Quantum Information,
Quantum Nonlocality and Spacetime

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Abstract

Basic principles of quantum information theory are considered. Modern quantum information deals with an idealized situation when the spacetime dependence of quantum phenomena is neglected. However the transmission of information is a physical process in spacetime. Therefore such basic notions in quantum information as qubit, channel and entangled states should be formulated in space and time. Entangled states in space and time are considered. It is shown that any reasonable quantum state becomes disentangled at large spacelike distances if one makes local observations. As a result a violation of Bell's inequalities can be observed without inconsistency with principles of quantum theory only if the distance between detectors is rather small. Loopholes in Bell type experiments are unavoidable. A modification of the Bell equations which includes the spacetime variables is suggested. Applications of these equations to the security of quantum key distribution in quantum cryptography are considered. Quantum nonlocality and noncommutative spectral theorem are discussed.

Contents

1 Introduction

Remarkable experimental and theoretical results obtained in quantum computing, teleportation and cryptography (these topics sometimes are considered as belonging to quantum information theory) are based on the investigation of basic properties of quantum mechanics, see for example [1]-[4] and references therein. Especially important are properties of nonfactorized entangled states introduced by Einstein, Podolsky and Rosen [5, 6] which were named by Schrodinger as the most characteristic feature of quantum mechanics.

Ideas of Shannon‘s classical information theory are important for the modern quantum information theory as well as the notions of qubit, quantum relative entropy, quantum channel, and entangled states.

The spacetime dependence is not explicitly indicated in this approach. As a result, many important achievements in modern quantum information theory have been obtained for an idealized situation when the spacetime dependence of quantum phenomena is neglected.

However transmission and processing of (quantum) information is a physical process in spacetime. Therefore a formulation of such basic notions in quantum information theory as composite systems, entangled states and the channel should include the spacetime variables [7].

Ultimately, quantum information theory should become a part of quantum field theory (perhaps, in future, a part of superstring theory) since quantum field theory is our most fundamental physical theory.

Entangled states, i.e. the states of two particles with the wave function which is not a product of the wave functions of single particles, have been studied in many theoretical and experimental works starting from the paper of Einstein, Podolsky and Rosen, see e.g. [6].

J. Bell proved [8] that there are quantum spin correlation functions in entangled states that can not be represented as classical correlation functions of separated random variables. Bell's theorem reads, see [9]:

$$\cos(\alpha - \beta) \neq E\xi_\alpha \eta_\beta$$

where ξ_α and η_β are two random processes such that $|\xi_\alpha| \leq 1$, $|\eta_\beta| \leq 1$ and E is the expectation. Here the function $\cos(\alpha - \beta)$ describes the quantum mechanical correlation of spins of two entangled particles. Bell‘s theorem has been interpreted as incompatibility of the requirement of locality with the statistical predictions of quantum mechanics [8], the so called quantum nonlocality.

However if we want to speak about locality in quantum theory then we have to localize somehow our particles. For example we could measure the density of the energy or the position of the particles simultaneously with the spin. Only then we could come to some conclusions about a relevance of the spin correlation function to the problem of locality.

The function $\cos(\alpha - \beta)$ describes quantum correlations of two spins in the two qubit Hilbert space when the spacetime dependence of the wave functions of the particles is neglected. Let us note however that the very formulation of the problem of locality in quantum mechanics prescribes a special role to the position in ordinary three-dimensional space. It is rather strange therefore that the problem of *local in space observations* was neglected in discussions of the problem of locality in relation to Bell's inequalities .

Let us stress that we discuss here not a problem of interpretation of quantum theory but a problem of how to make correct quantum mechanical computations describing an experiment with two detectors localized in space. Recently it was pointed out [9] that if we make *local* observations of spins then the spacetime part of the wave function leads to an extra factor in quantum correlations and as a result the ordinary conclusion from the Bell theorem about the nonlocality of quantum theory fails.

We present a modification of Bell‘s equation which includes space and time variables. The function $\cos(\alpha - \beta)$ describes the quantum mechanical correlation of spins of two entangled particles if we neglect the spacetime dependence of the wave function. It was shown in [9] that if one takes into account the space part of the wave function then the quantum correlation describing local observations of spins in the simplest case will take the form $g \cos(\alpha - \beta)$ instead of just $\cos(\alpha - \beta)$. Here the parameter g describes the location of the system in space and time. In this case one gets a modified equation

$$g \cos(\alpha - \beta) = E\xi_\alpha \eta_\beta$$

One can prove that if the distance between detectors is large enough then the factor g becomes small and there exists a solution of the modified equation. We will show that in fact at large distances all reasonable quantum states become disentangled. This fact leads also to important consequences for quantum teleportation and quantum cryptography, see below.

It is important to study also a more general question: which class of functions $f(s,t)$ admits a representation of the form

$$f(s,t) = Ex_s y_t$$

where x_s and y_t are bounded stochastic processes and also analogous question for the functions of several variables $f(t_1, ..., t_n)$.

Such considerations could provide a *noncommutative* generalization of von Neumann‘s spectral theorem.

