rank-one matrix or an approximate rank-one matrix leading to the realistic quark mass hierarchy, except $\mathrm{Im}\tau \to \infty$. In addition, each of the $Y^{(u),(d)}_{ij\ell}$ matrices with $\ell = 0, 1, 2, 3, 4$ has many zero elements, which originate from the coupling selection rule due to $\delta_{k,i+j+|M_1M_2|m}$ in Eq. (28). Therefore, a single VEV direction is not realistic.

In particular, we set the modulus value at $\tau = \omega$ in order to study the quark mass matrices. At this fixed point, the residual Z_3 symmetry, which is generated by ST, remains. At the compactification energy scale, all five pairs of Higgs fields are massless. We expect that they generate their mass terms,

$$\mu(\tau)_{ij} H^u_i H^d_j, \tag{42}$$

below the compactification scale. Then, one pair remains light, and they develop their VEVs. Such mass terms would be generated by non-perurbative effects such as D-brane instanton effects.[5] Also, coupling terms such as $Y(\tau)_{ij\ell} H^u_i H^d_j S_\ell$ may be the origin of the mass terms when the S_ℓ develop their VEVs like the next-to-minimal supersymmetric standard model. Furthermore, higher-order terms such as $Y(\tau)_{ij\ell_1\cdots\ell_n} H^u_i H^d_j S_{\ell_1}\cdots S_{\ell_n}$ may be the origin when the S_{ℓ_i} develop their VEVs. These depend on the details of the model. Thus, we take a phenomenological approach. That is, we study which VEV directions lead to realistic results in quark mass matrices assuming that such directions correspond to the light mode in the above mass matrices.

Quark masses are strongly hierarchical. This means that quark mass matrices are approximate rank-one matrices, and realistic mass matrices deviate slightly from such rank-one matrices. In Ref. [83], VEV directions leading to rank-one mass matrices were studied. We follow this analysis. For example, the Z_3 symmetry remains at the fixed point $\tau = \omega$. In particular, VEV directions, which lead to rank-one mass matrices and Z_3-invariant vacuum, were studied in Ref. [83]. Although the VEV directions in the Z_3 eigenbasis shown in Ref. [83] include non-zero for only one Z_3-invariant direction and zeros for the other directions, i.e., $A(1, 0, 0, 0, 0)$, the directions in our basis become

$$\langle H^u_\ell\rangle = \langle H^d_\ell\rangle = h_\ell,$$
$$h_\ell = A(0.6254e^{0.04567i}, 0.6295e^{-0.1507i}, 0.2269e^{-0.7397i}, 0.04126e^{-1.721i}, 0.005421e^{-3.096i}). \tag{43}$$

This means that, even if we consider this Z_3-invariant Higgs mode to be the lightest and that only this Higgs field develops its real VEV, this direction is constructed by the mixing of Higgs directions with different $Z_N^{(T)}$ charges, and each of the VEV phases is different from its $Z_N^{(T)}$ charge. Along this VEV direction, only the third generations gain masses; the first and second

[5] Non-perturbative effects such as D-brane instanton effects may break part of the modular symmetry [86].

Table 3. The mass ratios of the quarks, the values of the CKM matrix elements, and the Jarlskog invariant at $\tau = \omega$ under the vacuum alignments of Higgs fields in Eq. (44). Reference values of mass ratios are shown in Ref. [87]. Those of the CKM matrix elements and the Jarlskog invariant are shown in Ref. [88].

	Obtained values	Reference values
$(m_u, m_c, m_t)/m_t$	$(1.64 \times 10^{-5}, 6.22 \times 10^{-3}, 1)$	$(5.58 \times 10^{-6}, 2.69 \times 10^{-3}, 1)$
$(m_d, m_s, m_b)/m_b$	$(1.57 \times 10^{-3}, 1.32 \times 10^{-2}, 1)$	$(6.86 \times 10^{-4}, 1.37 \times 10^{-2}, 1)$
$\lvert V_{\mathrm{CKM}}\rvert$	$\begin{pmatrix} 0.974 & 0.225 & 0.004\,05 \\ 0.225 & 0.974 & 0.0353 \\ 0.007\,19 & 0.0348 & 0.999 \end{pmatrix}$	$\begin{pmatrix} 0.974 & 0.227 & 0.003\,61 \\ 0.226 & 0.973 & 0.0405 \\ 0.008\,54 & 0.0398 & 0.999 \end{pmatrix}$
J_{CP}	2.83×10^{-5}	2.80×10^{-5}

