A field theory approach to string theory

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Contents

Abstract

String field theory (SFT) is a reformulation of the worldsheet string theory (first-quantization) in the field theory language (second-quantization). The main motivation is to provide a rigorous and constructive formulation of string theory and to address questions which are not natural or even impossible to answer in the worldsheet description – off-shell continuation and renormalization to handle IR divergences; momentum analyticity; collective, thermal and non-perturbative effects; vacuum dynamics and properties. Two important research directions of string field theory are, first, to establish consistency of string theory, second, to obtain a description which can be used for explicit computations.

In this thesis, I will summarize my contributions to both topics. On the first aspect, I will prove that the worldsheet computation of the tree-level 2-point string amplitudes matches the expected field theory result. I will also derive the analyticity domain of n-point superstring amplitudes at all loops and show that it implies crossing symmetry of 4-point amplitudes. Finally, I will also show how string field theory techniques can be used to define the timelike Liouville two-dimensional quantum gravity. On the second aspect, I will describe general properties of effective string field actions at the classical level. Then, focusing on the quartic contribution to the heterotic string effective action, I will show that some computations can be performed without requiring a complete characterization of the string field action.

Chapter 1

Introduction

1.1 Background

The Standard Model of particle physics and general relativity are described in terms of fields whose fundamental quanta are point particles. A natural question is to ask whether fundamental extended objects can exist. Since the simplest example is the string, it makes sense to start by studying its properties. Then, one discovers that string theory provides a candidate theory of quantum gravity and grand unification which is UV finite. It also naturally contains higher-dimensional objects i.e. branes. Moreover, it offers a more general framework to think about quantum field theories, black holes and mathematics. For these reasons, string theory became a major landmark of the theoretical physics landscape and it is important to understand and clarify its properties.

In the same way that classical and quantum mechanics are the most direct approaches to study the behavior of a single point particle, the worldsheet description is the natural starting point to describe string theory. It characterizes the evolution of a string by considering the embedding of its worldsheet (i.e. spatial extension and time evolution) in spacetime. The worldsheet action corresponds to a $2d$ quantum field theory, the fields depending on the worldsheet coordinates and describing the embedding of the wosrldsheet in spacetime and the internal structure of the string. In fact, consistency of the theory implies that the $2d$ theory is a conformal field theory (CFT) when the background metric is flat. The corresponding path integral is equivalent to first quantization, and n-point scattering amplitudes can be computed by adding interactions by hand – which is equivalent to specify the possible interaction vertices and to sum over all worldsheet topologies with n legs having these internal vertices. An n-point g-loop amplitude corresponds to the integral of a CFT correlation functions over the moduli space $\mathcal{M}_{g,n}$ of genus-g Riemann surfaces with n punctures.

While this approach is intuitive and could be pushed far due to the exceptional properties of 2d theories, it becomes cumbersome for some questions or even fails at a more fundamental level [1–4]. We will just mention some of the problems which are the most interesting for this thesis. A first difficulty is that the worldsheet description is naturally an on-shell formalism. This leads to divergences in amplitudes when internal particles go on-shell. This is expected since quantum corrections typically modify the masses and shift the vacuum. The solution is renormalization, which requires going off-shell. More generally, most techniques used in QFT for proving consistency properties (such as unitarity, analyticity, crossing symmetry. . .) require to continue amplitudes off-shell. Another problem is the need to find consistent backgrounds: superstring theory is consistent only in $D = 10$ dimensions ($D = 26$ for the bosonic string), which means that $D - 4$ dimensions must be compactified to recover a 4-dimensional world. However, the worldsheet description does not provide a potential which can be minimized to find vacua. Moreover, the worldsheet path integral is intrinsically perturbative (nonetheless, string dualities have allowed understanding some of the non-perturbative aspects). Finally, one would expect a theory of quantum gravity to be background independent, which is not the case for the worldsheet theory.

For a point particle, all these problems are addressed by using the field (and second-quantized) description.[1] Hence, it is natural to construct a string field theory (SFT), that is, a field theory whose field quanta are strings. Writing such a theory is a formidable task because fields are not functions but functionals as quanta are extended objects. Moreover, special relativity implies that extended objects must interact non-locally. It is thus better to give up the position representation and to use the ket representation or the momentum representation. In the latter, the spatial extension of the string is replaced by an infinite number of discrete modes, each associated with a spacetime field. Hence, SFT can be understood simply as a standard QFT with an infinite number of fields and non-local interactions. This picture is the most intuitive as it allows to bring one's own knowledge from usual QFT to understand string theory and study its properties. To summarize, the original motivations for building a SFT are: off-shell continuation and renormalization to handle IR and UV divergences; momentum analyticity; collective, thermal and non-perturbative effects; vacuum dynamics and properties.

However, the SFT action is highly complicated because it is non-polynomial: there are n-point interactions for all $n \geq 3$, but also at all loops $g \geq 0$ (the latter correspond to finite counter-terms needed to ensure gauge invariance at the quantum level) [5]. Moreover, no explicit parametrization is known, except for the open super- and bosonic string [6, 7].[2] The reason is that a lot of off-shell unphysical data is needed to define the vertices, and there is no known method to derive them. Then, even though the SFT action is formally background independent [5, 12], obtaining a form amenable to explicit computations requires to specify a background, which brings back some of the limitations from the worldsheet formulation since the string field is expressed as a linear combination of first-quantized states. Since a background is a $2d$ worldsheet QFT, the space of all possible backgrounds is the space of $2d$ QFT, with $2d$ conformal field theories corresponding to classical solutions [5, 13–15]. Writing a field theory on a field theory space is definitely challenging; there is a proposal [13] for the open string, but it is still largely to be understood. Finally, there are reasons to think that the current form of closed SFT is only an effective theory, which means that non-perturbative aspects would be out of reach [16].[3]

Despite these challenges, SFT is the main avenue for understanding better string theory as it provides a rigorous and field theoretical construction. Moreover, several important results have been obtained using SFT. Currently, one can distinguish four main research directions:

1. Construction and structure of the action.

   The first step is to build the SFT action. The usual approach is to identify the propagator from the free SFT and to decompose the amplitudes in Feynman diagrams, identifying fundamental interaction vertices as the contribution which does not contain any propagator. The action enjoys a gauge invariance described by a L_∞ algebra which in turns ensure that the quantum BV master equation is satisfied. The closed and open-closed super- and bosonic SFT have all been constructed [5, 20–22]. This

[1] Background independence is manifest only for dynamical fields present in the Lagrangian.

[2] The 3-point vertex is straightforward. Tree-level closed bosonic vertices for $n = 4, 5$ have been constructed numerically in [8–10]. The 1-loop 1-point vertex has been constructed (up to an explicit choice of local coordinates) for the bosonic and heterotic closed SFT [11].

[3] On this aspect, the open SFT is on a better footing, however, its quantization has not been completely understood since closed strings can be created in loops. However, some limits may be consistent [17] and there is even some prospect for defining closed SFT from open SFT [18, 19].

involves many interesting mathematical structures, such as homotopy algebras (e.g. L_∞), BV algebra and moduli spaces of Riemann surface.

However, as mentioned earlier, this is a formal construction, useful to describe the structure of the action but not giving an explicit parametrization. Currently, research focuses on this point using different strategies. A first aspect is to find an optimal form of the action: indeed, several forms have been found for the super-SFT action and proved to be equivalent. The idea behind the two main formulations is to dress the bosonic products to obtain the superstring products, either using constraints from homotopy algebras or using Berkovits' WZW approach [7, 23–39]. However, it is necessary in any case to build the bosonic vertices explicitly, and only the first few have been found [8–11, 40, 41]. A last avenue is to study effective actions [18, 19, 42–45] and physical observables.

2. Consistency of string theory.

   Even if the explicit form of the action is not known, it is sufficient to know the general structure for assessing its consistency – and from there, the one of string theory in general – as a QFT. Indeed, it is expected that general properties are independent of the details of the theory. The idea is to write a general QFT, which contains SFT as a special case, and analyze its properties. This approach, initiated by A. Sen, could extend many fundamental results from local QFT to SFT, such as Cutkosky's rules, unitarity, soft theorems, generalized Wick rotation and $i\epsilon$ prescription etc. [46–54]. The added value of this approach is to extend also our knowledge of non-stringy QFT since the type of theory considered is more general than in the previous literature (for example, due to non-local interactions).

3. Improvement of worldsheet computations.

   As explained previously, the worldsheet approach has different problems. Many of them can be solved with various procedures, whose justifications ultimately come from SFT. A first example is the off-shell continuation of string amplitudes, which needs SFT for being unambiguously defined [5, 55]. More generally, the typical approach is to start with a worldsheet expression, rephrase it using SFT and then eliminate the off-shell data to write everything in terms of on-shell quantities – avoiding the problem related to the absence of explicit parametrization of the string vertices. Some examples of applications include mass renormalization and vacuum shift, background with RR fluxes, marginal deformations and instantons, etc. [56–64].

4. Classical solutions to open SFT and tachyon vacuum.

   Since the classical open string is exactly known, it is possible to derive solutions to the classical equations of motion, i.e. to investigate open string backgrounds. After pioneering works [65–67] investigating tachyon condensation numerically, Schnabl found an analytic solution of the open bosonic SFT in a seminal paper [68, 69]. Both approaches have been used extensively to investigate classical solutions [70–73], including lump solutions (decay of a p-brane to a p'-brane with $p' < p$) and marginal deformations. Moreover, analytic solutions can also be found for the open superstring [74].

There are very few modern reviews of SFT [3, 4, 73] (older but still useful reviews on open – and mostly bosonic – SFT with a focus on classical solutions are [75–81]). No textbook completely dedicated to the subject exists for the moment.[4] However, two will appear soon. First, a volume from Doubek, Jurčo, Markl and Sachs will appear, with a focus on the mathematical structures of SFT [83]. Second, I have written an introductory book which is currently in revision with the editor [**Erbin:2020:StringFieldTheory**].

[4] The book [82] appeared before SFT reached his modern form.

1.2 Research topics

My main area of research is string field theory (SFT), and I have decided to focus this thesis on the results I have obtained on this topic. However, I have also worked on various topics related to string theory and I will summarize the corresponding results in Chapter 7. The goal is to simplify the task of the reader by displaying a single research direction, while showing the connections with the other topics.

In this section, I describe the different topics and my motivations. The content of the thesis and the main results I have obtained are outlined in Section 1.3.