We shall consider entangled states in space and time. We point out a simple but the important fact that the vacuum state ω_0 in a free quantum field theory is a nonfactorized (entangled) state for observables belonging to spacelike separated regions:

$$\omega_0(\varphi(x)\varphi(y)) - \omega_0(\varphi(x))\omega_0(\varphi(y)) \neq 0$$

Here $\varphi(x)$ is a free scalar field in the Minkowski spacetime and $(x-y)^2 < 0$. Hence there is a statistical dependence between causally disconnected regions.

However one has an asymptotic factorization of the vacuum state for large separations of the spacelike regions. Moreover one proves that in quantum field theory *there is an asymptotic factorization for any reasonable state and any local observables. Therefore at large distances any reasonable state becomes disentangled.* We have the relation

$$\lim_{|l|\to\infty} [\omega(A(l)B) - \omega(A(l))\omega(B)] = 0$$

Here ω is a state from a rather wide class of the states which includes entangled states, A and B are two local observables, and $A(l)$ is the translation of the observable A along the 3 dim vector l. As a result a violation of Bell‘s inequalities (see below) can be observed without inconsistency with principles of relativistic quantum theory only if the distance between detectors is rather small. We suggest a further experimental study of entangled states in spacetime by studying the dependence of the correlation functions on the distance between detectors.

There is no a factorization of the expectation value $\omega_0(\varphi(x)\varphi(y))$ even for the space-like separation of the variables x and y if the distance between x and y is not large enough. However we will prove that there exist a representation of the form

$$\omega_0(\varphi(x)\varphi(y)) = E\xi(x)\xi^*(y)$$

which is valid for all x and y. Here $\xi(x)$ is a classical (generalized) complex random field and E is the expectation value. Therefore the quantum correlation function is represented as a classical correlation function of separated random fields. This representation can be called a local realistic representation by analogy with the Bell approach to the spin correlation functions.

In the next section Bell‘s theorem is discussed and a simple generalization of the known CHSH result is proved. In Sect.3 the locality in space is considered for entangled states and the asymptotic factorization of the states is proved. A hidden variable representation for quantum correlation which is local in the space is also obtained. Noncommutative spectral theory and local realism are considered in Sect.4. The disentanglement at large distances in quantum field theory is considered in Sect.5. Quantum cryptography in space is discussed in Sect.6.

2 Bell's Theorem

2.1 Bell‘s Theorem and Stochastic Processes

In the presentation of Bell's theorem we will follow [9] where one can find also more references. Bell's theorem reads:

$$\cos(\alpha - \beta) \neq E\xi_\alpha\eta_\beta \tag{2.1}$$

where ξ_α and η_β are two random processes such that $|\xi_\alpha| \leq 1, \ \ |\eta_\beta| \leq 1$ and E is the expectation. In more details:

Theorem 1. There exists no probability space $(\Lambda, \mathcal{F}, d\rho(\lambda))$ and a pair of stochastic processes $\xi_\alpha = \xi_\alpha(\lambda), \ \eta_\beta = \eta_\beta(\lambda), \ 0 \leq \alpha, \beta \leq 2\pi$ which obey $|\xi_\alpha(\lambda)| \leq 1, \ \ |\eta_\beta(\lambda)| \leq 1$ such that the following equation is valid

$$\cos(\alpha - \beta) = E\xi_\alpha\eta_\beta \tag{2.2}$$

for all α and β.

Here Λ is a set, $\mathcal{F}$ is a sigma-algebra of subsets and $d\rho(\lambda)$ is a probability measure, i.e. $d\rho(\lambda) \geq 0, \ \int d\rho(\lambda) = 1$. The expectation is

$$E\xi_\alpha\eta_\beta = \int_\Lambda \xi_\alpha(\lambda)\eta_\beta(\lambda)d\rho(\lambda)$$

One can write Eq. (2.2) as an integral equation

$$\cos(\alpha - \beta) = \int_\Lambda \xi_\alpha(\lambda)\eta_\beta(\lambda)d\rho(\lambda) \tag{2.3}$$

We say that the integral equation (2.3) has no solutions $(\Lambda, \mathcal{F}, d\rho(\lambda), \xi_\alpha, \eta_\beta)$ with the bound $|\xi_\alpha| \leq 1, \ \ |\eta_\beta| \leq 1$.

We will prove the theorem below. Let us discuss now the physical interpretation of this result.

Consider a pair of spin one-half particles formed in the singlet spin state and moving freely towards two detectors. If one neglects the space part of the wave function then one has the Hilbert space $C^2 \otimes C^2$ and the quantum mechanical correlation of two spins in the singlet state $\psi_{spin} \in C^2 \otimes C^2$ is

$$D_{spin}(a,b) = \langle \psi_{spin} | \sigma \cdot a \otimes \sigma \cdot b | \psi_{spin} \rangle = -a \cdot b \tag{2.4}$$

