Advanced Studies in Theoretical Physics
Vol. 19, 2025, no. 2, 43 - 54
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/astp.2025.92247

Dirac Equation in Newman Penrose Formalism: Separation in Schwarzschild Space Time with Torsion

Antonio Zecca

Dipartimento di Fisica dell' Universita' degli Studi di Milano (Retired)
Via Celoria, 16 20133 Milano
GNFM *Gruppo Nazionale per la Fisica Matematica* of the INdAM, Italy

Abstract

The Dirac equation, previously formulated in a general space time with torsion by the Newman Penrose formalism, is considered in the context of the Schwarzschild space time with torsion. Based on a suitable null tetrad frame, the equation is separated by a variable separation method. The separated angular equations are integrated. The separated radial dependence is reduced, on the base of elementary properties of the solutions, to the solution of a single non linear one dimensional differential equation in the wave function and in its complex conjugate.

Keywords: Schwarzschild metric, Newman Penrose formalism, Dirac equation with torsion, Separation

1 Introduction

The consideration of torsion in the formulation of field equations in flat and curved space time is of interest. It generally introduces new degrees of freedom that are useful to describe further physical interactions [10, 4]. On the other hand the torsion induced terms are in general non linear terms that heavily limit the explicit solution of the equations [11, 16]. From a mathematical point of view, this is indeed a difficult problem.

The extensions of field equation to include torsion, is of interest on account of the wide field of theoretical applications that run from cosmology (se,e e.g. [11, 15]) to particle physics (e. g., see [7, 9]).

In particular, the role of torsion is evident in physical applications of spin 1/2 particle with torsion. Torsion effect on neutrino oscillations can be evaluated [1], in particular in case of axial symmetry [27]. Moreover, by extending the result of ref. [5] obtained in Schwarzschild metric, resonance and adiabatic propagation seem possible in vacuum for cosmological neutrinos with torsional self interaction [18]. Also the modification of the spectrum of the Hydrogen atom induced by torsion can be evaluated perturbatively [17].

On theoretical grounds, the Dirac equation with torsion is generally formulated by the 4-dimensional spinor formalism [11, 2, 8]. Its form can be obtained, in the coordinate formalism, from a total action sum of the Einstein-Hilbert-Cartan action and the Dirac one, by varying with respect to the Dirac spinor and to the independent components into which the torsion tensor can be decomposed. The resulting equation, can then be converted into the Newman Penrose formalism [19]. In a more general way the Dirac equation with torsion was finally completely derived within the two spinor formalism [20], suitable to be extended to arbitrary spin field equation [24].
The equation is also of cosmological interest. In that connection, one can see that the effect of torsion on the Hydrogen energy spectrum in the standard cosmology is practically negligible at the present time [17]. Some aspects of the possible solutions in Robertson Walker space time have been discussed in [26]. Further application, such as the interaction of the gravitational and Dirac field with torsion, have been considered in general [21] as well as in Minkowski space time [23, 22], where standing wave solutions have been determined.

In this paper the starting formulation of Dirac equation with torsion is the one derived from a suitable Lagrangian directly formulated in the two spinor language [25]. The equation is studied in the Schwarzschild space time with torsion by the Newman Penrose formalism. The separation method employed is an application of the one previously adopted to separate the Dirac equation in RW and Minkowski space-time with torsion [23, 26] that in turn is an extension of Chandrasekhar method [6], adopted to separate the Dirac equation in the torsion free Kerr space time case. The Schwarzschild metric being static, the time dependence of the Dirac wave function factors out also in presence of torsion and it is easily integrated. The separated angular equations are integrated on the base of the results in the torsion free spherically symmetric space time case. The radial separated dependence amounts to two non linear differential equations in the two radial functions. The study of the radial equations can be reduced to the study of single first order non linear differential equation involving however a radial function and its complex conjugate function. Such reduction is possible on the base of some simple results that relate the two radial functions solutions. The problem of solving the final separated radial equation as well of providing particular solutions, is not answered here.

