Studies of five dimensional Superconformal Field Theories, Localization and Geometry

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A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2023

To my family.

Abstract

In this thesis, we study the U-plane of rank-one 5d $\mathcal{N}=1$ superconformal field theories (SCFTs), which is the one-dimensional Coulomb branch (CB) of such theories on $\mathbb{R}^4 \times S^1$. This circle compactification gives us 4d $\mathcal{N}=2$ supersymmetric field theories of 'Kaluza-Klein (KK) type', and thus the Coulomb branch physics can be described by its Seiberg-Witten (SW) geometry. The total space of the SW geometry, which consists of the SW curve fibered over the CB, can be modelled as a rational elliptic surface. As such, a classification of all possible Coulomb branch configurations, for the 5d field theories and their 4d descendants, is given by Persson's classification of rational elliptic surfaces. This formalism trivializes renormalization group flows, while also containing information about the global form of the flavour symmetry group and the one-form symmetry, through the Mordell-Weil group of the SW geometry. Moreover, in a number of important instances, the U-plane is a modular curve, and we use its beautiful modular properties to investigate aspects of the low-energy physics, such as the spectrum of light particles at strong coupling and the associated BPS quivers.

A related approach to studying the strong-coupling dynamics of these theories is to consider them on curved backgrounds $\mathcal{M}_5$. Our focus is on closed five-manifolds $\mathcal{M}_5$ which are principal circle bundles over simply-connected Kähler four-manifolds, $\mathcal{M}_4$. Starting with the topologically twisted 4d $\mathcal{N}=2$ KK theory on $\mathcal{M}_4$, we propose a new approach to compute the supersymmetric partition function on $\mathcal{M}_5$ through the insertion of a fibering operator in the 4d theory, which introduces a non-trivial fibration over $\mathcal{M}_4$. We determine the so-called Coulomb branch partition function on any such $\mathcal{M}_5$, which is conjectured to be the holomorphic 'integrand' of the full partition function. We precisely match the low-energy effective field theory approach to explicit one-loop computations, and we discuss the effect of non-perturbative 5d BPS particles in this context. When $\mathcal{M}_4$ is toric, we also reconstruct our CB partition function by appropriately gluing Nekrasov partition functions.

Statement of Originality

This thesis is based on results from the following publications, to which the author contributed substantially:

[1] C. Closset and H. Magureanu - *The U-plane of rank-one 4d* $\mathcal{N}=2$ *KK theories*, *SciPost Phys. 12, 065 (2022)*, arXiv:2107.03509

[2] H. Magureanu - *Seiberg-Witten geometry, modular rational elliptic surfaces and BPS quivers*, *Springer JHEP 05, 163 (2022)*, arXiv:2203.03755

[3] C. Closset and H. Magureanu - *Partition Functions and Fibering Operators on the Coulomb Branch of 5d SCFTs*, *Springer JHEP 01, 035 (2023)*, arXiv:2209.13564

In particular, part I of the thesis is mostly based on [1], with chapter 4 discussing the results of [2]. Part II of the thesis is based on [3]. The author also contributed to the following works, which are referenced in chapter 4:

[4] C. Closset and H. Magureanu - *Reading between the rational sections: Global structures of 4d* $\mathcal{N}=2$ *KK theories*, arXiv:2308.10225

[5] J. Aspman, C. Closset, E. Furrer, H. Magureanu and J. Manschot - *U-plane surgery: Galois covers, holonomy saddles and discrete symmetry gaugings, to appear*

[6] C. Closset, M. Del Zotto, A. Grossutti and H. Magureanu - *On 5d SCFTs and their BPS quivers. Part 2: Mirror 2d* $\mathcal{N}=(2,2)$ *LG models, to appear*

Acknowledgements

First and foremost, I would like to thank my supervisors Cyril Closset and Sakura Schäfer-Nameki for their continued support and for their great guidance throughout the last four years. Cyril, I am forever thankful for the trust that you put in me from the very beginning and for giving me all the freedom that I could ask for to find my own path and develop as a researcher. Sakura, thank you for making me feel part of the group especially during the challenging times in the pandemic, and for all the feedback during the years.

During my DPhil, I had the chance to meet and work with some amazing people. Elias and Johannes, thank you for introducing me to the world of modularity, for the long meetings that occasionally turned into chess viewing parties, and, of course, for the many pints. I would also like to thank my other collaborators Andrea, Jan, and Michele for the many interesting discussions, from which I have learned a great deal.

My time in Oxford has been filled with many joyful events, which were only made possible by the many brilliant people that surrounded me. Let me start by mentioning my fellow students and postdocs of the Mathematical Physics group, which made the Mathematical Institute a great place to work at. I have enjoyed the chess games at the happy hours, the countless jokes and funny stories, the pizza Thursdays, and the football matches. Many thanks to my great office mates as well for acknowledging the non-existent beeping sound. I would also like to thank the Romanian football group for eight years of weekly games through heat and rain, Ioan for being a source of inspiration, Mircea for being my tennis partner, Ben and Liz for being amazing neighbours, and my friends and colleagues from Somerville College for the many hours spent at the pool table. Thank you Henry for motivating me to become a better cook and for making me love cycling. Thank you to my good friends known as 'Salai' for your friendship.

In my path towards completing my DPhil, I have benefited from interactions with many great minds and caring people, who helped me discover my passions. I am grateful to Francesco, Graeme, Steve, and Roman for their patience and for motivating me to pursue this path. As vrea sa le multumesc in mod special si doamnelor Basoc, Iancu, Dobrescu, dar si domnului Sorta pentru increderea si tot sprijinul acordat de-a lungul anilor.

Multe dintre realizarile mele au fost posibile datorita eforturilor parintilor mei din momentul in care m-am decis sa studiez in strainatate. Nu pot sa imi exprim multumirea in cuvinte pentru tot ce ati facut pentru mine si sper ca am reusit sa va fac mandri. Tudor si Irina, ma bucur de fiecare data cand petrecem timp impreuna, iar teza asta este si pentru voi.

Nu in ultimul rand, as vrea sa ii multumesc Andreei, pentru ca a fost alaturi de mine in tot acest timp. Am invatat enorm de la tine si alaturi de tine, si m-am dezvoltat pe plan personal cum nu as fi crezut vreodata. Iti multumesc pentru ca ma impingi inafara zonei mele de confort, pentru spontaneitatea ta, si pentru lungile vacante petrecute impreuna. Multe multumiri, de asemenea, si familiei tale, dar si Mariei.

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Chapter 1

Introduction

Many questions regarding the nature of our universe have been answered over the last century, and many more have emerged due to the development of quantum field theory (QFT). This provides a powerful framework for studying interactions of elementary particles and their fields, the strength of which is controlled by a *coupling constant*. This quantity is what determines the different regimes of a theory. In the weak-coupling regime, perturbation theory leads to satisfying results, but non-perturbative effects remain elusive with this technique. As such, the quest to finding novel approaches to describe the strong-coupling regime of QFTs remains an active area of research.

Within the framework of quantum field theory, one is able to describe three of the four fundamental forces, but gravity is not easily described in this way. One of the best candidates for a theory of quantum gravity is string theory, which can often offer insights into the strong-coupling regimes of certain quantum field theories, upon compactifications and decoupling of gravity. These QFTs typically preserve an additional spacetime symmetry, called supersymmetry, which further constrains the models and leads to more tractable problems.

Supersymmetric quantum field theories with at least eight supercharges are particularly amenable to exact, non-perturbative methods. In the case of 4d $\mathcal{N} = 2$ supersymmetric field theories, in particular, the strong-coupling physics is encoded in the so-called 'Seiberg-Witten (SW) geometry' [7, 8]. Historically, the SW geometry also led to the discovery of the first non-Lagrangian strongly coupled 4d $\mathcal{N} = 2$ superconformal field theories (SCFTs) – the so-called Argyres-Douglas theories – through the analysis of the full parameter space of some of the simplest supersymmetric gauge theories [9, 10].

A large part of the thesis is devoted to such models, namely rank-one theories, where the Coulomb branch (CB) of the vacuum moduli space is one-dimensional, meaning that the low-energy physics is described by a $U(1)$ gauge theory. In these cases, the SW geometry consists of an elliptic fibration over the Coulomb branch, and the effective gauge coupling for the low-energy effective field theory is identified with the modular parameter of the elliptic fiber. These geometries are rational elliptic surfaces (RES), for which a complete classification exists in the mathematical literature [11, 12]. As such, an obvious question that we address is how this rich mathematical formalism translates into the physics of supersymmetric quantum field theories.

The framework of geometric engineering in Type-IIA string theory on local Calabi-Yau threefold singularities [13, 14] provides another perspective on 4d $\mathcal{N} = 2$ theories, which turns out to be the optimal perspective for analysing the relation to rational elliptic surfaces. In this language, the natural supersymmetric field theories are, in fact, not the purely 4d $\mathcal{N} = 2$ models, but the circle compactification of the *five-dimensional* superconformal field theories with the same amount of supersymmetry [15, 16], which is a simple consequence of the Type-IIA/M-theory duality [17, 18]. These Kaluza-Klein (KK) theories can be thus viewed as effective 4d theories, with an infinite number of 'fields' organised in KK towers.

The main upshot of our analysis of the SW geometries is threefold. Firstly, we reach a systematic understanding of the possible Coulomb branch configurations of all rank-one theories, which allows us to find new renormalization group (RG) flows between five-dimensional SCFTs and Argyres-Douglas theories. Additionally, we show how the global form of the flavour symmetry group, as well as the one-form symmetries [19], is encoded in the so-called Mordell-Weil group of the associated rational elliptic surface. Finally, we use modularity to simplify the low-energy effective theory description on the CB and to determine the BPS quivers for all rank-one theories that admit such a quiver description.

Another motivation behind the study of Seiberg-Witten geometry lies within the 'sphere' of supersymmetric localization, which can lead to further insights into the strong-coupling dynamics of supersymmetric QFTs. Since the computation of the partition function of 4d $\mathcal{N} = 2$ theories on S^4 in [20], there has been considerable progress in studying supersymmetric QFTs on curved manifolds. A fruitful approach for constructing these models is topological twisting, which, for 4d $\mathcal{N} = 2$ theories, is implemented by turning on a background for the R-symmetry and identifying it with the spin connection. The topologically twisted theory is known as Donaldson-Witten (DW) theory, due to the connection to the Donaldson invariants of four-manifolds $\mathcal{M}_4$ [21, 22].

Given this second layer of motivation, it is thus natural to consider five-dimensional supersymmetric QFTs on curved backgrounds $\mathcal{M}_5$, and, in particular, to uplift the Donaldson-Witten twist to circle-fibrations over four-manifolds. In the case of trivial fibrations $\mathcal{M}_4 \times S^1$, the problem reduces to studying the resulting KK theory on $\mathcal{M}_4$, whose features are described by its SW geometry, analogously to other 4d $\mathcal{N} = 2$ theories. We then show that the non-trivial fibration is captured by a so-called *fibering operator*, whose insertion in the topologically twisted 4d $\mathcal{N} = 2$ theory on $\mathcal{M}_4$ corresponds to introducing a non-trivial fibration of the circle over $\mathcal{M}_4$. Consequently, correlators on the five-dimensional curved background $\mathcal{M}_5$ become observables in the 4d Donaldson-Witten theory on $\mathcal{M}_4$.

Our work focuses on computing the 'integrand' of the partition function of 5d SCFTs on a special class of five-manifolds $\mathcal{M}_5$ which are principal bundles over Kähler four-manifolds $\mathcal{M}_4$. We will refer to this quantity as the *Coulomb branch partition function*. As such, we use three complementary methods, which ultimately yield the same answer. The simplest approach is a one-loop determinant computation, which, naively, would appear to only capture the 'perturbative' part of the full 5d SCFT partition function. However, as argued by Lockhart and Vafa [23], the partition function factorises into contributions from higher-spin BPS particles, which generalise the free hypermultiplet perturbative contribution. In the second approach, we study the low-energy effective couplings on the Coulomb branch and we define the previously mentioned fibering operator as a background flux insertion for the $U(1)_{\text{KK}}$ symmetry (that is, the conserved momentum along the circle). Thirdly, we construct the five-dimensional partition function as a gluing of $\chi(\mathcal{M}_4)$ distinct Nekrasov partition functions on $\mathbb{C}^2 \times S^1$, an approach which has been discussed extensively in the literature, see *e.g.* [23–28].[1] This approach is only valid for circle fibrations over toric Kähler $\mathcal{M}_4$ but is in perfect agreement with the other computations.

For the rest of this chapter, we give more background details on the areas covered in

[1] The gluing approach is more general and can also be applied beyond the topological twist by gluing 'topological' and 'anti-topological' Nekrasov partition functions, as first discussed by Pestun for S^4 [20] and later generalised in various directions [29–33].

the thesis and elaborate on the new results summarised above.

1.1 Seiberg-Witten geometry

1.1.1 $\mathcal{N} = 2$ supersymmetry

The simplest representation of the 4d $\mathcal{N} = 2$ superalgebra is the vector multiplet, which contains a scalar field ϕ, the gauge connection, as well as their fermionic superpartners. The $\mathcal{N} = 2$ super Yang-Mills (SYM) theory consists of a vector multiplet transforming in the adjoint representation of some gauge group G, which, for the remainder of this section, we will take to be $G = SU(2)$. The Lagrangian of this theory contains the scalar potential:

$$V(\phi) = \frac{1}{2g^2}\mathrm{tr}\left([\phi, \bar{\phi}]^2\right) \ , \tag{1.1}$$

where g is the SYM coupling. Generic vacua of such theories are described by the vanishing of the scalar potential, in which the gauge group is broken to its maximal torus $SU(2) \to U(1)$, while the scalar field ϕ receives a VEV via the Higgs mechanism. Semi-classically, one has:

$$\langle\phi\rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \ , \qquad a \in \mathbb{C} \ . \tag{1.2}$$

The vacuum moduli space (called, in this example, the *Coulomb branch*) is then parametrised by the gauge invariant operator:

$$u = \langle \mathrm{tr}\, \phi^2 \rangle \approx 2a^2 + \dots \ . \tag{1.3}$$

Note, however, that this expression is only valid in the weak-coupling limit, $g^2 \to 0$, where the path integral is dominated by the field configurations minimising the action. At the next level of complexity, one needs to consider loop contributions, but also possibly non-perturbative effects. Let us focus for now on the latter. In the full quantum theory, it is important to consider the topological θ-term, which schematically reads:[2]

$$S_{\text{top}} = i\frac{\theta}{16\pi^2}\int \text{tr}\,(F \wedge F) \ , \tag{1.4}$$

where F is the field strength. This term is locally a total derivative and, thus, does not affect the classical equations of motion. However, that is not to say that such a term is not relevant in the quantum theory, where the Feynman path integral is a summation over all possible field configurations. The field configurations affected by this term are the *instantons*, for which the topological action evaluates to $S_{\text{top}} = -i\theta k$, for some integer $k \in \mathbb{Z}$. Thus, θ behaves as an angle, as 2π shifts do not change the path integral.

As such, in an instanton background, any supersymmetric vector-multiplet configuration is weighted by a factor $e^{-S_{\text{SYM}}-S_{\text{top}}} = e^{2\pi i\tau k}$, where we introduced the holomorphic/complexified gauge coupling τ:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \ . \tag{1.5}$$

In fact, this quantity is rather natural in supersymmetric theories, as it enters the Lagrangian when expressed in superspace coordinates.

Let us now return to the perturbative contributions to the path integral. As mentioned earlier, this analysis relies on an expansion around classical vacua, being valid at weak-coupling, or, equivalently, at high energies. The low-energy effective action of a 4d $\mathcal{N} = 2$ theory on the Coulomb branch is obtained by integrating out the degrees of freedom above some ultraviolet (UV) cut-off scale Λ_{uv}. The dependence on this energy scale can be reabsorbed in a 'renormalization' of the couplings, leading to a one-loop running coupling at energy scale a:

$$\tau(a) = 2\tau_{uv} - \frac{b_0}{2\pi i}\log\frac{a}{\Lambda_{uv}} + \dots \ . \tag{1.6}$$

Here b_0 is the one-loop beta function coefficient, which depends on the gauge group and the matter content.[3] The factor of 2 in front of τ_{uv} is due to the fact that the UV theory

[2] Here we are working in flat-space $\mathbb{R}^4$. The full $\mathcal{N} = 2$ action, as well as curved backgrounds, will be discussed in more detail in chapter 5.

[3] For the pure $SU(2)$ SYM theory, we have $b_0 = 4$, and the theory is asymptotically free.

has gauge group $SU(2)$, while the effective action describes a $U(1)$ gauge theory. It is usually more convenient to introduce the so-called *dynamical scale* $\Lambda^{b_0} = \Lambda_{uv}^{b_0} e^{2\pi i \tau_{uv}}$, which is invariant to all orders in perturbation theory. $\mathcal{N} = 2$ gauge theories are one-loop exact perturbatively, but can still receive non-perturbative corrections. These corrections can be expressed in powers of Λ^{kb}, for some $k \in \mathbb{Z}$, which thus carry a phase $e^{ik\theta_{uv}}$. As such, these corrections correspond to configurations with instanton number k. The exact coefficients of these contributions can be, in principle, determined from a path-integral computation, but such methods are not very tractable for higher instanton numbers. However, Seiberg-Witten geometry provides a much faster approach, as we discuss below.

The $U(1)$ low-energy effective action is highly constrained by the $\mathcal{N} = 2$ supersymmetry. The vacuum moduli space is a Kähler manifold, being fully determined by a holomorphic function called the prepotential $\mathcal{F}(a)$, where, as before, a is the scalar of the abelian vector multiplet on the CB. In terms of the prepotential, the effective gauge coupling and the CB metric are given by [34]:

$$\tau = \frac{\partial^2 \mathcal{F}(a)}{\partial a^2} \ , \qquad ds^2(\mathcal{M}_C) = \operatorname{Im}\tau \ da d\bar{a} \ . \tag{1.7}$$

Note that the physical requirement of unitarity implies that $\operatorname{Im}\tau > 0$. But since this is also a harmonic function, $\operatorname{Im}\tau$ (and, implicitly, the prepotential $\mathcal{F}(a)$) can only be locally defined.

1.1.2 Electromagnetic duality and BPS states

Before discussing the Seiberg-Witten solution, let us mention a very important feature of the low-energy effective $U(1)$ action. This feature is, in fact, already present in the non-supersymmetric abelian Maxwell theory: there, the equations of motion are invariant under the exchange $F \leftrightarrow \star F$, where F is the abelian field strength and $\star$ is the Hodge star operator. This transformation has the effect of exchanging the electric and magnetic fields, and is often referred to as S-duality. As a result, the magnetic and electric charges of particles, $\gamma_i = (m_i, q_i)$, are not preserved under such a duality transformation, but it is their Dirac pairing that remains unchanged:

$$\langle \gamma_i, \gamma_j \rangle \equiv m_i q_j - q_i m_j \ . \tag{1.8}$$

For the low-energy effective theory on the Coulomb branch, it is useful to define the quantity $a_D = \partial\mathcal{F}(a)/\partial a$, such that the metric on the CB becomes [8]:

$$ds^2 = \mathrm{Im}\, da_D\, d\bar{a} = -\frac{i}{2}(da_D d\bar{a} - da\, d\bar{a}_D) \ . \tag{1.9}$$

In this form, the metric is completely symmetric in a and a_D, and, as a result, we could describe the theory using a_D as the local parameter instead of a. This is a strong indication that the theory possesses an intrinsic duality, similar in nature to the above S-duality. For this reason, a_D is usually referred to as the 'dual-photon', offering another description of the same theory, but at a different value of the coupling constant:

$$\tau_D = -\frac{1}{\tau} \ . \tag{1.10}$$

This is an example of a strong-weak coupling duality and allows us to gain further insight into the strong-coupling regime of the theory.

Let us also note that due to the periodicity of the θ angle, the theory also has a symmetry $\tau \to \tau+1$. Note that this transformation is different from the S-transformation, being a true symmetry of the theory. These two transformations are implemented through the action of the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ , \tag{1.11}$$

on τ by linear fractional transformations. These are, in fact, the generators of the $\mathrm{SL}(2,\mathbb{Z})$ group, which is the full duality group of the low-energy effective field theory. Note that the metric (1.9) is also invariant under this group. The $\mathcal{N}=2$ extended supersymmetry algebra also involves a *central charge* $Z \in \mathbb{C}$. The massive representations of this supersymmetry algebra satisfy the so-called Bogomonlyi-Prasad-Sommerfeld (BPS) inequality:

$$M \geq |Z| \ , \tag{1.12}$$

where M is the mass of the multiplet. The multiplets satisfying the BPS bound are *short multiplets* and, thus, they also satisfy this bound once quantum corrections are taken into account. Let us also note that the states becoming massive after the Higgs mechanism (1.2) will also be in short multiplets, as the symmetry-breaking pattern does not generate the additional helicity states for a long multiplet. The mass of the W-bosons generated by the Higgs mechanism is then $M_W = |a|$. This generalizes to states with both electric and magnetic charges (called *dyons*) as follows:

$$Z = m\, a_D + q\, a \ , \tag{1.13}$$

which, more generally, might also include flavour contributions. As such, in order to ensure that the mass of the BPS states remains unchanged under the duality transformations acting as $(a_D, a)^T \to \mathbb{M}(a_D, a)^T$, for some $\mathbb{M} \in \mathrm{SL}(2, \mathbb{Z})$ the charge vectors also need to transform as $(m, q) \to (m, q)\mathbb{M}$. Note that in the quantum theory, the quantities a and a_D should be viewed as functions of the CB parameter u, obtained by inverting (1.3).

1.1.3 The Seiberg-Witten solution

The Seiberg-Witten proposal assigns to each point on the Coulomb branch an elliptic curve. An elliptic curve is a non-singular (projective) cubic curve (over $\mathbb{C}$, in our case), with a marked point, described as

$$y^2 = 4x^3 - g_2(u)x - g_3(u) \ , \tag{1.14}$$

where g_2 and g_3 are functions of the CB parameters, which might also include masses of the matter multiplets, for instance. This form is known as the *Weierstrass form* of an elliptic curve. An elliptic curve over $\mathbb{C}$ is, in fact, equivalent to a complex torus. That is, given a lattice $L = \mathbb{Z} + \tau\mathbb{Z}$, for $\mathrm{Im}\, \tau > 0$, a complex torus is the quotient $\mathbb{C}/L$. In the SW proposal, the complex structure parameter of the elliptic fiber is identified with the complexified gauge coupling τ. Viewing the Coulomb branch as a complex projective plane $\mathbb{P}^1$, by compactifying the point at infinity, the Seiberg-Witten geometry then corresponds to the elliptic fibration:

$$E \longrightarrow \mathcal{S} \longrightarrow \mathbb{P}^1 \cong \{u\} \ . \tag{1.15}$$

We will see, in fact, that this corresponds to a rational elliptic surface $\mathcal{S}$, which have been thoroughly studied in the mathematical literature [11, 12].[4] An elliptic curve is singular whenever its discriminant vanishes:

$$\Delta(u) \equiv g_2(u)^3 - 27 g_3(u)^2 \ . \tag{1.16}$$

As such, points on the Coulomb branch where the discriminant vanishes need special treatment. Coulomb branch singularities are rather ubiquitous, as the low-energy effective field theory description breaks down at the loci where certain BPS states become massless. This is a consequence of the fact that the moduli space metric $\mathrm{Im}(\tau)$ is only locally defined, as already alluded to. The elliptic fibers above these loci are thus singular and can be found as the loci where the discriminant of the curve vanishes. Note that CB singularities can emanate a *Higgs branch*, which is a distinct branch of the vacuum moduli space of the theory.

The singularities can change as one varies the (mass) parameters of a theory, leading to interesting strong-coupling phenomena. Thus, it is crucial to understand what configurations might appear under such changes. The possible types of singular fibers are given by the Kodaira classification, as shown in table 1.1, where we also indicate how the effective gauge coupling τ transforms by some elements of $\mathrm{SL}(2,\mathbb{Z})$ under closed loops around these singularities. We postpone a more detailed discussion of this classification to chapter 2.

Recall, first, that for the $SU(2)$ gauge theory analysed in the previous subsection, in the weak coupling regime $u \to \infty$ where the semi-classical picture holds, the one-loop correction to the effective gauge coupling in (1.6) shows a logarithmic behaviour. As a result, closed 'paths' around this point will pick up a non-trivial monodromy. This can be interpreted as

[4] A rational elliptic surface is the special rank-one case of the complex integrable system of Donagi-Witten [35].

fiber	τ	$\mathrm{ord}(g_2)$	$\mathrm{ord}(g_3)$	$\mathrm{ord}(\Delta)$	$\mathbb{M}_*$	4d physics	$\mathfrak{g}$ flavour
I_k	$i\infty$	0	0	k	T^k	SQED	$\mathfrak{su}(k)$
I_k^*	$i\infty$	2	3	$k+6$	PT^k	$SU(2)$, $N_f = 4+k > 4$	$\mathfrak{so}(2k+8)$
I_0^*	τ_0	≥ 2	≥ 3	6	P	$SU(2)$, $N_f = 4$	$\mathfrak{so}(8)$
II	$e^{\frac{2\pi i}{3}}$	≥ 1	1	2	$(ST)^{-1}$	$\mathrm{AD}[A_1, A_2] = H_0$	-
II^*	$e^{\frac{2\pi i}{3}}$	≥ 4	5	10	ST	MN E_8	$\mathfrak{e}_8$
III	i	1	≥ 2	3	S^{-1}	$\mathrm{AD}[A_1, A_3] = H_1$	$\mathfrak{su}(2)$
III^*	i	3	≥ 5	9	S	MN E_7	$\mathfrak{e}_7$
IV	$e^{\frac{2\pi i}{3}}$	≥ 2	2	4	$(ST)^{-2}$	$\mathrm{AD}[A_1, D_4] = H_2$	$\mathfrak{su}(3)$
IV^*	$e^{\frac{2\pi i}{3}}$	≥ 3	4	8	$(ST)^2$	MN E_6	$\mathfrak{e}_6$

Table 1.1: Kodaira classification of singular fibers and associated 4d low-energy physics. The I_k fibers are also-called 'multiplicative' or 'semi-stable' fibers. (I_0 is the 'stable' generic smooth fiber.) All the other types of fibers are called 'additive' or 'unstable'.

a singularity at 'infinity', F_∞, which is characteristic of each theory. For a gauge theory, for instance, this monodromy depends on the beta function coefficient, b_0. This serves as the starting point of our work: *we identify a theory by its fiber at infinity*, a proposal also considered in [36,37]. As such, we identify the 4d $SU(2)$ gauge theories with N_f fundamental hypermultiplets as follows:

$$\text{4d } SU(2) \oplus N_f \quad : \qquad F_\infty = I^*_{4-N_f} \ . \tag{1.17}$$

This prescription generalizes to theories that do not admit a gauge theory description in the ultraviolet regime. The Coulomb branches of the 4d $\mathcal{N}=2$ rank-one SCFTs contain only one singularity in the bulk due to scale invariance [38–42] and it is thus equivalent to identify them by either this bulk singularity or by the singularity at infinity. One has:

$$\begin{aligned} II\,,\, III \text{ and } IV \;\; \text{SCFTs} \;: \qquad & F_\infty = II^*\,,\, III^* \text{ and } IV^* \text{ respectively}\,, \\ II^*\,,\, III^* \text{ and } IV^* \;\; \text{SCFTs} \;: \qquad & F_\infty = II\,,\, III \text{ and } IV \text{ respectively}\,, \end{aligned} \tag{1.18}$$

while the I_0^* SCFTs have $F_\infty = I_0^*$. These can be deduced from the rational elliptic surfaces with only two singular fibers, which are: (II^*, II), (III^*, III), (IV^*, IV) and (I_0^*, I_0^*). For instance, the $E_{8,7,6}$ Minahan-Nemeschansky SCFTs [43,44] are described by the bottom row in (1.18). Let us note that a choice of F_∞ should be accompanied by a choice of deformation pattern, to fully specify a 4d $\mathcal{N}=2$ SCFT [38–41].

The only choice of F_∞ missing at this stage corresponds to the I_n-type fibers. We will show in chapter 2 that this choice extends the space of theories considered above to the Kaluza-Klein theories obtained from the circle compactification of 5d $\mathcal{N}=1$ SCFTs.

1.2 5d SCFTs on a circle

In this section, we consider five-dimensional gauge theories with $\mathcal{N}=1$ supersymmetry. Such theories are very similar to 4d $\mathcal{N}=2$ theories, with the important distinction that the scalar field in the vector multiplet is now a real field. Moreover, it is not hard to notice that the effective gauge coupling has negative mass dimensions, thus rendering the theory non-renormalizable. As a result, such theories are not well-defined QFTs by themselves, but, rather interestingly, they can be realised as deformations of five-dimensional superconformal field theories. From the gauge theory perspective, however, the existence of a (super)conformal fixed point is highly non-trivial. Thankfully, the string theory embedding of these theories, to be discussed below, allows us to probe the UV behaviour more directly.

1.2.1 Geometric engineering

We are interested in the small family of 10 distinct rank-one 5d SCFTs with flavour symmetry algebra E_n [15, 16], namely:

$$\begin{aligned}
&E_0 = \varnothing \ , && E_2 = \mathfrak{su}(2)\oplus\mathfrak{u}(1) \ , && E_5 = \mathfrak{so}(10) \ , \\
&\widetilde{E}_1 = \mathfrak{u}(1) \ , && E_3 = \mathfrak{su}(3)\oplus\mathfrak{su}(2) \ , && E_n = \mathfrak{e}_n \quad (n=6,7,8) \ . \\
&E_1 = \mathfrak{su}(2) \ , && E_4 = \mathfrak{su}(5) \ , &&
\end{aligned} \tag{1.19}$$

These 5d fixed points are all related to each other by five-dimensional RG flows, starting from the E_8 model and breaking down the flavour symmetry to $E_{n<8}$ by appropriate real-mass deformations [15, 16, 45]. These rank-one 5d SCFTs can be 'geometrically engineered' as the low-energy limit of M-theory on $\mathbb{R}^5\times\mathbf{X}_{E_n}$, where $\mathbf{X}_{E_n}$ is a canonical singularity that admits a crepant resolution with a single exceptional divisor [16, 46]. Let $\mathcal{B}_4$ denote a Fano surface – that is, either a del Pezzo surface or the Hirzebruch surface $\mathbb{F}_0\cong\mathbb{P}^1\times\mathbb{P}^1$. We consider the local Calabi-Yau threefold obtained as the total space of the canonical line

E_n	E_0	E_1	$\widetilde{E}_1$	$E_n\,(n=2,\cdots,8)$
$\mathcal{B}_4$	$\mathbb{P}^2$	$\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$	$\mathbb{F}_1 \cong dP_1 = \mathrm{Bl}_1(\mathbb{P}^2)$	$dP_n = \mathrm{Bl}_n(\mathbb{P}^2) \cong \mathrm{Bl}_{n-1}(\mathbb{F}_0)$

Table 1.2: Correspondence between E_n SCFTs and del Pezzo surfaces. Here, $\mathrm{Bl}_k(\mathcal{B}_4)$ denotes the blow-up of the complex surface $\mathcal{B}_4$ at k generic points. Note that dP_1 can also be viewed as the Hirzebruch surface $\mathbb{F}_1$.

bundle over $\mathcal{B}_4$:

$$\widetilde{\mathbf{X}}_{E_n} \cong \mathrm{Tot}\,(\mathcal{K} \to \mathcal{B}_4)\ . \tag{1.20}$$

By blowing down the zero section, one obtains the canonical singularity $\mathbf{X}_{E_n}$. The correspondence between del Pezzo surfaces and E_n theories is summarized in table 1.2.

The smooth threefold (1.20) provides a crepant resolution of $\mathbf{X}_{E_n}$, which corresponds physically to going onto the extended Coulomb branch (ECB) of the 5d SCFT, by turning on the real Coulomb branch VEV, $\langle\sigma\rangle \neq 0$, as well as n real mass parameters m_i $(i = 1, \cdots, n)$. The n real masses should be understood as VEVs for real scalars in vector multiplets valued in the Cartan subalgebra $\oplus_{i=1}^n \mathfrak{u}(1)$ of E_n. In the M-theory geometric point of view, the full ECB is identified with the extended Kähler cone of $\widetilde{\mathbf{X}}_{E_n}$ [46]. The E_n symmetry at the fixed point arises because of M2-branes wrapping vanishing curves. Indeed, it is a beautiful mathematical fact that the second homology lattice of dP_n can be decomposed as:

$$H_2(\mathcal{B}_4, \mathbb{Z}) \cong \Lambda_{-\mathcal{K}} \oplus E_n^-\ , \tag{1.21}$$

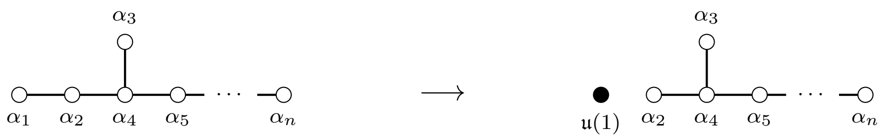
with $\Lambda_{-\mathcal{K}} \cong \mathbb{Z}$ generated by a choice of anticanonical divisor, $-\mathcal{K}$, of $\mathcal{B}_4$ [47]. Here, E_n^- denotes 'minus' the E_n root lattice,[5] which is generated by the curves orthogonal to $-\mathcal{K}$. One can pick a basis of curves, $\mathcal{C}_{\alpha_i}$, which are in one-to-one correspondence with the simple roots α_i of the flavour algebra E_n and intersect according to its Dynkin diagram [1].

The E_n fixed point is also the UV completion of a non-normalizable 5d gauge theory with $\mathcal{N} = 1$ supersymmetry, consisting of an $SU(2)$ vector multiplet coupled to $N_f = n-1$ hypermultiplets[6] with inverse gauge coupling $m_0 = 8\pi^2 g_{5\mathrm{d}}^{-2}$ [15]. This gauge theory

[5] In some appropriate basis, the intersection pairing is minus the Cartan matrix of E_n.

[6] For $n = 1$, we have $SU(2)$ with θ angle 0 or π, corresponding to E_1 or $\widetilde{E}_1$, respectively. The E_0 fixed point does not have a gauge theory interpretation but can be obtained as a deformation of the $\widetilde{E}_1$ theory [16].

description is obtained by a mass deformation of the SCFT that breaks E_n down to $\mathfrak{so}(2n-2) \oplus \mathfrak{u}(1)$:



(1.22)

The $\mathfrak{u}(1)$ factors appearing in the gauge theory description are specific to five-dimensions and correspond to a topological symmetry. That is, the associated current $j = \star\,\mathrm{tr}(F \wedge F)$ is automatically conserved due to the Bianchi identity. The particles charged under this symmetry are the 5d uplift of the 4d instantons and are usually referred to as *instanton particles*. To describe the $SU(2)$, $N_f = n-1$ gauge theory geometrically, one should pick a ruling of the exceptional divisor $\mathcal{B}_4$. This consists of a choice of 'fiber' and 'base' rational curves, $\mathcal{C}_f \cong \mathbb{P}^1$ and $\mathcal{C}_b \cong \mathbb{P}^1$ respectively. For $n = 1$, we have the Hirzebruch surfaces:

$$\mathcal{C}_f \to \mathbb{F}_p \to \mathcal{C}_b \ , \qquad p = 0, 1 \ . \tag{1.23}$$

The trivial ($p = 0$) or non-trivial ($p = 1$) fibration of $\mathcal{C}_f$ over $\mathcal{C}_b$ gives us the $SU(2)_0$ or $SU(2)_\pi$ gauge theory in the limit where the fiber curve collapses to a point;[7] the M2-brane wrapping $\mathcal{C}_f$ gives the $SU(2)$ W-boson, and the M2-brane wrapping $\mathcal{C}_b$ gives the 5d instanton particle. For $n > 1$, we view $\mathcal{B}_4 = dP_n$ as the blow-up of $\mathbb{F}_0$ at $N_f = n-1$ generic points. By a slight abuse of notation, we then denote by $\mathcal{C}_f$, $\mathcal{C}_b$ the same curves pulled back through the blow-down map $dP_n \to \mathbb{F}_0$. The N_f exceptional curves are denoted by E_i, $i = 1, \cdots, n-1$, and the corresponding wrapped M2-branes give us the hypermultiplets.

1.2.2 4d $\mathcal{N} = 2$ KK theories

We will be interested in the 5d E_n SCFTs compactified on a finite-size circle with radius β. This gives us 4d $\mathcal{N} = 2$ supersymmetric theories of Kaluza-Klein type, which we denote by $D_{S^1}E_n$. By the M-theory/Type-IIA duality, we can engineer these theories as the low-

[7] The 5d θ angle is a $\mathbb{Z}_2$ analog of the 4d θ angle. In 4d, this comes from $\pi_3(SU(2)) = \mathbb{Z}$, leading to non-trivial field configurations with integer-valued instanton numbers. Meanwhile, in 5d the relevant homotopy group is $\pi_4(SU(2)) = \mathbb{Z}_2$.

energy limit of Type-IIA string theory on $\mathbb{R}^4 \times \mathbf{X}_{E_n}$. The Coulomb branch physics of $D_{S^1}E_n$ is rather more subtle and interesting. This is due to quantum corrections, which kick in as soon as we compactify on a circle. In the geometric-engineering picture, we have worldsheet instanton corrections in Type-IIA. Equivalently, in M-theory, we have to account for M2-branes wrapping $\mathcal{C} \times S^1$, with $\mathcal{C}$ some curve inside $\widetilde{\mathbf{X}}_{E_n}$.

Note that the 4d $\mathcal{N} = 2$ theory $D_{S^1}E_n$ is a massive theory since we introduced the KK-scale $m_{\text{KK}} = 1/\beta$. For generic values of the parameters, this is an 'abstract' strongly coupled quantum field theory defined by the IIA geometry. In some particular limit on the Kähler parameters, called the geometric engineering limit [14, 48], we recover the 4d $\mathcal{N} = 2$ $SU(2)$ theory with N_f flavours, at least when $N_f \leq 4$, and the Coulomb branch physics is then governed by the celebrated Seiberg-Witten solution [7, 8]. More generally, the 5d gauge theory description remains useful for $m_0 \gg m_{\text{KK}}$ [18, 49].

The CB description of the KK theories is, in fact, very similar to the usual 4d $\mathcal{N} = 2$ picture presented in section 1.1.1, but there are some important differences. First, the real scalar σ in the 5d $\mathcal{N} = 1$ vector multiplet is paired with the $U(1)$ holonomy along the circle direction to form a complex scalar which, by abuse of notation, we will also denote by a. Due to the five-dimensional large gauge transformations along the circle direction, the gauge-invariant parameter becomes instead [1, 49]:

$$U = e^{2\pi i a} + e^{-2\pi i a} + \dots \ , \tag{1.24}$$

which corresponds to the expectation value of a Wilson line in the fundamental representation of $SU(2)$ in five dimensions.[8] We will show in chapter 2 that, for the $D_{S^1}E_n$ theories, the analogue of (1.17) is given by:

$$D_{S^1}E_n \quad (5\text{d}\, SU(2) \oplus N_f = n-1) \ : \qquad F_\infty = I_{9-n} \ . \tag{1.25}$$

The identification (1.25), together with (1.17) and (1.18) exhausts all possibilities for the fiber at infinity.[9] Let us note, however, that the $D_{S^1}E_n$ theories are the five-dimensional

[8] We will often use U for both the 5d and 4d Coulomb branch parameters.

[9] The only remaining possibility is $F_\infty = I_0$, which is a smooth fiber, corresponding to a six-

equivalent of the so-called 'I_1-series' in the classification of 4d $\mathcal{N} = 2$ SCFTs [38–41] – that is, their maximally deformed Coulomb branches involve only I_1 singularities. More general SW geometries are allowed, as classified by [50], which we briefly comment on in chapter 4.

1.2.3 Rational elliptic surfaces

The Seiberg-Witten geometries of 4d $\mathcal{N} = 2$ (KK) theories are modelled by the elliptic fibration (1.15). This construction corresponds to a rational elliptic surface, which have been thoroughly studied in the mathematical literature. Moreover, Persson and Miranda provided a classification of the allowed configurations of singular fibers for such surfaces [11, 12]. Consider a theory $\mathcal{T}_{F_\infty}$ described by the fiber at infinity F_∞. Then, any CB configuration of $\mathcal{T}_{F_\infty}$ corresponds to a RES with a fixed set of singular fibers:

$$U\text{-plane of } \mathcal{T}_{F_\infty} \text{ at fixed } M_F \qquad \longleftrightarrow \qquad \mathcal{S} \text{ with } \{F_\infty \,;\, F_1 \,,\, \cdots \,,\, F_k\} \,. \tag{1.26}$$

As previously mentioned, the singular fibers of $\mathcal{T}_{F_\infty}$ can change as we vary its mass parameters M_F. The classification due to Persson and Miranda allows us to find all such possible changes, by fixing the fiber at infinity F_∞ and then looking at the allowed configurations containing this particular fiber. The existence of fibers of the type (1.18) on the CB can also trigger non-trivial RG flows to known 4d $\mathcal{N} = 2$ SCFTs – this is, in fact, the way that the simplest Argyres-Douglas theories were first determined, by analysing the SW geometries of 4d $SU(2)$ gauge theories [9,10]. Using the formalism of rational elliptic surfaces, we find new RG flows from the $D_{S^1}E_n$ theories to 4d $\mathcal{N} = 2$ SCFTs, some of which have been suggested by the relation between 5d BPS quivers and the gauge/Painlevé correspondence [51–54].

Consider, for now, the theories whose maximally deformed Coulomb branches involve only I_1 singularities. In such cases, the flavour symmetry algebra can be deduced directly from the singular fibers of the SW geometry. This follows from Type-IIB mirror geometry, which can be equivalently described in terms of a single D3-brane probing a collection of 7-branes in an F-theory construction [55–58]. In the F-theory language, the Kodaira singularities on the U-plane are non-compact 7-branes, which therefore give rise to flavour

dimensional theory on a torus.

symmetry algebras of ADE type. Note also that the Kodaira singularity at infinity, F_∞, does not contribute to the infrared (IR) flavour symmetry.

The astute reader might have realized by now that fixing the fiber at infinity F_∞ might not be enough to fully determine a theory. In particular, this is the case for $F_\infty = I_8$, where there are two possibilities, namely the $D_{S^1}E_1$ and $D_{S^1}\widetilde{E}_1$ theories, both having the maximally deformed Coulomb branch $(I_8; 4I_1)$. In such cases, one needs to also specify the *Mordell-Weil group* (MW) of the RES. For an elliptic curve E, this group law is obtained by declaring that three points on E add up to the marked point at infinity if and only if they are collinear. This group law can then be uplifted to elliptic surfaces, where the rational points of the elliptic fibers are in one-to-one correspondence with the rational sections of the RES [59]. The Mordell-Weil theorem then states that this group is a finitely generated abelian group, which we will denote by Φ.

In the Weierstrass model (1.14), all singular fibers are either rational curves with a node or with a cusp. Resolving the singularity of the fiber by blow-ups introduces new exceptional curves, which, ultimately, intersect according to the Dynkin diagram associated with the singular fiber, as shown in table 1.1. We say that a rational section P of the RES – that is, $P = (x(U), y(U))$ satisfying (1.14), with $x(U)$, $y(U)$ rational functions of U – intersects a singular fiber 'trivially' if it does not intersect the singular point of the fiber. We will define $\mathcal{Z}^{[1]} \subset \Phi_{\text{tor}}$ to be the maximal subgroup of torsion sections that intersect 'trivially' all the fibers in the interior of the U-plane, and we will define the abelian group $\mathscr{F}$ to be the cokernel of the inclusion map:

$$0 \to \mathcal{Z}^{[1]} \to \Phi_{\text{tor}} \to \mathscr{F} \to 0\ . \tag{1.27}$$

Then, we claim that:

- $\mathcal{Z}^{[1]}$ gives the one-form symmetry of the 4d field theory [19]. In particular, it does not change as we vary the mass parameters. Incidentally, this distinguishes between the E_1 and $\widetilde{E}_1$ configurations for $F_\infty = I_8$, in which case $\mathcal{Z}^{[1]} \cong \mathbb{Z}_2$ for E_1 while it is trivial for $\widetilde{E}_1$.

- The IR flavour symmetry group G_F, for any given CB configuration $\mathcal{S}$ of $\mathcal{T}_{F_\infty}$, takes the schematic form:

$$G_F \cong \left(U(1)^{\mathrm{rk}(\Phi)} \times \prod_{v\neq\infty} \widetilde{G}_v\right) \Big/ \mathscr{F} \ , \tag{1.28}$$

  where $\widetilde{G}_v$ is the simply-connected group with Lie algebra $\mathfrak{g}_v$, associated to each non-reducible Kodaira fiber F_v in the interior of the U-plane and $\mathrm{rk}(\Phi) \in \mathbb{Z}_{\geq 0}$ is the rank of the MW group.

Note that, here, the flavour symmetry *group* is the group acting faithfully on gauge-invariant states. We will prove these statements in chapter 3.

1.3 BPS quivers and Modularity

Consider one of the 5d theories on S^1, $D_{S^1}\mathcal{T}_{\mathbf{X}}$, determined by its fiber at infinity F_∞, engineered in M-theory compactifications on some non-compact Calabi-Yau (CY) threefold $\mathbf{X}$. As already mentioned, the extended Coloumb branch can be explored by considering a crepant resolution $\widetilde{\mathbf{X}}$ of $\mathbf{X}$, being identified with the extended Kähler cone of $\widetilde{\mathbf{X}}$ [60]. In this section, we discuss some aspects about the BPS quivers of such theories and how modularity can be used to find these quivers.

1.3.1 A brief review of BPS quivers

As already alluded to in the previous section, Coulomb branch singularities are due to BPS states becoming massless. BPS particles $\{\gamma\}$ correspond to D-branes wrapped over holomorphic cycles inside $\widetilde{\mathbf{X}}$ [61, 62], but determining the spectrum of *stable* BPS states for such theories is generally a very difficult problem. Mathematically, the BPS states are objects in the (bounded) derived category of coherent sheaves on $\widetilde{\mathbf{X}}$ [63]. Alternatively, one can introduce the *BPS quiver* $\mathcal{Q}_{\mathbf{X}}$ of $D_{S^1}\mathcal{T}_{\mathbf{X}}$, which contains the same information through its category of 'quiver representations'.

To give a rough intuition of what the BPS quiver represents, consider, for simplicity, a Type-IIB setup, where BPS states arise from D3-branes wrapping special Lagrangian 3-cycles $S^3_{\gamma_i}$ inside a Calabi-Yau threefold. Then, the Dirac pairing between the two states is given by the intersection pairing of the 3-cycles in the CY threefold. When wrapping N_i D3-branes on each $S^3_{\gamma_i}$ cycle, on the remaining ('time') direction of the D3-branes, which is transverse to the CY, we are left with a $\mathcal{N} = 4$ supersymmetric quantum mechanics (SQM) [64]. This SQM is the worldline of the BPS particle formed as a formal linear combination of the basis of 3-cycles of the CY manifold.

The 1d theory formed this way is described by a quiver with gauge group $G = \prod_i U(N_i)$, where each gauge factor arises from open strings stretching from a stack of D3-branes to itself. Moreover, open strings stretching between different stacks give rise to bifundamental matter; for the nodes $U(N_i)$ and $U(N_j)$ in the quiver, the number of bifundamentals is then given by the intersection pairing $S^3_{\gamma_i} \cdot S^3_{\gamma_j}$. Let us also note that loops in the quiver, corresponding to collections of special Lagrangian 3-cycles bounding a holomorphic disk inside the threefold, give rise to superpotential terms in the SQM. These are due to disk instantons in the Type-IIB picture and are generally difficult to compute – see *e.g.* [65] for the case of the mirror to a toric threefold.

The BPS quiver is just the 'abstract' quiver corresponding to this supersymmetric quantum mechanics. That is, given a basis of elementary particles (corresponding to the $S^3_{\gamma_i}$ cycles in the above picture), with magnetic-electric charges $\gamma_i = (m_i, q_i)$, the BPS quiver is determined by assigning a quiver node $(i) \sim \mathcal{E}_{\gamma_i}$ to each light dyon, and a (effective) number n_{ij} of arrows from node (i) to (j) given by the Dirac pairing:

$$n_{ij} = \langle \gamma_i, \gamma_j \rangle = m_i q_j - q_i m_j \ . \tag{1.29}$$

In this picture, the above SQM describes the worldline of a BPS state $\gamma = \sum_i N_i \gamma_i$, where the sum is over all elementary particles, with N_i being the number of D3-branes wrapping each $S^3_{\gamma_i}$ cycle. This construction assumes the existence of a *quiver point*, where the central charges of the elementary particles are almost aligned. BPS quivers and their superpotentials for the theories associated to toric geometries can be derived using brane-tiling techniques [66–68], or exceptional collections [69] – see *e.g.* [51,70–78] for more recent works on these subjects.

We should also mention that the $U(N)$ gauge groups in the SQM picture also admit

Fayet-Iliopoulos (FI) terms. These are controlled by the central charges of the wrapped D3-branes, determined as the periods of the holomorphic 3-form Ω of the CY on the S^3_γ cycles. As such, we can slightly modify the geometry such that the strength of the FI terms is affected while preserving the quiver and superpotential. The FI terms change the D-term equations of motion, changing thus the moduli space of vacua $\mathcal{M}_\gamma$ of the supersymmetric quantum mechanics. Thus, the FI terms are responsible for creating certain *walls of marginal stability* where BPS states can decay – namely, this happens when the Euler number of $\mathcal{M}_\gamma$ changes, and the state $\gamma = \sum_i N_i \gamma_i$ is no longer stable.

1.3.2 Modular Coulomb branches

We propose a different approach for determining BPS quivers using the major simplification of the low-energy dynamics provided by the modularity of the underlying Seiberg-Witten geometry, as we now explain. Given the SW geometry (1.14), we define the J-function and J-invariant of the curve as:

$$J(u) = \frac{1}{1728} j(u) = \frac{g_2(u)^3}{\Delta(u)} , \qquad J(\tau) = \frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2} , \tag{1.30}$$

where, by abuse of notation, we have $u = (U, M_F, \ldots)$. The former is a rational function of the CB parameter, U, while the modular J-invariant, $J(\tau)$, is a function of the complex structure of the curve. Let us note that the zeroes of the Eisenstein series on the canonical fundamental domain of the upper half-plane are at $\tau = \zeta_3$ and $\tau = i$ for E_4 and E_6, respectively, with $\zeta_3 = e^{\frac{2\pi i}{3}}$. These are the *elliptic points* of the $\mathrm{SL}(2, \mathbb{Z})$ group – which are the points with a non-trivial stabilizer – where we find $J(\zeta_3) = 0$ and $J(i) = 1$. The other special point of the canonical fundamental domain for the modular group is the *cusp* at $\tau = i\infty$; cusps are defined as equivalence classes in $\mathbb{Q} \cup \{\infty\}$ under the group action, where $J(\tau)$ diverges.

It is then natural to identify the two functions in (1.30), $J(u) = J(\tau)$, leading to a polynomial equation for $U(\tau)$ [79–81], with coefficients in the field of modular functions of $\Gamma(1) = \mathrm{SL}(2, \mathbb{Z})$, $\mathbb{C}(\Gamma(1))$, for fixed mass parameters. That is, the roots $U = U(\tau)$ will generally involve fractional powers of the $J(\tau)$ function. Occasionally, the splitting field of this polynomial turns out to be $\mathbb{C}(\Gamma)$, for Γ some subgroup of $\mathrm{SL}(2,\mathbb{Z})$. In such cases, the order parameter $U(\tau)$ becomes a modular function for the subgroup Γ, and the Coulomb branch is said to be *modular*.

A modular rational elliptic surface can be constructed by starting with a particular subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ [82]. In this construction, the special points of Γ – that is, the elliptic points and the cusps – become singular fibers in Kodaira's classification. We will construct explicitly this map and, based on the classification of subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ [83–85], we are able to reproduce the classification of the modular rational elliptic surfaces of [86]. However, our approach provides more insight into the modular properties of rational elliptic surfaces.

The modularity condition is usually quite restrictive and there is no a priori reason why a 4d $\mathcal{N}=2$ theory should contain a modular configuration. Recall from section 1.1.1 that the CB singularities induce monodromies $\mathbb{M} \in \mathrm{SL}(2,\mathbb{Z})$ when considering closed loops around them. These monodromies act on the physical periods (a_D, a), and, thus, on the effective gauge coupling τ, being symmetries of the quantum theory. Thus, when the CB is modular, the monodromies generate a subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$. Let us also note that in such cases, the U-plane is isomorphic to a region of the upper half-plane $\mathbb{H}$, called the fundamental domain of Γ. This can be obtained from a list of coset representatives $\{\alpha_i\}$ of Γ in the modular group as:

$$\mathcal{F}_\Gamma = \bigsqcup_{i=1}^{n_\Gamma} \alpha_i \mathcal{F}_0 \ , \tag{1.31}$$

where $\mathcal{F}_0$ is the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ and n_Γ is the index of Γ inside the modular group. Then, given the choice of coset representatives, the monodromies around the special points of Γ can be easily determined, without the need of evaluating the SW periods around these points. These monodromies are directly linked to the electromagnetic charges of the light BPS states, and, thus, we ultimately find a BPS quiver construction from the singular fibers of the SW geometry.

Consider, for instance, an I_1 singular fiber on the CB of a theory $\mathcal{T}$, corresponding to a single light BPS state becoming massless. This singularity can be mapped to a rational number on the boundary of the upper half-plane, $\tau = \frac{q}{m}$, for some $q, m \in \mathbb{Z}$ mutually prime. We will then show that such a singularity is generated by a BPS state with magnetic-electric charges given by:

$$\tau = \frac{q}{m} \in \mathbb{Q} \quad \longleftrightarrow \quad \pm(m, -q) \ . \tag{1.32}$$

Using this identification and assuming the existence of a quiver point, we are then able to construct the BPS quiver of $\mathcal{T}$ from a basis of BPS states.

1.4 Field theories on curved backgrounds

The second part of the thesis will be dedicated to studying five-dimensional SCFTs on curved backgrounds. As such, in this section, we provide more details on topological twisting and partition function computations.

1.4.1 Topological twisting

A general procedure for defining supersymmetric QFTs in curved backgrounds is to first couple the flat space field theory to supergravity, in which the metric is allowed to fluctuate, and then take a *rigid limit* where the metric and its superpartners become fixed background fields [87]. This approach leads to generalised Killing spinor equations, whose solutions are the preserved supercharges on the curved background. Schematically, one has:

$$(\nabla_\mu - iA_\mu)\, \xi = 0 \ , \tag{1.33}$$

where ∇_μ is the covariant derivative on the curved background, while in A_μ we include any background fields, such as the R-symmetry background.

There is no general approach to solving these Killing spinor equations, but topological twisting provides a very elegant alternative. Consider a 4d $\mathcal{N} = 2$ theory with a global symmetry that includes the Euclidean rotation $\text{Spin}(4) \cong SU(2)_l \times SU(2)_r$ and the $SU(2)_R$ R-symmetry. Witten's definition of the topological twist [21] consists of relabelling the spins of fields according to a new 'twisted spin':

$$SU(2)_l \times SU(2)_D \ , \qquad SU(2)_D \equiv \text{diag}(SU(2)_r \times SU(2)_R) \ . \tag{1.34}$$

That is, we couple the fields to the $SU(2)_r$ spin connection according to the way they transform under the $SU(2)_R$ R-symmetry. The twist is then implemented by identifying the $SU(2)_R$ indices with those of the $SU(2)_r$ group. In physical terms, we turn on some $SU(2)_R$ R-symmetry background that precisely cancels the spin connection in (1.33). Such a background can be shown to exist on any Riemannian four-manifold [21].

On general four-manifolds $\mathcal{M}_4$, the topological twist preserves only one supercharge – namely, there is a *topological supercharge* $\bar{\mathcal{Q}}$ that is a scalar with respect to the new rotation group $SU(2)_l \times SU(2)_D$ [21]. Moreover, this charge is nilpotent, *i.e.* $\bar{\mathcal{Q}}^2 = 0$ on all fields. Note that when $\mathcal{M}_4$ is Kähler, only a $U(1)_R \subset SU(2)_R$ background is needed, due to the reduced holonomy of $\mathcal{M}_4$, leading to two conserved supercharges on the curved background. As such, computations on Kähler manifolds become more tractable.

The Donaldson-Witten theory is an example of a *topological quantum field theory* (TQFT). In such models, the action and the operators can depend on the metric of the manifold, but due to the topological symmetry arising from the conserved supercharge, the correlation functions are metric-independent. Additionally, observables are $\bar{\mathcal{Q}}$-invariant operators, while the $\bar{\mathcal{Q}}$-exact operators decouple from the theory since their correlations functions vanish – see *e.g.* [88] for a nice review. For these reasons, the DW theory is usually referred to as a *cohomological* TQFT.

1.4.2 Fibering operator

To further study the strong-coupling behaviour of 5d SCFTs, we are interested in defining these theories on five-manifolds $\mathcal{M}_5$ that can be constructed as circle fibrations over a Kähler four-manifold:

$$S^1 \longrightarrow \mathcal{M}_5 \xrightarrow{\pi} \mathcal{M}_4 \ . \tag{1.35}$$

We initiate a new approach to computing the $\mathcal{M}_5$ supersymmetric partition function, $\mathbf{Z}_{\mathcal{M}_5}$, following a line of ideas which was successfully applied to 3d $\mathcal{N} = 2$ theories on Seifert manifolds [89] – see also [90–94]. For this, we first consider the 5d theory on the trivial fibration $\mathcal{M}_4 \times S^1$, where the presence of the S^1 factor allows us to consider the 4d $\mathcal{N} = 2$ KK theory that one obtains by compactifying the 5d SCFT on a circle [17, 49]. This KK theory can then be defined on $\mathcal{M}_4$ using the usual DW twist.