Here $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are two unit vectors in three-dimensional space R^3, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma \cdot a = \sum_{i=1}^{3} \sigma_i a_i$$

and

$$\psi_{spin} = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

If the vectors a and b belong to the same plane then one can write $-a \cdot b = \cos(\alpha - \beta)$ and hence Bell's theorem states that the function $D_{spin}(a,b)$ Eq. (2.4) can not be represented in the form

$$P(a,b) = \int \xi(a,\lambda) \eta(b,\lambda) d\rho(\lambda) \tag{2.5}$$

i.e.

$$D_{spin}(a,b) \neq P(a,b) \tag{2.6}$$

Here $\xi(a,\lambda)$ and $\eta(b,\lambda)$ are random fields on the sphere, $|\xi(a,\lambda)| \leq 1, \; |\eta(b,\lambda)| \leq 1$ and $d\rho(\lambda)$ is a positive probability measure, $\int d\rho(\lambda) = 1$. The parameters λ are interpreted as hidden variables in a realist theory. It is clear that Eq. (2.6) can be reduced to Eq. (2.1).

2.2 CHSH Inequality

To prove Theorem 1 we will use the following theorem which is a slightly generalized Clauser-Horn-Shimony-Holt (CHSH) result.

Theorem 2. Let $f_1, \; f_2, \; g_1$ and g_2 be random variables (i.e. measured functions) on the probability space $(\Lambda, \mathcal{F}, d\rho(\lambda))$ such that

$$|f_i(\lambda) g_j(\lambda)| \leq 1, \quad i,j = 1,2.$$

Denote

$$P_{ij} = E f_i g_j, \quad i,j = 1,2.$$

Then

$$|P_{11} - P_{12}| + |P_{21} + P_{22}| \leq 2.$$

Proof of Theorem 2. One has

$$P_{11} - P_{12} = Ef_1g_1 - Ef_1g_2 = E(f_1g_1(1 \pm f_2g_2)) - E(f_1g_2(1 \pm f_2g_1))$$

Hence

$$|P_{11} - P_{12}| \leq E(1 \pm f_2g_2) + E(1 \pm f_2g_1) = 2 \pm (P_{22} + P_{21})$$

Now let us note that if x and y are two real numbers then

$$|x| \leq 2 \pm y \quad \rightarrow \quad |x| + |y| \leq 2.$$

Therefore taking $x = P_{11} - P_{12}$ and $y = P_{22} + P_{21}$ one gets the bound

$$|P_{11} - P_{12}| + |P_{21} + P_{22}| \leq 2.$$

The theorem is proved.

The last inequality is called the CHSH inequality. By using notations of Eq. (2.5) one has

$$|P(a,b) - P(a,b')| + |P(a',b) + P(a',b')| \leq 2 \tag{2.7}$$

for any four unit vectors a, b, a', b'.

Proof of Theorem 1. Let us denote

$$f_i(\lambda) = \xi_{\alpha_i}(\lambda), \quad g_j(\lambda) = \eta_{\beta_j}(\lambda), \quad i,j = 1,2$$

for some α_i, β_j. If one would have

$$\cos(\alpha_i - \beta_j) = Ef_ig_j$$

then due to Theorem 2 one should have

$$|\cos(\alpha_1 - \beta_1) - \cos(\alpha_1 - \beta_2)| + |\cos(\alpha_2 - \beta_1) + \cos(\alpha_2 - \beta_2)| \leq 2.$$

However for $\alpha_1 = \pi/2,\ \alpha_2 = 0,\ \beta_1 = \pi/4, \beta_2 = -\pi/4$ we obtain

$$|\cos(\alpha_1 - \beta_1) - \cos(\alpha_1 - \beta_2)| + |\cos(\alpha_2 - \beta_1) + \cos(\alpha_2 - \beta_2)| = 2\sqrt{2}$$

which is greater than 2. This contradiction proves Theorem 1.

It will be shown below that if one takes into account the space part of the wave function then the quantum correlation in the simplest case will take the form $g\cos(\alpha-\beta)$ instead of just $\cos(\alpha-\beta)$ where the parameter g describes the location of the system in space and time. In this case one can get a representation

$$g\cos(\alpha-\beta)=E\xi_\alpha\eta_\beta \tag{2.8}$$

if g is small enough. The factor g gives a contribution to visibility or efficiency of detectors that are used in the phenomenological description of detectors.

3 Local Observations

3.1 Modified Bell‘s equation

In the previous section the space part of the wave function of the particles was neglected. However exactly the space part is relevant to the discussion of locality. The Hilbert space assigned to one particle with spin 1/2 is $C^2\otimes L^2(R^3)$ and the Hilbert space of two particles is $C^2\otimes L^2(R^3)\otimes C^2\otimes L^2(R^3)$. The complete wave function is $\psi=(\psi_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2,t))$ where α and β are spinor indices, t is time and $\mathbf{r}_1$ and $\mathbf{r}_2$ are vectors in three-dimensional space.

We suppose that there are two detectors (A and B) which are located in space R^3 within the two localized regions $\mathcal{O}_A$ and $\mathcal{O}_B$ respectively, well separated from one another. If one makes a local observation in the region $\mathcal{O}_A$ then this means that one measures not only the spin observable σ_i but also some another observable which describes the localization of the particle like the energy density or the projection operator $P_{\mathcal{O}}$ to the region $\mathcal{O}$. We will consider here correlation functions of the projection operators $P_{\mathcal{O}}$.

