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Grassmann Numbers and Clifford-Jordan-Wigner Representation of Supersymmetry

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Abstract. The elementary particles of Physics are classified according to the behavior of the multi-particle states under exchange of identical particles: bosonic states are symmetric while fermionic states are antisymmetric. This manifests itself also in the commutation properties of the respective creation operators: bosonic creation operators commute while fermionic ones anticommute. It is natural therefore to study bosons using commuting entities (e.g. complex variables), whereas to describe fermions, anticommuting variables are more naturally suited. In this paper we introduce these anticommuting- and at first sight unfamiliar- variables (Grassmann numbers) and investigate their properties. In particular, we briefly discuss differential and integral calculus on Grassmann numbers. Work supported in part by DOE contracts No. DE-AC-0276-ER 03074 and 03075; NSF Grant No. DMS-8917754.

1. Fermionic Analysis

1.1. Grassmann Numbers

Grassmann numbers (θ_m) are defined to satisfy the following anticommutation rule:

$$\{\theta_m, \theta_n\} \equiv \theta_m\theta_n + \theta_n\theta_m \equiv 0 \tag{1}$$

Note that this implies (by setting $m = n$) that the square of a Grassmann number is zero. In addition Grassmann numbers are associative:

$$[\theta_l, \theta_m, \theta_n] \equiv (\theta_l\theta_m)\theta_n - \theta_l(\theta_m\theta_n) \equiv 0 \tag{2}$$

This second property is usually implied and not written explicitly, but it is important since it is possible to construct a system that satisfies eq.(1) but doesn't satisfy eq.(2). It also allows the θ's to have a matrix representation: for instance, in field theory, a fermion operator such as an electron operator can be represented by an infinite matrix. In general, the matrix representation can be either finite or infinite.

This abstract number system was introduced by Grassmann[1] who didn't give a representation for it. He introduced the system in terms of differential operators in the process of inventing differential forms, which provide an example of Grassmann variables. For instance, we see from Maxwell's equations, expressed in terms of differential forms,

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$$df = 0 \quad \text{and} \quad f = dA \quad \Rightarrow d^2 = 0 \tag{3}$$

where A is the four-potential, f is the electromagnetic field tensor and d is the differential operator of the form calculus. We therefore see that d is also a Grassmann operator, although it can't be represented by a finite matrix.

There are simpler systems with finite number of θ's which can be represented by finite matrices. This was first shown by Clifford[2][3] in 1878. Clifford also discovered Clifford algebra; in fact, he invented the algebra in order to find a matrix representation of Grassmann numbers. We will see this explicitly below.

1.2. Grassmann Numbers and the Clifford Algebra

Clifford was a young English mathematician who became a professor at King's college at a very early age. He was years ahead of his time in many respects- he thought that gravitation should be represented as a curvature of Riemannian space, long before Einstein. He also thought that these anticommutation relations must have a deep physical meaning, and was looking to develop them when he died at the age of thirty, postponing the advancement of physics by thirty years or more.

Clifford found that for every Grassmann system, there was a conjugate Grassmann system that obeyed the equations given above, but did not anticommute with the original set of Grassmann numbers. That is, Clifford's conjugate system consists of π's such that:

$$\theta_m \leftrightarrow \pi_m \tag{4}$$

$$\{\pi_m, \pi_n\} = 0 \tag{5}$$

$$[\pi_l, \pi_m, \pi_n] = 0 \tag{6}$$

but where:

$$\{\theta_m, \pi_n\} = \delta_{mn} \tag{7}$$

We recognize this last relation as the algebra of fermion operators (i.e. the Heisenberg uncertainty principle for fermions) where the π's are the momentum conjugates to the fermion operator θ's. Thus Clifford invented the algebra of fermionic operators long before the development of quantum theory. This principle also contains the Pauli exclusion principle. If we identify fermionic creation operators with Grassmann numbers, $\theta^2 = 0$ implies that a single state cannot contain two identical fermions. We will see that in general the total fermionic wave function will naturally be antisymmetrized.

As discussed above, associativity allows us to represent Grassmann numbers in terms of matrices[4] [5][6][7]. Clifford developed a representation of θ's and π's in terms of the elements of the quaternion algebra. Pauli discovered his matrices not knowing that they were essentially quaternions (and therefore the building blocks of the Grassmann matrix representation). He has a footnote to the effect that his friend, Pascual Jordan told him that the matrices he used are quaternion units. Then, around 1928, Jordan and Wigner[8] found a matrix representation of the fermion operators. They introduced fermion creation and annihilation operators which satisfied the Grassmann algebra exactly. They were looking for a fermionic matrix representation, and rediscovered the Clifford algebra, not knowing that Clifford had done it 50 years previously.

We will employ the Jordan-Wigner matrix representation to explicitly demonstrate the connection between the Clifford and Grassmann algebras. This name is unfair to Clifford, so we will refer to it here as the Clifford-Jordan-Wigner (CJW) representation. We will also see how this leads to the Berezin calculus.

We start with the Clifford algebra. A special case of this was constructed by Dirac, known as the Dirac algebra, which corresponds to a Clifford algebra with two sets of quaternion units. (The CJW representation can be built out of any number of quaternion units).

The Clifford algebra consists of non-anti-commuting associative elements:

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \tag{8}$$

$$[\gamma_\mu, \gamma_\nu, \gamma_\lambda] = 0 \tag{9}$$

where $\mu, \nu = 1, \cdots, M$. Again, the associative condition is usually not written, but is necessary for the existence of a matrix representation.

For the purposes of the CJW representation we divide Clifford algebras into two classes: $M = 2N$ (even) and $M = 2N + 1$ (odd). We will show that this is an unnecessary distinction, since every even Clifford algebra ($2N$) corrsponds to an odd algebra with ($2N + 1$) defining elements.