Observables in the four-dimensional topological quantum field theory (TQFT) obtained by the DW twist have been computed in a direct path integral approach, which is however a famously challenging problem [95–100]. See also [101–111] for more recent progress on the topic, and especially [112] for the case of 4d $\mathcal{N}=2$ KK theories. Our aim is to introduce a so-called *fibering operator*, $\mathscr{F}$, such that the $\mathcal{M}_5$ partition function becomes a particular observable in the DW theory:

$$\mathbf{Z}_{\mathcal{M}_5} = \langle \mathscr{F}_{\mathfrak{p}} \rangle^{\rm DW}_{\mathcal{M}_4 \times S^1} \ , \tag{1.36}$$

where $\mathfrak{p}$ denotes the first Chern class of the principal circle bundle (1.35).

Thanks to topological invariance, one can use the low-energy Seiberg-Witten description of the KK theory to compute the partition function on $\mathcal{M}_4 \times S^1$ and hence, upon insertion of the fibering operator, on $\mathcal{M}_5$. It is particularly useful to consider the theory at any given point on the Coulomb branch, with $\boldsymbol{a}$ denoting the scalars in the low-energy abelian vector multiplets.[10] We will denote by

$$Z_{\mathcal{M}_5}(\boldsymbol{a})_{\mathfrak{m}} \ , \tag{1.37}$$

the 'partition function' on $\mathcal{M}_5$ at a fixed value of $\boldsymbol{a}$, and with some fixed background fluxes $\mathfrak{m}$ for the abelian gauge fields turned on, and refer to this quantity as the *CB partition function*, by a slight abuse of terminology. The partition function $\mathbf{Z}_{\mathcal{M}_5}$ is then found by integrating out the dynamical low-energy vector multiplets and, based on a number of previous results and conjectures in the literature (see *e.g.* [99, 107, 113, 114]), we expect it to be schematically given as:

$$\mathbf{Z}_{\mathcal{M}_5} = \sum_{\mathfrak{m}} \oint_{\mathcal{C}} d\boldsymbol{a}\, Z_{\mathcal{M}_5}(\boldsymbol{a})_{\mathfrak{m}} \ , \tag{1.38}$$

where the precise form of the sum over fluxes and of the integration contour have to be determined. Our work will focus on the CB partition function $Z_{\mathcal{M}_5}$, which already entails a number of subtleties and leads to new and interesting results.

[10] We will also use $\boldsymbol{a}$ to denote mass terms, which arise as background vector multiplets.

1.5 Outline of the thesis

The thesis is organised in two separate parts. In the first part, we discuss the Seiberg-Witten geometry of 4d $\mathcal{N} = 2$ (KK) theories, making extensive use of the mathematical formalism of rational elliptic surfaces. Chapter 2 reviews certain aspects of the geometric engineering of the 5d SCFTs of rank-one, as well as their Seiberg-Witten geometries found using local mirror symmetry. In chapter 3 we introduce the formalism of rational elliptic surfaces and describe the SW geometries of the rank-one theories in this context. Moreover, we show how global aspects of the theories can be determined using this formalism. Chapter 4 discusses modular RES and how these can be used to find BPS quivers.

In the second part of the thesis, we provide three distinct approaches for computing the CB partition function (1.37). In chapter 5, we develop a five-dimensional uplift of the Donaldson-Witten twist to five-manifolds of the type (1.35), and use this to compute the CB partition function using standard supersymmetric localization computations [115]. Then, in chapter 6 we introduce the fibering operator from a low-energy effective field theory perspective, which provides an alternative to the supersymmetric localization computation. Finally, in chapter 7, we study $\mathcal{M}_5$ partition functions as the gluing of Nekrasov instanton partition functions. This approach is only valid for toric $\mathcal{M}_4$ manifolds but provides consistency checks for the previous methods.

Part I

Seiberg-Witten geometry and BPS quivers

Chapter 2

Local mirror symmetry and Seiberg-Witten geometry

In this chapter, we review some aspects of the Seiberg E_n 5d SCFTs [7, 8]. These are of course the simplest 5d SCFTs we could consider – the geometric engineering of general 5d SCFTs has attracted a lot of interest in recent years, see *e.g.* [70, 116–148]. The circle compactification of the E_n theory is described by Type-IIA string theory on the same dP_n singularity, and the local mirror description in Type-IIB gives us the Seiberg-Witten geometry we are interested in. After reviewing some standard facts about families of elliptic curves and Seiberg-Witten geometry, we discuss the E_n Seiberg-Witten curves. We also refer to chapter 1.2.1 for a brief introduction to these models.

2.1 The U-plane: a gauge theory perspective

One can gain some useful intuition about the Coulomb branch physics of $D_{S^1}E_n$ from their 5d gauge-theory description, as discussed in [49]. Firstly and most importantly, the Coulomb branch is a one-complex dimensional variety because the 5d real scalar σ in the abelian vector multiplet for $U(1) \subset SU(2)$ is paired with the $U(1)$ holonomy along the circle. Let us then define the dimensionless scalar:

$$a = i\left(\beta\sigma + iA_5\right) \ , \qquad\qquad A_5 \equiv \frac{1}{2\pi}\int_{S^1} A_M dx^M \ . \tag{2.1}$$

The classical $SU(2)$ Coulomb branch is then of the form $(\mathbb{R} \times S^1)/\mathbb{Z}_2$, which is spanned by $a \in \mathbb{C}$ modulo $a \to -a$ (the $SU(2)$ Weyl group action) and $a \to a + 1$ (the five-dimensional large gauge tranformations along S^1). It will be useful to parameterize the Coulomb branch in a gauge invariant way, as:

$$U = e^{2\pi i a} + e^{-2\pi i a} \ . \tag{2.2}$$

This corresponds to the classical expectation value of a five-dimensional supersymmetric Wilson line in the fundamental representation of $SU(2)$, wrapping the circle:

$$U \equiv \langle W \rangle \ , \qquad\qquad W \equiv \mathrm{Tr}\, \mathrm{P} \exp\left(i \int_{S^1} (A - i\sigma d\psi) \right) \ . \tag{2.3}$$

For each $U(1)_i \subset E_n$ symmetry on the ECB, we similarly introduce the complexified flavour parameters:

$$\nu_i = i\left(\beta m_i^{(F)} + i A_{i,5}^{(F)}\right) \ , \qquad\qquad M_{Fi} \equiv e^{2\pi i \nu_i} \ , \tag{2.4}$$

which include flavour Wilson lines along the circle. In this basis, each ν_i corresponds to an exceptional curve E_i of the dP_n geometry. We will see in section 2.3 that a more appropriate description for the gauge theory phase involves a change of basis to the complexified mass parameters μ_i of the hypermultiplets, which also involves the (complexified) inverse gauge coupling μ_0.

The classical relation (2.2) will be modified by quantum corrections. Let us consider (2.3) as the intrinsic definition of U, valid in the full quantum theory. Recall that the 4d $\mathcal{N} = 2$ low-energy description on the CB is fully determined, in flat space, by the holomorphic prepotential $\mathcal{F}(a)$, with the effective gauge coupling determined by (1.7) at any given point on the Coulomb branch. The challenge is then to write down the low-energy parameter a in terms of the VEV U in (2.3), $a = a(U, M_F)$, where we include a dependence on the flavour parameters M_{Fi}. Then, (1.7) gives us the effective gauge coupling on the CB as a function of U and M_{Fi}.

At large distance on the CB, namely for $U \to \infty$, one can compute the prepotential at the one-loop order similarly to the 4d gauge-theory case, by resumming the KK towers [49].

For $SU(2)$ with N_f flavours, one finds:

$$\begin{aligned}\mathcal{F} &= \mathcal{F}_0 + \frac{2}{(2\pi i)^3}\mathrm{Li}_3\big(e^{4\pi i a}\big) - \frac{1}{(2\pi i)^3}\sum_{i=1}^{n-1}\sum_{\pm}\mathrm{Li}_3\big(e^{2\pi i(\pm a+\mu_i)}\big) \\ &\approx \mathcal{F}_0 + \frac{2}{(2\pi i)^3}\mathrm{Li}_3\left(\frac{1}{U^2}\right) - \frac{1}{(2\pi i)^3}\sum_{i=1}^{n-1}\mathrm{Li}_3\left(\frac{1}{U}\right) ,\end{aligned} \tag{2.5}$$

with $\mathcal{F}_0 = \frac{1}{2}\mu_0 a^2$ a classical contribution, and the trilogarithms arising at one-loop. Here we also assumed $|a| \gg |\mu_i|$ on the second line.

2.2 Monodromies, periods and Seiberg-Witten geometry

The low-energy scalar field a is not a single-valued function of the parameter U. This is already true, in a somewhat trivial way, in the large distance approximation, where we have:

$$a = \frac{1}{2\pi i}\log\left(\frac{1}{U}\right) + \mathcal{O}\left(\frac{1}{U}\right) . \tag{2.6}$$

The presence of a logarithmic branch cut is equivalent to the statement that a and $a+1$ are gauge equivalent. More importantly, the effective gauge coupling itself is not single-valued. As we go around the point at infinity, $U^{-1} = 0$, we have:

$$U^{-1} \to e^{2\pi i}U^{-1} \qquad : \qquad \tau \to \tau + 9 - n \tag{2.7}$$

which follows from (2.5). This gives us a shift of the effective 4d θ-angle by $2\pi b_0$, with b_0 the β-function coefficient of the 5d gauge theory [15]:

$$b_0 = 8 - N_f = 9 - n . \tag{2.8}$$

In the interior of the U-plane, one should then have more singularities, around which the effective gauge coupling τ transforms by some non-trivial elements of $\mathrm{SL}(2,\mathbb{Z})$, exactly like in the case of purely four-dimensional $SU(2)$ theories [7,8]. Such singularities are physically allowed because of the electric-magnetic duality of the 4d $\mathcal{N}=2$ abelian vector multiplet. Let a_D denote the scalar field magnetic dual to a, defined as $a_D = \partial\mathcal{F}/\partial a$, as in (1.9).

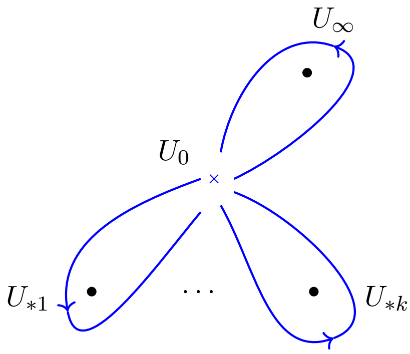


Figure 2.1: Paths γ_v generating the fundamental group of the U-plane. The path around infinity is equal to minus the sum of all the other paths, $\gamma_\infty = -(\gamma_1 + \cdots + \gamma_k)$.

Semi-classically, at large distance on the U-plane, the field a_D describes a BPS monopole. The low-energy effective theory is fully determined by the data of a section (a_D, a) of a rank-two holomorphic affine bundle[11] over the U-plane, with structure group $\mathbb{C}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$, such that the effective gauge coupling is given by:

$$\tau = \frac{\partial a_D}{\partial a} \ . \tag{2.9}$$

The low-energy scalars a and a_D are called the electric and magnetic 'periods', respectively. As we go around any singularity $U = U_*$ on the U-plane (including the point at infinity) in a clockwise fashion, the periods transform as:

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \mathbb{M}_* \begin{pmatrix} a_D \\ a \end{pmatrix} \ , \qquad \mathbb{M}_* \in \mathrm{SL}(2,\mathbb{Z}) \ . \tag{2.10}$$

The $\mathrm{SL}(2,\mathbb{Z})$ matrix $\mathbb{M}_*$ is the so-called monodromy around that point. Let us denote the $k+1$ singularities on the U plane (including the point at infinity, U_∞) by $\widehat{\Delta} = \{U_{*1}, \cdots, U_{*k}, U_\infty\}$, and let:

$$\mathcal{M}_C = \{U\} - \widehat{\Delta} \tag{2.11}$$

be the Coulomb branch with its singular points removed. Given one of our rank-one theories with fixed mass parameters M_{Fi}, the quantum Coulomb branch data is an affine bundle $\boldsymbol{E}$

[11]This is an affine bundle instead of a vector bundle because of the presence of masses, as we will discuss momentarily. For now, let us focus on the $\mathrm{SL}(2,\mathbb{Z})$ part of the structure group.

with $\mathbb{C}^2$ fibers:

$$\mathbb{C}^2 \hookrightarrow \boldsymbol{E} \xrightarrow{\pi} \mathcal{M}_C \ . \tag{2.12}$$

By definition, the monodromy group at some base point U_0 is a representation of the fundamental group $\pi_1(\mathcal{M}_C, U_0)$ on the fiber $\mathbb{C}^2 \cong \pi^{-1}(U_0)$. It is generated by the matrices $\mathbb{M}_{*l}$, for some convenient choice of base point and of paths $\boldsymbol{\gamma}_v$, where each 'vanishing path' goes once along a single singularity as shown in figure 2.1. We then have the obvious constraint:

$$\mathbb{M}_\infty \prod_{l=1}^{k} \mathbb{M}_{*l} = \mathbf{1} \ . \tag{2.13}$$

Part of this work is dedicated to a thorough study of this elementary structure for the $D_{S^1}E_n$ theories. In particular, we would like to give a detailed account of the Coulomb branch singularities, and of the corresponding low-energy physics. Recall that the modular group $\mathrm{SL}(2,\mathbb{Z})$ is generated by the S and T matrices (1.11). Let us also denote by $P = S^2 = -\mathbf{1}$ the generator of the $\mathbb{Z}_2$ center of $\mathrm{SL}(2,\mathbb{Z})$. The monodromy at $U = \infty$ can be computed from (2.5) and (2.6), which gives:

$$a_D \to a_D + (9-n)a + \mu_0 - \sum_{i=1}^{n-1} \mu_i \ , \qquad a \to a + 1 \ . \tag{2.14}$$

We then have the following $\mathrm{SL}(2,\mathbb{Z})$ monodromy at infinity:

$$\mathbb{M}_\infty = T^{9-n} = \begin{pmatrix} 1 & 9-n \\ 0 & 1 \end{pmatrix} \ . \tag{2.15}$$

Note that this is tied to the Witten effect [149]: a shift of the 4d θ-angle as in (2.7) induces an electric charge for the monopole, turning it into a dyon. This monodromy corresponds to a fiber of type I_{9-n} in Kodaira classification, as summarised in table 1.1.

2.2.1 Central charge, massless BPS particles and T^k monodromies

Half-BPS massive particle excitations on the Coulomb branch of $D_{S^1}E_n$ have a mass determined by their electromagnetic and flavour charges:

$$\gamma \in \Gamma \cong \mathbb{Z}^{n+3} , \qquad \gamma \cong (m, q, q_F^i, n_{\mathrm{KK}}) , \tag{2.16}$$

according to the central-charge formula, $m_\gamma = |Z_\gamma|$. The integer-quantized charges consist of the magnetic and electric charges, (m, q), the E_n flavour charges q_F^i, and the KK momentum n_{KK} [71]. Using the KK scale as the unit of mass, let us define the dimensionless central charge $\mathcal{Z} \equiv \beta Z$. At any given point on the extended Coulomb branch, the central charge is a map $\mathcal{Z} : \Gamma \to \mathbb{C}$, which is given explicitly by:

$$\mathcal{Z}_\gamma(U, M) = \mathbf{q}\,\mathbf{\Pi} = m a_D + q a + q_F^i \mu_i + n_{\mathrm{KK}} , \tag{2.17}$$

in terms of the electromagnetic periods. The parameters μ_i and μ_{KK}=1 are 'exact periods', as we will review below. Around any singularity on the U-plane, we have an enlarged monodromy of the form:

$$\mathbf{\Pi} \to \mathbf{M}_* \mathbf{\Pi} , \qquad \mathbf{\Pi} \equiv \begin{pmatrix} a_D \\ a \\ \mu_i \\ 1 \end{pmatrix} , \qquad \mathbf{M}_* = \begin{pmatrix} m_{11} & m_{12} & n_1^i & n_1^0 \\ m_{21} & m_{22} & n_2^i & n_2^0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{2.18}$$

with $m_{11}m_{22} - m_{12}m_{21} = 1$, the upper-left corner of $\mathbf{M}_*$ being the electromagnetic monodromy (2.10). Note that the monodromy can be equivalently understood as acting on the electromagnetic and flavour charges as:

$$\mathbf{q} \to \mathbf{q}\mathbf{M}_* , \qquad \mathbf{q} \equiv (m, q, q_F^i, n_{\mathrm{KK}}) . \tag{2.19}$$

In general, we keep to the 'electric' duality frame dictated by the non-abelian gauge theory limit (or, more precisely, the large volume limit, as we will see below). The local infrared physics is invariant under electric-magnetic duality, however. When analysing the physics at a given point on the U-plane, it can be convenient to change the duality frame. This change of basis leaves the central charge invariant and therefore acts on the charges and periods as:

$$\mathbf{q} \to \mathbf{q}\mathbf{B}^{-1} \,, \qquad \mathbf{\Pi} \to \mathbf{B}\mathbf{\Pi} \,, \tag{2.20}$$

with $\mathbf{B}$ the basis-change matrix.

The simplest type of singularity that can occur in the interior of the U-plane is when a single charged particle becomes massless. In the appropriate duality frame, the low-energy physics at that point is then governed by SQED, namely a $U(1)$ gauge field coupled to a single massless hypermultiplet of charge 1, denoted by $\widetilde{a}$. Let us assume that a dyon of electromagnetic charge (m, q) becomes massless at U_*, with m and q mutually prime, so that $\widetilde{a} = ma_D + qa$. Due to the β-function of SQED, the local monodromy is given by T, in the variables $(\widetilde{a}_D, \widetilde{a})^T = B(a_D, a)^T$, with B a submatrix of $\mathbf{B}$ acting only on the (a_D, a) periods. It thus follows that a massless dyon at U_* induces a monodromy:

$$\mathbb{M}_*^{(m,q)} = B^{-1}TB = \begin{pmatrix} 1+mq & q^2 \\ -m^2 & 1-mq \end{pmatrix} \,. \tag{2.21}$$

Any such singularity with a monodromy conjugate to T is called an I_1 singularity. Similarly, we could have SQED with k electrons (or some hypermultiplets of charges q_j such that $\sum_j q_j^2 = k$), with a monodromy conjugate to T^k, which is an I_k singularity [7, 8]. Other types of singularities are possible, as we will review momentarily.

For the $D_{S^1}E_n$ theory, at generic values of the mass parameters M_F, there are $k = n+3$ singularities of type I_1 in the interior of the U-plane, at each of which a single BPS particle becomes massless. This number of Seiberg-Witten points can be understood in various ways. From the perspective of local mirror symmetry, which we take below, $n+3$ is the number of generators of the third homology of the Type-IIB mirror threefold, which equals the total number of generators of the even homology of $\widetilde{\mathbf{X}}$. From the point of view of the 5d gauge theory, if we admit that the pure 5d $SU(2)$ gauge theory on a circle has 4 CB singularities [49], then adding $N_f = n-1$ massive flavours adds N_f singularities, which are semi-classical in some particular large-mass regime.

2.2.2 Kodaira singularities and low-energy physics

For any rank-one 4d $\mathcal{N}=2$ field theory, the physical problem is to find exact expressions for the electromagnetic periods (a_D, a) such that the CB metric (1.7) is positive definite, and which otherwise match the known asymptotics. The original Seiberg-Witten solution for 4d $\mathcal{N}=2$ $SU(2)$ gauge theories was obtained by realising that, given some physical ansatz for the singularities and monodromies on the Coulomb branch, a positive-definite metric can be elegantly obtained by viewing the low-energy fields (a_D, a) as the periods of a meromorphic one-form, $\lambda_{\rm SW}$, on a family of elliptic curves [7,8].

At fixed mass parameters, we wish to consider a one-parameter family of elliptic curves, which we generally call 'the Seiberg-Witten geometry':

$$\Sigma \to \mathcal{S}_{\rm CB} \to \overline{\mathcal{M}}_C \cong \{U\} \ . \tag{2.22}$$

Here, $\mathcal{S}_{\rm CB}$ denotes a one-parameter family of elliptic curves over the U-plane, including the singularities. At each smooth point $U \in \mathcal{M}_C$ on the Coulomb branch, we have an elliptic curve $\Sigma_U \cong E$. One then identifies $\tau(U)$ with the modular parameter of that curve. The latter is computed as $\tau = \frac{\omega_D}{\omega_a}$, where ω_D and ω_a are the periods of the holomorphic one-form $\boldsymbol{\omega}$ along the A- and B-cycles in Σ_U:

$$\omega_D = \int_{\gamma_B} \boldsymbol{\omega} \ , \qquad \omega_a = \int_{\gamma_A} \boldsymbol{\omega} \ . \tag{2.23}$$

We call these periods the 'geometric periods'. The holomorphic one-form of an elliptic curve is unique up to rescaling. The Seiberg-Witten differential $\lambda_{\rm SW}$ is a meromorphic one-form such that:

$$\frac{d\lambda_{\rm SW}}{dU} = \boldsymbol{\omega} \ , \tag{2.24}$$

modulo an exact 1-form. The 'physical periods' are then defined as:

$$a_D = \int_{\gamma_B} \lambda_{\rm SW} \ , \qquad a = \int_{\gamma_A} \lambda_{\rm SW} \ . \tag{2.25}$$

We then indeed have:

$$\omega_D = \frac{da_D}{dU}\ , \qquad \omega_a = \frac{da}{dU}\ , \qquad\qquad \tau = \frac{\omega_D}{\omega_a} = \frac{\partial a_D}{\partial a}\ . \tag{2.26}$$

The SW curve of $D_{S^1}E_n$, similarly to the case of the massive 4d $SU(2)$ gauge theories [8], can be viewed as a genus-one Riemann surface with (generically) $n+1$ punctures, where the SW differential has simple poles with residues given by the masses (or 'flavour periods') μ_i and $\mu_{\rm KK}$. For our purpose, however, we can mainly bypass an explicit determination of the SW differential. It will often be enough to determine the geometric periods before using (2.26) to determine the electromagnetic periods up to integration constants. The latter will be fixed by matching to known asymptotics.

Kodaira classification and infrared physics. All the SW curves considered in this work can be brought to the standard Weierstrass form (1.14) by a change of coordinates.[12] The possible singularities of the rank-one Seiberg-Witten geometries are captured by the Kodaira classification of singular fibers. The singularity type can be read off from the Weierstrass form of the curve by looking at the order of vanishing at $U = U_*$ of g_2, g_3 and of the discriminant:

$$g_2 \sim (U-U_*)^{\mathrm{ord}(g_2)}\ , \qquad g_3 \sim (U-U_*)^{\mathrm{ord}(g_3)}\ , \qquad \Delta \sim (U-U_*)^{\mathrm{ord}(\Delta)}\ . \tag{2.27}$$

The different types of fibers, in Kodaira's notation, are listed in table 1.1. This gives us a crucial tool to identify the types of singularities in the low-energy physical description, given the Seiberg-Witten geometry [17].

We already discussed in section 2.2.1 that the I_k singularities can be due to k BPS particles of the same charge becoming massless, with the local physics being that of massless

[12]When viewing the SW curve as a compact curve, the Weierstrass equation can be read as the cubic $Y^2 = 4X^3 - g_2 X Z^4 - g_3 Z^6$ with $[X,Y,Z] = \mathbb{P}^2_{[2,3,1]}$. Here we are working on the patch $Z=1$. In fact, even though we call Σ_U 'an elliptic curve', we will remain somewhat agnostic about the precise mathematical definition. In some string-theory geometric engineering scenarios, it appears more natural to view the SW curve as an affine curve in $(\mathbb{C}^*)^2$ instead of a curve in projective space. These subtle differences of perspective will not affect our physical discussion. We used SAGE [150] to find the explicit Weierstrass form of various curves.

SQED with k electrons. This is an IR free theory, consistent with the fact that the effective inverse gauge coupling is $\tau = i\infty$ at that point. Moreover, this theory has a Higgs branch which is isomorphic to the moduli space of one $SU(k)$ instanton.[13] Therefore, there is a 'quantum Higgs branch' emanating from such a point on the U-plane, and, in particular, there is an $\mathfrak{su}(k)$ flavour symmetry associated with this type of singularity.

The second and third type of singularity in table 1.1, called I_k^*, has a monodromy conjugate to PT^k. The low-energy physics is that of a 4d $\mathcal{N} = 2$ $SU(2)$ gauge theory with $N_f = 4 + k$ flavours, which is IR-free for $k > 0$, and conformal for $k = 0$. Its Higgs branch is the moduli space of one $SO(8 + 2k)$ instanton, and the flavour symmetry algebra is $\mathfrak{so}(8 + 2k)$.

The Kodaira singularities of type II, III and IV realise the three 'classic' rank-one Argyres-Douglas theories [9, 10]. These are non-trivial 4d $\mathcal{N} = 2$ SCFTs with fractional scaling dimensions for the Coulomb branch operator ($\frac{6}{5}$, $\frac{4}{3}$ and $\frac{3}{2}$, respectively). The flavour symmetry of the H_0 theory (Kodaira fiber II) is trivial, while the flavour symmetry of the H_1 and H_2 theories is $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$, respectively. The latter two have a Higgs branch which is the moduli space of one $SU(2)$ or $SU(3)$ instanton, respectively.

Finally, the Kodaira singularities of type II^*, III^* and IV^* correspond to the E_n Minahan-Nemeschansky theories [43, 44], for $n = 6, 7, 8$, as indicated in the table. These 4d SCFTs have a Higgs branch isomorphic to the moduli space of one E_n instanton.

Picard-Fuchs equations. Consider a one-parameter family of curves, Σ_U, by setting the various mass parameters to definite values. We consider the Weierstrass form (1.14), with $g_2(U)$ and $g_3(U)$ some polynomials in U, and we would like to determine the geometric periods:

$$\omega = \int_\gamma \boldsymbol{\omega} \, , \qquad \boldsymbol{\omega} = \frac{dx}{y} \, , \tag{2.28}$$

[13] This follows, for instance, from compactification to 3d together with 3d $\mathcal{N} = 4$ mirror symmetry [60].

with γ any one-cycle $\gamma \in H_1(\Sigma_U)$. These periods satisfy a second-order linear differential equation, the Picard-Fuchs equation, which can be expressed in a universal form [151,152]:

$$\frac{d^2\Omega}{dJ^2} + \frac{1}{J}\frac{d\Omega}{dJ} + \frac{31J-4}{144J^2(1-J)^2}\Omega = 0 \ , \qquad \omega(U) = \sqrt{\frac{g_2(U)}{g_3(U)}}\,\Omega(J(U)) \ . \tag{2.29}$$

2.3 Large-volume limit and mirror Calabi-Yau threefold

In the Type-IIA description of the $D_{S^1}E_n$ Coulomb branch, the BPS particles are D-branes wrapping holomorphic cycles inside the local del Pezzo geometry, at least semi-classically. (More generally, they are Π-stable objects in the derived category of coherent sheaves of $\widetilde{\mathbf{X}}_{E_n}$ [153].) The associated 'exact periods' are the 'quantum volumes' of the D0-, D2-, and D4-branes. In the large volume limit, we have:

$$\Pi_{\rm D4} = \int_{\mathcal{B}_4} e^{(B+iJ)}{\rm ch}(L_\varepsilon)\sqrt{\frac{{\rm Td}(T\mathcal{B}_4)}{{\rm Td}(N\mathcal{B}_4)}} + \mathcal{O}(\alpha') \ , \tag{2.30}$$

for the wrapped D4-brane. Here J is the Kähler form, which is complexified by the B-field, and L_ε is a (spin^c) line bundle, which must often be non-trivial [154]. The period of a D2-brane wrapped on any 2-cycle $\mathcal{C}^k \subset \mathcal{B}_4$ is given by the corresponding complexified Kähler parameter:

$$\Pi_{{\rm D2}_{\mathcal{C}^k}} = t_k \equiv \int_{\mathcal{C}^k}(B+iJ) \ . \tag{2.31}$$

We also have $\Pi_{\rm D0} = 1$, the D0-brane being stable at any point on the Kähler moduli space. For $n > 0$, we can view dP_n as the blow up of $\mathbb{F}_0 \cong \mathcal{C}_f \times \mathcal{C}_b$ at $n-1$ points, with exceptional curves E_i, for $i = 1, \cdots, n-1$. We then choose a basis of Kähler parameters:

$$t_f = \int_{\mathcal{C}_f}(B+iJ) \ , \qquad t_b = \int_{\mathcal{C}_b}(B+iJ) \ , \qquad t_{E_i} = \int_{E_i}(B+iJ) \ , \quad i = 1, \cdots, n-1 \ . \tag{2.32}$$

Note that these parameters are only defined up to shifts by integers, due to large gauge transformations of the B-field. For any curve $\mathcal{C}$, we also define the single-valued parameter:

$$Q_{\mathcal{C}} \equiv e^{2\pi i t_{\mathcal{C}}} \ . \tag{2.33}$$

Thus, the large Kähler volume limit for any effective curve $\mathcal{C}$ is equivalent to $Q_{\mathcal{C}} \to 0$. Let $\{\mathcal{C}^k\}$ be some basis of $H_2(\mathcal{B}_4, \mathbb{Z})$, with the intersection pairing:

$$\mathcal{C}^k \cdot \mathcal{C}^l = \mathcal{I}^{kl} \ . \tag{2.34}$$

We also choose the worldvolume flux on the D4-brane to be:

$$\frac{1}{2\pi} \int_{\mathcal{C}^k} F = \varepsilon_k \ . \tag{2.35}$$

These fluxes must generally be non-zero and half-integer, due to the Freed-Witten anomaly cancellation condition [154]:

$$c_1(L_\varepsilon) + \frac{1}{2} c_1(\mathcal{K}) \in H^2(\mathcal{B}_4, \mathbb{Z}) \ . \tag{2.36}$$

On the other hand, any integer-quantized flux on the D4-brane could be set to zero by a large gauge transformation of the B-field. The latter transformation corresponds to a large-volume monodromy. We then have:

$$\Pi_{\mathrm{D4}} = \frac{1}{2} \sum_{k,l} (t_k + \varepsilon_k) \mathcal{I}^{kl} (t_l + \varepsilon_l) + \frac{\chi(\mathcal{B}_4)}{24} + \mathcal{O}(\alpha') \ . \tag{2.37}$$

Note that the parameters ε_k just amount to shifting t_k by some half-integers. For the IIA geometries that are obtained by blowing up $\mathbb{F}_0$,[14] it will be convenient to choose another basis of Kähler parameters, denoted by a, μ_0 and μ_i $(i = 1, \cdots, n-1)$, with:

$$t_f = 2a \ , \qquad t_b = 2a + \mu_0 \ , \qquad t_{E_i} = a + \mu_i \ . \tag{2.38}$$

The parameter a is the low-energy photon in the 'electric' frame. In the $SU(2)$ gauge-theory limit, the D2-brane wrapped on $\mathcal{C}_f$ is identified with the W-boson, and the factor of 2 in (2.38) corresponds to the '$SU(2)$' normalisation of the electric charge such that it has charge 2; similarly, the other identifications in (2.38) corresponds to the electric and

[14] Thus, in all cases except for E_0 and $\widetilde{E}_1$, which we can treat separately.

flavour charges of the other D2 particles, *i.e.* five-dimensional instanton particles and flavour hypermultiplets. Note that the parameters μ_0, μ_i are pure flavour parameters, in that the corresponding (non-effective) curves $\mathcal{C}_\mu$ have vanishing intersection with the compact four-cycle $\mathcal{B}_4 \subset \widetilde{\mathbf{X}}$. Consequently, they lie along the E_n^- lattice in (1.21). From (2.37), we then find:

$$\Pi_{\rm D4} = 2a(2a+\mu_0) - \frac{1}{2}\sum_{i=1}^{n-1}(a+\mu_i+\varepsilon_i)^2 + \frac{n+3}{24} + \mathcal{O}(Q) \ , \tag{2.39}$$

where we chose $\varepsilon_f = \varepsilon_b = 0$. Once we identify the W-boson as coming from a D2-brane wrapping $\mathcal{C}_f$ (and, more generally, the 'electric' particles as being the wrapped D2-branes), then the wrapped D4-brane is identified with the magnetic monopole. We then have:

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = \Pi_{\rm D4} \ , \tag{2.40}$$

and the large volume result (2.39) then corresponds to a prepotential:

$$\mathcal{F} = \left(\mu_0 - \frac{1}{2}\sum_{i=1}^{n-1}\widetilde{\mu}_i\right)a^2 + \frac{9-n}{6}a^3 + \left(\frac{n+3}{24} - \frac{1}{2}\sum_{i=1}^{n-1}\widetilde{\mu}_i^2\right)a + \mathcal{O}(Q) \ , \tag{2.41}$$

where we defined the shifted masses $\widetilde{\mu}_i \equiv \mu_i + \varepsilon_i$. This should be compared to the 5d prepotential for $SU(2)$ with $N_f = n-1$, which reads [46]:

$$\mathcal{F}_{\rm 5d} = m_0\sigma^2 + \frac{4}{3}\sigma^3 - \frac{1}{6}\sum_{i=1}^{n-1}\sum_{\pm}\Theta(\pm\sigma+m_i)(\pm\sigma+m_i)^3 \ , \tag{2.42}$$

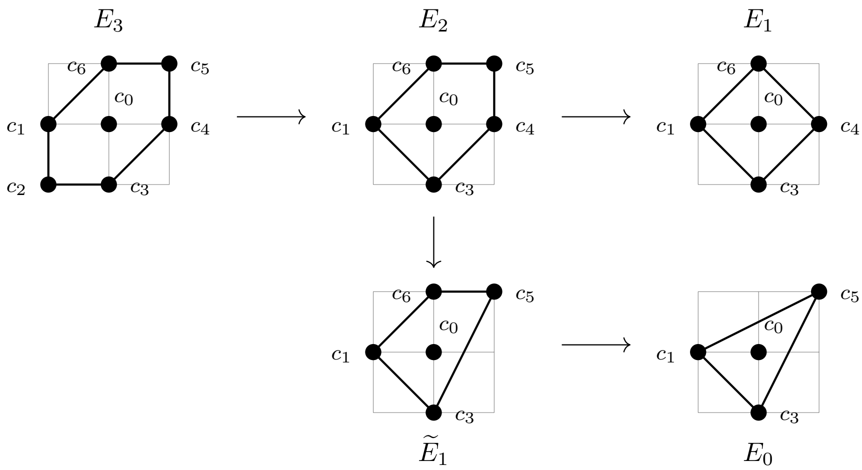
in the conventions of [70]. We indeed recover the 5d prepotential from (2.41), in the appropriate limit and in a specific Kähler chamber:

$$\mathcal{F}_{\rm 5d} = \lim_{\beta\to\infty}\frac{i}{\beta^3}\mathcal{F} \ , \qquad \sigma > |m_j| \ , \quad j = 0, \cdots, n-1 \ , \tag{2.43}$$

using the fact that ${\rm Im}(a) = \beta\sigma$ and ${\rm Im}(\mu_j) = \beta m_j$.

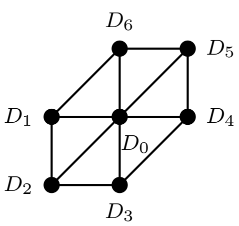
2.3.1 Local mirror symmetry for the toric models

For $n \leq 3$, the E_n singularity in Type-IIA is also a toric Calabi-Yau threefold. The corresponding toric diagrams are:



$$(2.44)$$

Here, the arrows denote the possible partial resolutions of the singularities, which correspond to massive deformations of the 5d SCFTs. Let us then consider the E_3 singularity first, since the other toric singularities can be obtained from it by this partial resolution process. The internal point in the toric diagram, indicated by c_0 in (2.44), corresponds to the compact divisor $D_0 \cong \mathcal{B}_4 = dP_3$. Associated to each external point, indicated by c_i, $i = 1, \cdots, 6$, we have a non-compact toric divisor D_i of the threefold, which intersects the compact divisor along curves $\mathcal{C}_i$ inside the resolved singularity, $\mathcal{C}_i \cong D_0 \cdot D_i$, The intersection numbers between toric divisors and curves are captured by the following table, which is equivalent to the data of a gauged linear sigma-model (GLSM) [155]:



	D_1	D_2	D_3	D_4	D_5	D_6	D_0	FI
$\mathcal{C}_1$	-1	1	0	0	0	1	-1	ξ_1
$\mathcal{C}_2$	1	-1	1	0	0	0	-1	ξ_2
$\mathcal{C}_3$	0	1	-1	1	0	0	-1	ξ_3
$\mathcal{C}_5$	0	0	0	1	-1	1	-1	ξ_5
$\mathcal{C}_4$	0	0	1	-1	1	0	-1	ξ_4
$\mathcal{C}_6$	1	0	0	0	1	-1	-1	ξ_6

$$(2.45)$$

Note the linear equivalences $\mathcal{C}_4 \cong \mathcal{C}_1 + \mathcal{C}_2 - \mathcal{C}_5$ and $\mathcal{C}_6 \cong \mathcal{C}_2 + \mathcal{C}_3 - \mathcal{C}_5$. The triangulated toric diagram shown in (2.45) corresponds to the smooth local dP_3 geometry. The real parameters ξ_i are the Kähler volumes of the curves $\mathcal{C}_i$ in the local threefold – they are the 'FI parameters' in the GLSM language. The Kähler cone is spanned by $(\xi_1, \xi_2, \xi_3, \xi_5) \in \mathbb{R}^4$ satisfying:

$$\xi_1 \geq 0 \; , \quad \xi_2 \geq 0 \; , \quad \xi_3 \geq 0 \; , \quad \xi_5 \geq 0 \; , \quad \xi_1 + \xi_2 - \xi_5 \geq 0 \; , \quad \xi_2 + \xi_3 - \xi_5 \geq 0 \; . \tag{2.46}$$

Other phases can be obtained by successive flops, therefore moving onto the extended Kähler cone of the singularity, which maps out the full extended Coulomb branch of the 5d SCFT E_3 [70]. Viewing dP_3 as the blow-up of $\mathbb{F}_0$ at two points, we have the natural basis of curves discussed in subsection 1.2.1: $\mathcal{C}_f$ and $\mathcal{C}_b$ are the 'fiber' and 'base' curves, respectively, and E_1 and E_2 are the two exceptional curves. This basis is related to the curves shown in (2.45) by:

$$\mathcal{C}_f = \mathcal{C}_1 + \mathcal{C}_2 \; , \qquad \mathcal{C}_b = \mathcal{C}_2 + \mathcal{C}_3 \; , \qquad E_1 = \mathcal{C}_5 \; , \qquad E_2 = \mathcal{C}_2 \; . \tag{2.47}$$

In the 5d $SU(2)$, $N_f = 2$ gauge-theory description, the M2-branes wrapped over $\mathcal{C}_f$ and $\mathcal{C}_b$ give us the W-boson and the instanton particle, respectively, while the M2-branes wrapped over E_1 or E_2 give rise to hypermultiplets.[15] The fixed point of the theory has an enhanced $E_3 = \mathfrak{su}(3) \oplus \mathfrak{su}(2)$ symmetry. The simple roots of E_3 are in one-to-one correspondence with the curves:

$$\mathcal{C}_{\alpha_1} = \mathcal{C}_b - \mathcal{C}_f \; , \qquad \mathcal{C}_{\alpha_2} = \mathcal{C}_f - E_1 - E_2 \; , \qquad \mathcal{C}_{\alpha_3} = E_1 - E_2 \; , \tag{2.48}$$

which intersect inside dP_3 according to the E_3 Cartan matrix. :

$$\mathcal{C}_{\alpha_i} \cdot \mathcal{C}_{\alpha_j} = -A_{ij} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \; . \tag{2.49}$$

[15] We are following the analysis of [70], where the gauge theory description is read off from a 'vertical reduction' of the toric diagram.

Note that, using the 5d gauge-theory parameters (2.38), the Kähler parameters associated to the roots are:

$$t_{\alpha_1} = \mu_0 \ , \qquad t_{\alpha_2} = -\mu_1 - \mu_2 \ , \qquad t_{\alpha_3} = \mu_1 - \mu_2 \ . \tag{2.50}$$

We refer the reader to [1] for more details on the 5d gauge-theory parameterisation.

Local mirror description. Let us now consider the mirror description of the extended Coulomb branch, as the complex structure deformations of the mirror threefold in IIB:

$$D_{S^1} E_n \quad \longleftrightarrow \quad \text{IIA on } \mathbb{R}^4 \times \widetilde{\mathbf{X}} \quad \longleftrightarrow \quad \text{IIB on } \mathbb{R}^4 \times \widehat{\mathbf{Y}} \tag{2.51}$$

For any toric singularity, the local mirror threefold, $\widehat{\mathbf{Y}}$, is given by a hypersurface in $\mathbb{C}^2 \times (\mathbb{C}^*)^2$, with equation [156]:

$$\widehat{\mathbf{Y}} = \left\{ v_1 v_2 + F(t,w) = 0 \,\middle|\, (v_1, v_2) \in \mathbb{C}^2 \ , \ \ (t,w) \in (\mathbb{C}^*)^2 \right\} \ . \tag{2.52}$$

Here, $F(t,w)$ is the Newton polynomial associated with the toric diagram, which takes the general form:

$$F(t,w) = \sum_{m \in \Gamma_0} c_m t^{x_m} w^{y_m} \ , \tag{2.53}$$

where the sum runs over all the points in the toric diagram $\Gamma_0 \subset \mathbb{Z}^2$, with coordinates (x_m, y_m). The coefficients c_m are the complex structure parameters of the mirror, modulo the gauge equivalences:

$$F(t,w) \sim s_0 F(s_1 t, s_2 w) \ , \qquad (s_0, s_1, s_2) \in (\mathbb{C}^*)^3 \ . \tag{2.54}$$

Let us associate to each effective curve $\mathcal{C} \subset \widetilde{\mathbf{X}}$ a complexified Kähler parameter $Q_{\mathcal{C}} = e^{2\pi i t_{\mathcal{C}}}$ as in (2.33). Given a GLSM description of $\widetilde{\mathbf{X}}$, as in (2.45), the mirror parameter $z_{\mathcal{C}}$ is given by:

$$z_{\mathcal{C}} = \prod_{m \in \Gamma_0} (c_m)^{q^m} \ , \qquad q^m \equiv \mathcal{C} \cdot D_m \ . \tag{2.55}$$

Here, D_m is the toric divisor associated to the point $m \in \Gamma_0$. This is normalized such that we have $z_\mathcal{C} \approx Q_f$ in the large volume limit, or equivalently:

$$t_f = \frac{1}{2\pi i} \log(z_f) + \mathcal{O}(z) \ . \tag{2.56}$$

The hypersurface (2.52) is a so-called suspension of the affine curve:

$$\Sigma = \{F(t, w) = 0\} \subset (\mathbb{C}^*)^2 \ , \tag{2.57}$$

and we may focus on the latter. One may view the threefold $\widehat{\mathbf{Y}}$ as a double fibration of Σ and $\mathbb{C}^*$ over some complex plane $\{W\} \cong \mathbb{C}$, as we will review in section 3.1.2. The BPS particles arise from D3-branes wrapping supersymmetric 3-cycles which can be constructed explicitly in a standard way [58, 157]. The exact periods are then given by the classical periods of the holomorphic 3-form on $\widehat{\mathbf{Y}}$, which can be reduced to a line integral along a one-cycle $\gamma = S^3_\gamma \cap \Sigma$ on the curve Σ:

$$\Pi_\gamma = \int_{S^3_\gamma} \Omega = \int_\gamma \lambda_{\rm SW} \ . \tag{2.58}$$

From these considerations, one finds the following Seiberg-Witten differential:

$$\lambda_{\rm SW} = \log t \, \frac{dw}{w} \ , \tag{2.59}$$

up to an overall numerical constant.

The E_3 curve. The mirror curve for the local dP_3 geometry is given by:

$$F_{dP_3}(w, t) = \frac{1}{t}\left(c_1 + \frac{c_2}{w}\right) + \frac{c_3}{w} + c_6 w + c_0 + t\,(c_4 + c_5 w) \ . \tag{2.60}$$

We denote by:

$$z_f = \frac{c_3 c_6}{c_0^2} \ , \qquad z_b = \frac{c_1 c_4}{c_0^2} \ , \qquad z_{E_1} = \frac{c_4 c_6}{c_5 c_0} \ , \qquad z_{E_2} = \frac{c_1 c_3}{c_2 c_0} \ , \tag{2.61}$$

the complex-structure parameters mirror to the Kähler volume of the curves (2.47). We find it useful to introduce the parameters U, λ, M_1 and M_2 such that:

$$z_f = \frac{1}{U^2} \;, \qquad z_b = \frac{\lambda}{U^2} \;, \qquad z_{E_1} = -\frac{1}{UM_1} \;, \qquad z_{E_2} = -\frac{1}{UM_2} \;. \tag{2.62}$$

Using the gauge freedom (2.54), we may set $c_3 = c_6 = 1$, $c_1 = c_4$, and choose $c_0 = -U$, so that the E_3 Seiberg-Witten curve reads:

$$E_3 \;: \qquad \frac{\sqrt{\lambda}}{t}\left(1 + \frac{M_2}{w}\right) + \frac{1}{w} + w - U + t\sqrt{\lambda}\,(1 + M_1 w) = 0 \;. \tag{2.63}$$

The CB parameter U is chosen such that

$$U \approx \frac{1}{\sqrt{Q_f}} = e^{-2\pi i a} \;, \tag{2.64}$$

at large volume, while the mass parameters λ, M_1, M_2 are related to the 5d gauge parameters as by the mirror map:

$$\lambda = \frac{Q_b}{Q_f} = e^{2\pi i m_0} \;, \qquad M_i = -\frac{\sqrt{Q_f}}{Q_{E_i}} = e^{-2\pi i \widetilde{\mu}_i} = -e^{-2\pi i \mu_i} \;, \quad i = 1, 2 \;, \tag{2.65}$$

setting $\varepsilon_i = \frac{1}{2}$ (mod 1) for the exceptional 2-cycles E_i in $\mathcal{B}_4 \cong \mathrm{Bl}_{n-1}(\mathbb{F}_0)$. Here, λ corresponds to the 5d gauge coupling, and M_1, M_2 correspond to the two hypermultiplet masses. These 'flavour' complex-structure parameters, which we will often call 'the masses' by a slight abuse of terminology, are such that the massless limit corresponds to $\lambda = M_i = 1$. Unlike the relation (2.64) between U and a, which is corrected by worldsheet instantons from the IIA point of view, the large-volume relations (2.65) are exact in α', as is the case for any Kähler parameter $t_{\mathcal{C}}$ in $\widetilde{\mathbf{X}}$ Poincaré dual to a non-compact divisor.

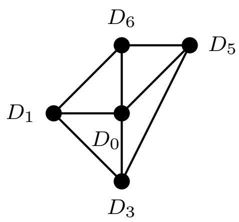
The other toric curves. Let us consider the successive 5d mass deformations shown in (2.44), to obtain the curves for the other toric E_n singularities. To obtain the E_2 geometry, we need to flop the curve $\mathcal{C}_2 \subset dP_3$ and take it to large negative volume. This corresponds to the limit of large negative 5d mass, $m_2 \to -\infty$, which is the limit $M_2 \to 0$. This is equivalent to setting $c_2 = 0$ in (2.60). We then obtain the curve:

$$E_2 \ : \qquad \frac{\sqrt{\lambda}}{t} + \frac{1}{w} + w - U + t\sqrt{\lambda}\,(1 + M_1 w) = 0 \ , \tag{2.66}$$

The 5d gauge-theory phase is $SU(2)$ with $N_f = 1$. From the E_2 theory in its gauge-theory phase, we can integrate out the hypermultiplet with either $m_1 \to -\infty$ or $m_1 \to \infty$ in 5d, which gives us the $SU(2)_0$ or the $SU(2)_\pi$ 5d gauge theory, respectively. These limits correspond to $M_1 \to 0$ and $M_1 \to \infty$, respectively. It is thus straightforward to find the E_1 curve:

$$E_1 \ : \qquad \sqrt{\lambda}\left(\frac{1}{t} + t\right) + \frac{1}{w} + w - U = 0 \ . \tag{2.67}$$

The $\widetilde{E}_1$ case is distinct from the all other E_n with $n > 0$, since $dP_1 \cong \mathbb{F}_1$ is not a blow-up of $\mathbb{F}_0$. That is, our basis choice is more suitable for blow-ups of $\mathbb{F}_0$. The GLSM description reads:



	D_1	D_3	D_5	D_6	D_0	FI
$\mathcal{C}_f = \mathcal{C}_1 \cong \mathcal{C}_5$	0	1	0	1	-2	ξ_1
$\mathcal{C}_b = \mathcal{C}_3$	1	1	1	0	-3	ξ_3
$\mathcal{C}_6 \cong \mathcal{C}_3 - \mathcal{C}_5$	1	0	1	-1	-1	ξ_5

(2.68)

Let us note that the instanton particle, which is the D2-brane wrapping $\mathcal{C}_b$, has electromagnetic charge $(m, q) = (0, 3)$, since $D_0 \cdot \mathcal{C}_b = -3$. We then have the identification:

$$t_f = 2a \ , \qquad t_b = 3a + \mu_0 \ , \tag{2.69}$$

which is distinct from (2.38). Starting from the E_2 curve, we first use the gauge freedom (2.54), to rescale $t \to t/\sqrt{M_1}$ and then redefine $\lambda \to \lambda/\sqrt{M_1}$. Then, in the $M \to \infty$ limit, we have:

$$\widetilde{E}_1 \ : \qquad \sqrt{\lambda}\left(\frac{1}{t} + tw\right) + \frac{1}{w} + w - U = 0 \ . \tag{2.70}$$

Finally, we can take the limit from $\widetilde{E}_1$ to E_0, which corresponds to a 'negative 5d gauge coupling', $\lambda \to \infty$. We should first perform a gauge transformation (2.54) with $(s_0, s_1, s_2) =$

$(\lambda^{-\frac{1}{3}}, \lambda^{-\frac{1}{3}}, \lambda^{\frac{1}{6}})$, rescale $U \to 3\lambda^{\frac{1}{3}} U$ (the factor 3 being there for future convenience), and then take the limit $\lambda \to \infty$. One then obtains:

$$E_0 \; : \qquad \frac{1}{t} + \frac{1}{w} + tw - 3U = 0 \; . \tag{2.71}$$

Geometric-engineering limit. It is also interesting to consider the four-dimensional 'geometric-engineering' limit of the E_3 curve (2.63), given by the small-β limit. We pick:

$$w = e^{-2\pi\beta x} \; , \tag{2.72}$$

for the coordinate w, as well as:

$$\lambda = \left(2\pi i \beta \Lambda_{(2)}\right)^2 \; , \qquad M_1 = -e^{2\pi\beta m_1} \; , \qquad M_2 = -e^{-2\pi\beta m_2} \; , \tag{2.73}$$

for the mass parameters,[16] keeping $\Lambda_{(2)}$ fixed. This scale is identified with the dynamical scale of 4d $\mathcal{N} = 2$ $SU(2)$ with two flavours. Recall, that, for $SU(2)$ with N_f flavours, we have:

$$\Lambda_{(N_f)}^{b_0} = \mu^{b_0} e^{2\pi i \tau(\mu)} \; , \qquad b_0 = 4 - N_f \; , \qquad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{4\text{d}}^2} \; . \tag{2.74}$$

We identify the 5d and 4d gauge couplings at the threshold scale $\mu \sim \frac{1}{\beta}$, according to $\beta m_0 \propto \frac{1}{g_{4\text{d}}^2}$. The 5d U-parameter and the 4d u-parameter can be matched as:

$$U = 2 + 4\pi^2 u \beta^2 + O(\beta^3) \; , \qquad u = \langle \text{Tr}(\Phi^2) \rangle \approx -a^2 \; , \qquad \Phi = -\frac{i}{\sqrt{2}} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \; . \tag{2.75}$$

We then obtain the 4d curve:

$$\frac{\Lambda(x + m_1)}{t} + \Lambda t (x + m_2) + x^2 - u = 0 \; , \tag{2.76}$$

with the replacement $t \to -it$ done for convenience. Due to the change of coordinate (2.72), the 4d curve is now a curve in $\mathbb{C} \times \mathbb{C}^*$. The residual $\mathbb{Z}_2$ symmetry of the 4d u-plane for the

[16]The sign change of m_2 is such that the SW curve agrees with instanton partition function computations.

$N_f = 2$ curve is restored by shifting u by an a-independent term, namely: $\widetilde{u} = u - \Lambda^2/2$. As pointed out in [158], this leads to a-independent terms in the prepotential, which have no effect on the low-energy effective action. From the five-dimensional curve perspective, we can view this as a mixing of the $\mathcal{O}(\beta^2)$ term in (2.75) with λ, due to the fact that the parameters u and $\Lambda_{(2)}^2$ have the same scaling dimension. Such mixings will be a general feature of 4d limits.

Similar 4d limits can be taken from the E_2, E_1 and $\widetilde{E}_1$ curves, with:

$$\lambda_{E_2} = -i\left(2\pi i\beta\Lambda_{(1)}\right)^3 \ , \qquad \lambda_{E_1} = \lambda_{\widetilde{E}_1} = \left(2\pi i\beta\Lambda_{(0)}\right)^4 \ . \tag{2.77}$$

For both the E_1 and the $\widetilde{E}_1$ theory, this gives us the curve corresponding to the pure $SU(2)$ gauge theory in four-dimensions:

$$\frac{\Lambda_{(0)}^2}{t} + \Lambda_{(0)}^2 t + x^2 - u = 0 \ . \tag{2.78}$$

We give the Weierstrass form of all these curves in appendix A.

While the mirror curves for the local toric dP_n geometries (*i.e.* $n \leq 3$) can be found from the toric data, the curves for the non-toric cases ($n \geq 4$) can be determined as limits of the E-string theory SW curve [17,159,160], or, alternatively, using toric-like diagrams [161]. The curves are written explicitly in Weierstrass form in appendix A and we will review them in chapter 3.3.2.

Chapter 3

Rational elliptic surfaces, Mordell-Weil group and global symmetries

In the previous chapter, we discussed the flavour symmetry *algebra* of various rank-one theories, but it is natural to ask whether one can also determine the global form of the *flavour symmetry group* – that is, the group that acts faithfully on gauge-invariant states – directly from the SW geometry. For the massless E_n theories, the Higgs branch is always isomorphic to the moduli space of one E_n-instanton, or equivalently to the minimal nilpotent orbit of E_n. (Except for $\widetilde{E}_1$ and E_2, which one should discuss separately.) These Higgs branches are consistent with the actual flavour symmetry group of the massless theory being:

$$G_F = \mathrm{E}_n/\mathrm{Z}(\mathrm{E}_n) \ , \tag{3.1}$$

where E_n denotes the simply-connected Lie group with Lie algebra E_n, and $\mathrm{Z}(\mathrm{E}_n)$ denotes its center – see table 3.1. Very recently, the flavour symmetry group was determined to be precisely the centerless (3.1) by looking at the 5d BPS states in M-theory [162] – see also [163] and the index computation in [164]. In this work, we will give complementary derivations of that same fact, from the 4d Coulomb branch point of view. In addition, we will discuss the abelian symmetries, and any flavour symmetry-breaking pattern, in a unified manner, by taking full advantage of the elliptic fibration structure of the rank-one SW geometry.

In order to do so, it is useful to introduce some additional formalism, namely the theory of rational elliptic surfaces.[17] From that more global perspective, one can study the physics of $D_{S^1}E_n$ throughout its whole parameter space rather systematically and efficiently. This

[17] For further background on this subject, we refer to the very accessible book by Schütt and Shioda [59], from which much of the mathematical discussion in this section is taken.

n	1	3	4	5	6	7	8
E_n	$SU(2)$	$SU(3)\times SU(2)$	$SU(5)$	Spin(10)	E_6	E_7	E_8
$\mathrm{Z}(\mathrm{E}_n)$	$\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2\cong\mathbb{Z}_6$	$\mathbb{Z}_5$	$\mathbb{Z}_4$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$-$

Table 3.1: Simply-connected E_n groups and their centers.

perspective also leads to an improved understanding of the 'well-known' 4d gauge theories and SCFTs, as we will discuss in the next section.