Quantum correlation describing the localized measurements of spins in the regions $\mathcal{O}_A$ and $\mathcal{O}_B$ is

$$\omega(\sigma\cdot aP_{\mathcal{O}_A}\otimes\sigma\cdot bP_{\mathcal{O}_B})=\langle\psi|\sigma\cdot aP_{\mathcal{O}_A}\otimes\sigma\cdot bP_{\mathcal{O}_B}|\psi\rangle \tag{3.9}$$

Let us consider the simplest case when the wave function has the form of the product of the spin function and the space function $\psi=\psi_{spin}\phi(\mathbf{r}_1,\mathbf{r}_2)$. Then one has

$$\omega(\sigma\cdot aP_{\mathcal{O}_A}\otimes\sigma\cdot bP_{\mathcal{O}_B})==g(\mathcal{O}_A,\mathcal{O}_B)D_{spin}(a,b) \tag{3.10}$$

where the function

$$g(\mathcal{O}_A,\mathcal{O}_B)=\int_{\mathcal{O}_A\times\mathcal{O}_B}|\phi(\mathbf{r}_1,\mathbf{r}_2)|^2d\mathbf{r}_1d\mathbf{r}_2 \tag{3.11}$$

describes correlation of particles in space. It is the probability to find one particle in the region $\mathcal{O}_A$ and another particle in the region $\mathcal{O}_B$.

One has

$$0 \leq g(\mathcal{O}_A, \mathcal{O}_B) \leq 1. \tag{3.12}$$

If $\mathcal{O}_A$ is a bounded region and $\mathcal{O}_A(l)$ is a translation of $\mathcal{O}_A$ to the 3-vector l then one has

$$\lim_{|l|\to\infty} g(\mathcal{O}_A(l), \mathcal{O}_B) = 0. \tag{3.13}$$

Since

$$\langle\psi_{spin}|\sigma \cdot a \otimes I|\psi_{spin}\rangle = 0$$

we have

$$\omega(\sigma \cdot aP_{\mathcal{O}_A} \otimes I) = 0.$$

Therefore we have proved the following proposition which says that the state $\psi = \psi_{spin}\phi(\mathbf{r}_1, \mathbf{r}_2)$ becomes disentangled at large distances.

Proposition. One has the following property of the asymptotic factorization (disentanglement) at large distances:

$$\lim_{|l|\to\infty} [\omega(\sigma \cdot aP_{\mathcal{O}_A(l)} \otimes \sigma \cdot bP_{\mathcal{O}_B}) - \omega(\sigma \cdot aP_{\mathcal{O}_A(l)} \otimes I)\omega(I \otimes \sigma \cdot bP_{\mathcal{O}_B})] = 0 \tag{3.14}$$

or

$$\lim_{|l|\to\infty} \omega(\sigma \cdot aP_{\mathcal{O}_A(l)} \otimes \sigma \cdot bP_{\mathcal{O}_B}) = 0.$$

Now one inquires whether one can write a representation

$$\omega(\sigma \cdot aP_{\mathcal{O}_A(l)} \otimes \sigma \cdot bP_{\mathcal{O}_B}) = \int \xi(a, \mathcal{O}_A, \lambda)\eta(b, \mathcal{O}_B, \lambda)d\rho(\lambda) \tag{3.15}$$

where $|\xi(a, \mathcal{O}_A(l), \lambda)| \leq 1, \quad |\eta(b, \mathcal{O}_B, \lambda)| \leq 1.$

Remark. A local modified equation reads

$$|\phi(\mathbf{r_1}, \mathbf{r_2}, t)|^2 \cos(\alpha - \beta) = E\xi(\alpha, \mathbf{r_1}, t)\eta(\beta, \mathbf{r_2}, t).$$

If we are interested in the conditional probability of finding the projection of spin along vector a for the particle 1 in the region $\mathcal{O}_A(l)$ and the projection of spin along the vector b for the particle 2 in the region $\mathcal{O}_B$ then we have to divide both sides of Eq. (3.15) by $g(\mathcal{O}_A(l), \mathcal{O}_B)$.

Note that here the classical random variable $\xi = \xi(a, \mathcal{O}_A(l), \lambda)$ is not only separated in the sense of Bell (i.e. it depends only on a) but it is also local in the 3 dim space since it depends only on the region $\mathcal{O}_A(l)$. The classical random variable η is also local in 3 dim space since it depends only on $\mathcal{O}_B$. Note also that since the eigenvalues of the projector $P_{\mathcal{O}}$ are 0 or 1 then one should have $|\xi(a,\mathcal{O}_A)| \leq 1$.

Due to the property of the asymptotic factorization and the vanishing of the quantum correlation for large $|l|$ there exists a trivial asymptotic classical representation of the form (3.15) with $\xi = \eta = 0$.

We can do even better and find a classical representation which will be valid uniformly for large $|l|$.