For the case of an even algebra, we divide the set in half:

$$\gamma_1, \cdots, \gamma_N \qquad \text{and} \qquad \gamma_{N+1}, \cdots, \gamma_{2N} \tag{10}$$

We can now explicitly construct two Grassmann sets, by letting:

$$\theta_m = \frac{1}{2}(\gamma_m + i\gamma_{N+m}) \tag{11}$$

and

$$\pi_m = \frac{1}{2}(\gamma_m - i\gamma_{N+m}) \tag{12}$$

It is clear from the anticommutation relations satisfied by the γ_m's (note the normalization factor) that the θ's and π's will independently form Grassmann algebras, but will not mutually anticommute- they will satisfy the conjugate relations above eq.(7). This is the connection between the Grassmann and Clifford Algebras - the Clifford Algebra has embedded in it two sets of conjugate Grassmann Algebras, which satisfy the conjugate relations above.

We can always form one or more γ that anticommutes with the first $2N$ by forming the product of all $2N$ γ's:

$$\gamma_{2N+1} = \varepsilon(N)\gamma_1\gamma_2 \cdots \gamma_{2N} \tag{13}$$

where $\varepsilon(N) = \pm 1, \pm i$ is a normalization factor discussed below. It is clear that $\{\gamma_{2N+1}, \gamma_\mu\} = 0$ from the anticommutation relations eq.(8) and the fact that there are even number of γ's. We will also use:

$$(\gamma_{2N+1})^2 = 1 \tag{14}$$

For the case of an odd number of members in the Clifford algebra, we perform the same construction on the first $2N$ of them. Then we identify γ_{2N+1} as the final anticommuting member. Note that we cannot pull the same trick here and construct another matrix which anticommutes with all $2N+1$ γ's from a product of these - if we use all $2N+1$ in the product, it will contain an odd number of γ's and won't anticommute with them. If we use an even subset to form a product, any member of the algebra not in the subset will commute with the new γ, rather than anticommute. In fact, the product of all $2N + 1$ γ's will be proportional to the unit matrix.

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1.3. Clifford - Jordan - Wigner Construction

A systematic method of constructing representations of Clifford Algebras in N dimensions was devised by Clifford in 1878 and rediscovered by Jordan and Wigner 50 years later. We will now consider the first few cases in detail, and then give the general scheme.

For the $N = 1$ case, we define

$$\gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{15}$$

$$\gamma_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{16}$$

$$\gamma_3 = \sigma_3 = -i\sigma_1\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{17}$$

These are the familiar Pauli matrices which satisfy:

$$[\sigma_i, \sigma_j] = 2\epsilon_{ijk}\sigma_k \tag{18}$$

and

$$(\sigma_i)^2 = 1 \tag{19}$$

The two conjugate Grassmann algebras contain one element each:

$$\theta = \theta_1 = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{20}$$

and

$$\pi = \pi_1 = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{21}$$

where, clearly,

$$\theta^2 = \pi^2 = 0 \tag{22}$$

and

$$\{\theta, \pi\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{23}$$

A mathematical digression is in order. We see that for $N = 1$, the γ's are the generators of $SO(2)$. In general, for $2N + 1$, the γ's will be generators of $SO(2N)$. For more physically interesting cases, such as systems invariant under $SO(3, 1)$, we must also deal with antihermitian matrices. If we modify the Clifford anticommutation relation eq.(8) to

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{24}$$

where $\eta_{\mu\nu}$ is a metric with signature (p, q), we will have p hermitian and q antihermitian γ's, which are invariant under $SO(p, q)$. This corresponds, for the Dirac algebra, to taking $i\gamma_4 \to \gamma_0$.

If γ has a matrix representation, then a similarity transformation

$$\gamma \to S\gamma S^{-1} \tag{25}$$

is another matrix representation of the Clifford Algebra. Similarity transformations do not in general preserve positivity conditions. If γ's are hermitian ($\gamma^\dagger = \gamma$), then in order to preserve hermiticity, the matrices must be transformed by a unitary matrix:

$$\gamma \to U\gamma U^{-1} \tag{26}$$

where

$$UU^\dagger = 1 \tag{27}$$

If hermiticity is unimportant, S can be any matrix. From the construction above, if γ_{2N+1} is to be hermitian, then it is necessary to put a factor of i in front of the product of $2N$ γ matrices. It's always possible to find a matrix representation where the θ's and π's are chosen to be real (this corresponds to Majorana representation) although in this representation they will not be hermitian, and thus cannot represent observables.

Incidentally, the $N = 1$ case shows clearly the connection between Pauli matrices, the Clifford Algebra and quaternions. We construct the three antihermitian matrices from the hermitian Pauli matrices:

$$e_i = -i\sigma_i \tag{28}$$

These satisfy

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \tag{29}$$

This is precisely the quaternion algebra. It also follows from eq.(8) that

$$\{e_i, e_j\} = -\delta_{ij} \tag{30}$$

so that two antihermitian matrices give us a metric $(0, 2)$ as expected, and we see that Pauli matrices are just the hermitian version of quaternion units.

The special case discussed by Dirac[10] in 1928 corresponds to $N = 2$, or four elements in the algebra: γ_1, γ_2, γ_3, γ_4. These are the Euclidean analog of the more familiar Dirac matrices, where $\gamma_0 = i\gamma_4$. Dirac also constructed the fifth anticommuting matrix,

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = i\gamma_0\gamma_1\gamma_2\gamma_3 \tag{31}$$

Dirac needed 4×4 matrices to describe the case $N = 2$. For the general case we will need $2^N \times 2^N$ dimensional matrices.