1.2.1 String field theory

SFT is arguably the most rigorous approach to string theory and the best path to understand its most fundamental structure – in the same way as QFT has established itself as the most universal framework of theoretical physics. This explains largely my interest for SFT. More generally, it is interesting to understand how to build field theories of extended objects, as they could be useful for other purposes (like effective theories in condensed matter, etc.). In fact, understanding SFT could even help making progress towards M-theory. Indeed, the strong-coupling limit of type IIA is conjectured to give M-theory. Another possibility is that building a membrane field theory could help bypass the problems associated with the worldvolume approach for $(p > 1)$-branes.

My research in SFT is focused on the first two topics described at the end of Section 1.1: 1) constructing and determining the structure of SFT, and 2) determining if string theory is consistent. Indeed, my first goal is to understand the deepest aspects of SFT in order to deduce the principles governing our Universe – if string theory is the appropriate description. But, for this, it is important to know if string theory is consistent. However, as explained in Section 1.1, the current description of SFT displays various conceptual problems: I think that understanding better its structure and how to compute explicitly with it, may give hints on how to improve SFT.

1.2.2 Other topics

Beside string field theory, I have worked on four other topics:

1. machine learning for theoretical physics;
2. two-dimensional quantum gravity;
3. tensor and SYK models;
4. black holes and supergravity.

Beside being interesting by themselves, all four topics are related to string (field) theory and provide useful tools and alternative points of view. I am of firm conviction that research cannot be too specialized and must progress on different fronts, in order to favorize inspiration and flexibility.

Machine learning (ML) is a set of powerful techniques which allows the computer to make computations without being explicitly programmed. In the recent years, its development progressed at a very rapid pace and it emerged as one of the main technology powering the leading industry companies. More generally, it started to pervade most scientific fields as an essential tool. In fact, Andrew Ng, one of the leading protagonist in ML research, stated that “Artificial intelligence is the new electricity.”

As theoretical physics becomes more and more complex, relying on advanced mathematical theories and computationally expensive simulations, ML techniques offer a new perspective on handling these problems (see [84–86] for selected reviews). Indeed, ML can be recognized as a general toolbox for pattern recognition – which includes many concepts such as statistical sampling, function approximation, optimization, conjecture generation, etc.

A first application is the classification of consistent string vacua which produce low-energy theories close to the Standard Model. As explained previously, this is a question which could be addressed systematically with SFT, but the latter is not developed sufficiently to allow it. As a consequence, it is necessary to classify mathematically the possible manifolds and to check the properties of the induced low-energy effective theory. Fortunately, the properties of the latter are generically determined by the topology of the manifold used for the compactification, which requires less data than specifying the full geometry. However, even computing only topological properties is difficult: the techniques from algebraic topology (such as spectral sequences) are complicated and rarely provide closed-form formulas. Moreover, some problems are NP-complete or even undecidable [86, 87]. Starting with the seminal papers [88–91], ML techniques have been applied with success to this topic in the recent years (see [86] for a review). Since one of my motivation for developing SFT is to classify consistent string backgrounds, it makes sense to investigate other techniques for this aim.

Another application is lattice computations (for gauge theories, condensed matter or statistical physics). Indeed, the latter use Monte Carlo (MC) algorithms which are extremely costly in terms of computational powers [92]. Moreover, some regions of the parameter space are not accessible. Recent works starting with [93, 94] have demonstrated that ML techniques can be employed to support and extend MC simulations. Because such simulations are currently the best approach to understand the fundamental aspects of QCD – with a focus on the questions of the confinement and the properties of the quark-gluon plasma –, ML provides exciting possibilities for new developments. Since I am interested in understanding better properties of QFT, both in view of improving our understanding of the phenomenology of the Standard Model and for applications to string field theory, it is a natural step to study lattice simulations and the possibilities offered by ML. Furthermore, note that lattice simulations have been performed within the AdS/CFT correspondence [95–104].[5] Moreover, an attempt at a lattice formulation of SFT has been proposed in [105]. Finally, to make contact with the next topic, extensive lattice computations have been performed for different theories and definitions of gravity, in particular, using causal dynamical triangulations (see [106, 107] for selected reviews).

Two-dimensional quantum gravity provides a toy model for quantizing gravity. Indeed, gravity is much simpler in two dimensions thanks to larger symmetries (for example, conformal and superconformal algebras are infinite-dimensional), milder divergences (whereas gravity is perturbatively non-renormalizable in four dimensions) and a simpler path integral (only the conformal mode is dynamical). Since I am interested more generally in understanding quantum gravity, it is logical to also study useful toy models. In fact, $2d$ gravity shares many aspects with string theory since its definition from the functional integral generalizes the string worldsheet path integral, as will be reviewed later in this thesis. For example, $2d$ gravity coupled to conformal matter also describes non-critical string theories. Such theories are particularly useful as toy model for string (field) theory since computations can be performed more easily, but also compared to dual matrix models [62–

[5] The AdS/CFT correspondence can be understood as providing a non-perturbative definition of the closed string theory on an AdS background. While only the low-energy limit – supergravity on the gravity side – of the correspondence is usually studied, a recent proposal [18, 19] extends the correspondence to full string field theory.

64, 108–112]. Another motivation is the observation that many theories of quantum gravity (including string theory) display a phase where spacetime is spontaneously reduced to two dimensions [113–115]. It has also been shown that Jackiw–Teitelboim gravity ($2d$ dilatonic gravity, see [116, 117] for reviews) is dual to the SYK model, which will be described below. Finally, $2d$ gravity is also related to the quantum Hall effect on $2d$ curved surfaces [118–120] (the latter using also the theory of Riemann surfaces).

The most direct approach to $2d$ quantum gravity coupled to matter is to investigate the functional integral where the measure is weighted by the classical matter action and cosmological constant term (since the Einstein–Hilbert term is topological in $2d$). Integrating out the matter yields an effective action for gravity, which has a single dynamical degree of freedom. However, since this action is difficult to compute directly, it is in general simpler to compute the gravitational action, which corresponds to the WZW action obtained by parametrizing the metric as the conformal factor times a background metric. In this parametrization, the dynamics of gravity decouples from the one of the matter (up to integration over moduli parameters when spacetime has a non-trivial topology). The gravitational action always contains a universal term given by the Liouville functional and whose coefficient is given by the conformal anomaly of the matter action [121]. When the matter is a CFT on flat space, there are no other terms and this coefficient corresponds to the matter central charge.

From that starting point, there are two main research directions. The first consists in understanding better the Liouville quantum gravity since it is universal. While the spacelike Liouville theory (positive- definite kinetic term) is well understood following different approaches [122–125], this is not the case of the timelike theory (negative-definite kinetic term). A second aspect concerns the computation of the gravitational action when the matter is not conformal. Indeed, since it seems that our $4d$ world is not conformal, a good toy model should also share this property. Surprisingly, this topic has been mostly ignored in the literature which focused mostly on conformal matter. Until recently, the few papers making exception have employed the David–Distler–Kawai (DDK) ansatz for the case where the matter action is a CFT deformed by primary operators. More recently, the gravitational action for a free massive scalar field has been rigorously derived [118, 126–129]. The leading correction to the Liouville action is provided by the Mabuchi action: most properties of this action are still unknown and it is important to study them. Moreover, other cases beyond must be investigated to better understand the form of the gravitational action in general.

Tensor and SYK models Random tensor models are generalizations of random matrix models in any dimension: the functional integral of interacting tensors of rank d is perturbatively equivalent to a sum over d-angulations providing simplicial decompositions of d-dimensional pseudo-manifolds. While dormant for several decades, the discovery of a large N expansion led to a burst of interest [130].

Matrix models are one of the most useful mathematical tools and found applications in many fields (including string theory, $2d$ gravity and machine learning). Since tensor models are even more general, they will likely find a place in many applications. Because tensors models provide a sum over discrete d-dimensional geometries, it can be expected that the continuum limit yields a functional integral over smooth manifolds, i.e. quantum gravity [131, 132]. Moreover, in the same way that matrix models provide a triangulation of the string worldsheet, tensor models could provide a discretization of brane worldvolumes and help understand better their properties. This could be useful to provide an alternative worldvolume description, since the simplest approach based on the Nambu–Goto and Polyakov actions fail in most cases for $(p > 1)$-branes.

The SYK model [116, 117, 133, 134] is a quantum mechanical model of N Majorana fermions which is solvable in the strong coupling regime, enjoys a near-conformal invariance and displays maximal chaos. This makes it a perfect candidate for the dual of the AdS_2 near-horizon geometry of black holes. Moreover, as mentioned earlier, it has been understood that the dual theory is related to the Jackiw–Teitelboim $2d$ gravitational model. Since both models are tractable, it makes them a good setup for studying the AdS/CFT correspondence. In fact, this duality has been generalized and been related to all random matrix ensembles and moduli spaces of Riemann surfaces [135]. Finally, it was understood that the SYK and tensor models share the same Feynman graphs at leading order [136].

These different aspects make the tensor and SYK models important tools to be familiar with, and which could be useful for my research topics.

Black holes and supergravity Supergravity describes the different supersymmetric extensions of Einstein gravity. This larger amounts of symmetry makes it better behaved than Einstein gravity [137–139]. Hence, it is an interesting attempt to solve the non-renormalizability problem.[6] Moreover, as reviewed earlier, supergravity corresponds to the low-energy limit of string theory, with the field content determined by the compactification under consideration. It also knows about non-perturbative aspects of string theory since D-branes are found among its classical solutions. Finally, microstate counting of supersymmetric black holes is one of the most convincing test of string theory as a theory of quantum gravity (beyond perturbative scattering of gravitons).

The Plebański–Demiański (PD) metric is the most general black hole solution in the Einstein–Maxwell theory with a cosmological constant [143, 144]. Since black holes are one of the few phenomena allowing us to glimpse in the realm of quantum gravity, it is of prime importance to cartography the possible black holes in gravitational theories and to study their properties. Other solutions such as domain walls are interesting within the AdS/CFT correspondence [145]. Hence, one of the main question is to embed the PD black hole and domain walls in supergravity. Until now, the only complete embedding is in pure $N = 2, D = 4$ supergravity [146].

1.3 Content

1.3.1 Structure and results

This thesis starts by the introductory Chapter 2. SFT has the reputation to look esoteric at first sight: this chapter is the occasion to clarify the formalism and to describe the most important properties which we will need in this thesis. The next four chapters focus on the results I have obtained in SFT.