generations are massless. We take the following VEV directions:

$$\langle H_u^k \rangle = v_u(0.6228e^{0.1169i}, 0.6273e^{-0.1738i}, 0.2425e^{-1.055i}, 0.057\,74e^{-2.441i}, 0.011\,86e^{2.088i}),$$
$$\langle H_d^k \rangle = v_d(0.6201e^{0.1713i}, 0.6259e^{-0.2009i}, 0.2605e^{-1.215i}, 0.063\,49e^{-2.538i}, 0.009\,710e^{1.930i}), \tag{44}$$

which deviate slightly from the above rank-one directions. When the above directions of Higgs modes are light and they develop their VEVs, we realize the following quark mass matrices:

$$M_u = m_t \begin{pmatrix} 0.6228e^{0.1167i} & 0.4444e^{-0.3663i} & 0.087\,99e^{-1.840i} \\ 0.4444e^{-0.3663i} & 0.3219e^{-0.8600i} & 0.066\,26e^{-2.350i} \\ 0.087\,99e^{-1.840i} & 0.066\,26e^{-2.350i} & 0.014\,82e^{2.430i} \end{pmatrix}, \tag{45}$$

$$M_d = m_b \begin{pmatrix} 0.6201e^{0.1712i} & 0.4433e^{-0.3926i} & 0.094\,52e^{-2.000i} \\ 0.4433e^{-0.3926i} & 0.3253e^{-0.9292i} & 0.069\,25e^{-2.438i} \\ 0.094\,52e^{-2.000i} & 0.069\,25e^{-2.438i} & 0.011\,94e^{2.388i} \end{pmatrix}, \tag{46}$$

at $\tau = \omega$ in the orbifold wave function basis of Table 2. These mass matrices lead to the quark mass ratios

$$\frac{m_u}{m_t} = 1.64 \times 10^{-5}, \qquad \frac{m_c}{m_t} = 6.22 \times 10^{-3},$$
$$\frac{m_d}{m_b} = 1.57 \times 10^{-3}, \qquad \frac{m_s}{m_b} = 1.32 \times 10^{-2}. \tag{47}$$

These ratios are obtained at the compactification scale, which may be quite high. When we compare them with experimental values, we have to evaluate the renormalization group effects. Renormalization group effects depend on the breaking scale of supersymmetry and $\tan\beta$. For example, we compare them with mass ratios at the GUT scale by assuming the low-energy supersymmetric model with $\tan\beta = 5$ [87]. The Cabibbo–Kobayashi–Maskawa (CKM) matrix is also obtained in our model:

$$|V_{\mathrm{CKM}}| = \begin{pmatrix} 0.974 & 0.225 & 0.004\,05 \\ 0.225 & 0.974 & 0.0353 \\ 0.007\,19 & 0.0348 & 0.999 \end{pmatrix}. \tag{48}$$

Furthermore, our model leads to the Jarlskog invariant:

$$J = 2.83 \times 10^{-5}. \tag{49}$$

These results are shown in Table 3. Thus, our model can realize almost experimental values.

The important point in our results is that we can realize non-vanishing CP phase even at the fixed point $\tau = \omega$. In Sect. 2, it was found that the CP symmetry remains at the fixed point τ

$= \omega$ because of the T-symmetry. Now, let us investigate the T-symmetry in our model from the viewpoint of the CP violation. At $\tau = \omega$, the Yukawa couplings have the following phases:

$$X_0 = |X_0|, \quad X_1 = e^{-2\pi i/8}|X_1|, \qquad X_2 = -|X_2|,$$
$$X_3 = e^{-2\pi i/32}|X_3|, \qquad X_4 = e^{-2\pi i9/32}|X_4|. \tag{50}$$

Also, the 4D quark fields $q_j = (Q_j, u_j, d_j)$ with $j = 0, 1, 2$ in our model transform as

$$T : q_j \;\to\; e^{-2\pi i j^2/8} q_j, \tag{51}$$

under $Z_N^{(T)}$, while the Higgs modes transform as

$$T : H_\ell^{u,d} \;\to\; e^{-2\pi i \ell^2/16} H_\ell^{u,d}, \tag{52}$$

with $\ell = 0, 1, 2, 3, 4$ under the $Z_N^{(T)}$ symmetry. For simplicity, we consider the case in which only the $H_0^{u,d}$ and $H_1^{u,d}$ develop their VEVs. Then, the mass matrices can be written as

$$M_{u,d} = c\begin{pmatrix} v_0|X_0| & v_1 e^{-2\pi i/32}|X_3| & 0 \\ v_1 e^{-2\pi i/32}|X_3| & v_0 e^{-2\pi i/8}|X_1| & v_1 e^{-2\pi i9/32}|X_4| \\ 0 & v_1 e^{-2\pi i9/32}|X_4| & -v_0|X_2| \end{pmatrix}. \tag{53}$$