For the $N = 2$ case, Dirac took two sets of $N = 1$ Clifford algebras (i.e. Pauli matrices, σ_i, ρ_i) which mutually commute:

$$[\sigma_i, \rho_j] = 0 \tag{32}$$

The resulting Clifford matrices will have dimension $2^2 = 4$. we construct them first by taking $\sigma_i = \sigma_i \times I$ and $\rho_i = I \times \rho_i$:

$$\sigma_i = \begin{pmatrix} \sigma_i & o \\ 0 & \sigma_i \end{pmatrix} = \sigma_i I \quad , \tag{33}$$

$$\rho_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \rho_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{34}$$

where

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$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{35}$$

The σ's will obviously commute with the ρ's. for the $N = 2$ case, if we identify

$$\sigma_i^{(1)} = \sigma_i \tag{36}$$

$$\sigma_i^{(2)} = \rho_i \tag{37}$$

we can set up the table

$$\begin{array}{cccc} \gamma_1 & \sigma_1^{(1)} & \sigma_2^{(1)} & \gamma_3 \\ \gamma_2 & \sigma_3^{(1)}\sigma_1^{(2)} & \sigma_3^{(1)}\sigma_2^{(2)} & \gamma_4 \\ & \sigma_3^{(1)}\sigma_3^{(2)} & \gamma_5 & \end{array} \tag{38}$$

where the γ's are identified with the product of γ's in the column they are closest to.

We have used these matrix representations for N=2 case in supersymmetric chiral field theories developed earlier (see Catto references in the biblography). The matrix forms allow separation of equations and make calculations clear and more easier. Particular generalization for quark-diquark symmetries, multiquark systems and their mass predictions as well as applications to multi instanton processes were shown in Catto and Gürsey's many papers.

There are actually many different representations possible. Dirac used a unitary transformation of these, such that his γ's were

$$\gamma_1 \cdots \gamma_5 \to \rho_1, \rho_3, \rho_2\sigma_i \tag{39}$$

Other possibilities are

$$\gamma_1 \cdots \gamma_5 \to \sigma_1, \sigma_3, \sigma_3\rho_i \tag{40}$$

$$\gamma_1 \cdots \gamma_5 \to \rho_1, \rho_2, \rho_3\sigma_i \tag{41}$$

Dirac identified γ_4 with ρ_3; Weyl used $\gamma_5 = \rho_3$. All of these are specific representations of the algebra, but the theory is independent of the one used. It is only when an explicit representation of a wavefunction is needed that a particular representation will be used.

We can continue this construction technique for the $N = 3$ case. We use three sets of commuting Pauli matrices (i.e. $\sigma \times I \times I$, $I \times \sigma \times I$, $I \times I \times \sigma$):

$$\begin{array}{cccc} \gamma_1 & \sigma_1^{(1)} & \sigma_2^{(1)} & \gamma_4 \\ \gamma_2 & \sigma_3^{(1)}\sigma_1^{(2)} & \sigma_3^{(1)}\sigma_2^{(2)} & \gamma_5 \\ \gamma_3 & \sigma_3^{(1)}\sigma_3^{(2)}\sigma_1^{(3)} & \sigma_3^{(1)}\sigma_3^{(2)}\sigma_2^{(3)} & \gamma_6 \\ & \sigma_3^{(1)}\sigma_3^{(2)}\sigma_3^{(3)} & \gamma_7 & \end{array} \tag{42}$$

Recall that σ_1, σ_3 are real, σ_2 is imaginary and all three are hermitian. We see that all the entries in the first column are real and hermitian, while all the entries in the second column are imaginary and hermitian. Thus $\gamma_{1,2,3,7}$ are real and symmetric, while $\gamma_{4,5,6}$ are imaginary and antisymmetric.

When these are plugged into the relations defining the θ's and π's in terms of the γ's (eqs.(11,12) it is clear that, by construction, the θ's and π's will all be real. This procedure can be extended indefinitely, yielding the Clifford - Jordan - Wigner construction:

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$$\begin{array}{cccc} \gamma_1 & \sigma_1^{(1)} & \sigma_2^{(1)} & \gamma_{N+1} \\ \gamma_2 & \sigma_3^{(1)}\sigma_1^{(2)} & \sigma_3^{(1)}\sigma_2^{(2)} & \gamma_{N+2} \\ \gamma_3 & \sigma_3^{(1)}\sigma_3^{(2)}\sigma_1^{(3)} & \sigma_3^{(1)}\sigma_3^{(2)}\sigma_2^{(3)} & \gamma_{N+3} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_N & \sigma_3^{(1)}\cdots\sigma_3^{(N-1)}\sigma_1^{(N)} & \sigma_3^{(N-1)}\cdots\sigma_3^{(N-1)}\sigma_2^{(N)} & \gamma_{2N} \\ & \sigma_3^{(1)}\sigma_3^{(2)}\cdots\sigma_3^{(N)} & \gamma_{2N+1} & \end{array} \tag{43}$$

where, in general, we will have $N+1$ real and N imaginary matrices. We can also allow $N \to \infty$ to have infinite matrices which obey Grassmann Algebra conditions.

There is an alternative formula for γ_{2N+1} in terms of the θ's and π's, rather than the γ's:

$$\gamma_{2N+1} = \prod_{m=1}^{N} [\theta_m, \pi_m] = (-i)^{N^2} \gamma_1 \gamma_2 \cdots \gamma_{2N} \tag{44}$$

This identity including the prefactor can be derived explicitly, although we will not do it here. As an example, for $N = 1$,

$$\gamma_3 = [\frac{\sigma_1 + i\sigma_2}{2}, \frac{\sigma_1 - i\sigma_2}{2}] = \frac{-i}{2}[\sigma_1, \sigma_2] = -i\sigma_1\sigma_2 = \sigma_3 \tag{45}$$

1.4. Calculus of Grassmann Numbers

We will now begin the study of the calculus of Grassmann numbers, also known as the the Berezin calculus. Specifically, we will construct a differential operator representation of conjugate Grassmann (i.e. Heisenberg) algebras. Since we will construct this by analogy to the bosonic case, let's consider the bosonic Heisenberg Algebra. We have elements x_i and their conjugates p_i such that they obey the uncertainty relation:

$$[x_i, x_j] = [p_i, p_j] = 0 \tag{46}$$

$$[x_i, p_j] = i\delta_{ij} \tag{47}$$

With this definition, we see that the p's can be represented as derivatives:

$$p_j \leftrightarrow -i\frac{\partial}{\partial x_j} \tag{48}$$

where the operators now act on the space of all functions of x. If we want to work without "i"s, we can use the associated creation and annihilation operators:

$$a_i = \frac{1}{\sqrt{2}}(x_i + ip_i) \tag{49}$$

$$a_i^\dagger = \frac{1}{\sqrt{2}}(x_i + ip_i) \tag{50}$$

So that

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \tag{51}$$

$$[a_i, a_j^\dagger] = \delta_{ij} \tag{52}$$

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The $[a_j^\dagger]$ can also be represented as derivative operators:

$$a_j^\dagger \leftrightarrow \frac{\partial}{\partial a_j} \tag{53}$$

where the operators act on the space of all functions of a.