In Chapter 3, I will describe the proper computation of the tree-level 2-point string amplitude. It has been assumed for decades that it vanishes, which would imply that the worldsheet path integral computes truncated Green functions and not amplitudes. However, we found that a rigorous computation gives a non-vanishing result in agreement with the one required by general QFT principles. As a consequence, the worldsheet path integral does compute amplitudes. This closes a problem which remained open since the inception of string theory.

Next, in Chapter 4, I describe the proof that n-point superstring amplitudes are analytic in a subset of the primitive domain at all loops. This implies that 4-point amplitudes are crossing symmetric, a fact used by Veneziano to motivate his model [147] but which had never been proved. This also provides an alternative derivation of analyticity in QFT which

[6] However, only supergravities with a high number of supersymmetries may be finite, but these theories are likely unphysical (they are not chiral, have large gauge groups and too many fields). Nonetheless, nothing prevents finding a non-Gaussian fixed point for supergravities with less supersymmetries, making them non-perturbatively renormalizable (asymptotic safety [140–142]).

is more general than the previous proofs. Primitive analyticity and crossing symmetry are two of the most important consistency properties found in QFT as they are consequences of locality and causality. It was not clear at first that they would still hold (completely or partially) in string theory since it is non-local, but our results show that, on this aspect, string theory behaves like a local QFT.

Chapter 5 does not treat directly of SFT, but instead describes the construction of the timelike Liouville $2d$ quantum gravity. Nonetheless, the two main ideas which made the construction possible come from SFT, which motivates the inclusion of this work as part of the main presentation. Because the path integral of the timelike Liouville action is not bounded from below, its definition has presented a major challenge and has been the source of many confusions. However, the Liouville theory being a CFT, it can be studied using the conformal bootstrap. The main insight from my work has been to reuse a generalization of the Wick rotation introduced in SFT [46] to obtain the correlation functions in the timelike theory from the one of the bootstrap. Moreover, we have obtained a complete characterization of the physical states using BRST cohomology, following standard methods from string theory.

Then, Chapter 6 describes several works on effective SFT. I will start the chapter by describing different aspects of effective actions in SFT. This includes: effective gauge invariance, the difference between gauge fixing and integrating out auxiliary fields (like Nakanishi–Lautrup), and the inclusion of a source. Then, I will show that the heterotic effective action of the zero-momentum massless fields reproduces the commutator-squared term from the Yang–Mills action. The interesting point is that the computation displays a localization effect [148, 149]: the quartic effective interaction is completely given by a Feynman diagram with an infinitely long propagator connecting two cubic vertices. As a consequence, the knowledge of the quartic vertex is not needed. This is a promising way for performing computations since most vertices are not known (see the discussion in Section 1.1).

The penultimate Chapter 7 presents the results I have obtained in the other research fields described in Section 1.2.2. Finally, Chapter 8 concludes this thesis with a summary of short- and mid-term perspectives.

1.3.2 Publications

The content of the thesis is based on my works after the PhD:

- Chapter 2 [**Erbin:2020:StringFieldTheory**, 3]

  H. Erbin. “Introduction to string field theory”. Lectures Notes in Physics, Springer (to appear). Draft available online: `http://www.lpthe.jussieu.fr/~erbin/files/reviews/book_string_theory.pdf`

  C. de Lacroix, H. Erbin, S. P. Kashyap, A. Sen, M. Verma. “Closed Superstring Field Theory and Its Applications”. International Journal of Modern Physics A 32.28n29 (Oct. 2017), p. 1730021. 137 pages. arXiv: 1703.06410.

- Chapter 3 [150]

  H. Erbin, J. Maldacena, D. Skliros. “Two-Point String Amplitudes”. Journal of High Energy Physics 2019:7 (July 2019), p. 139. 7 pages. arXiv: 1906.06051.

- Chapter 4 [151]

  C. de Lacroix, H. Erbin, A. Sen. “Analyticity and Crossing Symmetry of Superstring Loop Amplitudes”. Journal of High Energy Physics 2019:5 (May 2019), p. 139. 28 pages. arXiv: 1810.07197.

- Chapter 5 [152, 153]

  T. Bautista, H. Erbin, M. Kudrna. “BRST Cohomology of Timelike Liouville Theory”. Journal of High Energy Physics 2020:05 (May 2020). 29 pages. arXiv: 2002.01722.

  T. Bautista, A. Dabholkar, H. Erbin. “Quantum Gravity from Timelike Liouville theory”. Journal of High Energy Physics 2019:10 (Oct. 2019), p. 284. 43 pages. arXiv: 1905.12689.

- Chapter 6 [154, 155]

  H. Erbin, C. Maccaferri, M. Schnabl, J. Vošmera. “Classical algebraic structures in string theory effective actions”. 85 pages. arXiv: 2006.16270.

  H. Erbin, C. Maccaferri, J. Vošmera. “Localization of Effective Actions in Heterotic String Field Theory”. Journal of High Energy Physics 2020:02 (Feb. 2020). 48 pages. arXiv: 1912.05463.

- Section 7.1 [**Erbin:2020:InceptionNeuralNetwork**, 156–159]

  H. Erbin, R. Finotello. “Machine learning for complete intersection Calabi-Yau manifolds: a methodological study” (July 2020). arXiv: 2007.15706.

  H. Erbin, R. Finotello. “Inception Neural Network for Complete Intersection Calabi-Yau 3-folds” (July 2020). arXiv: 2007.13379.

  M. N. Chernodub, H. Erbin, V. A. Goy, A. V. Molochkov. “Topological defects and confinement with machine learning: the case of monopoles in compact electrodynamics”. Accepted in: Physical Review D (Aug. 2020). arXiv: 2006.09113.

  M. N. Chernodub, H. Erbin, I. V. Grishmanovskii, V. A. Goy, A. V. Molochkov. “Casimir effect with machine learning”. Accepted in: Physical Review Research (Aug. 2020). arXiv: 1911.07571.

  H. Erbin, S. Krippendorf. “GANs for generating EFT models”. Accepted in: Physics Letter B (Sep. 2020). arXiv: 1809.02612.

- Section 7.2 [160–162]

  C. de Lacroix, H. Erbin. “A Short Note on Dynamics and Degrees of Freedom in $2d$ Classical Gravity”. General Relativity and Gravitation 52.9 (Jan. 2020). 13 pages. arXiv: 1612.04097.

  C. de Lacroix, H. Erbin, E. E. Svanes. “Minisuperspace Computation of the Mabuchi Spectrum”. Classical and Quantum Gravity 35.18 (Aug. 2018), p. 185011. 24 pages. arXiv: 1704.05855.

  C. de Lacroix, H. Erbin, E. E. Svanes. “Mabuchi Spectrum from the Minisuperspace”. Physics Letters B 758 (July 2016), pp. 186–189. 5 pages. arXiv: 1511.06150.

- Section 7.3 [**Erbin:2019:ConstructiveExpansionQuartic**, 163]

  S. Dartois, H. Erbin, S. Mondal. “Conformality of $1/N$ Corrections in SYK-like Models”. Physical Review D 100 (Dec. 2019), p. 125005. 31 pages. arXiv: 1706.00412.

  H. Erbin, V. Lahoche, M. Tamaazousti. “Constructive expansion for quartic vector fields theories. I. Low dimensions” (Apr. 2019). arXiv: 1904.05933.

- Section 7.4 [164, 165]

  H. Erbin, V. Lahoche. “Universality of Tunnelling Particles in Hawking Radiation”. Physical Review D 98 (Nov. 2018), p. 104001. 12 pages. arXiv: 1708.00661.

  J. D. Dorronsoro, H. Erbin, T. Van Riet. “Domain Wall Seeds in CSO-Gauged Supergravity”. Journal of High Energy Physics 17.06 (June 2017), p. 097. 17 pages. arXiv: 1704.06534.

Chapter 2

String field theory

This chapter introduces to the main concepts of string field theory (SFT) to facilitate the description of the results in the next chapters. In the introduction, I have indicated that SFT can be understood simply as a normal QFT. However, in its modern form, SFT does not use the standard language from QFT textbooks and may look intimidating. For this reason, I start this chapter by rewriting a simple scalar field theory in the same way as SFT (Section 2.1). Then, I review the worldsheet description of string theory (Section 2.2), describe the free SFT (Section 2.3), and motivate from both the form of the interacting SFT action (Section 2.4). I conclude this chapter with the momentum representation of SFT (Section 2.5).

To avoid repetitions, I will focus on the closed bosonic string, outlining the differences with the classical open string when needed. The construction of superstring field theory follows the same steps, up to few additional complications which I will not review [3, 55, 166, 167].

This chapter is partly based on the review [3], and on a book (in preparation) expanding on the lecture notes from the course I taught within the "Elite Master Program – Theoretical and Mathematical Physics" at Ludwig-Maximilians-Universität in 2017–2019 [**Erbin:2020:StringFieldTheory**].

2.1 A detour by a scalar field theory

Let's consider a field theory of N real scalar fields ϕ_i, $i = 1, \ldots, N$, with non-local cubic and quartic interactions in position representation (Euclidean signature):

$$\begin{aligned}
S := \frac{1}{2} \int \mathrm{d}^D x \, \phi_i(x) \big(- \delta_{ij} \Delta + M_{ij}^2 \big) \phi_j(x) \\
+ \frac{g}{3!} \int \mathrm{d}^D x_1 \mathrm{d}^D x_2 \mathrm{d}^D x_3 \, V^{(3)}_{i_1 i_2 i_3}(x_1, x_2, x_3) \phi_{i_1}(x_1) \phi_{i_2}(x_2) \phi_{i_3}(x_3) \\
+ \frac{g^2}{4!} \int \mathrm{d}^D x_1 \mathrm{d}^D x_2 \mathrm{d}^D x_3 \mathrm{d}^D x_4 \, V^{(4)}_{i_1 i_2 i_3 i_4}(x_1, x_2, x_3, x_4) \phi_{i_1}(x_1) \phi_{i_2}(x_2) \phi_{i_3}(x_3) \phi_{i_4}(x_4).
\end{aligned} \tag{2.1}$$

where the sum over repeated indices is understood, g is a coupling constant counting the interaction order (other constants can appear in the vertices), $\Delta = \partial^2$ is the Laplace–Beltrami operator on flat space, and M_{ij} is the mass matrix. For simplicity, we assume that the fields ϕ_i form a basis of mass eigenstates, such that $M_{ij} = m_i \, \delta_{ij}$. Since they are real and satisfy no constraints, the internal space is $\mathbb{R}^N$. The cubic and quartic vertex functions $V^{(3)}$ and $V^{(4)}$ are symmetric in their indices and the arguments x_i appear only as derivatives.

