



Exact overlaps in the Lieb-Liniger model from coordinate Bethe ansatz

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ABSTRACT

In the paper [1], the authors developed a new method to compute the exact overlap formulas between integrable boundary states and on-shell Bethe states in integrable spin chains. This method utilizes the coordinate Bethe ansatz representation of wave functions and singularity property of the off-shell overlaps. In this paper, we use this new method to derive the formula for overlaps between the Lieb-Liniger Bethe states and the Bose-Einstein condensate (BEC) state. As a simple application this method, we obtained the overlaps between the Lieb-Liniger eigenstates and the free particle states with pair structure.

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1. Introduction

In the study of non-equilibrium statistical physics and high energy physics, in particular in the homogeneous quantum quench problem [2] and one-point function in the defect AdS/CFT correspondence [3–5] (see [6] for a review), the exact on-shell overlap is an important quantity. In the quantum quench framework, a system is prepared in some initial state and then is allowed to evolve in time with some Hamiltonian for which the initial state is not an eigenstate. Overlaps between the initial state and the eigenstates of the model are starting point to obtain analytical results for the post-quench time evolution [7]. This quantity is also related to exact g -function or boundary entropy [8,9] in integrable quantum field theory. Recently, people find that under some interesting set-up, the three-point function in AdS_5/CFT_4 can be viewed as an exact g -function in some integrable quantum field theory on the string world sheet [10].

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The first overlap formula of the on-shell Bethe states with some particular initial state was found in [11], it has a factorized form and is similar to Gaudin formula for the norm of the Bethe state [12]. People found that this factorizable property still survives in a large set of states-integrable boundary states [13] (or integrable states for short). In [13] integrable states are defined as the states which are annihilated by all odd local conserved charges for integrable lattice systems. In this paper, we also use this definition for the integrable continuum model.

In [14] the authors studied the Bose-Einstein condensate (BEC) to the Lieb-Liniger quench using the quench action method where the conjectured exact overlap formula for the BEC state with the Lieb-Liniger energy eigenstate is a starting point. This conjectured overlap formula was rigorously proved in [15], using the fact that the Lieb-Liniger model quantities can be obtained from a scaling limit of the XXZ spin chain [16] where a generalized Quantum Transfer Matrix (QTM) method can be applied [17,18].

Although this overlap formula is rather important, a direct proof is still missing. In a very recent paper [1], the author proposed a new method to directly calculate the overlap between on-shell Bethe states and integrable states in integrable spin chain models using the Coordinate Bethe Ansatz (CBA) formalism. This method is very simple and inspiring and can directly apply to the Lieb-Liniger model where the Bethe wave function is known. In fact, using coordinate Bethe ansatz to calculate the overlaps between integrable states and Bethe states was firstly proposed in [3]. Coordinate Bethe ansatz was also applied to calculating the correlation functions in non-equilibrium problems [19,20].

In this paper, we use this newly proposed method to derive the overlap for the BEC state with the Lieb-Liniger energy eigenstate, which agrees with the early result. We also obtained overlap formulas for a special kind of N -particle states with the Lieb-Liniger Bethe states. For $N = 2$, the state is integrable and we obtained the exact overlap. For $N \geq 4$, the states are non-integrable, we only give the overlap in the large volume (length) limit by focusing on the singular part of the formal (off-shell) overlap. Overlaps of the Bethe states with non-integrable states have been studied previously [21,22]. Such kind formulas are of great importance since overlap formulas serve as a starting point to the study of the related quench problem. The remaining part of this paper is organized as follows: In section 2, we review some basic facts about the Lieb-Liniger model and fix some notations. In section 3, we study the overlap formula using coordinate Bethe ansatz method started with the one- and two-particle cases which serve as warm-up examples and give hints to deal with generic cases. In the next subsection, we derive the overlap formula of the BEC state and the generic parity symmetric on-shell Bethe state. In section 4, we obtained some new overlap formulas using the same techniques. Finally, we conclude in section 5 and the proof of the vanishing of the overlap of generic on-shell Bethe states with the BEC state is given in appendix A.

2. The Lieb-Liniger model

We consider one-dimensional Boson gas on a ring of circumference L with δ -function repulsive potential and impose periodic boundary condition. This is the famous Lieb-Liniger model, whose Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j>k} \delta(x_j - x_k). \quad (2.1)$$

We will set $2m = \hbar = 1$ in the following for simplicity.

2.1. The Bethe equations

We can write down the energy eigenstates of the Lieb-Liniger Hamiltonian in the coordinate space as

$$\langle \mathbf{x}_N | \lambda_N \rangle = \frac{c^{N/2}}{\sqrt{N!}} \sum_{\sigma \in S_N} A(\mathbf{x}_N | \sigma \lambda_N) \exp\{i \sum_{k=1}^N \lambda_{\sigma_k} x_k\}, \quad (2.2)$$

where

$$A(\mathbf{x}_N | \sigma \lambda_N) = \prod_{j>k} \left[1 - \frac{ic \operatorname{sign}(x_j - x_k)}{\lambda_{\sigma_j} - \lambda_{\sigma_k}} \right], \quad (2.3)$$

and $|\mathbf{x}_N\rangle$ and $|\lambda_N\rangle$ is the shorthand notations of $|x_1, x_2, \dots, x_N\rangle$ and $|\lambda_1, \lambda_2, \dots, \lambda_N\rangle$ respectively. In the above expressions, we also have used the general notation

$$\sigma \lambda_N = \{\lambda_{\sigma_1}, \lambda_{\sigma_2}, \dots, \lambda_{\sigma_N}\}. \quad (2.4)$$

The requirement of the wave function $\langle \mathbf{x}_N | \lambda_N \rangle$ should be periodic in each of its arguments x_j results in the Bethe equations

$$e^{i\lambda_j L} = - \prod_{k=1}^N \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}, \quad j = 1, 2, \dots, N. \quad (2.5)$$

The corresponding energy eigenvalue is

$$E = \sum_{j=1}^N \lambda_j^2. \quad (2.6)$$

When the Bethe equations (2.5) are not satisfied, the wave function eq. (2.2) still defines a state which we call it off-shell. On the contrary, when the Bethe equations are instead satisfied, we call the state on-shell.