When the phases of the VEVs satisfy Eq. (22), i.e.,

$$(v_0, v_1) = |a|e^{i\phi}(1, e^{-2\pi i/32}), \tag{54}$$

which are related to the $Z_N^{(T)}$ charges of $H_0^{u,d}$ and $H_1^{u,d}$, the phases in mass matrices can be canceled by the following rephasing of fields:

$$q'_j = e^{-2\pi i j^2/16} q_j, \tag{55}$$

which are related to the $Z_N^{(T)}$ charges of q_j. Similarly, we can discuss the case in which more Higgs modes develop their VEVs. What is important is the difference of the VEV phases from the $Z_N^{(T)}$ charge pattern. On the other hand, if a single mode among $H_\ell^{u,d}$ develops its VEV, we cannot realize the physical CP violation as discussed in Sect. 2. In the above example, it corresponds to $v_1 = 0$, and such a case just leads to an overall phase, as discussed in Sect. 2. In order to realize a non-vanishing CP phase, we need a linear combination of more than one Higgs modes corresponding to the light Higgs mode and it to develop its VEV, where the VEV phases must be different from the $Z_N^{(T)}$ charge pattern. If the VEV phases coincide with the $Z_N^{(T)}$ phase pattern, the physical CP does not appear. Thus, the $Z_N^{(T)}$ phase pattern is the reference for the physical CP phase. Note that we need such a linear combination to realize the realistic mass matrices, which deviate slightly from the rank-one mass matrices, in our model. The VEV of a single mode cannot lead to the realistic mass matrices. Phenomenologically, we need Higgs modes along a generic VEV direction leading to rank-one mass matrices and the slight deviation from rank-one mass matrices. When we require such a direction for phenomenological purposes, we can automatically realize the CP violation at the fixed point $\tau = \omega$ for VEV phases different from the $Z_N^{(T)}$ charge pattern. This is an interesting scenario of the CP violation in modular flavor symmetric models.

Also, we comment an obvious example. If the quark mass matrices are diagonal, we can always remove phases by $U(1)^9$ rotation, which may be independent of the $Z_N^{(T)}$ rotation. Such an accidental symmetry may forbid the physical CP phase.

Table 4. The mass ratios of the quarks, the values of the CKM matrix elements, and the Jarlskog invariant at $\tau = \omega$ under the vacuum alignments of Higgs fields in Eq. (44). Reference values of mass ratios are shown in Ref. [87]. Those of the CKM matrix elements and the Jarlskog invariant are shown in Ref. [88].

	Obtained values	Reference values
$(m_u, m_c, m_t)/m_t$	$(5.34 \times 10^{-5}, 4.68 \times 10^{-2}, 1)$	$(5.58 \times 10^{-6}, 2.69 \times 10^{-3}, 1)$
$(m_d, m_s, m_b)/m_b$	$(1.38 \times 10^{-3}, 4.10 \times 10^{-2}, 1)$	$(6.86 \times 10^{-4}, 1.37 \times 10^{-2}, 1)$
$\lvert V_{\mathrm{CKM}}\rvert$	$\begin{pmatrix} 0.975 & 0.223 & 0.003\,23 \\ 0.223 & 0.974 & 0.0410 \\ 0.0103 & 0.0398 & 0.999 \end{pmatrix}$	$\begin{pmatrix} 0.974 & 0.227 & 0.003\,61 \\ 0.226 & 0.973 & 0.0405 \\ 0.008\,54 & 0.0398 & 0.999 \end{pmatrix}$
J	2.80×10^{-5}	2.80×10^{-5}

We also show an example with the real VEV direction in our field basis. We take the following VEV directions:

$$\langle H_\ell^u \rangle = v_u(0.7911, -0.6040, 0.096\,92, -0.000\,095\,78, -0.000\,2417),$$
$$\langle H_\ell^d \rangle = v_d(0.7471, -0.6512, 0.1333, -0.001\,655, -0.004\,176). \tag{56}$$