2 Preliminary assumptions

The following considerations are developed within a four dimensional (curved) space-time of metric tensor g_{ik} with associated spinor and tensor formalism. For notations and mathematical conventions we refer to [15]. The standard correspondence between complex tensors of rank n and spinors of type (n, n) (that will be denoted by $\leftrightarrow$) can be realized by the van derWaerden σ-matrices. In particular the covariant tensor and spinor derivatives are related by $\nabla_{AA'} = \sigma^{\alpha}_{AA'}\nabla_{\alpha}$. Further more the Newman Penrose formalism [14] is assumed. Accordingly one considers a null tetrad frame $\{l^i, n^i, m^i, m^{*i}\}$, with l^μ, n^μ and such that $m^{\star\mu} = m^{\mu\star}$. Therefore $l^\mu l_\mu = n^\mu n_\mu = m^\mu m_\mu = m^{\star\mu} m^\star_\mu = 0,\ l^\mu n_\mu = 1,\ m^\mu m^\star_\mu = -1$. Associated to the null tetrad frame there are then the directional derivatives:

$$D = \partial_{00'} = l^i\partial_i, \quad \delta = \partial_{01'} = m^i\partial_i, \quad \delta^* = \partial_{10'} = m^{*i}\partial_i \quad \Delta = \partial_{11'} = n^i\partial_i \quad (1)$$

The σ-matrices can be represented by (e.g., [15, 12]):

$$G_\mu \equiv \sigma^{A}_{\mu B'} \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} m^\star_\mu & n_\mu \\ -l_\mu & -m_\mu \end{bmatrix}, \quad G^+_\mu G_\nu + G_\nu G^+_\mu = -2g_{\mu\nu} I_2 \quad (2)$$

in terms of which the 4-dimensional Dirac gamma matrices can be represented:

$$\gamma_\mu = \sqrt{2} \begin{bmatrix} 0 & \sigma^{AB'}_{\mu} \\ \sigma^+_{\mu CD'} & 0 \end{bmatrix}, \quad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\, I_4 \quad (3)$$

In many applications, as in the case of the present paper, the σ-matrices result to be selfadjoint. Under such property one can directly check that the expression $J_{AB'} = P_A \bar{P}_{B'} + \bar{Q}_A Q_{B'}$ is the corresponding of the Dirac 4-current in the two spinor formalism: $J_{AB'} = \sigma^\mu_{AB'} J_\mu$ where $J_\mu = \psi^\dagger\gamma_0\gamma_\mu\psi$, $\gamma_0 = \left(\begin{smallmatrix} 0 & I_2 \\ I_2 & 0 \end{smallmatrix}\right)$.

In the following the object is of separating the Dirac equation in the Schwarzschild space time with torsion. The equation will be derived by the two spinor Lagrangian formalism. Hence the four dimensional space time is endowed by a covariant derivative with torsion $\widetilde{\nabla}$ and by the Levvi-Civita connection ∇. They act in the same way on scalars, $\widetilde{\nabla}_{AB'} f = \nabla_{AB'} f$, ∇ the Levvi-Civita connection. On the contravariant spinor they are related by:

$$\widetilde{\nabla}_{AB'}\xi^C = \nabla_{AB'}\xi^C + \Theta_{AB'}{}^{C}{}_{D}\xi^D \quad (4)$$

$$\widetilde{\nabla}_{AA'}\chi^{PS'} = \nabla_{AA'}\chi^{PS'} + \Theta_{AA'X}{}^{P}\chi^{XS'} + \overline{\Theta}_{A'AX'}{}^{S'}\chi^{PX'} \quad (5)$$

$$\Theta_{AB'CD} = \Theta_{AB'DC} \quad (\widetilde{\nabla}_{AB'}\,\epsilon_{CD} = 0, \quad \nabla_{AB'}\,\epsilon_{CD} = 0) \quad (6)$$

$$\overline{\Theta}_{A'BC'D'} = \overline{\Theta}_{BA'C'D'} \quad (7)$$

Θ the torsion spinor.

As to the total Lagrangian L, it is defined by the sum of the gravitational and of the Dirac Lagrangian: $L = L_g + L_d$.