The Lieb-Liniger model has infinite many conserved charges Q_1, Q_2, Q_3, \dots . We can identify

$$Q_1 = P = -i \sum_{j=1}^N \frac{\partial}{\partial x_j}, \quad Q_2 = H = - \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j>k} \delta(x_j - x_k). \quad (2.7)$$

The higher charges can have very complicated expressions in the coordinate space representation [23]. For example:

$$Q_3 = i \sum_{j=1}^N \frac{\partial^3}{\partial x_j^3} - 3ic \sum_{j>k} \delta(x_j - x_k) \left(\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_k} \right). \quad (2.8)$$

However their actions on the Bethe states $|\lambda_N\rangle$ are extremely simple:

$$Q_n |\lambda_1, \lambda_2, \dots, \lambda_N\rangle = \left(\sum_{j=1}^N \lambda_j^n \right) |\lambda_1, \lambda_2, \dots, \lambda_N\rangle, \quad n = 1, 2, 3, \dots \quad (2.9)$$

2.2. The parity symmetric Bethe state

The norm of the Bethe states is given by the Gaudin formula [12]

$$\langle \lambda_N | \lambda_N \rangle = \int d^N \mathbf{x}_N \langle \lambda_N | \mathbf{x}_N \rangle \langle \mathbf{x}_N | \lambda_N \rangle = c^N \prod_{j \neq k} f(\lambda_j, \lambda_k) \det G, \quad (2.10)$$

where G is an $N \times N$ matrix whose elements are

$$G_{jk} = \delta_{jk} [L + \sum_{l=1}^N \varphi(\lambda_j - \lambda_l)] - \varphi(\lambda_j - \lambda_k), \quad (2.11)$$

and the function $f(\lambda_1, \lambda_2)$ is defined as

$$f(\lambda_1, \lambda_2) = \frac{\lambda_1 - \lambda_2 + ic}{\lambda_1 - \lambda_2}. \quad (2.12)$$

The function $\varphi(\lambda)$ is given by

$$\varphi(\lambda) = \frac{2c}{\lambda^2 + c^2}. \quad (2.13)$$

In this paper, we will consider the overlaps of the Lieb-Liniger energy eigenstates $|\lambda_N\rangle$ with both integrable and non-integrable states. As mentioned in the introduction, a state $|\Psi\rangle$ is called an integrable state when [13]

$$Q_{2k+1} |\Psi\rangle = 0, \quad k = 0, 1, 2, \dots \quad (2.14)$$

Integrable states can only have non-vanishing overlaps with Bethe states with parity symmetry:

$$\{\lambda_N\} = \{\lambda_1^+, -\lambda_1^+, \dots, \lambda_{N/2}^+, -\lambda_{N/2}^+\} \equiv \{\lambda_{N/2}^+, -\lambda_{N/2}^+\}, \quad (2.15)$$

where we have assumed N is an even number.

The norm of the on-shell Bethe states having the above pair structure can be factorized further

$$\| |\lambda_{N/2}^+, -\lambda_{N/2}^+\rangle \|^2 = c^N \prod_{j=1}^{N/2} f(\lambda_j^+, -\lambda_j^+) f(-\lambda_j^+, \lambda_j^+) \prod_{1 \leq j < k \leq N/2} [\bar{f}(\lambda_j^+, \lambda_k^+)]^2 \det G^+ \det G^-, \quad (2.16)$$

where¹

$$\bar{f}(\lambda_1, \lambda_2) = f(\lambda_1, \lambda_2) f(\lambda_1, -\lambda_2) f(-\lambda_1, \lambda_2) f(-\lambda_1, -\lambda_2), \quad (2.17)$$

and G^\pm are $\frac{N}{2} \times \frac{N}{2}$ matrices with elements

$$G_{jk}^\pm = \delta_{jk} [L + \sum_{l=1}^{N/2} \varphi^\pm(\lambda_j^+, \lambda_l^+)] - \varphi^\pm(\lambda_j^+, \lambda_k^+), \quad (2.18)$$

with

$$\varphi^\pm(\lambda_1, \lambda_2) = \varphi(\lambda_1 - \lambda_2) \pm \varphi(\lambda_1 + \lambda_2). \quad (2.19)$$

¹ The function $\bar{f}(\lambda_1, \lambda_2)$ is symmetric in our case: $\bar{f}(\lambda_1, \lambda_2) = \bar{f}(\lambda_2, \lambda_1)$.

For future use, we introduce the variables

$$a(\lambda) = e^{i\lambda L}, \quad a_j \equiv a(\lambda_j), \quad j = 1, 2, \dots, N. \quad (2.20)$$

Thus the Bethe equations (2.5) can be written as

$$a_j = \prod_{k \neq j} \frac{f(\lambda_j, \lambda_k)}{f(\lambda_k, \lambda_j)}. \quad (2.21)$$

3. Overlap between the Bethe state and the BEC state

In this section, we will consider the overlap between the Bethe state and the BEC state $|BEC\rangle$, whose wave function is constant: $\langle x_N | BEC \rangle = \frac{1}{L^{N/2}}$.

3.1. One-particle states

As a warm-up, let's first consider the one-particle states

$$\begin{aligned} \langle BEC | \lambda_1 \rangle &= \int_0^L dx_1 \langle BEC | x_1 \rangle \langle x_1 | \lambda_1 \rangle = \sqrt{\frac{c}{L}} \int_0^L e^{i\lambda_1 x_1} dx_1 \\ &= \sqrt{\frac{c}{L}} \frac{e^{i\lambda_1 L} - 1}{i\lambda_1} = \sqrt{\frac{c}{L}} \frac{a_1 - 1}{i\lambda_1} \quad (\lambda \neq 0), \end{aligned} \quad (3.1)$$

where the definition eq. (2.20) has been used. For on-shell states, which means the Bethe equation $a_1 = e^{i\lambda_1 L} = 1$ is satisfied, the above overlap is non-vanishing only when $\lambda = 0$. The integral is trivially integrated to give the result $\langle BEC | \lambda_1 = 0 \rangle = \sqrt{cL}$.

We can obtain the same result using a limiting procedure:

$$\langle BEC | \lambda_1 = 0 \rangle = \lim_{\lambda_1 \rightarrow 0} \sqrt{\frac{c}{L}} \frac{e^{i\lambda_1 L} - 1}{i\lambda_1} = \sqrt{cL}. \quad (3.2)$$

3.2. Two-particle states

We now consider the two-particle case. For the readers' convenience and to gain more experiences for computing overlap of Bethe state with multi-particle states, we work out the details.