For example, the usual ϕ^4 theory of a single scalar field is given by the vertex function:

$$V^{(4)}(x_1, x_2, x_3, x_4) := \delta^{(D)}(x_1 - x_2)\, \delta^{(D)}(x_1 - x_3)\, \delta^{(D)}(x_1 - x_4). \tag{2.2}$$

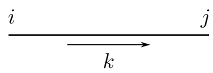
In order to write the Feynman rules – and make contact with SFT –, we perform a Fourier transform to work in momentum representation:

$$\begin{aligned} S = \frac{1}{2} \int \frac{\mathrm{d}^D k}{(2\pi)^D}\, \phi_i(k)\big(\delta_{ij} k^2 + M^2_{ij}\big)\phi_j(-k) \\ + \frac{g}{3!} \int \frac{\mathrm{d}^D k_1}{(2\pi)^D} \frac{\mathrm{d}^D k_2}{(2\pi)^D} \frac{\mathrm{d}^D k_3}{(2\pi)^D}\, V^{(3)}_{i_1 i_2 i_3}(k_1, k_2, k_3) \phi_{i_1}(k_1)\phi_{i_2}(k_2)\phi_{i_3}(k_3) \\ + \frac{g^2}{4!} \int \frac{\mathrm{d}^D k_1}{(2\pi)^D} \frac{\mathrm{d}^D k_2}{(2\pi)^D} \frac{\mathrm{d}^D k_3}{(2\pi)^D} \frac{\mathrm{d}^D k_4}{(2\pi)^D}\, V^{(4)}_{i_1 i_2 i_3 i_4}(k_1, k_2, k_3, k_4) \phi_{i_1}(k_1)\phi_{i_2}(k_2)\phi_{i_3}(k_3)\phi_{i_4}(k_4). \end{aligned} \tag{2.3}$$

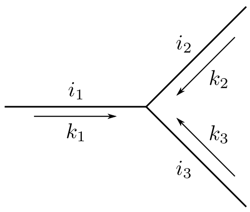
The main advantage of this representation is to turn $V^{(n)}$ into functions of k_i instead of containing derivatives. Moreover, Poincaré invariance implies conservation of momentum, such that

$$V^{(n)}(k_1, \ldots, k_n) \propto \delta^{(n)}(k_1 + \cdots + k_n). \tag{2.4}$$

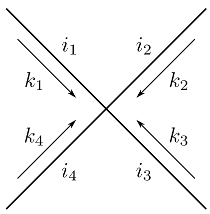
Feynman rules and amplitudes Next, we can derive the Feynman rules. The propagator is easily found since the kinetic term is diagonal:

$$\Delta_{ij}(k) :=$$  $$= \frac{\delta_{ij}}{k^2 + m_i^2}\,. \tag{2.5}$$

Since the interactions are already symmetrized in terms of the indices and momenta, the vertices are simply given by $V^{(3)}$ and $V^{(4)}$:



$$= -V^{(3)}_{i_1 i_2 i_3}(k_1, k_2, k_3), \tag{2.6a}$$



$$= -V^{(4)}_{i_1 i_2 i_3 i_4}(k_1, k_2, k_3, k_4). \tag{2.6b}$$

From the Feynman rules, we can compute the scattering amplitudes of $n \geq 3$ particles with quantum numbers i_j and momenta k_j ($k = 1, \ldots, n$), which are given by the truncated Green functions as follows from the LSZ prescription (for more details, see the discussion in Section 3.1). The 3-point amplitude is simply given by the 3-point vertex:

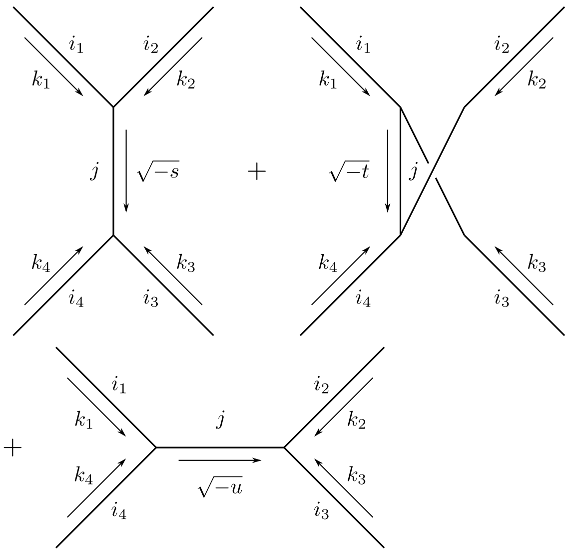
$$A_3(k_1, k_2, k_3)_{i_1 i_2 i_3} = -g\, V^{(3)}_{i_1 i_2 i_3}(k_1, k_2, k_3). \tag{2.7}$$

Next, the 4-point amplitude is given by the sum of four contributions: the 4-point vertex and the s-, t- and u-channel diagrams where an internal particle propagates between two cubic vertices:

$$A_4(k_1, k_2, k_3, k_4)_{i_1 i_2 i_3 i_4} = -g^2\, V^{(4)}_{i_1 i_2 i_3 i_4}(k_1, k_2, k_3, k_4) + g^2\, \mathcal{F}_4(k_1, k_2, k_3, k_4)_{i_1 i_2 i_3 i_4}, \tag{2.8}$$

where $\mathcal{F}_4$ is the contribution from the propagator graphs (which are 1PR at tree-level):

$$\mathcal{F}_4(k_1, k_2, k_3, k_4)_{i_1 i_2 i_3 i_4} :=$$



$$\begin{aligned}
&= V^{(3)}_{i_1 i_2 j}\big(k_1, k_2, \sqrt{-s}\big)\, \frac{1}{-s + m_j^2}\, V^{(3)}_{j i_3 i_4}\big(\sqrt{-s}, k_3, k_4\big) \\
&\quad + V^{(3)}_{i_1 i_3 j}\big(k_1, k_2, \sqrt{-t}\big)\, \frac{1}{-t + m_j^2}\, V^{(3)}_{j i_2 i_4}\big(\sqrt{-t}, k_3, k_4\big) \\
&\quad + V^{(3)}_{i_1 i_4 j}\big(k_1, k_2, \sqrt{-u}\big)\, \frac{1}{-u + m_j^2}\, V^{(3)}_{j i_2 i_3}\big(\sqrt{-u}, k_3, k_4\big)
\end{aligned} \tag{2.9}$$

and where the Mandelstam variables are:

$$s = -(p_1 + p_2)^2, \qquad t = -(p_1 + p_3)^2, \qquad u = -(p_1 + p_4)^2. \tag{2.10}$$

Now, imagine that you can get the full $A^{(n)}$ with $n \geq 3$ in some way,[1], and you want to reconstruct the action. Knowing the propagator, it is possible to reverse-engineer the interactions vertices recursively. Indeed, the tree-level 3-point amplitude equals directly the 3-point vertex as in (2.7). Knowing the 3-point vertex, we can build all propagator graphs $\mathcal{F}_4$ in (2.9) and define the 4-point vertex as what remains after subtracting $\mathcal{F}_4$ from A_4, see (2.8). Next, we build all 5-point propagator graphs from the 3- and 4-point vertices and find that it matches A_5, which indicates that there is no 5-point vertex. This procedure can be continued at all orders n, but also at all loops. In this case, one would build all 1PI and 1PR graphs which contain propagators. The corresponding loop interaction vertices correspond to counter-terms. This is exactly how the SFT action is deduced from the worldsheet path integral.

Physical states and ket representation The action has been written using the position and momentum representations. According to the interpretation of field theory as a second quantization of quantum mechanics, (classical) fields can be viewed as wave functions. As

[1] This is for example the goal in the S-matrix and scattering amplitude programs. In string theory, A_4 is given by the worldsheet path integral over a single surface. However, the worldline path integral would not give A_4 for a point particle in a simple way. The reason is that the former is much more constrained: there are very few possible interactions respecting Lorentz invariance, and higher-order interactions are completely determined by the 3-point interaction (ultimately, because worldsheets are Riemann surfaces). This is not the case for the point particle for which all interactions and graphs must be specified by hand, with the latter proliferating quickly with the number of external states and loops.

such, they can be replaced by a ket field, which is constructed as a linear superposition of first-quantized states forming a complete basis. This formulation has the advantage of changing the space on which the fields live, which is helpful when this space is hard to characterize, as in SFT. However, this introduces a dependence in first-quantized objects alongside the problems associated with this formulation (in particular, apparent lack of non-perturbative description and background dependence).

Denoting the space of first-quantized states by $\mathcal{H}_{\mathrm{m}}$, we introduce a complete basis in momentum space:

$$\mathcal{H}_{\mathrm{m}} = \mathrm{Span}\{|\varphi_i(k)\rangle\}, \tag{2.11}$$

since quantum numbers are in general better given by Fourier modes. These states are orthonormal and define an inner product $\langle\cdot|\cdot\rangle$:

$$\langle\varphi_i(k)|\varphi_j(k')\rangle = (2\pi)^D\, \delta_{ij}\, \delta^{(D)}(k+k'). \tag{2.12}$$

An explicit construction of the basis can be found from the worldline description.[2] The states (2.11) correspond to eigenstates of the worldline momentum p and mass m operators:

$$p\,|\varphi_i(k)\rangle = k\,|\varphi_i(k)\rangle\,, \qquad m\,|\varphi_i(k)\rangle = m_i\,|\varphi_i(k)\rangle\,. \tag{2.13}$$

Therefore, they are eigenstates of the wordline Hamiltonian:

$$H := p^2 + m^2. \tag{2.14}$$

Denoting by X the operator conjugated to p, the basis states are vertex operators, given by the product of a plane-wave with momentum k and a polarization tensor $\epsilon_i \in \mathbb{R}^N$:

$$|\varphi_i(k)\rangle = \epsilon_i\, \mathrm{e}^{\mathrm{i}k\cdot X}\,|0\rangle\,, \tag{2.15}$$

where $|0\rangle$ is the vacuum. In this representation, the momentum operator reads $p := -\mathrm{i}\frac{\partial}{\partial X}$.

Before continuing, let's ask what is the interpretation of the propagator (2.5) in the worldline description? To find out, let's rewrite it using Schwinger parametrization:[3]

$$\frac{1}{k^2+m_i^2} = \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-s(k^2+m_i^2)}. \tag{2.16}$$

A graph containing such a term can be interpreted as a collection of processes where an internal particle propagates in a proper-time s. This makes sense since the Hamiltonian (2.14) generates proper-time evolution.