If g would not depend on $\mathcal{O}_A$ and $\mathcal{O}_B$ then instead of Eq (2.2) in Theorem 1 we could have a modified equation

$$g\cos(\alpha-\beta) = E\xi_\alpha\eta_\beta \tag{3.16}$$

The factor g is important. In particular one can write the following representation for $0 \leq g \leq 1/2$:

$$g\cos(\alpha-\beta) = \int_0^{2\pi} \sqrt{2g}\cos(\alpha-\lambda)\sqrt{2g}\cos(\beta-\lambda)\frac{d\lambda}{2\pi} \tag{3.17}$$

Therefore if $0 \leq g \leq 1/2$ then there exists a solution of Eq. (3.16) where

$$\xi_\alpha(\lambda) = \sqrt{2g}\cos(\alpha-\lambda), \quad \eta_\beta(\lambda) = \sqrt{2g}\cos(\beta-\lambda)$$

and $|\xi_\alpha| \leq 1,\ |\eta_\beta| \leq 1$. If $g > 1/\sqrt{2}$ then it follows from Theorem 2 that there is no solution to Eq. (3.16). We have obtained

Theorem 3. If $g > 1/\sqrt{2}$ then there is no solution $(\Lambda, \mathcal{F}, d\rho(\lambda), \xi_\alpha, \eta_\beta)$ to Eq. (3.16) with the bounds $|\xi_\alpha| \leq 1,\ |\eta_\beta| \leq 1$. If $0 \leq g \leq 1/2$ then there exists a solution to Eq. (3.16) with the bounds $|\xi_\alpha| \leq 1,\ |\eta_\beta| \leq 1$.

Let us take now the wave function ϕ of the form $\phi = \psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)$ where

$$\int_{R^3} |\psi_1(\mathbf{r}_1)|^2 d\mathbf{r}_1 = 1, \quad \int_{R^3} |\psi_2(\mathbf{r}_2)|^2 d\mathbf{r}_2 = 1$$

In this case

$$g(\mathcal{O}_A(l),\mathcal{O}_B) = \int_{\mathcal{O}_A(l)} |\psi_1(\mathbf{r}_1)|^2 d\mathbf{r}_1 \cdot \int_{\mathcal{O}_B} |\psi_2(\mathbf{r}_2)|^2 d\mathbf{r}_2$$

There exists such $L > 0$ that

$$\int_{B_L} |\psi_1(\mathbf{r}_1)|^2 d\mathbf{r}_1 = \epsilon < 1/2,$$

where $B_L = \{\mathbf{r} \in R^3 : |\mathbf{r}| \geq L\}$. Let us make an additional assumption that the classical random variable has the form of a product of two independent classical random variables $\xi(a, \mathcal{O}_A) = \xi_{space}(\mathcal{O}_A)\xi_{spin}(a)$ and similarly for η. We have the following

Theorem 4. Under the above assumptions and for large enough $|l|$ there exists the following representation of the quantum correlation function

$$g(\mathcal{O}_A(l), \mathcal{O}_B)\cos(\alpha - \beta) = (E\xi_{space}(\mathcal{O}_A)(l))(E\eta_{space}(\mathcal{O}_B))E\xi_{spin}(\alpha)\xi_{spin}(\beta)$$

where all classical random variables are bounded by 1.

Proof. To prove the theorem we write

$$g(\mathcal{O}_A(l), \mathcal{O}_B)\cos(\alpha - \beta) = \int_{\mathcal{O}_A(l)} \frac{1}{\epsilon}|\psi_1(\mathbf{r}_1)|^2 d\mathbf{r}_1 \cdot \int_{\mathcal{O}_B} |\psi_2(\mathbf{r}_2)|^2 d\mathbf{r}_2 \cdot \epsilon \cos(\alpha - \beta)$$

$$= (E\xi_{space}(\mathcal{O}_A(l))(E\eta_{space}(\mathcal{O}_B))E\xi_{spin}(\alpha)\xi_{spin}(\beta)$$

Here $\xi_{space}(\mathcal{O}_A(l))$ and $\eta_{space}(\mathcal{O}_B)$ are random variables on the probability space $B_L \times R^3$ with the probability measure

$$dP(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\epsilon}|\psi_1(\mathbf{r}_1)|^2 \cdot |\psi_2(\mathbf{r}_2)|^2 d\mathbf{r}_1 d\mathbf{r}_2$$

of the form

$$\xi_{space}(\mathcal{O}_A(l), \mathbf{r}_1, \mathbf{r}_2) = \chi_{\mathcal{O}_A(l)}(\mathbf{r}_1), \quad \eta_{space}(\mathcal{O}_B, \mathbf{r}_1, \mathbf{r}_2) = \chi_{\mathcal{O}_B}(\mathbf{r}_2)$$

where $\chi_{\mathcal{O}}(\mathbf{r})$ is the characteristic function of the region $\mathcal{O}$. We assume that $\mathcal{O}_A(l)$ belongs to B_L. Further $\xi_{spin}(\alpha)$ is a random process on the circle $0 \leq \varphi \leq 2\pi$ with the probability measure $d\varphi/2\pi$ of the form

$$\xi_{spin}(\alpha, \varphi) = \sqrt{2\epsilon}\cos(\alpha - \varphi)$$

The theorem is proved.