For Grassmann Algebras we have anticommutation relations, rather than commutation relation, but Berezin showed how it was possible to represent the π's as derivatives of θ_i's.

We begin by considering a simple algebra with two θ's. Then the most general function of these will be

$$F(\theta_1, \theta_{2)} = A + B^1\theta_1 + B^2\theta_2 + C\theta_1\theta_2 \tag{54}$$

All higher powers of θ's in the power series expansion vanish as a result of $\theta_i^2 = 0$. We see then that if $N < \infty$, F is a polynomial, terminating after a finite number of terms. Eq.(1) also allows us to write

$$F(\theta_1, \theta_{2)} = A + B^1\theta_1 + B^2\theta_2 - C\theta_1\theta_2 \tag{55}$$

For the moment we will assume that A, B, C are ordinary numbers which commute with θ_i. We define left and right derivatives:

$$\frac{\partial F}{\partial \theta_1} = B^1 + C\theta_2 \tag{56}$$

$$F\frac{\overleftarrow{\partial}}{\partial \theta_1} = B^1 - C\theta_2 \tag{57}$$

where there is a minus sign difference. In everything that follows, we will discuss only the left derivative, but we could have equivalently selected the right derivative and performed and analogous derivation. We can first show that the derivatives satisfy a Grassmann Algebra. First, operating on the eq. (55) from the left we have

$$\frac{\partial F}{\partial \theta_2} = B^2 - C\theta_1 \tag{58}$$

Differentiating eq.(56) with respect θ_2 and eq.(58) with respect to θ_1 we get

$$\frac{\partial}{\partial \theta_1}\frac{\partial F}{\partial \theta_2} = -C \quad \text{and} \quad \frac{\partial}{\partial \theta_2}\frac{\partial F}{\partial \theta_1} = C \tag{59}$$

Thus we have:

$$\{\frac{\partial}{\partial \theta_1}\frac{\partial}{\partial \theta_2} + \frac{\partial}{\partial \theta_2}\frac{\partial}{\partial \theta_1}\}F = 0 \tag{60}$$

It is also obvious that:

$$\frac{\partial}{\partial \theta_1}\frac{\partial}{\partial \theta_1}F = \frac{\partial}{\partial \theta_2}\frac{\partial}{\partial \theta_2}F = 0 \tag{61}$$

All these lead to the fact that the derivatives form a Grassmann Algebra, when acting on the space of all functions:

$$\{\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}\}F(\theta_1, \theta_2) = 0 \tag{62}$$

Now we check that the derivatives anticommute with the $\theta' s$:

$$\theta_1 \frac{\partial}{\partial\theta_1} F = B^1\theta_1 + C\theta_1\theta_2 \tag{63}$$

and

$$\frac{\partial}{\partial\theta_1}\theta_1 F = A + B^2\theta_2 \tag{64}$$

so that

$$\{\theta_1, \frac{\partial}{\partial\theta_1}\}F = A + B^1\theta_1 + B^2\theta_2 + C\theta_1\theta_2 = F \tag{65}$$

By going through all the remaining relations in the same way we find that the derivatives are the conjugates of the θ's.

$$\{\theta_i, \frac{\partial}{\partial\theta_j}\}F = \delta_{ij}F \tag{66}$$

We have shown that the derivatives form a conjugate set for the case of two θ's, and it generalizes to any number by induction.

We can work backwards from the Grassmann sets to reconstruct the Clifford Algebra, in terms of the θ's and $\frac{\partial}{\partial\theta}$'s. We can invert eq.(11) and (12) so that:

$$\gamma_m = \theta_m + \frac{\partial}{\partial\theta_m} \tag{67}$$

and

$$\gamma_{N+m} = -i(\theta_m - \frac{\partial}{\partial\theta_m}) \tag{68}$$

To construct the final anticommuting matrix γ_{2N+1}, we use eq.(44):

$$\gamma_{2N+1} = \prod_{m=1}^{N} [\theta_m, \frac{\partial}{\partial\theta_m}] \tag{69}$$

This completes the brief discussion of the Berezin differential calculus.

Now that we have discussed the differential calculus we can consider integration. This is actually rather difficult. We will follow in Berezin's footsteps, and give a set of rules which, when applied, yield the right result. Along the way we will try to motivate some of the more seemingly mysterious aspects of this procedure.

We start by generating the calculus for a set of two Grassmann numbers, which can be easily extrapolated to an arbitrary set. We have fundamentally,

$$\int d\theta_i = 0 \tag{70}$$

$$\int \theta_i d\theta_i = 1 \tag{71}$$

where this is for each i (no sum).

At this point we don't say whether this is a definite or an indefinite integral. We just write an integral without worrying about it - we cannot take θ from $-\infty$ to $+\infty$; it doesn't make sense. We will discuss below what we mean by integrals over Grassmann variables.