The field ket is obtained by taking a general linear superposition of (2.11), with the coefficients given by the spacetime fields:

$$|\phi\rangle := \int \frac{\mathrm{d}^D k}{(2\pi)^D}\, \phi_i(k)\,|\varphi_i(k)\rangle\,. \tag{2.17}$$

Given two fields ϕ and ϕ', we see that the inner product (2.12) correctly induces the spacetime inner product used in the action:

$$\langle\phi|\phi'\rangle = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\mathrm{d}^D k'}{(2\pi)^D}\, \phi_i'(k)\phi_j(k')\,\langle\varphi_i(k)|\varphi_j(k')\rangle = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\, \phi_i'(k)\phi_i(-k). \tag{2.18}$$

[2] For lack of space, we cannot enter in the details of the worldline quantization and the reader is referred to the literature for more details [1, 83, 168].

[3] Note that this parametrization is divergent when $k^2 \leq -m_i^2$. This explains the presence and the nature of the UV divergences in the wordline and worldsheet since amplitudes are naturally given in this parametrization.

Conversely, the spacetime field at a given momentum can be expressed from the field ket by contracting with the appropriate basis state:

$$\phi_i(k) = \langle \varphi_i(k) | \phi \rangle \, . \tag{2.19}$$

Note that $\langle \phi | \phi' \rangle = \langle \phi' | \phi \rangle$ since the fields are real.

The field is said to be on-shell if:

$$H \, |\phi\rangle = 0, \tag{2.20}$$

which reproduces the linearized equation of motion from (2.3):

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \, \phi_i(k)(p^2 + m^2) \, |\varphi_i(k)\rangle = 0 \quad \Longrightarrow \quad (k^2 + m_i^2)\phi_i(k) = 0. \tag{2.21}$$

Physical states are solutions to this equation and obey the mass-shell condition:

$$k^2 = -m_i^2. \tag{2.22}$$

We could have motivated the basis (2.11) by starting from the linearized equation of motion and continuing the momentum off-shell to get a complete basis. This teaches that the worldline quantization contains all information about the kinetic term of the field theory. From the equation of motion, the action can be reconstructed given an inner product. The latter is determined by the basis states (2.11) and the associated orthonormality condition (2.19).

Hence, the free action is obtained from the linearized equation of motion by taking the inner product of the state $H \, |\phi\rangle$ with ϕ:

$$S_{\text{free}} := \frac{1}{2} \, \langle \phi | \, H \, |\phi\rangle \, . \tag{2.23}$$

It is straightforward to recover the kinetic term from (2.3) using (2.17) and (2.19). How to describe interactions? They can be implemented by linear operators $\hat{\mathcal{V}}_n : \mathcal{H}_{\mathrm{m}}^{\otimes n} \to \mathbb{C}$ defined by their actions on the basis states (2.11):

$$\hat{\mathcal{V}}_n\big(\, |\varphi_{i_1}(k_1)\rangle \, , \dots , |\varphi_{i_n}(k_n)\rangle \, \big) := V^{(n)}_{i_1 \cdots i_n}(k_1, \dots, k_n). \tag{2.24}$$

For simplicity, we will omit the symbol of the ket inside the arguments of vertex functions. Note that $\hat{\mathcal{V}}_n$ is symmetric in its arguments. Then, the action (2.3) can simply be written as:

$$S = \frac{1}{2} \, \langle \phi | \, H \, |\phi\rangle + \frac{g}{3!} \, \hat{\mathcal{V}}_3(\phi^3) + \frac{g^2}{4!} \, \hat{\mathcal{V}}_4(\phi^4), \tag{2.25}$$

where $\hat{\mathcal{V}}_n(\phi^n) := \hat{\mathcal{V}}_n(\phi, \dots, \phi)$. In fact, it is also possible to define a quadratic vertex function:

$$\hat{\mathcal{V}}_2(\phi, \phi') := \langle \phi | \, H \, |\phi'\rangle \, . \tag{2.26}$$

One sees that vertex functions correspond to classical n-point truncated Green functions without internal propagators.

There is a last modification to make in order to reach a full equivalence with SFT: considering a field with a dependence in a Grassmann-odd coordinate c. It corresponds to the wordline reparametrization ghost, whose conjugate variable is another ghost b. They arise by fixing the worldline reparametrization invariance through the Faddeev–Popov procedure: by following the second-quantization procedure, all worldline variables are promoted to field dependence. Since c is just a Grassmann number, the field can be represented as the sum of two terms:

$$\Phi(k, c) := \phi_\downarrow(k) + c \, \phi_\uparrow(k). \tag{2.27}$$

The full first-quantized Hilbert space $\mathcal{H}$ corresponds to:

$$\mathcal{H} := \mathcal{H}_{\mathrm{m}} \otimes c\mathcal{H}_{\mathrm{m}}, \tag{2.28}$$

since c defines a two-state Hilbert space $\{|\downarrow\rangle, |\uparrow\rangle\}$ such that

$$|\uparrow\rangle = c\,|\downarrow\rangle, \qquad \langle\uparrow|\downarrow\rangle = 1. \tag{2.29}$$

There is a wordline BRST charge associated to the reparametrization invariance:

$$Q := cH, \tag{2.30}$$

where H is the Hamiltonian (2.14).

The action (2.25) can be rewritten in the Hilbert space $\mathcal{H}$ by multiplying each term with $\langle\uparrow|\downarrow\rangle = 1$ and replacing the field ϕ by Φ: the component $\phi_\uparrow$ cannot contribute to the action as c is nilpotent. Hence, we find the action:

$$S = \frac{1}{2}\langle\Phi|\,Q\,|\Phi\rangle + \frac{g}{3!}\,\mathcal{V}_3(\Phi^3) + \frac{g^2}{4!}\,\mathcal{V}_4(\Phi^4), \tag{2.31}$$

where $\mathcal{V}_n := c\hat{\mathcal{V}}_n$. This is the natural form of field theory as derived from the wordline.

Note that the kinetic term is invariant under the gauge transformation

$$\delta\,|\Phi\rangle = Q\,|\Lambda\rangle \tag{2.32}$$

since Q is nilpotent. The gauge transformation can be modified to take into account interactions such that the full action is gauge invariant. The conditions of gauge invariance can be rephrased in terms of L_∞ algebra [169, 170], which we will discuss in more details later.

In order to get a form of the action from which it is simpler to take variations (to derive the equation of motion or to study the gauge invariance), it is useful to introduce products $\ell_n : \mathcal{H}^{\otimes n} \to \mathcal{H}$ defined by the inner product:

$$\mathcal{V}_{n+1}(\Phi_0, \ldots, \Phi_n) := \langle\Phi_0|\ell_n(\Phi_1, \ldots, \Phi_n)\rangle\,. \tag{2.33}$$

We have $\ell_1 = Q$ and for $n \geq 2$:

$$\ell_n(\Phi_1, \ldots, \Phi_n) = \int \prod_{i=0}^{n} \frac{\mathrm{d}^D k_i}{(2\pi)^D}\, V^{(n+1)}_{i_0 i_1 \cdots i_n}(k_0, k_1, \ldots, k_n)\, \phi_{i_1}(k_1) \cdots \phi_{i_n}(k_n)\, c\,|\varphi_{i_0}(k_0)\rangle\,. \tag{2.34}$$

This study of scalar field theory illustrates how to construct a field theory from the worldline path integral. We can now apply this method to string theory.

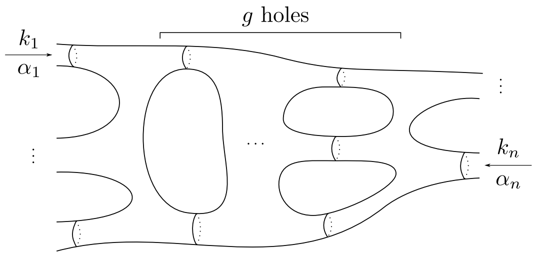
2.2 Worldsheet path integral

The objective is to build the SFT action by reverse engineering of the worldsheet description. In this section, we describe the worldsheet path integral and its gauge fixing to compute amplitudes, and the BRST quantization to obtain the physical states. We will outline the main steps of the construction, referring the reader to the literature for more details [**Erbin:2020:StringFieldTheory**, 171–173].

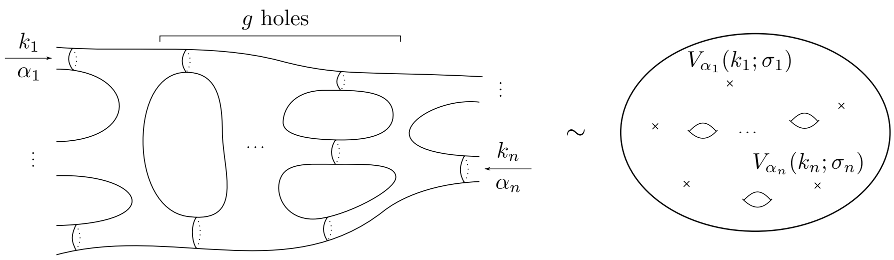
2.2.1 Worldsheet path integral and amplitudes

An amplitude is obtained by summing over all possible processes with the same external states. Since the closed string is topologically a circle S^1, its time evolution is a 2-dimensional surface, called the worldsheet. Hence, the interaction of n strings corresponds to a 2-dimensional manifold with n legs. Internal closed string loops make holes in the surface. Thus, the g-loop amplitude of n external states with quantum numbers α_i and momenta k_i can be represented as:

$$A_{g,n}(k_1,\ldots,k_n)_{\alpha_1,\ldots,\alpha_n} = \sum_{\text{surfaces}} \tag{2.35}$$



where the sum is over all inequivalent surfaces and the legs should be understood as semi-infinite tubes. The worldsheet is topologically a genus-g Riemann surfaces with n punctures denoted as $\Sigma_{g,n}$. Thus, we can obtain a canonical representation by mapping the external states to points σ_i on the surfaces (punctures):



(2.36)

In this case, the quantum numbers of the external states are carried by a vertex operators inserted at the puncture.[4] To write the path integral, we introduce a metric g_{ab} on the worldsheet and the embedding of the string in spacetime is described by non-compact scalar fields X^μ with $\mu = 0,\ldots,D-1$. Free propagation in the worldsheet is described by the Polyakov action

$$S_{\mathrm{P}}[g,X^\mu] = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{g}\,g^{ab}\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}, \tag{2.37}$$

such that the worldsheet path integral reads:

$$A_{g,n}(\{k_i\})_{\{\alpha_i\}} := \int \frac{\mathrm{d}_g g_{ab}\mathrm{d}_g X}{\mathrm{Vol}\,G}\,\mathrm{e}^{-S_{\mathrm{P}}[g,X]-\Phi_0 S_{\mathrm{EH}}[g]}\prod_{i=1}^{n}\left(\int \mathrm{d}^2\sigma_i\sqrt{g}\,V_{\alpha_i}(k_i;\sigma_i)\right), \tag{2.38}$$

where G is the gauge group of the worldsheet theory (diffeomorphisms and Weyl symmetry). The vertex operators are built from g_{ab} and X: the associated physical states will be described below. In the Polyakov action, α' is the Regge slope which is a dimensionful constant converting from the worldsheet to spacetime units. The integral is over all metrics: since many metrics describe topological equivalent surfaces, we have divided by the volume of the corresponding gauge group to sum only over inequivalent surfaces. The vertex operators are

[4] This is an advantage of using the momentum representation, where external states are described by inserting appropriate wave functions in the path integral. In the position representation, they are described by boundary conditions on the field in the path integral.

integrated over the surfaces because the mapping to a puncture can be done at any point. We omit an overall normalization C_g which can be found from unitarity [172].