3.2 Expansion of Wave Packet

Let us remind that there is a well known effect of expansion of wave packets due to the free time evolution. If ϵ is the characteristic length of the Gaussian wave packet describing a particle of mass M at time $t = 0$ then at time t the characteristic length ϵ_t will be

$$\epsilon_t = \epsilon\sqrt{1 + \frac{\hbar^2 t^2}{M^2 \epsilon^4}}. \tag{3.18}$$

It tends to $(\hbar/M\epsilon)t$ as $t \to \infty$. Therefore the locality criterion is always satisfied for nonrelativistic particles if regions $\mathcal{O}_A$ and $\mathcal{O}_B$ are far enough from each other.

3.3 Relativistic Particles

We can not immediately apply the previous considerations to the case of relativistic particles such as photons and the Dirac particles because in these cases the wave function can not be represented as a product of the spin part and the spacetime part. Let us show that the wave function of photon can not be represented in the product form. Let $A_i(k)$ be the wave function of photon, where $i = 1, 2, 3$ and $k \in R^3$. One has the gauge condition $k^i A_i(k) = 0$ [10]. If one supposes that the wave function has a product form $A_i(k) = \phi_i f(k)$ then from the gauge condition one gets $A_i(k) = 0$. Therefore the case of relativistic particles requires a separate investigation (see below).

4 Noncommutative Spectral Theory and Local Realism

As a generalisation of the previous discussion we would like to suggest here a general relation between quantum theory and theory of classical stochastic processes which expresses the condition of local realism. Let $\mathcal{H}$ be a Hilbert space, ρ is the density operator, $\{A_\alpha\}$ is a family of self-adjoint operators in $\mathcal{H}$. One says that the family of observables $\{A_\alpha\}$ and the state ρ satisfy to *the condition of local realism* if there exists a probability space $(\Lambda, \mathcal{F}, d\rho(\lambda))$ and a family of random variables $\{\xi_\alpha\}$ such that the range of ξ_α belongs to the spectrum of A_α and for any subset $\{A_i\}$ of mutually commutative operators one has a representation

$$Tr(\rho A_{i_1} ... A_{i_n}) = E\xi_{i_1} ... \xi_{i_n}$$

The physical meaning of the representation is that it describes the quantum-classical correspondence. If the family $\{A_\alpha\}$ would be a maximal commutative family of self-adjoint operators then for pure states the previous representation can be reduced to the von Neumann spectral theorem [11]. In our case the family $\{A_\alpha\}$ consists from not necessary commuting operators. Hence we will call such a representation a *noncommutative spectral representation.* Of course one has a question for which families of operators and states a *noncommutative spectral theorem* is valid, i.e. when we can write the noncommutative spectral representation. We need a noncommutative generalization of von Neumann's spectral theorem.

It would be helpful to study the following problem: describe the class of functions $f(t_1, ..., t_n)$ which admits the representation of the form

$$f(t_1, ..., t_n) = Ex_{t_1} ... z_{t_n}$$

where $x_t, ..., z_t$ are random processes which obey the bounds $|x_t| \leq 1, ..., |z_t| \leq 1$.

From the previous discussion we know that there are such families of operators and such states which do not admit the noncommutative spectral representation and therefore they do not satisfy the condition of local realism. Indeed let us take the Hilbert space $\mathcal{H} = C^2 \otimes C^2$ and four operators A_1, A_2, A_3, A_4 of the form (we denote $A_3 = B_1, A_4 = B_2$)

$$A_1 = \begin{pmatrix} \sin\alpha_1 & \cos\alpha_1 \\ \cos\alpha_1 & -\sin\alpha_1 \end{pmatrix} \otimes I, \quad A_2 = \begin{pmatrix} \sin\alpha_2 & \cos\alpha_2 \\ \cos\alpha_2 & -\sin\alpha_2 \end{pmatrix} \otimes I$$

and

$$B_1 = I \otimes \begin{pmatrix} -\sin\beta_1 & -\cos\beta_1 \\ -\cos\beta_1 & \sin\beta_1 \end{pmatrix}, \quad B_2 = I \otimes \begin{pmatrix} -\sin\beta_2 & -\cos\beta_2 \\ -\cos\beta_2 & \sin\beta_2 \end{pmatrix}$$

Here operators A_i correspond to operators $\sigma \cdot a$ and operators B_i corresponds to operators $\sigma \cdot b$ where $a = (\cos\alpha, 0, \sin\alpha),\ b = (-\cos\beta, 0, -\sin\beta)$. Operators A_i commute with operators $B_j,\ \ [A_i, B_j] = 0,\ i, j = 1, 2$ and one has

$$\langle\psi_{spin}|A_i B_j|\psi_{spin}\rangle = \cos(\alpha_i - \beta_j), \quad i, j = 1, 2$$

We know from Theorem 2 that this function can not be represented as the expected value $E\xi_i\eta_j$ of random variables with the bounds $|\xi_i| \leq 1,\ |\eta_j| \leq 1$.