Firstly, we write

$$\begin{aligned} \langle BEC | \lambda_1, \lambda_2 \rangle &= \frac{c}{L\sqrt{2!}} \left\{ \int_0^L dx_1 \int_0^L dx_2 e^{i\lambda_1 x_1 + i\lambda_2 x_2} \left[1 - \frac{ic \operatorname{sign}(x_2 - x_1)}{\lambda_2 - \lambda_1} \right] \right. \\ &\quad \left. + \int_0^L dx_1 \int_0^L dx_2 e^{i\lambda_2 x_1 + i\lambda_1 x_2} \left[1 - \frac{ic \operatorname{sign}(x_2 - x_1)}{\lambda_1 - \lambda_2} \right] \right\} \\ &= \frac{c}{L\sqrt{2!}} \left\{ f(\lambda_1, \lambda_2) \int_0^L dx_2 \int_0^{x_2} dx_1 e^{i\lambda_1 x_1 + i\lambda_2 x_2} + f(\lambda_2, \lambda_1) \int_0^L dx_1 \int_0^{x_1} dx_2 e^{i\lambda_1 x_1 + i\lambda_2 x_2} \right. \\ &\quad \left. + f(\lambda_2, \lambda_1) \int_0^L dx_2 \int_0^{x_2} dx_1 e^{i\lambda_2 x_1 + i\lambda_1 x_2} + f(\lambda_1, \lambda_2) \int_0^L dx_1 \int_0^{x_1} dx_2 e^{i\lambda_2 x_1 + i\lambda_1 x_2} \right\}. \end{aligned} \quad (3.3)$$

In order to evaluate the above integral, it's useful to introduce the basic integral in the region $0 < x_1 < x_2 < L$:

$$B_2(\lambda_1, \lambda_2; a_1, a_2) = \int_0^L dx_2 \int_0^{x_2} dx_1 e^{i\lambda_1 x_1 + i\lambda_2 x_2}, \quad (3.4)$$

where the a variables dependence of the function B_2 is obtained only after the integral is worked out and the definition eq. (2.20) is substituted. Other integral can be obtained by permutating the arguments of this function.

Note that the last two lines in the curly bracket in eq. (3.3) give the same result. It's easy to see that this property also holds in the multi-particle case. Thus the two-particle formal (off-shell) overlap can be written as

$$\langle BEC | \lambda_1, \lambda_2 \rangle = \frac{2!c}{L\sqrt{2!}} [f(\lambda_1, \lambda_2) B_2(\lambda_1, \lambda_2; a_1, a_2) + f(\lambda_2, \lambda_1) B_2(\lambda_2, \lambda_1; a_2, a_1)], \quad (3.5)$$

where

$$B_2(\lambda_1, \lambda_2; a_1, a_2) = -\frac{a_1 a_2 - 1}{\lambda_1(\lambda_1 + \lambda_2)} + \frac{a_2 - 1}{\lambda_1 \lambda_2}. \quad (3.6)$$

It's easy to check that when the Bethe state is on-shell, i.e. when we do the following replacement in above two-particle formal overlap formula eq. (3.5)

$$a_1 = \frac{f(\lambda_1, \lambda_2)}{f(\lambda_2, \lambda_1)}, \quad a_2 = \frac{f(\lambda_2, \lambda_1)}{f(\lambda_1, \lambda_2)}, \quad (3.7)$$

we get identically zero.

To obtain non-vanishing on-shell overlaps, we need to consider the parity-invariant states with $\lambda_2 = -\lambda_1$. The formal overlap eq. (3.5) has a pole at $\lambda_2 = -\lambda_1$. Around this pole, we have

$$\langle BEC | \lambda_1, \lambda_2 \rangle \sim \frac{a_1 a_2 - 1}{i(\lambda_1 + \lambda_2)} \frac{c\sqrt{2!}}{L} \left[\frac{f(\lambda_1, \lambda_2)}{i\lambda_1} + \frac{f(\lambda_2, \lambda_1)}{i\lambda_2} \right]. \quad (3.8)$$

Note that when the state is on-shell, we have $a_1 a_2 = 1$. The singular part can have a proper limit. The non-vanishing on-shell overlap only comes from this singular part. In other words, the exact parity-invariant two-particle overlap can be obtained by taking the following limiting procedure

$$\begin{aligned} \langle BEC | \lambda_1, -\lambda_1 \rangle &= \lim_{\lambda_2 \rightarrow -\lambda_1} \frac{e^{i(\lambda_1 + \lambda_2)L} - 1}{i(\lambda_1 + \lambda_2)} \frac{c\sqrt{2!}}{L} \left[\frac{f(\lambda_1, \lambda_2)}{i\lambda_1} + \frac{f(\lambda_2, \lambda_1)}{i\lambda_2} \right] \\ &= c\sqrt{2!} \left[\frac{f(\lambda_1, -\lambda_1)}{i\lambda_1} + \frac{f(-\lambda_1, \lambda_1)}{-i\lambda_1} \right] \\ &= \sqrt{2} \frac{c^2}{\lambda_1^2}. \end{aligned} \quad (3.9)$$

3.3. Multi-particle states

Having the above explicit calculation in mind. Now we turn to the multi-particle case. It's obvious that we have the following relation

$$\begin{aligned} &\int_0^L dx_N \int_0^{x_N} dx_{N-1} \cdots \int_0^{x_2} dx_1 A(x_1, \dots, x_N | \lambda_1, \dots, \lambda_N) \\ &= B_N(\lambda_1, \dots, \lambda_N; a_1, \dots, a_N) \prod_{j < k} f(\lambda_j, \lambda_k), \end{aligned} \quad (3.10)$$

where we have defined the basic integral in the region $0 < x_1 < x_2 < \dots < x_N < L$ as before

$$B_N(\lambda_1, \dots, \lambda_N; a_1, \dots, a_N) = \int_0^L dx_N \int_0^{x_N} dx_{N-1} \cdots \int_0^{x_2} dx_1 e^{i(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_N x_N)}. \quad (3.11)$$

For lower N , B_N can be integrated out directly. For example

$$B_3 = \frac{a_1 a_2 a_3 - 1}{i\lambda_1 i(\lambda_1 + \lambda_2) i(\lambda_1 + \lambda_2 + \lambda_3)} - \frac{a_2 a_3 - 1}{i\lambda_1 i\lambda_2 i(\lambda_2 + \lambda_3)} + \frac{a_3 - 1}{i\lambda_2 i\lambda_3 i(\lambda_1 + \lambda_2)}. \quad (3.12)$$