As usual, L_g is assumed to be the Einstein-Hilbert-Cartan lagrangian density defined by $L_g = \sqrt{g}\,\widetilde{R}$, $(g = |det\, g_{ik}|,\ \widetilde{R} = \widetilde{R}_{\alpha\beta}^{\ \ \alpha\beta},\ R = R_{\alpha\beta}^{\ \ \alpha\beta})$. ($\tilde{R}$ the total scalar curvature, R the torsion free part of $\widetilde{R}$). By decomposing the scalar curvature into the torsion and into the torsion free part [15] one obtains:

$$L_g = \sqrt{g}[R + Q_{ac}{}^{b}Q_{b}{}^{ac} - Q_{bc}{}^{b}Q_{a}{}^{ac}] \tag{8}$$
$$Q_{abc} \leftrightarrow \Theta_{AA'BC}\,\epsilon_{B'C'} + \overline{\Theta}_{AA'B'C'}\,\epsilon_{BC}\ \ (a \equiv AA',\ b \equiv BB',\ c \equiv CC') \tag{9}$$

Inserting the spinor expression of Q into L_g and using (6), (7) one finally obtains

$$L_g = \sqrt{g}\Big\{\ R + \frac{4}{3}(Z_{B'D}Z^{B'D} + \bar{Z}_{BD'}\bar{Z}^{BD'}) \\ -\Theta_{(A|A'|BC)}\Theta^{(A|A'|BC)} - \bar{\Theta}_{(A|A'|B'C')}\bar{\Theta}^{(A|A'|B'C')}\Big\} \tag{10}$$

where $Z_{A'B} = \Theta_{XAB'}{}^{X}$ ([25], see also [3]). A divergence term in the complete expression of $\widetilde{R}$ has been neglected because no variation of the boundary will be considered when applying the action principle.

For what concerns the Dirac Lagrangian with torsion in the two spinor formalism we follow Ref. [25]. Accordingly a spin-(1/2) particle of mass m_o is described by two two spinors $P \equiv (P^A)$ and $Q \equiv (Q_{A'})$ of generalized Dirac Lagrangian

$$L_d = \sqrt{g}\Big\{i\sqrt{2}\Big[-\overline{Q}^{A}\nabla_{AX'}Q^{X'} + \overline{P}^{X'}\nabla_{AX'}P^{A}\Big] \\ +\alpha\Big[\overline{Q}^{A}Z_{B'A}Q^{B'} + P^{A}Z_{B'A}\overline{P}^{B'}\Big] + \overline{\alpha}\Big[Q^{A'}\overline{Z}_{BA'}\overline{Q}^{B} + \overline{P}^{A'}\overline{Z}_{BA'}P^{B}\Big] \\ -m_o\big(-\overline{Q}_{A}P^{A} + \overline{P}_{X'}Q^{X'}\big)\Big\} \tag{11}$$

α a complex parameter, m_o the mass of the Dirac particle. Such expression is in the line of Ref. [25], but it is not exactly the same.

3 Dirac equation with torsion

The equation of motion can now be obtained by the Euler-Lagrange equation:

$$\frac{\partial L}{\partial \eta} - \nabla_{XY'}\left(\frac{\partial L}{\partial \nabla_{XY'}\eta}\right) = 0, \qquad L = L_g + L_d \tag{12}$$

for each one of the spinor involved. By applying (12) to L for $\eta = \overline{P}^{A'}$, $\overline{Q}^{A}$, $\Theta_{(A|A'|BC)}$, $Z_{A'A}$ and $\overline{Z}_{AA'}$ one has, respectively,

$$\Big[\nabla_{AB'} - \frac{i}{\sqrt{2}}\Big(\alpha Z_{B'A} + \overline{\alpha}\overline{Z}_{AB'}\Big)\Big]P^A = -i\mu_* Q_{B'} \tag{13}$$

$$\Big[\nabla_{AB'} + \frac{i}{\sqrt{2}}\Big(\alpha Z_{B'A} + \overline{\alpha}\overline{Z}_{AB'}\Big)\Big]Q^{B'} = -i\mu_* P_A \tag{14}$$

$$\Theta_{(A|A'|BC)} = 0 \tag{15}$$

$$Z^{A'A} = -\frac{3}{8}\alpha(P^A\overline{P}^{A'} + Q^{A'}\overline{Q}^{A}) \tag{16}$$

$$\overline{Z}^{AA'} = -\frac{3}{8}\overline{\alpha}(P^A\overline{P}^{A'} + Q^{A'}\overline{Q}^{A}) \tag{17}$$

If instead one chooses $\eta = P^A$, $Q^{A'}$ in (12), one obtains the complex conjugate equations of eqs. (13), (14) respectively.