However, as it was discussed above, the space part of the wave function was neglected in the previous consideration. We suggest that *in physics one could prepare only such states and observables which satisfy the condition of local realism.* Perhaps we should restrict ourself in this proposal to the consideration of only such families of observables which satisfy the condition of relativistic local causality. If there are physical phenomena which do not satisfy this proposal then it would be important to *describe quantum processes which satisfy the above formulated condition of local realism and also processes which do not satisfy this condition.*

5 Quantum Probability and Quantum Field Theory

In quantum probability (see [12]) we are given a * - algebra $\mathcal{A}$ and a state (i.e. a linear positive normalized functional) ω on $\mathcal{A}$. Elements from $\mathcal{A}$ are called random variables. Two random variables A and B are called (statistically) independent if $\omega(AB) = \omega(A)\omega(B)$.

First we will prove the following

Proposition. *There is a statistical dependence between two spacelike separated regions in the theory of free scalar quantum field.*

Proof. Let us consider a free scalar quantum field $\varphi(x)$:

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int_{R^3} \frac{d\mathbf{k}}{\sqrt{2k^0}} (e^{ikx} a^*(\mathbf{k}) + e^{-ikx} a(\mathbf{k}))$$

Here $kx = k^0 x^0 - \mathbf{kx}$, $k^0 = \sqrt{\mathbf{k}^2 + m^2}, m \leq 0$ and $a(\mathbf{k})$ and $a^*(\mathbf{k})$ are annihilation and creation operators,

$$[a(\mathbf{k}), a^*(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}')$$

The field $\varphi(x)$ is an operator valued distribution acting in the Fock space $\mathcal{F}$ with the vacuum $|0>$,

$$a(\mathbf{k})|0>= 0$$

The vacuum expectation value of two fields is

$$\omega_0(\varphi(x)\varphi(y)) =< 0|\varphi(x)\varphi(y)|0 >= W_0(x - y, m^2)$$

where

$$W_0(x - y, m^2) = \frac{1}{(2\pi)^3} \int_{R^3} \frac{d\mathbf{k}}{2k^0} e^{-ik(x-y)}$$

The statistical independence of two spacelike separated regions in particular would lead to the relation

$$\omega_0(\varphi(x)\varphi(y)) - \omega_0(\varphi(x))\omega_0(\varphi(y)) = 0$$

if $(x - y)^2 < 0$. But since $\omega_0(\varphi(x)) = 0$ in fact we have

$$\omega_0(\varphi(x)\varphi(y)) - \omega_0(\varphi(x))\omega_0(\varphi(y)) = W_0(x - y, m^2) \neq 0$$

However the violation of the statistical independence vanish exponentially with the spacial separation of x and y since for large $\lambda = m\sqrt{-x^2}$ the function $W_0(x, m^2)$ behaves like

$$\frac{m^2}{4\pi\lambda} \left(\frac{\pi}{2\lambda}\right)^{1/2} e^{-\lambda}$$

Let us prove that any polynomial state is asymptotically disentangled (factorized) for large spacelike distances. Let $\mathcal{A}$ be the algebra of polynomials in the Fock space $\mathcal{F}$ at the field $\varphi(f)$ with the test functions f . Let

$C \in \mathcal{A}$ and $|\psi >= C|0>$. Denote the state $\omega(A) = \langle\psi|A|\psi\rangle /||\psi||^2$ for $A \in \mathcal{A}$.

Theorem 5. One has the following asymptotic disentanglement property

$$\lim_{|l|\to\infty} [\omega(A(l)B) - \omega(A(l))\omega(B)] = 0$$

Here A and B belong to $\mathcal{A}$ and $A(l)$ is the translation of A along the 3 dim vector l. One has also

$$\lim_{|l|\to\infty} [\omega(A(l)) - \langle 0|A(l)|0\rangle] = 0$$

The proof of the theorem is based on the Wick theorem and the Riemann-Lebesgue lemma.

Similar theorems take place also for the Dirac and the Maxwell fields. In particular for the Dirac field $\psi(x)$ one can prove the asymptotic factorization for the local spin operator

$$\mathbf{S}(\mathcal{O}) = \int_{\mathcal{O}} \psi^* \mathbf{\Sigma} \psi dx$$

Here Σ is made from the Dirac matrices. Asymptotic factorization in quantum field theory is discussed in [13].

Finally let us show that some correlation functions in the relativistic quantum field theory can be represented as mathematical expectations of the classical (generalized) random fields.

Theorem 6. If $\varphi(x)$ is a scalar complex quantum field then one has a representation

$$\langle 0|\varphi(x_1)...\varphi(x_n)\varphi^*(y_1)...\varphi^*(y_n)|0\rangle = E\xi(x_1)...\xi(x_n)\xi^*(y_1)...\xi^*(y_n).$$

Here $\xi(x)$ is a complex random field.

The proof of the theorem follows from the positivity of the quantum correlation functions. It is interesting that we have obtained a functional integral representation for the quantum correlation functions in real time. Similar representation is valid also for the 2-point correlation function of an interacting scalar field. It follows from the Kallen-Lehmann representation, [13].

6 Quantum Cryptography in Space

Let us now apply these considerations to quantum cryptography. For a general discussion of quantum cryptography see for example [1, 2, 4, 14, 15].