When the above directions of Higgs modes are light and they develop their VEVs, we realize the following quark mass matrices:

$$M_u = m_t \begin{pmatrix} 0.7675 & 0.4170e^{0.9375\pi i} & 0.034\,118e^{-\pi i/4} \\ 0.4170e^{0.9375\pi i} & 0.2479e^{-0.1898\pi i} & 0.027\,33e^{0.4383\pi i} \\ 0.034\,118e^{-\pi i/4} & 0.027\,33e^{0.4383\pi i} & -0.006\,886 \end{pmatrix}, \tag{57}$$

$$M_d = m_b \begin{pmatrix} 0.7247 & 0.4495e^{0.9374\pi i} & 0.046\,91e^{-\pi i/4} \\ 0.4495e^{0.9374\pi i} & 0.2571e^{-0.1698\pi i} & 0.029\,49e^{0.4498\pi i} \\ 0.046\,91e^{-\pi i/4} & 0.029\,49e^{0.4498\pi i} & -0.010\,33 \end{pmatrix} \tag{58}$$

at $\tau = \omega$ in the orbifold wave function basis of Table 2. These mass matrices lead to mass rations, mixing angles, and the Jarlskog invariant shown in Table 4. Also, this VEV direction can realize almost experimental values except for the ratio m_u/m_t.

We have concentrated on the modulus value $\mathrm{Re}\tau = \pm 1/2$, in particular the fixed point $\tau = \omega$. This fixed point has the highest provability in the moduli stabilization analysis [65], and is phenomenologically interesting. The next favorable values in the moduli stabilization analysis [65] are $\mathrm{Re}\tau = \pm 1/4$ and 0. Even at $\mathrm{Re}\tau = -1/4$, the phase structure is controlled by the T-symmetry. The Yukawa couplings at this point have the following phase:

$$X_0 = |X_0|, \quad X_1 = e^{-2\pi i/16}|X_1|, \quad X_2 = -i|X_2|,$$
$$X_3 = e^{-2\pi i/64}|X_3|, \quad X_4 = e^{-2\pi i 9/64}|X_4|. \tag{59}$$

Again for simplicity, we consider the case in which only the $H_0^{u,d}$ and $H_1^{u,d}$ develop their VEVs:

$$M_{u,d} = c \begin{pmatrix} v_0|X_0| & v_1 e^{-2\pi i/64}|X_3| & 0 \\ v_1 e^{-2\pi i/64}|X_3| & v_0 e^{-2\pi i/16}|X_1| & v_1 e^{-2\pi i 9/64}|X_4| \\ 0 & v_1 e^{-2\pi i 9/64}|X_4| & -i v_0|X_2| \end{pmatrix}. \tag{60}$$

When the phases of the VEVs satisfy

$$(v_0, v_1) = |a|e^{i\phi}(1, e^{-2\pi i/64}), \tag{61}$$

which are related to the $Z_N^{(T)}$ charges of $H_0^{u,d}$ and $H_1^{u,d}$, the phases in mass matrices can be canceled by the $Z_N^{(T)}$ rotation. Thus, the $Z_N^{(T)}$ charge pattern is the reference to judge whether the non-trivial CP phase appears or not for $\mathrm{Re}\tau = \pm 1/4$, too.

One can expand our discussions for the modulus value Re$\tau = \pm 1/n$, although we may not have a clear motivation to set Re$\tau = \pm 1/n$ from the viewpoint of the moduli stabilization or phenomenology. Also, CP violation occurs at $\tau = i$ along the Higgs VEV directions, where the VEVs have relatively different phases.

4. Conclusion

We have studied the CP phase of quark mass matrices in modular flavor symmetric models at the fixed point of τ. The CP symmetry remains at Re$\tau = \pm 1/2$, although Re$\tau = -1/2$ transforms to Re$\tau = 1/2$ under CP. The reason for this is that these transform each other under the T-symmetry. This may suggest that, if the T-symmetry is broken, the CP is also violated. However, simple breaking of T-symmetry is not enough to lead to CP violation.

We have studied quark mass matrices in magnetized orbifold models. Our model has five pairs of Higgs fields. In general, string compactification leads to more than one candidate for Higgs modes. We have computed quark mass matrices at the fixed point $\tau = \omega$. This point is favorable from the viewpoint of moduli stabilization and is also phenomenologically interesting in flavor physics. The CP is not violated at the fixed point $\tau = \omega$ if the T-symmetry remains. In our model, a non-vanishing physical CP phase can be realized. The important point is that the Higgs modes mix with each other. Such mixing is required for phenomenological purposes to realize realistic quark mass matrices, which are approximate rank-one mass matrices. The physical CP phase can appear when the phases of the VEVs differ from the $Z_N^{(T)}$ charge.