From (16), (17) the spinorial current $J^{AA'} = P^A\overline{P}^{A'} + Q^{A'}\overline{Q}^{A}$ associated to the Dirac field results, for α real, such that :

$$Z^{A'A} = -\frac{3\alpha}{4}J^{AA'} = \overline{Z}^{AA'} \qquad (\overline{\alpha} = \alpha) \tag{18}$$

Finally, by the further substitution $P^A \to -P^A$ into eqs. (13)-(17) one is left with the equations

$$(\nabla_{AB'} - icJ_{AB'})P^A = \frac{im_0}{\sqrt{2}}Q_{B'} \tag{19}$$

$$(\nabla_{AB'} + icJ_{AB'})Q^{B'} = \frac{im_0}{\sqrt{2}}P_A, \qquad c = -(3\alpha^2)/(4\sqrt{2}) \tag{20}$$

$$J_{AB'} = P_A\bar{P}_{B'} + \bar{Q}_A Q_{B'}, \tag{21}$$

m_0 the mass of the particle. It will be called Dirac equation with torsion. As to physical interpretation, since $J_{AB'}$ is the Dirac current in the two spinor form, the interpretation is that the effect of torsion amount to a self interaction of the particle with its own current. Indeed if the current term is dropped in eqs. (19), (20), one is left with the canonical two spinor form of the spin 1/2 particle equation in curved space time (e.g. [15]). One can also check that the Dirac spinor current is conserved, even in presence of the non trivial torsion term:

$$\nabla_{AB'}J^{AB'} = 0 \tag{22}$$

as it can be verified from the solution of the Dirac equation.

The equations (19, 20) can be made explicit in the Newman Penrose formalism. The covariant spinor derivatives can be expanded in terms of the directional derivatives and of the spin coefficients [15, 6]. The equation (19), (20) can be conveniently written:

$$(D-\rho+\epsilon)P_1-(\delta^*-\pi-\alpha)P_0=i\Big(\frac{m_0}{\sqrt{2}}+c\bar{Q}_E P^E\Big)Q_0 \qquad (23)$$

$$(\delta-\tau+\beta)P_1-(\Delta-\gamma+\mu)P_0=i\Big(\frac{m_0}{\sqrt{2}}+c\bar{Q}_E P^E\Big)Q_1 \qquad (24)$$

$$(D-\bar{\rho}+\bar{\epsilon})Q_1-(\delta+\bar{\pi}-\bar{\alpha})Q_0=i\Big(\frac{m_0}{\sqrt{2}}-cQ^E\bar{P}_E\Big)P_0 \qquad (25)$$

$$(\delta^*-\bar{\tau}+\bar{\beta})Q_1-(\Delta-\bar{\gamma}+\bar{\mu})Q_0=i\Big(\frac{m_0}{\sqrt{2}}-cQ^E\bar{P}_E\Big)P_1 \qquad (26)$$

In view of the following developments it is useful to set (see [6]):

$$(P_0,P_1)=(\,-F_2,F_1), \qquad (Q_0,Q_1)=(G_1,G_2) \qquad (27)$$

so that equations (23)-(26) read:

$$(D-\rho+\epsilon)F_1+(\delta^*+\pi-\alpha)F_2=i\Big[\frac{m_0}{\sqrt{2}}+c(\bar{G}_1F_1+\bar{G}_2F_2)\Big]G_1 \qquad (28)$$

$$(\Delta-\gamma+\mu)F_2+(\delta-\tau+\beta)F_1=i\Big[\frac{m_0}{\sqrt{2}}+c(\bar{G}_1F_1+\bar{G}_2F_2)\Big]G_2 \qquad (29)$$