In the path integral, we have added the Einstein–Hilbert action (with boundary terms and normalized by $1/4\pi$) with coupling Φ_0: the action is topological and proportional to the Euler characteristics $\chi_{g,n}$:

$$S_{\text{EH}} = \chi_{g,n} := 2 - 2g - n. \tag{2.39}$$

What is the role of this contribution? In fact, the coupling constant Φ_0 corresponds to a constant dilaton background whose value, in turn, is related to the string coupling constant g_s as

$$\Phi_0 = \ln g_s. \tag{2.40}$$

Thus, this factor can be rewritten as

$$\mathrm{e}^{-\Phi_0 S_{\text{EH}}} = g_s^{2g+n-2} \tag{2.41}$$

which counts the number of cubic interactions in the amplitude. Indeed, we see from the first form of the worldsheet that any interaction of order higher than 3 can be recast as a series of cubic interactions since the surfaces are topologically equivalent. This explains largely why the worldsheet path integral is so much simpler than the worldline.

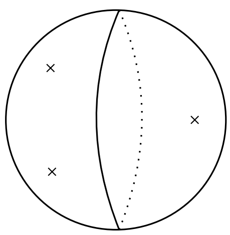
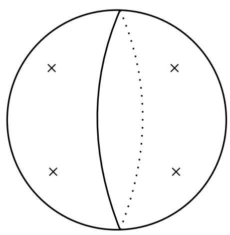
The Polyakov action is invariant under diffeomorphisms and Weyl transformations (local rescalings of the metric and fields). To proceed, we gauge fix these symmetries by writing the metric in the conformal gauge:

$$g_{ab} = \mathrm{e}^{2\phi}\, \hat{g}_{ab}, \tag{2.42}$$

where ϕ is the Liouville field (conformal mode) and $\hat{g}_{ab}$ the background metric. Since the metric has 3 components and there are 3 gauge parameters (2 for diffeomorphisms, 1 for Weyl transformations), one could think that the integral over the metrics completely cancels the volume of the gauge group and that $\hat{g}_{ab}$ is completely fixed. This would be correct if all Riemann surfaces at a given genus were topologically equivalent, but this is not the case. The space of inequivalent genus-g Riemann surfaces is denoted by $\mathcal{M}_g$. The infinite-dimensional integral over the metric reduces to a finite-dimensional over the moduli parameters t_i which parametrize $\mathcal{M}_g$, but also the background metric : $\hat{g}_{ab} = \hat{g}_{ab}(t)$. The real dimension M_g of the moduli space reads

$$\mathsf{M}_g := \dim \mathcal{M}_g = \begin{cases} 0 & g = 0, \\ 2 & g = 1, \\ 6g - 6 & g > 1. \end{cases} \tag{2.43}$$

There is one caveat: the integral over the Liouville field disappears only when spacetime has the critical dimension $D = 26$ (otherwise, one gets $2d$ gravity, see Chapter 5). As an example, 3- and 4 punctured spheres are represented as:

$\Sigma_{0,3} =$  $\Sigma_{0,4} =$  (2.44)

and the corresponding moduli spaces are:

$$\mathcal{M}_{0,3} = \{1\}, \qquad \mathcal{M}_{0,4} = \mathbb{C}. \tag{2.45}$$

However, the conformal gauge does not completely fix the gauge symmetry when there are conformal Killing vectors (CKV). The latter form a group denoted by $\mathcal{K}_g$ and its real dimension K_g is:

$$\mathsf{K}_g := 2\mathsf{K}_g^c := \dim_{\mathbb{R}} \mathcal{K}_g = \begin{cases} 6 & g = 0, \\ 2 & g = 1, \\ 0 & g > 1. \end{cases} \tag{2.46}$$

More specifically, we have:

$$\mathcal{K}_0 = \mathrm{SL}(2, \mathbb{C}), \qquad \mathcal{K}_1 = \mathrm{U}(1) \times \mathrm{U}(1). \tag{2.47}$$

The CKV are gauge fixed by specifying the locations of K_g^c vertex operators: this limits the current procedure to the cases where $\chi_{g,n} < 0$. However, a simple extension allows to treat also the case $g = 1$ and $n = 0$ since the CKV group is compact [172]. The remaining interesting case of the tree-level 2-point amplitude ($g = 0, n = 2$) is special and will be discussed in Chapter 3.

The gauge fixing also introduces a Faddeev–Popov determinant rewritten in terms of ghosts c^a and b_{ab} (symmetric traceless). After several simplifications, the amplitude becomes:

$$A_{g,n}(\{k_i\})_{\{\alpha_i\}} = g_s^{-\chi_{g,n}} \int_{\mathcal{M}_g} \mathrm{d}^{\mathsf{M}_g} t \int \prod_{i=\mathsf{K}_g^c+1}^{n} \mathrm{d}^2\sigma_i \sqrt{\hat{g}} \left\langle \prod_{i=1}^{\mathsf{M}_g} \hat{B}_i \prod_{j=1}^{\mathsf{K}_g^c} \hat{\mathscr{V}}_{\alpha_j}(k_j; \sigma_j^0) \prod_{i=\mathsf{K}_g^c+1}^{n} \hat{V}_{\alpha_i}(k_i; \sigma_i) \right\rangle_{\hat{g}}, \tag{2.48}$$

where

$$\hat{\mathscr{V}}_{\alpha_j}(k_j; \sigma_j^0) := \frac{\epsilon_{ab}}{2} c^a(\sigma_j^0) c^b(\sigma_j^0)\, \hat{V}_{\alpha_j}(k_j; \sigma_j^0), \qquad \hat{B}_i := (\hat{\mu}_i, b)_{\hat{g}}. \tag{2.49}$$

The hat on the different quantities indicates that they are evaluated in the background metric $\hat{g}_{ab}$. The operators $\hat{B}_i$ denotes inner products of the ghost b_{ab} with the Beltrami differentials μ_{iab}, which parametrize the possible deformations of the background metric under variations of the moduli parameters. Only $n - \mathsf{K}_n^c$ matter vertex operators are integrated over the surfaces. The remaining K_n^c matter operators are dressed with c ghosts and called *unintegrated operators*. These are the natural elements of the BRST cohomology. Finally, the correlation function is evaluated over the matter and ghost theories in the background metric $\hat{g}$:

$$\langle \mathcal{O} \rangle_{\hat{g}} := \int \mathrm{d}_{\hat{g}} X \, \mathrm{d}_{\hat{g}} b \, \mathrm{d}_{\hat{g}} c \, \mathrm{e}^{-S_{\mathrm{P}}[\hat{g}, X] - S_{\mathrm{gh}}[\hat{g}, b, c]} \, \mathcal{O}, \tag{2.50}$$

where the ghost action reads:

$$S_{\mathrm{gh}}[g, b, c] := \frac{1}{4\pi} \int \mathrm{d}^2\sigma \sqrt{g}\, g^{ab} \big(b_{ac} \nabla_b c^c + b_{bc} \nabla_a c^c - b_{ab} \nabla_c c^c \big). \tag{2.51}$$

The ghost action is invariant under a U(1) symmetry whose charge is called the ghost number:

$$N_{\mathrm{gh}}(b) = -1, \qquad N_{\mathrm{gh}}(c) = 1. \tag{2.52}$$

This symmetry is anomalous on curved spaces, which implies that the total ghost number of a correlation function must be:

$$N_{\mathrm{gh}} = 3\chi_{g,0}. \tag{2.53}$$

The simplest choice of background metric is the flat metric $\hat{g}_{ab} = \delta_{ab}$. In this case, the diffeomorphism and Weyl invariances of the matter and ghost actions imply that they are conformal field theories (CFT). This allows to use all techniques from CFT to compute the correlation functions appearing in the amplitudes. More generally, we can consider more general matter by changing the boundary conditions of the fields X^μ or considering other CFTs. This would change the properties of spacetime and of the string, for example, by describing compact dimensions or modifying the spectrum. The only condition is that the total central charge must be $c_{\mathrm{m}} = 26$ (scalar fields have $c = 1$).