We have shown a scenario to realize the CP violation in modular flavor symmetric models. It is interesting to apply this scenario to other modular flavor symmetric models including the lepton sector, e.g., at the fixed point $\tau = \omega$. For phenomenological purposes, we required that the light Higgs modes correspond to linear combinations of Higgs modes. It is important to show this point by computation of the μ mass matrices theoretically. However, that is beyond the scope of this paper and we will study it elsewhere. In addition, heavier Higgs modes may contribute to flavor-changing processes and CP-violating processes, if they are light enough. Such studies are phenomenologically important if the Higgs mass spectrum is calculated, although sufficiently heavy Higgs modes may not contribute.

We have found that the breaking of T-symmetry is important, and have shown a scenario of T-symmetry breaking leading to CP violation. It would also be important to study whether there is another way to break the T-symmetry leading to CP violation.

Acknowledgement

This work was supported by JSPS KAKENHI Grant Numbers JP22J10172 (S.K.) and JP20J20388 (H.U.).

Funding

Open Access funding: SCOAP^3.

References

[1] M. B. Green, J. H. Schwarz, and E. Witten, Cambridge Monographs On Mathematical Physics, "Superstring Theory" Cambridge University Press, Cambridge, UK, (1987), p. 596.
[2] A. Strominger and E. Witten, Commun. Math. Phys. **101**, 341 (1985).
[3] M. Dine, R. G. Leigh, and D. A. MacIntire, Phys. Rev. Lett. **69**, 2030 (1992) [arXiv:hep-th/9205011] [Search inSPIRE].