$$(D-\bar{\rho}+\bar{\epsilon})G_2-(\delta+\bar{\pi}-\bar{\alpha})G_1=-i\Big[\frac{m_0}{\sqrt{2}}+c(G_1\bar{F}_1+G_2\bar{F}_2)\Big]F_2 \qquad (30)$$

$$(\Delta-\bar{\gamma}+\bar{\mu})G_1-(\delta^*-\bar{\tau}+\bar{\beta})G2=-i\Big[\frac{m_0}{\sqrt{2}}+c(G_1\bar{F}_1+G_2\bar{F}_2)\Big]F_1 \qquad (31)$$

4 Schwarzschild space time with torsion

The object is now to study Dirac equation with torsions in the Schwarzschild space time metric $g_{\mu\nu}$ of the form

$$ds^2=\Big(1-\frac{2M}{r}\Big)dt^2-\Big(1-\frac{2M}{r}\Big)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2) \qquad (32)$$

As to the Newman Penrose formalism, it will be based on the null tetrad frame whose corresponding directional derivatives and non zero spin coefficients are given by (see, e.g., [6]):

$$D=\frac{r}{r-2M}\partial_t+\partial_r, \qquad \Delta=\frac{1}{2}\partial_t-\frac{r-2M}{2r}\partial_r \qquad (33)$$

$$\delta=\frac{1}{r\sqrt{2}}\partial_\theta+\frac{i}{\sin\theta\sqrt{2}}\partial_\varphi, \qquad \delta^*=\overline{(\delta)}=\frac{1}{r\sqrt{2}}-\frac{i}{r\sin\theta\sqrt{2}}\partial_\varphi \qquad (34)$$

$$\rho=-\frac{1}{r}, \qquad \mu=\frac{2M-r}{2r^2}, \quad \gamma=\frac{M}{2r^2}, \quad \beta=-\alpha=\frac{1}{2\sqrt{2}}\frac{\cot\theta}{r} \qquad (35)$$

The equations (28)-(31) can be separated by setting

$$(F_1, F_2) \equiv \Big(R_1(r,t)S_1(\theta),\ R_2(r,t)S_2(\theta)\Big)e^{im\varphi} \tag{36}$$

$$(G_1, G_2) \equiv \Big(R_2(r,t)S_1(\theta),\ R_2 r,t)S_2(\theta)\Big)e^{im\varphi} \tag{37}$$

$$S_1(\theta)\overline{S_1(\theta)} = S_2(\theta)\overline{S_2(\theta)} = 1 \tag{38}$$

Accordingly the angular dependence separates . By defining $L_\pm = \partial_\theta \mp \frac{m}{\sin\theta} + \frac{1}{2}\cot\theta$, from equation (28), (30) one obtains

$$L_- S_2 = -\lambda S_1, \quad L_+ S_1 = \lambda S_2, \tag{39}$$

λ the angular separation constant that is the same for the two equation for which the r, t dependence remains the same. Similarly the equations (29), (31) separate giving again equation (39) and identical r, t separated equations but different from those of (28), (30) so that the separated r, t dependent equations remain ($R = R(r,t)$):

$$(D - \rho)R_1 = \frac{1}{\sqrt{2}}\Big[im_0 + ic\sqrt{2}\Big(\bar{R}_2 R_1 + \bar{R}_1 R_2\Big) + \frac{\lambda}{r}\Big]R_2 \tag{40}$$

$$(\Delta - \gamma + \mu)R_2 = \frac{1}{\sqrt{2}}\Big[im_0 + ic\sqrt{2}\Big(\bar{R}_2 R_1 + \bar{R}_1 R_2\Big) - \frac{\lambda}{r}\Big]R_1 \tag{41}$$