For general N , it's not easy to directly work out this integral. However, we have the following recursion relation

$$\frac{\partial B_N(\lambda_N; \mathbf{a}_N)}{\partial L} = a_N B_{N-1}(\lambda_{N-1}; \mathbf{a}_{N-1}), \quad (3.13)$$

where we view $B_N(\lambda_N; \mathbf{a}_N)$ as a function of the system size L inexplicitly as the L -dependence is encoded in the definition of the variables a_i through eq. (2.20). We also have the initial conditions for each N

$$B_N(\lambda_N; \mathbf{a}_N)|_{L=0} = 0, \quad N = 1, 2, 3, \dots. \quad (3.14)$$

For example, from

$$B_1(\lambda_1; a_1) = \frac{a_1 - 1}{i\lambda_1}, \quad (3.15)$$

we can get B_2 as

$$\begin{aligned} B_2(\lambda_1, \lambda_2; a_1, a_2) &= \int_0^L a_2 B_1(\lambda_1; a_1)|_{L \rightarrow L'} dL' = \frac{1}{i\lambda_1} \int_0^L [e^{i(\lambda_1 + \lambda_2)L'} - e^{i\lambda_2 L'}] dL' \\ &= \frac{e^{i(\lambda_1 + \lambda_2)L} - 1}{i\lambda_1 i(\lambda_1 + \lambda_2)} - \frac{e^{i\lambda_2 L} - 1}{i\lambda_1 i\lambda_2} = \frac{a_1 a_2 - 1}{i\lambda_1 i(\lambda_1 + \lambda_2)} - \frac{a_2 - 1}{i\lambda_1 i\lambda_2}, \end{aligned} \quad (3.16)$$

which gives the same result with eq. (3.6). In a similar way, we can reproduce the result of eq. (3.12).

In the above calculation, it is not hard to figure out the pattern. Using the recursion relation eq. (3.13) and the initial condition eq. (3.13) repeatedly, we obtain the general expressions for B_N .

$$B_N(\lambda_N; \mathbf{a}_N) = \sum_{j=1}^N (-1)^{j+1} \frac{\prod_{k=j}^N a_k - 1}{(\prod_{k=j}^N \sum_{m=j}^k (i\lambda_m)) (\prod_{k=1}^{j-1} \sum_{m=k}^{j-1} (i\lambda_m))} \quad (3.17)$$

or

$$\begin{aligned} B_N(\lambda_N; \mathbf{a}_N) &= \sum_{j=0}^N (-1)^j \frac{\prod_{k=j+1}^N a_k}{(\prod_{k=j+1}^N \sum_{m=j+1}^k (i\lambda_m)) (\prod_{k=1}^j \sum_{m=k}^j (i\lambda_m))} \\ &= \sum_{j=0}^N B_{N,j}(\lambda_N; \mathbf{a}_N), \end{aligned} \quad (3.18)$$

where we define

$$B_{N,j}(\lambda_N; \mathbf{a}_N) = (-1)^j \frac{\prod_{k=j+1}^N a_k}{(\prod_{k=j+1}^N \sum_{m=j+1}^k (i\lambda_m)) (\prod_{k=1}^j \sum_{m=k}^j (i\lambda_m))}. \quad (3.19)$$

These two formulas for B_N in eq. (3.17) and eq. (3.18) have slight differences but are equivalent to each other. However, we found the latter is more useful for our convenience.

The N -particle formal overlap can be written as

$$\begin{aligned} \langle BEC | \lambda_N \rangle &= \int d^N \mathbf{x}_N \langle BEC | \mathbf{x}_N \rangle \langle \mathbf{x}_N | \lambda_N \rangle \\ &= \frac{c^{N/2}}{L^{N/2} \sqrt{N!}} \sum_{\sigma \in S_N} \int d^N \mathbf{x}_N A(\mathbf{x}_N | \sigma \lambda_N) \exp\{i \sum_{k=1}^N \lambda_{\sigma_k} x_k\} \\ &= \frac{c^{N/2} N!}{L^{N/2} \sqrt{N!}} \int d^N \mathbf{x}_N A(\mathbf{x}_N | \lambda_N) \exp\{i \sum_{k=1}^N \lambda_k x_k\} \\ &= \sqrt{\frac{N! c^N}{L^N}} \sum_{\sigma \in S_N} B_N(\lambda_{\sigma_1}, \dots, \lambda_{\sigma_N}; a_{\sigma_1}, \dots, a_{\sigma_N}) \prod_{j < k} f(\lambda_{\sigma_j}, \lambda_{\sigma_k}) \\ &= \sqrt{\frac{N! c^N}{L^N}} \mathcal{S}_N(\lambda_N; \mathbf{a}_N). \end{aligned} \quad (3.20)$$

In the above expression, up to some constant factor, we have found the expression for N -particle formal overlap

$$\mathcal{S}_N(\lambda_N; \mathbf{a}_N) = \sum_{\sigma \in S_N} B_N(\lambda_{\sigma_1}, \dots, \lambda_{\sigma_N}; a_{\sigma_1}, \dots, a_{\sigma_N}) \prod_{j < k} f(\lambda_{\sigma_j}, \lambda_{\sigma_k}). \quad (3.21)$$

In [14], it was argued that the BEC state is an integrable state. Hence for generic (non-parity-symmetric) on-shell Bethe state, $\mathcal{S}_N(\lambda_N; \mathbf{a}_N)$ vanishes. Only for the parity symmetric on-shell states $|\lambda_{N/2}^+, -\lambda_{N/2}^+\rangle$, we have finite overlaps. See Appendix A for a direct proof of this statement.

3.3.1. Determining the singular part

In order to find out the non-vanishing overlap of on-shell parity symmetric Bethe states with the BEC state, we only need to figure out the singular part of $\mathcal{S}_N(\lambda_N; \mathbf{a}_N)$ and taking the limiting procedure.