[4] K. w. Choi, D. B. Kaplan, and A. E. Nelson, Nucl. Phys. B **391**, 515 (1993) [arXiv:hep-ph/9205202] [Search inSPIRE].
[5] C. S. Lim, Phys. Lett. B **256**, 233 (1991).
[6] T. Kobayashi and C. S. Lim, Phys. Lett. B **343**, 122 (1995) [arXiv:hep-th/9410023] [Search inSPIRE].
[7] T. Kobayashi, S. Raby, and R. J. Zhang, Nucl. Phys. B **704**, 3 (2005) [arXiv:hep-ph/0409098] [Search inSPIRE].
[8] T. Kobayashi, H. P. Nilles, F. Ploger, S. Raby, and M. Ratz, Nucl. Phys. B **768**, 135 (2007) [arXiv:hep-ph/0611020] [Search inSPIRE].
[9] P. Ko, T. Kobayashi, J. h. Park, and S. Raby, Phys. Rev. D **76**, 035005 (2007); **76**, 059901 (2007) [erratum][arXiv:0704.2807 [hep-ph]] [Search inSPIRE].
[10] F. Beye, T. Kobayashi, and S. Kuwakino, Phys. Lett. B **736**, 433 (2014) [arXiv:1406.4660 [hep-th]] [Search inSPIRE].
[11] H. Abe, K. S. Choi, T. Kobayashi, and H. Ohki, Nucl. Phys. B **820**, 317 (2009) [arXiv:0904.2631 [hep-ph]] [Search inSPIRE].
[12] G. Altarelli and F. Feruglio, Rev. Mod. Phys. **82**, 2701 (2010) [arXiv:1002.0211 [hep-ph]] [Search inSPIRE].
[13] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, Prog. Theor. Phys. Suppl. **183**, 1 (2010) [arXiv:1003.3552 [hep-th]] [Search inSPIRE].
[14] H. Ishimori, T. Kobayashi, H. Ohki, H. Okada, Y. Shimizu, and M. Tanimoto, Lecture Notes in Physics, An Introduction to Non-Abelian Discrete Symmetries for Particle Physicists, Springer, Berlin, (2012), Vol. 858, p. 1.
[15] D. Hernandez and A. Y. Smirnov, Phys. Rev. D **86**, 053014 (2012) [arXiv:1204.0445 [hep-ph]] [Search inSPIRE].
[16] S. F. King and C. Luhn, Rept. Prog. Phys. **76**, 056201 (2013) [arXiv:1301.1340 [hep-ph]] [Search inSPIRE].
[17] S. F. King, A. Merle, S. Morisi, Y. Shimizu, and M. Tanimoto, New J. Phys. **16**, 045018 (2014) [arXiv:1402.4271 [hep-ph]] [Search inSPIRE].
[18] M. Tanimoto, AIP Conf. Proc. **1666**, 120002 (2015).
[19] S. F. King, Prog. Part. Nucl. Phys. **94**, 217 (2017) [arXiv:1701.04413 [hep-ph]] [Search inSPIRE].
[20] S. T. Petcov, Eur. Phys. J. C **78**, 709 (2018) [arXiv:1711.10806 [hep-ph]] [Search inSPIRE].
[21] T. Kobayashi, H. Ohki, H. Okada, Y. Shimizu, and M. Tanimoto, Lecture Notes in Physics, "An Introduction to Non-Abelian Discrete Symmetries for Particle Physicists, Second Edition" , Springer, Berlin, (2022), Vol. 995, p. 1.
[22] S. Ferrara, D. Lust, and S. Theisen, Phys. Lett. B **233**, 147 (1989).
[23] W. Lerche, D. Lust, and N. P. Warner, Phys. Lett. B **231**, 417 (1989).
[24] J. Lauer, J. Mas, and H. P. Nilles, Nucl. Phys. B **351**, 353 (1991).
[25] T. Kobayashi, S. Nagamoto, S. Takada, S. Tamba, and T. H. Tatsuishi, Phys. Rev. D **97**, 116002 (2018) [arXiv:1804.06644 [hep-th]] [Search inSPIRE].
[26] T. Kobayashi and S. Tamba, Phys. Rev. D **99**, 046001 (2019).
[27] H. Ohki, S. Uemura, and R. Watanabe, Phys. Rev. D **102**, 085008 (2020) [arXiv:2003.04174 [hep-th]] [Search inSPIRE].
[28] S. Kikuchi, T. Kobayashi, S. Takada, T. H. Tatsuishi, and H. Uchida, Phys. Rev. D **102**, 105010 (2020) [arXiv:2005.12642 [hep-th]] [Search inSPIRE].
[29] S. Kikuchi, T. Kobayashi, H. Otsuka, S. Takada, and H. Uchida, J. High Energy Phys. **2011**, 101 (2020) [arXiv:2007.06188 [hep-th]] [Search inSPIRE].
[30] S. Kikuchi, T. Kobayashi, and H. Uchida, Phys. Rev. D **104**, 065008 (2021) [arXiv:2101.00826 [hep-th]] [Search inSPIRE].
[31] Y. Almumin, M. C. Chen, V. Knapp-Perez, S. Ramos-Sanchez, M. Ratz, and S. Shukla, J. High Energy Phys. **2105**, 078 (2021) [arXiv:2102.11286 [hep-th]] [Search inSPIRE].
[32] A. Strominger, Commun. Math. Phys. **133**, 163 (1990).
[33] P. Candelas and X. de la Ossa, Nucl. Phys. B **355**, 455 (1991).
[34] K. Ishiguro, T. Kobayashi, and H. Otsuka, Nucl. Phys. B **973**, 115598 (2021) [arXiv:2010.10782 [hep-th]] [Search inSPIRE].
[35] K. Ishiguro, T. Kobayashi, and H. Otsuka, J. High Energy Phys. **2201**, 020 (2022) [arXiv:2107.00487 [hep-th]] [Search inSPIRE].