3.1 Seiberg-Witten geometry as a rational elliptic surface

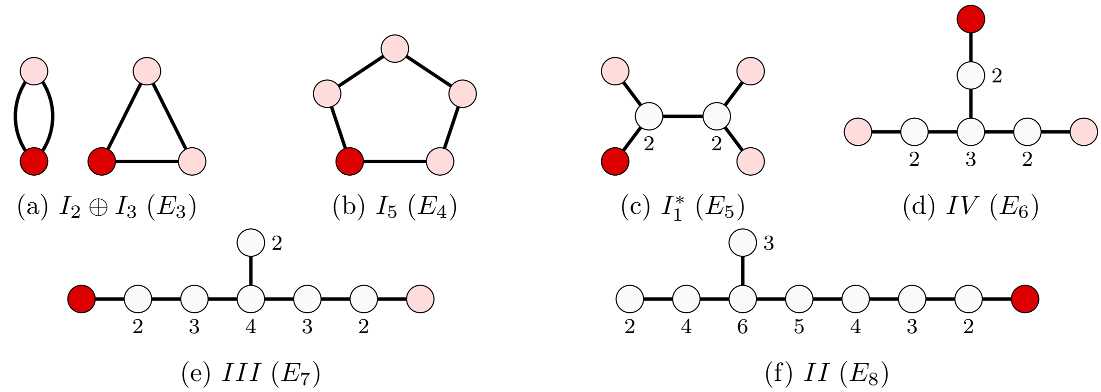
Consider the SW geometry (2.22) at fixed mass parameters, viewed as an elliptic fibration over a genus-zero base $\overline{\mathcal{M}}_C\cong\mathbb{P}^1$, which is the U-plane with the point at infinity added, while the fiber E is the Seiberg-Witten curve. Its minimal Weierstrass model (1.14) is a single equation in the complex variables (x,y,U), thus describing a dimension-two complex variety. By using homogeneous coordinates (as in footnote 12), this can be interpreted as a projective variety. Importantly, this rational elliptic fibration has a section, called the zero *section* O, which is given explicitly by the point 'at infinity', $O=(x,y)=(\infty,\infty)$ on each elliptic fiber.[18]

The Weierstrass model (1.14) has codimension-one singularities along the discriminant locus $\Delta(U)=0$, which look locally like ADE singularities. Each singular Kodaira fiber F_v at $U=U_{*,v}$ can then be resolved in a canonical fashion, giving smooth reducible fibers:

$$\pi^{-1}(U_{*,v})=F_v\cong\sum_{i=0}^{m_v-1}\widehat{m}_{v,i}\Theta_{v,i}\ ,\tag{3.2}$$

where $\Theta_{v,i}$ are the m_v irreducible fiber components, of multiplicity $\widehat{m}_{v,i}$, in F_v. If $m_v=1$, the irreducible fiber $F_v=\Theta_{v,0}$ is a genus-zero curve (a rational curve with a node or with a cusp, for F_v of type I_1 or II, respectively). In all other cases, F_v is reducible and the exceptional fibers together with $\Theta_{0,v}$ (all of genus zero) intersect according to the *affine* Dynkin diagram of $\mathfrak{g}$, where $\mathfrak{g}$ is the flavour algebra listed in table 1.1, and $\widehat{m}_{v,i}$ are the

[18]In the notation of footnote 12, the zero section is $[X,Y,Z]=[1,1,0]$. At smooth fibers, this defines the 'origin' of the elliptic curve $E\cong T^2$.



(a) $I_2 \oplus I_3$ (E_3) (b) I_5 (E_4) (c) I_1^* (E_5) (d) IV (E_6) (e) III (E_7) (f) II (E_8)

Figure 3.1: Examples of affine Dynkin diagrams corresponding to resolved Kodaira fibers. These are the ones that correspond to the semi-simple E_n Lie algebras. The affine node $\Theta_{v,0}$ is indicated in dark red, and the nodes with unit multiplicity ($\widehat{m}_{v,i} = 1$) are all the nodes in (dark or light) red. The multiplicities $\widehat{m}_{v,i} > 1$ are indicated next to the nodes.

Coxeter labels; in particular, every irreducible component $\Theta_{v,i}$ has self-intersection -2 and corresponds to a simple root of $\mathfrak{g}_v$. For every resolved fiber F_v, the zero section O intersects F_v only through the fiber component $\Theta_{v,0}$ (which corresponds to the affine node in the ADE Dynkin diagram of F_v). Some of the relevant affine Dynkin diagrams are shown in figure 3.1. The resulting smooth surface $\widetilde{\mathcal{S}} \to \mathcal{S}$, called the Kodaira-Neron model, is birational to the Weierstrass model $\mathcal{S}$ of the SW geometry.

For future reference, to each reducible fiber F_v, let us associate the finite abelian group:

$$Z(F_v) \equiv R_v^\vee / R_v \ , \tag{3.3}$$

where R_v is the root lattice of $\mathfrak{g}_v$ and $R_v^\vee$ is its dual lattice.[19] It is isomorphic to the center $Z(\widetilde{G}_v)$ of the simply-connected Lie group $\widetilde{G}_v$ associated with that algebra, and it has order:

$$N_v = |Z(F_v)| = |\det(A_{\mathfrak{g}_v})| \ , \tag{3.4}$$

where $A_{\mathfrak{g}_v}$ denotes the Cartan matrix of the Lie algebra $\mathfrak{g}_v$. Note that N_v is the number of components $\Theta_{v,i}$ of F_v with $\widehat{m}_{v,i} = 1$ in the decomposition (3.2).

[19] In the present case of an ADE algebra, we have $R_v^\vee \cong \Lambda_v$, with Λ_v the weight lattice of $\widetilde{G}_v$ such that $\mathrm{Lie}(\widetilde{G}_v) = \mathfrak{g}_v$ and $\pi_1(\widetilde{G}_v) = 0$.

3.1.1 Mathematical interlude (I): rational elliptic surfaces

What we have described so far is a rational complex surface that admits an elliptic fibration with a section, which we will take as the definition[20] of a *rational elliptic surface*. Such surfaces are tightly constrained, and a full classification exists [11,12]. Any rational elliptic surface can be obtained by blowing up $\mathbb{P}^2$ at nine points – in other words, a RES is an almost del Pezzo surface dP_9. In particular, it is also Kähler.

The most important topological fact about $\widetilde{\mathcal{S}}$ is that it is simply-connected and that its topological Euler characteristic $e(\widetilde{\mathcal{S}})$ is equal to 12. Another very important set of global constraints is as follows. To each exceptional fiber F_v, one associates its Euler number, which is given by:

$$e(F_v) = \begin{cases} m_v = k\,, & \text{if } F_v \text{ is of type } I_{k>0}\,, \\ m_v + 1\,, & \text{otherwise} \end{cases} \quad = \ \text{ord}(\Delta) \text{ at } U_{*,v}\ , \tag{3.5}$$

where $\text{ord}(\Delta)$ is as listed in table 1.1. We also associate an ADE Lie algebra $\mathfrak{g}_v$ to each fiber F_v, including the trivial algebra for F_v of type I_1 or II, with rank:

$$\text{rank}(F_v) \equiv \text{rank}(\mathfrak{g}_v) = m_v - 1\ . \tag{3.6}$$

Given these definitions, we have the two conditions:

$$\sum_v e(F_v) = 12\ , \qquad\qquad \sum_v \text{rank}(F_v) \leq 8\ , \tag{3.7}$$

which severely restrict the possible configurations of singular fibers. Using these and some more subtle geometric constraints, the complete list of all rational elliptic surfaces was first constructed by Persson [11] and further checked by Miranda [12]. There are exactly 289 distinct RES and we will see that a given surface can be interpreted as the Coulomb branch of several distinct E_n theories on a circle.

[20] To be precise, we should also require that the fibration be relatively minimal, meaning that one should blow down any exceptional curve (*i.e.* any (-1)-curve) in the fiber.

Quadratic twist and 'transfer of *' operation. The allowed coordinate transformations that preserve the Weierstrass form are $(x,y) \to (f^2 x, f^3 y)$, with $f = \mathbb{C}(U)$. On the other hand, a 'quadratic twist' is a rescaling of the form:

$$(x,y) \to (fx, f^{\frac{3}{2}} y) \ , \qquad f \in \mathbb{C}(U) \ , \tag{3.8}$$

which is equivalent to the rescaling:

$$(g_2, g_3) \to (f^{-2} g_2, f^{-3} g_3) \ , \qquad f \in \mathbb{C}(U) \ . \tag{3.9}$$

A quadratic twist induces a so-called 'transfer of *' amongst the singular fibers, wherever $\sqrt{f}$ has branch cuts (which can be at a smooth fiber, I_0). The corresponding changes in fiber types are:

$$I_k \leftrightarrow I_k^* \quad (k \geq 0) \ , \qquad II \leftrightarrow IV^* \ , \qquad III \leftrightarrow III^* \ , \qquad IV \leftrightarrow II^* \ . \tag{3.10}$$

This simple operation relates many distinct rational elliptic surfaces amongst themselves.

3.1.2 Local mirror, rational elliptic surfaces and the F-theory picture

Recall that the local Calabi-Yau threefold $\widehat{\mathbf{Y}}$ mirror to the local dP_n geometry $\widetilde{\mathbf{X}}_{E_n}$ is a suspension of the E_n Seiberg-Witten curve. In the toric case, in particular, it is given by (2.52). Let $F(x,y;U) = 0$ denote the SW curve at a particular value of $U \in \mathbb{C}$. By introducing some complex variables v_1, v_2 and W, one can write down the threefold as a complete intersection in five variables (x, y, v_1, v_2, W) [58]:

$$F(x,y;W) = 0 \ , \qquad v_1 v_2 = U - W \ . \tag{3.11}$$

This describes the mirror threefold as a double fibration over the W-plane, at fixed U (and, implicitly, fixed mass parameters M):

$$E \times \mathbb{C}^* \to \widehat{\mathbf{Y}} \to \mathbb{C} \cong \{W\} \ . \tag{3.12}$$

The SW curve fibered over the W-plane is again our RES $\mathcal{S}$, with W substituted for U. The $\mathbb{C}^* \cong \mathbb{R} \times S^1_*$ fiber contains a non-trivial one-cycle S^1_* which degenerates precisely when $W = U$. Then, the Coulomb branch BPS states arise from D3-branes wrapping Lagrangian 3-cycles S^3_γ calibrated by the holomorphic 3-form Ω.[21]

The 3-cycle S^3_γ can be constructed explicitly as follows [58]. Consider a path on the W-plane from a singularity $W = W_*$, where the elliptic fiber E degenerates along some one-cycle $\gamma \in E$, to $W = U$, where the $\mathbb{C}^*$ fiber degenerates. By fibering the torus $T^2 \cong \gamma \times S^1_*$ over that path, one spans out the closed 3-cycle S^3_γ, which is topologically a three-sphere. Let $\Gamma_2 \subset S^3_\gamma$ be the two-chain with boundary along $\gamma \in E_U$ above the fiber at $W = U$, obtained by forgetting the S^1_* fiber. We then have the periods:

$$\Pi_\gamma = \int_{S^3_\gamma \subset \widehat{\mathbf{Y}}} \Omega = \int_{\Gamma \subset \mathcal{S}} \Omega_2 = \int_{\gamma \in E} \lambda_{\rm SW} \ , \tag{3.13}$$

with $\partial\Gamma = \gamma$, provided that:

$$\Omega_2 = d\lambda_{\rm SW} \ , \tag{3.14}$$

inside $\mathcal{S}$. Here, the closed (and exact) 2-form Ω_2 is the holomorphic symplectic 2-form on $\mathcal{S}$ that appears in the integrable-system description of Seiberg-Witten theory [35]. Note that we simply have:

$$\Omega_2 = \boldsymbol{\omega} \wedge dU \ , \tag{3.15}$$

with $\boldsymbol{\omega}$ the holomorphic one-form of the elliptic fiber, as follows from (2.24); here and in the following, we freely switch back and forth between W and U to describe the 'U-plane' base of the rational elliptic surface $\mathcal{S}$. It is important to note, however, that W is a coordinate on the IIB geometry while U is a complex structure parameter. It is the double fibration structure (3.11) that allows us to substitute one for the other in the obvious way. In general, one should also consider more general paths on the W-plane to construct supersymmetric 3-cycles. The electro-magnetic charge of the BPS state is fixed by a choice of γ at the 'base point' $W = U$, but the path can branch out and meet several Kodaira singularities, as long as the total charge γ is conserved. More formally, we may also consider candidate 'pure

[21] This is of the form $\Omega = \Omega_2 \wedge \frac{dv_1}{v_1}$.

flavour states', which are closed 2-cycles Γ with $\partial\Gamma = 0$, constructed by connecting directly different Kodaira singularities in the appropriate manner. In all cases, it follows from (3.13) and (3.15) that a necessary topological condition for a 2-cycle or 2-chain $\Gamma \subset \mathcal{S}$ to give rise to a BPS state is that it has 'one leg along the base and one leg along the fiber'.

Correspondence with F-theory. Since part of the IIB mirror symmetry appears to have an elliptic fibration, it is useful to think about it in the language of F-theory. In our original setup, we have a pure geometry in Type-IIB with constant axio-dilaton, which is then 'F-theory' on $\mathbb{R}^4 \times \widehat{\mathbf{Y}} \times T^2$. If we now interpret the elliptic fiber E as the axio-dilaton, instead of the trivial T^2 factor, the Kodaira singularities of the Weierstrass model correspond to 7-branes in the standard way. In this picture, the singularity of the $\mathbb{C}^*$ fibration at $W = U$ is interpreted as the position of a probe D3-brane on the W-plane [58]. This gives a nice alternative description of the U-plane as the geometry seen by a D3-brane in the background of some fixed 7-branes.

The F-theory language offers some additional physical intuition. Firstly, it is clear in this picture that the Kodaira singularities of the SW geometry realize the non-abelian ADE-type *flavour symmetries* of the theory, simply because the 7-branes wrap non-compact cycles $\mathbb{C}^* \times T^2 \subset \widehat{\mathbf{Y}} \times T^2$. The BPS states from the 2-chains $\Gamma \subset \mathcal{S}$ here correspond to *string junctions* on the W-plane, which are open-string networks connecting the D3-brane to the 7-branes in a supersymmetric fashion. Such string junctions have been extensively studied in the literature, in this very same context [165–171]. Secondly, it is well-known in F-theory that sections of the elliptic fibration are related to abelian symmetries and to the global form of the 'gauge group' – see *e.g.* the review [172]. In the rest of this section, we will argue, not surprisingly given what we have written so far, that essentially the same conclusions can be reached when interpreting sections of the rank-one Seiberg-Witten geometries in terms of the 4d flavour symmetry.

Let us also recall that the F-theory perspective leads to a nice interpretation of the Higgs branch that emanates from a Kodaira singularity with reducible components [56]. Indeed, moving onto that Higgs branch corresponds to moving the D3-brane probe on top of the 7-brane stack at $W = U_{*,v}$ before 'dissolving' it into the 7-branes, which gives the

Higgs branch as the $\widetilde{G}_v$ one-instanton moduli space.[22]

Fixing F_∞, the fiber at infinity. Consider a fixed rank-one 4d $\mathcal{N}=2$ supersymmetric field theory $\mathcal{T}_{F_\infty}$, which is either a 5d SCFT on a circle, a 4d SCFT, or a 4d $\mathcal{N}=2$ asymptotically-free theory. For each theory, we are interested in the class of all rational elliptic surfaces with a fixed singularity at $U=\infty$, whose corresponding (resolved) Kodaira fiber is denoted by F_∞. The choice of F_∞ fixes the 'UV definition' of the field theory:[23]

$$\mathcal{T}_{F_\infty} \qquad \longleftrightarrow \qquad \{\mathcal{S} \mid \pi^{-1}(\infty) = F_\infty\} \ . \tag{3.16}$$

For purely four-dimensional theories, this point of view was emphasized in [36]. As we reviewed in the previous section, the SW geometry for the KK theory $D_{S^1}E_n$ has an I_{9-n} fiber at infinity, as determined by the large volume monodromy in Type-IIA. We can then obtain the strictly four-dimensional theories by additional limits, thus 'growing' the singularity at infinity. The 4d limits from the 5d E_n SCFT to the 4d E_n MN SCFT for $n=6,7,8$ correspond to the degenerations:

$$F_\infty^{\rm 5d} \to F_\infty^{\rm 4d} \ : \qquad I_3 \to IV \quad (E_6) \ , \qquad I_2 \to III \quad (E_7) \ , \qquad I_1 \to II \quad (E_8) \ , \tag{3.17}$$

at infinity, wherein one I_1 collides with the '5d' fiber at infinity $F_\infty^{\rm 5d}$ to give the '4d' fiber $F_\infty^{\rm 4d}$. Similarly, the geometric-engineering limit from the $D_{S^1}E_n$ theory with $1 \leq n \leq 5$ to the 4d $SU(2)$ gauge theory with $N_f = n-1$ corresponds to:

$$F_\infty^{\rm 5d} \to F_\infty^{\rm 4d} \ : \qquad I_{8-N_f} \to I^*_{4-N_f} \quad (E_{N_f-1},\, N_f = 0,1,2,3,4) \ , \tag{3.18}$$

wherein two I_1's are brought in to merge with the I_{8-N_f} fiber at infinity. The remaining choices, $F_\infty = II^*$, III^* or IV^* correspond to the Argyres-Douglas theories H_0, H_1 and H_2, respectively, as also discussed in [36].

Finally, we should mention that one may also consider the 'generic' situation for which

[22] When a perturbative open-string description of this process exists (in particular, for k D7-branes in the case of an I_k singularity), it reproduces exactly the ADHM construction.

[23] With the important exception of $F_\infty = I_8$, which includes both E_1 and $\widetilde{E}_1$.

the fiber at infinity is trivial. The interpretation of that configuration is that we are considering the 6d $\mathcal{N} = (1, 0)$ E-string SCFT with E_8 symmetry compactified on T^2, whose U-plane has the singularities [17]:

$$\text{6d } E\text{-string } (F_\infty^{\text{6d}} = I_0): \qquad II^* \oplus I_1 \oplus I_1 \ , \tag{3.19}$$

in the massless limit. This curve is discussed from our perspective in [4]. The 5d E_8 theory with $F_\infty = I_1$ is obtained from the E-string theory by sending one I_1 singularity to infinity, which corresponds to shrinking the T^2 to S^1 [160].

3.2 Mordell-Weil group and global symmetries

Let us finally explain how the flavour symmetry group is encoded by the rank-one Seiberg-Witten geometry. This involves reviewing some very interesting mathematical results, following closely [59].

3.2.1 Mathematical interlude (II): Mordell-Weil group and Shioda map

Any elliptic curve famously has the structure of an additive group; viewing the curve as the torus $E \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, the neutral element is the origin, and the addition operation is simply the addition of complex numbers. This becomes more interesting for an elliptic curve defined over the field $\mathbb{Q}$, in which case the equation $F(x, y) = 0$ for the curve has a finite number of rational solutions, which form a finitely generated abelian group. More generally, we are here considering the equation (1.14) where g_2, g_3 are valued in $\mathbb{C}(U)$, the field of rational functions of U.

A *rational section* of this elliptic fibration is a rational solution to the Weierstrass equation (1.14) $P = (x(U), y(U))$, with $x(U), y(U) \in \mathbb{C}(U)$. By the Mordell-Weil theorem, the sections of $\mathcal{S}$ form a finitely generated abelian group, which we denote by either $\text{MW}(\mathcal{S})$ or Φ.[24] We then have:

$$\Phi = \text{MW}(\mathcal{S}) \cong \mathbb{Z}^{\text{rk}(\Phi)} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_t} \ . \tag{3.20}$$

[24]We will denote the MW *group* by Φ, and use the symbol $\text{MW}(\mathcal{S})$ for the MW *lattice*, to be defined below.

Here, $\text{rk}(\Phi)$ is the rank of the MW group – that is, the number of independent generators of the free part of Φ. Note that the point 'at infinity' $O = (\infty, \infty)$ is the neutral element of the MW group, and therefore does not contribute to the rank. The MW group also generally has a torsion component, which we denote by Φ_{tor}. The addition of sections in Φ is given by the standard addition of rational points of an elliptic curve. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two distinct points in Φ. Their sum is given by:

$$P = P_1 + P_2 = (x, y) \ , \qquad \begin{cases} x = -(x_1 + x_2) + \frac{1}{4}\left(\frac{y_1 - y_2}{x_1 - x_2}\right)^2 \ , \\ y = -\frac{y_1 - y_2}{x_1 - x_2}(x - x_1) - y_1 \ . \end{cases} \tag{3.21}$$

Meanwhile, for $P_1 = P_2$, we have the duplication formula:

$$P = 2P_1 = (x, y) \ , \qquad \begin{cases} x = -2x_1 + \xi^2 \ , & \xi \equiv \frac{12x_1^2 - g_2}{4y_1} \ , \\ y = -2\xi(x - x_1) - y_1 \ . \end{cases} \tag{3.22}$$

The inverse of a point $P = (x, y)$ is given by $-P = (x, -y)$, so that $P - P = O$. A section P is $\mathbb{Z}_k$ torsion if $kP = P + P + \cdots + P = O$. Each section P defines a divisor (P) in the Neron-Severi group $\text{NS}(\widetilde{\mathcal{S}})$, *i.e.* the group of divisors modulo linear equivalences. Note that the NS group is naturally endowed with an integral *lattice structure*, with the bilinear form defined as the intersection number of the divisors.

Vertical and horizontal divisors. Let U be the dimension-2 lattice generated by the zero section (O) and the generic fiber $F \cong E$, with intersection pairing:

$$U \cong \text{Span}((O), F) \ , \qquad I_U = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{3.23}$$

The *trivial lattice* of *vertical divisors* in $\widetilde{\mathcal{S}}$ is defined as the sublattice $\text{Triv}(\widetilde{\mathcal{S}}) \subset \text{NS}(\widetilde{\mathcal{S}})$ generated by the zero section, (O), and by the fiber components, such that:

$$\text{Triv}(\widetilde{\mathcal{S}}) \cong U \oplus T^- \ , \qquad T \equiv \bigoplus_v R_v \ . \tag{3.24}$$

Here R_v is the root lattice of the Lie algebra $\mathfrak{g}_v$ associated to the reducible fiber F_v in the Kodaira-Neron model, with the intersection form given by the Cartan matrix. Note that:

$$(I_v)_{ij} = (-A_{\mathfrak{g}_v})_{ij} = \Theta_{v,i} \cdot \Theta_{v,j} \ , \tag{3.25}$$

is the intersection pairing for T^-. We will refer to T as 'the 7-brane root lattice', as a nod to the F-theory picture. Note also that:

$$\mathrm{rank}(T) = \sum_v \mathrm{rank}(\mathfrak{g}_v) \ , \tag{3.26}$$

with $\mathrm{rank}(\mathfrak{g}_v)$ as in (3.6). The Neron-Severi lattice of a RES can be expressed as the direct sum $U \oplus E_8^-$ [59], from which it follows that T is a sublattice of the E_8 lattice. The 'non-trivial' divisors, or *horizontal divisors*, must then span the complement of T in E_8. They are generated by the (non-zero) sections P; each divisor (P) decomposes into a horizontal and a vertical component, but there are enough sections to generate all vertical divisors. More precisely, we have the following theorem:

$$\Phi \cong \mathrm{NS}(\widetilde{\mathcal{S}})/\mathrm{Triv}(\widetilde{\mathcal{S}}) \ , \tag{3.27}$$

as an isomorphism of abelian groups. It follows, in particular, that:

$$\mathrm{rk}(\Phi) = 8 - \mathrm{rank}(T) \ , \tag{3.28}$$

which implies the second condition in (3.7). The simple relation (3.28) will be important to understand the flavour symmetry on the U-plane.

The Mordell-Weil lattice. While the Neron-Severi and trivial lattices come with a 'natural pairing', this is no longer the case for the Mordell-Weil group, due to possibly non-trivial torsion in Φ. For this, one makes use of the Shioda homomorphism [173]:

$$\varphi \ : \ \Phi \to \mathrm{NS}(\widetilde{\mathcal{S}}) \otimes \mathbb{Q} \ , \tag{3.29}$$

which maps sections to horizontal divisors with rational coefficients. In other words, we must have that $\varphi(P) = (P) \bmod \mathrm{Triv}(\widetilde{\mathcal{S}}) \otimes \mathbb{Q}$ and that:

$$\varphi(P) \cdot (O) = 0 \ , \qquad \varphi(P) \cdot F = 0 \ , \qquad \varphi(P) \cdot \Theta_{v,i} = 0 \ , \ \forall v, i \ . \tag{3.30}$$

The Shioda map is given explicitly by:

$$\varphi(P) = (P) - (O) - ((P) \cdot (O) + 1)F + \sum_v \sum_{i=1}^{\mathrm{rank}(\mathfrak{g}_v)} \lambda_{v,i}^{(P)} \Theta_{v,i} \ , \tag{3.31}$$

with the rational coefficients:

$$\lambda_{v,i}^{(P)} = \sum_{j=1}^{\mathrm{rank}(\mathfrak{g}_v)} (A_{\mathfrak{g}_v}^{-1})_{ij} \, \Theta_{v,j} \cdot (P) \ , \tag{3.32}$$

given in terms of the inverse of the Cartan matrix of $\mathfrak{g}_v$. In particular, for each F_v, the coefficients $\lambda_{v,i}$ are valued in $\frac{1}{N_v}\mathbb{Z}$, with N_v defined in (3.4). Note also that $\lambda_{v,i} = 0$, $\forall i$, if P intersects the resolved Kodaira fiber F_v at the 'trivial' affine node $\Theta_{v,0}$. Given this map, we then define the $\mathbb{Q}$-valued bilinear form:

$$\langle P, Q \rangle = -(\varphi(P) \cdot \varphi(Q)) \ . \tag{3.33}$$

In this way, the intersection pairing induces a (positive-definite) lattice structure on the free part of the MW group:

$$\mathrm{MW}(\widetilde{\mathcal{S}})_{\mathrm{free}} \equiv \Phi / \Phi_{\mathrm{tor}} \ . \tag{3.34}$$

This defines the *Mordell-Weil lattice* (MWL). The intersection pairing on sections is called the height pairing. It is often useful to define some natural sublattices of the MW lattice. In particular, one defines the *narrow Mordell-Weil lattice* $\mathrm{MS}(\mathcal{S})_0$ as:

$$\mathrm{MS}(\widetilde{\mathcal{S}})_0 = \left\{ P \in \mathrm{MW}(\mathcal{S}) \;\middle|\; (P) \text{ intersects } \Theta_{v,0} \text{ for all } F_v \ \right\} \ , \tag{3.35}$$

with the lattice structure defined by the height pairing. Since $\lambda_{v,i} = 0$ for narrow sections, the narrow MW lattice is an integral lattice. In practice, we can find the narrow MW lattice without computing the intersections of the sections with the singular fibers; this is due to the fact that the narrow MW lattice is isomorphic to minus the complement of the trivial lattice inside the NS lattice:

$$\mathrm{NS}(\widetilde{\mathcal{S}}) = \mathrm{MW}(\widetilde{\mathcal{S}})_0^- \oplus \mathrm{Triv}(\widetilde{\mathcal{S}}) \ . \tag{3.36}$$

Moreover, the narrow MW lattice is the orthogonal complement of the 7-brane root lattice T inside the E_8 lattice:

$$\mathrm{MW}(\widetilde{\mathcal{S}})_0 = T^\perp \text{ in } E_8 \ . \tag{3.37}$$

Thus, while $T = \oplus_v R_v$ is the root lattice of a semi-simple subalgebra $\mathfrak{g}_T = \oplus_v \mathfrak{g}_v$ of E_8, the narrow MW lattice depends not only on $\mathfrak{g}_T$ as a Lie algebra but on its particular embedding inside E_8.

Torsional sections. The kernel of the Shioda map is precisely the torsion part of the Mordell-Weil group:

$$\ker(\varphi) = \Phi_{\rm tor} \ . \tag{3.38}$$

Equivalently, a section P is torsion if and only if $\langle P, P\rangle = 0$. It follows that, if P is torsion, we have $\varphi(P)\cdot\Gamma = 0$ for any divisor $\Gamma \in \mathrm{NS}(\widetilde{\mathcal{S}})$, and therefore we have the non-trivial integrality condition:

$$\sum_v \sum_{i=1}^{\mathrm{rank}(\mathfrak{g}_v)} \lambda_{v,i}^{(P)} \Theta_{v,i}\cdot\Gamma \in \mathbb{Z} \ . \tag{3.39}$$

Let T' denote the primitive closure of the 7-brane root lattice T inside the E_8 lattice:[25], namely $T' = (T\otimes\mathbb{Q})\cap E_8$. One can prove that $\Phi_{\rm tor} \cong T'/T$, and, moreover, since T' is a sublattice of the dual lattice $T^\vee$, we have the important property that the torsion subgroup of the Mordell-Weil group is injective onto the center group $Z(T) = T^\vee/T$:

$$\Phi_{\rm tor} \hookrightarrow Z(T) = \bigoplus_v Z(F_v) \ , \tag{3.40}$$

[25] A sublattice $M\subset N$ is called primitive if N/M is torsion-free. The primitive closure of any sublattice N in M is the smallest primitive sublattice $N'\subset M$ that contains N.

with $Z(F_v)$ defined in (3.3). This embedding can be determined by explicit computation in the Kodaira-Neron model $\widetilde{\mathcal{S}}$.

3.2.2 Flavour symmetry group from the SW elliptic fibration

To study the flavour symmetry of a theory $\mathcal{T}_{F_\infty}$ with a Coulomb branch described by a family of rational elliptic surfaces as in (3.16), it is useful to consider two opposite limits. We first consider the 'massless curve' – in particular, we have then $M_F = 1$ for the $D_{S^1}E_n$ theories. In the massless limit, the full flavour symmetry of the UV theory should be manifest. The other limit is the 'maximally massive curve', wherein the UV flavour symmetry G_F is broken explicitly to a maximal torus, $U(1)^f$.

Structure of the flavour symmetry algebra. Consider the U-plane of a 4d $\mathcal{N} = 2$ theory $\mathcal{T}_{F_\infty}$ with fixed masses (and/or relevant deformations) turned on, which is described by a particular RES $\mathcal{S}$ with Kodaira fibers:

$$F_v = F_\infty \oplus F_1 \oplus \cdots \oplus F_k \ . \tag{3.41}$$

We decompose the 7-brane root lattice in terms of the contribution from infinity and of the contribution from the interior:

$$T = R_\infty \oplus R_F \ , \qquad\qquad R_F = \bigoplus_{v=1}^{k} R_v \ , \tag{3.42}$$

Here, the 'flavour 7-brane root lattice' R_F is the root lattice of the non-abelian flavour algebra of the theory $\mathcal{T}_{F_\infty}$ for some fixed values of the masses:

$$\mathfrak{g}_F^{\rm NA} = \bigoplus_{v=1}^{k} \mathfrak{g}_v \ . \tag{3.43}$$

On the other hand, the fiber at infinity does not contribute to the flavour symmetry. The reason for this is perhaps easiest to explain in the F-theory picture: BPS states charged under the flavour symmetry are open strings stretched between the probe D3-brane and stacks of 7-branes, which have a mass proportional to the distance between the D3- and the

7-branes. Modes of open strings stretching all the way to infinity have infinite mass and are therefore not part of the 4d $\mathcal{N} = 2$ theory under consideration.

In addition, the flavour group generally includes abelian factors. They are precisely generated by infinite-order sections, $P \in \Phi_{\rm free}$. Indeed, that is how $U(1)$ gauge fields arise in F-theory [174, 175]. Consider the E_n theories, for definiteness (the other 4d $\mathcal{N} = 2$ theories being obtained from them in appropriate limits). In the IIB description on $\widehat{\mathbf{Y}}$, we have 3-cycles of the schematic form $\varphi(P) \times S^1_*$, which are mirror to 'flavour' two-cycles in the E_n sublattice of $H_2(\widetilde{\mathbf{X}}, \mathbb{Z})$ [176]. Reducing the C_4 RR gauge field of IIB on that 3-cycle, we obtain a background $U(1)$ gauge field in the low-energy description. The horizontality conditions (3.30) ensure that the abelian gauge field is massless and neutral under the non-abelian flavour symmetry $\mathfrak{g}_F^{\rm NA}$. The number of abelian factors in the low-energy flavour symmetry is then given by the rank of the Mordell-Weil group, and we have the full flavour algebra:

$$\mathfrak{g}_F = \bigoplus_{s=1}^{\mathrm{rk}(\Phi)} \mathfrak{u}(1)_s \oplus \bigoplus_{v=1}^{k} \mathfrak{g}_v \ , \tag{3.44}$$

for any extended CB configuration described by a particular RES $\mathcal{S}$. In particular, we see from (3.28) that:

$$\mathrm{rank}(\mathfrak{g}_F) = 8 - \mathrm{rank}(F_\infty) \ . \tag{3.45}$$

This equation only depends on the fiber at infinity, and gives the rank of the flavour symmetry G_F of $\mathcal{T}_{F_\infty}$, as indicated. The physical reason for this is clear: as we vary the mass parameters of a given theory $\mathcal{T}_{F_\infty}$, we may break the UV symmetry group G_F to its maximal torus, or to any allowed subgroup, while keeping the rank fixed. This is precisely what being on the extended Coulomb branch, as opposed to the Higgs or mixed branches, means. Such extended CB deformations are realised by 'fusing' or 'splitting' 7-branes by continuously varying the complex structure parameters of the mirror threefold $\widehat{\mathbf{Y}}$ or, equivalently, the parameters of the Weierstrass model $\mathcal{S}$ over the W-plane.

Flavour charges of the BPS states. Consider any BPS state on the Coulomb branch, corresponding to a 2-chain Γ in $\widetilde{\mathcal{S}} \subset \widehat{\mathbf{Y}}$. Its flavour charges under the non-abelian flavour symmetry $\mathfrak{g}_v \subset \mathfrak{g}_F$ associated to the Kodaira fiber F_v are determined by the intersection numbers:

$$w_i^{(\mathfrak{g}_v)}(\Gamma) = \Theta_{v,i} \cdot \Gamma \ . \tag{3.46}$$

The integers $w_i^{(\mathfrak{g}_v)}$ give us the weight vectors in the Dynkin basis and thus determine which representations of $\mathfrak{g}_v$ are spanned by the BPS states. Any physical state of the theory $\mathcal{T}_{F_\infty}$ should have finite mass, and therefore its corresponding 2-chain Γ should not intersect the fiber at infinity. We then have:

$$\Gamma \text{ physical} \qquad \Leftrightarrow \qquad w_i^{(F_\infty)}(\Gamma) = \Theta_{\infty,i} \cdot \Gamma = 0 \ , \tag{3.47}$$

which can be taken as a 'topological' definition of what we mean by a physical state.

Massless limit with G_F semi-simple. Consider a theory $\mathcal{T}_{F_\infty}$ in the massless limit such that $\mathfrak{g}_F$ is semi-simple, and let $\widetilde{G}_F$ denote the corresponding simply-connected group. That is the case, in particular, for all the E_n KK theories with the exception of $\widetilde{E}_1$ and E_2. This means that the Mordell-Weil group of $\mathcal{S}$ is purely torsion, $\Phi = \Phi_{\rm tor}$, and so $\text{rk}(\Phi) = 0$. Such rational elliptic surfaces are called *extremal* – we will discuss them further in subsection 3.3.3. The flavour algebra $\mathfrak{g}_F = \mathfrak{g}_F^{\rm NA}$ is a maximal semi-simple Dynkin subalgebra of E_8. As explained above, $\Phi_{\rm tor}$ injects into the finite abelian group $Z(T) = T^\vee/T$, which is:

$$\Phi_{\rm tor} \hookrightarrow Z(T) = Z(F_\infty) \oplus Z(\widetilde{G}_F) \ . \tag{3.48}$$

In the extremal case, $T' = E_8$ and the torsion group is related to the embedding of the full 7-brane lattice inside the E_8 lattice $\Phi_{\rm tor} \cong E_8/T$. Let us denote by $\mathcal{Z}^{[1]}$ the subgroup of sections that are narrow in the interior of the U-plane:

$$\mathcal{Z}^{[1]} = \left\{ P \in \Phi_{\rm tor} \ \middle| \ (P) \text{ intersects } \Theta_{v,0} \text{ for all } F_{v\neq\infty} \right\} \ , \tag{3.49}$$

and let us denote by $\mathscr{F}$ the cokernel of the inclusion map $\mathcal{Z}^{[1]} \to \Phi_{\rm tor}$. In other words, $\mathscr{F}$ is the abelian group defined by the short exact sequence:

$$0 \to \mathcal{Z}^{[1]} \to \Phi_{\rm tor} \to \mathscr{F} \to 0 \ . \tag{3.50}$$

Note that $\mathscr{F}$ is a subgroup of $Z(\widetilde{G}_F)$. Given the injection (3.48), we can write any element of Φ_{tor} as $P \sim (z_\infty, z_F)$, where $z_\infty \in Z(F_\infty)$ and $z_F \in Z(\widetilde{G})_F$. The subgroup $\mathcal{Z}^{[1]}$ corresponds to elements of the form $P \sim (z_\infty, 0)$, while the group $\mathscr{F}$ contains all the elements in the image of the projection map $(z_\infty, z_F) \mapsto z_F$. We then claim that the *flavour symmetry group* of the theory $\mathcal{T}_{F_\infty}$ is given by:

$$G_F = \widetilde{G}_F / \mathscr{F} \ . \tag{3.51}$$

The argument for (3.51) is similar to the one given in the F-theory context [177–179]. One should consider all possible *closed* 2-cycles $\Gamma \in \text{NS}(\widetilde{\mathcal{S}})$, which give rise to formal 'pure flavour' states. The existence of torsion sections P_{tor} constrains the allowed weights of the pure flavour states due to the integrability condition (3.39), which gives:

$$\sum_{l=1}^{\text{rank}(F_\infty)} \lambda_{\infty,l}^{(P_{\text{tor}})} w_l^{(F_\infty)} + \sum_{i=1}^{\text{rank}(\mathfrak{g}_F^{\text{NA}})} \lambda_{v,i}^{(P_{\text{tor}})} w_i^{(\mathfrak{g}_F^{\text{NA}})} \in \mathbb{Z} \ . \tag{3.52}$$

For the pure flavour states that satisfy the physical state condition (3.47), we have:

$$\sum_{i=1}^{\text{rank}(\mathfrak{g}_F^{\text{NA}})} \lambda_{v,i}^{(P_{\text{tor}})} w_i^{(\mathfrak{g}_F^{\text{NA}})} \in \mathbb{Z} \ , \qquad \forall \, P_{\text{tor}} \in \mathscr{F} \ . \tag{3.53}$$

The only sections that contribute to the constraint (3.53) are the elements of $\mathscr{F}$ since, by definition, the 'interior-narrow' sections in $\mathcal{Z}^{[1]} \subset \Phi_{\text{tor}}$ lead to the constraint:

$$\sum_{i=1}^{\text{rank}(F_\infty)} \lambda_{\infty,i}^{(P_{\text{tor}})} w_i^{(F_\infty)} \in \mathbb{Z} \ , \qquad \forall \, P_{\text{tor}} \in \mathcal{Z}^{[1]} \ , \tag{3.54}$$

which is trivial on physical states. This determines (3.51) as the effectively acting non-abelian group on pure flavour states. We should note that the actual BPS states, which correspond to two-chains ending on the fiber above $W = U$ and thus carry electro-magnetic charge, will typically be charged under the center of $\widetilde{G}_F$, but the heuristic argument above shows that the 'gauge invariant states' are only charged under the smaller group G_F. We will also check this claim explicitly in many examples, using a more direct but essentially equivalent argument presented in subsection 3.2.3.

We should also note that the 'interior-narrow' section constraint (3.54) would be non-trivial when dealing with defect states, which are BPS D3-branes on non-compact 3-cycles stretching all the way to infinity. This leads us to the natural *conjecture* that this group is isomorphic to the one-form symmetry of the field theory:

$$\mathscr{Z}^{[1]} \cong \text{1-form symmetry of } \mathcal{T}_{F_\infty} . \tag{3.55}$$

These have received much interest in recent years, and are only a particular case of generalized symmetries – see *e.g.* [140, 180–192] for recent developments. We will show that this agrees with all the known results. For instance, if the conjecture holds, it must be true that, for a fixed F_∞, $\mathscr{Z}^{[1]}$ remains the same for any configuration of the singular fibers $\{F_v\}$ in the interior, which is a very strong constraint.

As we will shortly see, the $D_{S^1}E_1$ theory shows a unique feature: the short exact sequence (3.50) does not split. We expect that this is related to the two-group symmetry of the theory, recently discovered in [162], which involves the mixing of 0- and 1-form symmetries.

Non-abelian flavour symmetry $G_F^{\rm NA}$ in general. In any theory $\mathcal{T}_{F_\infty}$ with a flavour algebra (3.43) for some fixed values of the masses, the same argument as above determines the global form of the non-abelian part of the flavour symmetry group:

$$G_F^{\rm NA} = \widetilde{G}_F^{\rm NA}/\mathscr{F} \ , \tag{3.56}$$

where $\mathscr{F}$ is defined as in (3.51) in terms of the torsion part of the Mordell-Weil group. Of course, the conjecture (3.55) should still hold as well.

Abelian limit with generic masses. The opposite limit to the extremal limit is when the rank of the Mordell-Weil group is the maximal one allowed by the fiber at infinity:

$$\text{rk}(\Phi) = 8 - \text{rank}(F_\infty) = \text{rank}(G_F) \ . \tag{3.57}$$

In that limit, the flavour group is abelian and thus entirely generated by sections. The singular fibers in the interior are then irreducible (that is, of type I_1 or II). This corresponds to the maximal symmetry breaking allowed on the extended CB, *i.e.* with generic masses turned on:

$$G_F \to \prod_{s=1}^{\mathrm{rank}(G_F)} U(1)_s \ . \tag{3.58}$$

Let the sections P_s be the corresponding generators of Φ_{free}. The divisor dual to $U(1)_s$ is given by $\varphi(P_s)$. Then, the $U(1)_s$ charge of any 'pure flavour' state Γ is given by:

$$q_s(\Gamma) \equiv \varphi(P_s) \cdot \Gamma \ . \tag{3.59}$$

From the Shioda map, we then obtain an integrality condition:

$$q_s - \sum_{i=1}^{\mathrm{rank}(F_\infty)} \lambda_{\infty,i}^{(P_s)} w_i^{(F_\infty)} \in \mathbb{Z} \ . \tag{3.60}$$

On states satisfying the physical condition (3.47), the second contribution is trivial, and we simply have:

$$q_s(\Gamma) \in \mathbb{Z} \quad \text{if } \Gamma \text{ is 'physical'}. \tag{3.61}$$

Since there are no reducible fibers in this abelian configuration, the physical states actually span the narrow Mordell-Weil lattice (3.35). Let Λ_{phys} denote the weight lattice of flavour charges for the physical states, which is then isomorphic to the narrow MWL – in particular, it is an integral lattice. Then, according to (3.37), this physical flavour weight lattice is isomorphic to the complement of the 7-brane lattice at infinity inside the E_8 lattice:

$$\Lambda_{\mathrm{phys}} \cong R_\infty^\perp \ \text{ in } E_8 \ . \tag{3.62}$$

For $\mathfrak{g}_F$ semi-simple in the UV, we can check in each case, according to the general classification results [59,193], that Λ_{phys} is the root lattice of $\mathfrak{g}_F$. Therefore, since $Z(G_F) \cong \Lambda_{\mathrm{phys}}/\Lambda_r$, the actual flavour group is the centerless group, $G_F = \widetilde{G}_F/Z(\widetilde{G}_F)$. This gives a complementary derivation of (3.51) which avoids having to carefully compute the intersection of torsion sections with the reducible fibers.[26]

Symmetry group G_F in the general case. In the general case of a flavour algebra (3.44), physical states Γ carry both weights under $\mathfrak{g}_F^{\rm NA}$ and abelian charges:

$$w_i^{(\mathfrak{g}_v)}(\Gamma) = \Theta_{v,i} \cdot \Gamma \;, \qquad\qquad q_s(\Gamma) \equiv \varphi(P_s) \cdot \Gamma \;. \tag{3.63}$$

The allowed weights are constrained by torsion sections as in (3.53), and the abelian charges satisfy the conditions:

$$q_s - \sum_{i=1}^{\mathrm{rank}(\mathfrak{g}_F^{\rm NA})} \lambda_{v,i}^{(P_s)} w_i^{(\mathfrak{g}_F^{\rm NA})} \in \mathbb{Z} \;, \qquad \forall\, P_s \in \Phi_{\rm free} \;. \tag{3.64}$$

Thus, for any given RES $\widetilde{\mathcal{S}}$ corresponding to an extended CB configuration of $\mathcal{T}_{F_\infty}$, the global form of the IR flavour symmetry takes the schematic form:

$$G_F = \frac{U(1)^{\mathrm{rk}(\Phi)} \times \widetilde{G}_F^{\rm NA}}{\prod_{s=1}^{\mathrm{rk}(\Phi)} \mathbb{Z}_{m_s} \times \prod_{j=1}^{p} \mathbb{Z}_{k_p}} \;, \tag{3.65}$$

where the two factors in the denominator are determined by the conditions (3.64) and by the torsion sections, respectively. In this work, we will mostly focus on the case of G_F semi-simple. The detailed form of (3.64) can also be deduced from the general classification of Mordell-Weil lattices [59, 193], in principle, by mass-deforming into a purely abelian flavour phase.

3.2.3 Global symmetries from the BPS spectrum

As a consistency check of the above discussion, it is interesting to also compute the flavour group more directly, which can be done if we know the low-energy spectrum $\mathscr{S}$, similarly to the recent discussion in [162]. As a reasonably good approximation of the strong-coupling spectrum, for our purpose, we can often consider $\mathcal{S}$ to be the set of dyons that become massless at the SW singularities U_*. This is closely related to the existence of quiver point,

[26] See [194] for a related argument in the context of F-theory on an elliptically fibered Calabi-Yau threefold.

which we will discuss in chapter 4.

At a generic point on the Coulomb branch, there is a $U(1)^{[1]}_m \times U(1)^{[1]}_e$ one-form symmetry in the strict IR limit, which is the one-form symmetry of a pure $U(1)$ gauge theory [19]. One can think of $U(1)^{[1]}_e$ as the group of global gauge transformations in the electric frame, and similarly for $U(1)^{[1]}_m$ in the magnetic frame. This accidental continuous one-form symmetry is broken explicitly to a discrete subgroup (which can be trivial) by the spectrum of charged massive BPS particles $\mathscr{S}$. The one-form symmetry of the full 4d $\mathcal{N}=2$ theory is then given by that subgroup.[27] See also [195] for further discussion.

Given a theory at fixed masses with a flavour symmetry algebra $\mathfrak{g}_F$ which is non-abelian, for simplicity, let $\widetilde{G}_F$ denote the corresponding simply-connected group, and let $Z(\widetilde{G}_F)$ be its center. For concreteness, let us have $Z(\widetilde{G}_F) = \mathbb{Z}_{n_1} \times \cdots \mathbb{Z}_{n_p}$. The dyons in $\mathscr{S}$ fall in representations $\mathfrak{R}_\psi$ of $\mathfrak{g}_F$. Let us denote these states ψ by the charges:

$$\psi \ : \ (m, q; l_1, \cdots, l_p) \ , \qquad l_1 \in \mathbb{Z}_{n_1} \ , \cdots \ , l_p \in \mathbb{Z}_{n_p} \ , \tag{3.66}$$

where (m, q) are the electromagnetic charges, and the integers l_j mod n_j give the charges of ψ under the center $Z(\widetilde{G}_F)$. Let us define the subgroup:

$$\mathscr{E} \ \subset \ U(1)^{[1]}_m \times U(1)^{[1]}_e \times Z(\widetilde{G}_F) \ , \tag{3.67}$$

as the maximal subgroup that leaves the spectrum $\mathscr{S}$ invariant. We will denote the generators of $\mathscr{E}$ by:

$$g^{\mathscr{E}} = (k_m, k_q; z_1, \cdots, z_p) \ , \qquad k_m \in \mathbb{Q} \ , \ k_q \in \mathbb{Q} \ , \ \ z_j \in \mathbb{Z}_{n_j} \ . \tag{3.68}$$

This is a generator that acts on a state (3.66) as:

$$g^{\mathscr{E}} \ : \quad \psi \to e^{2\pi i \left(k_m m + k_q q + \sum_{j=1}^{p} \frac{z_i l_i}{n_i}\right)} \psi \ . \tag{3.69}$$

Let $\mathcal{Z}^{[1]}$ denote the subgroup of $\mathcal{E}$ generated by $g^{\mathcal{Z}^{[1]}} = (k_m, k_e; 0, \cdots, 0)$. In addition, the

[27]We are very grateful to M. Del Zotto for explaining this to us.

projection $\pi_F : U(1)_m^{[1]} \times U(1)_e^{[1]} \times Z(\widetilde{G}_F) \to Z(\widetilde{G}_F)$ gives a subgroup $\mathscr{F}$ of $Z(\widetilde{G}_F)$ generated by $g^{\mathscr{F}} = (z_1, \cdots, z_p)$, for each generator (3.68). These three groups are related by a short exact sequence:

$$0 \to \mathcal{Z}^{[1]} \to \mathcal{E} \to \mathscr{F} \to 0 \,, \tag{3.70}$$

precisely as in (3.50). Here, $\mathcal{Z}^{[1]}$ is exactly the one-form symmetry. On the other hand, $\mathscr{F}$ is the subgroup of the flavour center $Z(\widetilde{G}_F)$ that can be compensated by gauge transformations, and therefore the actual non-abelian flavour group of the theory is $G_F = \widetilde{G}_F/\mathscr{F}$, as in (3.51).

As an example, consider the pure 4d $SU(2)$ gauge theory, with the SW curve as given in (2.78) and appendix A.1. At strong coupling, the BPS spectrum consists of the monopole $\pm(1,0)$ and dyon $\pm(-1,2)$, while at weak coupling we have a tower of dyons and the W-boson [7, 196]:

$$\mathscr{S}_S \;:\; (1,0)\,, \quad (1,\pm 2)\,, \qquad\qquad \mathscr{S}_W \;:\; (0,2)\,, \quad (1,2n)\,,\; n \in \mathbb{Z}\,. \tag{3.71}$$

In either regime, the spectrum is left invariant by $g^{\mathscr{E}} = \left(0, \frac{1}{2}\right)$, following the notation (3.68). We therefore have an electric one-form symmetry $\mathbb{Z}_2 \subset U(1)_e^{[1]}$, as expected from the UV description [19]. This result is also in agreement with the $\mathbb{Z}_2$ torsion section of the SW geometry $(I_4^*; 2I_1)$.

3.3 Coulomb branch configurations

3.3.1 A first example: the E_1 and $\widetilde{E}_1$ theories

As a first example, we explore the U-planes of the E_1 and $\widetilde{E}_1$ theories. The corresponding toric geometries in Type-IIA, and their Type-IIB mirrors, have been well studied in the literature – see *e.g.* [18, 58, 62, 197]. Here, we focus on the 5d interpretation and conduct a systematic analysis of the possible Coulomb branch configurations. We will solve the PF equations for the physical periods as explicitly as possible for some interesting values of the masses in chapter 4, where we also discuss the modular properties of the U-plane.

The E_1 theory – 5d $SU(2)_0$. Let us first consider the E_1 theory, which is the UV completion of the five-dimensional $SU(2)_0$ gauge theory [15]. Its SW curve was first derived and studied in [18, 49]. The 'toric' expression (2.67) for the curve can be brought to the Weierstrass form (1.14), with:

$$\begin{aligned} g_2(U) &= \frac{1}{12}\Big(U^4 - 8(1+\lambda)U^2 + 16\left(1-\lambda+\lambda^2\right)\Big)\ , \\ g_3(U) &= -\frac{1}{216}\Big(U^6 - 12(1+\lambda)U^4 + 24\left(2+\lambda+2\lambda^2\right)U^2 - 32\left(2-3\lambda-3\lambda^2+2\lambda^3\right)\Big)\ , \end{aligned} \tag{3.72}$$

and with discriminant:

$$\Delta(U) = \lambda^2\Big(U^4 - 8(1+\lambda)U^2 + 16(1-\lambda)^2\Big)\ . \tag{3.73}$$

At generic values of λ, the discriminant has four distinct roots, and we have four distinct I_1 singularities in the interior of the U-plane, plus the I_8 singularity at infinity – see figure 3.2. Note that g_2 and g_3 in (3.72) depend on U^2 instead of U, and therefore the $\mathbb{Z}_2$ action:

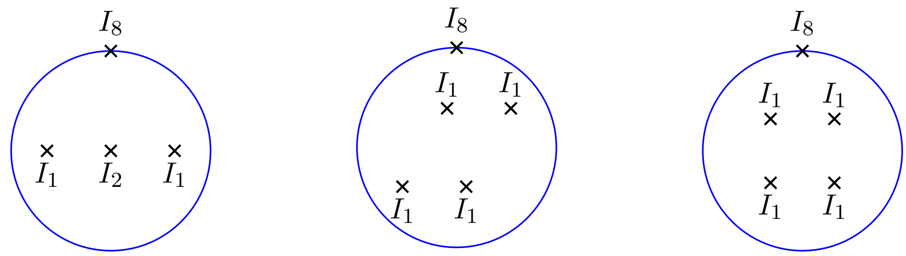
$$\mathbb{Z}_2\ :\quad U \to -U\ , \tag{3.74}$$

is a symmetry of the U-plane for any value of the complexified 5d gauge coupling, λ. This symmetry has a simple physical explanation. Recall that U is defined as the expectation value of the five-dimensional fundamental Wilson loop wrapped on S^1. Then (3.74) is precisely the action of the $\mathbb{Z}_2$ one-form symmetry of the E_1 theory [135, 136], which gives rise to both a one-form and an ordinary (zero-form) symmetry of the KK theory $D_{S^1}E_1$. Both are spontaneously broken on the Coulomb branch. More details on gauging these symmetries from a CB perspective can be found in [4].

Let us study the U-plane in some detail. There are two configurations of singular fibers depending on the value of the parameter λ, as shown below:

$\{F_v\}$	λ	$\mathfrak{g}_F$	$\mathrm{rk}(\Phi)$	Φ_{tor}
$(I_8^\infty; 2I_1, I_2)$	$\lambda = 1$	$\mathfrak{su}(2)$	0	$\mathbb{Z}_4$
$(I_8^\infty; 4I_1)$	$\lambda \neq 1$	$\mathfrak{u}(1)$	1	$\mathbb{Z}_2$

(3.75)



(a) Massless case, $\lambda = 1$. (b) $\lambda = e^{\frac{2\pi i}{3}}$. (c) $\lambda = -1$.

Figure 3.2: The U-plane of the $D_{S^1}E_1$ theory for some values of λ. Notice the $\mathbb{Z}_2$ symmetry, which is enhanced to $\mathbb{Z}_4$ at $\lambda = -1$.

The case $\lambda = 1$ is the massless point, which gives us the low-energy description of the 5d SCFT on $\mathbb{R}^4 \times S^1$, with vanishing real masses and without any non-trivial flavour Wilson line. For $\lambda \neq 1$, the corresponding configuration $(I_8, 4I_1)$ breaks the $\mathfrak{su}(2)$ flavour algebra to the Cartan subalgebra. The point $\lambda = -1$ corresponds to setting to zero the fundamental flavour Wilson line for $E_1 = \mathfrak{su}(2)$:

$$\chi_1^{E_1} = \sqrt{\lambda} + \frac{1}{\sqrt{\lambda}} = 0 \ . \tag{3.76}$$

In this case, the U-plane turns out to be $\mathbb{Z}_4$ symmetric. The massless E_1 curve has three non-trivial sections:

$$P_1 = \left(\frac{1}{12}(U^2+4), -U\right) \ , \quad P_2 = \left(\frac{1}{12}(U^2-8), 0\right) \ , \quad P_3 = \left(\frac{1}{12}(U^2+4), U\right) \ , \tag{3.77}$$

which span a $\mathbb{Z}_4$ torsion group with $P_k + P_l = P_{k+l \ (\mathrm{mod}\ 4)}$. Let us note that the sections P_1 and P_3 intersect non-trivially the I_2 singular fiber at $U = 0$. The remaining section, P_2, only intersects the 'trivial' component of this fiber and therefore generates a $\mathbb{Z}_2^{[1]}$ subgroup which injects in the torsion group $\mathbb{Z}_4$ according to (3.50). The group $\mathscr{F} = \mathbb{Z}_2^{(f)} = \mathbb{Z}_4/\mathbb{Z}_2^{[1]}$ then constrains the global form of the flavour group to be:

$$G_F = SU(2)/\mathbb{Z}_2^{(f)} \cong SO(3) \ , \tag{3.78}$$

in agreement with [162]. We also identify the $\mathbb{Z}_2^{[1]}$ subgroup of the MW group as the one-form symmetry of the E_1 theory. In fact, for generic mass deformations $\lambda \neq 1$, a $\mathbb{Z}_2$ torsion section is still preserved, with the MW group for the $(I_8; 4I_1)$ configuration being $\Phi = \mathbb{Z} \oplus \mathbb{Z}_2$. The $U(1)$ symmetry is generated by the horizontal divisor $\varphi(P)$ associated to the section:

$$P = \left(\frac{U^2 + 4(2-\lambda)}{12} \; , -U \right) \; , \tag{3.79}$$

which generates the free part of Φ, and reduces to the $\mathbb{Z}_4$ generator P_1 in (3.77) when $\lambda = 1$. The $\mathbb{Z}_2$ torsion section reads

$$P_{\rm tor} = \left(\frac{U^2 - 4(1+\lambda)}{12} \; , 0 \right) \; , \tag{3.80}$$

which reduces to P_2 in (3.77) when $\lambda = 1$. For any $\lambda \neq 1$, we have $\Phi_{\rm tor} = \mathcal{Z}^{[1]} = \mathbb{Z}_2^{[1]}$, consistent with our identification of $\mathcal{Z}^{[1]}$ with the one-form symmetry of the field theory.

The $\widetilde{E}_1$ theory – 5d $SU(2)_\pi$. The $\widetilde{E}_1$ theory is the UV completion of the parity-violating $SU(2)_\pi$ gauge theory in 5d. Let us briefly discuss its U-plane. The Weierstrass form of the curve (2.70) reads:

$$\begin{aligned} g_2(U) &= \frac{1}{12}\left(U^4 - 8U^2 - 24\lambda U + 16\right) \; , \\ g_3(U) &= -\frac{1}{216}\left(U^6 - 12U^4 - 36\lambda U^3 + 48U^2 + 216\lambda^2 - 64\right) \; , \end{aligned} \tag{3.81}$$

with the massless limit corresponding to $\lambda = 1$. By direct inspection, we find the following allowed configurations of singular fibers:

(3.82)

$\{F_v\}$	λ	$\mathfrak{g}_F$	rk(Φ)	$\Phi_{\rm tor}$
$(I_8^\infty; 2I_1, II)$	$\lambda = \pm\frac{16i}{3\sqrt{3}}$	$\mathfrak{u}(1)$	1	–
$(I_8^\infty, 4I_1)$	$\lambda \neq \pm\frac{16i}{3\sqrt{3}}$	$\mathfrak{u}(1)$	1	–

This is of course in agreement with the Persson classification [11]. As for E_1, the generic point on the Coulomb branch of $\widetilde{E}_1$ has $4I_1$-type singularities. It is worth pointing out that the classification of rational elliptic surfaces includes two distinct configurations with singular fibers $(I_8; 4I_1)$, which are distinguished by their MW torsion. That mathematical fact dovetails nicely with the existence of two distinct theories with T^8 monodromy at large volume, E_1 and $\widetilde{E}_1$, with only the former having a non-trivial one-form symmetry [135,136].