We define multiple integrals by:

$$\int d\theta_1 d\theta_2 = \int d\theta_1 \int d\theta_2 \tag{72}$$

We also use the fact that the differentials anticommute with one another and with the θ's:

$$d\theta_m d\theta_n = -d\theta_n \theta_m \tag{73}$$

and

$$d\theta_m \; \theta_n = -\theta_n \; d\theta_m \tag{74}$$

We consider again the most general function of two Grassmann numbers, eq.(54):

$$F(\theta_1, \theta_2) = A + B^1\theta_1 + B^2\theta_2 + C\theta_1\theta_2 \tag{75}$$

When we integrate this over both variables, by eq.(70) above it is clear that only the term with both $\theta_1$and θ_2 will be non-zero:

$$\int F d\theta_1 d\theta_2 = C \int \theta_1\theta_2 d\theta_1 d\theta_2 = -C \int \theta_1 d\theta_1 \int \theta_2 d\theta_2 = -C \tag{76}$$

where the minus sign is due to eq.(74).

We note here the interesting fact that

$$\int F d\theta_1 d\theta_2 = \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2} F = -C \tag{77}$$

so that differentiation and integration are essentially the same thing. This result is true for all N.

We will now discuss the connection between Berezin integration and complex integration. Recall that the γ's are defined to be hermitian. Hermitian operators correspond to observables and have real eigenvalues - the γ's are the generalizations of real numbers.

In the CJW representation, the γ_j's are real, while the γ_{j+N}'s are imaginary, but all have real eigenvalues. The θ's are real in this representation, but are not hermitian, so they do not correspond to observables. They have complex eigenvalues and are the generalization of complex numbers. This hints at the possibility of analogies between Berezin rules and the rules of integration of complex variables. For instance, consider Cauchy's theorem:

$$\frac{1}{2\pi i} \oint_C F(z) dz = 0 \tag{78}$$

where this is true if $F(z)$ is analytic inside the contour C. For example with $F(z) = 1$,

$$\frac{1}{2\pi i} \oint_C dz = 0 \tag{79}$$

This can be compared to the Berezin rule (eq.(70)) above:

$$\int d\theta = 0 \tag{80}$$

The general correspondence will be:

$$\frac{1}{2\pi i} \oint \quad \text{for } z \quad \leftrightarrow \quad \int \quad \text{for } \theta \tag{81}$$

Is there a complex analog of eq.(71) $\int \theta d\theta = 1$? We could naively try:

$$\oint_C z dz \tag{82}$$

but this is identically zero. On the other hand,

$$-\frac{1}{2\pi i}\oint_C \frac{dz}{1-z} = 1 \tag{83}$$

if the contour C encloses the point $z = 1$. Can we find an analog of this in the Berezin Calculus? Consider the integral:

$$-\int d\theta \frac{1}{1-\theta} \tag{84}$$

where we choose $d\theta$ to the left of $\frac{1}{1-\theta}$(remember that the order matters for Grassmann variables). θ has no inverse ($\theta^2 = 0$) but $\frac{1}{1-\theta}$ does:

$$\frac{1}{1-\theta} = 1 + \theta \tag{85}$$

This can be seen either by multiplying directly, or by expanding $\frac{1}{1-\theta}$in a power series where all terms of θ^2can higher are zero. We can write:

$$-\int d\theta \frac{1}{1-\theta} = -\int d\theta(1+\theta) = -\int d\theta\theta = \int \theta d\theta = 1 \tag{86}$$

We see that the second Berezin rule is the analog of contour integration. This is why we didn't write limits on integrals of Grassmann variables, or determine whether they were definite or indefinite.

We can consider this issue a little further What do we man by $d\theta$? In the case of Riemannian integration we divide an interval $L = x_n - x_o$ into N parts and put

$$x_j = x_0 + j\epsilon \ , \qquad \epsilon = L/N \tag{87}$$

Then the Riemann integral is defined as the limit of a sum, namely

$$\begin{aligned} \int_{x_0}^{x_N} F(x)dx &= \lim_{N\to\infty}\sum_{j=1}^{N} F(x_j)(x_j - x_{j-1}) \\ &= \lim_{N\to\infty}\sum_{j=1}^{N} F(x_j)\Delta x \end{aligned} \tag{88}$$

where

$$\Delta x = \epsilon = L/N \tag{89}$$

In the case of Grassmann variables we can start by defining the difference of two Grassmann numbers θ_j and θ_{j-1} by

$$\Delta\theta = \theta_j - \theta_{j-1} \tag{90}$$

Since θ_j and θ_{j-1} anticommute, $\Delta\theta$ also anticommutes with both θ_j and θ_{j-1}. Indeed,

$$\{\theta_j - \theta_{j-1}, \theta_j\} = \{\theta_j - \theta_{j-1}, \theta_{j-1}\} = 0 \tag{91}$$

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We generalize this by inserting $N-1$ anticommuting Grassmann numbers between θ_0 and θ_N:

$$\theta_j = \theta_0 + j\Delta\theta, \quad \Delta\theta = (\theta_N - \theta_0)/N \tag{92}$$

Let us also define the magnitude of a Grassmann number θ represented by a $k \times k$ matrix by the equation

$$\mid \theta \mid^2 = \frac{Tr(\theta^\dagger \theta)}{k} \tag{93}$$

We see that if θ is a Grassmann number with conjugate π rather then $\lambda\theta$ is also a Grassmann number with conjugate $\lambda^{-1}\pi$. We also have

$$\mid \lambda\theta \mid = \mid \lambda \mid\mid \theta \mid \tag{94}$$

so that

$$\mid \Delta\theta \mid = \frac{1}{N} \mid \theta_N - \theta_0 \mid \tag{95}$$

and as $N \to \infty$ the magnitude of $\Delta\theta$ tend to zero just like Δx in the Riemannian integral.

After these preliminaries it is clear that there is a possibility of defining the Berezin integral as a sum of terms

$$\phi = \sum_j F(\theta_j)\Delta\theta \tag{96}$$

defined on a contour in the θ plane with anticommuting points and taken in the limit when the number N of points on the contour increases indefinitely so that the magnitude of the Grassmann number $\Delta\theta$ tends to zero.