The amplitude (2.48) can be further improved. Indeed, it would be better if all vertices were treated on an equal footing. Moreover, we have argued at the beginning of this subsection that the worldsheet is a genus-g Riemann surfaces with n punctures, but the current form treats differently the moduli associated with punctures (vertex operators, $\mathbb{C}^{n-\mathsf{K}_g^c}$) and holes (metric, $\mathcal{M}_g$). The solution is to move the dependence on the moduli from the metric and vertex operators to transition functions. This is achieved by introducing local coordinate patches: there is a canonical construction of Riemann surfaces once such patches and transition functions are specified. Let's introduce a local coordinate patch ς_i^a around each puncture, and transition functions f_i^a such that the puncture located at $\sigma^a = \sigma_i^a$ lies at $\varsigma_i^a = 0$ in the patch:

$$\sigma^a = f_i^a(\varsigma_i^a), \qquad \sigma_i^a = f_i^a(0), \tag{2.54}$$

where σ^a denotes the coordinate on the surface used until now. To describe the surface away from the punctures, we need additional patches but, since we don't need to explicit them for our purposes, we will not describe them further. Then, the amplitude (2.48) can be rewritten as:

$$A_{g,n}(\{k_i\})_{\{\alpha_i\}} = g_s^{-\chi_{g,n}} \int_{\mathcal{M}_{g,n}} \mathrm{d}^{\mathsf{M}_{g,n}} t \left\langle \prod_{\lambda=1}^{\mathsf{M}_{g,n}} \hat{B}_\lambda \prod_{i=1}^{n} f_i \circ \hat{\mathscr{V}}_{\alpha_i}(k_i; 0) \right\rangle_{\hat{g}}, \tag{2.55}$$

where $\mathcal{M}_{g,n}$ is the moduli space of genus-g Riemann surfaces with n punctures of real dimension $\mathsf{M}_{g,n}$:

$$\mathsf{M}_{g,n} := \dim \mathcal{M}_{g,n} = 6g - 6 + 2n, \qquad \text{for} \quad \begin{cases} g \geq 2, \\ g = 1, n \geq 1, \\ g = 0, n \geq 3. \end{cases} \tag{2.56}$$

Note that the conditions written on the side are equivalent to $\chi_{g,n} < 0$. Similarly, we also introduce the CKV groups $\mathcal{K}_{g,n}$ of $\Sigma_{g,n}$ which dimensions are:

$$\mathsf{K}_{g,n} := \dim \mathcal{K}_{g,n} = \begin{cases} 6 - 2n & g = 2, n \leq 2, \\ 2 & g = 1, n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.57}$$

The groups are not empty when $\chi_{g,n} \geq 0$. The insertions $\hat{B}_\lambda$ are now written in terms of the transition functions, without reference to the metric. Note that there are $n - \mathsf{K}_g^c$ additional such insertions. In turn, all vertex operators are unintegrated: the notation $f_i\circ$ indicate that they are evaluated in the corresponding patch, with f_i acting as a change of coordinates. Finally, we note that the amplitude is graded-symmetric under exchange of its external states (because the punctures can move freely on the surfaces without overlapping).

It can be shown that the result of the amplitude is independent from the choice of the transition functions f_i for *on-shell* external states. In fact, this formulation opens the door for defining off-shell amplitudes: instead of consider the moduli space $\mathcal{M}_{g,n}$, one considers the infinite-dimensional fiber bundle $\mathcal{P}_{g,n}$ with $\mathcal{M}_{g,n}$ as the base space and the possible local coordinates as the fiber. Then, the amplitude is defined by integrating over a $\mathsf{M}_{g,n}$-dimensional section over $\mathcal{P}_{g,n}$. This formulation allows to characterize precisely the off-shell dependence in local coordinates and to ensure a consistent factorization of amplitudes.

The formalism extends directly to the classical open string. In this case, the Riemann surfaces is a disk with punctures on its boundary describing the external open string states. However, since the punctures are located on the boundary and cannot be reordered without overlapping, the contribution from a given family of surfaces is only invariant under cyclic permutations of the punctures. The total amplitude is obtained by summing over all permutations. All dimensions are divided by two (so there are twice less moduli and CKV), and vertex operators are dressed by a single linear combination of c^a. Finally, the ghost number anomaly on the disk implies that the total ghost number of correlation functions must be $N_{\text{gh}} = 3$.

2.2.2 BRST quantization

The Faddeev–Popov gauge fixing introduces a BRST symmetry for the combined action of the ghost and matter:

$$\delta_\epsilon X = \mathrm{i}\epsilon\, \mathcal{L}_c X, \qquad \delta_\epsilon c^a = \mathrm{i}\epsilon\, \mathcal{L}_c c^a, \qquad \delta_\epsilon b_{ab} = \mathrm{i}\epsilon\, T_{ab}. \tag{2.58}$$

where T_{ab} is the total energy–momentum tensor and $\mathcal{L}_c$ is the Lie derivative along the vector given by the ghost field c^a. The symmetry is generated by the BRST charge Q which is nilpotent (in the critical dimension) and has ghost number 1:

$$Q^2 = 0, \qquad N_{\text{gh}} = 1. \tag{2.59}$$

We want to characterize the physical states to know which amplitudes to compute, but also because it allows to build the kinetic term of the field action, as reviewed with the toy model from Section 2.1. As usual for gauge theories, physical states $|\psi\rangle$ are given by the BRST cohomology $\mathcal{H}(Q)$:

$$|\psi\rangle \in \mathcal{H}(Q) := \frac{\ker Q}{\operatorname{Im} Q}, \tag{2.60}$$

that is, states which are closed but not exact:

$$Q\,|\psi\rangle = 0, \qquad \nexists\, |\chi\rangle : |\psi\rangle = Q\,|\chi\rangle\,. \tag{2.61}$$

The second condition is equivalent to saying that two states in the cohomology differing by an exact state are equivalent:

$$|\psi\rangle \sim \psi + Q\,|\lambda\rangle\,. \tag{2.62}$$

It is possible to show that physical states have respectively ghost numbers 1 and 2 for the open and closed strings:

$$\text{open}: \quad N_{\text{gh}}(\psi) = 1, \qquad \text{closed}: \quad N_{\text{gh}}(\psi) = 2. \tag{2.63}$$

This is everything we need to describe SFT: a description of the BRST cohomology in the more general case of $2d$ gravity will be given in Chapter 5.

2.3 Free string field theory

The previous section has provided everything which is needed to build the SFT action. As explained in Section 2.1, we first need the free action in order to derive the propagator, from which we can decompose amplitudes.

Starting from this section, we will adopt a flat background metric with complex coordinates z and $\bar{z}$ and use the CFT language to be more explicit. In complex coordinates, the ghosts and energy–momentum tensor are described by holomorphic $b(z)$, $c(z)$ and $T(z)$ and anti-holomorphic components $\bar{b}(\bar{z})$ and $\bar{c}(\bar{z})$, $\bar{T}(\bar{z})$. The holomorphic components have the following mode expansions:

$$b(z) = \sum_{n\in\mathbb{Z}} \frac{b_n}{z^{n+2}}, \qquad c(z) = \sum_{n\in\mathbb{Z}} \frac{c_n}{z^{n-1}}, \qquad T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}, \tag{2.64}$$

and similarly for the anti-holomorphic components. We define the following combinations:

$$b_0 := b_0 \pm \bar{b}_0, \qquad c_0^{\pm} := \frac{1}{2}\,(c_0 \pm \bar{c}_0), \qquad L_0^{\pm} := L_0 \pm \bar{L}_0. \tag{2.65}$$

We refer to the appendix Appendix A.1 for more details on CFT.

The string field $|\Psi\rangle$ is given by a general linear combination of all string states in the CFT Hilbert space $\mathcal{H}$. Since going off-shell should not change the ghost number, the fields must have the same ghost number (2.63). Then, the first-quantized physical state condition (2.61) can be reinterpreted as the equation of motion for the free string field:

$$Q\,|\Psi\rangle = 0. \tag{2.66}$$

To recover the action, we need an inner product $\langle \cdot, \cdot\rangle$ on $\mathcal{H}$ such that

$$S_{\text{free}} := \frac{1}{2}\,\mathcal{V}_{0,2}(\Psi^2) = \frac{1}{2}\,\langle \Psi, Q\Psi\rangle. \tag{2.67}$$

We have introduced the notation $\mathcal{V}_{0,2}$ for later purpose.

A natural inner product on the CFT Hilbert space is the BPZ product defined as:

$$\langle A|B\rangle = \lim_{\substack{z\to\infty \\ w\to 0}} z^{2h_A}\,\langle 0|\, A(z)B(w)\,|0\rangle \tag{2.68}$$

for any holomorphic operators A and B. However, one has to be careful with the ghost number. For the open string, $N_{\text{gh}}(\Psi) = 1$ and $N_{\text{gh}}(Q) = 1$ such that the action has correctly $N_{\text{gh}} = 3$ and the inner product is simply the BPZ product. This does not work for the closed string since $N_{\text{gh}}(\Psi) = 2$ and the action would have $N_{\text{gh}} = 5$ instead of $N_{\text{gh}} = 6$ as dictated by the ghost number anomaly (2.53). This means that the inner product must contain an additional c ghost insertion. The simplest solution is to insert a combination of zero-modes, and the only possibility is c_0^-. Indeed, the decomposition (A.47) shows that $Q = c_0^+ L_0^+ + c_0^- L_0^- + \cdots$, but since $L_0^+ = \alpha'(k^2 + m^2)$ is the operator which reproduces the mass-shell condition, it must be present. We thus have:

$$\text{open}: \quad \langle A, B\rangle := \langle A|B\rangle\,, \qquad \text{closed}: \quad \langle A, B\rangle := \langle A|\,c_0^-\,|B\rangle\,. \tag{2.69}$$

But the closed string inner product defined in (2.69) has a problem because it is degenerate on $\mathcal{H}$. To fix this problem, it is necessary to consider a subspace where the inner product is non-degenerate. This is achieved by imposing the *level-matching conditions*:

$$b_0^-\,|\Psi\rangle = 0, \qquad L_0^-\,|\Psi\rangle = 0. \tag{2.70}$$

To avoid the use of new notations, we will not introduce a new symbol for this subspace, but it is always understood that the closed string field satisfies these conditions. The physical interpretation is as follows: since the closed string is a circle, all points are equivalent and the worldsheet theory is invariant under spatial translations along the circle. However, using an explicit parametrization breaks this translation symmetry and the corresponding generator L_0^- must be set to zero. Correspondingly, the ghost b_0^- associated to this symmetry must also be removed from the description.

The equation of motion (2.66) is not sufficient to characterize physical states: there is also the equivalence (2.62) of states in the cohomology which differ by an exact state. In fact, this equivalence can be extended off-shell to the string field and interpreted as a gauge transformation with parameter $\Lambda \in \mathcal{H}$

$$\delta_\Lambda \left|\Psi\right\rangle = Q \left|\Lambda\right\rangle \tag{2.71}$$

which leaves the action invariant since Q is nilpotent:

$$\delta_\Lambda S_{\text{free}} = 0. \tag{2.72}$$

The parameter has ghost number:

$$\text{open}: \quad N_{\text{gh}}(\Lambda) = 0, \qquad \text{closed}: \quad N_{\text{gh}}(\Lambda) = 1 \tag{2.73}$$

and satisfies the level-matching condition (2.70) for the closed string.