The $\widetilde{E}_1$ Coulomb branch exhibits a feature that did not appear on the CB of the E_1 theory, however: there exists an allowed configuration with a singularity of type II, whose low-energy description is in terms of H_0, the Argyres-Douglas theory without flavour symmetry. H_0 also appears on the Coulomb branch of the $SU(2)$ theory with $N_f = 1$, at a point where two mutually non-local BPS particles $\mathcal{E}_{1,2}$ with Dirac pairing $\langle \mathcal{E}_1, \mathcal{E}_2 \rangle = 1$ become simultaneously massless. This leads to a rather intriguing RG flow from $\widetilde{E}_1$ to the H_0 AD model. We review such flows in more detail in the following section.

3.3.2 Non-toric Seiberg-Witten curves and RG flows

The SW curves for the non-toric dP_n geometries can be determined as limits of the E-string theory SW curve [17, 159, 160]. These curves are most easily written in terms of the E_n characters:

$$\chi_{\mathcal{R}}^{E_n}(\nu) = \sum_{\rho \in \mathcal{R}} e^{2\pi i \rho(\nu)} , \tag{3.83}$$

for $\rho = (\rho_i)$ the weights of the representation $\mathcal{R}$, and $\nu = (\nu_i)$ the E_n flavour parameters, with the index $i \in \{1, \dots, n\}$. The explicit relation between these parameters and the 5d gauge-theory parameters is explained in [1]. We give the explicit form of the curves in appendix A. The massless $D_{S^1}E_n$ curves correspond to the S^1 reduction of the 5d SCFTs, with no mass deformations turned on. The massless limit of these curves is obtained by setting the characters equal to the dimension of the corresponding representation. For the $D_{S^1}E_{6,7,8}$ theories, they read:

$$\begin{aligned} D_{S^1}E_8 \ &: \ y^2 = 4x^3 - \frac{1}{12}U^4 x + \frac{1}{216}(U-864)U^5 \ , \\ D_{S^1}E_7 \ &: \ y^2 = 4x^3 - \frac{1}{12}(U-36)(U+12)^3 x + \frac{1}{216}(U-60)(U+12)^5 \ , \\ D_{S^1}E_6 \ &: \ y^2 = 4x^3 - \frac{1}{12}(U-18)(U+6)^3 x + \frac{1}{216}(U^2-24U+36)(U+6)^4 \ , \end{aligned} \tag{3.84}$$

with the following singular fibers on the U-plane:

$$\begin{aligned} D_{S^1}E_8 &: \quad (I_1)^\infty \oplus II^* \oplus I_1 \ , \\ D_{S^1}E_7 &: \quad (I_2)^\infty \oplus III^* \oplus I_1 \ , \\ D_{S^1}E_6 &: \quad (I_3)^\infty \oplus IV^* \oplus I_1 \ . \end{aligned} \tag{3.85}$$

Note, in particular, that the E_n flavour symmetry can be directly observed from the singular fibers of the Seiberg-Witten geometry in the massless limit. This manifestation of the flavour symmetry in the mirror threefold occurs for all $D_{S^1}E_n$ theories. Furthermore, from this configuration of singular fibers it is straightforward to obtain the four-dimensional limit of these theories. This is done by identifying the I_1 singularities with the KK charge and decoupling it from the bulk by 'zooming in' around the E_n type Kodaira singularity on the U-plane. It is well known that these theories flow in 4d to the Minahan-Nemeschansky (MN) theories [43,44], which have the following scaling dimensions for the Coulomb branch parameter: $(\Delta_{E_8}, \Delta_{E_7}, \Delta_{E_6}) = (6, 4, 3)$. Thus, we have:

$$\begin{aligned} D_{S^1}E_8 &: \quad (U, x, y) \longrightarrow (\beta^6 u, \beta^{10} x, \beta^{15} y) \ , \\ D_{S^1}E_7 &: \quad (U, x, y) \longrightarrow (\beta^4 u - 12, \beta^6 x, \beta^9 y) \ , \\ D_{S^1}E_6 &: \quad (U, x, y) \longrightarrow (\beta^3 u - 6, \beta^4 x, \beta^6 y) \ , \end{aligned} \tag{3.86}$$

including constant shifts to bring the relevant singularity to the origin of the 4d u-plane. This leads to the massless SW curves for the 4d MN theories:

$$\begin{aligned} E_8^{(4\mathrm{d})} &: \quad y^2 = 4x^3 - 4u^5 \ , \\ E_7^{(4\mathrm{d})} &: \quad y^2 = 4x^3 + 4u^3 x \ , \\ E_6^{(4\mathrm{d})} &: \quad y^2 = 4x^3 + u^4 \ , \end{aligned} \tag{3.87}$$

which are standard $D_{S^1}E_n$ double-point singularities at the origin of $(x, y, u) \in \mathbb{C}^3$. One can also reproduce the deformation patterns of these singularities by keeping track of the various 5d mass parameters [159,160].

The other massless E_n curves can be analysed in a similar way. One finds that the

U-plane has the following singularities, in addition to the I_{9-n} singularity at infinity [17]:

$$\begin{aligned}
D_{S^1}E_5 &: \quad (I_4)^\infty \oplus I_1^* \oplus I_1 \ , \\
D_{S^1}E_4 &: \quad (I_5)^\infty \oplus I_5 \oplus I_1 \oplus I_1 \ , \\
D_{S^1}E_3 &: \quad (I_6)^\infty \oplus I_3 \oplus I_2 \oplus I_1 \ , \\
D_{S^1}E_2 &: \quad (I_7)^\infty \oplus I_2 \oplus I_1 \oplus I_1 \oplus I_1 \ , \\
D_{S^1}E_1 &: \quad (I_8)^\infty \oplus I_2 \oplus I_1 \oplus I_1 \ , \\
D_{S^1}\widetilde{E}_1 &: \quad (I_8)^\infty \oplus I_1 \oplus I_1 \oplus I_1 \oplus I_1 \ , \\
D_{S^1}E_0 &: \quad (I_9)^\infty \oplus I_1 \oplus I_1 \oplus I_1 \ .
\end{aligned} \tag{3.88}$$

The 4d low-energy effective field theories obtained from the circle compactification of the 5d E_n SCFTs are IR free for $n < 6$. Interestingly, the E_5 theory, which has a gauge-theory phase corresponding to $SU(2)$, $N_f = 4$ in five dimensions, becomes an $SU(2)$ theory with $N_f = 5$ upon S^1 reduction, which matches the $E_5 = \mathfrak{so}(10)$ symmetry of the UV theory. In some sense, the 'instanton particle' becomes a perturbative hypermultiplet in four-dimensions, but it is more accurate to say that the full IR-free $SU(2)$ description is a magnetic dual description of the UV theory.

For the E_4 theory, we have an I_5 point, corresponding to SQED with five flavours, which again reproduces the $E_4 = \mathfrak{su}(5)$ symmetry. Note that the E_3 theory is special in that there are now two distinct points with a non-trivial Higgs branch. This matches with the fact that the Higgs branch of the 5d SCFT E_3 is the union of two cones, on which each of the two factors in $E_3 = \mathfrak{su}(3) \oplus \mathfrak{su}(2)$ act independently. In 4d, the instanton corrections separate the two Higgs branch cones along the complexified Coulomb branch.

Similarly, the E_2 and E_1 theories both have an $\mathfrak{su}(2)$ symmetry that is reproduced by an I_2 singularity. On the other hand, the abelian part of $E_2 = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ and $\widetilde{E}_1 = \mathfrak{u}(1)$ is encoded in the Seiberg-Witten geometry in a more subtle manner, through the Mordell-Weil group, as discussed earlier in the chapter.

An advantage of the formalism of rational elliptic surfaces is that RG flows become rather natural. Searching through Persson's list based on the distinguished fiber at infinity allows us to find some rather peculiar flows, as shown in figure 3.3. Consider, for instance,

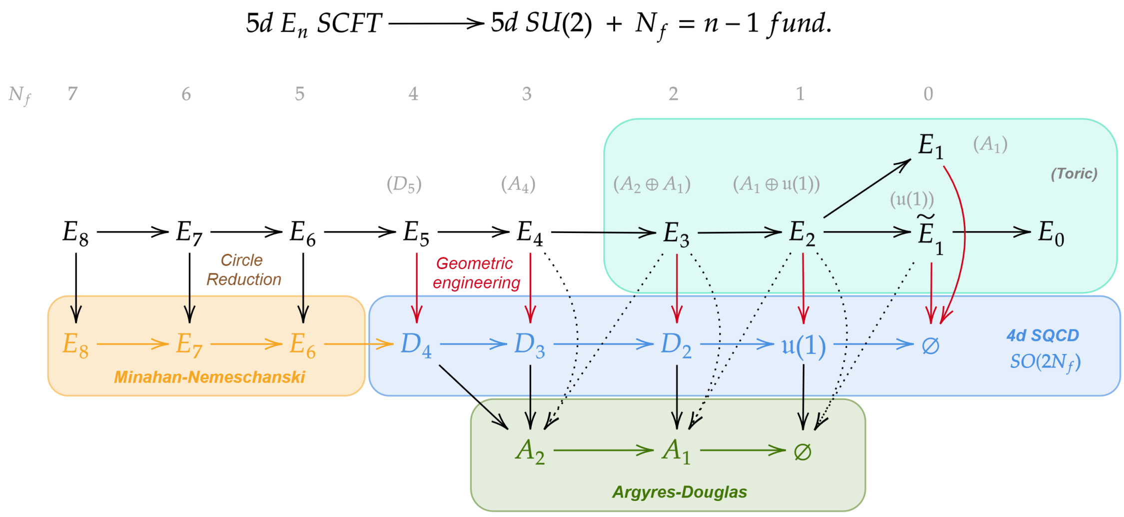


Figure 3.3: RG flows among 4d $\mathcal{N} = 2$ (KK) theories. The top rows contain the $D_{S^1}E_n$ theories, while the bottom two rows involve the $E_{6,7,8}$ Minahan-Nemeschansky theories, 4d SQCD theories, and the rank-one Argyres-Douglas theories.

the $D_{S^1}E_3$ theory, with $F_\infty = I_6$. This theory has a gauge theory phase corresponding to 5d $SU(2)$ $N_f = 2$ and, in the geometric engineering limit (2.73), it flows to the 4d $SU(2)$ $N_f = 2$ theory, which has $F_\infty = I_2^*$. Schematically, in the geometric engineering limit, two I_1 fibers from the bulk of the $D_{S^1}E_3$ theory are decoupled, such that upon merging with the I_6 fiber the I_2^* singularity arises at large U.

The simplest 4d $\mathcal{N} = 2$ SCFTs have been found by searching on the Coulomb branches of SQCD theories, and it is by now well-known that the rank-one Argyres-Douglas SCFTs arise at special values of the mass parameters on the Coulomb branches of $SU(2)$ gauge theories with fundamental matter. From the perspective of rational elliptic surfaces, we can have the following configurations of singular fibers for 4d SQCD, for example, where the SW curves are listed in appendix A:

$\mathcal{T}$	F_∞	$\{F_v\}$	m_i
$SU(2)\ N_f = 1$	I_3^*	$II \oplus I_1$	$m_1^3 = \frac{27}{16}\Lambda^3$
$SU(2)\ N_f = 2$	I_2^*	$III \oplus I_1$	$m_1 = m_2 = \pm\Lambda$
$SU(2)\ N_f = 3$	I_1^*	$IV \oplus I_1$	$m_1 = m_2 = m_3 = \Lambda/2$

(3.89)

Then, taking a 'zooming in' limit, similar to what we have done for the massless $D_{S^1}E_{6,7,8}$ curves, we recover the curves for the Argyres-Douglas models H_0, H_1 and H_2, respectively. These are sometimes also referred to as (A_1, A_2), (A_1, A_3) and (A_1, D_4), respectively, with their flavour symmetry algebras being $\emptyset, A_1$ and A_2. Note that an RG flow from the $N_f = 3$ theory to any of the other AD models is clearly possible, by decomposing the IV singularity in the above configuration.

Let us now return to the 5d $D_{S^1}E_3$ theory. Since the geometric engineering limit leads to the 4d $SU(2)$ $N_f = 2$ theory, we thus expect that a flow to the AD H_1 model should be possible. What is perhaps more surprising, however, is that a flow to the AD H_2 model can be realised by turning on some holonomy around the S^1 direction. This is quite remarkable, as the latter can only be found on the Coulomb branch of the 4d $SU(2)$ theory with $N_f = 3$ fundamentals. The relevant configuration for the $D_{S^1}E_3$ theory is given below:

$$D_{S^1}E_3 : \quad (I_6)^\infty \oplus IV \oplus 2I_1 \ , \tag{3.90}$$

which can be achieved by setting $\lambda = 1$ and $M_1 = -M_2 = i$. The limit to the 4d AD H_2 model is realised by:

$$U \to u\ \beta^{\frac{3}{2}} + 2c\ \beta^{\frac{1}{2}} \ , \qquad \lambda \to 1 - \mu_1\beta \ , \qquad M_{1,2} \to \pm i - c\ \beta^{\frac{1}{2}} \pm \frac{i\ \mu_2}{2}\beta \ , \tag{3.91}$$

where β is the dimension of the S^1 direction and (c, μ_1, μ_2) are the parameters of the AD model, having scaling dimensions $\left(\frac{1}{2}, 1, 1\right)$. The existence of this type of RG flow was implied by the work of Bonelli, Del Monte, and Tanzini [51] relating the 5d BPS quiver of $D_{S^1}E_3$ to the gauge/Painlevé correspondence [52–54]. In this example, the $\mathfrak{su}(3)$ flavour symmetry of H_2 is inherited from the symmetry $E_3 = \mathfrak{su}(3) \oplus \mathfrak{su}(2)$ of the larger theory, which arises due to 'infinite-coupling effects' from the 5d gauge theory point of view and is related to the condensation of instanton particles – see *e.g.* [198]. The AD points are, in fact, ubiquitous on the extended Coulomb branch of the $D_{S^1}E_n$ theories as we tune the mass parameters, rendering such flows rather natural. These are shown explicitly in figure 3.3, with the dashed arrows being the 'new' RG flows that we discover. For the explicit form of all these

$\{F_v\}$	*Notation*	$\Phi_{\rm tor}$	*4d theory*	$\mathfrak{g}_F$	G_F
II^*, II	X_{22}	-	AD H_0	-	-
			E_8 MN	E_8	E_8
III^*, III	X_{33}	$\mathbb{Z}_2$	AD H_1	A_1	$SO(3)$
			E_7 MN	E_7	$E_7/\mathbb{Z}_2$
IV^*, IV	X_{44}	$\mathbb{Z}_3$	AD H_2	A_2	$SU(3)/\mathbb{Z}_3$
			E_8 MN	E_6	$E_6/\mathbb{Z}_3$
I_0^*, I_0^*	$X_{11}(j)$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$SU(2), N_f=4$	D_4	$\mathrm{Spin}(8)/\mathbb{Z}_2\times\mathbb{Z}_2$

Table 3.2: Extremal rational elliptic surfaces without multiplicative (*i.e.* I_k) fibers.

RG flows, we refer to [1].

3.3.3 Extremal rational elliptic surfaces

A small and particularly interesting subset of all rational elliptic surfaces consists of those with a Mordell-Weil group of rank zero, $\mathrm{rk}(\Phi) = 0$, which are called *extremal*. There are only 16 of them, as classified by Miranda and Persson [199]. We list them in tables 3.2 and 3.3. By our general discussion, they correspond to Coulomb branch configurations with a semi-simple flavour symmetry. A given extremal RES generally corresponds to several 4d $\mathcal{N}=2$ theories $\mathcal{T}_{F_\infty}$, simply by choosing which of the Kodaira fibers sits 'at infinity' on the one-dimensional Coulomb branch.

The four surfaces listed in table 3.2 do not have any multiplicative fibers, and therefore they cannot correspond to the $D_{S^1}E_n$ theories, which have $F_\infty = I_{9-n}$. Instead, they correspond to the seven 'classic' 4d SCFTs associated to the 7 additive Kodaira singularities II, III, IV, II^*, III^*, IV^* and I_0^* – this was previously discussed in [36]. In each case, the massless curve has a single Kodaira singularity at the origin, and therefore the singularity at infinity is such that $\mathbb{M}_0\mathbb{M}_\infty = \mathbf{1}$. Thus, the first three extremal surfaces in table 3.2 describe both the E_n Minahan-Nemeschansky theories [43, 44] and the three rank-one AD theories. The last surface, $X_{11}(j)$, describes $SU(2)$ with four flavours. It is the only extremal surface that comes in a one-dimensional family [199] (all the other extremal fibrations are unique), corresponding to the marginal gauge coupling of this 4d SCFT.

The remaining 12 extremal RES are listed in table 3.3. These are also all the extremal

$\{F_v\}$	*Notation*	Φ_{tor}	*Field theory*	$\mathfrak{g}_F$	*Modularity*
II^*, I_1, I_1	X_{211}	–	$D_{S^1}E_8$	E_8	–
			AD H_0	–	
III^*, I_2, I_1	X_{321}	$\mathbb{Z}_2$	$D_{S^1}E_8$	$E_7 \oplus A_1$	$\Gamma_0(2)$
			$D_{S^1}E_7$	E_7	
			AD H_1	A_1	
IV^*, I_3, I_1	X_{431}	$\mathbb{Z}_3$	$D_{S^1}E_8$	$E_6 \oplus A_2$	$\Gamma_0(3)$
			$D_{S^1}E_6$	E_6	
			AD H_2	A_2	
I_4^*, I_1, I_1	X_{411}	$\mathbb{Z}_2$	$D_{S^1}E_8$	D_8	$\Gamma_0(4)$
			4d pure $SU(2)$	–	
I_1^*, I_4, I_1	X_{141}	$\mathbb{Z}_4$	$D_{S^1}E_8$	$D_5 \oplus A_3$	$\Gamma_0(4)$
			$D_{S^1}E_5$	D_5	
			4d $SU(2)\, N_f = 3$	A_3	
I_2^*, I_2, I_2	X_{222}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D_{S^1}E_7$	$D_6 \oplus A_1$	$\Gamma(2)$
			4d $SU(2)\, N_f = 2$	$A_1 \oplus A_1$	
I_9, I_1, I_1, I_1	X_{9111}	$\mathbb{Z}_3$	$D_{S^1}E_8$	A_8	$\Gamma_0(9)$
			$D_{S^1}E_0$	–	
I_8, I_2, I_1, I_1	X_{8211}	$\mathbb{Z}_4$	$D_{S^1}E_8$	$A_7 \oplus A_1$	$\Gamma_0(8)$
			$D_{S^1}E_7$	A_7	
			$D_{S^1}E_1$	A_1	
I_5, I_5, I_1, I_1	X_{5511}	$\mathbb{Z}_5$	$D_{S^1}E_8$	$A_4 \oplus A_4$	$\Gamma_1(5)$
			$D_{S^1}E_4$	A_4	
I_6, I_3, I_2, I_1	X_{6321}	$\mathbb{Z}_6$	$D_{S^1}E_8$	$A_5 \oplus A_2 \oplus A_1$	$\Gamma_0(6)$
			$D_{S^1}E_7$	$A_5 \oplus A_2$	
			$D_{S^1}E_6$	$A_5 \oplus A_1$	
			$D_{S^1}E_3$	$A_2 \oplus A_1$	
I_4, I_4, I_2, I_2	X_{4422}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$D_{S^1}E_7$	$A_3 \oplus A_3 \oplus A_1$	$\Gamma_0(4) \cap \Gamma(2)$
			$D_{S^1}E_5$	$A_3 \oplus A_1 \oplus A_1$	
I_3, I_3, I_3, I_3	X_{3333}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$D_{S^1}E_6$	$A_2 \oplus A_2 \oplus A_2$	$\Gamma(3)$

Table 3.3: Extremal rational elliptic surfaces with I_k fibers and corresponding field theories. The configurations corresponding to massless limits of the theories are emphasized in blue.

RES that have more than 2 singular fibers – in fact, they can only have 3 or 4 singular fibers. The first and second columns in table 3.3 indicate the singular fibers and the names of the corresponding surfaces in the notation of [199]. The third column gives the MW group of the elliptic fibration, which is purely torsion. The fourth column lists the 4d $\mathcal{N} = 2$ (KK) field theories for which this extremal RES describes a CB configuration, while the fifth column gives the unbroken flavour symmetry algebra in each case. The last column in table 3.3 indicates the modular group of the surface, up to conjugacy. We will discuss these modular properties in more detail in the following chapter.

From the MW torsion of these surfaces, one can also deduce the flavour symmetry group, according to (3.50). Let us exemplify this using 4d SQCD. As we explained in the previous section, the SW geometry of the 4d $\mathcal{N} = 2$ $SU(2)$ gauge theory coupled to N_f fundamental hypermultiplets is described by rational elliptic surfaces with $F_\infty = I^*_{4-N_f}$. In the limit of vanishing quark masses, the flavour symmetry group for $N_f > 1$ is the quotient of $\text{Spin}(2N_f)$ by its center, namely:

N_f	2	3	4
G_F	$(SU(2)/\mathbb{Z}_2) \times (SU(2)/\mathbb{Z}_2)$	$SU(4)/\mathbb{Z}_4$	$\text{Spin}(8)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

(3.92)

This can be shown by computing the intersections of the torsion sections with the singular fibers of the SW curves, as we have done for $D_{S^1}E_1$ in the previous section.

For $N_f = 1$, the flavour symmetry is abelian. For $N_f = 0$, the flavour symmetry group is trivial and we have a $\mathbb{Z}_2$ electric one-form symmetry, $\mathcal{Z}^{[1]} = \mathbb{Z}_2$. The flavour symmetry groups (3.92) are easily understood in the free UV description: there is an $SO(2N_f)$ symmetry acting on $2N_f$ half-hypermultiplets in the fundamental of the $SU(2)$ gauge group, but the action of the $\mathbb{Z}_2$ center of $SO(2N_f)$ on the matter fields is equivalent to the action of the center of the gauge $SU(2)$. Therefore, the actual *flavour* symmetry is $SO(2N_f)/\mathbb{Z}_2$, which the same as (3.92). At first sight, this appears to be in tension with the discussion in [8], where it is shown that various dyons sit in spinors of $\text{Spin}(2N_f)$. These are not gauge-invariant states, however, thus there is no contradiction. As a further confirmation, note that this global form of the flavour symmetry group is in perfect agreement with the Schur index as given in [200].

All the massless E_n KK theories other than E_2 and $\widetilde{E}_1$ appear in table 3.3. These last two are the exceptions because their flavour group includes one $U(1)$ factor, and therefore the corresponding rational elliptic surfaces have $\mathrm{rk}(\Phi) = 1$. (Similarly so for 4d $SU(2)$ with $N_f = 1$.) Finally, let us note that the last configuration in table 3.3, X_{3333}, gives the so-called T_3 description of the E_6 theory, in which only an A_2^3 algebra is manifest; similarly, the configuration X_{4422} for E_7 with $A_3^2 \oplus A_1$ realised, and the configuration X_{6321} for E_8 with $A_5 \oplus A_2 \oplus A_1$ realised, can be obtained by Higgsing from the T_4 and T_6 theories, respectively [201].

Chapter 4

Modular Coulomb branches and BPS quivers

An important subset of rational elliptic surfaces consists of the modular RES, in which case the U-plane is isomorphic to a region of the upper half-plane $\mathbb{H}$, which is a fundamental domain for some subgroup Γ of the modular group $\mathrm{PSL}(2,\mathbb{Z})$.[28] In this construction, the singular fibers of the Seiberg-Witten geometry are mapped to the cusps and elliptic points of Γ, while $U(\tau)$ is a modular function for Γ.

As the SW geometry is a *rational* elliptic surface, the constraint on its Euler number translates into a constraint on the index of Γ inside $\mathrm{PSL}(2,\mathbb{Z})$, namely, this index must be at most equal to 12, which occurs when all singular fibers are multiplicative (*i.e.* of I_n-type). Using this constraint together with the explicit mapping of the singular fibers to the special points of Γ, as well as the classification of finite index subgroups of the modular group [83–85], we are able to provide a list of all modular rational elliptic surfaces [2].

Modularity provides a major simplification of the low-energy dynamics of a 4d $\mathcal{N}=2$ QFT. In particular, it allows one to find the monodromies around the CB singularities directly from the properties of the group Γ, as its fundamental domain is isomorphic to the U-plane. As such, we propose a simple prescription for obtaining BPS quivers [64, 71, 202–204] from fundamental domains of 4d $\mathcal{N}=2$ (KK) theories.

4.1 Modular elliptic surfaces

It has already been known since the original work of Seiberg and Witten [7, 8] that the curve of the pure 4d $SU(2)$ theory is the modular curve $\mathbb{H}/\Gamma^0(4)$, where $\Gamma^0(4)$ is a congruence

[28] Note that $\mathrm{PSL}(2,\mathbb{Z})$, which is the quotient of $\mathrm{SL}(2,\mathbb{Z})$ by its center, is the preferred version of the modular group for our purposes, due to the quadratic twist operation introduced in (3.10), as we will shortly see.

subgroup of $\mathrm{PSL}(2,\mathbb{Z})$. The modular properties of the other SQCD theories were found in [205], and more recently discussed in great detail in [79–81]. Furthermore, this latter work also includes fundamental domains of configurations that are not modular, in which case the map $U \mapsto \tau$ is not one-to-one anymore. In such cases, the fundamental domains cannot be constructed as quotients $\mathbb{H}/\Gamma$ due to the existence of branch points and branch cuts on the upper half-plane.

4.1.1 Subgroups of the modular group

For our purposes, it will be useful to introduce certain subgroups of the modular group $\mathrm{PSL}(2,\mathbb{Z})$. We first introduce the principal congruence subgroups of level N, described as:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\} , \tag{4.1}$$

which can be viewed as the kernel of the group homomorphism $\mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z}_N)$. The subgroups Γ of $\mathrm{PSL}(2,\mathbb{Z})$ containing the principal congruence subgroup $\Gamma(N)$ are called *congruence subgroups*, with the *level* being the smallest such positive integer N. The most common level-N congruence subgroups are:

$$\begin{aligned} \Gamma_0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) : c = 0 \mod N \right\} . \\ \Gamma_1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod N \right\} . \end{aligned} \tag{4.2}$$

Similarly, we define the groups $\Gamma^0(N)$ and $\Gamma^1(N)$, by imposing $b = 0 \mod N$ instead of the above constraints. Note that these are related to the $\Gamma_0(N)$ and $\Gamma_1(N)$ groups by conjugation by S. We also have the following inclusions:

$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) , \tag{4.3}$$

with $\Gamma_1(N) \cong \Gamma_0(N)$ for $N = 2, 3, 4$ and 6. Note that these congruences are no longer satisfied in $\mathrm{SL}(2,\mathbb{Z})$, unless $N = 2$.

Non-congruence subgroups are those that do not contain $\Gamma(N)$ as a subgroup and are much less studied in the mathematical literature. In most cases their modular forms do not have closed-form expressions; moreover, it was conjectured that the Fourier coefficients of modular forms of non-congruence subgroups have unbounded denominators, a conjecture only recently proved in [206].

Given a subgroup $\Gamma \in \mathrm{PSL}(2,\mathbb{Z})$, its index $n_\Gamma = [\Gamma(1) : \Gamma]$ in $\Gamma(1) \cong \mathrm{PSL}(2,\mathbb{Z})$ is the number of right-cosets of Γ in the modular group. As a result, we have:

$$\mathrm{PSL}(2,\mathbb{Z}) = \bigsqcup_{i=1}^{n_\Gamma} \Gamma\ \alpha_i \ , \qquad \alpha_i \in \mathrm{PSL}(2,\mathbb{Z}) \ , \tag{4.4}$$

for a list of coset representatives $\{\alpha_i\}$. The elements of the modular group act on the upper half-plane $\mathbb{H}$ as:

$$\tau \mapsto \gamma\tau = \frac{a\tau + b}{c\tau + d} \ , \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) \ , \qquad \forall \tau \in \mathbb{H} \ . \tag{4.5}$$

It then follows that a fundamental domain for the subgroup Γ is defined as an open subset $\mathcal{F}_\Gamma \subset \mathbb{H}$ of the upper half-plane, such that no two distinct points are equivalent under the action of Γ unless they are on the boundary of $\mathcal{F}_\Gamma$; furthermore, under the action of Γ, any point of $\mathbb{H}$ is mapped to the closure of $\mathcal{F}_\Gamma$. Let us denote the fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$ by $\mathcal{F}_0$. The upper half-plane $\mathbb{H}$ is then obtained by the action of the modular group on $\mathcal{F}_0$. The fundamental domain of $\Gamma \subset \mathrm{PSL}(2,\mathbb{Z})$ can be obtained from a list of coset representatives $\{\alpha_i\}$, since:

$$\mathbb{H} = \left(\bigsqcup_{i=1}^{n_\Gamma} \Gamma\ \alpha_i \right) \mathcal{F}_0 = \Gamma \left(\bigsqcup_{i=1}^{n_\Gamma} \alpha_i \mathcal{F}_0 \right) \ . \tag{4.6}$$

Thus, the fundamental domain of Γ is the disjoint union $\mathcal{F}_\Gamma = \bigsqcup \alpha_i \mathcal{F}_0$, with the coset representatives chosen such that $\mathcal{F}_\Gamma$ has a connected interior.[29]

A *cusp* of Γ is defined as an equivalence class in $\mathbb{Q} \cup \{\infty\}$ under the action of Γ. The $\mathrm{PSL}(2,\mathbb{Z})$ group has only one cusp, with the representative usually chosen as $\tau_\infty = i\infty$. The *width* of the cusp τ_∞ in Γ is the smallest integer w such that $T^w \in \Gamma$. More generally,

[29]For the standard groups introduced above, fundamental domains can be drawn with Helena Verrill's Java applet [207].

for a cusp at $\widetilde{\tau} = \gamma\tau_\infty$, the width is defined as the width of τ_∞ for the group $\gamma^{-1}\Gamma\gamma$. The cusps other than τ_∞ are typically chosen as the points of intersection of the fundamental domain with the real axis.

The other special points in the fundamental domain are the *elliptic* points, which are those points with non-trivial stabilizer, *i.e.* $\gamma_e\tau_* = \tau_*$ for some non-trivial element $\gamma_e \in \Gamma$. The elements γ_e are called the elliptic elements of Γ. It can be shown that the elliptic points always lie on the boundary of the fundamental domain. Finally, the order of an elliptic point τ_* is the order of the stabilizing subgroup of τ_* in Γ. For $\mathrm{PSL}(2,\mathbb{Z})$ the only elliptic points are $\tau_0 \in \{i, e^{\frac{2\pi i}{3}}\}$, with stabilizers $\langle S\rangle$ and $\langle ST\rangle$, of order 2 and 3, respectively. One can prove that, for a given finite index subgroup Γ with fundamental domain $\mathcal{F}$, the elliptic points $\tau_* \in \mathcal{F}$ are always in the $\mathrm{PSL}(2,\mathbb{Z})$ orbit of the above elliptic points, *i.e.* $\tau_* = \gamma\tau_0$, and thus must have orders 2 or 3.

4.1.2 Shioda's construction

Modular elliptic surfaces are constructed based on subgroups of $\Gamma \subset \mathrm{PSL}(2,\mathbb{Z})$, as first discussed by Shioda [82]. Let n_Γ, ϵ_2, ϵ_3 and ϵ_∞ be the index of Γ in $\mathrm{PSL}(2,\mathbb{Z})$, the number of elliptic elements of order two and three, and the number of cusps of Γ, respectively. The quotient $\mathbb{H}/\Gamma$ together with a finite number of cusps ϵ_∞ forms a compact Riemann surface Δ_Γ [208]. Then, there exists a holomorphic map onto the projective line:

$$J_\Gamma : \Delta_\Gamma \to \mathbb{P}^1\ , \tag{4.7}$$

as follows. Let $\Gamma \subset \Gamma_1$ with a canonical map between the Riemann surfaces $\mathbb{H}/\Gamma \to \mathbb{H}/\Gamma_1$, *i.e.* the map arising naturally from the surjection $\Gamma \to \Gamma_1$. Taking $\Gamma_1 = \mathrm{PSL}(2,\mathbb{Z})$, one has $\Delta_{\Gamma_1} \cong \mathbb{P}^1 \cong S^2$, where the J_{Γ_1}-map is the usual J-invariant defined on $\mathbb{H}$, which due to $\mathrm{PSL}(2,\mathbb{Z})$ invariance descends to a holomorphic map $\Delta_{\Gamma_1} \to \mathbb{P}^1$. The map (4.7) is then simply the canonical map between the Riemann surfaces.

Next, we would like to define an elliptic surface $\Phi : \mathcal{S}_\Gamma \to \Delta_\Gamma$ with the singular fibers residing above the 'special points' of Γ. These points are the ones that require special treatment when defining a complex structure on the Riemann surface Δ_Γ, being thus the cusps and elliptic points of Γ. Following [82], let Σ be the set of these points, and let:

$$\Delta'_\Gamma = \Delta_\Gamma \setminus \Sigma \subset \mathbb{H}/\Gamma \ . \tag{4.8}$$

Then, for a universal covering $\mathcal{U}$ of Δ'_Γ, there is a holomorphic map $\varpi : \mathcal{U} \to \mathbb{H}$ such that:

$$J_\Gamma(\pi(u)) = j(\varpi(u)) \ , \qquad u \in \mathcal{U} \ , \tag{4.9}$$

with j the elliptic modular function on $\mathbb{H}$. That is, the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{U} & \xrightarrow{\pi} & \Delta'_\Gamma & \longrightarrow & \Delta_\Gamma \\ {\scriptstyle\varpi}\downarrow & & \downarrow{\scriptstyle J_\Gamma} & & \\ \mathbb{H} & \xrightarrow{j} & \mathbb{P}^1 & & \end{array} \tag{4.10}$$

Using Kodaira's construction, one can then define an elliptic surface $\mathcal{S}_\Gamma$ over Δ_Γ, with a global section having J_Γ as a functional invariant. Additionally, the holomorphic map introduced before, $\varpi : \mathcal{U} \to \mathbb{H}$ becomes the 'period' map of the elliptic fiber at $u \in \mathcal{U}$. Note that $\varpi(u)$ is a single-valued function on the universal covering $\mathcal{U}$, being a multi-valued function on Δ_Γ.

The singular fibers lie above the set Σ of elliptic points and cusps. The type of singular fiber (as shown in table 1.1) is determined by some additional data. That is, there is a unique representation φ of the fundamental group $\pi_1(\Delta'_\Gamma)$ of Δ'_Γ:

$$\varphi : \pi_1(\Delta'_\Gamma) \to \Gamma \subset \mathrm{PSL}(2, \mathbb{Z}) \ , \tag{4.11}$$

satisfying:

$$\varpi(\gamma \cdot u) = \varphi(\gamma) \cdot \varpi(u) \ , \qquad \gamma \in \pi_1(\Delta'_\Gamma) \ . \tag{4.12}$$

Note that on the right-hand side of the above equation, we have the usual action of an element of Γ on a point on the upper half-plane, as $\varphi(\gamma) \in \Gamma$. Thus, for γ_u a positively oriented loop around $u \in \Delta_\Gamma$, the matrices $\varphi(\gamma_u)$ determine the type of singular fibers [82].

The modular group has two elements of order 2 and 3, namely the generators S and

ST, respectively. Thus, if u is an elliptic point of order 2, for instance, then $\varphi(\gamma_u)$ must be conjugate to S. Let us also note that based on Kodaira's classification of singular fibers in table 1.1, the J_Γ map 'forgets' about starred fibers when we restrict to subgroups of $\mathrm{PSL}(2,\mathbb{Z})$; that is, monodromy matrices S and S^{-1}, for example, correspond to the same type of special point of Γ. This is, in fact, related to the notion of 'quadratic twisting', in which the J-map remains the same, but the starred singular fibers change according to (3.10).

The same argument holds for elliptic points of order 3, as well as for the cusps of width n of Γ, where $\varphi(\gamma_u)$ must be conjugate to ST and T^n, respectively. To summarise, we have the following map between the 'special' points of Γ and the singular elliptic fibers:

$$\begin{aligned} \textit{elliptic point of order 2} \quad (J=1) \ &: \qquad III \ , \ III^* \ , \\ \textit{elliptic point of order 3} \quad (J=0) \ &: \qquad II \ , \ IV^* \ , \\ \textit{cusp of width } n \quad (J\to\infty) \ &: \qquad I_n \ , \ I_n^* \ . \end{aligned} \tag{4.13}$$

It is worth pointing out that to a subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{Z})$, we can thus associate more than one elliptic surface. As we are interested in *rational* elliptic surfaces, a necessary condition is that the sum of the Euler numbers associated to all fibers is 12, as indicated before in (3.7). This constraint can be satisfied in more than one way: we have the (rational) elliptic surfaces that differ by a quadratic twist, with the singular fibers related as in (3.10).

Let us also note that we should, in principle, include in (4.13) the singular fibers of the type II^* and IV as corresponding to elliptic points of order 3. However, such singular fibers do not actually appear in *rational* elliptic surfaces, as shown in [86].

4.1.3 Classification of Coulomb branches

Doran showed that there are 33 modular RES up to quadratic twists [86], which, however, do not change the modular properties of the elliptic surface. As a result, we can extend this number to 47 distinct rational elliptic surfaces. Our approach, however, uses a different perspective compared to [86]. The two main ingredients are the classification of subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ [83–85] and the map (4.13).

As shown earlier, the rationality condition imposes a constraint on the maximal value of the index inside the modular group. Thus, we only need a complete classification of subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ up to index 12. Let us give some examples of how this process works.

Index 1 subgroups. The first group that we have to consider is the modular group $\mathrm{PSL}(2,\mathbb{Z})$ itself. This has a single cusp of width 1, and two elliptic points of orders 2 and 3, respectively. As a result, using (4.13) we can build the rational elliptic surfaces

$$(I_1, III^*, II)\,, \qquad (I_1, III, IV^*)\,, \qquad (I_1^*, III, II)\,, \qquad (I_1, III, II, I_0^*)\,. \tag{4.14}$$

All these configurations are related by a quadratic twist, where in the last case the star is transferred to a 'smooth fiber'. As such, the modular group does not distinguish between an I_0 fiber and an I_0^* fiber.

Depending on which fiber is chosen as the fiber at infinity, these surfaces can describe multiple 4d $\mathcal{N}=2$ theories, including also the $D_{S^1}E_8$ theory. For any of these configurations, the modular function (hauptmodul) $U(\tau)$ is $J(\tau)$ itself, or, more generally, a linear function of $J(\tau)$.

Higher index subgroups. For index 2, there is a unique subgroup of $\mathrm{PSL}(2,\mathbb{Z})$, which has a cusp of width 2 and two elliptic points of order of order 3 [85]. We will denote this group by Γ^2. The possible rational elliptic surfaces are (I_2, II, IV^*) and its quadratic twists. Let us note that the massless $D_{S^1}E_8$ configuration (as well as the massless 4d $SU(2)$ $N_f=1$ theory), having the singular fibers $(I_1^\infty; II^*, I_1)$ is distinct from the Γ^2 configuration, and, thus, is not modular.

The analysis for the higher index subgroups follows in a similar fashion. In table 4.1 we list all *congruence* subgroups – up to conjugation inside $\mathrm{PSL}(2,\mathbb{Z})$ – from which we can construct rational elliptic surfaces, as well as the modular functions for these groups. Rather interestingly, the series expansion of the modular functions of these groups reproduce certain McKay-Thompson series of the Monster group [209]. The modular extremal rational elliptic surfaces have already appeared in table 3.3. These turn out to be extremely useful for deriving BPS quivers, as we will discuss next.

Let us also briefly comment on the non-congruence subgroups. Non-congruence subgroups are much less studied compared to congruence subgroups. For a nice review of the existing literature, see [210]. Their systematic study was initiated by Atkin and Swinnerton-Dyer in [211], when it was observed that the Fourier coefficients of the associated modular forms have *unbounded denominators*, a conjecture only recently proved [206]. Closed-form expressions for these modular forms can still be found in certain cases when the non-congruence subgroup of interest is also a subgroup of a proper congruence subgroup [212–214], but they involve fractional powers of the usual modular functions. This procedure is based on a connection to Galois theory and only works for 2 out of the 11 non-congruence subgroups from which rational elliptic surfaces can be built. We will not discuss these here, and refer the reader to [2].

4.2 BPS quivers from fundamental domains

The goal of this section is to derive the identification between singular fibers and nodes of the BPS quiver. Building on chapter 1.3, we propose an identification of the nodes of the BPS quiver with the singular fibers of the SW geometry, making use of modularity.

4.2.1 BPS quivers from a basis of BPS states

Let us return to the study of 4d $\mathcal{N}=2$ KK theories, $D_{S^1}\mathcal{T}_{\boldsymbol{X}}$, obtained from the circle compactification of a 5d theory SCFT of E_n type. Hence, here $\boldsymbol{X}$ is a CY3 where a 4-cycle $\mathcal{B}_4$ shrinks to zero volume. At generic points on the Coulomb branch, we have a spectrum of massive half-BPS particles, with masses:

$$M = |Z_\gamma|\,, \qquad Z_\gamma = m a_D + q a + q_F \mu_F \ , \tag{4.15}$$

where γ is the charge of the BPS particle, being valued in the charge lattice $\gamma \in \mathbb{Z}^{2r+f+1} \cong \mathbb{Z}^{n+3}$, where $r = 1$ is the rank of the theory and $f = n$ is the rank of the flavour symmetry algebra, with the additional direction accounting for the $U(1)_{\rm KK}$ symmetry. At *quiver* *points* on the U-plane, the central charges of $n+3$ 'light' BPS particles almost align – for the $D_{S^1}E_n$ theories, they become real – and, conjecturally, the full BPS spectrum can be obtained as bound-states of the $n+3$ elementary particles. In such cases, the quiver arrows are determined by the Dirac pairing of the 'simple objects', *i.e.* the states forming the basis of light BPS states. All the stable particles arise then as bound states of these simple objects.

The problem of finding the spectrum at such a quiver point can be formulated in terms of a BPS quiver – see *e.g.* [64,71,203,215]. The rough intuition for quiver points, and an explicit way to compute the resulting quivers, follows from considering the IIB mirror geometry, $\widehat{\mathbf{Y}}$. We mentioned that BPS particles correspond to D3-branes wrapping Lagrangian 3-cycles. In the IIA description, the full $\mathcal{B}_4$ collapses to zero-volume in the classical picture, and the (derived) category of quiver representations is expected to accurately describe the category of B-branes in that regime. In the mirror IIB description, we have 'light' wrapped D3-branes on the 'small' 3-cycles mirror to the shrinking D0/D2/D4 bound states, that correspond to string junctions connecting a base point $W=U_0$ near the origin of the W-plane to the '7-branes' around it. In many cases, the fractional branes are then simply the smallest 'vanishing paths' (in the sense of Picard-Lefschetz theory) on the W-plane [58]. In other words, in an ideal situation, the fractional branes are the dyons that become massless at the U-plane singularities around the base point.

Once we have identified the electromagnetic charge $\gamma_i=(m_i,q_i)$ of these dyons, the BPS quiver is obtained by assigning a quiver node $(i)\sim\mathcal{E}_{\gamma_i}$ to each light dyon, and a number n_{ij} or arrows from node (i) to (j) given by the Dirac pairing, which is also the oriented intersection number between the 3-cycles inside $\widehat{\mathbf{Y}}$, as discussed in chapter 1.3. For the $D_{S^1}E_n$ theories, we recover in this way many known 'fractional brane quivers' – note that as emphasised in [71], fractional-brane quivers *are* 5d BPS quivers.

Let us finally point out that a quiver description depends on the basis of BPS states and, thus, is not unique. Such a basis choice splits the spectrum into particles and anti-particles, which have central charge vectors of equal magnitude but opposite directions. A change of the basis states can be implemented through a quiver mutation, which effectively rotates the central charge half-plane, leading to a relabelling of the particles and anti-particles [203].

4.2.2 Fundamental domains of modular configurations

We will limit our analysis to the modular rational elliptic surfaces, and denote by $\mathcal{F}_{\mathcal{T}}$ the fundamental domain of $\mathcal{T}$. Here, we relax the definition of $\mathcal{T}$, allowing it to be any 4d $\mathcal{N}=2$ (possibly KK) theory. The simplest type of singularity occurring in the interior of the U-plane is when a single charged particle becomes massless. In the appropriate duality frame, the low-energy physics at that point is then governed by SQED, as discussed in chapter 2.2.1. The, massless dyon of charges (m,q) at this point U_* induces a monodromy $\mathbb{M}_*^{(m,q)}$ that is conjugate to T, given by (2.21).

These are the I_1 singularities in Kodaira's classification, as listed in table 1.1. More generally, we can have n electrons becoming massless at the same point, with the monodromy being conjugate to T^n [7, 8], leading thus to an I_n singularity. The additive singularities appear due to mutually non-local light BPS states becoming massless simultaneously. As a result, the monodromy induced by the additive fibers should be viewed as a product of monodromies of the type (2.21). Thus, we will typically deform the Coulomb branches containing additive singular fibers, such that the singularities are broken to multiplicative fibers only.

The classification programme of 4d $\mathcal{N}=2$ rank-one SCFTs [38–41], recently reviewed in [37,216], also involves theories whose Coulomb branches include *undeformable* (or *frozen*) singularities. In this context, a frozen I_n singularity is to be interpreted as due to a single massless hypermultiplet, of charge $Q=\sqrt{n}$, in a purely electric duality frame. This leads to certain constraints on the configurations of singular fibers, due to Dirac charge quantization.

The flavour algebra of these theories can be understood from the so-called flavour root system of [36]. Yet, there is no clear picture of how the MW group restricts the global form of the flavour symmetry. However, as the SW geometries of these theories are already known in 4d (see [39], for instance), our approach can bypass these issues by analysing their modular properties as well.

Fundamental domains are based on the Dedekind tessellation of the upper half-plane $\mathbb{H}$, which is obtained by the Möbius action on the modular group $\mathrm{PSL}(2,\mathbb{Z})$. Fundamental domains for subgroups Γ of the modular group can be constructed using a set of coset representatives $\{\alpha_i\}$, for $i = 1, \dots, n_\Gamma$ from the disjoint union $\mathcal{F}_\Gamma = \bigsqcup_{i=1}^{n_\Gamma} \alpha_i \mathcal{F}_0$, as already discussed around (4.6), where $\mathcal{F}_0$ is the fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$. When the SW geometry is modular, there is a homeomorphism from the CB to a subregion of the upper half-plane. Focus for now on a modular RES, with monodromy group Γ with no elliptic points. The cusps of the monodromy group of the theory correspond to U-plane singularities and can be mapped to the real axis $\mathbb{R}$ of the upper half-plane (except the cusp F_∞ at ∞, which is fixed). The cusps are in the $\mathrm{PSL}(2,\mathbb{Z})$ orbit of $\tau_\infty = i\infty$, which, by convention, is the natural position of the unique width-one cusp of the modular group. The action of an element of the modular group on the cusp τ_∞ leads to rational numbers of the form $\frac{q}{m}$, with $q, m \in \mathbb{Z}$, as follows:

$$\sigma\tau_\infty = \frac{a\tau_\infty + b}{c\tau_\infty + d} = \frac{q}{m} \ , \qquad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) \ . \tag{4.16}$$

Here σ should be viewed as one of the coset representatives α_i and $\frac{q}{m}$ is an *irreducible* fraction. For this equality to be satisfied, one requires:

$$\frac{a}{c} = \frac{q}{m} \ . \tag{4.17}$$

If such a cusp corresponds to an I_1 singular fiber, we have the monodromy matrix:

$$\sigma T \sigma^{-1} = \mathbb{M}_{(c,-a)} = \mathbb{M}_{(km,-kq)} \in \Gamma \subset \mathrm{PSL}(2,\mathbb{Z}) \ , \tag{4.18}$$

which can be generated by a light dyon of charge $(c, -a) = k(m, -q)$, for some non-zero $k \in \mathbb{C}$. For the theories that do not contain undeformable singularities, it generally suffices to restrict attention to $k = 1$, with the states having $k > 1$ being unstable. Thus, we have the following correspondence:

> To an I_n singularity – which can be fully deformed to I_1 singularities – that corresponds to a width n cusp at $\tau = \frac{q}{m} \in \mathbb{Q}$ on the upper half-plane we assign n light BPS states of charge $\pm(m, -q)$.

Note that since this approach is solely based on the monodromies, there is a sign ambiguity in choosing the charges of the BPS states. A constraint that follows from the central charges is that one needs to make the same sign choice for all n states forming an I_n cusp. In principle, this sign ambiguity could be fixed by solving the associated Picard-Fuchs equation and evaluating the central charges at the quiver point.

All BPS quivers studied in this work that arise from fundamental domains with no elliptic points satisfy a curious property. This observation leads to the following conjecture that remedies the sign ambiguity of the BPS charges. Let (τ_i) be the *ordered* vector of the positions of the distinct (I_{n_i}) cusps:

$$(\tau_1 \ , \ldots , \tau_k) \ , \qquad \tau_1 < \ldots < \tau_k \ . \tag{4.19}$$

and let $j_{(i)}$ be the position of the cusp τ_i in this vector. As shown above, we can write each such τ_i as a rational number:

$$\tau_i = \frac{q_i}{m_i} \in \mathbb{Q} \ , \qquad m_i \in \mathbb{Z}_{>0} \ , \tag{4.20}$$

where we make a choice for the sign of the denominator, which also fixes the sign of the numerator. Then, we conjecture that for modular configurations containing only multiplicative cusps (I_{n_i}), the assignment of n_i light BPS states of charges:

$$(-1)^{j_{(i)}}(m_i, -q_i) \ . \tag{4.21}$$

leads to the correct BPS quiver description. Note that an overall sign change of all BPS states will still lead to a consistent quiver, as this simply replaces all particles with their anti-particles.

These statements can be further generalized to include undeformable multiplicative singularities as follows. Recall that an undeformable I_n singularity corresponds to a single massless hypermultiplet of charge $Q = \sqrt{n}$ in an electric frame. Thus, the proportionality factor $k \in \mathbb{C}$ precisely amounts for this charge renormalization, leading to:

To an I_n singularity that corresponds to a width n cusp at $\tau = \frac{q}{m} \in \mathbb{Q}$ on the upper half-plane, with deformation pattern $I_n \to \bigoplus_j I_{k_j}$, where each I_{k_j} is undeformable, we assign light BPS states of charges $\left(\pm \sqrt{k_j}(m, -q)\right)_j$.

One of the assumptions behind the above identification is that there exists a BPS chamber that contains the states associated to the I_n cusps. The U-plane has generally walls of marginal stability connecting the singularities, which separate these BPS chambers. While for massless 4d SQCD this assumption is known to be true as there are only two BPS chambers [196, 217, 218], the structure of the U-plane for the KK theories is much more intricate and, in general, there might not be such a chamber. However, in all the checks that we have performed for the theories of the 'I_1-series',[30] we managed to relate the different modular configurations to known BPS quivers of the theories of interest.

Let us finally comment on the matrices $\sigma \in \mathrm{PSL}(2, \mathbb{Z})$ appearing in (4.18) that satisfy $\sigma T \sigma^{-1} = \mathbb{M}_{(m,-q)}$. These are some of the coset representatives of $\Gamma \subset \mathrm{PSL}(2, \mathbb{Z})$, and do play an important role in the identification (1.32). Finding such matrices is a non-trivial task, but there exists an algorithmic way of solving this problem using continued fractions [219]. One has:[31]

$$\frac{q}{m} = p_1 + \cfrac{1}{p_2 + \cfrac{1}{p_3 + \cfrac{1}{p_4 + \dots}}} \ , \tag{4.22}$$

for some $p_i \in \mathbb{Z}$. Note that since $\frac{q}{m}$ is rational, the sequence $\{p_1, p_2, \dots\}$ must terminate. This list of integers will determine a possible matrix σ as follows:

$$\sigma = \prod_{i=1}^{N} T^{(-1)^{i+1} p_i} S = T^{p_1} S T^{-p_2} S \dots T^{(-1)^{N+1} p_N} S \ . \tag{4.23}$$

Note that this matrix is not unique, with the same monodromy being reproduced by σT^k, for some integer $k \in \mathbb{Z}$. Let us also mention that the continued fraction representation of the cusp-positions can be used to find accumulation rays of the BPS quivers of the KK theories, which, as opposed to 4d SQCD theories do not necessarily lie along the real axis

[30]These are the theories whose maximally deformed Coulomb branches contain only I_1 singularities in the bulk.

[31]For this, we use `Mathematica`'s inbuilt function `ContinuedFraction[]`.

of the central charge plane. One might also notice that a choice of fundamental domain for a given group Γ is not unique. We have shown in [2], that such changes can be interpreted as quiver mutations.

4.2.3 The $D_{S^1}E_1$ theory revisited

As a first application of our formalism, let us revisit the $D_{S^1}E_1$ theory, with the SW curve in Weierstrass form given by (3.72). We will focus here on the massless curve, obtained by setting $\lambda = 1$. Then, the resulting configuration is $(I_8; I_2, 2I_1)$, with the I_2 fiber at the origin $U = 0$. This is also a modular configuration for $\Gamma^0(8)$, as we will shortly see. We first solve the PF equation satisfied by the periods and compute the monodromies and central charges at the quiver point explicitly. Then, we show how modularity can lead to the same monodromies in a much faster way.

PF equation. Let (a_D, a) be the physical periods of the SW curve, which are related to the D4- and D2-brane periods as discussed in chapter 2.3, namely:

$$\Pi_{D4} = a_D \ , \qquad \Pi_{D2_f} = 2a \ , \qquad \omega_D = \frac{da_D}{dU} \ , \qquad \omega_a = \frac{da}{dU} \ , \tag{4.24}$$

where (ω_D, ω_a) are the 'geometric periods'. The D4 period as given by (2.37) becomes:

$$\Pi_{\rm D4} = \Pi_{{\rm D2}_f}\Pi_{{\rm D2}_b} + \frac{1}{6} = 2a\left(2a + \frac{1}{2\pi i}\log(\lambda)\right) + \frac{1}{6} \ , \tag{4.25}$$

Introducing the variable $w = U^2/16$, the Picard-Fuchs equation (2.29) reduces to:

$$\frac{d^2\omega}{dU^2} + \frac{3U^2 - 16}{U(U^2 - 16)}\frac{d\omega}{dU} + \frac{1}{U^2 - 16}\omega = 0 \ . \tag{4.26}$$

One can analyse the solutions to this equation, and their monodromies, rather explicitly. In particular, in terms of the w coordinate, the differential equation is the same differential equation arising in the 4d $SU(2)$ theory. Defining $\widetilde{\omega} \equiv \sqrt{16w}\,\omega = U\frac{d\Pi}{dU}$, the correct basis choice turns out to be given by:

$$\widetilde{\omega}_a(w) = -\frac{1}{2\pi i}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;\frac{1}{w}\right) \ , \qquad \widetilde{\omega}_D(w) = -\frac{1}{\pi}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;1-\frac{1}{w}\right) \ . \qquad (4.27)$$

Here the ${}_2F_1$ functions are the standard hypergeometric functions, with branch points at $w = 0, 1$ and ∞. The period $\widetilde{\omega}_a$ is regular in the large volume limit, $w = \infty$, while the 'dual period' $\widetilde{\omega}_D$ is regular at the 'conifold point', $w = 1$. Analytic continuation past the region of convergence can be done using the Gauss-Ramanujan identity and the Barnes integral representation of these hypergeometric functions. This analysis is rather cumbersome and was worked out in great detail in [1].