Firstly, we must try to find out the residue

$$\text{Res}_{\lambda_{m+1} \rightarrow -\lambda_m} B_N(\lambda_N; \mathbf{a}_N). \quad (3.22)$$

It's easy to find that there are two terms $B_{N,m-1}$ and $B_{N,m+1}$ contribute

$$\begin{aligned} &\frac{(-1)^{m-1} \prod_{k=m}^N a_k}{(\prod_{k=m}^N \sum_{l=m}^k (i\lambda_l)) (\prod_{k=1}^{m-1} \sum_{l=k}^{m-1} (i\lambda_l))} + \frac{(-1)^{m+1} \prod_{k=m+2}^N a_k}{(\prod_{k=m+2}^N \sum_{l=m+2}^k (i\lambda_l)) (\prod_{k=1}^{m+1} \sum_{l=k}^{m+1} (i\lambda_l))} \\ &= \frac{(-1)^{m-1} a_m a_{m+1} \prod_{k=m+2}^N a_k}{i\lambda_m i(\lambda_m + \lambda_{m+1}) (\prod_{k=m+2}^N \sum_{l=m+2}^k (i\lambda_l)) (\prod_{k=1}^{m-1} \sum_{l=k}^{m-1} (i\lambda_l))} \\ &+ \frac{(-1)^{m-1} \prod_{k=m+2}^N a_k}{i\lambda_{m+1} i(\lambda_m + \lambda_{m+1}) (\prod_{k=m+2}^N \sum_{l=m+2}^k (i\lambda_l)) (\prod_{k=1}^{m-1} \sum_{l=k}^{m-1} (i\lambda_l))}. \end{aligned} \quad (3.23)$$

Then the singular part of B_N is

$$B_N(\lambda_N; \mathbf{a}_N) \sim \frac{a_m a_{m+1} - 1}{i(\lambda_m + \lambda_{m+1})} \frac{1}{i\lambda_m} \frac{(-1)^{m-1} \prod_{k=m+2}^N a_k}{(\prod_{k=m+2}^N \sum_{l=m+2}^k (i\lambda_l)) (\prod_{k=1}^{m-1} \sum_{l=k}^{m-1} (i\lambda_l))}. \quad (3.24)$$

The relation eq. (3.24) can be rewritten as

$$B_N(\{\lambda_j\}; \{a_j\} | j \in \{1, \dots, N\}) \sim \frac{a_m a_{m+1} - 1}{i(\lambda_m + \lambda_{m+1})} \frac{1}{i\lambda_m} \times B_{N-2, m-1}(\{\lambda_j\}; \{a_j\} | j \in \{1, \dots, m, m+1, \dots, N\}). \quad (3.25)$$

After the summation of all permutations that make particles m and $m+1$ in neighbouring position, we find the singularity of the formal overlap at $\lambda_{m+1} = -\lambda_m$ is

$$S_N \sim \frac{a_m a_{m+1} - 1}{i(\lambda_m + \lambda_{m+1})} F(\lambda_m) \prod_{\substack{j=1 \\ j \neq m, m+1}}^N f(\lambda_j, \lambda_m) f(\lambda_j, -\lambda_m) S_{N-2}^{\text{mod}}(m, m+1), \quad (3.26)$$

where $S_{N-2}^{\text{mod}}(m, m+1)$ is the formal overlap for $N-2$ particles which does not include particles m and $m+1$, and is evaluated with the modified a-variables²:

$$a_j^{\text{mod}} = \frac{f(\lambda_m, \lambda_j)}{f(\lambda_j, \lambda_m)} \frac{f(-\lambda_m, \lambda_j)}{f(\lambda_j, -\lambda_m)} a_j. \quad (3.27)$$

The function $F(\lambda)$ can be found as

$$F(\lambda) = \frac{f(\lambda, -\lambda)}{i\lambda} + \frac{f(-\lambda, \lambda)}{-i\lambda} = \frac{c}{\lambda^2}. \quad (3.28)$$

3.3.2. Taking the limiting procedure

The exact on-shell parity symmetric Bethe state with the BEC state overlap is obtained only after the following limiting procedure is taken

$$\lambda_{2j-1} \rightarrow \lambda_j^+, \quad \lambda_{2j} \rightarrow -\lambda_j^+, \quad j = 1, 2, \dots, N. \quad (3.29)$$

Note that

$$\begin{aligned} \lim_{\substack{\lambda_{2j-1} \rightarrow \lambda_j^+ \\ \lambda_{2j} \rightarrow -\lambda_j^+}} \frac{a_{2j-1} a_{2j} - 1}{i(\lambda_{2j-1} + \lambda_{2j})} &= \lim_{\lambda \rightarrow \lambda_j^+} \frac{a(\lambda) a(-\lambda_j^+) - 1}{i(\lambda - \lambda_j^+)} \\ &= \lim_{\lambda \rightarrow \lambda_j^+} \frac{a'(\lambda) a(-\lambda_j^+) a(\lambda)}{i a(\lambda)} = -i \frac{d}{d\lambda} \log(a(\lambda)) \Big|_{\lambda=\lambda_j^+}. \end{aligned} \quad (3.30)$$

Then it's useful to introduce new variables

$$m(\lambda) = -i \frac{d}{d\lambda} \log(a(\lambda)), \quad m_j^+ \equiv m(\lambda_j^+), \quad j = 1, 2, \dots, \frac{N}{2}. \quad (3.31)$$

We denote $D(\lambda_{N/2}^+, \mathbf{m}_{N/2}^+)$ as the limit of $S_N(\lambda_N; \mathbf{a}_N)$ describe in eq. (3.29).

It's easy to show that the function $D(\lambda_{N/2}^+, \mathbf{m}_{N/2}^+)$ satisfies the following recursion relation

$$\frac{\partial D(\lambda_{N/2}^+, \mathbf{m}_{N/2}^+)}{\partial m_j^+} = F(\lambda_j^+) \prod_{k=1, k \neq j}^{N/2} \bar{f}(\lambda_k^+, \lambda_j^+) D(\lambda_{N/2-1}^+, \mathbf{m}_{N/2-1}^{+, \text{mod}}), \quad (3.32)$$

where the modification rule for the m-parameters is

$$m^{\text{mod}}(\lambda) = -i \frac{d}{d\lambda} \log(a^{\text{mod}}(\lambda)) = m(\lambda) + \varphi^+(\lambda, \lambda_j^+), \quad (3.33)$$

and $D(\lambda_{N/2-1}^+, \mathbf{m}_{N/2-1}^{+, \text{mod}})$ is understood as the j -th variables have removed from its arguments. The solution of the recursion relation eq. (3.32) can be found as [1]

$$D(\lambda_{N/2}^+, \mathbf{m}_{N/2}^+) = \prod_{j=1}^{N/2} F(\lambda_j^+) \prod_{1 \leq k < j \leq N/2} \bar{f}(\lambda_k^+, \lambda_j^+) \det G^+. \quad (3.34)$$

² Here, the definition of the modified a-variables is slightly different from the definition given in [1].