Note that when $N \to \infty$, the Grassmann numbers θ_j becomes infinite matrices. In this case θ can vary quasicontinuosly between θ_0 and θ_N and behave like a continuous bosonic variable x. To each θ_j corresponds the function $F(\theta_j)$ which in the $N \to \infty$ limit can be regarded as a continuous function of the fermionic variable θ in the given interval.

1.5. Digression: Grassmann Numbers and Fermion Annihilation and Creation Operators

We now make a point concerning the identification of Grassmann numbers as fermion creation and annihilation operators. The relation between conjugate Grassmann sets is the same, you will recall, as the relation between fermionic annihilation and creation operators:

$$\{b_i, b_j\} = \{b_i^\dagger, b_j^\dagger\} = 0 \tag{97}$$

$$\{b_i, b_j^\dagger\} = \delta_{ij} \tag{98}$$

There is a complete symmetry between b and $b^\dagger$; how we decide which is a creation and which an annihilation operator? We are only able to decide after we define the vacuum state by:

$$b_i \mid 0 >= 0 \tag{99}$$

$$b_j^\dagger \mid 0 >=\mid j > \tag{100}$$

where $\mid j >$ is a one particle state with label j. Eqs.(97) and (98) tell us that b is an annihilation and $b^\dagger$ is a creation operator. We can extend eq.(97) to any number of annihilation operators

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$$b_i b_j ... b_n \mid 0 >= 0 \tag{101}$$

But if we act with creation operators on the vacuum we don't get 0. Instead, extending eq.(98) and using the anticommutation relations:

$$b_i^\dagger b_j^\dagger \mid 0 >= -b_j^\dagger b_i^\dagger \mid 0 >=\mid ij >= - \mid ji > \tag{102}$$

Thus the two-particle state is totally antisymmetric. Similarly, we can construct n-particle states:

$$b_{i^{(1)}}^\dagger, b_{i^{(2)}}^\dagger ... b_{i^{(n)}}^\dagger \mid 0 >=\mid i^{(1)} i^{(2)} ... i^{(n)} > \tag{103}$$

which will also be totally antisymmetric. Anticommunitation of the Grassmann numbers leads directly to the Pauli exclusion principle for fermionic states:

$$b_i^\dagger b_i^\dagger \mid 0 >= 0 \tag{104}$$

such that we cannot have to identical particles in the same state. Dirac discovered this relation between the Pauli exclusion principle, the antisymmetry of the state, and the anticommutation relations.

We see that in order to interpret Grassmann variables as creation and annihilation operators, we need to find a vacuum. Is it possible for us to construct the vacuum? To do this we will consider an alternate formulation of quantum mechanics.

In quantum mechanics, physical systems are described in the language of states and operators which act on states. Quantum operators can be represented by matrices or by differential operators. States are then represented by vectors (column matrices). In the Dirac notation, $\mid \alpha >$ is a state and $M_{\alpha\beta}$ are the matrix elements of an operator M. If we have an orthonormal basis of states, we can describe these matrix elements by:

$$M_{\alpha\beta} =< \alpha \mid M \mid \beta > \tag{105}$$

If M is a hermitian operator, it can always be written as the sum of projection operators:

$$M = \sum_i \lambda_i \mid i >< i \tag{106}$$

where the λ_i are real and

$$P_i =\mid i >< i \mid \tag{107}$$

is a projection operator which projects out state $\mid i >$. All eigenstates are orthonormal:

$$< i \mid j >= \delta_{ij} \tag{108}$$

If we expand a general state:

$$\mid \alpha >= \sum_i \Psi_i \mid i > \tag{109}$$

then P_i projects out the component along the ith eigenstate:

$$\begin{aligned} P_i \mid \alpha &= \mid i >< i \mid \alpha > \\ &= \mid i >< i \mid \sum_j \Psi_j \mid j > \\ &= \mid i > \sum_j \Psi_j < i \mid j > \\ &= \mid i > \Psi_i \end{aligned} \tag{110}$$

By construction, P_i is Hermitian:

$$P_i = \mid i >< i \mid = P_i^\dagger \tag{111}$$

Projection operators have the unique trait that they are idempotent -they equal themselves when squared:

$$P_i^2 = \mid i >< i \mid i >< i \mid = P_i \tag{112}$$

We can thus associate a projection operator with each state: rather then representing states by vectors, we can represent them by their projection operators. Then instead of having operators and observables on a different footing, we have hermitian operators that represent observables, and hermitian projection operators that represent states. This formulation is useful because both operators and states can be represented by hermitian matrices.

As an example, consider a two state (i.e. spin) system. The most general state will be a linear combination of spin up and spin down:

$$\mid \Psi >= a \mid\uparrow> + b \mid\downarrow> \tag{113}$$

$$<\uparrow\mid\downarrow>= 0 \qquad\qquad <\uparrow\mid\uparrow>=<\downarrow\mid\downarrow>= 1 \tag{114}$$

$$\mid a \mid^2 + \mid b \mid^2 = 1 \tag{115}$$

We usually represent these as column vectors:

$$\mid\uparrow>= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mid\downarrow>= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{116}$$

$$\mid \Psi >= a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{117}$$

Alternatively, we could express these states by their projection operators. The projection operators for the up state is:

$$P_\uparrow = \mid\uparrow><\uparrow\mid = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{118}$$

Equivalently, the down state projection operator is:

$$P_\downarrow = \mid\downarrow><\downarrow\mid = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{119}$$

These can be expressed in terms of the Pauli matrices:

$$P_\uparrow = \frac{1+\sigma_3}{2}, \qquad P_\downarrow = \frac{1-\sigma_3}{2} \tag{120}$$