Another difficult aspect of the computation is to determine the correct form of the periods on the U-plane. For this, note first that the period $\widetilde{\omega}_D$ has a branch cut stretching from $w = 0$ to $-\infty$, while $\widetilde{\omega}_a$ has a branch cut from $w = 0$ to $w = 1$. Thus, the principal branch cuts of the two periods differ. This subtlety is very important when considering linear combinations of the two periods. To be able to analytically continue the periods on the U-plane, we introduce the functions:

$$f_a(w) = -\frac{1}{2\pi i}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{16}{U^2}\right) \ , \qquad f_D(w) = -\frac{1}{\pi}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\frac{16}{U^2}\right) \ . \qquad (4.28)$$

Since the map $U \mapsto w$ is 2 to 1, we will split the U-plane into two regions separated by the imaginary axis, which we denote by A (for $\mathrm{Re}(U) > 0$) and B (for $\mathrm{Re}(U) < 0$). The above functions have branch cuts inherited from the hypergeometric function. Thus, it is not directly obvious what their expressions throughout the whole U-plane are. Here, we will interpret f_D as a *local* function, which is well defined only around one of the two 'conifold' singularities at $U^2 = 16$. The branch cuts of f_a connect the singularities at $U = \pm 4$ to the $U = 0$ singularity. We choose the branch cut of f_D to run along $U \in [0, i\infty)$, in agreement with the w-plane branch cut. The large volume asymptotics on the w plane read:

$$f_a(w) \approx -\frac{1}{2\pi i} + \mathcal{O}\left(\frac{1}{w}\right) \ , \qquad f_D(w) \ \ \approx -\frac{1}{\pi^2}\log(16w) + \mathcal{O}\left(\frac{1}{w}\right) \ . \qquad (4.29)$$

The geometric periods in the A and B regions will be linear combinations of f_a and f_D, with the large volume ($U \to \infty$) asymptotics:

$$\widetilde{\omega}_a(U) \approx -\frac{1}{2\pi i} \ , \qquad \widetilde{\omega}_D(U) \approx \frac{2}{\pi^2} \log\left(\frac{1}{U}\right) \ , \tag{4.30}$$

which reproduce the large volume monodromy:

$$\mathbb{M}_{U=\infty} = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix} = T^8 \ . \tag{4.31}$$

In fact, f_a can be used for the $\widetilde{\omega}_a$ period in both regions of the U-plane, since the large volume expression is a regular function. The $\widetilde{\omega}_a$ period will thus have two branch cuts, running along $U \in (0_+, 4]$ and $U \in [-4, 0_-)$. We can choose $U_* = -4$ to be the cusp where $a_D(U_*) = 0$. Thus, in region B, the dual period $\widetilde{\omega}_D$ will be given by f_D. The mapping of the angles between the w-sheets and the U-plane is:

$$\begin{aligned} U \ &: \ -\frac{3\pi}{2} \to -\pi \ , \qquad -\pi \ \to -\frac{\pi}{2} \ , \qquad -\frac{\pi}{2} \to 0 \ , \qquad 0 \to \frac{\pi}{2} \ , \\ w \ &: \ -3\pi \to -2\pi \ , \qquad -2\pi \to -\pi \ , \qquad -\pi \to 0 \ , \qquad 0 \to \pi \ . \end{aligned} \tag{4.32}$$

Recall that $\arg(w) \in (-\pi, \pi)$ was the principal branch of $\widetilde{\omega}_D$ in the w-plane. Now, consider the function f_D in region A. Analytic continuation to $U \to \infty$ leads to:

$$f_D^{(A)} \approx -\frac{1}{\pi^2} \log(16 w^{(A)}) = -\frac{1}{\pi^2}\Big(\log(16 w^{(B)}) - 2\pi i\Big) \ . \tag{4.33}$$

In order for this to match with the asymptotic expansion of $\widetilde{\omega}_D$ in all regions, we must subtract a factor of $4 f_a$. We then have:

$$A \ : \ \widetilde{\omega}_D(U) = f_D(U) + 4 f_a(U) \ , \qquad B \ : \ \widetilde{\omega}_D(U) = f_D(U) \ . \tag{4.34}$$

while $\widetilde{\omega}_a(U) = f_a(U)$ for the entire U-plane. The branch cuts of the geometric periods $\widetilde{\omega}_D$ and $\widetilde{\omega}_a$ are shown in figure 4.1. The monodromies around the two singularities at $U = \pm 4$ read:

$$\mathbb{M}_{U=-4} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = STS^{-1} \ , \qquad \mathbb{M}_{U=4} = \begin{pmatrix} -3 & 16 \\ -1 & 5 \end{pmatrix} = (T^4 S) T (T^4 S)^{-1} \ . \tag{4.35}$$

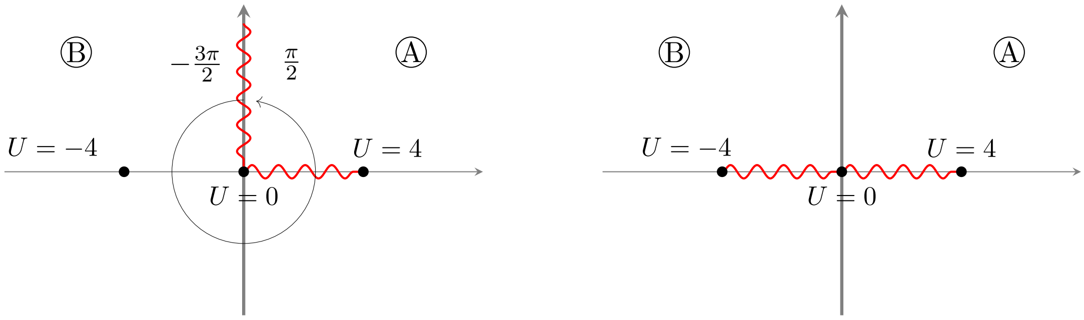


Figure 4.1: Branch cuts for $\widetilde{\omega}_D(U)$ and $\widetilde{\omega}_a(U)$, respectively.

For the series expansion around $U = 0$, one needs to take into account the various branch cuts of $\widetilde{\omega}_a$ and $\widetilde{\omega}_D$. In the region where $\mathrm{Re}(U) > 0$, and $\mathrm{Im}(U) < 0$, for instance, the asymptotics are [1]:

$$\widetilde{\omega}_a(U) \approx -\frac{U}{2\pi^2}\log\frac{U}{4} + i\frac{U}{2\pi} + \mathcal{O}\left(U^3\right) \ , \qquad \widetilde{\omega}_D(U) \approx -\frac{U}{4\pi^2}\log\frac{U}{4} + i\frac{U}{8\pi} + \mathcal{O}\left(U^3\right) \ , \tag{4.36}$$

leading to the monodromy:

$$\mathbb{M}_{U=0} = \begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix} = (T^2S)T^2(T^2S) \ . \tag{4.37}$$

which, in particular, agrees with the fact that $U = 0$ is an I_2 singularity. These monodromies satisfy the consistency condition (2.13). Finally, by integrating the geometric periods once, we can obtain the physical periods on the U-plane, similarly to the analysis on the w-plane. One can determine in that way which BPS particles become massless at which points. This can also be understood, more simply, from the explicit monodromy matrices that we just derived.

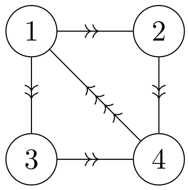
Explicitly, one finds that the following dyons of the KK theory $D_{S^1}E_1$ become massless at these points:

$$\begin{aligned} &U=-4 \quad (I_1) \quad : \quad \text{a monopole of charge } \gamma_1=(1,0)\text{, becomes massless,} \\ &U=0 \quad\ \ (I_2) \quad : \quad \text{two dyons of charge } \gamma_{2,3}=(-1,2)\text{, become massless,} \\ &U=4 \quad\ \ (I_1) \quad : \quad \text{a dyon of charge } \gamma_4=(1,-4)\text{, becomes massless.} \end{aligned} \tag{4.38}$$

Here, we fixed the overall signs of the electromagnetic charges such that the total charge vanishes. Interestingly, the point $U=0$ is also a *quiver point*, as the phases of the central charges of the BPS particles align. Using the fact that $a=\frac{1}{4}$ and $a_D=\frac{1}{2}$ at the origin [1], we find the central charges:

$$\mathcal{Z}_{\gamma=(1,0)}(U=0)=\frac{1}{2}\ , \qquad \mathcal{Z}_{\gamma=(-1,2)}(U=0)=0\ , \qquad \mathcal{Z}_{\gamma=(1,-4)}(U=0)=\frac{1}{2}\ . \tag{4.39}$$

The central charge of the $\gamma=(1,-4)$ also carries a contribution from one unit of KK momentum (D0-brane charge) [71]. The associated 5d BPS quiver is obtained by assigning one node $\mathcal{E}_\gamma$ to each of the four dyons, and by drawing a net number of arrows n_{ij} from $\mathcal{E}_{\gamma_i}$ to $\mathcal{E}_{\gamma_j}$ given by the Dirac pairing, $n_{ij}=\det(\gamma_i,\gamma_j)$. The resulting quiver reads:



(4.40)

This is a well-known 'toric' quiver for the local $\mathbb{F}_0$ – see *e.g.* [68, 220]. This same quiver can be also found more easily from the modular properties of the curve. The Coulomb branch of the massless $D_{S^1}E_1$ theory is, in particular, a modular curve for the congruence subgroup $\Gamma^0(8)$. To see this explicitly, we should look at the explicit map $U=U(\tau)$. This is determined from the $J=J(U)$, expression, combining with the $J=J(\tau)$ relation:

$$U(\tau)=\frac{\eta\left(\frac{\tau}{2}\right)^{12}}{\eta\left(\frac{\tau}{4}\right)^{4}\ \eta(\tau)^8}=q^{-\frac{1}{8}}+4q^{\frac{1}{8}}+2q^{\frac{3}{8}}-8q^{\frac{5}{8}}-q^{\frac{7}{8}}+\mathcal{O}(q)\ . \tag{4.41}$$

The η-quotient (4.41) is the Hauptmodul for $\Gamma^0(8)$, being obviously invariant under the action of T^8. Its series expansion reproduces the coefficients of the McKay-Thompson series of class $8E$ of the Monster group [209]. Using the transformation properties of the Dedekind η function we find that the cusp at $\tau=0$ corresponds to the type-I_1 Kodaira singularity at $U=4$ on the U-plane, the cusp at $\tau=2$ corresponds to the I_2 type singularity located at $U=0$, while the cusp at $\tau=4$ corresponds to $U=-4$. There is therefore a

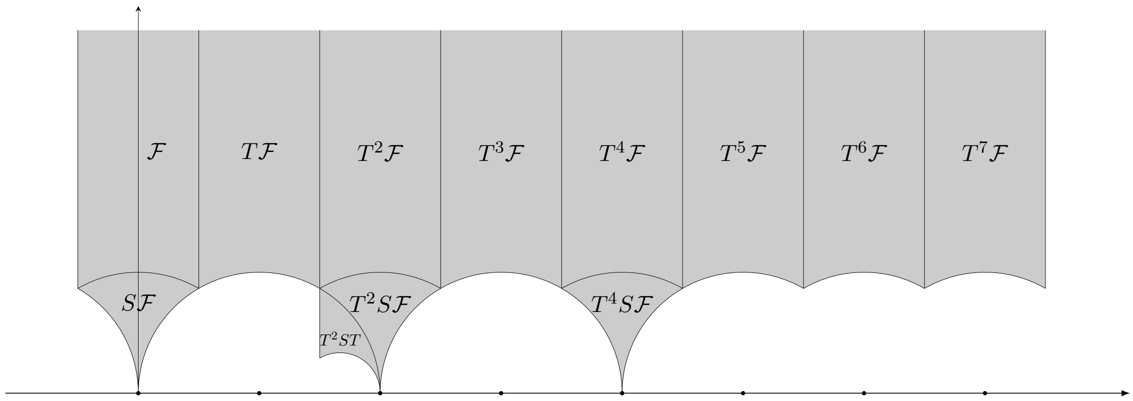


Figure 4.2: A fundamental domain for $\Gamma^0(8)$ on the upper half-plane. The four cusps at $\tau = 0, 2, 4, i\infty$ have widths 1, 2, 1 and 8, respectively. The modular curve $\mathbb{H}/\Gamma^0(8)$ is isomorphic to the Coulomb branch of the massless $D_{S^1}E_1$ theory.

one-to-one mapping between the cusps of $\Gamma^0(8)$ and the U-plane singularities.

This mapping then realizes the monodromy matrices computed from the explicit geometric periods, leading thus to the same BPS quiver as above. Thus, modularity allows us to compute the monodromies without requiring the explicit form of the periods. We will see more examples of this in the following section.

We have already seen that the $\mathbb{Z}_2^{[1]}$ one-form symmetry of the $D_{S^1}E_1$ theory is encoded in the MW torsion of the SW geometry. We can also see this symmetry from the light BPS states. Namely, we have the states:

$$\mathscr{S} \quad : \quad (1,0;0) \ , \quad (-1,2;1) \ , \quad (1,-4;0) \ , \tag{4.42}$$

where the charges $(m, q; l)$ are given as in (3.66), with $l \in \mathbb{Z}_2$ the charge under the center of the flavour $\widetilde{G}_F \cong SU(2)$. The spectrum is left invariant by a group $\mathscr{E} = \mathbb{Z}_4$ generated by:

$$g^{\mathscr{E}} = \left(0, \frac{1}{4}; 1\right) \ . \tag{4.43}$$

This $\mathbb{Z}_4$ contains a $\mathbb{Z}_2^{[1]}$ subgroup generated by $g^{\mathcal{Z}^{[1]}} = \left(0, \frac{1}{2}; 0\right)$, which implies that the theory has an electric one-form symmetry $\mathbb{Z}_2^{[1]}$, as expected [135, 136]. We also have the cokernel $\mathscr{F} = \mathbb{Z}_2^{(f)}$ as above, which implies (3.78).

4.3 Other examples and new BPS quivers

Our discussion, so far, has focused on the theories whose maximally deformed Coulomb branches involve only I_1-type singularities. In this section, we will consider other examples, where the CB contains undeformable singularities.

4.3.1 Undeformable singularities

As we know, by now, the SW geometries of 4d $\mathcal{N} = 2$ SCFTs can include undeformable singularities [38–41]. The I_n-type frozen singularities, for instance, are generated by massless hypermultiplets with a different charge normalization. This leads to a refined version of our identification (1.32), as presented in section 4.2.2.

The language of rational elliptic surfaces should include the theories whose Coulomb branches involve such frozen singularities, but it is not entirely clear what the interpretation of the Mordell-Weil group should be in this context. Additionally, the flavour algebra cannot be directly obtained using the standard F-theory arguments, as in table 1.1, but it was still determined in [36] from the so-called *flavour root system* of the SW geometry. Recently, a (partial) classification of SW geometries for the KK theories arising from 5d theories on S^1 was proposed in [50], which is based on a similar approach to the 4d $\mathcal{N} = 2$ classification programme. This work includes 5d geometries with frozen singularities, some of which we shall consider below.

Let us first note that the language of rational elliptic surfaces can still be used to find the allowed configurations of singular fibers as follows. One first associates to the maximally deformed Coulomb branch a 'naive' flavour lattice T_{def}, using the data in table 1.1. Note again that this is not the correct flavour symmetry of such theories [36, 221]. Then, we claim that the allowed configurations for these theories are those that not only contain the singular fiber F_∞, but, additionally, their associated naive flavour lattice contains T_{def} as a sublattice. This argument might be subject to slight changes if the theory has a non-trivial one-form symmetry, in which case the MW torsion can also play a role.

We will discuss only some of the new 5d SW geometries found in [50], which are listed below, where $\mathcal{S}_{\max}$ is the maximally deformed CB and the MW listed is the one for this maximal deformation:

$$\begin{array}{|c||c|c|c|c||c|}\hline F_\infty & \mathcal{S}_{\max} & \text{rk}(\Phi) & \Phi_{\text{tor}}(\mathcal{S}) & \mathfrak{g}_F & \mathfrak{g}_F \subset \mathfrak{e}_n \\ \hline I_5 & (I_5; I_4, 3I_1) & 1 & - & \mathfrak{u}(1) & \mathfrak{e}_4 \\ \hline I_4 & (I_4; I_4, 4I_1) & 2 & \mathbb{Z}_2 & \mathfrak{sp}(2) & \mathfrak{e}_5 \\ \hline I_3 & (I_3; I_4, 5I_1) & 3 & - & \mathfrak{sp}(2) \oplus \mathfrak{u}(1) & \mathfrak{e}_6 \\ \hline \end{array} \tag{4.44}$$

Here, the flavour symmetry algebra $\mathfrak{g}_F$ is determined using the flavour root system of [36]. To analyse these geometries in more detail, we would like to derive SW curves with an explicit $\mathfrak{g}_F$ symmetry. As such, we look for embeddings $\mathfrak{g}_F \oplus k\,\mathfrak{u}(1) \subset \mathfrak{e}_n$, with maximal k, where the $\mathfrak{e}_n$ symmetry corresponds to a curve having the same fiber at infinity F_∞ as the theory with algebra $\mathfrak{g}_F$.[32] Then, to ensure that the theory has the correct deformation pattern, one needs to fix the k free parameters that correspond to the $\mathfrak{u}(1)$ factors in the above embedding.

4.3.2 Quiver shrinking

In this section, we derive the SW geometries for the models in (4.44), and then propose BPS quivers for these models based on modularity. Certain aspects of these quivers will be further discussed in [4, 6].

$\boldsymbol{F_\infty = I_5}$**.** Let us start with the theory having $F_\infty = I_5$. To find the SW curve we look at the (unique) embedding of $\mathfrak{su}(4) \oplus \mathfrak{u}(1)$ inside $\mathfrak{e}_4 \cong \mathfrak{su}(5)$. Under this embedding, the $\mathfrak{e}_4$ characters split as:

$$\chi_1 \to \frac{1}{L^4} + 4L\ , \qquad \chi_2 \to \frac{4}{L^3} + 6L^2\ , \qquad \chi_3 \to \frac{4}{L} + L^4\ , \qquad \chi_4 \to \frac{6}{L^2} + 4L^3\ , \tag{4.45}$$

where L is the $\mathfrak{u}(1)$ parameter, which, in this case, is also the flavour symmetry of the curve. Thus, the curve is simply given by the $D_{S^1}E_4$ curve, with these particular characters. The

[32] To find such embeddings, we use the `GroupMath` package in `Mathematica` [222].

allowed configurations of singular fibers are listed below, in terms of the $\mathfrak{u}(1)$ parameter:

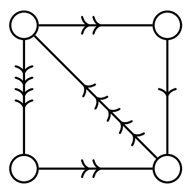
$\mathcal{S}$	L	rk(Φ)	$\Phi_{\text{tor}}(\mathcal{S})$	$\mathfrak{g}_F$
$(I_5; I_4, 3I_1)$	generic	1	-	$\mathfrak{u}(1)$
$(I_5; I_4, II, I_1)$	$L^5 = \frac{32}{27}$	1	-	$\mathfrak{u}(1)$
$(I_5; I_5, 2I_1)$	$L^5 = 1$	0	$\mathbb{Z}_5$	$\mathfrak{u}(1)$

(4.46)

We then note that the 'massless' configuration is the same as the massless $D_{S^1}E_4$ configuration, being modular, with monodromy group $\Gamma^1(5)$, as pointed out in table 4.1. A fundamental domain for this configuration is shown in figure 4.3a. From a field theory perspective, the frozen I_4 corresponds to a hypermultiplet of charge $\sqrt{4}$, in a purely electric frame. Thus, the I_5 singularity corresponds to a massless hyper of charge 1 and one of charge $\sqrt{4}$, such that there is no further flavour enhancement. Thus, we find that a basis of BPS states for this geometry is given by:

$$\gamma_1 = (1, 0) \ , \qquad \gamma_2 = (2, -5) \ , \qquad \gamma_3 = (-1, 2) \ , \qquad \gamma_4 = \sqrt{4}\,(-1, 2) \ , \tag{4.47}$$

with the corresponding BPS quiver shown below:



(4.48)

This quiver has, in fact, already appeared in [223], in the context of 4d $\mathcal{N} = 1$ superconformal field theories living on the worldvolume of D3-branes probing CY singularities. This quiver was obtained from a *shrinking* procedure applied to the dP_4 quiver, being thus referred to as the 'shrunk dP_4' quiver, or $shdP_4$. The shrinking is, of course, not arbitrary, being a procedure that preserves conformality in the 4d $\mathcal{N} = 1$ theory.[33] Imposing this condition leads to a set of Diophantine equations, which turn out to be very similar to the Diophantine equations appearing in the context of 2d $\mathcal{N} = (2, 2)$ Landau-Ginzburg

[33] Another procedure which preserves this condition is *orbifolding*. In this case, there is a beautiful interpretation at the level of the BPS category, in terms of *Galois covers*, which is analysed in [5]. However, it is not clear how the shrinking procedure manifests on the category of BPS states.

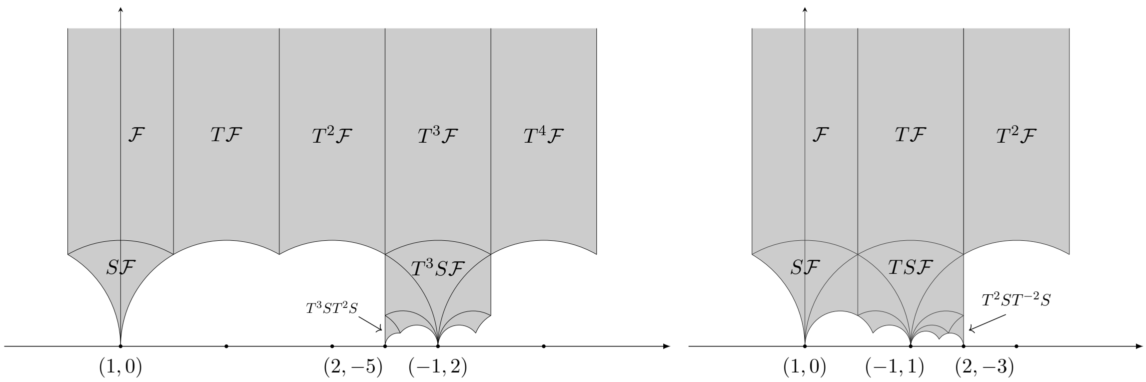
models [224]. It can be shown that for the shrunk models considered here, this condition translates to a condition on the BPS category – namely, the BPS category is *numerically CY*. For more details on these aspects, we refer to [37, 225, 226].

Let us mention that the CY singularity engineering this quiver is not known and nor is the theory that is described by this BPS quiver. We can, however, make some claims based on the SW geometry and on the (possibly incomplete) classification of 5d $\mathcal{N} = 1$ SCFTs of [118–120, 126]. Recall first that the E_0 SCFT does not admit a gauge theory deformation; however, there is an RG flow from the $\widetilde{E}_1$ theory, which is the UV completion of the $SU(2)_\pi$ gauge theory. Hence, the E_0 theory is sometimes referred to as $SU(2)_\pi \ominus 1\,\mathrm{F}$, by an abuse of notation, to emphasize this RG flow. Note also that the E_0 theory has no flavour symmetry.

A rather intriguing 5d SCFT, originally proposed in [126], is the so-called local $\mathbb{P}^2 \oplus 1\,\mathrm{Adj}$, or, equivalently, $SU(2)_\pi \oplus 1\,\mathrm{Adj} \ominus 1\,\mathrm{F}$, where F stands for fundamental matter. For ease of notation, we will simply refer to it as the Bhardwaj SCFT. As the name suggests, this theory can flow to the E_0 SCFT and, naively, there is a $\mathfrak{u}(1)$ flavour symmetry coming from the adjoint matter. Let us note, however, that our proposal for the SW curve is in disagreement with the recent works of [227] and [228]. In [227], for instance, the SW curve computed from the brane web of the Bhardwaj theory is the one with maximal deformation $(I_1; 2I_4, 3I_1)$. However, the flavour root system of this curve turns out to be the rank-two algebra $\mathfrak{sp}(2)$, which appears to be too large for this theory. Meanwhile, the superconformal index computations of [228] suggest that the naive $\mathfrak{u}(1)$ flavour symmetry might enhance to $\mathfrak{su}(2)$, which we do not see from our curve.

$F_\infty = I_4$ and $F_\infty = I_3$. Let us also consider the two models in (4.44) with $F_\infty = I_4$ and $F_\infty = I_3$. For these, we want to find SW curves with manifest $\mathfrak{g}_F$ symmetry. For the first case, we look at embeddings of $\mathfrak{sp}(2) \oplus 2\mathfrak{u}(1)$ inside $\mathfrak{e}_5 \cong \mathfrak{so}(10)$. There are two such embeddings, leading to distinct curves. However, only one of them turns out to have non-trivial torsion in the MW group.[34] We can then tune the $\mathfrak{u}(1)$ parameters such that

[34] The non-torsion curve is also part of the classification of [50], but we cannot find a BPS quiver using modularity.



(a) (b)

Figure 4.3: Fundamental domains for $\Gamma^1(5)$ and $\Gamma^0(3) \cap \Gamma_0(2)$, which are modular configurations for $D_{S^1}E_4$ and $D_{S^1}E_6$, respectively.

the curve has a frozen I_4 singularity, leading to the character decomposition:

$$\begin{aligned} \chi_1 &= 5 + \chi_2^{\mathfrak{sp}(2)} , & \chi_3 &= \chi_5 = 4\chi_1^{\mathfrak{sp}(2)} , \\ \chi_2 &= \chi_1^{\mathfrak{sp}(2)}\chi_1^{\mathfrak{sp}(2)} + 4\chi_2^{\mathfrak{sp}(2)} + 9 , & \chi_4 &= 6\chi_1^{\mathfrak{sp}(2)}\chi_1^{\mathfrak{sp}(2)} + 4\chi_2^{\mathfrak{sp}(2)} + 4 . \end{aligned} \tag{4.49}$$

Meanwhile, for $F_\infty = I_3$, we are interested in the unique embedding of $\mathfrak{su}(4) \oplus 3\mathfrak{u}(1)$ inside $\mathfrak{e}_6$. Fixing 2 of the 3 free parameters we can recover the desired curved. To find the BPS quivers, we can use the modular configurations $(I_4; I_4, 2I_2)$ and $(I_3; I_6, I_2, I_1)$, which correspond to the congruence subgroups $\Gamma^0(4) \cap \Gamma(2)$ and $\Gamma^0(3) \cap \Gamma_0(2)$, respectively. These are also allowed configurations for $D_{S^1}E_5$ and $D_{S^1}E_6$, and lead to the quivers:

(4.50)

Here, quiver nodes within the same *block* share the same incidence information, as in [68, 223]. These are also known as the $shdP_5$ and $shdP_6$ quivers, respectively. Note that the $shdP_5$ model has a $\mathbb{Z}_2^{[1]}$ one-form symmetry, which is embedded in the Smith normal form of the intersection matrix of the quiver, as shown in [195]. We do not have good candidates for what theories these quivers might describe.

Γ	n_Γ	(w_i)	(e_2, e_3)	$f(\tau)$	*Monster*
$\Gamma(1)$	1	(1)	$(1,1)$	$j(\tau)$	$1A$
Γ^2	2	(2)	$(0,2)$	$\frac{E_6(\tau)}{\eta(\tau)^{12}}$	$2a$
Γ^3	3	(3)	$(0,2)$	$\frac{\vartheta_2(\tau)^8+\vartheta_3(\tau)^8+\vartheta_4(\tau)^8}{2\eta(\tau)^8}$	$3C$
$\Gamma_0(2)$	3	$(2,1)$	$(1,0)$	$\left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{24}$	$2B$
$\Gamma_0(3)$	4	$(3,1)$	$(0,1)$	$\left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^{12}$	$3B$
$4A^0$	4	(4)	$(2,1)$	$\frac{\vartheta_2(\tau)^6+i\vartheta_3(\tau)^6+\vartheta_4(\tau)^6}{\vartheta_2(\tau)^2\vartheta_3(\tau)^2\vartheta_4(\tau)^2}$	–
$5A^0$	5	(5)	$(1,2)$	[2]	$5a$
$6A^0$	6	(6)	$(0,3)$	$\frac{\vartheta_3(\tau)^4+e^{\frac{2\pi i}{3}}\vartheta_2(\tau)^4}{\eta(\tau)^4}$	–
$\Gamma_0(4)$	6	$(4,1,1)$	$(0,0)$	$\left(\frac{\eta(\tau)}{\eta(4\tau)}\right)^8$	$4B$
$\Gamma(2)$	6	$(2,2,2)$	$(0,0)$	$\left(\frac{\eta(\frac{\tau}{2})}{\eta(2\tau)}\right)^8$	$4C$
$\Gamma_0(5)$	6	$(5,1)$	$(2,0)$	$\left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6$	$5B$
$3C^0$	6	$(3,3)$	$(2,0)$	$\left(\frac{\eta(\tau)^2}{\eta(\frac{\tau}{3})\eta(3\tau)}\right)^6$	$9A$
$4C^0$	6	$(4,2)$	$(2,0)$	$\left(\frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})\eta(2\tau)}\right)^{12}$	$4D$
$\Gamma_0(7)$	8	$(7,1)$	$(0,2)$	$\left(\frac{\eta(\tau)}{\eta(7\tau)}\right)^4$	$7B$
$6C^0$	8	$(6,2)$	$(0,2)$	$\left(\frac{\eta(2\tau)}{\eta(6\tau)}\right)^6$	$6c$
$4D^0$	8	$(4,4)$	$(0,2)$	[2]	–
$\Gamma_1(5)$	12	$(5,5,1,1)$	$(0,0)$	$\frac{1}{q}\prod_{n=1}^{\infty}(1-q^n)^{-5\left(\frac{n}{5}\right)}$	–
$\Gamma_0(6)$	12	$(6,3,2,1)$	$(0,0)$	$\left(\frac{\eta(\tau)^5\eta(3\tau)}{\eta(6\tau)^5\eta(2\tau)}\right)$	$6E$
$\Gamma_0(8)$	12	$(8,2,1,1)$	$(0,0)$	$\left(\frac{\eta(4\tau)^3}{\eta(2\tau)\eta(8\tau)^2}\right)^4$	$8E$
$\Gamma_0(4)\cap\Gamma(2)$	12	$(4,4,2,2)$	$(0,0)$	$\left(\frac{\eta(2\tau)^3}{\eta(\tau)\eta(4\tau)^2}\right)^4$	$8D$
$\Gamma_0(9)$	12	$(9,1,1,1)$	$(0,0)$	$\left(\frac{\eta(\tau)}{\eta(9\tau)}\right)^3$	$9B$
$\Gamma(3)$	12	$(3,3,3,3)$	$(0,0)$	$\left(\frac{\eta(\frac{\tau}{3})}{\eta(3\tau)}\right)^3$	$9B$

Table 4.1: Modular functions of congruence subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ that correspond to rational elliptic surfaces; n_Γ is the index, (w_i) are the widths of the cusps and (e_2, e_3) are the number of elliptic elements. For $\Gamma_1(5)$, $\left(\frac{n}{p}\right)$ is the Legendre symbol.

Part II

Fibering operators in 5d SCFTs

Chapter 5

Donaldson-Witten theory and its five-dimensional uplift

In this chapter, we review the topological twist of four-dimensional $\mathcal{N}=2$ supersymmetric gauge theories [21, 22]. We focus on Kähler four-manifolds, on which the topological twist preserves two supercharges. By a slight abuse of terminology, we call this the Donaldson-Witten (DW) twist. We will then consider the uplift of this construction to five-dimensions, where the five-manifolds are principal S^1 bundles over Kähler four-manifolds.

5.1 Topological twist on Kähler surfaces

To begin with, we will consider a Kähler four-manifold $\mathcal{M}_4$, and introduce the 'twisted fields' which are most useful in the context of the topological twist. Of particular interest is the case of a hypermultiplet coupled to a background vector multiplet.

5.1.1 Kähler surfaces and the $\mathcal{N}=2$ topological twist

Let us view the Kähler four-manifold $\mathcal{M}_4$ as a hermitian manifold $(\mathcal{M}_4, J, g)$ whose complex structure is covariantly constant, $\nabla_\mu J^\nu{}_\rho = 0$. In local complex coordinates $(z^i) = (z^1, z^2)$, the Kähler metric g reads:

$$ds^2 = 2g_{i\bar{j}}dz^i d\bar{z}^{\bar{j}} \ , \qquad g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^{\bar{j}}} \ , \tag{5.1}$$

where K is the Kähler potential.[35] We will follow the geometry and supersymmetry conventions of [3]. We are interested in 4d $\mathcal{N}=2$ quantum field theories in Euclidean space-time with an exact $SU(2)_R$ R-symmetry. The topological twist consists of relabelling the spins

[35] In the complex frame, the metric simply reads $ds^2 = e^1 e^{\bar{1}} + e^2 e^{\bar{2}}$, with the vierbein $e^a_\mu, e^{\bar{a}}_\mu$, where $a = 1, 2$.

of fields according to a 'twisted spin', as reviewed in chapter 1.4.1. This twisted spin is just the diagonal subgroup of the $SU(2)_R$ symmetry and the $SU(2)_r$ factor of the Euclidean rotation $\text{Spin}(4) \cong SU(2)_l \times SU(2)_r$. Alternatively, the DW twist can be understood as a supergravity background (*i.e.* a rigid limit of some 4d $\mathcal{N} = 2$ supergravity), consisting of a metric (5.1) and of a background gauge field $\mathbf{A}^{(R)}$ for the R-symmetry, preserving some fraction of the flat-space supersymmetry (see *e.g.* [87, 229, 230]). Given the flat space supercharges $Q_\alpha^I, \widetilde{Q}_I^{\dot\alpha}$, one preserves a right-chiral supersymmetry:

$$\delta_{\widetilde{\xi}} \equiv \widetilde{\xi}^I_{\dot\alpha} \widetilde{Q}_I^{\dot\alpha} \ , \tag{5.2}$$

on any background $(\mathcal{M}_4, g, \mathbf{A}^{(R)})$ that admits a covariantly-constant spinor $\widetilde{\xi}_I$:

$$D_\mu \widetilde{\xi}_I \equiv \left(\nabla_\mu \delta_I{}^J - i(\mathbf{A}_\mu^{(R)})_I{}^J\right) \widetilde{\xi}_J = 0 \ . \tag{5.3}$$

Here and in the following, $I, J \in \{1, 2\}$ are $SU(2)_R$ indices.[36] Such a background exists on any Riemannian four-manifold: one obtains a solution to (5.3) by identifying the $SU(2)_R$ connection with the spin connection [21]. This leads to a 'trivial' solution of (5.3):

$$(\widetilde{\xi}_I^{\dot\alpha}) = (\delta^{\dot\alpha}{}_I) \ . \tag{5.4}$$

More formally, the Killing spinor $\widetilde{\xi}_I^{\dot\alpha}$ is a section of a complex vector bundle $S_+ \otimes E_R$, where E_R is a rank-2 $SU(2)_R$ vector bundle. Then, the topological twist consists in choosing $E_R \cong S_+$, in which case $S_+ \otimes E_R$ decomposes as a direct sum

$$S_+ \otimes E_R \cong \mathcal{O} \oplus \Omega^+ \ , \tag{5.5}$$

where Ω^+ is the rank-3 vector bundle of self-dual 2-forms.[37] Then our Killing spinor $\widetilde{\xi}$ is simply the constant section of the trivial line bundle $\mathcal{O}$. It is also important to note that the topological twist is defined on any four-manifold, irrespective of whether it is a spin manifold, because the bundle (5.5) is well-defined even when S_+ is not. For Kähler surfaces,

[36] We usually keep the $SU(2)_R$ indices explicit, while suppressing the spinor indices $\alpha, \dot\alpha$.
[37] At the level of Spin(4) representations, we have $(0, \frac{1}{2}) \otimes (0, \frac{1}{2}) = (0, 0) \oplus (0, 1)$.

we preserve two distinct supersymmetries:

$$\delta_1 \equiv \widetilde{\xi}^{I}_{(1)\dot{\alpha}} \widetilde{Q}^{\dot{\alpha}}_{I} \;, \qquad\qquad \delta_2 \equiv \widetilde{\xi}^{I}_{(2)\dot{\alpha}} \widetilde{Q}^{\dot{\alpha}}_{I} \;. \tag{5.6}$$

The Levi-Civita connection on a Kähler manifold has reduced holonomy $U(2) \cong SU(2)_l \times U(1)_r \subset SU(2)_l \times SU(2)_r$, and we then only need to 'twist' $U(1)_r$ by turning on a non-trivial gauge field for the R-symmetry subgroup $U(1)_R \subset SU(2)_R$. By choosing an appropriate background $SU(2)_R$ gauge field [3], in the complex frame basis we preserve the two Killing spinors:

$$(\widetilde{\xi}^{\dot{\alpha}}_{(1)I}) = (\delta^{\dot{\alpha}\dot{1}} \delta_{I1}) \;, \qquad\qquad (\widetilde{\xi}^{\dot{\alpha}}_{(2)I}) = (\delta^{\dot{\alpha}\dot{2}} \delta_{I2}) \;. \tag{5.7}$$

Note that the Killing spinor (5.4) is the sum of these two Killing spinors, $\widetilde{\xi} = \widetilde{\xi}_{(1)} + \widetilde{\xi}_{(2)}$. Correspondingly, we preserve the flat-space supercharges $\widetilde{Q}^{\dot{2}}_{2}$ and $\widetilde{Q}^{\dot{1}}_{1}$ on any Kähler manifold, while on a generic four-manifold, we only preserve their sum, $\widetilde{Q}^{\dot{1}}_{1} + \widetilde{Q}^{\dot{2}}_{2}$. More covariantly, on any Kähler surface $\mathcal{M}_4$, the spin bundle $\mathbf{S} \equiv S_- \oplus S_+$ formally decomposes as:

$$S_- \cong \mathcal{K}^{\frac{1}{2}} \otimes \Omega^{0,1} \;, \qquad\qquad S_+ \cong \mathcal{K}^{\frac{1}{2}} \oplus \mathcal{K}^{-\frac{1}{2}} \;, \tag{5.8}$$

with $\mathcal{K}$ the canonical line bundle. Here, $\mathcal{M}_4$ is spin if and only if the 'square-root' $\mathcal{K}^{\frac{1}{2}}$ actually exists. Recall that the second Stiefel-Withney class of a complex surface $\mathcal{M}_4$ is related to its first Chern class, namely $w_2(\mathcal{M}_4) \cong c_1(\mathcal{K}) \mod 2$. Let us choose an $SU(2)_R$ vector bundle of the form

$$E_R = L_R^{-1} \oplus L_R \;, \tag{5.9}$$

for L_R some $U(1)_R$ line bundle. The Killing spinors (5.7) are really sections $\widetilde{\xi}_{(1)} \in \Gamma[S_+ \otimes L_R]$ and $\widetilde{\xi}_{(2)} \in \Gamma[S_+ \otimes L_R^{-1}]$, while the DW twist amounts to the formal identification

$$L_R \cong \mathcal{K}^{-\frac{1}{2}} \;. \tag{5.10}$$

In general, $\mathcal{M}_4$ is not spin, and therefore $\mathcal{K}^{\frac{1}{2}}$ does not exist, but the bundles $S_+ \otimes L_R^{\pm 1}$ are nonetheless well-defined spin^c bundles. We will further comment on this point in section 5.1.3 below, where we discuss the topological twist of the hypermultiplet.

Spinor bilinears and Kähler structure. Given the Killing spinors introduced so far, one can construct well-defined two-forms on $\mathcal{M}_4$. First of all, given any solution $\widetilde{\xi}$ to the Killing spinor equation (5.3), we can define the $SU(2)_R$-neutral anti-self-dual two-form:

$$\mathcal{J}_{\mu\nu}[\widetilde{\xi}] \equiv -2i\frac{\widetilde{\xi}^{\dagger I}\widetilde{\sigma}_{\mu\nu}\widetilde{\xi}_I}{|\widetilde{\xi}|^2} , \qquad |\widetilde{\xi}|^2 \equiv \widetilde{\xi}^{\dagger I}\widetilde{\xi}_I , \tag{5.11}$$

where the sum over repeated indices is understood. For the Killing spinor (5.4) on a general four-manifold, the bilinear (5.11) identically vanishes. On the other hand, from the Killing spinors (5.7), we obtain $J_{\mu\nu} \equiv \mathcal{J}_{\mu\nu}[\widetilde{\xi}_{(1)}] = -\mathcal{J}_{\mu\nu}[\widetilde{\xi}_{(2)}]$, which satisfies:

$$J^\mu{}_\nu J^\nu{}_\rho = -\delta^\mu{}_\rho , \qquad \nabla_\mu J_{\nu\rho} = 0 . \tag{5.12}$$

Thus, (5.12) gives us the complex structure (and the associated Kähler form) of the hermitian Kähler manifold $\mathcal{M}_4$. In this way, one can show that there are two linearly independent solutions to (5.3) if and only $\mathcal{M}_4$ is Kähler [231]. Given the two Killing spinors (5.7), we may also write down the bilinears:[38]

$$p^{2,0}_{(1)} \equiv \widetilde{\xi}_{(1)}\sigma_{\mu\nu}\widetilde{\xi}_{(1)}\, dx^\mu \wedge dx^\nu \qquad p^{0,2}_{(2)} \equiv \widetilde{\xi}_{(2)}\sigma_{\mu\nu}\widetilde{\xi}_{(2)}\, dx^\mu \wedge dx^\nu \tag{5.13}$$

These are nowhere-vanishing sections of the line bundles $\mathcal{K}\otimes L_R^2$ and $\mathcal{K}^{-1}\otimes L_R^{-2}$, respectively, and therefore the corresponding line bundles are trivial. This is another way to see that (5.10) must hold, or more precisely, $L_R^2 \cong \mathcal{K}^{-1}$ if $\mathcal{M}_4$ is not spin.

5.1.2 The vector multiplet on $\mathcal{M}_4$

Let us consider the $\mathcal{N} = 2$ vector multiplet $\mathcal{V}$, in the adjoint representation of some Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, on $\mathcal{M}_4$ a Kähler manifold. It consists of a gauge field A_μ, two sets of gauginos λ^I, and a triplet of auxiliary scalar fields $D_{IJ} = D_{JI}$:

$$\mathcal{V} = \Big(A_\mu, \phi, \widetilde{\phi}, \lambda^I, \widetilde{\lambda}_I, D_{IJ}\Big) . \tag{5.14}$$

[38] In the frame basis, we have $p^{2,0}_{(1)} = -e^1 \wedge e^2$ and $p^{0,2}_{(2)} = -e^{\bar{1}} \wedge e^{\bar{2}}$ for the solutions (5.7).

The gauge connection $A = A_\mu dx^\mu$ is well defined on any four-manifold. It is also customary to introduce the Dolbeault operators ∂_A and $\bar{\partial}_A$ twisted by the gauge field $A = A_\mu dx^\mu$, as $d_A = d - iA = \partial_A + \bar{\partial}_A$. Moreover, let $F^{2,0}$ and $F^{0,2}$ denote the $(2,0)$ and $(0,2)$ projectionS of the field strength $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$. After the topological twist, the gauginos are also well-defined on $\mathcal{M}_4$, being sections of

$$\begin{aligned} &\lambda \in \Gamma[S_- \otimes E_R] \cong \Gamma[\Omega^{0,1} \oplus (\mathcal{K} \otimes \Omega^{0,1})] \cong \Gamma[\Omega^{0,1} \oplus \Omega^{1,0}] \ , \\ &\widetilde{\lambda} \in \Gamma[S_+ \otimes \bar{E}_R] \cong \Gamma[\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{K} \oplus \mathcal{K}^{-1}] \ . \end{aligned} \tag{5.15}$$

Here we use the Hodge star operator to map $(2,1)$-forms (the sections of $\mathcal{K}\otimes\Omega^{1,0}$) to $(1,0)$-forms, according to $\omega^{1,0} = \star\,\omega^{2,1}$. These (p,q)-forms can be constructed explicitly from the ordinary (flat-space) spinors, by contracting the gauginos with the Killing spinors (5.7) to form $SU(2)_R$-neutral tensors. We can also define the $SU(2)_R$-neutral auxiliary fields:

$$\mathcal{D}^{2,0} \equiv -iD_{22}p^{2,0}_{(1)} \ , \qquad \mathcal{D}^{0,2} \equiv -iD_{11}p^{0,2}_{(2)} \ , \qquad \mathcal{D}^{0,0} = D_{12} + \widehat{F} \ , \tag{5.16}$$

where we introduced $\widehat{F} \equiv \frac{1}{2}J^{\mu\nu}F_{\mu\nu}$, At this stage, one can, in principle, write down the supersymmetry transformations for the twisted fields using the flat-space transformations of the vector multiplet. However, given that we will not use these explicitly here, we refer the reader to [3] for the explicit variations. These realize the supersymmetry algebra:

$$\delta_1^2 = 0 \ , \qquad \delta_2^2 = 0 \ , \qquad \{\delta_1, \delta_2\} = 2\sqrt{2}\delta_{g(\phi)} \ , \tag{5.17}$$

where $\delta_{g(\phi)}$ is a gauge transformation with parameter ϕ. In particular, we have $\delta_{g(\phi)}A = d_A\phi = d\phi + i[\varphi, A]$ for the gauge field, and $\delta_{g(\phi)}\varphi = i[\phi,\varphi]$ for any field φ transforming in the adjoint representation of $\mathfrak{g}$. The 4d $\mathcal{N}=2$ SYM Lagrangian can be written compactly as

$$\mathscr{L}_{\text{SYM}} = \frac{1}{g^2}\delta_1\delta_2 \operatorname{tr}\left(\frac{1}{8}\star\left(\widetilde{\Lambda}^{2,0}\wedge\widetilde{\Lambda}^{0,2}\right) - i\frac{\sqrt{2}}{4}\widetilde{\phi}\left(\mathcal{D}^{0,0} - 2\widehat{F}\right)\right) - \frac{1}{2g^2}\star\operatorname{tr}\left(F\wedge F\right) \ , \tag{5.18}$$

which is “mostly” $\mathcal{Q}$-exact. Here, $\widetilde{\Lambda}^{2,0}$ and $\widetilde{\Lambda}^{0,2}$ are the $(2,0)$ and $(0,2)$-components of the twisted $\widetilde{\lambda}$ field, as in (5.15). Recall also that on supersymmetric configurations, the $\mathcal{Q}$-exact terms in (5.18) evaluate to zero. Defining the ‘instanton number’ – more precisely, (minus) the second Chern class of any holomorphic vector bundle associated to a principal G bundle – as

$$k = -\frac{1}{16\pi^2}\int_{\mathcal{M}_4} \mathrm{tr}\,(F\wedge F)\ , \tag{5.19}$$

and adding the topological coupling S_{top} defined in (1.4) to the Lagrangian, any supersymmetric vector-multiplet configuration is weighted by a factor:

$$e^{-S_{\text{SYM}}-S_{\text{top}}} = e^{2\pi i\tau k}\ , \tag{5.20}$$

where τ is the holomorphic gauge coupling as (1.5). In particular, the classical saddles are Yang-Mills instantons[39] and they contribute in this way.

5.1.3 The hypermultiplet on $\mathcal{M}_4$

Let us also briefly discuss the 4d $\mathcal{N}=2$ hypermultiplet on $\mathcal{M}_4$. As for the vector multiplet, we will not write down explicitly the supersymmetry variations, as they will not be needed for the rest of this work. We will later on compute one-loop determinants for the 5d uplift of the DW twist, which, for consistency, should give the same result as that of the 4d computations, after resumming the KK tower. This check is performed in great detail in [3]. We will limit the discussion in this section to some general features of 4d $\mathcal{N}=2$ hypermultiplets, and see later on how these are reflected in the 5d ‘twisted’ theory.

Consider a hypermultiplet $\mathcal{H}$ charged under a Lie group G. When considered as part of a larger gauge theory, G will include both the gauge group, with its dynamical gauge fields, and the flavour symmetry group, with its background gauge fields. On-shell, this multiplet consists of two complex scalars, q^I, forming a doublet of $SU(2)_R$, and of two

[39] More precisely, *anti-instantons*, satisfying the anti-self-duality condition $F=-\star F$. We can call them ‘instantons’ since self-dual instantons do not play a role in Donaldson-Witten theory.

$SU(2)_R$-neutral Dirac fermions:

$$\mathcal{H} = \left(q^I \ , \widetilde{q}_I \ , \eta \ , \widetilde{\eta} \ , \chi \ , \widetilde{\chi}\right) \ . \tag{5.21}$$

In addition, we will need to introduce some auxiliary fields in order to realize the two supersymmetries δ_1 and δ_2 off-shell.[40] The fields q^I, η and $\widetilde{\chi}$ transform in some representation $\mathfrak{R}$ of the gauge algebra $\mathfrak{g}$, and the fields $\widetilde{q}_I$, χ and $\widetilde{\eta}$ transform in the conjugate representation $\bar{\mathfrak{R}}$. After the topological twist, the scalars q^I, $\widetilde{q}_I$ become right-chiral spinors, which are therefore not well-defined unless $\mathcal{M}_4$ is a spin manifold [232]. For charged hypermultiplets, this issue can be remedied by introducing a spin^c structure [88, 95, 233]. Such a structure exists on any oriented closed four-manifold, but it is important to emphasise that this is an additional choice that we make when considering hypermultiplets. We thus call this an 'extended DW twist'.

Without too much loss of generality, let us consider $\mathcal{H}$ charged under some gauge group $\widetilde{G} = U(1) \times G$, where the $U(1)$ gauge field is really a spin^c connection. It is associated with a line bundle $\mathcal{L}_0$ such that $\mathcal{L}_0^{\frac{1}{2}} \otimes S_+$ is well-defined. On a Kähler manifold, with the spin bundle formally given by (5.8), we will choose:

$$\mathcal{L}_0 \cong \mathcal{K}^{-1} \ . \tag{5.22}$$

We insist on the fact that this is a somewhat arbitrary choice, however natural it appears on a Kähler manifold. For our purposes, it will also be important to consider the more general case:

$$\mathcal{L}_0 \cong \mathcal{K}^{2\varepsilon} \ , \tag{5.23}$$

with $\varepsilon \in \frac{1}{2} + \mathbb{Z}$ a free parameter.[41] Roughly speaking, the extended topological twist is simply a choice of ε for each hypermultiplet in a theory; this must be done in a consistent way, as we will discuss further in later sections.

[40] It is well-known that one cannot realize the full flat-space $\mathcal{N} = 2$ supersymmetry off-shell with a finite number of auxiliary fields, but there is no problem with realizing these two particular supercharges off-shell.

[41] This should not be confused with the ε parameter used in chapter 2.3.

The ordinary DW twist of the hypermultiplet scalars gives us the right-chiral spinors $\mathbf{q} \equiv \widetilde{\xi}_I q^I$ and $\widetilde{\mathbf{q}} \equiv \epsilon^{IJ}\widetilde{\xi}_I \widetilde{q}_J$. On an arbitrary Kähler manifold, the extended topological twist exists when these spinors are further valued in $\mathcal{L}_0^{\frac{1}{2}}$, namely:

$$\mathbf{q} \in \Gamma(S_+ \otimes \mathcal{K}^{\varepsilon}) \ , \qquad \widetilde{\mathbf{q}} \in \Gamma(S_+ \otimes \mathcal{K}^{-\varepsilon}) \ . \tag{5.24}$$

In the rest of this section, we will set $\varepsilon = -\frac{1}{2}$. Reinstating a general ε will simply correspond to having the twisted hypermultiplet, as described below, also valued in a line bundle $\mathcal{K}^{\varepsilon+\frac{1}{2}}$. Thus, setting $\varepsilon = -\frac{1}{2}$, the scalars become (p,q)-forms, with $\mathbf{q} \in \Gamma(\mathcal{O} \oplus \mathcal{K}^{-1})$ and $\widetilde{\mathbf{q}} \in \Gamma(\mathcal{K} \oplus \mathcal{O})$. That is, $\mathbf{q} = (Q^{0,0}, Q^{0,2})$, while $\widetilde{\mathbf{q}} = (\widetilde{Q}^{0,0}, \widetilde{Q}^{2,0})$.

All fields are also valued in the appropriate vector bundles $E_{\mathfrak{R}}$ or $E_{\bar{\mathfrak{R}}}$ determined by the representation $\mathfrak{R}$ – we omitted this from the notation to avoid clutter. We also have the two Dirac spinors

$$\Psi = (\eta_\alpha, \widetilde{\chi}^{\dot{\alpha}}) \ , \qquad \widetilde{\Psi} = (\chi_\alpha, \widetilde{\eta}^{\dot{\alpha}}) \ , \tag{5.25}$$

which are sections of spin^c bundles $S \otimes \mathcal{K}^{-\frac{1}{2}}$ and $S \otimes \mathcal{K}^{\frac{1}{2}}$, respectively. They can be conveniently decomposed into (p,q)-forms:

$$\Psi^{0,\bullet} = (\eta^{0,1}, \widetilde{\chi}^{0,0}, \widetilde{\chi}^{0,2}) \in \Omega^{0,\bullet} \ , \qquad \widetilde{\Psi}^{\bullet,0} = (\chi^{1,0}, \widetilde{\eta}^{0,0}, \widetilde{\eta}^{2,0}) \in \Omega^{\bullet,0} \ . \tag{5.26}$$

For instance, the spinor χ is a section of $S_- \otimes \mathcal{K}^{\frac{1}{2}} \cong \Omega^{0,1} \otimes \mathcal{K} \cong \Omega^{2,1} \cong \Omega^{1,0}$, where we find it convenient to use $\chi^{1,0} \equiv \star\chi^{2,1}$. Finally, we need to introduce the auxiliary one-forms $h^{0,1}$ and $\widetilde{h}^{1,0}$ in order to close the curved-space supersymmetry algebra off-shell. In fact, under the two supersymmetries δ_1, δ_2, the hypermultiplet splits into two off-shell multiplets (coupled to the vector multiplet):

$$\mathcal{H} \cong (Q^{0,0}, Q^{0,2}, \Psi^{0,\bullet}, h^{0,1}) \oplus (\widetilde{Q}^{0,0}, \widetilde{Q}^{2,0}, \widetilde{\Psi}^{\bullet,0}, \widetilde{h}^{1,0}) \ , \tag{5.27}$$

which consist of purely anti-holomorphic and holomorphic forms, respectively. The supersymmetry algebra (5.17) is then realized as

$$\{\delta_1, \delta_2\}\, f = 2i\sqrt{2}\,\phi\, f \ , \qquad \{\delta_1, \delta_2\}\, \widetilde{f} = -2i\sqrt{2}\, \widetilde{f}\, \phi \ , \tag{5.28}$$

on any fields $f = (Q, \widetilde{\chi}, \eta, h)$ and $\widetilde{f} = (\widetilde{Q}, \widetilde{\eta}, \chi, \widetilde{h})$ in the gauge representation $\mathfrak{R}$ and $\bar{\mathfrak{R}}$, respectively.[42] We refer to [3] for more details on the supersymmetry variations. The hypermultiplet Lagrangian on $\mathcal{M}_4$ can be obtained by starting from the flat-space Lagrangian, writing it in twisted variables, and adding in the auxiliary fields to ensure off-shell supersymmetry. The important fact is that it is $\mathcal{Q}_{\rm DW}$-exact. We find [3]:

$$\begin{aligned}\mathscr{L}_{\mathcal{H}} = \frac{1}{4}(\delta_1 + \delta_2) \star \Bigg(& \widetilde{h}^{1,0} \wedge \star \eta^{0,1} - 2i\sqrt{2}\, \chi^{1,0} \wedge \star \bar{\partial}_A Q^{0,0} + i\sqrt{2}\, \chi^{1,0} \wedge \partial_A Q^{0,2} \\ & - \frac{i}{4}\, \widetilde{\eta}^{0,0}\, \widetilde{\phi}\, Q^{0,0}\, d\mathrm{vol} + i\, \widetilde{\eta}^{2,0} \wedge \widetilde{\phi}\, Q^{0,2} - i\, \widetilde{Q}^{2,0} \wedge \widetilde{\Lambda}^{0,2}\, Q^{0,0} \\ & + \frac{i}{4}\, \widetilde{Q}^{0,0}\, \widetilde{\Lambda}^{0,0}_{(1)}\, Q^{0,0}\, d\mathrm{vol} + i\, \widetilde{Q}^{2,0} \wedge \widetilde{\Lambda}^{0,0}_{(2)}\, Q^{0,2} + i\, \widetilde{Q}^{0,0}\, \widetilde{\Lambda}^{2,0} \wedge Q^{0,2} \Bigg) \ , \end{aligned} \tag{5.29}$$

with $d\mathrm{vol} = \star 1$ being the volume form on $\mathcal{M}_4$.

5.2 Five dimensional uplift of DW twist

In this section, we uplift the topological twist of the previous section to a supersymmetric background for 5d $\mathcal{N} = 1$ supersymmetric field theories on any five-manifold $\mathcal{M}_5$ which is a principal circle bundle over a Kähler surface,

$$S^1 \longrightarrow \mathcal{M}_5 \xrightarrow{\pi} \mathcal{M}_4 \ . \tag{5.30}$$

Supersymmetric backgrounds of similar geometries were discussed by many authors – see *e.g.* [234–240]. Our approach here is limited to a supersymmetric background that reduces to the (extended) topological twist on $\mathcal{M}_4$.

[42] In our conventions expressions of the type ϕf refer to contractions of the type $\phi^a T^a_{ij} f^j$, where T^a_{ij} are generators for the representation under which the field f transforms, with indices i, j, while a are adjoint representation indices. We use the generators of the representation $\mathfrak{R}$ for all fields of the hypermultiplet, including for those transforming in $\bar{\mathfrak{R}}$.

5.2.1 Circle-bundle geometries and the 5d Killing spinor equation

Let the five-manifold $\mathcal{M}_5$ be a principal circle bundle over a Kähler four-manifold $\mathcal{M}_4$. This fibration is fully determined by the first Chern class:

$$c_1(\mathcal{L}_{\rm KK}) = \frac{1}{2\pi}\hat{\mathtt{F}} \in H^2(\mathcal{M}_4, \mathbb{Z}) \;, \tag{5.31}$$

where the 'defining line bundle' $\mathcal{L}_{\rm KK}$ is the complex line bundle associated to the S^1 bundle. We define the Chern numbers $\mathfrak{p}_k$ by:

$$c_1(\mathcal{L}_{\rm KK}) = \sum_k \mathfrak{p}_k [S_k] \;, \tag{5.32}$$

with the 2-cycles $\mathrm{S}_k \subset \mathcal{M}_4$ forming a basis of $H_2(\mathcal{M}_4, \mathbb{Z})$, and $[S_k] \in H^2(\mathcal{M}_4, \mathbb{Z})$ their Poincaré duals. We have:

$$\mathbf{I}_{kl} = \int_{\mathrm{S}_k} [\mathrm{S}_l] = \mathrm{S}_k \cdot \mathrm{S}_l \;, \tag{5.33}$$

the intersection numbers on $\mathcal{M}_4$. Given a Kähler metric (5.1) on $\mathcal{M}_4$ with local coordinates (z^1, z^2), we choose the five-dimensional metric

$$ds^2(\mathcal{M}_5) = ds^2(\mathcal{M}_4) + \eta^2 \;, \qquad \eta \equiv \beta(d\psi + \mathrm{C}) \;, \tag{5.34}$$

with the fiber coordinate ψ subject to the identification $\psi \sim \psi + 2\pi$, and the connection C on $\mathcal{M}_4$ such that:

$$d\mathrm{C} = \hat{\mathtt{F}} = 2\pi\, c_1(\mathcal{L}_{\rm KK}) \;. \tag{5.35}$$

In (5.34) we also introduced β, the radius of the circle fiber. It has been shown in [240] that theories with $\mathcal{N} = 1$ supersymmetry can be defined on five-manifolds that admit such a metric. The existence of curved-space supersymmetry is related to the existence of a transversely holomorphic foliation (THF) structure defined by the one-form η, similarly to the three-dimensional geometries studied in [89, 241].