Ekert [16] showed that one can use the Einstein-Podolsky-Rosen correlations to establish a secret random key between two parties ("Alice" and "Bob"). Bell's inequalities are used to check the presence of an intermediate eavesdropper ("Eve"). We will call this method the Einstein-Podolsky-Rosen-Bell-Ekert (EPRBE) quantum cryptographic protocol. There are two stages to the EPRBE protocol, the first stage over a quantum channel, the second over a public channel.

The quantum channel consists of a source that emits pairs of spin one-half particles, in a singlet state. The particles fly apart towards Alice and Bob, who, after the particles have separated, perform measurements on spin components along one of three directions, given by unit vectors a and b. In the second stage Alice and Bob communicate over a public channel.They announce in public the orientation of the detectors they have chosen for particular measurements. Then they divide the measurement results into two separate groups: a first group for which they used different orientation of the detectors, and a second group for which they used the same orientation of the detectors. Now Alice and Bob can reveal publicly the results they obtained but within the first group of measurements only. This allows them, by using Bell's inequality, to establish the presence of an eavesdropper (Eve). The results of the second group of measurements can be converted into a secret key. One supposes that Eve has a detector which is located within the region $\mathcal{O}_E$ and she is described by hidden variables λ.

We will interpret Eve as a hidden variable in a realist theory and will study whether quantum correlations could be represented as classical correlations. From the previous discussion one can see that if the following inequality

$$g(\mathcal{O}_A, \mathcal{O}_B) \leq 1/\sqrt{2} \tag{6.19}$$

is valid for regions $\mathcal{O}_A$ and $\mathcal{O}_B$ which are well separated from one another then there is no violation of the CHSH inequalities (2.7) and therefore Alice and Bob can not detect the presence of an eavesdropper. On the other side, if for a pair of well separated regions $\mathcal{O}_A$ and $\mathcal{O}_B$ one has

$$g(\mathcal{O}_A, \mathcal{O}_B) > 1/\sqrt{2} \tag{6.20}$$

then it could be a violation of the realist locality in these regions for a given state. Then, in principle, one can hope to detect an eavesdropper in these circumstances.

Note that if we set $g(\mathcal{O}_A, \mathcal{O}_B) = 1$ as it was done in the original proof of Bell's theorem, then it means we did a special preparation of the states of particles to be completely localized inside of detectors. There exist such well localized states (see however the previous Remark) but there exist also another states, with the wave functions which are not very well localized inside the detectors, and still particles in such states are also observed in detectors. The fact that a particle is observed inside the detector does not mean, of course, that its wave function is strictly localized inside the detector before the measurement. Actually one has to perform a thorough investigation of the preparation and the evolution of our entangled states in space and time if one needs to estimate the function $g(\mathcal{O}_A, \mathcal{O}_B)$.

Let us remind that one has the expansion of wave packets due to the free time evolution and the locality criterion is always satisfied for nonrelativistic particles if regions $\mathcal{O}_A$ and $\mathcal{O}_B$ are far enough from each other.

7 Conclusions

We have discussed some problems in quantum information theory which requires the inclusion of spacetime variables. In particular entangled states in space and time were considered. A modification of Bell‘s equation which includes the spacetime variables is suggested and investigated. A general relation between quantum theory and theory of classical stochastic processes is proposed which expresses the condition of local realism in the form of a noncommutative spectral theorem. Applications of this relation to the security of quantum key distribution in quantum cryptography is mentioned.

There are many interesting open problems in the approach to quantum information in space and time discussed in this paper. Some of them related with the noncommutative spectral theory and theory of classical stochastic processes have been discussed above.

Entangled states in space and time have been considered. It is shown that any reasonable state in relativistic quantum field theory becomes disentangled at large spacelike distances if one makes local observations. As a result a violation of Bell‘s inequalities can be observed without inconsistency with principles of relativistic quantum theory only if the distance between detectors is rather small. We suggest a further experimental study of entangled states in spacetime by studying the dependence of the correlation functions on the distance between detectors. More considerations of these problems see in [17, 18].

In quantum cryptography there are many interesting open problems which require further investigations. In quantum cryptographic protocols with two entangled photons (such as the EPRBE protocol) to detect the eavesdropper's presence by using Bell's inequality we have to estimate the function $g(\mathcal{O}_A, \mathcal{O}_B)$. In order to increase the detectability of the eavesdropper one has to do a thorough investigation of the process of preparation of the entangled state and then its evolution in space and time towards Alice and Bob. One has to develop a proof of the security of such a protocol.

In the previous section Eve was interpreted as an abstract hidden variable. However one can assume that more information about Eve is available. In particular one can assume that she is located somewhere in space in a region $\mathcal{O}_E$. It seems that one has to study a generalization of the function $g(\mathcal{O}_A, \mathcal{O}_B)$, which depends not only on the Alice and Bob locations $\mathcal{O}_A$ and $\mathcal{O}_B$ but also on Eve's location $\mathcal{O}_E$. Then one can try to find a strategy which leads to an optimal value of this function.

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