[36] R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, Nucl. Phys. B **858**, 437 (2012) [arXiv:1112.1340 [hep-ph]] [Search inSPIRE].
[37] F. Feruglio, in From My Vast Repertoire...: Guido Altarelli's Legacy, eds. S. Forte, A. Levy, and G. Ridolfi. World Scientific Publishing, Singapore (2019), p. 227[arXiv:1706.08749 [hep-ph]] [Search inSPIRE].
[38] T. Kobayashi, K. Tanaka, and T. H. Tatsuishi, Phys. Rev. D **98**, 016004 (2018) [arXiv:1803.10391 [hep-ph]] [Search inSPIRE].
[39] J. T. Penedo and S. T. Petcov, Nucl. Phys. B **939**, 292 (2019) [arXiv:1806.11040 [hep-ph]] [Search inSPIRE].
[40] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, J. High Energy Phys. **1904**, 174 (2019) [arXiv:1812.02158 [hep-ph]] [Search inSPIRE].
[41] G. J. Ding, S. F. King, and X. G. Liu, Phys. Rev. D **100**, 115005 (2019) [arXiv:1903.12588 [hep-ph]] [Search inSPIRE].
[42] J. C. Criado and F. Feruglio, SciPost Phys. **5**, 042 (2018) [arXiv:1807.01125 [hep-ph]] [Search inSPIRE].
[43] T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, J. High Energy Phys. **1811**, 196 (2018) [arXiv:1808.03012 [hep-ph]] [Search inSPIRE].
[44] G. J. Ding, S. F. King, and X. G. Liu, J. High Energy Phys. **1909**, 074 (2019) [arXiv:1907.11714 [hep-ph]] [Search inSPIRE].
[45] H. Okada and M. Tanimoto, J. High Energy Phys. **2103**, 010 (2021) [arXiv:2012.01688 [hep-ph]] [Search inSPIRE].
[46] C. Y. Yao, J. N. Lu, and G. J. Ding, J. High Energy Phys. **2105**, 102 (2021) [arXiv:2012.13390 [hep-ph]] [Search inSPIRE].
[47] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, J. High Energy Phys. **1904**, 005 (2019) [arXiv:1811.04933 [hep-ph]] [Search inSPIRE].
[48] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, J. High Energy Phys. **2002**, 097 (2020) [arXiv:1907.09141 [hep-ph]] [Search inSPIRE].
[49] X. Wang and S. Zhou, J. High Energy Phys. **2005**, 017 (2020) [arXiv:1910.09473 [hep-ph]] [Search inSPIRE].
[50] H. Okada and M. Tanimoto, Phys. Lett. B **791**, 54 (2019) [arXiv:1812.09677 [hep-ph]] [Search inSPIRE].
[51] H. Okada and M. Tanimoto, Eur. Phys. J. C **81**, 52 (2021) [arXiv:1905.13421 [hep-ph]] [Search inSPIRE].
[52] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi, and H. Uchida, Phys. Rev. D **101**, 055046 (2020) [arXiv:1910.11553 [hep-ph]] [Search inSPIRE].
[53] A. Baur, H. P. Nilles, A. Trautner, and P. K. S. Vaudrevange, Phys. Lett. B **795**, 7 (2019) [arXiv:1901.03251 [hep-th]] [Search inSPIRE].
[54] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, J. High Energy Phys. **1907**, 165 (2019) [arXiv:1905.11970 [hep-ph]] [Search inSPIRE].
[55] A. Baur, H. P. Nilles, A. Trautner, and P. K. S. Vaudrevange, Nucl. Phys. B **947** (2019), 114737[arXiv:1908.00805 [hep-th]] [Search inSPIRE].
[56] A. Baur, M. Kade, H. P. Nilles, S. Ramos-Sanchez, and P. K. S. Vaudrevange, J. High Energy Phys. **2102**, 018 (2021) [arXiv:2008.07534 [hep-th]] [Search inSPIRE].
[57] H. P. Nilles, S. Ramos–Sánchez, and P. K. S. Vaudrevange, Nucl. Phys. B **966**, 115367 (2021) [arXiv:2010.13798 [hep-th]] [Search inSPIRE].
[58] K. Bönisch, M. Elmi, A. K. Kashani-Poor, and A. Klemm,arXiv:2204.06506 [hep-th] [Search inSPIRE].
[59] T. Kobayashi, K. Nishiwaki, and Y. Tatsuta, J. High Energy Phys. **1704**, 080 (2017) [arXiv:1609.08608 [hep-th]] [Search inSPIRE].
[60] B. S. Acharya, D. Bailin, A. Love, W. A. Sabra, and S. Thomas, Phys. Lett. B **357**, 387 (1995); **407**, 451 (1997) [erratum] [arXiv:hep-th/9506143] [Search inSPIRE].
[61] T. Dent, Phys. Rev. D **64**, 056005 (2001) [arXiv:hep-ph/0105285] [Search inSPIRE].
[62] S. Khalil, O. Lebedev, and S. Morris, Phys. Rev. D **65**, 115014 (2002) [arXiv:hep-th/0110063] [Search inSPIRE].
[63] J. Giedt, Mod. Phys. Lett. A **17**, 1465 (2002) [arXiv:hep-ph/0204017] [Search inSPIRE].

[64] T. Kobayashi and H. Otsuka, Phys. Rev. D **102**, 026004 (2020) [arXiv:2004.04518 [hep-th]] [Search inSPIRE].

[65] K. Ishiguro, T. Kobayashi, and H. Otsuka, J. High Energy Phys. **2103**, 161 (2021) [arXiv:2011.09154 [hep-ph]] [Search inSPIRE].

[66] H. Abe, T. Kobayashi, S. Uemura, and J. Yamamoto, Phys. Rev. D **102**, 045005 (2020) [arXiv:2003.03512 [hep-th]] [Search inSPIRE].