For what concerns the solution of the angular separated equation (39) they give rise to the closed equation in S_1, S_2:

$$L_- L_+ S_1 = -\lambda S_1, \qquad L_+ L_- S_2 = -\lambda S_2 \tag{42}$$

Those equations represent the angular eigenvalue problem for spin 1/2 particle in spherically symmetric space time (e. g., [13]). The solutions are essentially given by $S_i = S_{lm}(\theta), i = 1, 2$, under the usual regularity condition, the S_{lm}'s being the Jacobi ($m \neq 0$) and Chebicheff ($m = 0$) polynomials [13]. In order to satisfy the condition (38) we set $S_2(\theta) = \exp(is_2(\theta), s_2 \in R$. Then, e. g., $\cos s_2(\theta)$ satisfies the same equation of S_2. One can then define $\cos s_2(\theta) = S_{lm}(\theta)$ by choosing the integration constant such that $|S_{lm}(\theta)| \leq 1$. Hence we assume:

$$S_2(\theta) = \exp(i\cos^{-1}(S_{lm}(\theta)) \tag{43}$$

Similarly we define $S_1 = \exp(i\cos^{-1}(S_{l-m}(\theta))$ since the equations (39) interchange themselves by the substitution $m \to -m$. Moreover [13]:

$$\lambda^2 = (l + 1/2)^2, \quad |m| \geq 1, \quad l = |m|, |m|+1, |m|+2, ... \tag{44}$$

$$\lambda^2 = (l+1)^2, \quad m = 0, \quad l = 0, 1, 2, ... \tag{45}$$

4.1 Properties of the radial equations

By further setting $R_i = R_i(r)e^{ikt}$, $i = 1, 2$, the equations (40), (41) read:

$$\Big(ik\frac{r^2}{\triangle} + \frac{1}{r} + \partial_r\Big)R_1 = M^+ R_2 \qquad (\triangle = r^2 - 2Mr) \tag{46}$$

$$\Big(\frac{ik}{2} + \frac{M-r}{2r^2} - \frac{\triangle}{2r^2}\partial_r\Big)R_2 = M^- R_1 \tag{47}$$

$$M_\pm = \frac{1}{\sqrt{2}}\Big[im_0 + ic\sqrt{2}(\bar{R}_2 R_1 + \bar{R}_1 R_2) \pm \frac{\lambda}{r}\Big] \tag{48}$$

Note that $\bar{M}_+ = -M_-$ and $\bar{M}_- = -M_+$- We have the following two results:

i) By comparing (46) and the complex conjugate of equation (47) gives

$$\frac{\Big(ik\frac{r^2}{\triangle} + \frac{1}{r}\Big)R_1 + R_1'}{R_2} = -\frac{\Big(-\frac{ik}{2} + \frac{M-r}{2r^2}\Big)\bar{R}_2 - \frac{\triangle}{2r^2}\overline{R}_2'}{\overline{R}_1} \tag{49}$$

By developing the equation (49) and summing the result with its complex conjugate, one finally obtains ($' = d/dr$)

$$\Big[r^2 R_1 \bar{R}_1\Big]' = \Big[\frac{\triangle}{2} R_2 \bar{R}_2\Big]' \tag{50}$$

and by choosing a vanishing integration constant

$$|R_1|^2 = \frac{\triangle}{2r^2}|R_2|^2 \tag{51}$$

ii) By multiplying (46) by $\bar{R}_2\triangle/(2r^2)$ one obtains:

$$\frac{\triangle}{2r^2}\Big[\frac{ikr^2}{\triangle} + \frac{1}{r}\Big]R_1\bar{R}_2 + R_1'\bar{R}_2\Big] = M_+ R_1 \bar{R}_1 \tag{52}$$

On the other hand , by multiplying the complex conjugate of (47) by R_1one has

$$\Big(-\frac{ik}{2} + \frac{M-r}{2r^2}\Big)R_1\bar{R}_2 - \frac{\triangle}{2r^2}R_2'\bar{R}_2 = -M_+ R_1 \bar{R}_1 \tag{53}$$

Summing up (52), (53) one finally obtains

$$MR_1\overline{R}_2 = \triangle\Big[R_1'\overline{R}_2 - \overline{R}_2' R_1\Big] \tag{54}$$

whose integration, gives

$$R_1 = \overline{R}_2\sqrt{\frac{\triangle}{2r^2}}, \qquad \overline{R}_2 = R_1\sqrt{\frac{2r^2}{\triangle}}, \tag{55}$$

The multiplicative constant integration factor has being chosen to be $\frac{1}{\sqrt{2}}$ so that the result implies exactly (51).