Now, getting away from quantum mechanics, consider the case of a single Grassmann number θ with a conjugate π, and $N = 1$. In the CJW construction, θ and π are represented by linear combinations of the Pauli matrices:

$$\theta = \frac{\sigma_1+\sigma_2}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \leftrightarrow b \tag{121}$$

$$\pi = \frac{\sigma_1 - i\sigma_2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \leftrightarrow b^\dagger \tag{122}$$

where we show that we wish to interpret θ as an annihilation and π has a creation operator. We need the vacuum state, which will satisfy

$$\theta \mid 0 >= 0 \tag{123}$$

with normalization

$$< 0 \mid 0 >= 1 \tag{124}$$

In terms of matrices:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \tag{125}$$

which gives (keeping in mind eq.(124))

$$\begin{pmatrix} a_2 \\ 0 \end{pmatrix} = 0 \quad \Rightarrow \quad a_2 = 0 \quad \Rightarrow \quad a_1 = 1 \tag{126}$$

so that the vacuum state is

$$\mid 0 >= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{127}$$

Can we descibe the vacuum state by a projection operator? We have:

$$P^{(0)} =\mid 0 >< 0 \mid= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1+\sigma_3}{2} \tag{128}$$

Using the representation of b from eq.(122) we have:

$$bP^{(0)} = 0 \tag{129}$$

and its hermitian conjugate:

$$P^{(0)} b^\dagger = 0 \tag{130}$$

Eq.(129) correspond to eq.(97), which is one of the defining relations of the vacuum. Eq.(130) is the mathematical representation of the statement that the 1-particle state is orthogonal to the vacuum:

$$b^\dagger \mid 0 >= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{131}$$

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which is clearly orthogonal to the vacuum (eq.(127)).

We can now identify this with our spin system. The vacuum corresponds to the spin up state, and one particle corresponds to the spin down state. This state is obtained from the vacuum by acting with a fermionic creation operator defined in eq.(122).

This generalizes very nicely when we use the CJW construction of N dimensions (eq.(43)). We can generalize the vacuum projection operator to:

$$\mid 0 >< 0 \mid = P^{(0)} = \Omega = \frac{1+\sigma_3^{(1)}}{2}\frac{1+\sigma_3^{(2)}}{2}...\frac{1+\sigma_3^{(N)}}{2} \tag{132}$$

The general form for this will be an $(N+1)\times(N+1)$ matrix:

$$\begin{pmatrix} 1 & 0 & 0 & . & . & . & . & & \\ 0 & 0 & & & & & & & \\ 0 & & . & & & & & & \\ . & & & . & & & & & \\ . & & & & . & & & & \\ . & & & & & . & & & \\ . & & & & & & . & & \\ & & & & & & & & 0 \end{pmatrix} \tag{133}$$

This vacuum state projection operator satisfies relation that are generalizations of eqs.(129) and (130):

$$\theta_i \Omega = 0 \, , \quad \Omega \pi_i = 0 \tag{134}$$

These can be checked using the explicit form of θ and π in terms of the Pauli matrices from the CJW construction (eq.(43)) inserted into eqs.(11) and (12) and by using the form of Ω from eq.(132).

Consider a state with r particles in it. We can represent this by:

$$\mid r >= \pi_{i(1)}\pi_{i(2)}...\pi_{i(n)} \mid 0 > \tag{135}$$

To build a projection operator, we need to take $\mid r >< r \mid$. Using the fact that, by construction and by identification as creation and annihilation operators, $\pi^\dagger = \theta$, we have:

$$P_r = \mid r >< r \mid = \pi_{i(1)}...\pi_{i(n)}\Omega\theta_{i(n)}...\theta_{i(1)} \tag{136}$$

Here we have a general projection operator that clearly demonstrates the symmetry between θ and π.

We have shown by construction that the vacuum state always exists, allowing us to interpret a system of Grassmann numbers as the annihilation operator of some particle. Identifying the vacuum completes the connection between fermions and Grassmann numbers.

1.6. Spinors with Grassmann Components

We have seen that N Grassmann numbers can be constructed out of the Clifford algebra generated by $2N$ matrices γ_μ which satisfy the anticommutation relation eq.(24). This relation is invariant under $SO(p,q)$ which becomes $SO(2N)$ in the hermitian case. Note that the θ_j $(j = 1, ..., N)$ and π_j belong respectively to the (N) and $(\overline{N})$ representations of the $SU(N)$ subgroup of $SO(2N)$. Passage from the Clifford numbers to the Grassmann numbers corresponds to the decomposition of the rotation group $SO(2N)$ with respect to its $SU(N)$ subgroup. The generators of the Lie algebra of this rotation group are the spin matrices

$$S_{\mu\nu} = \frac{1}{4i}[\gamma_\mu, \gamma_\nu] \tag{137}$$

where $\gamma, \nu = 1, ..., 2N$. As we have seen from the CJW construction, these are $M \times M$ matrices, where

$$M = 2^N \tag{138}$$

since N commuting sets of Pauli matrices must be accommodated in the γ matrices. The corresponding $SO(2N)$ group element associated with an M-dimensional representation is

$$R(\omega) = e^{\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}} \tag{139}$$

which acts on the M-dimensional column ψ, the spinor representation of $SO(2N)$. When a rotation of 2π is performed around any axis, the spinor changes sign, so that strictly speaking it is a representation of the double covering of $SO(2N)$ which is called Spin$(2N)$.

In relativistic local quantum field theory there is a theorem (the Spin-Statistics Theorem) which states that if a local field is associated with an integer spin representation of the Poincaré group it must be quantized as a boson field, while a local field corresponding to a half-integer spin representation of the same space-time group must be quantized as a fermion. It follows that symmetrical or antisymmetrical tensor fields like scalar or vector fields are bosonic, while spinor fields ($s = 1/2$ or $s = 3/2$) are fermionic.

In relativistic theories, spinor fields like the neutrino field or the electron-positron Dirac field $\psi(x)$ must have Grassmann components that vary "continuously" with x.

In the $2N$ dimensions the 2^N-dimensional spinor representation Ψ is reducible since γ_{2N+1} commutes with the $SO(2N)$ generators $S_{\mu\nu}$. It decomposes into left handed and right handed irreducible spinors (Weyl spinors) given by

$$\Psi_L = \frac{1}{2}(1 + \gamma_{2N+1})\Psi \tag{140}$$

$$\Psi_R = \frac{1}{2}(1 - \gamma_{2N+1})\Psi \tag{141}$$

each having dimensions $M/2$.