By assumption, since we have a fibration structure, the metric (5.34) admits a Killing vector K with dual one-form given by η, $K^M \equiv \eta^M$, namely:[43]

$$K = \frac{1}{\beta}\partial_\psi \ . \tag{5.36}$$

Note that we have $d\eta = \beta\hat{\mathtt{F}} = 2\pi\beta\sum_k \mathfrak{p}_i[\mathrm{S}_k]$ and $\nabla_M\eta_N + \nabla_N\eta_M = 0$, which both follow from the relation:

$$\nabla_M\eta_N = \frac{\beta}{2}\hat{\mathtt{F}}_{MN} \ . \tag{5.37}$$

Here ∇_M is the 5d Levi-Civita connection. We would like to construct a supersymmetric background on $\mathcal{M}_5$ which is the uplift of the four-dimensional DW twist. In particular, such a background should admit two five-dimensional Killing spinors $\zeta^I_{(i)}$, for $i = 1, 2$, related to the four-dimensional Killing spinors (5.7) by[44]

$$\zeta_{(i)I} = \begin{pmatrix} 0 \\ \widetilde{\xi}_{(i)I} \end{pmatrix} \ , \tag{5.38}$$

with $\widetilde{\xi}^{\dot\alpha}_{(1)I} = \delta^{\dot\alpha\dot 1}\delta_{I1}$ and $\widetilde{\xi}^{\dot\alpha}_{(2)I} = \delta^{\dot\alpha\dot 2}\delta_{I2}$. Once we posit the Killing spinors (5.38), we must reconstruct the 5d Killing spinor equations that they satisfy. The trivial uplift of the 4d Killing spinor equation,

$$\left(\nabla_M \delta_I{}^J - i(A^R_M)_I{}^J\right)\zeta_J = 0 \ , \tag{5.39}$$

only holds for the trivial circle fibration. This is related to the fact that the connection ∇_M does not preserve the decomposition of tensors into vertical and horizontal components with respect to the fibration, since

$$\nabla_M\eta_N \neq 0 \ , \tag{5.40}$$

unless $\mathfrak{p}_i = 0$. To correct this, we can simply introduce a new connection that preserves the fibration structure. The price to pay is that such a connection will have non-zero torsion.

[43]We use five-dimensional conventions that naturally reduce to our four-dimensional conventions upon circle reduction along the fifth coordinate. In flat space, we have $x^M = (x^\mu, x^5)$, with the index $M = (\mu, 5)$, for $\mu = 1, \ldots, 4$. In curved space, we choose a complex frame adapted to the fibration structure, such that $ds^2 = e^1 e^{\bar 1} + e^2 e^{\bar 2} + \eta^2$.

[44]A five-dimensional Dirac spinor transforms in the $\bar{\mathbf{4}}$ of USp(4) $\cong$ Spin(5). Spinor indices are raised with $\Omega^{\mathbf{ab}} = \mathrm{diag}(\epsilon^{\alpha\beta}, \epsilon_{\dot\alpha\dot\beta})$. One can still impose a Majorana-Weyl condition on the 5d spinors, which, upon dimensional reduction to 4d lead to two distinct Majorana spinors. For more details on our conventions, see [3].

Transversely holomorphic foliation and adapted connection. The five-dimensional manifolds that we are considering here are fibrations with a Kähler base – in particular, they are transversely holomorphic foliations (THF) with an adapted metric. A one-dimensional foliation structure on the five-manifold $\mathcal{M}_5$ is generated by a nowhere-vanishing vector field $K^M = g^{MN}\eta_N$. The foliation is transversely holomorphic if there exists a tangent bundle endomorphism Φ – *i.e.* a two-tensor $\Phi^M{}_N$ – whose restriction to the kernel of the one-form η gives an integrable complex structure,

$$\Phi\Big|_{\ker(\eta)} = J \ . \tag{5.41}$$

In particular, we have the relation

$$\Phi^M{}_N\Phi^N{}_P = -\delta^M{}_P + K^M\eta_P \ . \tag{5.42}$$

As shown in [240], the existence of one supercharge on a five-manifold[45] implies the existence of a THF. In the present case, we have two supercharges and the foliation is actually a fibration. We further restrict ourselves to the case when the holomorphic base manifold is Kähler, as required by the DW twist with two supercharges.

Focussing then on the class of fibered five-manifolds introduced above, with the adapted metric (5.34), it is convenient to introduce a modified connection $\hat{\nabla}$ that preserves the THF and fibration structure,

$$\hat{\nabla}_M g_{NP} = 0 \ , \qquad \hat{\nabla}_M \eta_N = 0 \ , \qquad \hat{\nabla}_M \Phi^N{}_P = 0 \ . \tag{5.43}$$

This connection can be expressed in terms of the Levi-Civita connection as

$$\hat{\Gamma}^P_{MN} = \Gamma^P_{MN} + K^P_{MN} \ , \tag{5.44}$$

[45]Together with a few other assumptions about the type of theories considered.

where K is the cotorsion tensor. In terms of the circle-bundle curvature $\hat{\mathtt{F}}_{MN}$, this becomes:

$$K_{PMN} = \frac{\beta}{2}\left(\eta_P \hat{\mathtt{F}}_{MN} - \eta_N \hat{\mathtt{F}}_{MP} - \eta_M \hat{\mathtt{F}}_{NP}\right) \ . \tag{5.45}$$

The torsion tensor of the modified connection reads

$$T^P{}_{MN} = K^P{}_{MN} - K^P{}_{NM} = \beta \eta^P \hat{\mathtt{F}}_{MN} \ . \tag{5.46}$$

From here on, we will denote by $\hat{D}_M$ the covariant derivative with respect to the modified connection (which is also $SU(2)_R$- and gauge-covariant, as the case may be). For instance, for a scalar field ϕ we have

$$\left[\hat{D}_M, \hat{D}_N\right] \phi = -T^P{}_{MN} \hat{D}_P \phi \ . \tag{5.47}$$

Killing spinor equation and spinor bilinears. Given the adapted connection $\hat{D}_M$ on $\mathcal{M}_5$, we choose the Killing spinor equation

$$\hat{D}_M \zeta_I \equiv \left(\hat{\nabla}_M \delta^I{}_J - i(\mathbf{A}^{(R)}_M)^I{}_J\right) \zeta_J = 0 \ , \tag{5.48}$$

with $\mathbf{A}^{(R)}_M$ the $SU(2)_R$ background gauge field. One can easily check that the 5d Killing spinors (5.38) are indeed solutions to (5.48), once we take $\mathbf{A}^{(R)}_M$ to be the pull-back of the corresponding DW-twist connection on $\mathcal{M}_4$. In fact, we only need to turn on a $U(1)_R \subset SU(2)_R$ background, as in the four-dimensional case. As a result, we can introduce the spinors:

$$\zeta_{(1)} = \zeta_{(1)\, I=1} \ , \qquad \zeta_{(2)} = \zeta_{(2)\, I=2} \ , \tag{5.49}$$

for which we have:

$$\left(\hat{\nabla}_M - i A^{(R)}_M\right) \zeta_{(1)} = 0 \ , \qquad \left(\hat{\nabla}_M + i A^{(R)}_M\right) \zeta_{(2)} = 0 \ , \tag{5.50}$$

where the $U(1)_R$ gauge field is essentially the same as the four-dimensional background in (5.7) – see [3] for the explicit form.

THF from spinors. Given two distinct nowhere-vanishing solutions to (5.48), we can reconstruct THF tensors. We define the one-form

$$\eta_M = -\frac{1}{|\zeta_{(1)}|^2}\zeta^{\dagger\,I}_{(1)}\gamma_M\zeta_{(1)\,I} = -\frac{1}{|\zeta_{(2)}|^2}\zeta^{\dagger\,I}_{(2)}\gamma_M\zeta_{(2)\,I} \ , \tag{5.51}$$

with the Hermitian conjugate defined as for four-dimensional spinors [231, 242]. Our conventions for the γ matrices are the same as in [3]. Additionally, similarly to the three-dimensional analysis of [241], the quantity defined as:

$$K^M = \zeta_{(1)I}\gamma^M\zeta^I_{(2)} = -\zeta_{(1)}\gamma^M\zeta_{(2)} \ , \tag{5.52}$$

is a non-vanishing Killing vector, whose orbits define a foliation of $\mathcal{M}_5$. Note that $K^M = \eta^M$ for our choice of metric. For future reference, let us also define the scalar

$$\kappa \equiv \zeta^I_{(1)}\zeta_{(2)\,I} \ . \tag{5.53}$$

Note that, when plugging in (5.38), we have $K^M = \delta^{M5}$ and $\kappa = 1$. Finally, we define a two-tensor

$$\Phi^{MN} = \frac{i\zeta^\dagger_{(1)}\gamma^{MN}\zeta_{(1)}}{|\zeta_{(1)}|^2} \ , \tag{5.54}$$

with $\gamma^{MN} \equiv \frac{1}{2}[\gamma^M, \gamma^N]$, which satisfies (5.41) and (5.42). The Killing spinor equation (5.48) also implies (5.43).

The 5d DW twist. The five-dimensional uplift of the DW twist on $\mathcal{M}_4$ can be formulated a little bit more covariantly. To do this, it is useful to consider the two-forms

$$\begin{aligned} \mathcal{P}_{(1)} &\equiv \zeta_{(1)}\Sigma_{MN}\zeta_{(1)}\, dx^M \wedge dx^N = ie^1 \wedge e^2 \ , \\ \mathcal{P}_{(2)} &\equiv \zeta_{(2)}\Sigma_{MN}\zeta_{(2)}\, dx^M \wedge dx^N = ie^{\bar{1}} \wedge e^{\bar{2}} \ , \end{aligned} \tag{5.55}$$

where $\Sigma^{MN} = \frac{i}{2}\gamma^{MN}$. Here, we use the complex frame mentioned in footnote 43. Let us define the canonical line bundle $\mathcal{K}_{\mathcal{M}_5}$ on $\mathcal{M}_5$ as the pull-back of the canonical line bundle

$\mathcal{K}_{\mathcal{M}_4}$ on the Kähler manifold $\mathcal{M}_4$, using the fibration structure $\pi : \mathcal{M}_5 \to \mathcal{M}_4$, namely:

$$\mathcal{K}_{\mathcal{M}_5} = \pi^* \mathcal{K}_{\mathcal{M}_4} \ . \tag{5.56}$$

Since the spinors $\zeta_{(1)}$ and $\zeta_{(2)}$ have $U(1)_R$ charges $+1$ and -1, respectively, the two forms (5.55) are nowhere-vanishing sections of $\mathcal{K} \otimes L_R^2$ and $\bar{\mathcal{K}} \otimes L_R^{-2}$, respectively, with L_R the $U(1)_R$ bundle on $\mathcal{M}_5$. We, therefore, have the 5d uplift of the DW twist,

$$L_R \cong \mathcal{K}^{-\frac{1}{2}} \ , \tag{5.57}$$

literally as in (5.10) but now written in terms of line bundles on $\mathcal{M}_5$. As before, $\mathcal{K}^{\frac{1}{2}}$ will not be well-defined unless $\mathcal{M}_4$ is spin, but the Killing spinors are well-defined sections of appropriate spin^c bundles nonetheless.

(p,q)-forms and twisted Dolbeault operators on $\mathcal{M}_5$. The 5d uplift of the DW twist remains independent of the choice of Kähler metric on $\mathcal{M}_4$. To make this property manifest, we express all fields in terms of differential forms, exactly like in 4d. On $\mathcal{M}_5$, differential forms can be further decomposed into horizontal and vertical forms (*i.e.* along the base $\mathcal{M}_4$ and the circle fiber, respectively). This can be done explicitly by using the projectors:

$$\begin{aligned}
\Pi^M{}_N =& \frac{1}{2}\left(\delta^M{}_N - i\Phi^M{}_N - K^M \eta_N\right) \ , \\
\bar{\Pi}^M{}_N =& \frac{1}{2}\left(\delta^M{}_N + i\Phi^M{}_N - K^M \eta_N\right) \ , \\
\Theta^M{}_N =& K^M \eta_N \ .
\end{aligned} \tag{5.58}$$

Any k-form on $\mathcal{M}_5$ decomposes into $(p,q|\ell)$-forms, with $p+q+\ell = k$. Here, ℓ denotes the form degree along the fiber. By abuse of notation, a five-dimensional $(p,q|0)$-form is called (p,q)-form, denoted by $\omega^{p,q}$. Any $(p,q|1)$-form can be written as $\omega^{(p,q|1)} = \omega^{p,q} \wedge \eta$. For instance, for any one form $\omega = \omega_M dx^M$, we have

$$\omega = \omega^{1,0} + \omega^{0,1} + \omega_5 \eta = \omega^{1,0}_i dz^i + \omega^{0,1}_{\bar{i}} d\bar{z}^{\bar{i}} + \omega_5 \eta \ , \tag{5.59}$$

where:

$$\omega_M \Pi^M{}_N = \omega_N^{1,0} \ , \qquad \omega_M \bar{\Pi}^M{}_N = \omega_N^{0,1} \ , \qquad \omega_5 \equiv K^M \omega_M \ . \tag{5.60}$$

In particular, the vertical component is defined by contracting with K. For future reference, we also note that any 2-form F decomposes as:

$$F = F^{2,0} + F^{0,2} + F^{1,1} + F^{1,0} \wedge \eta + F^{0,1} \wedge \eta \ . \tag{5.61}$$

In particular, $(2,0)$-forms are sections of the 5d canonical line bundle (5.56).

Dolbeault operators on $\mathcal{M}_5$. The differential $d : \Omega^k \to \Omega^{k+1}$ on $\mathcal{M}_5$ decomposes as $d = \partial + \bar{\partial} + \widehat{\partial}_5$, where ∂ and $\bar{\partial}$ denote the twisted Dolbeault operators:

$$\partial : \Omega^{(p,q|\ell)} \to \Omega^{(p+1,q|\ell)} \ , \qquad \bar{\partial} : \Omega^{(p,q|\ell)} \to \Omega^{(p,q+1|\ell)} \ , \tag{5.62}$$

and $\widehat{\partial}_5 : \Omega^{(p,q|\ell)} \to \Omega^{(p,q|\ell+1)}$ is given by:[46]

$$\widehat{\partial}_5 \equiv \eta \wedge \partial_5 \ , \qquad \partial_5 \equiv \mathcal{L}_K = K^M \partial_M \ . \tag{5.63}$$

In terms of the local coordinates $(x^M) = (z^i, \bar{z}^{\bar{i}}, \psi)$, the twisted Dolbeault operators are given explicitly by:

$$\partial = dz^i \wedge (\partial_i - \mathrm{C}_i \partial_\psi) \ , \qquad \bar{\partial} = d\bar{z}^{\bar{i}} \wedge (\partial_{\bar{i}} - \mathrm{C}_{\bar{i}} \partial_\psi) \ , \tag{5.64}$$

where C_i and $\mathrm{C}_{\bar{i}}$ and the holomorphic and anti-holomorphic component of the connection C introduced in (5.34). Whenever the fibration is non-trivial, the twisted Dolbeault operators are not nilpotent. Instead, they satisfy the relations:

$$\partial^2 = -\beta \hat{\mathrm{F}}^{2,0} \wedge \partial_5 \ , \qquad \bar{\partial}^2 = -\beta \hat{\mathrm{F}}^{0,2} \wedge \partial_5 \ , \qquad \{\partial, \bar{\partial}\} = -\beta \hat{\mathrm{F}}^{1,1} \wedge \partial_5 \ . \tag{5.65}$$

Of course, they reduce to the ordinary Dolbeault operators on $\mathcal{M}_4$ upon dimensional re-

[46] Note that the operator ∂_5 does not change the form degree. We denote this way the Lie derivative along K, which is equal to $K^M \partial_M$ on forms because $\iota_K \omega = 0$ for any horizontal form, and moreover $\mathcal{L}_K \eta = 0$ because $K^M = \eta^M$ is a Killing vector.

duction along the fiber. We also have that

$$\{\partial + \bar{\partial}, \widehat{\partial}_5\} = \beta \hat{\mathtt{F}} \wedge \partial_5 \ , \qquad\qquad \widehat{\partial}_5^2 = 0 \ , \tag{5.66}$$

where $\beta \hat{\mathtt{F}} = d\eta$. Note that $\hat{\mathtt{F}}$ is a horizontal 2-form on $\mathcal{M}_5$.

Background fluxes on $\mathcal{M}_5$. Let us consider supersymmetry-preserving background fluxes for gauge fields on $\mathcal{M}_5$. Equivalently, we consider line bundles $L_{\mathcal{M}_5}$ with first Chern class

$$c_1(L_{\mathcal{M}_5}) \in H^2(\mathcal{M}_5, \mathbb{Z}) \ . \tag{5.67}$$

The supersymmetry-preserving line bundles are pull-back of holomorphic line bundles on $\mathcal{M}_4$:

$$L_{\mathcal{M}_5} = \pi^* L_{\mathcal{M}_4} \ . \tag{5.68}$$

Given our assumption that $\mathcal{M}_4$ is simply connected, the Gysin sequence implies the following simple relation between the second cohomologies of $\mathcal{M}_4$ and $\mathcal{M}_5$:

$$H^2(\mathcal{M}_5, \mathbb{Z}) = \text{coker}\Big(c_1(\mathcal{L}_{\text{KK}}) \ : \ H^0(\mathcal{M}_4, \mathbb{Z}) \to H^2(\mathcal{M}_4, \mathbb{Z})\Big) \ . \tag{5.69}$$

Let us introduce the notation $\mathfrak{m}$ for the abelian flux on $\mathcal{M}_4$:

$$c_1(L_{\mathcal{M}_4}) = \sum_k \mathfrak{m}_k [S_k] \ , \tag{5.70}$$

as in (5.32).[47] The relation (5.69) means that we can write any five-dimensional flux, denoted by $\mathfrak{m}_{\text{5d}} \in H^2(\mathcal{M}_5, \mathbb{Z})$, as:

$$\mathfrak{m}_{\text{5d}} = \mathfrak{m} \mod \mathfrak{p} \ . \tag{5.71}$$

One important example is the lens space $S^5/\mathbb{Z}_p$ obtained as a degree-p fibration over $\mathbb{P}^2$ (hence $\mathfrak{p} = p$), in which case we have $\mathfrak{m}_{\text{5d}} \in \mathbb{Z}_p$, with $p = 1$ corresponding to the five-sphere.

[47] Assuming that $\mathcal{M}_4$ is simply connected, all 2-cycles in $\mathcal{M}_5$ are inherited from 2-cycles in $\mathcal{M}_4$. More generally, the same would remain true of supersymmetry-preserving fluxes.

For future reference, let us introduce the intersection pairing. Given two line bundles L and L' on $\mathcal{M}_4$ with fluxes $\mathfrak{m}$ and $\mathfrak{m}'$, respectively, we define:

$$(\mathfrak{m}, \mathfrak{m}') = \int_{\mathcal{M}_4} c_1(L) \wedge c_1(L') = \sum_{k,l} \mathfrak{m}_k \mathbf{I}_{kl} \mathfrak{m}_l \ , \tag{5.72}$$

with $\mathbf{I}_{kl}$ defined in (5.33).

5.2.2 The 5d $\mathcal{N} = 1$ vector multiplet on $\mathcal{M}_5$

Let us now consider the simplest supersymmetry multiplets on $\mathcal{M}_5$. The 5d vector multiplet contains a gauge field A_M, a real scalar σ, an $SU(2)_R$ doublet of gauginos, Λ_I, transforming as a Majorana-Weyl spinor, and an $SU(2)_R$ triplet of auxiliary scalars D_{IJ}. The flat-space supersymmetry transformations are reviewed in [3]. On our curved-space background $\mathcal{M}_5$, the supersymmetry transformations read:

$$\begin{aligned}
\delta A_M &= i\zeta_I \gamma_M \Lambda^I \ , \\
\delta\sigma &= -\zeta_I \Lambda^I \ , \\
\delta\Lambda_I &= -i\Sigma^{MN}\zeta_I \left(F_{MN} - i\beta\sigma\hat{\mathtt{F}}_{MN}\right) + i\gamma^M \zeta_I \hat{D}_M \sigma - iD_{IJ}\zeta^J \ , \\
\delta D_{IJ} &= \zeta_I \gamma^M \hat{D}_M \Lambda_J + \zeta_J \gamma^M \hat{D}_M \Lambda_I + \zeta_I [\sigma, \Lambda_J] + \zeta_J [\sigma, \Lambda_I] \ .
\end{aligned} \tag{5.73}$$

Note that the difference from the flat-space algebra arises due to the expression for the field strength, which, when written in terms of the new covariant derivative, reads:

$$F_{MN} = \hat{\nabla}_M A_N - \hat{\nabla}_N A_M - i[A_M, A_N] + \beta\eta^P \hat{\mathtt{F}}_{MN} A_P \ . \tag{5.74}$$

The curved-space supersymmetry algebra on $\mathcal{M}_5$ reads:

$$\delta_1^2 = 0 \ , \qquad \delta_2^2 = 0 \ , \qquad \{\delta_1, \delta_2\} = -2i\mathcal{L}_K^{(A)} + 2\kappa\delta_{g(\sigma)} \ , \tag{5.75}$$

where $\mathcal{L}_K^{(A)}$ is the gauge-covariant Lie derivative along K^M, κ is defined in (5.53), and $\delta_{g(\sigma)}$ denotes a gauge transformation with parameter σ. The supersymmetry algebra (5.75) reproduces (5.17) upon dimensional reduction along the fiber direction.

Upon topological twisting, the various fields become differential forms on $\mathcal{M}_5$, exactly like in 4d. We decompose them into $(p,q|\ell)$-forms, following the notation introduced in the previous section. In particular, the decomposition (5.59) holds for the 5d connection $A \equiv A_M dx^M$, while the field-strength 2-form (5.74) decomposes as in (5.61). We then write the supersymmetry variations in terms of the twisted Dolbeault operator (5.62), which need to be further twisted by the gauge fields:

$$\partial_A \equiv \partial - iA^{1,0} \ , \qquad \bar\partial_A \equiv \bar\partial - iA^{0,1} \ , \qquad \partial_{5,A} \equiv \partial_5 - iA_5 \ , \tag{5.76}$$

to preserve gauge covariance. Note that they satisfy:

$$\begin{aligned} \partial_A^2 &= -\, iF^{2,0} \wedge -\beta \hat{\mathtt{F}}^{2,0} \wedge \partial_5 \ , \\ \bar\partial_A^2 &= -\, iF^{0,2} \wedge -\beta \hat{\mathtt{F}}^{0,2} \wedge \partial_5 \ , \\ \{\partial_A, \bar\partial_A\} &= -\, iF^{1,1} \wedge -\beta \hat{\mathtt{F}}^{1,1} \wedge \partial_5 \ . \end{aligned} \tag{5.77}$$

We then have the vector-multiplet supersymmetry variations:

$$\begin{aligned} &\delta_1 \sigma = \widetilde\Lambda^{0,0}_{(1)} \ , && \delta_2 \sigma = \widetilde\Lambda^{0,0}_{(2)} \ , \\ &\delta_1 A = -i\Lambda^{1,0} + i\widetilde\Lambda^{0,0}_{(1)}\eta \ , && \delta_2 A = -i\Lambda^{0,1} + i\widetilde\Lambda^{0,0}_{(2)}\eta \ , \\ &\delta_1 \Lambda^{1,0} = 0 \ , && \delta_2 \Lambda^{1,0} = 2i\partial_A\sigma - 2F^{1,0} \ , \\ &\delta_1 \Lambda^{0,1} = 2i\bar\partial_A\sigma - 2F^{0,1} \ , && \delta_2 \Lambda^{0,1} = 0 \ , \\ &\delta_1 \widetilde\Lambda^{0,0}_{(1)} = 0 \ , && \delta_2 \widetilde\Lambda^{0,0}_{(1)} = i\widehat{\mathcal{D}}^{0,0} - i\partial_{5,A}\sigma \ , \\ &\delta_1 \widetilde\Lambda^{0,0}_{(2)} = -i\widehat{\mathcal{D}}^{0,0} - i\partial_{5,A}\sigma \ , && \delta_2 \widetilde\Lambda^{0,0}_{(2)} = 0 \ , \\ &\delta_1 \widetilde\Lambda^{2,0} = \mathcal{D}^{2,0} \ , && \delta_2 \widetilde\Lambda^{2,0} = 4(F^{2,0} - i\beta\sigma\hat{\mathtt{F}}^{2,0}) \ , \\ &\delta_1 \widetilde\Lambda^{0,2} = 4(F^{0,2} - i\beta\sigma\hat{\mathtt{F}}^{0,2}) \ , && \delta_2 \widetilde\Lambda^{0,2} = \mathcal{D}^{0,2} \ , \\ &\delta_1 \widehat{\mathcal{D}}^{0,0} = [\sigma, \widetilde\Lambda^{0,0}_{(1)}] - \partial_{5,A}\widetilde\Lambda^{0,0}_{(1)} \ , && \delta_2 \widehat{\mathcal{D}}^{0,0} = -[\sigma, \widetilde\Lambda^{0,0}_{(2)}] + \partial_{5,A}\widetilde\Lambda^{0,0}_{(2)} \ , \end{aligned} \tag{5.78}$$

and:

$$\begin{aligned} &\delta_1 \mathcal{D}^{2,0} = 0 \ , && \delta_2 \mathcal{D}^{2,0} = 4i\partial_A\Lambda^{1,0} + 2i[\sigma, \widetilde\Lambda^{2,0}] - 2i\partial_{5,A}\widetilde\Lambda^{2,0} \ , \\ &\delta_1 \mathcal{D}^{0,2} = 4i\bar\partial_A\Lambda^{0,1} + 2i[\sigma, \widetilde\Lambda^{0,2}] - 2i\partial_{5,A}\widetilde\Lambda^{0,2} \ , && \delta_2 \mathcal{D}^{0,2} = 0 \ . \end{aligned} \tag{5.79}$$

The scalar appearing in $\widehat{\mathcal{D}}^{0,0}$ in (5.78) is related to the 4d scalar $\mathcal{D}^{0,0}$ defined in (5.16) by:

$$\widehat{\mathcal{D}}^{0,0} \equiv \mathcal{D}^{0,0} + 2\beta\sigma\hat{\mathtt{F}}^{0,0} \ , \tag{5.80}$$

where $\hat{\mathtt{F}}^{0,0}$ is defined as $\hat{\mathtt{F}}^{0,0} \equiv \frac{1}{4}\Phi^{MN}\hat{\mathtt{F}}_{MN}$. One can then check that the supersymmetry algebra closes, such that $\delta_1^2 = 0 = \delta_2^2$, while $\{\delta_1, \delta_2\}f = 2i[\sigma, f] - 2i\partial_{5,A}f$ for any of the fields f in the vector multiplet.[48] The twisted vector multiplet therefore realises the supersymmetry algebra (5.75), namely:

$$\delta_1^2 = 0 \ , \qquad \delta_2^2 = 0 \ , \qquad \{\delta_1, \delta_2\} = -2i\mathcal{L}_K + 2\delta_{g(\sigma + i\,\iota_K A)} \ , \tag{5.81}$$

where $\mathcal{L}_K$ is the usual Lie derivative along K, $\iota_K A = K^M A_M = A_5$ is the contraction with the vector field K, while $\delta_{g(\epsilon)}$ is the gauge transformation with parameter ϵ introduced in (5.17). Finally, one can check that the 5d $\mathcal{N} = 1$ SYM Lagrangian on $\mathcal{M}_5$ is 'almost' $\mathcal{Q}$-exact, similarly to the 4d Lagrangian (5.18).

5.2.3 The 5d $\mathcal{N} = 1$ hypermultiplet on $\mathcal{M}_5$

The 5d $\mathcal{N} = 1$ hypermultiplet consists of an $SU(2)_R$ doublet of complex scalar fields q^I and of a Dirac spinor $\Psi, \widetilde{\Psi}$. The fields (q, Ψ) transform in some representation $\mathfrak{R}$ of the gauge group, while the fields $(\widetilde{q}, \widetilde{\Psi})$ in the complex conjugate representation $\bar{\mathfrak{R}}$. The reduction to the 4d $\mathcal{N} = 2$ hypermultiplet is done following the identification:

$$\Psi = \begin{pmatrix} -\eta \\ \widetilde{\chi} \end{pmatrix} \ , \qquad \widetilde{\Psi} = \begin{pmatrix} \chi \ , \widetilde{\eta} \end{pmatrix} \ . \tag{5.82}$$

As in the case of the 4d $\mathcal{N} = 2$ hypermultiplet, we can realise the two supercharges of the DW twist off-shell by introducing some appropriate auxiliary fields. In flat space, these are five-dimensional commuting spinors $h_{\mathbf{a}}, \widetilde{h}^{\mathbf{a}}$, with only two non-vanishing components $(h_1, h_2, 0, 0)$. For the curved space background, one then simply replaces the derivatives D_M with the torsionfull adapted connection $\hat{D}_M$. We will not give the details here, and instead work directly with twisted variables, and refer the reader to [3].

[48]This also holds for the field-strength F upon using the Bianchi identity.

The hypermultiplet can be recast in twisted variables, exactly as in 4d. The field content is formally the same as in (5.27), with the (p,q)-forms being now interpreted as forms on $\mathcal{M}_5$, following the discussion of the previous section. The supersymmetry variations read:

$$\begin{aligned}
&\delta_1 Q^{0,0} = 0 \ , && \delta_2 Q^{0,0} = \sqrt{2}\,\widetilde{\chi}^{0,0} \ , \\
&\delta_1 Q^{0,2} = \sqrt{2}\,\widetilde{\chi}^{0,2} \ , && \delta_2 Q^{0,2} = 0 \ , \\
&\delta_1 \eta^{0,1} = 2i\sqrt{2}\,\bar{\partial}_A Q^{0,0} + h^{0,1} \ , && \delta_2 \eta^{0,1} = i\sqrt{2}\star\left(\partial_A Q^{0,2}\right) \ , \\
&\delta_1 \widetilde{\chi}^{0,0} = i\sqrt{2}(\sigma - \partial_{5,A}) Q^{0,0} \ , && \delta_2 \widetilde{\chi}^{0,0} = 0 \ , \\
&\delta_1 \widetilde{\chi}^{0,2} = 0 \ , && \delta_2 \widetilde{\chi}^{0,2} = i\sqrt{2}(\sigma - \partial_{5,A}) Q^{0,2} \ , \\
&\delta_1 h^{0,1} = 0 \ , && \delta_2 h^{0,1} = X^{0,1} \ ,
\end{aligned} \tag{5.83}$$

and:

$$\begin{aligned}
&\delta_1 \widetilde{Q}^{0,0} = -\sqrt{2}\,\widetilde{\eta}^{0,0} \ , && \delta_2 \widetilde{Q}^{0,0} = 0 \ , \\
&\delta_1 \widetilde{Q}^{2,0} = 0 \ , && \delta_2 \widetilde{Q}^{2,0} = \sqrt{2}\,\widetilde{\eta}^{2,0} \ , \\
&\delta_1 \chi^{1,0} = i\sqrt{2}\star\left(\bar{\partial}_A \widetilde{Q}^{2,0}\right) \ , && \delta_2 \chi^{1,0} = 2i\sqrt{2}\,\partial_A \widetilde{Q}^{0,0} + \widetilde{h}^{1,0} \ , \\
&\delta_1 \widetilde{\eta}^{0,0} = 0 \ , && \delta_2 \widetilde{\eta}^{0,0} = i\sqrt{2}\left(\widetilde{Q}^{0,0}\sigma + \partial_{5,A}\widetilde{Q}^{0,0}\right) \ , \\
&\delta_1 \widetilde{\eta}^{2,0} = -i\sqrt{2}\left(\widetilde{Q}^{2,0}\sigma + \partial_{5,A}\widetilde{Q}^{2,0}\right) \ , && \delta_2 \widetilde{\eta}^{2,0} = 0 \ , \\
&\delta_1 \widetilde{h}^{1,0} = \widetilde{X}^{1,0} \ , && \delta_2 \widetilde{h}^{1,0} = 0 \ ,
\end{aligned} \tag{5.84}$$

where we defined:

$$\begin{aligned}
X^{0,1} \ &\equiv \ -4i\,\bar{\partial}_A \widetilde{\chi}^{0,0} - 2i\star(\partial_A \widetilde{\chi}^{0,2}) + 2i\sqrt{2}\,\Lambda^{0,1} Q^{0,0} + i\sqrt{2}\star(\Lambda^{1,0}\wedge Q^{0,2}) \\
&\quad + 2i(\sigma - \partial_{5,A})\eta^{0,1} \ , \\
\widetilde{X}^{1,0} \ &\equiv \ 4i\,\partial_A \widetilde{\eta}^{0,0} - 2i\star(\bar{\partial}_A \widetilde{\eta}^{2,0}) - 2i\sqrt{2}\,\widetilde{Q}^{0,0}\Lambda^{1,0} - i\sqrt{2}\star(\widetilde{Q}^{2,0}\wedge\Lambda^{0,1}) \\
&\quad - 2i\left(\chi^{1,0}\sigma + \partial_{5,A}\chi^{1,0}\right) \ .
\end{aligned} \tag{5.85}$$

Let us also point out that the Hodge star operator used above is in fact the Hodge dual on $\mathcal{M}_4$, obtained from the 5d Hodge dual by the contraction $\star \equiv \iota_K \star_5$. The four-dimensional scalar field ϕ is replaced in the supersymmetry variations by a differential operator:

$$\phi \to \frac{1}{\sqrt{2}}(\sigma \mp \partial_{5,A}) \ , \tag{5.86}$$

when acting on a field in the representation $\mathfrak{R}$ or $\bar{\mathfrak{R}}$, respectively. One can easily check that the supersymmetry algebra (5.81) is satisfied. The kinetic Lagrangian is again $\mathcal{Q}$-exact. The five-dimensional uplift of (5.29) reads:

$$\begin{aligned}\mathscr{L}_H = \frac{1}{4}\star(\delta_1+\delta_2)\Big(&\widetilde{h}^{1,0}\wedge\star\eta^{0,1} - 2i\sqrt{2}\,\chi^{1,0}\wedge\star\bar{\partial}_A Q^{0,0} + i\sqrt{2}\,\chi^{1,0}\wedge\partial_A Q^{0,2} \\ &- \frac{i\sqrt{2}}{8}\,\widetilde{\eta}^{0,0}(\sigma+\partial_{5,A})\,Q^{0,0}\,d\mathrm{vol} + \frac{i\sqrt{2}}{2}\,\widetilde{\eta}^{2,0}\wedge(\sigma+\partial_{5,A})\,Q^{0,2} - i\,\widetilde{Q}^{2,0}\wedge\widetilde{\Lambda}^{0,2}\,Q^{0,0} \\ &+ \frac{i}{4}\,\widetilde{Q}^{0,0}\,\widetilde{\Lambda}^{0,0}_{(1)}\,Q^{0,0}\,d\mathrm{vol} + i\,\widetilde{Q}^{2,0}\wedge\widetilde{\Lambda}^{0,0}_{(2)}\,Q^{0,2} + i\,\widetilde{Q}^{0,0}\,\widetilde{\Lambda}^{2,0}\wedge Q^{0,2}\Big) \ .\end{aligned} \tag{5.87}$$

5.3 One-loop determinants: hypermultiplet and higher-spin particles

In this section, we compute one-loop determinants on $\mathcal{M}_5$. One can first consider the contribution of a free hypermultiplet in 4d, and then obtain the 5d result by summing over the KK modes. We will not do this here, but comment on how our result precisely reflects this structure. We then generalise our result to derive the one-loop contribution of any 5d BPS particle running along the circle fiber.

Let us first sketch the standard supersymmetric localization argument [20]. First, it is not difficult to see that the expectation value of Q-exact operators vanishes in a supersymmetric QFT. Consequently, we can deform the action by a Q-exact term with an arbitrary coefficient, without changing the result of the path integral. In the limit where this coefficient is very large, the integrand is dominated by the saddle points of the *localizing action*. These are determined by the configurations for which the fermions and their supersymmetry variations vanish. Then, the one-loop determinant is nothing but the first-order fluctuation around this localizing configuration. In particular, this quantity is given by the ratio of the quadratic operators for the fermionic and bosonic fluctuations - see *e.g.* [243] for a review.

5.3.1 Free hypermultiplet on $\mathcal{M}_5$

Consider a charged 5d $\mathcal{N}=1$ hypermultiplet on $\mathcal{M}_5$. The simplest way to compute the partition function is to expand the 5d fields into 4d modes, through KK reduction along the S^1 fiber, which effectively reduces to evaluating the one-loop determinant of a 4d $\mathcal{N}=2$ hypermultiplet on $\mathcal{M}_4$, for the 4d modes of fixed KK charge. Due to the non-trivial fibration structure on $\mathcal{M}_5$, the supersymmetric background for the 5d vector multiplet is slightly more complicated than the equivalent 4d background, being determined by

$$\mathcal{D}^{2,0}=0\ , \qquad \mathcal{D}^{0,2}=0\ , \qquad F^{2,0}=i\beta\sigma\hat{\mathrm{F}}^{2,0}\ , \qquad F^{0,2}=i\beta\sigma\hat{\mathrm{F}}^{0,2}\ , \tag{5.88}$$

together with:

$$i\partial_A\sigma=F^{1,0}\ , \qquad i\bar{\partial}_A\sigma=F^{0,1}\ , \qquad \widehat{\mathcal{D}}^{0,0}=\partial_{5,A}\sigma=0\ , \tag{5.89}$$

in terms of the 5d twisted Dolbeault operators. These conditions are obtained by imposing that the gaugino variations vanish. The kinetic operators for the bosonic and fermionic fluctuations around the 5d background read:

$$\begin{aligned}
\Delta_{\text{bos}} &= \begin{pmatrix} -2\star\partial_A\star\bar{\partial}_A+(\sigma\sigma-\partial_{5,A}\partial_{5,A}) & *\partial_A\partial_A \\ -\star\bar{\partial}_A\bar{\partial}_A & -\frac{1}{2}\star\bar{\partial}_A\star\partial_A+\frac{1}{4}(\sigma\sigma-\partial_{5,A}\partial_{5,A}) \end{pmatrix}\ ,\\
\Delta_{\text{fer}} &= \begin{pmatrix} -\frac{i}{2}(\sigma-\partial_{5,A}) & i\,\bar{\partial}_A & \frac{i}{2}\star\partial_A \\ -\,i\star\partial_A\star & i\,(\sigma+\partial_{5,A}) & 0 \\ -\frac{i}{2}\star\bar{\partial}_A & 0 & -\,\frac{i}{4}(\sigma+\partial_{5,A}) \end{pmatrix}\ .
\end{aligned} \tag{5.90}$$

They are related as follows:

$$\Delta_{\text{fer}}\begin{pmatrix} 1 & -2i\bar{\partial}_A & i\star\partial_A \\ 0 & -i\,(\sigma-\partial_{5,A}) & 0 \\ 0 & 0 & i\,(\sigma-\partial_{5,A}) \end{pmatrix}=\begin{pmatrix} -\frac{i}{2}(\sigma-\partial_{5,A}) & 0 & 0 \\ -i\star\partial_A\star & & \\ \frac{i}{2}\star\bar{\partial}_A & & \Delta_{\text{bos}} \end{pmatrix}\ . \tag{5.91}$$

As a result, the one-loop determinant reduces to [20]:

$$Z^{\mathcal{H}}_{\mathcal{M}_5}=\frac{\det(\Delta_{\text{fer}})}{\det(\Delta_{\text{bos}})}=\frac{\det(\mathbb{L}^{(0,1)})}{\det(\mathbb{L}^{(0,0)})\det(\mathbb{L}^{(0,2)})}\ , \qquad \mathbb{L}=i\,(\sigma-\partial_{5,A})\ , \tag{5.92}$$

with the superscript indicating the type of (p,q)-forms that $\mathbb{L} : \Omega^{p,q} \to \Omega^{p,q}$ acts upon. Roughly speaking, this determinant counts the unpaired bosonic and fermionic modes. To evaluate this explicitly, we expand the 5d fields φ into 4d KK modes, as a Fourier decomposition along the S^1 fiber:

$$\varphi(z,\bar{z},\psi) = \sum_{n\in\mathbb{Z}} \varphi_{(n)}(z,\bar{z},\psi) \ , \qquad \varphi_{(n)}(z,\bar{z},\psi) \equiv e^{-in\psi}\varphi_n(z,\bar{z}) \ . \tag{5.93}$$

We then have:

$$\mathbb{L}\varphi_n = \lambda_n \varphi_n \ , \qquad \lambda_n = i\beta(\sigma + iA_5) + n \ . \tag{5.94}$$

In the following, let us fix a 4d KK mode, *i.e.* fix the value of n. Then, recall that the Dolbeault operators $\bar{\partial} : \Omega^{p,q-1} \to \Omega^{p,q}$ and $\partial : \Omega^{p-1,q} \to \Omega^{p,q}$ have adjoints $\bar{\partial}^* : \Omega^{p,q} \to \Omega^{p,q-1}$ and $\partial^* : \Omega^{p,q} \to \Omega^{p-1,q}$, respectively, defined as:

$$\bar{\partial}^* = - \star \partial \star \ , \qquad \partial^* = - \star \bar{\partial} \star \ , \tag{5.95}$$

and similarly for the gauge-covariant generalisation. Then, one can check that:

$$\begin{aligned} \ker(\bar{\partial}_A) &= \ker(\star\partial_A \star \bar{\partial}_A) \ , \qquad & \ker(\bar{\partial}_A^*) &= \ker(\bar{\partial}_A \star \partial_A \star) \ , \\ \ker(\partial_A) &= \ker(\star\bar{\partial}_A \star \partial_A) \ , \qquad & \ker(\partial_A^*) &= \ker(\partial_A \star \bar{\partial}_A \star) \ . \end{aligned} \tag{5.96}$$

As a result, the non-zero eigenvalues of $\bar{\partial}_A^* \bar{\partial}_A$ and those of $\bar{\partial}_A \bar{\partial}_A^*$ are in one-to-one correspondence, and similarly for the operators in the second line. Note that these operators do not change the degree of the differential form they act upon. Moreover, they clearly commute with the operators $\mathbb{L}$ introduced above and, as a result, the eigenvalues of $\mathbb{L}$ that lie outside these kernels will cancel in the one-loop determinant. Thus, we have:

$$Z^{\mathcal{H}}_{\mathcal{M}_4} = \frac{\det_{\ker(\bar{\partial}_A^*)\oplus\ker(\bar{\partial}_A)}(\mathbb{L}^{0,1})}{\det_{\ker(\bar{\partial}_A)}(\mathbb{L}^{0,0}) \det_{\ker(\bar{\partial}_A^*)}(\mathbb{L}^{0,2})} \ . \tag{5.97}$$

where the subscript $\mathcal{M}_4$ on the LHS indicates that this expression is evaluated for a fixed KK mode (*i.e.* fixed n). We are thus restricting attention to the zero modes:

$$\bar{\partial}_A Q^{0,0} = 0 \ , \qquad \bar{\partial}_A^*(\star Q^{0,2}) = 0 \ , \qquad \bar{\partial}_A^* \eta^{0,1} = 0 \ , \qquad \bar{\partial}_A \eta^{0,1} = 0 \ , \tag{5.98}$$

using the fact that $\star : \Omega^{0,2} \to \Omega^{0,2}$. Up to an irrelevant numerical factor, we find that:

$$Z^{\mathcal{H}}_{\mathcal{M}_4} = \lambda_n^{-\mathcal{I}} \ , \tag{5.99}$$

with $\mathcal{I}$ the net number of zero-modes of the twisted Dolbeault operator (5.98) contributing. It is given by:

$$\begin{aligned} \mathcal{I} = \quad & \dim\ker(\bar{\partial}_A : E^{0,0} \to E^{0,1}) + \dim\ker(\bar{\partial}_A^* : E^{0,2} \to E^{0,1}) \\ & - \dim\ker(\bar{\partial}_A^* : E^{0,1} \to E^{0,0}) - \dim\ker(\bar{\partial}_A : E^{0,1} \to E^{0,2}) \ , \end{aligned} \tag{5.100}$$

where we denoted by $E^{0,q} \equiv \Omega^{0,q} \otimes E$ the space of $(0,q)$-forms valued in the gauge bundle E with connection A. Let $\mathrm{ind}(\bar{\partial}_A)$ denote the index of the Dolbeault complex twisted by E:

$$0 \longrightarrow \Omega^{0,0} \otimes E \xrightarrow{\bar{\partial}_A} \Omega^{0,1} \otimes E \xrightarrow{\bar{\partial}_A} \Omega^{0,2} \otimes VE \longrightarrow 0 \ . \tag{5.101}$$

Formally, one finds:

$$\mathcal{I} = \mathrm{ind}(\bar{\partial}_A) - \dim(\Omega^{0,1} \otimes E) \ . \tag{5.102}$$

By the assumption that $\mathcal{M}_4$ is simply connected, however, we have $\dim(\Omega^{0,1} \otimes E) = 0$ and thus $\mathcal{I} = \mathrm{ind}(\bar{\partial}_A)$. Now, each KK mode $\varphi^{p,q}$ of a given (p,q)-degree can be thought of as a section of a bundle

$$\Omega^{p,q} \otimes V_n \ , \qquad V_n = E_{\mathfrak{R}} \otimes (\mathcal{L}_{\mathrm{KK}})^n \ , \tag{5.103}$$

where $E_{\mathfrak{R}}$ is the gauge bundle and $\mathcal{L}_{\mathrm{KK}}$ is the defining line bundle introduced in section 5.2.1. The 5d hypermultiplet partition function then takes the form:

$$Z^{\mathcal{H}}_{\mathcal{M}_5} = \prod_{n \in \mathbb{Z}} \lambda_n^{-\mathrm{ind}(\bar{\partial}_{V_n})} \ , \tag{5.104}$$

where $\text{ind}(\bar{\partial}_{V_n})$ is the index of the 4d Dolbeault complex twisted by the vector bundle V_n. This infinite product needs to be properly regularised, as we discuss next.

5.3.2 Regularisation: summing up the KK tower

For our purposes, we will only consider abelian gauge bundles, by choosing a maximal torus of the (background or dynamical) gauge group. Then, without loss of generality, we can consider the hypermultiplet coupled to a single $U(1)$ gauge field with background flux $\mathfrak{m}$, as discussed in section 5.2.1. The complex scalar in the effective 4d $\mathcal{N}=2$ vector multiplet is denoted by

$$a \equiv i\beta(\sigma + iA_5) \ , \tag{5.105}$$

with the identification $a \sim a+1$ under a $U(1)$ large gauge transformation. The 5d hypermultiplet partition function is given formally by the infinite product:

$$Z^{\mathcal{H}}_{\mathcal{M}_5} = \prod_{n\in\mathbb{Z}} \left(\frac{1}{a+n}\right)^{\text{ind}(\bar{\partial}_{V_{n,\varepsilon}})} \ , \tag{5.106}$$

in terms of the index of the $V_{n,\varepsilon}$-twisted Dolbeault complex. Here, we take $V_{n,\varepsilon}$ to be the line bundle:

$$V_{n,\varepsilon} \cong \mathcal{K}^{\varepsilon+\frac{1}{2}} \otimes L \otimes (\mathcal{L}_{\text{KK}})^n \ , \tag{5.107}$$

where the L connection is the background $U(1)$ gauge field with flux $\mathfrak{m}$, and ε indexes our choice of extended twist for the hypermultiplet, as discussed around (5.23). The canonical choice on a generic Kähler base $\mathcal{M}_4$ is $\varepsilon = -\frac{1}{2}$, while if $\mathcal{M}_4$ is spin it is also natural to choose $\varepsilon = 0$. Note that:

$$c_1(V_{n,\varepsilon}) = \sum_l \left(\left(\varepsilon + \frac{1}{2}\right) \mathbf{k}_l + \mathfrak{m}_l + n\,\mathfrak{p}_l \right) [\mathrm{S}_l] \ , \tag{5.108}$$

where $\mathbf{k}$ denotes the first Chern class of the canonical line bundle on $\mathcal{M}_4$,

$$c_1(\mathcal{K}) = \sum_l \mathbf{k}_l [S_l] \ , \tag{5.109}$$

and $\mathfrak{p}$ was defined in (5.32). For simplicity of notation, we can absorb the $(\varepsilon + \frac{1}{2})\mathbf{k}$ term into $\mathfrak{m}$, effectively setting $\varepsilon = -\frac{1}{2}$ in what follows. From the index theorem, we find:

$$\text{ind}(\bar{\partial}_{V_n}) = \int_{\mathcal{M}_4} \text{Td}(T\mathcal{M}_4) \wedge \text{ch}(V_n) = \chi_h + \frac{1}{2}(\mathfrak{m} + n\,\mathfrak{p} - \mathbf{k}, \mathfrak{m} + n\,\mathfrak{p}) \;, \tag{5.110}$$

with $\chi_h = \frac{\chi+\sigma}{4}$ the holomorphic Euler characteristic, and with the intersection pairing $(-,-)$ on $\mathcal{M}_4$ as defined in (5.72).

Regularisation of the result. Given (5.110), the infinite product to be regularised takes the explicit form:

$$Z^{\mathcal{H}}_{\mathcal{M}_5}(a)_{\mathfrak{m}} = \prod_{n\in\mathbb{Z}} \left(\frac{1}{a+n}\right)^{\chi_h + \frac{1}{2}(\mathfrak{m}+n\,\mathfrak{p}-\mathbf{k},\mathfrak{m}+n\,\mathfrak{p})} \;. \tag{5.111}$$

The notation $Z^{\mathcal{H}}_{\mathcal{M}_5}(a)_{\mathfrak{m}}$ makes the dependence on a and $\mathfrak{m}$ manifest. It is convenient to factor (5.111) as follows:

$$Z^{\mathcal{H}}_{\mathcal{M}_5}(a)_{\mathfrak{m}} = \mathbf{\Pi}^{\mathcal{H}}(a)^{\chi_h + \frac{1}{2}(\mathfrak{m}-\mathbf{k},\mathfrak{m})}\, \mathscr{K}^{\mathcal{H}}(a)^{(\mathfrak{m}-\frac{1}{2}\mathbf{k},\mathfrak{p})}\, \mathscr{F}^{\mathcal{H}}(a)^{\frac{1}{2}(\mathfrak{p},\mathfrak{p})} \;. \tag{5.112}$$

Here, we formally defined the following functions in terms of divergent products:

$$\mathbf{\Pi}^{\mathcal{H}}(a) \equiv \prod_{n\in\mathbb{Z}} \frac{1}{a+n} \;, \qquad \mathscr{K}^{\mathcal{H}}(a) \equiv \prod_{n\in\mathbb{Z}} \left(\frac{1}{a+n}\right)^{n} \;, \qquad \mathscr{F}^{\mathcal{H}}(a) \equiv \prod_{n\in\mathbb{Z}} \left(\frac{1}{a+n}\right)^{n^2} \;. \tag{5.113}$$

These formal products give us information on the analytic structure of the corresponding meromorphic functions, with poles or zeros at $a \in \mathbb{Z}$. Namely, $\mathbf{\Pi}^{\mathcal{H}}$ has poles of order 1 at any integer $a \in \mathbb{Z}$, $\mathscr{K}^{\mathcal{H}}$ has poles of order n at $a = n$ for every negative integer n (and zeros at the positive integers), and $\mathscr{F}^{\mathcal{H}}$ has poles of order n^2 at $a = n$ for any integer n. Following the discussion in [70, 89, 93], we choose the gauge-invariant regularisation, also known as the '$U(1)_{-\frac{1}{2}}$ quantisation'. We then find:

$$\begin{aligned} \mathbf{\Pi}^{\mathcal{H}}(a) &= \frac{1}{1-e^{2\pi i a}} \;, \\ \mathscr{K}^{\mathcal{H}}(a) &= \exp\left(\frac{1}{2\pi i}\text{Li}_2(e^{2\pi i a}) + a\log(1-e^{2\pi i a})\right) \;, \\ \mathscr{F}^{\mathcal{H}}(a) &= \exp\left(-\frac{1}{2\pi^2}\text{Li}_3(e^{2\pi i a}) - \frac{a}{\pi i}\text{Li}_2(e^{2\pi i a}) - a^2\log(1-e^{2\pi i a})\right) \;. \end{aligned} \tag{5.114}$$

Despite the appearance of polylogarithms, these functions are meromorphic in a, with the poles mentioned above. They also have simple transformation properties under large gauge transformations, $a \sim a+1$, with $\mathbf{\Pi}^{\mathcal{H}}(a+1) = \mathbf{\Pi}^{\mathcal{H}}(a)$ and

$$\mathscr{K}^{\mathcal{H}}(a+1) = \mathbf{\Pi}^{\mathcal{H}}(a)^{-1}\mathscr{K}^{\mathcal{H}}(a) \;, \qquad \mathscr{F}^{\mathcal{H}}(a+1) = \mathbf{\Pi}^{\mathcal{H}}(a)\mathscr{K}^{\mathcal{H}}(a)^{-2}\mathscr{F}^{\mathcal{H}}(a) \;. \tag{5.115}$$

Using these relations, we can check that the partition function is gauge invariant. Whenever the circle is non-trivially fibered over $\mathcal{M}_4$, a large gauge transformation amounts to the simultaneous shift $(a, \mathfrak{m}) \to (a+1, \mathfrak{m}+\mathfrak{p})$. More invariantly, this corresponds to tensoring the $U(1)$ line bundle with the defining line bundle, $L \to L \otimes \mathcal{L}_{\text{KK}}$. We indeed find that:

$$Z^{\mathcal{H}}_{\mathcal{M}_5}(a+1)_{\mathfrak{m}+\mathfrak{p}} = Z^{\mathcal{H}}_{\mathcal{M}_5}(a)_{\mathfrak{m}} \;, \tag{5.116}$$

as expected.

Example: Trivial fibrations. Let us first consider the case $\mathcal{M}_5 = \mathcal{M}_4 \times S^1$. Since $\mathfrak{p} = 0$, the partition function takes the simple form:

$$Z^{\mathcal{H}}_{\mathcal{M}_4\times S^1}(a)_{\mathfrak{m}} = \mathbf{\Pi}^{\mathcal{H}}(a)^{\chi_h + \frac{1}{2}(\mathfrak{m}-\mathbf{k},\mathfrak{m})} = \left(\frac{1}{1-e^{2\pi i a}}\right)^{\chi_h + \frac{1}{2}(\mathfrak{m}-\mathbf{k},\mathfrak{m})} \;, \tag{5.117}$$

for $\varepsilon = -\frac{1}{2}$. This agrees with previous results [114], up to some differences in conventions.[49] For a more general choice of extended DW twist, we find:

$$Z^{\mathcal{H}}_{\mathcal{M}_4\times S^1}(a;\varepsilon)_{\mathfrak{m}} = \left(\frac{1}{1-e^{2\pi i a}}\right)^{-\frac{\sigma}{8}+\frac{\varepsilon^2}{2}(2\chi+3\sigma)+\frac{1}{2}(\mathfrak{m}+2\varepsilon\mathbf{k},\mathfrak{m})} \;, \tag{5.118}$$

where we used the relation $(\mathbf{k}, \mathbf{k}) = 2\chi + 3\sigma$.

[49] The most important difference is in the choice of regularisation. As emphasised in [70], our choice is singled out by requiring gauge invariance under large gauge transformations.

Example: The five-sphere S^5. Let us now consider the example of S^5, fibered over $\mathbb{P}^2$, setting $\varepsilon = -\frac{1}{2}$ for simplicity. We have $\chi = 3$, $\sigma = 1$ and $\mathbf{k} = -3$, and thus (5.117) gives us:

$$Z^{\mathcal{H}}_{\mathbb{P}^2\times S^1}(a)_{\mathfrak{m}} = \left(\frac{1}{1-e^{2\pi i a}}\right)^{1+\frac{1}{2}\mathfrak{m}(\mathfrak{m}+3)} , \tag{5.119}$$

for any flux $\mathfrak{m} \in \mathbb{Z}$. The S^5 is obtained by a fibration with $\mathfrak{p} = 1$, so that:

$$Z^{\mathcal{H}}_{S^5}(a)_{\mathfrak{m}} = Z^{\mathcal{H}}_{\mathbb{P}^2\times S^1}(a)_{\mathfrak{m}}\, \mathscr{K}^{\mathcal{H}}(a)^{\mathfrak{m}-\frac{3}{2}}\, \mathscr{F}^{\mathcal{H}}(a)^{\frac{1}{2}} , \tag{5.120}$$

and we can set $\mathfrak{m} = 0$ by a large gauge transformation (5.116), reflecting the fact that $H^2(S^5)$=0. Hence we find:

$$Z^{\mathcal{H}}_{S^5}(a) = \exp\left(-\frac{1}{4\pi^2}\mathrm{Li}_3(e^{2\pi i a}) - \frac{2a-3}{4\pi i}\mathrm{Li}_2(e^{2\pi i a}) - \frac{a^2-3a+2}{2}\log(1-e^{2\pi i a})\right) . \tag{5.121}$$

This is in good agreement with previous results [236], with the distinction being in the choice of regularisation, as discussed above [70, 89, 93]. This different choice is due to our treatment of the 5d parity anomaly [70], and will also be reflected in the 5d prepotential which enters explicitly in the partition function through the 'fibering operator'.