[67] P. P. Novichkov, J. T. Penedo, and S. T. Petcov, J. High Energy Phys. **2203**, 149 (2022) [arXiv:2201.02020 [hep-ph]] [Search inSPIRE].

[68] K. Ishiguro, H. Okada, and H. Otsuka, arXiv:2206.04313 [hep-ph] [Search inSPIRE].

[69] H. Okada and M. Tanimoto, Phys. Rev. D **103**, 015005 (2021) [arXiv:2009.14242 [hep-ph]] [Search inSPIRE].

[70] S. F. King and Y. L. Zhou, Phys. Rev. D **101**, 015001 (2020) [arXiv:1908.02770 [hep-ph]] [Search inSPIRE].

[71] P. P. Novichkov, J. T. Penedo, and S. T. Petcov, J. High Energy Phys. **2104**, 206 (2021) [arXiv:2102.07488 [hep-ph]] [Search inSPIRE].

[72] F. Feruglio, V. Gherardi, A. Romanino, and A. Titov, J. High Energy Phys. **2105**, 242 (2021) [arXiv:2101.08718 [hep-ph]] [Search inSPIRE].

[73] T. Kobayashi, H. Okada, and Y. Orikasa, arXiv:2111.05674 [hep-ph] [Search inSPIRE].

[74] D. Cremades, L. E. Ibanez, and F. Marchesano, J. High Energy Phys. **0405**, 079 (2004) [arXiv:hep-th/0404229 [hep-th]] [Search inSPIRE].

[75] H. Abe, T. Kobayashi, and H. Ohki, J. High Energy Phys. **0809**, 043 (2008) [arXiv:0806.4748 [hep-th]] [Search inSPIRE].

[76] T. H. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, and M. Sakamoto, J. High Energy Phys. **1401**, 065 (2014) [arXiv:1309.4925 [hep-th]] [Search inSPIRE].

[77] T. h. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, and M. Sakamoto, Nucl. Phys. B **890**, 442 (2014) [arXiv:1409.5421 [hep-th]] [Search inSPIRE].

[78] H. Abe, K. S. Choi, T. Kobayashi, and H. Ohki, Nucl. Phys. B **814**, 265 (2009) [arXiv:0812.3534 [hep-th]] [Search inSPIRE].

[79] T. h. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, M. Sakamoto, and Y. Tatsuta, Nucl. Phys. B **894**, 374 (2015) [arXiv:1501.02787 [hep-ph]] [Search inSPIRE].

[80] H. Abe, T. Kobayashi, H. Ohki, A. Oikawa, and K. Sumita, Nucl. Phys. B **870**, 30 (2013) [arXiv:1211.4317 [hep-ph]] [Search inSPIRE].

[81] H. Abe, T. Kobayashi, K. Sumita, and Y. Tatsuta, Phys. Rev. D **90**, 105006 (2014) [arXiv:1405.5012 [hep-ph]] [Search inSPIRE].

[82] Y. Fujimoto, T. Kobayashi, K. Nishiwaki, M. Sakamoto, and Y. Tatsuta, Phys. Rev. D **94**, 035031 (2016) [arXiv:1605.00140 [hep-ph]] [Search inSPIRE].

[83] S. Kikuchi, T. Kobayashi, Y. Ogawa, and H. Uchida, Prog. Theor. Exp. Phys. **2022**, 033B10 (2022) [arXiv:2112.01680 [hep-ph]] [Search inSPIRE].

[84] S. Kikuchi, T. Kobayashi, H. Otsuka, M. Tanimoto, H. Uchida, and K. Yamamoto, arXiv:2201.04505 [hep-ph] [Search inSPIRE].

[85] H. Abe, T. Kobayashi, K. Sumita, and S. Uemura, Phys. Rev. D **96**, 026019 (2017) [arXiv:1703.03402 [hep-th]] [Search inSPIRE].

[86] S. Kikuchi, T. Kobayashi, K. Nasu, H. Uchida, and S. Uemura,arXiv:2202.05425 [hep-th] [Search inSPIRE].

[87] F. Björkeroth, F. J. de Anda, I. de Medeiros Varzielas, and S. F. King, J. High Energy Phys. **1506**, 141 (2015) [arXiv:1503.03306 [hep-ph]] [Search inSPIRE].

[88] P. A. Zyla et al. [Particle Data Group], Prog. Theor. Exp. Phys. **2020**, 083C01 (2020).