Then the exact on-shell parity symmetric Bethe state with the BEC state overlap is

$$\frac{|\langle BEC | \lambda_{N/2}^+, -\lambda_{N/2}^+ \rangle|}{\| |\lambda_{N/2}^+, -\lambda_{N/2}^+ \rangle \|} = \sqrt{\frac{N!}{L^N}} \prod_{j=1}^{N/2} \frac{F(\lambda_j^+)}{\sqrt{f(\lambda_j^+, -\lambda_j^+) f(-\lambda_j^+, \lambda_j^+)}} \sqrt{\frac{\det G^+}{\det G^-}}. \quad (3.35)$$

After the substitution of the expression eq. (3.28) and eq. (2.12) into eq. (3.35), we obtained the final result

$$\frac{|\langle BEC | \lambda_{N/2}^+, -\lambda_{N/2}^+ \rangle|}{\| |\lambda_{N/2}^+, -\lambda_{N/2}^+ \rangle \|} = \frac{\sqrt{N!(cL)^{-N}}}{\prod_{j=1}^{N/2} \frac{\lambda_j^+}{c} \sqrt{\frac{\lambda_j^{+2}}{c^2} + \frac{1}{4}}} \sqrt{\frac{\det G^+}{\det G^-}}. \quad (3.36)$$

This formula agrees with the earlier results [14,15].

4. Overlap between the Bethe state and free particles state with pair structure

As an application of our formula eq. (3.18) obtained in the last section, in this section we consider the overlap between the Bethe state and a special kind of free particle states $|\Psi_{\mathbf{k}_N}\rangle$. The momenta of the particles are paired up leading to zero total momentum of these states. The wave function of $|\Psi_{\mathbf{k}_N}\rangle$ can be written as

$$\langle \mathbf{x}_N | \Psi_{\mathbf{k}_N} \rangle = C_N \sum_{\sigma \in S_N} e^{i\mathbf{k}_N \cdot \sigma \mathbf{x}_N} = C_N \sum_{\sigma \in S_N} e^{i\sigma \mathbf{k}_N \cdot \mathbf{x}_N} = C_N \sum_{\sigma \in S_N} e^{i \sum_{j=1}^N k_{\sigma_j} x_j}, \quad (4.1)$$

where

$$\{\mathbf{k}_N\} = \{k, -k, \dots, k, -k\} \equiv \{\mathbf{k}_{N/2}, -\mathbf{k}_{N/2}\}, \quad (4.2)$$

and C_N is the normalization constant. In the following part, we will denote $|\Psi_{\mathbf{k}_N}\rangle$ as $|\Psi_{\mathbf{k}_{N/2}, -\mathbf{k}_{N/2}}\rangle$ to emphasize its pair structure.

Let's begin by two-particle state

$$\langle x_1, x_2 | \Psi_{k, -k} \rangle = C_2 (e^{ikx_1 - ikx_2} + e^{ikx_2 - ikx_1}). \quad (4.3)$$

The normalization constant is k -dependent: $C_2 = 1/\sqrt{2(L^2 + \frac{\sin^2(kL)}{k^2})}$. When $|\Psi_{\mathbf{k}_{N/2}, -\mathbf{k}_{N/2}}\rangle$ is on-shell, i.e. we impose periodic condition to these wave function eq. (4.1), which means $e^{ikL} = 1$. Then $C_2 = 1/\sqrt{2L^2}$.

Then the two-particle overlap can be computed directly

$$\begin{aligned} \langle \Psi_{k, -k} | \lambda_1, \lambda_2 \rangle &= \frac{2c}{\sqrt{2!} \sqrt{2L^2}} \left\{ f(\lambda_1, \lambda_2) \int_0^L dx_2 \int_0^{x_2} dx_1 (e^{i(\lambda_1+k)x_1 + i(\lambda_2-k)x_2} + e^{i(\lambda_1-k)x_1 + i(\lambda_2+k)x_2}) \right. \\ &\quad \left. + f(\lambda_2, \lambda_1) \int_0^L dx_1 \int_0^{x_1} dx_2 (e^{i(\lambda_1+k)x_1 + i(\lambda_2-k)x_2} + e^{i(\lambda_1-k)x_1 + i(\lambda_2+k)x_2}) \right\} \\ &= \frac{c}{L} \{ f(\lambda_1, \lambda_2) [B_2(\lambda_1+k, \lambda_2-k; a_1, a_2) + B_2(\lambda_1-k, \lambda_2+k; a_1, a_2)] + (\lambda_1 \leftrightarrow \lambda_2, a_1 \leftrightarrow a_2) \}, \end{aligned} \quad (4.4)$$

where we have assumed $|\Psi_{k, -k}\rangle$ is on-shell. It's then easily to check that when the Bethe state is also on-shell, i.e. when we substitute the Bethe equations (2.5) into eq. (4.4), we obtain zero. This implies that $|\Psi_{k, -k}\rangle$ is an integrable state.³

The non-vanishing overlap can be obtained only for Bethe states with parity symmetry as before. We only need to figure out the singular pieces in the formal overlap eq. (4.4) and taking the limiting procedure. Around the pole $\lambda_2 = -\lambda_1$, we have

$$\begin{aligned} \langle \Psi_{k, -k} | \lambda_1, \lambda_2 \rangle &\sim \frac{a_1 a_2 - 1}{i(\lambda_1 + \lambda_2)} \frac{c}{L} \left\{ f(\lambda_1, \lambda_2) \left[\frac{1}{i(\lambda_1 + k)} + \frac{1}{i(\lambda_1 - k)} \right] \right. \\ &\quad \left. + f(\lambda_2, \lambda_1) \left[\frac{1}{i(\lambda_2 + k)} + \frac{1}{i(\lambda_2 - k)} \right] \right\} \\ &= \frac{a_1 a_2 - 1}{i(\lambda_1 + \lambda_2)} \frac{c}{L} \left[f(\lambda_1, \lambda_2) \frac{-2i\lambda_1}{\lambda_1^2 - k^2} + f(\lambda_2, \lambda_1) \frac{-2i\lambda_2}{\lambda_2^2 - k^2} \right]. \end{aligned} \quad (4.5)$$