Example: The five-manifold T^{p_1,p_2}. As another example, consider the fibration over $\mathcal{M}_4 = \mathbb{F}_0 = \mathbb{P}^1\times\mathbb{P}^1$, with $\mathfrak{p} = (p_1, p_2)$, which is sometimes called T^{p_1,p_2}.[50] (We take the two $\mathbb{P}^1$ factors as our basis curves, $\mathbb{F}_0 \cong \mathrm{S}_1\times\mathrm{S}_2$.) Then we have $\chi = 4$, $\sigma = 0$ and $\mathbf{k} = (-2,-2)$, hence:

$$Z^{\mathcal{H}}_{\mathbb{P}^2\times S^1}(a)_{\mathfrak{m}} = \left(\frac{1}{1-e^{2\pi i a}}\right)^{4\varepsilon^2-2\varepsilon\mathfrak{m}_1\mathfrak{m}_2+\mathfrak{m}_1\mathfrak{m}_2} , \tag{5.122}$$

for any flux $\mathfrak{m} = (\mathfrak{m}_1, \mathfrak{m}_2)$, and keeping an arbitrary ε. (Since $\mathbb{F}_0$ is spin, we can choose it as we like, including choosing the DW twist value $\varepsilon = 0$.) We then have:

$$Z^{\mathcal{H}}_{T^{p_1,p_2}}(a)_{\mathfrak{m}} = Z^{\mathcal{H}}_{\mathbb{P}^2\times S^1}(a)_{\mathfrak{m}}\, \mathscr{K}^{\mathcal{H}}(a)^{(p_1\mathfrak{m}_2+p_2\mathfrak{m}_1)-2\varepsilon(\mathfrak{m}_1+\mathfrak{m}_2)}\, \mathscr{F}^{\mathcal{H}}(a)^{p_1p_2} . \tag{5.123}$$

[50] In particular, for $p_1 = p_2 = 1$, $T^{1,1}$ famously admits a Sasaki-Einstein metric [244].

For $T^{1,1}$ with $\varepsilon = 0$, for instance, this gives:

$$\begin{aligned} Z^{\mathcal{H}}_{T^{1,1}}(a)_{\mathfrak{m}} &= \exp\Bigg(-\frac{1}{2\pi^2}\text{Li}_3(e^{2\pi i a}) - \frac{2a-\mathfrak{m}_1-\mathfrak{m}_2}{4\pi i}\text{Li}_2(e^{2\pi i a}) \\ &\qquad - (a^2-(\mathfrak{m}_1+\mathfrak{m}_2)a-\mathfrak{m}_1\mathfrak{m}_2)\log(1-e^{2\pi i a})\Bigg) \ . \end{aligned} \tag{5.124}$$

Note that we have the gauge equivalence $(a, \mathfrak{m}_1, \mathfrak{m}_2) \sim (a+1, \mathfrak{m}_1+1, \mathfrak{m}_2+1)$. Using (5.69), one can check that $H^2(T^{1,1}, \mathbb{Z}) \cong \mathbb{Z}$.

5.3.3 Higher-spin particles on $\mathcal{M}_5$

By a small generalisation of the above computation, one can also capture the contribution of higher-spin states. Such electrically-charged states generally appear on the real Coulomb branch of 5d SCFTs. For instance, when we have an infrared non-abelian gauge theory phase, the W-bosons give spin-one states. More generally, 5d BPS particles of arbitrary spin can contribute. Following the approach of [23, 61, 245], we expect that, in the topologically-twisted theory, they contribute to the partition function on the Coulomb branch as KK towers of 4d off-shell hypermultiplets of $SU(2)_l \times SU(2)_r$ spin (j_l, j_r). In the 5d interpretation, (j_l, j_r) is the representation under the little group of the massive particle.

Let us first recall some elementary properties of the half-BPS massive representations of the 5d $\mathcal{N} = 1$ supersymmetry algebra. The BPS states saturate the BPS mass bound with $M = Z_{5\text{d}}$, where we take the fifth direction to be time, with $P_M = (0, 0, 0, 0, -M)$. Such states are annihilated by the 'right-chiral' supercharges (in the 4d notation), and the supersymmetry algebra:

$$\{\mathcal{Q}^I_{\mathbf{a}}, \mathcal{Q}^J_{\mathbf{b}}\} = 2\epsilon^{IJ}\left(\gamma^M_{\mathbf{ab}}P_M - i\Omega_{\mathbf{ab}}Z_{5\text{d}}\right) \ , \tag{5.125}$$

after Wick rotation, is realised as:

$$\{Q^I_\alpha, Q^J_\beta\} = -4iM\epsilon^{IJ}\epsilon_{\alpha\beta} \ , \qquad \widetilde{Q}^{\dot\alpha}_I = 0 \ , \tag{5.126}$$

where $\mathcal{Q}^I_{\mathbf{a}} = \left(Q^I_\alpha \, , \, -\epsilon^{IJ}\widetilde{Q}^{\dot\alpha}_J\right)$. Picking the supercharges $Q^{I=1}_\alpha$ and $Q^{I=2}_\alpha$ (of R-charge $R =$

± 1, under $U(1)_R \subset SU(2)_R$, respectively) as the creation and annihilation operators, we obtain the supermultiplet:

$$\left(j_l, j_r; \frac{1}{2}\right)^{(-1)^{2j_l+2j_r}} \oplus \left(j_l + \frac{1}{2}, j_r; 0\right)^{-(-1)^{2j_l+2j_r}} \oplus \left(j_l - \frac{1}{2}, j_r; 0\right)^{-(-1)^{2j_l+2j_r}} , \quad (5.127)$$

for any spin (j_l, j_r) for the 'ground state' – this is for $j_l > 0$, while for $j_l = 0$ there is no third summand in (5.127). Here $(j_l, j_r, s)^{\pm}$ denotes a 5d massive state of spin (j_l, j_r) and of $SU(2)_R$ 'isospin' s, with the superscript $\pm$ corresponding to bosons and fermions, respectively. The statistics is determined by the spin-statistics theorem.

Now, consider the standard DW twist of the multiplet (5.127). We obtain states of twisted $SU(2)_l \times SU(2)_D$ spins:

$$\left(j_l, j_r \pm \frac{1}{2}\right)^{(-1)^{2j_l+2j_r}} \oplus \left(j_l \pm \frac{1}{2}, j_r\right)^{-(-1)^{2j_l+2j_r}} , \quad (5.128)$$

which are most conveniently written as:

$$\left[\left(0, \frac{1}{2}\right)^{(-1)^{2j_l+2j_r}} \oplus \left(\frac{1}{2}, 0\right)^{-(-1)^{2j_l+2j_r}}\right] \otimes (j_l, j_r) \ . \quad (5.129)$$

The states in the bracket give us a standard massive hypermultiplet in the twisted theory (up to a choice of statistics), and we simply need to tensor by the general spin (j_l, j_r).

For $(j_l, j_r) = (0, 0)$, we recover the standard hypermultiplet. As a first non-trivial example, it is interesting to consider the massive vector multiplet after the DW twist. Such massive vectors appear as W-bosons through the Higgs mechanism on the CB, for instance. In this case, we can compute their contribution explicitly, as a one-loop computation, by considering the gauge-fixed SYM Lagrangian. This is discussed in some detail in [3]. After the topological twist, we have the multiplet:

$$\left[\left(0, \frac{1}{2}\right)^{-} \oplus \left(\frac{1}{2}, 0\right)^{+}\right] \otimes \left(0, \frac{1}{2}\right) = (0, 1)^{-} \oplus (0, 0)^{-} \oplus \left(\frac{1}{2}, \frac{1}{2}\right)^{+} , \quad (5.130)$$

corresponding to the on-shell gauginos and the massive vector, respectively. Thus the massive vector multiplet corresponds to $(j_l, j_r) = (0, \frac{1}{2})$.

Extended topological twist at higher spin. Consider a massive particle of spin (j_l, j_r) charged under some abelian gauge symmetry $\prod_K U(1)_K$ with charges $q_K \in \mathbb{Z}$, after the standard DW twist. The corresponding KK tower of fields on $\mathcal{M}_4$ is valued in the bundles:

$$[S_- \oplus S_+] \otimes \mathcal{K}^{\varepsilon} \otimes \bigotimes_K (L_K)^{q_K} \otimes S^{2j_l}(S_-) \otimes S^{2j_r}(S_+) \otimes (\mathcal{L}_{\rm KK})^n \ , \tag{5.131}$$

where L_K are $U(1)_K$ bundles (to be discussed further in section 6.1.1 below), and $S^k(E)$ denotes the symmetrised product of k copies of the bundle E. When the Kähler manifold $\mathcal{M}_4$ is not spin, the extended twist parameter ε cannot be zero unless $2j_l + 2j_r$ is odd. In general, we need to choose ε so that:

$$\varepsilon + j_l + j_r + \frac{1}{2} \in \mathbb{Z} \ , \tag{5.132}$$

which ensures that the bundle (5.131) is well-defined. This generalises the discussion of section 5.1.3. In a given 5d theory, there might be any number of massive particles of various spins that will contribute in this way, and the ε parameters for each cannot be chosen independently. We will come back to this important point in section 6.3.1 below.

Partition function at spin (j_l, j_r). We can now generalise the previous results for the partition function of a hypermultiplet with spin (j_l, j_r). It is determined in terms of the Dolbeault complex twisted by the KK tower of 'higher-spin' bundles:

$$V_{n,\varepsilon;(j_l,j_r)} = \mathcal{K}^{\frac{1}{2}} \otimes \bigotimes_K \left(\mathcal{K}^{\varepsilon^K} \otimes L_K\right)^{q_K} \otimes S^{2j_l}(S_-) \otimes S^{2j_r}(S_+) \otimes (\mathcal{L}_{\rm KK})^n \ . \tag{5.133}$$

For more details on the computation of that index, we refer to [3]. Using the notation $\mathfrak{m} \equiv q_K \mathfrak{m}^K$ and $\varepsilon = q_K \varepsilon^K$, one finds:

$$\begin{aligned} {\rm ind}(\bar\partial_{V_{n,\varepsilon;(j_l,j_r)}}) = (2j_l+1)(2j_r+1)\Bigg[& -\frac{\sigma}{8} + \frac{1}{2}\varepsilon^2(2\chi+3\sigma) - \frac{2}{3}j_l(j_l+1)\chi \\ & + \frac{j_l(j_l+1)+j_r(j_r+1)}{6}(2\chi+3\sigma) + \frac{1}{2}(\mathfrak{m}+n\mathfrak{p}+2\varepsilon\mathbf{k}, \mathfrak{m}+n\mathfrak{p})\Bigg] \ . \end{aligned} \tag{5.134}$$

Then, using the building blocks (5.114) and the notation $a = q_K a^K$, we obtain:

$$\begin{aligned} Z_{\mathcal{M}_5}^{(j_l,j_r)}(a)_{\mathfrak{m}} &= \mathbf{\Pi}^{\mathcal{H}}(a)^{c_{\mathcal{A}}\chi + c_{\mathcal{B}}\sigma + c_0\left[\frac{1}{2}\varepsilon^2(2\chi+3\sigma)+\frac{1}{2}(\mathfrak{m}+2\varepsilon\mathbf{k},\mathfrak{m})\right]} \\ &\quad \times \mathscr{K}^{\mathcal{H}}(a)^{c_0(\mathfrak{m}+\varepsilon\mathbf{k},\mathfrak{p})}\, \mathscr{F}^{\mathcal{H}}(a)^{\frac{1}{2}c_0(\mathfrak{p},\mathfrak{p})} , \end{aligned} \tag{5.135}$$

in terms of the following spin-dependent numbers:

$$\begin{aligned} c_{\mathcal{A}}^{(j_l,j_r)} &= (-1)^{2j_l+2j_r}(2j_l+1)(2j_r+1)\frac{j_r(j_r+1)-j_l(j_l+1)}{3} , \\ c_{\mathcal{B}}^{(j_l,j_r)} &= (-1)^{2j_l+2j_r}(2j_l+1)(2j_r+1)\left(-\frac{1}{8}+\frac{j_l(j_l+1)+j_r(j_r+1)}{2}\right) , \\ c_0^{(j_l,j_r)} &= (-1)^{2j_l+2j_r}(2j_l+1)(2j_r+1) , \end{aligned} \tag{5.136}$$

which are independent of the geometry.

Chapter 6

Fibering operators for principal circle bundles

In this chapter, we consider the low-energy effective action of a 5d $\mathcal{N} = 1$ field theory compactified on a circle. The 5d theories we have in mind are 5d SCFTs, but the following infrared approach is independent of the exact UV completion. We consider the effective 4d $\mathcal{N} = 2$ KK theory compactified on $\mathcal{M}_4$, at arbitrary fixed values of the extended Coulomb branch vector multiplets. As explained in the introduction, this is a crucial step towards a systematic computation of the U-plane integral.

6.1 KK theories on $\mathcal{M}_4 \times S^1$

Consider any 4d $\mathcal{N} = 2$ theory on $\mathcal{M}_4$. For definiteness, let us assume it is a KK theory so that we have a scale β^{-1} set by the inverse radius of the circle. We wish to study the Coulomb branch of this theory, where the low-energy degrees of freedom are r 4d $\mathcal{N} = 2$ abelian vector multiplets – r is the 'rank' of the 5d theory, by definition. We denote by a^i the scalars in the $U(1)^r$ vector multiplets, which are related to the 5d $\mathcal{N} = 1$ abelian vector multiplets as

$$a^i = i\beta \left(\sigma^i + iA_5^i\right) \ , \qquad\qquad i = 1, \cdots , r \ , \tag{6.1}$$

as already discussed in (2.1). Note that a^i is dimensionless, in our conventions. Furthermore, large-gauge transformations along the fifth direction give us the periodicity $a^i \sim a^i + 1$, $\forall i$. We also consider background vector multiplets for some maximal torus of the flavour symmetry group, $U(1)^{r_F} \subset G_F$, where r_F denotes the rank of the flavour group. The corresponding background scalars are simply complex masses, denoted by:

$$\mu^\alpha = i\beta\left(m^\alpha + iA^\alpha_{F,5}\right) ~, \qquad\qquad \alpha = 1, \cdots, r_F ~, \tag{6.2}$$

with the identification $\mu^\alpha \sim \mu^\alpha + 1$.[51] The total space of values for (a^i, μ^α) is called the extended Coulomb branch, of dimension $r + r_F$. It is convenient to introduce the notation:[52]

$$(\boldsymbol{a}^I) = (a^i, \mu^\alpha) ~, \qquad\qquad I = (i, \alpha) ~, \tag{6.3}$$

which treats dynamical and background vector multiplets democratically. We will furthermore assume that the vector multiplets are the only massless degrees of freedom at generic points on the (extended) CB.[53]

The low-energy 4d $\mathcal{N} = 2$ effective field theory in flat space is then governed by the effective prepotential, denoted by $\mathcal{F}(a, \mu)$. We define $\mathcal{F}(\boldsymbol{a})$ for the KK theory to be dimensionless (it is related to the usual 4d prepotential, $\mathcal{F}_{\rm 4d}$, by $\mathcal{F} = \beta^2 \mathcal{F}_{\rm 4d}$). The flat-space Lagrangian can be coupled to the DW-twist background on $\mathcal{M}_4$. Its key property is that it is 'almost' $\mathcal{Q}$-exact, similarly to (5.18). Discarding the $\mathcal{Q}$-exact pieces, we are left with the following topological action, which is well-defined on any $\mathcal{M}_4$ [99]:

$$\begin{aligned} S_{\rm flat} &= \frac{i}{4\pi}\int_{\mathcal{M}_4}\left(F^I \wedge F^I \frac{\partial^2 \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J} - \frac{i}{2} F^I \wedge \Lambda^J \wedge \Lambda^K \frac{\partial^3 \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J \partial \boldsymbol{a}^K}\right. \\ &\qquad\qquad \left. - \frac{1}{48}\Lambda^I \wedge \Lambda^J \wedge \Lambda^K \wedge \Lambda^L \frac{\partial^4 \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J \partial \boldsymbol{a}^K \partial \boldsymbol{a}^L}\right) ~, \end{aligned} \tag{6.4}$$

where the sum over repeated indices is understood. Here $F = dA$ for an abelian gauge field and we also introduced the one-form $\Lambda = \Lambda^{1,0} + \Lambda^{0,1}$, in the notation of (5.78). Formally,

[51] Technically, these are the ν parameters introduced in (2.4), which are related to the complex mass parameters μ appearing in the prepotential.

[52] *Beware the indices:* In this section and the next, the indices $I, J, \cdots$ run over the gauge and flavour maximal torus, while $i, j, \cdots$ are gauge indices and α, β are flavour indices. This is distinct from the conventions in other sections, for instance z^i denoted holomorphic coordinates on $\mathcal{M}_4$, and α, β are also 4d left-chiral spinor indices; no confusion is likely there. Note also that I, J were previously used as $SU(2)_R$ indices, but we are now dealing with DW-twisted fields which are $SU(2)_R$-neutral, therefore this notation switch should cause no confusion.

[53] More generally, there could be additional massless hypermultiplets, giving us a so-called enhanced CB. We will not consider this possibility in this paper.

(6.4) can be viewed as the fourth descendant, $\int_{\mathcal{M}_4} \mathcal{O}^{(4)}$, with respect to the DW supercharge $\delta = \delta_1 + \delta_2$, of the 0-form:

$$\mathcal{O}^{(0)} = -\frac{2i}{\pi}\mathcal{F}(\boldsymbol{a}) \ , \tag{6.5}$$

where we used the descent relations $\delta\mathcal{O}^{(n)} = d\mathcal{O}^{(n-1)}$ with the supersymmetry variations:

$$\delta a = 0 \ , \qquad \delta\Lambda = 2da \ , \qquad \delta F = -id\Lambda \ , \tag{6.6}$$

for an abelian vector multiplet, with $a = i\sqrt{2}\phi$. The fermionic terms in (6.4) only depend on the one-form Λ. Correspondingly, they will only affect the low-energy physics on $\mathcal{M}_4$ if the Λ fields have zero-modes, which is to say if $H^1(\mathcal{M}_4, \mathbb{R})$ is non-trivial. For simplicity, we assume that $H^1(\mathcal{M}_4, \mathbb{R}) = 0$ – *i.e.* $b_1 = 0$ – in this work. Thus, in the following, we can ignore the effect of these fermionic couplings.

6.1.1 Flux Operators

Let us now consider any background gauge field configuration for the $U(1)_I$ symmetries, assuming it preserves our two supercharges. We denote the corresponding fluxes on $\mathcal{M}_4$ by

$$c_1(F^I) = \frac{1}{2\pi}F^I = \sum_k \mathfrak{m}_k^I [\mathrm{S}_k] \ . \tag{6.7}$$

Recall that we denote the intersection pairing on $H_2(\mathcal{M}_4, \mathbb{Z})$ by $(-,-)$, so that we have:

$$(\mathfrak{m}^I, \mathfrak{m}^J) = \frac{1}{4\pi^2}\int_{\mathcal{M}_4} F^I \wedge F^J = \sum_{k,l} \mathfrak{m}_k^I \mathbf{I}_{kl} \mathfrak{m}_l^J \ , \tag{6.8}$$

with $\mathbf{I}_{kl}$ as in (5.33). At any generic point on the Coulomb branch, taking $\boldsymbol{a}^I$ to be constant, the action (6.4) evaluates to:

$$S_{\text{flux}} \equiv S_{\text{flat}}\Big|_{\text{CB}} = \pi i (\mathfrak{m}^I, \mathfrak{m}^J) \frac{\partial^2 \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J} \ . \tag{6.9}$$

In addition to (6.4), the infrared theory compactified on $\mathcal{M}_4$ is governed by well-studied gravitational couplings [96, 98, 99, 246, 247]. Up to $\mathcal{Q}$-exact terms and away from Seiberg-

Witten singularities, the topologically-twisted Coulomb branch theory takes the simple form:

$$S_{\rm TFT} = S_{\rm flat} + S_{\rm grav} \ . \tag{6.10}$$

The second term in (6.10) consists of couplings to the background metric:

$$\begin{aligned} S_{\rm grav} \ &= \frac{i}{64\pi}\int d^4x\sqrt{g}\,\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}R_{\mu\nu\alpha\beta}R_{\rho\sigma\gamma\delta}\,\mathcal{A}(\boldsymbol{a}) \\ &+\frac{i}{48\pi}\int d^4x\sqrt{g}\,\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\alpha}{}^{\beta}R_{\rho\sigma\beta}{}^{\alpha}\,\mathcal{B}(\boldsymbol{a}) \ . \end{aligned} \tag{6.11}$$

At constant values of the extended CB parameters, this evaluates to:

$$S_{\rm grav} = 2\pi i\Big(\chi\,\mathcal{A}(\boldsymbol{a}) + \sigma\,\mathcal{B}(\boldsymbol{a})\Big) \ , \tag{6.12}$$

where χ and σ are the topological Euler characteristic and the signature of $\mathcal{M}_4$, respectively. This gives the famous contribution [96, 98, 99, 246, 247]:

$$e^{-S_{\rm grav}} = \boldsymbol{A}(\boldsymbol{a})^{\chi}\boldsymbol{B}(\boldsymbol{a})^{\sigma} \ , \qquad \boldsymbol{A}(\boldsymbol{a}) \equiv e^{-2\pi i\mathcal{A}(\boldsymbol{a})} \ , \qquad \boldsymbol{B}(\boldsymbol{a}) \equiv e^{-2\pi i\mathcal{B}(\boldsymbol{a})} \ . \tag{6.13}$$

The prepotential $\mathcal{F}$ and the gravitational couplings $\mathcal{A}$ and $\mathcal{B}$ can be determined from the Seiberg-Witten geometry of the 5d theory on a circle, in principle, or else from an explicit instanton counting computation on the Ω-background.

On general grounds, the prepotential $\mathcal{F}$ suffers from branch-cut ambiguities:

$$\mathcal{F}(\boldsymbol{a}) \sim \mathcal{F}(\boldsymbol{a}) + \frac{n_2}{2}\boldsymbol{a}^2 + n_1\boldsymbol{a} + \frac{n_0}{2} \ , \qquad n_0, n_1, n_2 \in \mathbb{Z} \ . \tag{6.14}$$

Such shifts are incurred, in particular, when performing large gauge transformations along the 5d circle. It follows that the exponentiated action $\exp\left(-S_{\rm flux}\right)$ is singled-valued if and only if the intersection pairing is even, so that $(\mathfrak{m}, \mathfrak{m}) \in 2\mathbb{Z}$ for any integer-quantized flux $\mathfrak{m}$, which is true if $\mathcal{M}_4$ is spin. More generally, we need to modify the quantization condition on our fluxes, so that A^I describe spin^c connections rather than $U(1)$ gauge fields. For

$U(1)_I$ bundles, we would have $\mathfrak{m}^I \in \mathbb{Z}$, while more generally we may choose:

$$\frac{1}{2\pi} F^I = \sum_k \left(\varepsilon^I \mathbf{k}_k + \mathfrak{m}^I_k\right) [\mathrm{S}_k] \ . \tag{6.15}$$

Here, $\mathbf{k}$ was defined in (5.109), $\varepsilon^I \in \frac{1}{2}\mathbb{Z}$, and $\mathfrak{m}^I_k \in \mathbb{Z}$. The parameters ε^I must be carefully chosen depending on the theory so that it be well-defined on $\mathcal{M}_4$, as we will discuss in more detail in section 6.3.1 below. They are the infrared analogue of the extended DW-twist parameter ε introduced in section 5.1.3 for the hypermultiplet. The spin^c connections A^I can be formally viewed as connections on the ill-defined line bundles

$$\mathcal{L}_I = \mathcal{K}^{\varepsilon_I} \otimes L_I \ , \tag{6.16}$$

where L_I is a $U(1)$ line bundle with first Chern class $\mathfrak{m}^I$. The necessity of introducing spin^c connections arises from the fact that our 4d $\mathcal{N}=2$ KK theories generally contain spinors even after the standard DW twist – in the infrared description, these arise as massive BPS particles coupled to the low-energy (background and dynamical) photons, which can have arbitrary (twisted) spin. We will give the precise condition on ε^I in section 6.3 below. For now, we claim that the ε^I's can always be chosen so that the low-energy theory is well-defined; in particular, choosing these parameters correctly will render $e^{-S_{\mathrm{TFT}}(\boldsymbol{a})}$ fully gauge-invariant, single-valued and locally holomorphic in $\boldsymbol{a}$.[54]

Let us now define the 'flux operators':

$$\mathbf{\Pi}_{I,J}(\boldsymbol{a}) = \exp\left(-2\pi i \frac{\partial^2 \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}_I \partial \boldsymbol{a}_J}\right) \ , \tag{6.17}$$

which are meromorphic functions on the ECB parameters $\boldsymbol{a}_I$. Such insertions can be understood as local operators in the twisted infrared theory. Alternatively, we consider the insertion of (6.4) for specific fluxes, which can be viewed as the top-dimensional topological descendant of $\mathcal{F}(\boldsymbol{a})$, viewed itself as a local operator (at least formally). Using the topological invariance, we can localise $F^I \wedge F^J$ to have support at a point on $\mathcal{M}_4$, giving rise to

[54] Here, holomorphy is a formal consequence of supersymmetry since anti-holomorphic terms are $\mathcal{Q}$-exact.

the local insertion:[55]

$$e^{-S_{\rm flux}} = \prod_{I,J} \mathbf{\Pi}_{I,J}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{m}^I+\varepsilon^I\mathbf{k},\mathfrak{m}^J+\varepsilon^J\mathbf{k})} \ . \tag{6.18}$$

Using the fact that $(\mathbf{k},\mathbf{k}) = 2\chi+3\sigma$, it is convenient to factorise these contributions as:

$$e^{-S_{\rm flux}} = Z^{\rm flux}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}}\, \boldsymbol{G}(\boldsymbol{a};\varepsilon)^{2\chi+3\sigma} \ , \tag{6.19}$$

where we defined:

$$Z^{\rm flux}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}} \equiv \prod_{I,J} \mathbf{\Pi}_{I,J}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{m}^I+\varepsilon^I\mathbf{k},\mathfrak{m}^J)} = \exp\left(-\pi i \sum_{I,J}(\mathfrak{m}^I+2\varepsilon^I\mathbf{k},\mathfrak{m}^J)\frac{\partial^2\mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I\partial \boldsymbol{a}^J}\right) \ , \tag{6.20}$$

and:

$$\boldsymbol{G}(\boldsymbol{a};\varepsilon) \equiv e^{-2\pi i \mathcal{G}(\boldsymbol{a};\varepsilon)} \ , \qquad \mathcal{G}(\boldsymbol{a};\varepsilon) \equiv \frac{1}{2}\sum_{I,J}\varepsilon^I\varepsilon^J \frac{\partial^2\mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I\partial \boldsymbol{a}^J} \ . \tag{6.21}$$

The full exponentiated topological field theory action (6.10) evaluated on the CB then gives us the 'CB partition function' on $\mathcal{M}_4$ with gauge and flavour fluxes $\mathfrak{m}$:

$$Z_{\mathcal{M}_4\times S^1}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}} = Z^{\rm geom}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon)\, Z^{\rm flux}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}} \ . \tag{6.22}$$

This object is really the *holomorphic integrand* that will enter the U-plane integral of the 4d $\mathcal{N}=2$ KK theories, as discussed in the introduction. Here, we conjecture that the two factors in (6.22) are separately well-defined on any Kähler manifold (this is clearly true when $\mathcal{M}_4$ is spin, but not so obvious in the non-spin case). Consider first the "flux operator" contribution (6.20). The ε^I parameters should be such that (6.20) is single-valued. A sufficient set of conditions would be

$$\begin{aligned} &\frac{1}{2}(\mathfrak{m}^I,\mathfrak{m}^J)+\varepsilon^I(\mathbf{k},\mathfrak{m}^J)\in\mathbb{Z} \quad \text{if } I=J \ , \\ &\varepsilon^I(\mathbf{k},\mathfrak{m}^J)+\varepsilon^J(\mathbf{k},\mathfrak{m}^I)\in\mathbb{Z} \quad \text{if } I\neq J \ , \end{aligned} \tag{6.23}$$

for any $\mathfrak{m}^I_k\in\mathbb{Z}$, but this is much too strong in general. Instead, the correct condition on

[55] What we call the 'flux operator' has been denoted 'the C coupling' in recent works [108, 111]. The flux operator insertion can also be interpreted as a contact term localised at the intersection of the 2-cycles carrying the flux [112].

the ε^I's will depend on the 5d BPS spectrum of the field theory (see section 6.3.1).

The "geometrical" factor in (6.22) has contributions from the ordinary gravitational couplings (6.13) and from (6.21), which is dictated by our choice of (background) spin^c connections. It is given by:

$$Z^{\text{geom}}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon) = \boldsymbol{A}(\boldsymbol{a})^{\chi}\boldsymbol{B}(\boldsymbol{a})^{\sigma}\boldsymbol{G}(\boldsymbol{a};\varepsilon)^{2\chi+3\sigma} \; . \tag{6.24}$$

The $\boldsymbol{A}$ and $\boldsymbol{B}$ couplings are given in terms of the low-energy Seiberg-Witten geometry as [96, 99, 246]:

$$\boldsymbol{A} = \alpha\left(\det_{ij}\frac{dU_i}{da^j}\right)^{\frac{1}{2}} \; , \qquad \boldsymbol{B} = \beta\left(\Delta^{\text{phys}}\right)^{\frac{1}{8}} \; , \tag{6.25}$$

with α, β some numerical constants, which we determined explicitly in [1]. Here, $U_i(\boldsymbol{a})$ are the gauge-invariant U-parameters, which parametrise the Coulomb branch of the 4d $\mathcal{N} = 2$ KK theory, and Δ^{phys} is the so-called physical discriminant [247] of the Seiberg-Witten fibration. Our conjecture is then that the branch cuts ambiguities in $\boldsymbol{A}^{\chi}\boldsymbol{B}^{\sigma}$, that would generally arise from the expressions (6.25), are precisely cancelled by the third factor $\boldsymbol{G}^{2\chi+3\sigma}$ in (6.24).[56]

6.1.2 The free hypermultiplet

Let us consider the 5d hypermultiplet on $\mathcal{M}_4 \times S^1$ coupled to a single $U(1)$ vector multiplet with charge 1, whose partition function we computed in the previous section. In the present CB approach, we simply need to know the effective prepotential and gravitational couplings for the free hypermultiplet. They are given by:

$$\mathcal{F} = -\frac{1}{(2\pi i)^3}\text{Li}_3(Q) \; , \qquad \mathcal{A} = 0 \; , \qquad \mathcal{B} = -\frac{1}{16\pi i}\log(1-Q) \; , \tag{6.26}$$

[56] If the Kähler manifold $\mathcal{M}_4$ is spin, we have $\chi \in 4\mathbb{Z}$ and $\sigma \in 16\mathbb{Z}$. Then the $\boldsymbol{G}$ factor is well-defined by itself, and it can be reabsorbed into the flux operator.

with $Q \equiv e^{2\pi i a}$, as we will show in the next chapter, from the perspective of the Ω-background. We then have:

$$\boldsymbol{A} = 1 \;, \qquad\qquad \boldsymbol{B} = (1-Q)^{\frac{1}{8}} \;. \tag{6.27}$$

The non-trivial physical discriminant $\Delta^{\rm phys} = 1 - Q$ encodes the singularity on the (extended) Coulomb branch at $Q = 1$, where the hypermultiplet becomes massless. Taking the extended topological twist with $\varepsilon = -\frac{1}{2} + \delta$, and some background flux $\mathfrak{m}$, we also have

$$\boldsymbol{G} = (1-Q)^{-\frac{\varepsilon^2}{2}} = (1-Q)^{-\frac{1}{8}}(1-Q)^{-\frac{1}{2}\delta(\delta-1)} \;, \qquad Z^{\rm flux}_{\mathcal{M}_4} = (1-Q)^{-\frac{1}{2}(\mathfrak{m}+2\varepsilon\mathbf{k},\mathfrak{m})} \;. \tag{6.28}$$

Then, the formula (6.22) gives us:

$$Z^{\mathcal{H}}_{\mathcal{M}_4\times S^1}(a;\varepsilon)_{\mathfrak{m}} = \left(\frac{1}{1-Q}\right)^{\chi_h + \frac{1}{2}(\mathfrak{m}+\delta\mathbf{k}-\mathbf{k},\mathfrak{m}+\delta\mathbf{k})} \;, \tag{6.29}$$

in perfect agreement with (5.118).

6.2 KK theory on $\mathcal{M}_5$: the fibering operator

We now consider the non-trivial fibration $S^1 \to \mathcal{M}_5 \to \mathcal{M}_4$. From the 4d point of view, all 5d fields decompose in KK towers and there is always a distinguished $U(1)_{\rm KK}$ symmetry in 4d corresponding to the momentum along the fifth direction. A non-trivial fibration of the circle amounts to introducing background fluxes for the KK symmetry on $\mathcal{M}_4$:

$$\int_{\mathrm{S}_k} c_1(\mathcal{L}_{\rm KK}) = \frac{1}{2\pi}\int_{\mathrm{S}_k} \hat{\mathtt{F}} = \sum_l \mathbf{I}_{kl}\mathfrak{p}_l \;. \tag{6.30}$$

On the CB of the infrared topologically-twisted 4d $\mathcal{N} = 2$ KK theory, the non-trivial fibration of the fifth direction over $\mathcal{M}_4$ is then encoded in a 'flux operator' for $U(1)_{\rm KK}$, which we call the *fibering operator*. The expression for the latter is easily determined by dimensional analysis. Reinstating dimensions, the mass parameter for $U(1)_{\rm KK}$ is really

$\mu_{\rm KK} = 1/\beta$, so that $\mathcal{F}_{\rm 4d} = \mu_{\rm KK}^2 \mathcal{F}$ and one finds:

$$\frac{\partial^2 \mathcal{F}_{\rm 4d}}{\partial \mu_{\rm KK}^2} = 2\left(1 - \boldsymbol{a}^I \frac{\partial}{\partial \boldsymbol{a}^I} + \frac{1}{2}\boldsymbol{a}^I \boldsymbol{a}^J \frac{\partial^2}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J}\right) \mathcal{F}(\boldsymbol{a}) \;, \tag{6.31}$$

and:

$$\frac{\partial^2 \mathcal{F}_{\rm 4d}}{\partial \mu_{\rm KK} \partial(\mu_{\rm KK} \boldsymbol{a}^I)} = \left(1 - \boldsymbol{a}^J \frac{\partial}{\partial \boldsymbol{a}^J}\right) \frac{\partial \mathcal{F}}{\partial \boldsymbol{a}^I} \;. \tag{6.32}$$

For a principal circle bundle with first Chern numbers $\mathfrak{p}_k$, we then write down the fibering operator:

$$\widehat{\mathscr{F}_{\mathfrak{p}}}(\boldsymbol{a}; \varepsilon) \equiv \mathscr{F}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{p},\mathfrak{p})} \prod_I \mathscr{K}_I(\boldsymbol{a})^{(\mathfrak{p}, \mathfrak{m}^I + \varepsilon^I \mathbf{k})} \;, \tag{6.33}$$

where we defined:

$$\mathscr{F}(\boldsymbol{a}) \equiv \exp\left(-4\pi i \left(1 - \boldsymbol{a}^I \frac{\partial}{\partial \boldsymbol{a}^I} + \frac{1}{2}\boldsymbol{a}^I \boldsymbol{a}^J \frac{\partial^2}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J}\right) \mathcal{F}(\boldsymbol{a})\right) \;, \tag{6.34}$$

and:

$$\mathscr{K}_I(\boldsymbol{a}) \equiv \exp\left(-2\pi i \left(1 - \boldsymbol{a}^J \frac{\partial}{\partial \boldsymbol{a}^J}\right) \frac{\partial \mathcal{F}}{\partial \boldsymbol{a}^I}\right) \;. \tag{6.35}$$

The functions (6.34) and (6.35) are entirely determined by the exact effective prepotential of the 4d $\mathcal{N} = 2$ KK theory, and they are unaffected by the ambiguities (6.14). Moreover, while $\mathscr{F}(\boldsymbol{a})^{\frac{1}{2}}$ and $\mathscr{K}_I(\boldsymbol{a})^{\frac{1}{2}}$ suffer from branch-cut ambiguities, the product (6.33) is expected to be unambiguous. This is exactly like in the case of the flavour flux operators discussed above. For spin manifolds, the intersection form is even and the factors in (6.33) are individually well-defined, while on a non-spin $\mathcal{M}_4$ we again conjecture that the fibering operator (6.33) remains well-defined once the parameters ε^I are correctly chosen.

The $\mathcal{M}_5$ partition function and gauge invariance. Putting all the contributions together, we arrive at the full $\mathcal{M}_5$ partition function at fixed values of the (gauge and flavour) $U(1)_I$ vector multiplets. We have:

$$Z_{\mathcal{M}_5}(\boldsymbol{a}; \varepsilon)_{\mathfrak{m}} = Z_{\mathcal{M}_4}^{\rm geom}(\boldsymbol{a}; \varepsilon)\, Z_{\mathcal{M}_4}^{\rm flux}(\boldsymbol{a}; \varepsilon)_{\mathfrak{m}}\, \widehat{\mathscr{F}_{\mathfrak{p}}}(\boldsymbol{a}; \varepsilon)_{\mathfrak{m}} \;, \tag{6.36}$$

with:

$$
\begin{aligned}
&Z^{\rm geom}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon) = \boldsymbol{A}(\boldsymbol{a})^{\chi}\boldsymbol{B}(\boldsymbol{a})^{\sigma}\boldsymbol{G}(\boldsymbol{a};\varepsilon)^{2\chi+3\sigma}\;,\\
&Z^{\rm flux}_{\mathcal{M}_4}(\boldsymbol{a};\varepsilon) = \boldsymbol{\Pi}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{m}+2\varepsilon\mathbf{k},\mathfrak{m})}\;,\\
&\widehat{\mathscr{F}_{\mathfrak{p}}}(\boldsymbol{a};\varepsilon)_{\mathfrak{m}} = \mathscr{K}(\boldsymbol{a})^{(\mathfrak{p},\mathfrak{m}+\varepsilon\mathbf{k})}\,\mathscr{F}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{p},\mathfrak{p})}\;.
\end{aligned}
\tag{6.37}
$$

Here we suppressed the I, J indices.[57] Importantly, the partition function (6.36) is fully gauge invariant. Consider the large gauge transformations along $U(1)_I$:

$$
\boldsymbol{a}_J \to \boldsymbol{a}_J + \delta_{IJ}\;, \qquad \mathfrak{m}_J \to \mathfrak{m}_J + \delta_{IJ}\mathfrak{p}\;, \tag{6.38}
$$

which we denote by the shorthand $(\boldsymbol{a},\mathfrak{m}) \to (\boldsymbol{a}+\delta_I, \mathfrak{m}+\delta_I\mathfrak{p})$. Gauge invariance implies that:

$$
Z_{\mathcal{M}_5}(\boldsymbol{a}+\delta_I)_{\mathfrak{m}+\delta_I\mathfrak{p}} = Z_{\mathcal{M}_5}(\boldsymbol{a})_{\mathfrak{m}}\;. \tag{6.39}
$$

This is indeed the case. To check this, note that $Z^{\rm geom}_{\mathcal{M}_4}(\boldsymbol{a})$ is invariant by itself, and that we have the following large gauge transformations of the building blocks:

$$
\begin{aligned}
\boldsymbol{\Pi}_{J,K}(\boldsymbol{a}+\delta_I) &= \boldsymbol{\Pi}_{J,K}(\boldsymbol{a})\;,\\
\mathscr{K}_J(\boldsymbol{a}+\delta_I) &= \boldsymbol{\Pi}_{I,J}(\boldsymbol{a})^{-1}\,\mathscr{K}_J(\boldsymbol{a})\;,\\
\mathscr{F}(\boldsymbol{a}+\delta_I) &= \boldsymbol{\Pi}_{I,I}(\boldsymbol{a})\,\mathscr{K}_I(\boldsymbol{a})^{-2}\,\mathscr{F}(\boldsymbol{a})\;.
\end{aligned}
\tag{6.40}
$$

Matching the one-loop computation. Consider the free hypermultiplet coupled to a $U(1)$ vector multiplet. By an application of the general formulas (6.34)-(6.35), using the hypermultiplet prepotential (6.26), we find:

$$
\mathscr{K} = \mathscr{K}^{\mathcal{H}}(a)\;, \qquad \mathscr{F} = \mathscr{F}^{\mathcal{H}}(a)\;, \tag{6.41}
$$

[57] We will also often omit the ε from the notation, from now on, to avoid clutter.

in terms of the meromorphic functions introduced in (5.114), so that:

$$\widehat{\mathscr{F}_{\mathfrak{p}}}(a;\varepsilon)_{\mathfrak{m}} = \mathscr{F}^{\mathcal{H}}_{\mathfrak{p}}(a)_{\mathfrak{m}} \equiv \exp\Bigg(-\frac{(\mathfrak{p},\mathfrak{p})}{4\pi^2}\mathrm{Li}_3(e^{2\pi i a}) - \frac{(\mathfrak{p},\mathfrak{p})a - (\mathfrak{p},\mathfrak{m}+\varepsilon\mathbf{k})}{2\pi i}\mathrm{Li}_2(e^{2\pi i a}) \\ -\frac{a((\mathfrak{p},\mathfrak{p})a - 2(\mathfrak{p},\mathfrak{m}+\varepsilon\mathbf{k}))}{2}\log(1-e^{2\pi i a})\Bigg) . \tag{6.42}$$

By multiplying with (6.29), we obtain the full partition function of a free hypermultiplet on $\mathcal{M}_5$. This matches precisely with the direct one-loop computation of section 5.3.2.

6.3 Higher-spin state contributions

The prepotential of many five-dimensional superconformal field theories compactified on S^1 admits an expansion in terms of 5d BPS states:[58]

$$\mathcal{F} = -\frac{1}{(2\pi i)^3}\sum_{\boldsymbol{\beta}}\sum_{j_l,j_r} c_0^{(j_l,j_r)} N^{\boldsymbol{\beta}}_{j_l,j_r}\,\mathrm{Li}_3(Q^{\boldsymbol{\beta}}) . \tag{6.43}$$

Here, in keeping with common notation, we denote by $\boldsymbol{\beta}_I \equiv q_I$ the charges under the $U(1)^{r+r_F}$ symmetry on the extended Coulomb branch, with $Q^{\boldsymbol{\beta}} \equiv \prod_I Q_I^{\boldsymbol{\beta}_I}$ and $Q_I \equiv e^{2\pi i \boldsymbol{a}^I}$, and with the universal coefficients $c_0^{(j_l,j_r)}$ as in (5.136). In the context of geometrical engineering of 5d SCFTs in M-theory on a toric threefold, the theory-dependent non-negative integers $N^{\boldsymbol{\beta}}_{j_l,j_r}$ in (6.43) are the refined Gopakumar-Vafa invariants [248], as we will review momentarily. The expansion (6.44) can be written simply as:

$$\mathcal{F} = -\frac{1}{(2\pi i)^3}\sum_{\boldsymbol{\beta}} d_{\boldsymbol{\beta}}\,\mathrm{Li}_3(Q^{\boldsymbol{\beta}}) , \tag{6.44}$$

with

$$d_{\boldsymbol{\beta}} \equiv \sum_{j_l,j_r}(-1)^{2j_l+2j_r}(2j_l+1)(2j_r+1)N^{\boldsymbol{\beta}}_{j_l,j_r} , \tag{6.45}$$

the effective number of 5d BPS states of charge $\boldsymbol{\beta}$. Given the expression (6.44) for the prepotential, we can directly compute the CB fibering operator in terms of the hypermultiplet

[58] Here we ignore some possible 'classical' terms, which would contribute additional factors to the CB partition function.

result (6.42), at least formally, as a product over the charge lattice:

$$\mathscr{F}_{\mathfrak{p}}(\boldsymbol{a})_{\mathfrak{m}} = \prod_{\boldsymbol{\beta}} \left[\mathscr{F}_{\mathfrak{p}}^{\mathcal{H}}(\boldsymbol{\beta}(\boldsymbol{a}))_{\boldsymbol{\beta}(\mathfrak{m})}\right]^{d_{\boldsymbol{\beta}}} . \tag{6.46}$$

Thus, the higher-spin contributions are the same as for $d_{\boldsymbol{\beta}}$ hypermultiplets, in perfect agreement with the second line of (5.135).

One can similarly expand the flux operators. To obtain the full CB partition function, we should also consider the contribution of higher-spin particles to the gravitational couplings $\mathcal{A}$ and $\mathcal{B}$. In section 7.1.4 below, we will show that:

$$\begin{aligned} \mathcal{A} &= \frac{1}{2\pi i} \sum_{\boldsymbol{\beta}} \sum_{j_l, j_r} c_{\mathcal{A}}^{(j_l,j_r)} N^{\boldsymbol{\beta}}_{j_l,j_r} \log(1 - Q^{\boldsymbol{\beta}}) , \\ \mathcal{B} &= \frac{1}{2\pi i} \sum_{\boldsymbol{\beta}} \sum_{j_l, j_r} c_{\mathcal{B}}^{(j_l,j_r)} N^{\boldsymbol{\beta}}_{j_l,j_r} \log(1 - Q^{\boldsymbol{\beta}}) , \end{aligned} \tag{6.47}$$

when expanding in terms of the refined GV invariants, with the coefficients $c_{\mathcal{A},\mathcal{B}}^{(j_l,j_r)}$ given in (5.136). One then easily checks that the CB partition function on $\mathcal{M}_5$ can be written entirely in terms of the refined GV invariants of the 5d theory, as:

$$Z_{\mathcal{M}_5}(\boldsymbol{a})_{\mathfrak{m}} = \prod_{\boldsymbol{\beta}} \prod_{j_l, j_r} \left[Z_{\mathcal{M}_5}^{(j_l,j_r)}(\boldsymbol{\beta}(\boldsymbol{a}))_{\boldsymbol{\beta}(\mathfrak{m})}\right]^{N^{\boldsymbol{\beta}}_{j_l,j_r}} , \tag{6.48}$$

using the explicit expression (5.135). The expression (6.48) is the partition function that we would obtain by combining the localization results of section 5.3 with the assumption that the full partition function can be obtained as a product over the 5d BPS states, as argued by Lockhart and Vafa [23]. What we have just shown is that this factorisation is consistent with the low-energy approach of the present section. In fact, the factorisation (6.48) is simply equivalent to the expansions (6.43) and (6.47) of the low-energy effective couplings.

6.3.1 Spin/charge constraints on the 5d BPS spectrum

To conclude this section, let us mention an important constraint that arises when trying to put a general 5d SCFT on our supersymmetric $\mathcal{M}_5$, for a generic choice of our geometric background. Namely, every BPS particle of spin (j_l, j_r) and charge $\boldsymbol{\beta}$, at any point on the 5d CB, should be coupled consistently to the base manifold $\mathcal{M}_4$, *at the same time*. Given the CB (gauge and flavour) symmetry $\prod_I U(1)_I$, we need to choose the ε parameters ε^I, which define the spin^c connections as in (6.15), in such a way that

$$\frac{1}{2} + j_l + j_r + \boldsymbol{\beta}(\varepsilon) \in \mathbb{Z} \ , \qquad \forall\, j_l, j_r, \boldsymbol{\beta} \text{ with } N^{\boldsymbol{\beta}}_{j_l,j_r} \neq 0 \ . \tag{6.49}$$

Here, $\boldsymbol{\beta}(\varepsilon) \equiv q_I \varepsilon^I$ is the ε parameter of this particular BPS particle. For any fixed j_l, j_r, q_I, this condition is equivalent to the requirement that the vector bundle (5.133) be well-defined on any Kähler manifold $\mathcal{M}_4$. (Of course, if $\mathcal{M}_4$ is spin, then this condition is not necessary.)

Note that, once we fix ε^I, the condition (6.49) only holds if the spin and electric charges of the BPS states are appropriately correlated (mod 2). The theories for which this holds obey a "spin/charge" relation, which is somewhat reminiscent of the 3d spin/charge relation discussed in [249] for strongly-coupled electrons; this spin/charge relation for 4d $\mathcal{N} = 2$ theories was also discussed in [250].

In [3], we computed explicitly the Gopakumar-Vafa invariants for the toric E_n 5d SCFTs and showed that (6.49) is satisfied. For instance, for the E_1 theory, perturbatively in the 5d gauge-theory limit, we only have the massive W-boson, of spin $(0, \frac{1}{2})$, which satisfies the condition (6.49) with charge $\boldsymbol{\beta} = (0, 1)$ in the basis corresponding to the two factors $\mathbb{P}_b \times \mathbb{P}_f$ of the local $\mathbb{F}_0$ geometry in M-theory. Hence we need to have $\varepsilon^{I=2} \mod 1 = 0$ in this basis. Similarly, looking at the first instanton particle, $\boldsymbol{\beta} = (1, 0)$, we have $\varepsilon^{I=1} \mod 1 = 0$. (The SCFT has a symmetry exchanging the two charges, $\boldsymbol{\beta} = (m, n) \leftrightarrow (m, n)$.) Hence, by consistency, we should have $\frac{1}{2} + j_l + j_r \in \mathbb{Z}$ for any other particle in the spectrum, of any charge. This is indeed the case, at least to the order that we have checked it.

From our considerations, all the toric E_n models can be coupled consistently to our $\mathcal{M}_5$ background with the extended DW twist on $\mathcal{M}_4$. It would be interesting to understand whether the spin/charge relation must always hold, *a priori*, in any 5d SCFT.

Chapter 7

Five-dimensional partition functions

In this chapter, we give a complementary perspective on the Coulomb branch partition function (6.36), including the fibering operator, by building up $\mathcal{M}_5$ as a toric gluing of $\mathbb{C}^2 \times S^1$ patches, in the case when the base $\mathcal{M}_4$ is a toric four-manifold. We can then obtain the partition function $Z_{\mathcal{M}_5}$ as an appropriate gluing of 5d Nekrasov partition functions, generalising well-known results for the five-sphere [23–25, 237].

7.1 Nekrasov partition functions and topological strings

Partition functions of 4d $\mathcal{N} = 2$ field theories on toric four-manifolds can be computed in terms of the partition functions on toric patches $\mathbb{C}^2$ [113], and similarly for the 5d uplift. On each patch, one considers the so-called Nekrasov partition function on $\mathbb{C}^2 \times S^1$ with the Ω-background, which is obtained by the identification

$$(z_1, z_2, x_5) \sim (e^{2\pi i \tau_1} z_1, e^{2\pi i \tau_2} z_2, x_5 + \beta) \ , \tag{7.1}$$

where (z_1, z_2, x_5) are the $\mathbb{C}^2 \times S^1$ coordinates, and we also introduced the dimensionless Ω-deformation parameters:

$$\tau_1 = \beta \epsilon_1 \ , \qquad \qquad \tau_2 = \beta \epsilon_2 \ , \tag{7.2}$$

not to be confused with the gauge couplings. The Ω-background is a $U(1)^2$-equivariant deformation of the topological twist which effectively compactifies the non-compact $\mathbb{C}^2$, with a finite 'volume' $1/(\tau_1 \tau_2)$. Using topological invariance, one can equivalently consider a background geometry $D^2_{\tau_1} \times D^2_{\tau_2} \times S^1$, where $D^2_{\tau_{1,2}}$ are elongated cigars fibered over S^1 according to (7.1). Formally, we can assign the following Euler characteristic and signature to the Ω-deformed $\mathbb{C}^2$ geometry [251]:

$$\chi(\mathbb{C}^2) = \tau_1\tau_2 \ , \qquad \sigma(\mathbb{C}^2) = \frac{\tau_1^2+\tau_2^2}{3} \ . \tag{7.3}$$

Similarly, the first Chern class of the canonical line bundle over $\mathbb{C}^2$ is formally given by:

$$c_1(\mathcal{K}_{\mathbb{C}^2}) = \tau_1 + \tau_2 \ . \tag{7.4}$$

Note that we have $c_1(\mathcal{K})^2 = 2\chi + 3\sigma = (\tau_1+\tau_2)^2$. The partition function of a 5d $\mathcal{N}=1$ theory on $\mathbb{C}^2\times S^1$ is known as the (K-theoretic) Nekrasov partition function [252,253], and it will be denoted by:

$$Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a},\tau_1,\tau_2) \ . \tag{7.5}$$

Here, the CB parameters $\boldsymbol{a}^I$ arise as Dirichlet boundary conditions for the $U(1)_I$ vector multiplets at infinity. Whenever we have a four-dimensional gauge-theory interpretation, the Nekrasov partition function admits an expansion in some instanton counting parameter $\mathfrak{q} = e^{2\pi i \tau_{uv}}$, according to:

$$Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a},\tau_1,\tau_2) = Z^{\text{cl}}_{\mathbb{C}^2\times S^1}(\boldsymbol{a},\tau_1,\tau_2) Z^{\text{pert}}_{\mathbb{C}^2\times S^1}(\boldsymbol{a},\tau_1,\tau_2)\left(1+\sum_k \mathfrak{q}^k Z_k^{\text{Nek}}(\boldsymbol{a},\tau_1,\tau_2)\right) \ . \tag{7.6}$$

See *e.g.* [251,254,255] for reviews of instanton counting, and [256–258] for some more recent advances. When considering the Donaldson-Witten twist, we are interested in the non-equivariant limit $\tau_{1,2}\to 0$. In that limit, the partition function diverges in a way which precisely encodes the low-energy couplings $\mathcal{F}$, $\mathcal{A}$, and $\mathcal{B}$ of the CB theory, namely [113,259]:

$$\log Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a},\tau_1,\tau_2) \approx -\frac{2\pi i}{\tau_1\tau_2}\left(\mathcal{F}(\boldsymbol{a}) + (\tau_1+\tau_2)H(\boldsymbol{a}) + \tau_1\tau_2\mathcal{A}(\boldsymbol{a}) + \frac{\tau_1^2+\tau_2^2}{3}\mathcal{B}(\boldsymbol{a})\right) \ . \tag{7.7}$$

The term $H(\boldsymbol{a})$ in (7.7) is allowed by dimensional analysis, but it does not represent an additional effective coupling. In fact, for the $U(1)^2$-equivariant DW twist, we must have $H(\boldsymbol{a})=0$ because there are no supergravity background fields that could contribute to this coupling (see *e.g.* [260]). More generally, H is fully determined in terms of $\mathcal{F}$ by the choice of background $U(1)$ gauge fields, as we will see momentarily.

7.1.1 Nekrasov partition functions for the extended topological twist

When patching together Nekrasov partition functions into compact four- or five-manifolds, we will have to be careful about whether the base $\mathcal{M}_4$ is spin or not. In general, we should consider the possibility of an extended DW twist on $\mathbb{C}^2 \times S^1$, with parameters ε^I. We propose that this corresponds to twisting the background gauge fields at infinity according to:

$$\boldsymbol{a}^I \to \boldsymbol{a}^I + \varepsilon^I(\tau_1 + \tau_2) \ , \tag{7.8}$$

in agreement with the identification (7.4). Namely, the Nekrasov partition function for the extended DW twist is given by:

$$Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a}, \tau_1, \tau_2; \varepsilon) = Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a} + \varepsilon(\tau_1 + \tau_2), \tau_1, \tau_2) \ . \tag{7.9}$$

Hence, the non-equivariant limit of the partition function reads:

$$\log Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a}, \tau_1, \tau_2; \varepsilon) \approx \\ -\frac{2\pi i}{\tau_1\tau_2}\left(\mathcal{F}(\boldsymbol{a}) + (\tau_1+\tau_2)H(\boldsymbol{a};\varepsilon) + \tau_1\tau_2\mathcal{A}(\boldsymbol{a}) + \frac{\tau_1^2+\tau_2^2}{3}\mathcal{B}(\boldsymbol{a}) + (\tau_1+\tau_2)^2\mathcal{G}(\boldsymbol{a};\varepsilon)\right) \ , \tag{7.10}$$

with:

$$H(\boldsymbol{a};\varepsilon) = \varepsilon^I \frac{\partial \mathcal{F}}{\partial \boldsymbol{a}^I} \ , \qquad \mathcal{G}(\boldsymbol{a};\varepsilon) = \frac{1}{2}\varepsilon^I\varepsilon^J \frac{\partial^2\mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J} \ . \tag{7.11}$$

This parameterisation of the non-equivariant limit naturally parallels the discussion of section 6.1.1 for the CB effective couplings, with $\mathcal{G}$ being exactly as in (6.21).