In the $2N+1$ dimensions the $SO(2N+1)$ generators consist of the $(2N)^2$generators $S_{\mu\nu}$ from eq.(137) and the $2N$ additional generators

$$S_{2N+1,\nu} = \frac{1}{4i}[\gamma_{2N+1}, \gamma_\nu] \tag{142}$$

where $\nu = 1, ..., 2N$. Spin$(2N+1)$ also has a 2^N-dimensional representation Ψ but it is irreducible.

When the spinor representation of the $O(p, q)$ is real, it is called Majorana representation. In field theory it can have as components real Grassmann numbers. The Majorana condition depends on the dimension as well as the signature of the metric and cannot always be imposed.

As an example let us mention that in Minkowski space $O(3, 1)$ or its covering group $SL(2, C)$ can have either a 2-component complex Weyl spinor or a 4-component real Majorana spinor as representations. In the (9+1)-dimensional Minkowski space a spinor can be Weyl and Majorana spinor simultaneously so that an irreducible spinor of Spin(9,1) has only 16 real components instead of 32 complex components. In (1+1) dimensions we can choose

$$\gamma_0 = i\sigma_2, \quad \gamma_1 = \sigma_1, \quad \gamma_3 = \sigma_3 \tag{143}$$

Here there is no rotation; there is, however, a boost generator

$$S_{01} = \frac{1}{4i}[\gamma_0, \gamma_1] = \frac{i}{2}\sigma_3 = -\frac{i}{2}\gamma_3 \tag{144}$$

The corresponding group element

$$B = \exp(i\alpha S_{01}) = \exp(\frac{1}{2}\sigma_3\alpha) \tag{145}$$

is real thus

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{146}$$

can be taken as a real Majorana spinor. It is reducible, though, since we are in an even dimension. We have

$$\Psi_L = \frac{1}{2}(1+\gamma_3)\Psi = \frac{1}{2}(1+\sigma_3)\Psi = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} \tag{147}$$

$$\Psi_R = \frac{1}{2}(1-\gamma_3)\Psi = \frac{1}{2}(1-\sigma_3)\Psi = \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix} \tag{148}$$

as irreducible componets of Ψ. Hence in (1+1) dimensions an irreducbile spinor has one component $\psi_1 = \theta$, θ being a real Grassman number. In that sense $O(1,1)$ is like $O(9,1)$ since the Weyl and Majorana conditions can be imposed simultaneously. This is an illustration of the 8-fold Bott periodicity[12]. For the next Weyl-Majorana spinor we have to got to $O(17,1)$. The properties of the spinor in (1+1) and (9+1) Minkowski spaces are crucial in superstring theories.

Consider a function of Grassmann numbers such as eq.(54)

$$F(\theta_1, \theta_2) = A + B^\alpha\theta_\alpha + C\theta_1\theta_2 \quad (\alpha = 1,2) \tag{149}$$

We are now in a position to define the invariance properties of F under Lorentz transformations when the θ_α are considered to be the Grassmann components of a left-handed Weyl Spinor in (3+1) dimensions. Let

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \tag{150}$$

Under the $SL(2,C)$, the covering group of $SO(3,1)$, θ transforms as

$$\theta' = L\theta \tag{151}$$

with L a unimodular complex 2×2 matrix

$$Det\, L = 1, \quad L = e^{i\frac{\overrightarrow{\sigma}}{2}\cdot(\overrightarrow{\omega} - i\overrightarrow{\nu})} \tag{152}$$

where $\overrightarrow{\omega}$ and $\overrightarrow{\nu}$ are respectively the rotation and boost parameters of the Lorentz group. Define

$$\widehat{\theta} = -i\sigma_2\theta^* = \begin{pmatrix} -\theta_2^* \\ \theta_1^* \end{pmatrix} \tag{153}$$

It is easy to show that the $\widehat{\theta}$ transforms as a right-handed spinor, so that

$$\widehat{\theta}' = L^{\dagger-1}\widehat{\theta} \tag{154}$$

Hence

$$\widehat{\theta}^\dagger\theta' = \widehat{\theta}^\dagger L^{-1}L\theta = \widehat{\theta}^\dagger\theta \tag{155}$$

and

$$\widehat{\theta}^\dagger\theta = \begin{pmatrix} -\theta_2 & \theta_1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = [\theta_1, \theta_2] = 2\theta_1\theta_2 \tag{156}$$

is invariant under Lorentz transformations.

By introducing a metric in spinor space, $\epsilon^{\alpha\beta}$

$$\epsilon^{\alpha\beta} = (i\sigma_2)^{\alpha\beta} \tag{157}$$

we can also write

$$\widehat{\theta}^\dagger\theta = \theta_\alpha\theta^\alpha = \theta_\alpha\epsilon^{\alpha\beta}\theta_\beta = \theta^T(i\sigma_{2)}\theta \tag{158}$$

displaying the invariance properties of this quantity. Hence if B_α transforms like a spinor we can write

$$F(\theta_1, \theta_2) = A + B_\alpha\epsilon^{\alpha\beta}\theta_\beta + \frac{1}{2}C\theta_\alpha\epsilon^{\alpha\beta}\theta_\beta \tag{159}$$

so that if A is a scalar, $F(\theta_1, \theta_2)$ is a relativistic scalar. Because we have used a left-handed spinor, F is called chiral superfield when A, B_α and C are functions of space-time coordinate x. In order for F to be a bosonic scalar $B_\alpha(x)$ must also be Grassmann valued.

If the θ_α in relativistic field theories transform like scalars (or vectors) they contradict the spin-statistics theorem and behave like unphysical degrees of freedom that must disappear in the expressions of physical observables. In that case they are called ghost fields and are used as auxiliary quantities to cancel the effects of unphysical bosonic degrees of freedom in order to give positive energies and positive probabilities.

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