Then the two-particle overlap is

³ It's easy to check that $Q_3 |\Psi_{k, -k}\rangle = 0$.

$$\begin{aligned}
\langle \Psi_{k,-k} | \lambda_1, -\lambda_1 \rangle &= \lim_{\lambda_2 \rightarrow -\lambda_1} \frac{e^{i(\lambda_1 + \lambda_2)L} - 1}{i(\lambda_1 + \lambda_2)} \frac{c}{L} \left[f(\lambda_1, \lambda_2) \frac{-2i\lambda_1}{\lambda_1^2 - k^2} + f(\lambda_2, \lambda_1) \frac{-2i\lambda_2}{\lambda_2^2 - k^2} \right] \\
&= c \left[f(\lambda_1, -\lambda_1) \frac{-2i\lambda_1}{\lambda_1^2 - k^2} + f(-\lambda_1, \lambda_1) \frac{2i\lambda_1}{\lambda_1^2 - k^2} \right] \\
&= \frac{2c^2}{\lambda_1^2 - k^2}.
\end{aligned} \tag{4.6}$$

For the general N -particle states, the normalization constant can be fixed as $C_N = \frac{1}{(N/2)!\sqrt{N!L^{N/2}}}$ when $|\Psi_{k_{N/2}, -k_{N/2}}\rangle$ is on-shell. We can compute the N -particle formal overlap as

$$\begin{aligned}
\langle \Psi_{k_{N/2}, -k_{N/2}} | \lambda_N \rangle &= \int d^N \mathbf{x}_N \langle \Psi_{k_{N/2}, -k_{N/2}} | \mathbf{x}_N \rangle \langle \mathbf{x}_N | \lambda_N \rangle \\
&= \frac{c^{N/2}}{(N/2)!L^{N/2}\sqrt{N!}\sqrt{N!}} \sum_{\sigma \in S_N} \int d^N \mathbf{x}_N A(\mathbf{x}_N | \sigma \lambda_N) \exp\{i \sum_{k=1}^N \lambda_{\sigma_k} x_k\} \sum_{\tau \in S_N} e^{i \sum_{j=1}^N k_{\tau_j} x_j} \\
&= \frac{c^{N/2} N!}{(N/2)!L^{N/2}\sqrt{N!}\sqrt{N!}} \int d^N \mathbf{x}_N A(\mathbf{x}_N | \lambda_N) \exp\{i \sum_{k=1}^N \lambda_k x_k\} \sum_{\tau \in S_N} e^{i \sum_{j=1}^N k_{\tau_j} x_j} \\
&= \frac{c^{N/2}}{(N/2)!L^{N/2}} \int d^N \mathbf{x}_N \sum_{\tau \in S_N} e^{i(\lambda_N + \tau \mathbf{k}_N) \cdot \mathbf{x}_N} A(\mathbf{x}_N | \lambda_N) \\
&= \frac{c^{N/2}}{(N/2)!L^{N/2}} \sum_{\sigma \in S_N} \mathcal{B}_N(\sigma \lambda_N; \sigma \mathbf{a}_N) \prod_{j < k} f(\lambda_{\sigma_j}, \lambda_{\sigma_k}) \\
&= \frac{c^{N/2}}{(N/2)!L^{N/2}} \mathfrak{S}_N(\lambda_N; \mathbf{a}_N),
\end{aligned} \tag{4.7}$$

where

$$\mathcal{B}_N(\lambda_N; \mathbf{a}_N) = \sum_{j=0}^N \mathcal{B}_{N,j}(\lambda_N; \mathbf{a}_N) \tag{4.8}$$

with

$$\mathcal{B}_{N,j}(\lambda_N; \mathbf{a}_N) = \sum_{\tau \in S_N} B_{N,j}(\lambda_N + \tau \mathbf{k}_N; \mathbf{a}_N). \tag{4.9}$$

In the above expression, up to some constant factor, we have found that the N -particle formal overlap can be expressed as

$$\mathfrak{S}_N(\lambda_N; \mathbf{a}_N) = \sum_{\sigma \in S_N} \mathcal{B}_N(\sigma \lambda_N; \sigma \mathbf{a}_N) \prod_{j < k} f(\lambda_{\sigma_j}, \lambda_{\sigma_k}). \tag{4.10}$$

Unfortunately, for general N ($N \geq 4$), since $Q_3 |\Psi_{k_{N/2}, -k_{N/2}}\rangle \neq 0$, the states become non-integrable and they can have non-zero overlaps with non-parity-symmetric Bethe states. We still consider the overlaps with the parity symmetric Bethe states for simplicity. In this case, the strategy to obtain exact overlaps described in the last section becomes invalid. However, since the singular part in the formal overlap eq. (4.10) dominant in the large L limit [22], we can still rely on previous analysis to obtain the overlaps in the large L limit.

The singular part of \mathcal{B}_N at the pole $\lambda_{m+1} = -\lambda_m$ can be extracted from $\mathcal{B}_{N,m-1}$ and $\mathcal{B}_{N,m+1}$. Each contains a summation of permutation of \mathbf{k}_N in the argument of the corresponding $B_{N,j}$. Only the permutations that satisfy $k_{\tau_m} = k, k_{\tau_{m+1}} = -k$ or $k_{\tau_m} = -k, k_{\tau_{m+1}} = k$ contribute. Then the calculation is similar to that was presented in the last section. We find the following result

$$\begin{aligned}
\mathcal{B}_N(\{\lambda_j\}; \{a_j\} | j \in \{1, \dots, N\}) &\sim \frac{N^2(N-2)^2}{2^4} \frac{a_m a_{m+1} - 1}{i(\lambda_m + \lambda_{m+1})} \left[\frac{1}{i(\lambda_m + k)} + \frac{1}{i(\lambda_m - k)} \right] \\
&\times \mathcal{B}_{N-2,m-1}(\{\lambda_j\}; \{a_j\} | j \in \{1, \dots, m, m+1, \dots, N\}) \\
&= \frac{N^2(N-2)^2}{2^4} \frac{a_m a_{m+1} - 1}{i(\lambda_m + \lambda_{m+1})} \frac{2\lambda_m}{i(\lambda_m^2 - k^2)} \mathcal{B}_{N-2,m-1}(\{\lambda_j\}; \{a_j\} | j \in \{1, \dots, m, m+1, \dots, N\}).
\end{aligned} \tag{4.11}$$