7.1.2 Gluing transformations in the non-equivariant limit

We wish to glue together Nekrasov partition functions from different patches to obtain the CB partition function of a compact five-manifold $\mathcal{M}_5$. The most general gluing rules between two patches, for our purposes, are:[59]

$$\tau_i \to \widetilde{\tau}_i \equiv \frac{\widehat{\tau}_i}{\gamma} \;, \qquad\qquad \boldsymbol{a} \to \widetilde{\boldsymbol{a}} \equiv \frac{\boldsymbol{a} + \widehat{\mathfrak{n}}}{\gamma} \;, \tag{7.12}$$

where we defined $\widehat{\tau}_i \equiv \alpha_i \tau_1 + \beta_i \tau_2$ and $\gamma \equiv \gamma_1 \tau_1 + \gamma_2 \tau_2 + 1$, for some integers α_i, β_i, γ_i (with $i = 1, 2$), and:

$$\widehat{\mathfrak{n}} \equiv \widehat{\mathfrak{n}}_1 \widehat{\tau}_1 + \widehat{\mathfrak{n}}_2 \widehat{\tau}_2 \;. \tag{7.13}$$

The parameters $\widehat{\mathfrak{n}}_i$ allow us to introduce background fluxes. To recover the DW twist on $\mathcal{M}_5$, we need to consider the non-equivariant limit of the Nekrasov partition function in the variables (7.12). In the limit $\tau_i \to 0$ and using the ansatz (7.7), one finds:

$$\begin{aligned}
&\log Z_{\mathbb{C}^2 \times S^1}(\widetilde{\boldsymbol{a}}, \widetilde{\tau}_1, \widetilde{\tau}_2) \approx \\
&-\frac{2\pi i}{\widehat{\tau}_1 \widehat{\tau}_2} \Bigg(\mathcal{F} + (\widehat{\tau}_1 + \widehat{\tau}_2) H + 2(\gamma - 1) \left(\mathcal{F} - \boldsymbol{a}^I \partial_I \mathcal{F} \right) + \widehat{\tau}_1 \widehat{\tau}_2 \mathcal{A}(\boldsymbol{a}) + \frac{\widehat{\tau}_1^2 + \widehat{\tau}_2^2}{3} \mathcal{B}(\boldsymbol{a}) \\
&\qquad + (\gamma - 1)^2 \left(\mathcal{F} - \boldsymbol{a}^I \partial_I \mathcal{F} + \frac{1}{2} \boldsymbol{a}^I \boldsymbol{a}^J \partial_I \partial_J \mathcal{F} \right) + (\gamma - 1)(\widehat{\tau}_1 + \widehat{\tau}_2) \left(H - \boldsymbol{a}^I \partial_I H \right) \\
&\qquad + \widehat{\mathfrak{n}}^I \left(\gamma \partial_I \mathcal{F} + (\widehat{\tau}_1 + \widehat{\tau}_2) \partial_I H - (\gamma - 1) \boldsymbol{a}^J \partial_I \partial_J \mathcal{F} \right) + \frac{1}{2} \widehat{\mathfrak{n}}^I \widehat{\mathfrak{n}}^J \partial_I \partial_J \mathcal{F} \Bigg) \;,
\end{aligned} \tag{7.14}$$

where we used the notation $\partial_I = \frac{\partial}{\partial \boldsymbol{a}^I}$. When considering the extended topological twist as in (7.9), the general non-equivariant limit is obtained from (7.14) through the substitution:

$$H \to \varepsilon^I \partial_I \mathcal{F} \;, \qquad\qquad \mathcal{A} \to \mathcal{A} + 2\mathcal{G} \;, \qquad\qquad \mathcal{B} \to \mathcal{B} + 3\mathcal{G} \;, \tag{7.15}$$

with $\mathcal{G}$ defined in (7.11).

7.1.3 Refined topological string partition function

We are particularly interested in the 5d SCFTs that can be engineered at canonical singularities in M-theory [16, 46]; see *e.g.* [70, 119, 124, 125, 180] for recent studies. Then, the Coulomb branch low-energy effective theory on the Ω-background is obtained by considering

[59] More general gluings could be considered (similarly to the 3d computations in [93]), but this would go beyond the class of principal circle bundles that we consider in this paper.

the low-energy limit of M-theory on:

$$\mathbb{C}^2 \times S^1 \times \widetilde{\mathbf{X}} \ , \tag{7.16}$$

with the Ω-background turned on along $\mathbb{C}^2$. Here, $\widetilde{\mathbf{X}}$ denotes the crepant resolution of a threefold canonical singularity $\mathbf{X}$. Let us further choose $\mathbf{X}$ and its resolution to be toric. Then, the Nekrasov partition function of the five-dimensional theory can be computed using the refined topological vertex formalism [248, 261].

In the geometric-engineering picture, the various Coulomb branch parameters of the 5d theory are now Kähler parameters of the crepant resolution $\widetilde{\mathbf{X}}$. In keeping with standard notation, we use the fugacities q, $t = p^{-1}$ and Q, defined as:

$$q \equiv e^{2\pi i \tau_1} \ , \qquad p = t^{-1} \equiv e^{2\pi i \tau_2} \ , \qquad Q^{\boldsymbol{\beta}} \equiv e^{2\pi i \int_{\boldsymbol{\beta}} (B+iJ)} = e^{2\pi i \boldsymbol{\beta}(\boldsymbol{a})} \ , \tag{7.17}$$

where $\boldsymbol{\beta} \cong [\mathcal{C}] \in H_2(\widetilde{\mathbf{X}}, \mathbb{Z})$ denote the homology class of any effective curve in $\widetilde{\mathbf{X}}$, and $B+iJ$ is the complexified Kähler form in Type-IIA string theory.

The Nekrasov partition function of the 5d theory is expected to be equivalent to the refined topological string partition function for the threefold $\widetilde{\mathbf{X}}$ [248, 262]. In the M-theory approach, the 5d BPS states arise as M2-branes wrapped over curves. One can then write the Nekrasov partition function as a product over these BPS states, of electric charge $\boldsymbol{\beta}$ and spin (j_l, j_r) [23, 248]:

$$Z_{\mathbb{C}^2 \times S^1}(\boldsymbol{a}, \tau_1, \tau_2) = \prod_{\boldsymbol{\beta}} \prod_{j_l, j_r = 0}^{\infty} \left[\mathbf{Z}^{j_l, j_r}_{\mathbb{C}^2 \times S^1}(Q^{\boldsymbol{\beta}}, q, p) \right]^{N^{\beta}_{j_l, j_r}} \ , \tag{7.18}$$

where the non-negative integers $N^{\beta}_{j_l, j_r}$ are the refined Gopakumar-Vafa invariants. Here, the higher-spin particles contribute as:

$$\mathbf{Z}^{j_l, j_r}_{\mathbb{C}^2 \times S^1}(Q, q, p) \equiv \prod_{m_l = -j_l}^{j_l} \prod_{m_r = -j_r}^{j_r} (Q q^{\frac{1}{2} + m_r + m_l} p^{\frac{1}{2} + m_r - m_l}; q, p)_{\infty}^{(-1)^{1+2j_l+2j_r}} \ , \tag{7.19}$$

which is written in terms of the double-Pochhammer symbol:

$$(x;q,p)_\infty \equiv \prod_{j,k=0}^{\infty} (1 - xq^j p^k) \ . \tag{7.20}$$

Note that the definition (7.20) is only valid for $\text{Im}(\tau_i) > 0$. This can be analytically continued to $|q| \neq 1$, $|p| \neq 1$ [263], which gives us the formal identities:

$$(x;q^{-1},p)_\infty = (xq;q,p)_\infty^{-1} \ , \qquad (x;q,p^{-1})_\infty = (xp;q,p)_\infty^{-1} \ . \tag{7.21}$$

The expression (7.18) gives us the Nekrasov partition function for the ordinary Ω-deformed DW twist, and we can also obtain the extended DW twist expression by the substitution $Q \to Q(qp)^\varepsilon$. For completeness, let us also mention that the unrefined topological string limit corresponds to setting $t = p^{-1} = q$, giving us:

$$Z_{\text{top}}(Q,q) = \prod_{\boldsymbol{\beta}} \prod_{j_l=0}^{\infty} \widetilde{\mathbf{Z}}_{\text{top}}^{j_l}(Q^{\boldsymbol{\beta}},q)^{N_{j_l}^{\beta}} \ , \tag{7.22}$$

with:

$$\widetilde{\mathbf{Z}}_{\text{top}}^{j_l}(Q,q) = \prod_{m_l=-j_l}^{j_l} \prod_{k=1}^{\infty} \left[\left(1 - Qq^{k+2m_l}\right)^k \right]^{(-1)^{2j_l}} \ , \tag{7.23}$$

in terms of the unrefined GV invariants $N_{j_l}^{\beta} \equiv \sum_{j_r} (-1)^{2j_r}(2j_r+1)N_{j_l,j_r}^{\beta}$. The (refined) GV invariants have to be computed explicitly, for any given toric threefold $\widetilde{\mathbf{X}}$, for instance using the (refined) topological vertex formalism [248].

7.1.4 GV expansion in the non-equivariant limit

It is interesting to consider the non-equivariant limit of the expression (7.18). We find it useful to introduce the 'quantum trilogarithm' defined as:

$$\text{Li}_3(x;q,p) \equiv -\log(x;q,p)_\infty = \sum_{n=1}^{\infty} \frac{x^n}{n} \frac{1}{(1-q^n)(1-p^n)} \ . \tag{7.24}$$

In the small-τ_i limit, it admits an asymptotic expansion:

$$\mathrm{Li}_3(x;q,p) = \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m}}{n!\,m!} B_n B_m (2\pi i\tau_1)^{n-1}(2\pi i\tau_2)^{m-1}\mathrm{Li}_{3-n-m}(x) \ , \tag{7.25}$$

where B_n are the Bernoulli numbers.[60] We then have:

$$\mathrm{Li}_3(x;q,p) \approx \frac{1}{(2\pi i)^2\tau_1\tau_2}\mathrm{Li}_3(x) - \frac{1}{4\pi i}\frac{\tau_1+\tau_2}{\tau_1\tau_2}\mathrm{Li}_2(x) - \frac{1}{12}\left(3+\frac{\tau_1}{\tau_2}+\frac{\tau_2}{\tau_1}\right)\log(1-x) \ . \tag{7.26}$$

For a massive hypermultiplet with the extended DW twist, we have

$$\log Z^{\mathcal{H}}_{\mathbb{C}^2\times S^1}(\boldsymbol{a},\tau_1,\tau_2;\varepsilon) = \mathrm{Li}_3(Q(qp)^{\frac{1}{2}+\varepsilon};q,p) \ . \tag{7.27}$$

Setting $\varepsilon = 0$ for simplicity, the $\tau_i \to 0$ limit reads:

$$\log Z^{\mathcal{H}}_{\mathbb{C}^2\times S^1}(\boldsymbol{a},\tau_1,\tau_2) \approx -\frac{2\pi i}{\tau_1\tau_2}\left(-\frac{1}{(2\pi i)^3}\mathrm{Li}_3(Q) - \left(\frac{\tau_1^2+\tau_2^2}{3}\right)\frac{1}{16\pi i}\log(1-Q)\right) \ , \tag{7.28}$$

from which we can read off (6.26). More generally, for the equivariant DW twist ($\varepsilon = 0$), we have the refined GV expansion:

$$\log Z_{\mathbb{C}^2\times S^1} = \sum_{\boldsymbol{\beta}}\sum_{j_l,j_r} N^{\boldsymbol{\beta}}_{j_l,j_r} \log \mathbf{Z}^{j_l,j_r}_{\mathbb{C}^2\times S^1}(Q^{\boldsymbol{\beta}},q,p) \ , \tag{7.29}$$

with:

$$\log \mathbf{Z}^{j_l,j_r}_{\mathbb{C}^2\times S^1}(Q,q,p) = (-1)^{2j_l+2j_r}\sum_{m_l=-j_l}^{j_l}\sum_{m_r=-j_r}^{j_r} \mathrm{Li}_3(Qq^{\frac{1}{2}+m_r+m_l}p^{\frac{1}{2}+m_r-m_l};q,p) \ . \tag{7.30}$$

By taking the small-τ_i limit and comparing to (7.7), one can extract the contribution of a spin-(j_l,j_r) particle (of unit electric charge, $\boldsymbol{\beta}=1$) to the low-energy effective couplings.

[60] With the convention that $B_0 = 1$ and $B_1 = \frac{1}{2}$.

By a straightforward computation, one finds:

$$\begin{aligned}
\mathcal{F}^{j_l,j_r} &= -\frac{c_0^{(j_l,j_r)}}{(2\pi i)^3}\mathrm{Li}_3(Q) ~, \\
\mathcal{A}^{j_l,j_r} &= \frac{1}{2\pi i}c_{\mathcal{A}}^{(j_l,j_r)} \log(1-Q) ~, \qquad \mathcal{B}^{j_l,j_r} = \frac{1}{2\pi i}c_{\mathcal{B}}^{(j_l,j_r)} \log(1-Q) ~,
\end{aligned} \tag{7.31}$$

with the coefficients $c^{(j_l,j_r)}$ as defined in (5.136), exactly as anticipated in section 6.3, and in perfect agreement with the index computation of section 5.3.3. Note also that, for the extended topological twist, one has the additional terms H and $\mathcal{G}$ in (7.10), namely:

$$H^{j_l,j_r} = -\frac{c_0^{(j_l,j_r)}}{(2\pi i)^2}\varepsilon\,\mathrm{Li}_2(Q) ~, \qquad \mathcal{G}^{j_l,j_r} = \frac{c_0^{(j_l,j_r)}}{4\pi i}\varepsilon^2 \log(1-Q) ~, \tag{7.32}$$

according to (7.11).

7.2 Gluing Nekrasov partition functions

Let us now consider the explicit gluing of Nekrasov partition functions to obtain the circle-fibered five-manifold $\mathcal{M}_5$ in (5.30), where $\mathcal{M}_4$ is a toric four-manifold. For definiteness, we will mostly focus on the case when $\mathcal{M}_4$ is one of the five toric Fano surfaces, $\mathbb{P}^2$, $\mathbb{F}_0 \cong \mathbb{P}^1\times\mathbb{P}^1$, or dP_n (the blow-up of $\mathbb{P}^2$ at n points) with $n \leq 3$, whose Euler characteristic and signature are:

	$\mathbb{P}^2$	$\mathbb{F}_0$	dP_1	dP_2	dP_3
χ	3	4	4	5	6
σ	1	0	0	-1	-2

(7.33)

Let us first review the case of a trivial fibration, $\mathcal{M}_5 = \mathcal{M}_4 \times S^1$, before considering the case of a non-trivial fibration.

7.2.1 The $\mathcal{M}_4 \times S^1$ partition function

It was conjectured in [113] that the Ω-deformed Coulomb branch partition function on a toric manifold $\mathcal{M}_4$ can be obtained by gluing Nekrasov partition functions for each fixed point of the toric action. The full partition function is then obtained, in principle, by a particular contour integral over the CB parameters, together with a sum over fluxes, both of which one should determine by a more careful analysis. This approach was further developed in [101, 102, 264] (see also [265]), and generalised to 5d theories on $\mathcal{M}_4 \times S^1$ in [114]. The full partition function then reads:

$$\mathbf{Z}_{\mathcal{M}_4\times S^1} = \sum_{\mathfrak{n}_l} \oint d\boldsymbol{a} \prod_{l=1}^{\chi(\mathcal{M}_4)} Z_{\mathbb{C}^2\times S^1}(\boldsymbol{a} + \tau_1^{(l)}\mathfrak{n}_l + \tau_2^{(l)}\mathfrak{n}_{l+1}, \tau_1^{(l)}, \tau_2^{(l)}) \ , \tag{7.34}$$

with $\mathfrak{n}_l$ being fluxes associated with the toric divisors $D_l \subset \mathcal{M}_4$, corresponding to a line bundle:

$$L = \mathcal{O}(-\sum_l \mathfrak{n}_l D_l) \ , \tag{7.35}$$

over $\mathcal{M}_4$. Note that there are $\chi(\mathcal{M}_4)$ toric divisors, with 2 linear relations amongst them. The previously defined $U(1)_I$ background fluxes $\mathfrak{m}^I$ are then given by:

$$\sum_{k=1}^{\chi} \mathfrak{m}_k^I [\mathrm{S}_k] = -\sum_{l=1}^{\chi+2} \mathfrak{n}_l^I [D_l] \ . \tag{7.36}$$

The equivariant parameters $\tau_i^{(l)}$ are linear combinations of $\tau_{1,2}$, which we shall comment on momentarily. The non-equivariant limit of the integrand of (7.34) can be obtained by a direct computation using (7.14) (with $\gamma = 1$), wherein all divergent pieces cancel out between patches, leaving us with a finite quantity. One finds:

$$\begin{aligned}\log Z_{\mathcal{M}_4\times S^1}(\boldsymbol{a}) \approx -2\pi i \Bigg(& \chi\mathcal{A}(\boldsymbol{a}) + \sigma\mathcal{B}(\boldsymbol{a}) + \left(\sum D_l\right)\cdot\left(\sum \mathfrak{n}_l^I D_l\right)\frac{\partial H(\boldsymbol{a})}{\partial \boldsymbol{a}^I} + \\ & + \frac{1}{2}\left(\sum \mathfrak{n}_l^I D_l\right)\cdot\left(\sum \mathfrak{n}_l^J D_l\right)\frac{\partial^2 \mathcal{F}(\boldsymbol{a})}{\partial \boldsymbol{a}^I \partial \boldsymbol{a}^J}\Bigg) \ .\end{aligned} \tag{7.37}$$

For the DW twist, we have $H = 0$ and therefore:

$$Z_{\mathcal{M}_4\times S^1}(\boldsymbol{a}) = \boldsymbol{A}(\boldsymbol{a})^{\chi}\boldsymbol{B}(\boldsymbol{a})^{\sigma}\boldsymbol{\Pi}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{m},\mathfrak{m})} \ , \tag{7.38}$$

in terms of the quantities defined in section 6.1.1. This reproduces and generalises the results of [101, 107, 114]. More generally, as explained at length in previous sections, we should consider the extended DW twist with $\varepsilon \neq 0$, in which case we should substitute

(7.15) into (7.37). Then, using the fact that $\mathcal{K}_{\mathcal{M}_4} \cong \mathcal{O}(-\sum_l D_l)$, we find:

$$Z_{\mathcal{M}_4\times S^1}(\boldsymbol{a};\varepsilon) = \boldsymbol{A}(\boldsymbol{a})^{\chi}\boldsymbol{B}(\boldsymbol{a})^{\sigma}\boldsymbol{G}(\boldsymbol{a};\varepsilon)^{2\chi+3\sigma}\boldsymbol{\Pi}(\boldsymbol{a})^{\frac{1}{2}(\mathfrak{m}+2\varepsilon\mathbf{k},\mathfrak{m})} , \tag{7.39}$$

which exactly reproduces the formula (6.22). Next, let us explain how the Nekrasov partition functions have been glued together.

Equivariant parameters. The patch-dependent equivariant parameters $\tau^{(l)}$ in (7.34) are determined by the toric data, as follows (see *e.g.* [114] for a more detailed discussion). A compact toric surface $\mathcal{M}_4$ is described by a set of vectors $\vec{n}_l \in \mathbb{Z}^2$, with $l = 1, \dots, d$, which we order such that $\vec{n}_l$ and $\vec{n}_{l+1}$ are adjacent (with $\vec{n}_{d+1} \equiv \vec{n}_1$). Each such vector is associated to a non-compact divisor D_l.

Each pair of vectors $(\vec{n}_l, \vec{n}_{l+1})$ defines a two-dimensional cone σ_l, to which we can associate an affine variety V_{σ_l}. The construction is based on the dual cone $\hat{\sigma}_l$ generated by the primitive integer vectors $\vec{m}_l$ and $\vec{m}_{l+1}$, which are orthogonal to $\vec{n}_{l+1}$ and $\vec{n}_l$, respectively, and point inwards inside σ_l. The set of holomorphic functions on V_{σ_l} is given by monomials $z_1^{\mu_1} z_2^{\mu_2}$, for all $\vec{\mu} \in \hat{\sigma}_l$. Then, since $V_{\sigma_l} \cong \mathbb{C}^2$ by assumption that $\mathcal{M}_4$ be smooth, the local coordinates on V_{σ_l} can be chosen as:

$$\rho_1^{(l)} = z_1^{m_{l,1}} z_2^{m_{l,2}} , \qquad \rho_2^{(l)} = z_1^{m_{l+1,1}} z_2^{m_{l+1,2}} . \tag{7.40}$$

The toric variety $\mathcal{M}_4$ is obtained by gluing together the affine varieties V_{σ_l}, by identifying dense open subsets associated with the common vectors spanning the neighbouring cones σ_l. Due to the Ω-background, the $\mathcal{M}_4 \times S^1$ partition function only receives contributions from the $\chi(\mathcal{M}_4)$ fixed points of the toric action. There is then a single contribution from each chart V_{σ_l} of $\mathcal{M}_4$, as written explicitly in (7.34). Thus, the equivariant parameters will 'transform' under the $(\mathbb{C}^*)^2$ action similarly to (7.40), leading to:

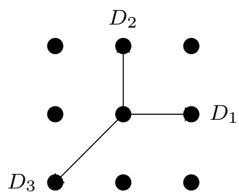
$$\tau_1^{(l)} = \vec{\tau} \cdot \vec{m}_l , \qquad \tau_2^{(l)} = \vec{\tau} \cdot \vec{m}_{l+1} . \tag{7.41}$$

Furthermore, the (background) gauge fluxes $\mathfrak{n}$, which appear as in (7.34), are similarly local contributions from each patch: Thus, the Coulomb branch VEVs change according to:

$$\boldsymbol{a}^{(l)} = \boldsymbol{a} + \tau_1^{(l)}\mathfrak{n}_l + \tau_2^{(l)}\mathfrak{n}_{l+1} \ . \tag{7.42}$$

At this stage, it is natural to wonder how this procedure can be modified to account for a non-trivial $U(1)_{\text{KK}}$ flux, leading to the non-trivial fibration $\mathcal{M}_5$. Before exploring this, let us briefly consider a couple of examples of toric gluings for $\mathcal{M}_4 \times S^1$.

The $\mathcal{M}_4 = \mathbb{P}^2$ case. The simplest example of a toric Kähler four manifold is that of $\mathbb{P}^2$, for which the toric fan and intersection numbers are:



	D_1	D_2	D_3	$\mathcal{K}$
S	1	1	1	-3

(7.43)

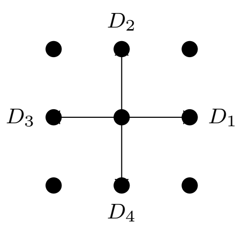
The toric divisors satisfy the linear relations $\mathrm{S} \cong D_1 \cong D_2 \cong D_3$, with $S \cong H$ the hyperplane class. Therefore, given the above triple intersection numbers, we have in our previous notation:

$$\mathfrak{m} = -(\mathfrak{n}_1 + \mathfrak{n}_2 + \mathfrak{n}_3) \ . \tag{7.44}$$

Note also that the canonical divisor is given by $\mathcal{K} \cong -\sum D_l \cong -3D_1$, and thus the Chern number is $\mathbf{k} = -3$. The equivariant parameters (7.41) are given by [114]:

$$\tau_1^{(l)} = (\tau_1, \, -\tau_1 + \tau_2, \, -\tau_2) \, , \qquad \tau_2^{(l)} = (\tau_2, \, -\tau_1, \, \tau_1 - \tau_2) \, . \tag{7.45}$$

The $\mathcal{M}_4 = \mathbb{F}_0$ case. Consider now the case of $\mathbb{F}_0 \cong S^1 \times S^1$, with the following toric data:



	D_1	D_2	D_3	D_4	$\mathcal{K}$
S_1	0	1	0	1	-2
S_2	1	0	1	0	-2

(7.46)

The toric divisors satisfy the linear relations $\mathrm{S}_1 \cong D_1 \cong D_3$ and $\mathrm{S}_2 \cong D_2 \cong D_4$ and, thus, there are two distinct compact curves S_1 and S_2, corresponding to the two $\mathbb{P}^1$ factors in $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. In this basis, the intersection form reads:

$$Q_{\mathbb{F}_0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{7.47}$$

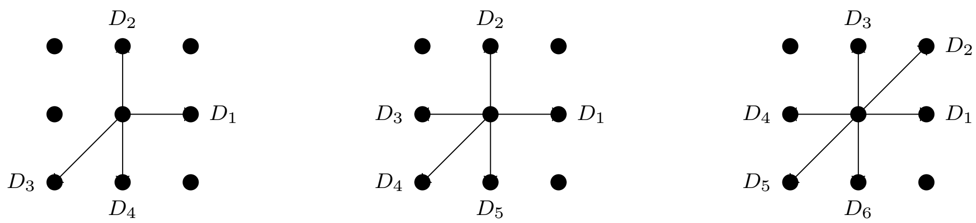
Furthermore, the canonical divisor is $\mathcal{K} = -2D_1 - 2D_2$, leading to the Chern numbers $\mathbf{k} = (-2, -2)$. Given the above triple intersection numbers, we also have the fluxes:

$$\mathfrak{m} = (\mathfrak{m}_1, \mathfrak{m}_2) = (-\mathfrak{n}_1 - \mathfrak{n}_3, \ -\mathfrak{n}_2 - \mathfrak{n}_4) \ . \tag{7.48}$$

Finally, the equivariant parameters are given by [114]:

$$\tau_1^{(l)} = (\tau_1, \ \tau_2, \ -\tau_1, \ -\tau_2) \, , \qquad \tau_2^{(l)} = (\tau_2, \ -\tau_1, \ -\tau_2, \ \tau_1) \, . \tag{7.49}$$

The dP_n cases. For completeness, let us also give the equivariant parameters for the remaining toric del Pezzo surfaces. The toric fans for dP_n with $n = 1, 2, 3$ read:



(7.50)

respectively, and the equivariant parameters are:

$$\begin{aligned}
dP_1 \ : \quad & \tau_1^{(l)} = (\tau_1, -\tau_1 + \tau_2, -\tau_1, -\tau_2) \, , & \tau_2^{(l)} &= (\tau_2, -\tau_1, \tau_1 - \tau_2, \tau_1) \, , \\
dP_2 \ : \quad & \tau_1^{(l)} = (\tau_1, \tau_2, -\tau_1 + \tau_2, -\tau_1, -\tau_2) \, , & \tau_2^{(l)} &= (\tau_2, -\tau_1, -\tau_2, \tau_1 - \tau_2, \tau_1) \, , \\
dP_3 \ : \quad & \tau_1^{(l)} = (\tau_1 - \tau_2, \tau_1, \tau_2, -\tau_1 + \tau_2, -\tau_1, -\tau_2) \, , & \tau_2^{(l)} &= (\tau_2, -\tau_1 + \tau_2, -\tau_1, -\tau_2, \tau_1 - \tau_2, \tau_1) \, .
\end{aligned} \tag{7.51}$$

7.2.2 The S^5 and $L(\mathfrak{p};1)$ partition functions

Let us now turn to the simplest and most important instance of a circle fibration $\mathcal{M}_5 \to \mathcal{M}_4$, which is the five-sphere viewed as a circle fibration over the complex projective plane:

$$S^1 \longrightarrow S^5 \longrightarrow \mathbb{P}^2 \ . \tag{7.52}$$

The gluing approach was first considered in [23], but it is worthwhile to discuss the argument in some detail. The metric of the round five-sphere can be written as $ds^2 = \sum dz_i d\bar{z}_i$, in terms of the coordinates $(z_i) \in \mathbb{C}^3$ subject to the constraint $\sum |z_i| = 1$. Alternatively, we can parametrise the five-sphere using the angles $\theta, \phi \in (0, \pi/2)$ and $\chi_i \in (0, 2\pi)$, as:

$$z_1 = e^{i\chi_1} \sin\theta \cos\phi \ , \qquad z_2 = e^{i\chi_2} \sin\theta \sin\phi \ , \qquad z_3 = e^{i\chi_3} \cos\theta \ . \tag{7.53}$$

In these coordinates, the S^5 metric reads:

$$ds^2 = d\theta^2 + \sin^2\theta \ d\phi^2 + \sin^2\theta \cos^2\phi \ d\chi_1^2 + \sin^2\theta \sin^2\phi \ d\chi_2^2 + \cos^2\theta \ d\chi_3 \ . \tag{7.54}$$

To apply our formalism, we should write this metric in the general form (5.34), namely as $ds^2(S^5) = ds^2(\mathbb{P}^2) + (d\psi + \mathrm{C})^2$, where $ds^2(\mathbb{P}^2)$ is the Fubini-Study metric on the base $\mathbb{P}^2$. An important requirement is that the connection C should be well defined on each coordinate patch on the base. Let us then consider the Fubini-Study metric on each patch and subtract it from the S^5 metric, in order to find the connection C on that patch. The z_i coordinates of the five-sphere descend to coordinates of the projective space $\mathbb{P}^2$. As such, let us denote by $V_i \cong \mathbb{C}^2$ the patch with coordinates $w_j = z_j/z_i$, for $j \neq i$ and $z_i \neq 0$, and define the corresponding azimuthal coordinates:

$$\begin{aligned} &\text{Patch } V_1 : \quad \rho_1^{(1)} = \chi_2 - \chi_1 \ , \qquad \rho_2^{(1)} = \chi_3 - \chi_1 \ , \\ &\text{Patch } V_2 : \quad \rho_1^{(2)} = \chi_3 - \chi_2 \ , \qquad \rho_2^{(2)} = \chi_1 - \chi_2 \ , \\ &\text{Patch } V_3 : \quad \rho_1^{(3)} = \chi_1 - \chi_3 \ , \qquad \rho_2^{(3)} = \chi_2 - \chi_3 \ . \end{aligned} \tag{7.55}$$

For each coordinate patch, the coordinate along the S^1 fiber will be a linear combination of the χ_i angles, $\psi = \alpha_i \chi_i$. With the normalisation $\sum_i \alpha_i = 1$, we find the $U(1)_{\mathrm{KK}}$ connection in each patch to be:

$$
\begin{aligned}
\mathrm{C}^{(1)} &= \frac{1}{4}\big(1 - 4\alpha_2 - \cos(2\theta) - 2\cos(2\phi)\sin(\theta)^2\big)d\rho_1^{(1)} + \frac{1}{2}\big(1 - 2\alpha_3 + \cos(2\theta)\big)d\rho_2^{(1)} \ , \\
\mathrm{C}^{(2)} &= \frac{1}{2}\big(1 - 2\alpha_3' + \cos(2\theta)\big)d\rho_1^{(2)} + \frac{1}{4}\big(1 - 4\alpha_1' - \cos(2\theta) + 2\cos(2\phi)\sin(\theta)^2\big)d\rho_2^{(2)} \ , \\
\mathrm{C}^{(3)} &= \frac{1}{4}\big(1 - 4\alpha_1'' - \cos(2\theta) + 2\cos(2\phi)\sin(\theta)^2\big)d\rho_1^{(3)} \\
&\quad + \frac{1}{4}\big(1 - 4\alpha_2'' - \cos(2\theta) - 2\cos(2\phi)\sin(\theta)^2\big)d\rho_2^{(3)} \ .
\end{aligned}
\tag{7.56}
$$

Let us note that the patch V_i is not defined at $z_i = 0$, at which point the differential $d\chi_i$ is ill-defined. Then, imposing continuity for the connection and well-definiteness on every coordinate patch (that is, the absence of 'Dirac string' singularities), we should pick the following coordinates along the S^1 fiber:

$$
\psi^{(1)} = \chi_1 \ , \qquad \psi^{(2)} = \chi_2 \ , \qquad \psi^{(3)} = \chi_3 \ . \tag{7.57}
$$

In this way, we find the following transformations between angles as we change coordinate patches of the $\mathbb{P}^2$ base:

$$
\begin{pmatrix} \rho_1^{(2)} \\ \rho_2^{(2)} \\ \psi^{(2)} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho_1^{(1)} \\ \rho_2^{(1)} \\ \psi^{(1)} \end{pmatrix} \ , \qquad \begin{pmatrix} \rho_1^{(3)} \\ \rho_2^{(3)} \\ \psi^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \rho_1^{(1)} \\ \rho_2^{(1)} \\ \psi^{(1)} \end{pmatrix} \ . \tag{7.58}
$$

In the toric description of S^5, the five-sphere is a T^3 fibration over a triangle [23]. Moreover, the Ω-background parameters $\tau_1^{(i)}$ and $\tau_2^{(i)}$ on each patch $V_l \cong \mathbb{C}^2$ can be interpreted as complex structure parameters for the tori $T^2 \subset T^3$ spanned by the angular coordinates $(\rho_1^{(i)}, \psi_i)$ and $(\rho_2^{(i)}, \psi_i)$, respectively. The $SL(3,\mathbb{Z})$ transformation matrices (7.58) then suggest the following gluing rules for the Nekrasov partition functions:

$$\begin{aligned}
V_1 &: \qquad \tau_1 \ , \qquad && \tau_2 \ , \qquad && \boldsymbol{a} \ , \\
V_2 &: \qquad \tau_1^* = \frac{-\tau_1+\tau_2}{\tau_1+1} \ , \qquad && \tau_2^* = \frac{-\tau_1}{\tau_1+1} \ , \qquad && \boldsymbol{a}^* = \frac{\boldsymbol{a}}{\tau_1+1} \ , \\
V_3 &: \qquad \widetilde{\tau}_1 = \frac{-\tau_2}{\tau_2+1} \ , \qquad && \widetilde{\tau}_2 = \frac{\tau_1-\tau_2}{\tau_2+1} \ , \qquad && \widetilde{\boldsymbol{a}} = \frac{\boldsymbol{a}}{\tau_2+1} \ ,
\end{aligned} \tag{7.59}$$

which generalises (7.45).

Lens spaces. It is also instructive to consider lens spaces $S^5/\mathbb{Z}_p$, as a simple generalisation of the above. The lens space $L(p; q_1, q_2, q_3)$ can be defined as a quotient of $S^5 \subset \mathbb{C}^3$ by the $\mathbb{Z}_p$ action generated by:

$$(z_1, z_2, z_3) \mapsto \left(e^{2\pi i \frac{q_1}{p}} z_1, e^{2\pi i \frac{q_2}{p}} z_2, e^{2\pi i \frac{q_3}{p}} z_3\right) \ , \tag{7.60}$$

with $q_i, p \in \mathbb{Z}$ and q_i coprime to p. For generic values of q_i, however, these five-manifolds are not fibrations over $\mathbb{P}^2$, but rather over singular quotients of $\mathbb{P}^2$, as one can see by considering the induced action on the $\mathbb{P}^2$ coordinates (7.55). In this paper, we restrict our attention to principal circle bundles over four-manifolds, which corresponds to the case $q_i = 1$. We denote the resulting lens space by $L(p; 1)$. It is simply a principal circle bundle:

$$S^1 \to L(\mathfrak{p}; 1) \to \mathbb{P}^2 \ , \tag{7.61}$$

with $\mathfrak{p} = p$. We can then derive the gluing rules in the same way as for the round S^5. One finds:

$$Z_{L(\mathfrak{p};1)}(\boldsymbol{a})_{\mathfrak{m}} = \prod_{l=1}^{\chi(\mathbb{P}^2)} Z_{\mathbb{C}^2\times S^1}\left(\frac{\boldsymbol{a} + \tau_1^{(l)}\mathfrak{n}_l + \tau_2^{(l)}\mathfrak{n}_{l+1}}{\gamma^{(l)}} \, , \, \frac{\tau_1^{(l)}}{\gamma^{(l)}} \, , \, \frac{\tau_2^{(l)}}{\gamma^{(l)}}\right) \ , \tag{7.62}$$

where $\tau_i^{(l)}$ are the equivariant parameters appearing in the $\mathbb{P}^2 \times S^1$ gluing (7.45), while the denominators γ are given by:

$$\gamma^{(l)} = (1, \, p\tau_1 + 1, \, p\tau_2 + 1) \ . \tag{7.63}$$

In (7.62) we also allowed for background fluxes, as in (7.34). In the non-equivariant limit (and turning on ε as before), this expression reproduces exactly the one expected from (6.36).

7.2.3 Fibrations over toric Kähler 4-manifolds

Having discussed fibrations over $\mathbb{P}^2$, it is natural to consider the generalisation to any toric $\mathcal{M}_4$. Here, we first derive the gluing formula for $\mathcal{M}_4 = \mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, which has an explicitly known Kähler metric. We then conjecture a gluing formula in the general case.

The $\mathcal{M}_4 = \mathbb{F}_0$ case. Consider circle fibrations over $\mathbb{F}_0$, with Chern numbers $\mathfrak{p} = (\mathfrak{p}_1, \mathfrak{p}_2)$, such that the metric of such a space is given by:

$$ds^2 = \sum_{i=1}^{2} \left(d\theta_i^2 + \sin^2\theta_i d\phi_1^2\right) + \left(d\psi + \frac{1}{2}\sum_{i=1}^{2} \mathfrak{p}_i(\pm 1 + \cos\theta_i)d\phi_i\right)^2 , \tag{7.64}$$

with $\theta_i \in [0, \pi)$, $\phi_i \in [0, 2\pi)$ and $\psi \in [0, 2\pi)$. This space has four coordinate patches, corresponding to the two patches of each of the $\mathbb{P}^1$ spaces. We proceed as before, by finding the well-defined coordinates on each patch. Defining:

$$\gamma^{(l)} = (1,\ \mathfrak{p}_1\tau_1 + 1,\ \mathfrak{p}_1\tau_1 + \mathfrak{p}_2\tau_2 + 1,\ \mathfrak{p}_2\tau_2 + 1) , \tag{7.65}$$

we then propose that the partition function on non-trivial fibrations over $\mathbb{F}_0$ is given by:

$$Z_{\mathbb{F}_0^{(\mathfrak{p})}}(\boldsymbol{a}) = \prod_{l=1}^{\chi(\mathbb{F}_0)} Z_{\mathbb{C}^2\times S^1}\left(\frac{\boldsymbol{a} + \tau_1^{(l)}\mathfrak{n}_l + \tau_2^{(l)}\mathfrak{n}_{l+1}}{\gamma^{(l)}}, \frac{\tau_1^{(l)}}{\gamma^{(l)}}, \frac{\tau_2^{(l)}}{\gamma^{(l)}}\right) , \tag{7.66}$$

where, as before, $\tau_i^{(l)}$ are the equivariant parameters appearing in the $\mathbb{F}_0 \times S^1$ case and $\mathfrak{n}_l$ are fluxes associated with the toric divisors.

General toric Kähler surfaces. We would like to generalise the above result to any principal circle bundle over a toric Kähler surface $\mathcal{M}_4$. The prescription used for non-trivial fibrations over $\mathbb{P}^2$ and $\mathbb{F}_0$ involved the coordinates $(\rho_1^{(l)}, \rho_2^{(l)})$ along the coordinate patches V_{σ_l} of the base four-manifold, as well as the coordinate along the fiber $\psi^{(l)}$. As explained in the previous sections, a non-trivial fibration can be viewed as a non-trivial flux for a $U(1)_{\rm KK}$ background symmetry on $\mathcal{M}_4$. We then propose that the denominators $\gamma^{(l)}$ should be given by:

$$\gamma^{(l)} = 1 + \tau_1^{(l)} p_l + \tau_2^{(l)} p_{l+1} \ , \tag{7.67}$$

where p_l are $U(1)_{\rm KK}$ fluxes associated with the non-compact toric divisors D_l, such that:

$$\sum_{k=1}^{\chi} \mathfrak{p}_k [\mathrm{S}_k] = - \sum_{l=1}^{\chi+2} p_l D_l \ , \tag{7.68}$$

as in (7.36). Thus, the CB partition function on non-trivial fibrations over $\mathcal{M}_4$ with Chern numbers $\mathfrak{p}$ should generalise to:

$$\mathbf{Z}_{\mathcal{M}_5} = \sum_{\mathfrak{n}_l} \oint d\boldsymbol{a} \prod_{l=1}^{\chi(\mathcal{M}_4)} Z_{\mathbb{C}^2\times S^1} \left(\frac{\boldsymbol{a} + \tau_1^{(l)} \mathfrak{n}_l + \tau_2^{(l)} \mathfrak{n}_{l+1}}{\gamma^{(l)}}, \frac{\tau_1^{(l)}}{\gamma^{(l)}}, \frac{\tau_2^{(l)}}{\gamma^{(l)}} \right) \ , \tag{7.69}$$

naturally generalising the Nekrasov conjecture [113]. (Here, as in the original conjecture, the precise form of the contour integration and of the sum over fluxes remain to be determined.)

Given the factorised integrand $Z_{\mathcal{M}_5}$ in (7.69) with the non-trivial Ω-background, we can again check our general formalism by taking the non-equivariant limit, generalising the formula (7.37). For every toric Fano four-manifold $\mathcal{M}_4$ in (7.33), using (7.14), we find the following expression:

$$\begin{aligned} \log Z_{\mathcal{M}_5}(\boldsymbol{a}) \approx &\log Z_{\mathcal{M}_4\times S^1}(\boldsymbol{a}) - 2\pi i \left(\sum D_l\right) \cdot \left(\sum p_l D_l\right) \left(H(\boldsymbol{a}) - \boldsymbol{a}^I \frac{\partial H(\boldsymbol{a})}{\partial \boldsymbol{a}^I} \right) \\ &+ \left(\sum p_l D_l\right) \cdot \left(\sum \mathfrak{n}_l^I D_l\right) \log \mathscr{K}_I(\boldsymbol{a}) + \frac{1}{2} \left(\sum p_l D_l\right) \cdot \left(\sum p_l D_l\right) \log \mathscr{F}(\boldsymbol{a}) \ , \end{aligned} \tag{7.70}$$

where $Z_{\mathcal{M}_4\times S^1}$ is given by (7.37) and $\mathscr{F}$, $\mathscr{K}$ are precisely the quantities defined in (6.34) and (6.35), respectively. Then, using the relation (7.68), as well as the substitution $H \to \varepsilon^I \partial_I \mathcal{F}$ for the extended topological twist, we recover the complete master formula (6.36) for the CB partition function on $\mathcal{M}_5$. We should note that the proposal (7.69) appears slightly different from the results above for $\mathbb{P}^2$ and $\mathbb{F}_0$, though all the formulas agree perfectly in the non-equivariant limit. For instance, for $\mathbb{P}^2$, the gluing (7.69) uses:

$$\gamma^{(l)}_{\text{new}} = \big(1 + p_1\tau_1 + p_2\tau_2,\, 1 - (p_2 + p_3)\tau_1 + p_2\tau_2,\, 1 + p_1\tau_1 - (p_1 + p_3)\tau_2\big) \;, \tag{7.71}$$

where $\mathfrak{p} = -(p_1 + p_2 + p_3)$, while previously we derived:

$$\gamma^{(l)} = \big(1,\, \mathfrak{p}\tau_1 + 1,\, \mathfrak{p}\tau_2 + 1\big) \;. \tag{7.72}$$

However, setting $p_1 = p_2 = 0$, the two expressions become identical. Similar comments hold true in general. It might be the case that the individual fluxes p_l (and $\mathfrak{n}_l$) have an intrinsic meaning on the Ω-deformed $\mathcal{M}_5$, which we did not explore. Our main motivation, here, was to provide a strong consistency check for our formulas for the fibering operator in the DW-twisted theory, hence our main interest was on the non-equivariant limit.

Chapter 8

Discussion and outlook

In part I of the thesis we studied the Coulomb branches of 4d $\mathcal{N} = 2$ and 5d $\mathcal{N} = 1$ rank-one theories through their Seiberg-Witten geometries. The novelty in our approach relies on the extensive use of the mathematical formalism of rational elliptic surfaces. Firstly, we were able to find new RG flows between the five-dimensional E_n SCFTs and four-dimensional Argyres-Douglas theories, realised by turning on certain Wilson lines along the circle direction. Some of these new flows have been inspired by the relation between 5d BPS quivers and the gauge/Painlevé correspondence [51–54]. However, not all 'flows' in the Sakai classification of q-Painlevé equations appear to have a physical realisation. It would be interesting, nevertheless, to study any possible relation between these remaining flows and 5d SCFTs.

The Mordell-Weil group of the SW geometry was shown to encode information about the $U(1)$ factors and the global form of the flavour symmetry, as well as the one-form symmetry. Our proofs relied on the Hori-Vafa mirrors for the toric Calabi-Yau singularities [156], without an obvious generalization to the non-toric cases. However, our results for the flavour symmetries of the non-toric E_n theories do agree with the known results from the literature. Perhaps even more challenging would be to extend this interpretation to theories whose Coulomb branches involve frozen singularities.

Thirdly, we have shown how modularity can simplify the computations in the low-energy effective field theory. Historically, modularity has been effectively used for the evaluation of the so-called 'u-plane integral', which we will comment on further below. In our work, we recover the classification of modular rational elliptic surfaces of [86], but from a perspective that gives more insight into the modular properties of the RES. Equipped with these tools, we then proposed a construction of BPS quivers directly from the singular fibers of the SW geometry, for the case when the latter is modular. Let us note, however, that we do not have a prescription for obtaining the superpotential directly from the SW geometry. This problem could have a solution for the toric cases, at least, where the mirror curve is known to be related to the brane-tiling construction [66–68]. Another open and very challenging question is finding the BPS spectrum of the theories considered in this work. The mutation algorithm provides an answer for the *complete* 4d $\mathcal{N} = 2$ theories [203,204], but the problem is much more involved for 5d quivers. An alternative approach to computing BPS spectra is from attractor indices, which are completely known (at least conjecturally) for 5d BPS quivers [74, 75, 266].

Another aspect that we have not touched upon in this work involves the automorphisms of the SW geometry. In particular, one can consider quotients of a rational elliptic surface $\mathcal{S}$ by subgroups $G \subset \text{Aut}(\mathcal{S})$, which turn out to be related to discrete symmetry gaugings [4, 5, 36, 42]. Moreover, there can exist SW geometries $\mathcal{S}$ and $\mathcal{S}/G$ corresponding to theories not related by a discrete gauging, for which such quotients have a clear interpretation at the level of the BPS quiver. Depending on the choice of G, these quotients are related to the Galois covers of [226], or to decompositions of the 5d BPS spectra into distinct copies of 4d spectra [71, 77, 267], at certain loci in the moduli space, as further discussed in [5].

Lastly, the theories having $F_\infty = I_0$ have not been considered in our work. These correspond to 6d theories compactified on a torus, and their SW curves are usually more involved compared to the 4d and 5d curves, due to the dependence on the complex structure of the torus. Nevertheless, our methods can be still used to study these theories, leading to certain proposals for 6d BPS quivers. Such aspects are considered in [4, 5].

Our initial motivation behind the study of SW geometry lies within the sphere of supersymmetric localization. It is well-known that the Donaldson-Witten partition function can be split into a 'Seiberg-Witten contribution' and the 'u-plane integral' [21, 95, 96]. The latter is a famously challenging computation, which, however, can be performed with the help of modularity, as previously alluded to.[61] In the context of five-dimensional theories, we have already mentioned in (1.38) that the full partition function should involve a similar integral over the whole Coulomb branch, which we leave for future work.

Part II of the thesis deals, instead, only with the 'integrand' of the full partition function,

[61] More recent work shows that the integral can be evaluated even without modularity [111].

which we computed in three different ways. Note that our methods apply to five-manifolds $\mathcal{M}_5$ which are principal $U(1)$ bundles over Kähler four-manifolds $\mathcal{M}_4$, and attempting generalisations to other five-manifolds would be a natural future direction. For the first approach, we studied ordinary 5d $\mathcal{N} = 1$ gauge theories on $\mathcal{M}_5$, in line with standard supersymmetric localization computations [115]. We computed, in particular, the one-loop determinant for a particle of spin (j_l, j_r), which serves as a building block for evaluating the Coulomb branch partition function. Then, for a 5d SCFT engineered at a CY threefold singularity in M-theory, the CB partition function is given by the product over all such particles arising from M2-branes wrapping holomorphic curves inside the CY threefold [23], which, are determined by the Gopakumar-Vafa invariants of the threefold.

The second approach was motivated by the $U(1)_{\mathrm{KK}}$ symmetry of the effective 4d $\mathcal{N} = 2$ theory obtained by compactifying the 5d SCFT on the circle. As such, we studied the low-energy effective action on the CB of the KK theory, which is controlled by the prepotential and the gravitational couplings. The advantage of this approach is that these quantities can be determined directly from the Seiberg-Witten geometry, without the need of an infinite sum over BPS states, as in the previous scenario. In both of these approaches, we assumed that the $\mathcal{M}_4$ base was simply-connected, as relaxing this assumption would lead to additional fermionic couplings in the low-energy effective action, whose effect would be interesting to consider. Furthermore, it would also be important to reconcile our approach with the Moore-Witten u-plane integral [96], which deals with a non-holomorphic integrand instead. We should also mention the condition that we needed to impose to consistently define a 5d theory on our curved backgrounds – the ‘spin/charge relation’ – which we hope to gain further insights on.

Finally, in the spirit of the fibering operator and of the $U(1)_{\mathrm{KK}}$ symmetry, we provided a generalisation of the Nekrasov conjecture for the partition function on a $\mathcal{M}_5$ manifold whose base is a toric Kähler manifold. This expression remains a conjecture due to the simple fact that the CB integration contour, as well as the precise sum over fluxes are not known. However, we hope to be able to shed some light on these aspects using similar methods to those developed in the context of 3d $\mathcal{N} = 2$ theories [89].

Appendix A

Seiberg-Witten curves

In this appendix, we list various Seiberg-Witten curves used throughout the main text.

A.1 SW Curves for the 4d $SU(2)$ gauge theories

The Weierstrass form of the four-dimensional $SU(2)$ SYM theories we use are given by [8, 268]:

$$
\begin{aligned}
N_f = 0 \ &: \ g_2(u) = \frac{4u^2}{3} - 4\Lambda^4 \ , \qquad g_3(u) = -\frac{8u^3}{27} + \frac{4}{3}u\Lambda^4 \ , \\
N_f = 1 \ &: \ g_2(u) = \frac{4u^2}{3} - 4m_1\Lambda^3 \ , \qquad g_3(u) = -\frac{8u^3}{27} + \frac{4}{3}m_1 u\Lambda^3 - \Lambda^6 \ , \\
N_f = 2 \ &: \ g_2(u) = \frac{4u^2}{3} - 4m_1 m_2\Lambda^2 + \Lambda^4 \ , \\
&\quad\ g_3(u) = -\frac{8u^3}{27} + \frac{4}{3}m_1 m_2 u\Lambda^2 - (m_1^2 + m_2^2)\Lambda^4 + \frac{2}{3}u\Lambda^4 \ , \\
N_f = 3 \ &: \ g_2(u) = \frac{4u^2}{3} - \frac{4u\Lambda^2}{3} - 4T_3\Lambda + T_2\Lambda^2 + \frac{\Lambda^4}{12} \ , \\
&\quad\ g_3(u) = -\frac{8u^3}{27} - \frac{5u^2\Lambda^2}{9} + \frac{u\Lambda}{9}\left(12T_3 + 6T_2\Lambda + \Lambda^3\right) \\
&\qquad\quad - T_4\Lambda^2 + \frac{1}{3}T_3\Lambda^3 - \frac{1}{12}T_2\Lambda^4 - \frac{1}{216}\Lambda^6 \ ,
\end{aligned}
\tag{A.1}
$$

where in the last line we introduce the $SO(6)$ Casimirs:

$$
T_2 = \sum_i^3 m_i^2 \ , \qquad T_4 = \sum_{i<j} m_i^2 m_j^2 \ , \qquad T_3 = \prod_i^3 m_i \ . \tag{A.2}
$$

These conventions are chosen such that the curves agree with the 4d Nekrasov partition function computations. These curves are isomorphic to:

$$\begin{aligned} N_f = 0 \ &: \quad \frac{\Lambda^2}{t} + \Lambda^2 t + x^2 - u = 0 \ , \\ N_f = 1 \ &: \quad \frac{\Lambda}{t}(x + m_1) + \Lambda^2 t + x^2 - u = 0 \ , \\ N_f = 2 \ &: \quad \frac{\Lambda}{t}(x + m_1) + \Lambda t(x + m_2) + x^2 - \widetilde{u} = 0 \ , \end{aligned} \tag{A.3}$$

where for $N_f = 2$ we have:

$$N_f = 2 \ : \quad \widetilde{u} = u - \frac{\Lambda^2}{2} \ . \tag{A.4}$$

The CB parameter $\widetilde{u}$ in the above notation breaks the $\mathbb{Z}_2$ symmetry but agrees with Nekrasov partition function considerations. These a-independent shifts do not change the low-energy effective action, as discussed in [158]. Finally, for the $N_f = 3$ theory, the curve:

$$\frac{1}{t}(x + \widetilde{m}_1)(x + \widetilde{m}_3) + \Lambda t(x + \widetilde{m}_2) + x^2 - \widetilde{u} = 0 \ , \tag{A.5}$$

has the same Weierstrass form as in (A.1), upon the identifications:

$$N_f = 3 \ : \qquad \widetilde{m}_i = m_i - \frac{\Lambda}{2} \ , \qquad \widetilde{u} = u - (m_1 + m_2 + m_3)\frac{\Lambda}{2} + \frac{\Lambda^2}{4} \ . \tag{A.6}$$

A.2 Seiberg-Witten curves for the E_n theories

In this section, we review the Seiberg-Witten curves for the non-toric (rank one) $D_{S^1}E_n$ theories, which are obtained as a limit of the E-string theory SW curve. The fully mass deformed curves were derived in [159, 160], and more recently reviewed in [161, 269]. In terms of the flavour characters χ_i, the E_8 curve can be written in Weierstrass form as:

$$\begin{aligned} &g_2(\widehat{U}, \chi) = \\ &\frac{\widehat{U}^4}{12} - \left(\frac{2}{3}\chi_1 - \frac{50}{3}\chi_8 + 1550\right)\widehat{U}^2 - (-70\chi_1 + 2\chi_3 - 12\chi_7 + 1840\chi_8 - 115010)\,\widehat{U} \\ &+ \frac{4}{3}\chi_1\chi_1 - \frac{8}{3}\chi_1\chi_8 - 1824\chi_1 + 112\chi_3 - 4\chi_2 + 4\chi_6 - 680\chi_7 + \frac{28}{3}\chi_8\chi_8 + 50744\chi_8 \\ &- 2399276 \ , \end{aligned} \tag{A.7}$$

and:

$$
\begin{aligned}
&g_3(\widehat{U},\chi) = \\
&-\frac{\widehat{U}^6}{216} + 4\widehat{U}^5 + \left(\frac{1}{18}\chi_1 + \frac{47}{18}\chi_8 - \frac{5177}{6}\right)\widehat{U}^4 \\
&+\left(-\frac{107}{6}\chi_1 + \frac{1}{6}\chi_3 + 3\chi_7 - \frac{1580}{3}\chi_8 + \frac{504215}{6}\right)\widehat{U}^3 + \Big(-\frac{2}{9}\chi_1\chi_1 - \frac{20}{9}\chi_1\chi_8 \\
&+\frac{5866}{3}\chi_1 - \frac{112}{3}\chi_3 + \frac{1}{3}\chi_2 + \frac{11}{3}\chi_6 - \frac{1450}{3}\chi_7 + \frac{196}{6}\chi_8\chi_8 + 39296\chi_8 - \frac{12673792}{3}\Big)\widehat{U}^2 \\
&+\Big(\frac{94}{3}\chi_1\chi_1 - \frac{2}{3}\chi_1\chi_3 + \frac{718}{3}\chi_1\chi_8 - \frac{270736}{3}\chi_1 - \frac{10}{3}\chi_3\chi_8 + 2630\chi_3 - 52\chi_2 + 4\chi_5 \\
&-416\chi_6 + 16\chi_7\chi_8 + 25880\chi_7 - \frac{7328}{3}\chi_8\chi_8 - \frac{3841382}{3}\chi_8 + 107263286\Big)\widehat{U} + \frac{8}{27}\chi_1\chi_1\chi_1 \\
&+\frac{28}{9}\chi_1\chi_1\chi_8 - 1065\chi_1\chi_1 + \frac{118}{3}\chi_1\chi_3 - \frac{4}{3}\chi_1\chi_2 + \frac{4}{3}\chi_1\chi_6 - \frac{8}{3}\chi_1\chi_7 - \frac{40}{9}\chi_1\chi_8\chi_8 \\
&-\frac{19264}{3}\chi_1\chi_8 + \frac{4521802}{3}\chi_1 - \chi_3\chi_3 + \frac{572}{3}\chi_3\chi_8 - 59482\chi_3 - \frac{20}{3}\chi_2\chi_8 + 1880\chi_2 + 4\chi_4 \\
&-232\chi_5 + \frac{8}{3}\chi_6\chi_8 + 11808\chi_6 - \frac{2740}{3}\chi_7\chi_8 - 460388\chi_7 + \frac{136}{27}\chi_8\chi_8\chi_8 + \frac{205492}{3}\chi_8\chi_8 \\
&+\frac{45856940}{3}\chi_8 - 1091057493\ .
\end{aligned}
\tag{A.8}
$$

The other $D_{S^1}E_n$ curves are recovered iteratively. Starting from the $D_{S^1}E_8$ curve, one should rescale the variables as:

$$
(\widehat{U}, x, y) \longrightarrow (\alpha\widehat{U}, \alpha^2 x, \alpha^3 x)\ , \tag{A.9}
$$

and the characters as:

$$
(\chi_1,\chi_2,\chi_3,\chi_4,\chi_5,\chi_6,\chi_7,\chi_8) \to (\alpha^2\chi_1,\alpha^4\chi_2,\alpha^3\chi_3,\alpha^6\chi_4,\alpha^5\chi_5,\alpha^4\chi_6,\alpha^3\chi_7,\alpha^2\chi_8)\ . \tag{A.10}
$$

Then, taking $\alpha \to \infty$ and setting $\chi_8 = 1$, one obtains the $D_{S^1}E_7$ curve, and similarly for the other $D_{S^1}E_n$ theories. Note that this statement is equivalent to decomposing E_n into $E_{n-1} \times U(1)$ and decoupling the $U(1)$ factor [160]. For the toric theories, we can also find the map between the $\widehat{U}$, χ variables used here and the U, λ, M_i variables used in the main text, by explicit comparison with the Weierstrass form of the 'toric' curves.

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