By the last results and by the identity $R_1^2 + \bar{R}_1^2 = 2|R_1|^2$, the equation (46) reduces to:

$$\Big(ik\frac{r^2}{\triangle} + \frac{1}{r} + \partial_r\Big)R_1 = \frac{1}{\sqrt{2}}\Big[im_0 + 2ic\sqrt{2}\sqrt{\frac{2r^2}{\triangle}}|R_1|^2 + \frac{\lambda}{r}\Big]\sqrt{\frac{2r^2}{\triangle}}\ \overline{R}_1 \tag{56}$$

One can indeed check that the complex conjugate of (47) gives equation (46) by the substitution $\bar{R}_2 \to R_1\sqrt{\frac{2r^2}{\triangle}}R_1,\ \ R_1 \to \bar{R}_2\sqrt{\frac{\triangle}{2r^2}}$.

Therefore, once the solution R_1 has been determined, the solution R_2 will follows from the second relation in (55). One is then left with one only radial equation in both R_1 and $\bar{R}_1$.

5 Remarks and comments

In this paper the Dirac equation with torsion, in the formulation considered in previous papers, has been studied in the Schwarzschild metric. The main result is that the equation can be separated in a similar way to the separation of the torsion free Dirac equation in spherically symmetrical static space time. Indeed the time dependence of the wave function can be easily obtained on account of the fact that the metric is static. The separated angular dependence is integrated, by a suitable assumption, on the base of the results of the torsion free spherically symmetrical case. As to the radial separated dependence, it has been reduced to the single equation (56) that however involves a radial function with its complex conjugates. [It must be remarked that the generality of the result depends however on the choice of the integration constant in (54)]. Once the final radial equation (56) has been separated into the real and imaginary part, the problem reduces to the solution of two real coupled non linear equation. This seems a better situation with respect to problem of solving the two non linear complex coupled equations (46), (47).

The present procedure of separation could also be applied to the study of the Dirac equation with torsion in case of Minkowski [23] and Robertson Walker metric [26]. In particular, in the Robertson Walker metric case, one would have, instead of (55), the further simplification $R_1 = \overline{R}_2$.

The object of the present paper was of providing the separation of the Dirac equation in the Schwarzschild metric by the Neuman Penrose formalism. This is of interest as non trivial application of the Neuman Penrose formalism. Of course the important purpose would be then of providing the final solution of the equation. Not less important it would be also of providing particular solutions of eq. (56). They could be of primary interest as they are the standing plane waves solutions determined in Minkowski space time with torsion (e. g., [23]).

References

[1] M. Alimohammadi, A. Shariati, Neutrino oscillation in a space-time with torsion, *Modern Physics Lett. A*, **14** (1999), no. 267.
https://doi.org/10.1142/s0217732399000316

[2] J. Audretsch, Dirac electron on space-tome with torsion: Spinor propagation, spin precession, and non geodesic orbits, *Physical Review D*, **24** (1981), 1470.
https://doi.org/10.1103/physrevd.24.1470

[3] H. Balasin, C. G., Bhmer, C. D. Grumiller, The spherically symmetric Standard Model with gravity, *Gen. Relativ. Gravit.*, **37** (2005), 1435.
https://doi.org/10.1007/s10714-005-0128-6

[4] IL Buchbinder, SD Odintsov, IL Shapiro, *Effective Action in Quantum Gravity*, Institue of Physics Publishing, London 1992.

[5] C. Y Cardall, F.G. Fuller, Neutrino oscillations in curved spacetime: A heuristic treatment, *Phys. Rev. D*, **55** (1997), no. 2760.
https://doi.org/10.1103/physrevd.55.7960

[6] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Press. New York, 1983.