After the summation of all permutations that make particles m and $m+1$ in neighbouring position, we find the singularity of the formal overlap \mathfrak{S}_N at $\lambda_{m+1} = -\lambda_m$ essentially has the same recursion relation with eq. (3.26)

$$\mathfrak{S}_N \sim \frac{N^2(N-2)^2}{2^4} \frac{a_m a_{m+1} - 1}{i(\lambda_m + \lambda_{m+1})} F(\lambda_m) \prod_{\substack{j=1 \\ j \neq m, m+1}}^N f(\lambda_j, \lambda_m) f(\lambda_j, -\lambda_m) \mathfrak{S}_{N-2}^{\text{mod}}(m, m+1), \tag{4.12}$$

but in this case

$$F(\lambda) = \frac{2\lambda}{i(\lambda^2 - k^2)} f(\lambda, -\lambda) + \frac{-2\lambda}{i(\lambda^2 - k^2)} f(-\lambda, \lambda) = \frac{2c}{\lambda^2 - k^2}. \quad (4.13)$$

Then applying the same discussion in the last section and noting that the ratio of the determinant $\sqrt{\frac{\det G^+}{\det G^-}}$ only gives $\mathcal{O}(1)$ pieces in the large L limit, we arrive at our final result

$$\frac{|\langle \Psi_{k_{N/2}, -k_{N/2}} | \lambda_{N/2}^+, -\lambda_{N/2}^+ \rangle|}{\| |\lambda_{N/2}^+, -\lambda_{N/2}^+ \rangle\|} = \frac{(N/2)!}{L^{N/2}} \left[\prod_{j=1}^{N/2} \frac{4c}{(\lambda_j^{+2} - k^2) \sqrt{\frac{c^2}{\lambda_j^{+2}} + 4}} + \mathcal{O}(L^{-1}) \right]. \quad (4.14)$$

The same strategy presented in this section can be applied to calculating overlaps of the Bethe states with other initial states with simple wave functions in the large L limit [21].

5. Conclusion

In this note, we derived the exact overlap formula between the Lieb-Liniger Bethe states and the Bose-Einstein condensate state using a recently developed method. This method is based on the coordinate Bethe ansatz which is available for the Lieb-Liniger model and does not rely on the complicated “rotation trick” which is not known in our case. This overlap formula is of great importance in the study of the BEC to Lieb-Liniger quench, but a transparent derivation is lacking. This paper gives a rather concise computation procedure based on the newly proposed method. We also obtained overlap for a special kind of free particle states in the large L limit. This formula serves as a starting point to the study of the related quench problem.

We hope more insight can be found in this new method and it's very interesting to find more applications such as the next integrable system where similar exact overlap formulae also exist (see for example [5,24,25]).

In integrable spin chain models, there also exist exact overlap formulae for integrable Matrix Product States (MPS) [4,5,26]. In [27] the authors introduced the continuous Matrix Product States (cMPS) in the Lieb-Liniger model. It would be very interesting to find out exact overlap formulae for integrable cMPS.

As you will see in the appendix, the proof of the overlap vanishes for generic (non-parity-symmetric) on-shell Bethe is very specific and is not inspiring (for an earlier proof, see [28]). A more transparent proof is needed.

It turns out that the coordinate Bethe ansatz and the calculation of exact overlap have some applications in the physics of the stochastic non-linear Kardar-Parisi-Zhang (KPZ) equation, where the computation of the exact generating function of the Kardar-Parisi-Zhang height requires to calculate the overlap between the Bethe wave functions and the initial condition of the equation [29,30]. We think our result and all the classification obtained in [13] could help to classify the integrable initial conditions in the KPZ equation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of $\mathcal{S}_N = 0$ for generic on-shell Bethe state

In this appendix, let's prove that the overlap of a generic on-shell Bethe state with the BEC state vanishes.

For a given σ , let's consider terms in eq. (3.21) containing the factor

$$F_\sigma \equiv \prod_{j < k} f(\lambda_{\sigma_j}, \lambda_{\sigma_k}). \quad (A.1)$$

If we define $\lambda_{\sigma_j} = ip_j^\sigma - ip_{j-1}^\sigma$, then we have

$$F_\sigma B_{N,j}(\lambda_{\sigma_1}, \dots, \lambda_{\sigma_N}; a_{\sigma_1}, \dots, a_{\sigma_N}) = F_{\tau_j \sigma} \prod_{\substack{k=0 \\ k \neq j}}^N \frac{1}{p_j^\sigma - p_k^\sigma}, \quad (A.2)$$

where $\tau_j \in S_N$ and is defined as

$$\tau_j = \begin{pmatrix} 1, \dots, j, j+1, \dots, N \\ j+1, \dots, N, 1, \dots, j \end{pmatrix}, \quad (A.3)$$

when $0 < j < N$, and τ_0 and τ_N are identity. Then, for a generic on-shell Bethe state, we have

$$\begin{aligned}
S_N(\lambda_N; \mathbf{a}_N) &= \sum_{\sigma \in S_N} \sum_{j=0}^N F_{\tau_j \sigma} \prod_{\substack{k=0 \\ k \neq j}}^N \frac{1}{p_j^\sigma - p_k^\sigma} \\
&= \sum_{\sigma \in S_N} F_\sigma \sum_{j=0}^N \prod_{\substack{k=0 \\ k \neq j}}^N \frac{1}{p_j^{\tau_j^{-1} \sigma} - p_k^{\tau_j^{-1} \sigma}}
\end{aligned} \tag{A.4}$$

It is not difficult to see that the coefficient of the F_σ term in eq. (A.4) vanish. In fact, for any permutation $\sigma \in S_N$, we have

$$\sum_{j=0}^N \prod_{\substack{k=0 \\ k \neq j}}^N \frac{1}{p_j^{\tau_j^{-1} \sigma} - p_k^{\tau_j^{-1} \sigma}} = 0. \tag{A.5}$$

The terms in eq. (A.5) with $j = 0$ and $j = N$ come from the summand in eq. (3.21) corresponding to the permutation σ . and the term with $0 < j < N$ comes from the summand in eq. (3.21) corresponding to the permutation $\tau_j^{-1} \sigma$.

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