[7] A. Dobado, A. L. Maroto, Standard model anomalies in curved space-time with torsion *Physical Review D*, **54** (1996), no. 5185.
https://doi.org/10.1103/PhysRevD.54.5185

[8] L.Fabbri, Torsion Gravity for Dirac fields and their effective phenomenology, *Modern Physics Letters A*, **29** (2014), 1450133.
https://doi.org/10.1142/s0217732314501338

[9] R. T. Hammond, Strings in Gravity with Torsion, *General Relativity and Gravitation*, **32** (2000). https://doi.org/10.1023/a:1001942301598

[10] F.W. Hehl, B.K. Datta, Nonlinear Spinor Equation and Asymmetric Connection in General Relativity, *J. Mat. Phys.*, **12** (1971), 1334.
https://doi.org/10.1063/1.1665738

[11] F.W. Hehl, P. von der Heyde, G. D. Kerlick and G.D. Nester, General relativity with spin and torsion: Foundations and Prospects, *Review of Modern Physics*, **48** (1976), 393. https://doi.org/10.1103/revmodphys.48.393

[12] R. Illge: Massive Fields of Arbitrary Spin in Curved Space-Times, *Communications in Mathematical Physics*, **158** (1993), 433.
https://doi.org/10.1007/bf02096798

[13] E. Montaldi, A. Zecca, Neutrino Wave Equation in the Robertson-Walker Geometry, *International Journal of Theoretical Physics*, **33** (1994), no. 1053. https://doi.org/10.1007/bf01882752

[14] E. T. Newman and R. Penrose, An Approach to Gravitational Radiation by a Method of Spin Coefficients, *J. Math. Phys.*, **3** (1962), no. 566. https://doi.org/10.1063/1.1724257

[15] R. Penrose & W. Rindler, *Spinors and Space-Time: Volume 1, Two-Spinor Calculus and Relativistic Fields*, Cambridge Monographs on Mathematical Physics. Cambridge 1984, Vol. I, II.

[16] I. L. Shapiro: Physical Aspects of the Space-Time Torsion. arXiv:hep-th/0103093.

[17] A. Zecca, Effect of torsion in Dirac equation for Coulomb like potential in Robertson Walker space-time, *International Journal of Theoretical Physics*, **41** (2002), no. 1145.

[18] A. Zecca, Neutrino oscillations in Robertson-Walker space-time with torsion, *Il Nuovo Cimento B*, **119** (2004), no. 81.

[19] A. Zecca, Dirac Equation in Space-time with Torsion, *International Journal of Theoretical Physics*, **41** (2002), no. 421.

[20] A. Zecca, The Dirac equation in the Newman-Penrose formalism with torsion, *Il Nuovo Cimento*, **B117** (2002), no. 197.

[21] A. Zecca, Interacting Gravitational and Dirac Fields with Torsion, *International Journal of Theoretical Physics*, **42** (2003), no. 290. https://doi.org/10.1023/b:ijtp.0000006017.23508.58

[22] A. Zecca, Elementary solutions of Dirac equation with torsion in flat-space-time, *Il Nuovo Cimento,* **118 B** (2003), no. 65.

[23] A. Zecca, Dirac equation with self interaction induced by torsion: Minkowski space-time, *Advanced Studies in Theoretical Physics*, **9** (2015), no. 15, 701 - 708. https://doi.org/10.12988/astp.2015.5986

[24] A. Zecca, Field Equations of Arbitrary Spin in Space-time with Torsion, *International Journal of Theoretical Physics*, **46** (2007), 1045-1054. https://doi.org/10.1007/s10773-006-9258-1

[25] A. Zecca, Dirac equation with self interaction induced by torsion, *Adv. Stud. Theor. Phys.*, **9** (2015), no. 12. https://doi.org/10.12988/astp.2015.5773

[26] A. Zecca, Nonlinear Dirac equation in two-spinor form: Separation in static RW space-time, *Eur. Phys. J. Plus.*, **131** (2016), no. 45. https://doi.org/10.1140/epjp/i2016-16045-3

[27] C. M. Zhang, Mass neutrino flavour evolution in space time with torsion, *Il Nuovo Cimento B*, **115** (2000), no. 437.

Received: March 7, 2025; Published: March 26, 2025