Since there is a gauge invariance, it is necessary to gauge fix the action in order to invert the kinetic term to find the propagator. The simplest gauge fixing condition is the Siegel gauge:[5]

$$\text{open}: \quad b_0 \left|\Psi\right\rangle = 0, \qquad \text{closed}: \quad b_0^+ \left|\Psi\right\rangle = 0. \tag{2.74}$$

Then, using the decomposition (A.47) of the BRST charge, the action reduces to:

$$\text{open}: \quad S_{\text{free}} = \frac{1}{2} \langle \Psi, c_0 L_0 \Psi \rangle, \qquad \text{closed}: \quad S_{\text{free}} = \frac{1}{2} \langle \Psi, c_0^+ L_0^+ \Psi \rangle. \tag{2.75}$$

The full procedure will be discussed in more details in Chapter 6. Then, we obtain the propagators:

$$\text{open}: \quad \Delta = \frac{b_0}{L_0}, \qquad \text{closed}: \quad \Delta = \frac{b_0^+}{L_0^+}. \tag{2.76}$$

To make contact with QFT, we introduce a complete basis of states:

$$\mathcal{H} = \operatorname{Span}\{\left|\phi_\alpha(k)\right\rangle\}. \tag{2.77}$$

which are eigenstates of L_0 and $\bar{L}_0$ such that:

$$\begin{aligned} \text{open}: \quad & L_0 \left|\phi_\alpha(k)\right\rangle = \alpha'(k^2 + m^2) \left|\phi_\alpha(k)\right\rangle, \\ \text{closed}: \quad & L_0^+ \left|\phi_\alpha(k)\right\rangle = \frac{\alpha'}{2}(k^2 + m^2) \left|\phi_\alpha(k)\right\rangle. \end{aligned} \tag{2.78}$$

Note that the mass are related to the total level operator as:

$$\text{open}: \quad m^2 = \frac{1}{\alpha'} \widehat{L}_0, \qquad \text{closed}: \quad m^2 = \frac{4}{\alpha'} \widehat{L}_0. \tag{2.79}$$

The gauge fixed equations of motions $L_0 = 0$ and $L_0^+ = 0$ match the on-shell condition (2.22) from point particle QFT.

Finally, we can check that the action is correctly normalized by truncating the string field to the tachyon (focusing on the open string for simplicity):

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{\alpha'}} \int \frac{\mathrm{d}^D k}{(2\pi)^D} T(k)\, c_1 \left|k\right\rangle. \tag{2.80}$$

The action reads:

$$S[T] = \frac{1}{2} \int \frac{\mathrm{d}^D k}{(2\pi)^D} T(-k) \left(k^2 - \frac{1}{\alpha'}\right) T(k) \tag{2.81}$$

which correctly describes a real scalar of mass $m^2 = -1/\alpha'$. The action for massless states, including the gauge field, will be discussed in Chapter 6. We will see that the Siegel gauge is a generalization of the Lorentz gauge in SFT.

[5] Note that these conditions are in fact identical since the level-matching conditions gives $b_0 = \bar{b}_0$ such that $b_0^+ = 2b_0$.

2.4 Interacting string field theory

2.4.1 Plumbing fixture and Feynman diagrams

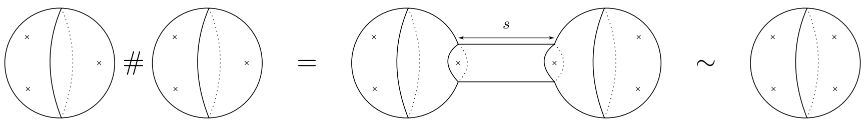
Having the propagator (2.76), it is possible to decompose the amplitudes in Feynman diagrams in order to identify the interaction vertices. Since the amplitudes are expressed as the sum over topologically distinct Riemann surfaces, it is expected that each Feynman diagrams will describe a subset of surfaces. The question is then what is the meaning of the propagator in terms of Riemann surfaces. Following what we did for the scalar QFT, we rewrite the propagator using Schwinger parametrization (we focus on the closed string):

$$\frac{1}{L_0^+} = \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-sL_0^+}. \tag{2.82}$$

Since L_0^+ (resp. L_0) is the generator of dilatations for the closed (resp. open) string, the integrand corresponds to a piece of worldsheet – a tube (resp. a strip) – of length s. In fact, there is an additional parameter for the closed string, since the tube can be twisted by an angle θ. This arises from the level matching condition which is enforced on propagating states:

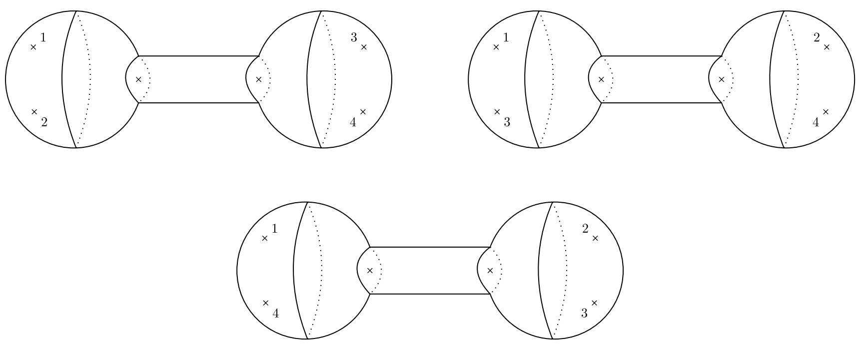
$$\delta(L_0^-) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, \mathrm{e}^{\mathrm{i}\theta L_0^-}. \tag{2.83}$$

The operation of adding a tube or a strip between two Riemann surfaces is called the *plumbing fixture* and denoted as #: it consists in cutting a disk (closed string) or half-disk (open string) around a puncture in each surface and identifying the boundaries. The plumbing fixture can be explicitly defined in terms of local coordinates, but we don't need it for our purpose. For example, gluing two 3-punctured spheres with s fixed yields a 4-punctured sphere:



(2.84)

Varying s and θ gives a 2-dimensional family of 4-punctured surfaces. Moreover, since the punctures are distinguishable, inequivalent surfaces are obtained by permuting the punctures. In the previous case, labeling the punctures by 1, 2, 3 and 4, we have the following three inequivalent (s-, t- and u-channel) diagrams:



(2.85)

The operation # is defined on moduli spaces by considering the plumbing fixture of all possible pairs of surfaces, permutations of punctures, and $s \in [0, \infty)$ and $\theta \in [0, 2\pi]$. The space obtained by gluing 3-punctured spheres together is denoted by $\mathcal{F}_{0,4}$:

$$\mathcal{F}_{0,4} := \mathcal{M}_{0,3} \# \mathcal{M}_{0,3} \subset \mathcal{M}_{0,4}. \tag{2.86}$$

This is called the propagator region of the moduli space. The question is whether this inclusion is an equality, which is the case if all 4-punctured spheres can be obtained from plumbing fixture of 3-punctured spheres. The answer is negative, and the remaining subspace of 4-punctured spheres which cannot be obtained in that way is denoted by $\mathcal{V}_{0,4}$, and is called the fundamental region:

$$\mathcal{V}_{0,4} := \mathcal{M}_{0,4} - \mathcal{F}_{0,4}. \tag{2.87}$$

This construction can be iterated to higher-dimensional moduli spaces, noting that the plumbing fixture can also be used to glue two punctures from the same surface (then, $\#$ acts as a unary operator). This provides a decomposition of the moduli spaces $\mathcal{M}_{g,n}$ and yields a series of fundamental regions $\mathcal{V}_{g,n}$ and propagator regions $\mathcal{F}_{g,n}$ for all $g, n \geq 0$. Since the plumbing fixture depends on the local coordinates, it means that the spaces $\mathcal{V}_{g,n}$ depend on the choice of local coordinates for all surfaces from moduli spaces of lower dimensions. Moreover, the local coordinates for surfaces in $\mathcal{F}_{g,n}$ are induced from the ones of lower-dimensional spaces. Constructing consistent sets of local coordinates for all $g, n \geq 0$ and determining the appropriate $\mathcal{V}_{g,n}$ is the main challenge of SFT as described in the introduction (Section 1.1).

Coming back to SFT, we have everything to reinterpret the amplitudes in terms of Feynman diagrams. It is now obvious that the set of surfaces which can be written as two surfaces connected by a tube must be identified with Feynman diagrams containing propagators. The corresponding contribution to the amplitude (2.55) is simply obtained by restricting the integration over the moduli parameters to the region $\mathcal{F}_{g,n}$ (and using the appropriate local coordinates obtained from the recurring plumbing fixtures). This contribution is denoted by the same name as the region of the moduli space:

$$\mathcal{F}_{g,n}(\{k_i\})_{\{\alpha_i\}} := g_s^{-\chi_{g,n}} \int_{\mathcal{F}_{g,n}} \mathrm{d}^{\mathsf{M}_{g,n}} t \left\langle \prod_{\lambda=1}^{\mathsf{M}_{g,n}} \hat{B}_\lambda \prod_{i=1}^{n} f_i \circ \hat{\mathscr{V}}_{\alpha_i}(k_i; 0) \right\rangle_{\hat{g}}. \tag{2.88}$$

Similarly, the contribution from the fundamental region is:

$$\mathcal{V}_{g,n}(\{k_i\})_{\{\alpha_i\}} := g_s^{-\chi_{g,n}} \int_{\mathcal{V}_{g,n}} \mathrm{d}^{\mathsf{M}_{g,n}} t \left\langle \prod_{\lambda=1}^{\mathsf{M}_{g,n}} \hat{B}_\lambda \prod_{i=1}^{n} F_i \circ \hat{\mathscr{V}}_{\alpha_i}(k_i; 0) \right\rangle_{\hat{g}}, \tag{2.89}$$

where the local coordinates F_i are different from the ones used in $\mathcal{F}_{g,n}$. Since it cannot be interpreted as a Feynlan diagrams with propagators, it must be an interaction vertex.

To conclude this section, we briefly discuss the notion of *stub*. Since there is some arbitrariness in which Riemann surfaces are considered to have a tube sufficiently long to be called a propagator, it is possible to rearrange the respective contributions from $\mathcal{F}_{g,n}$ and $\mathcal{V}_{g,n}$. This amounts to set a cut-off $s \geq s_0$ on the Schwinger parameter, such that all tubes below this length are viewed as part of the interaction vertex instead of the propagator. This allows to obtain better parametrizations of the string vertices and, in particular, is useful to show that SFT is UV finite to all orders in perturbation theory (Section 2.5).

2.4.2 String field action

Following the example of the scalar QFT, we know that we obtain the interactions in the action simply by replacing the n states in the vertex by n copies of the field. The above constructions thus provides tree-level interactions $\mathcal{V}_{0,n}$ for $n \geq 3$, but also quantum terms $\mathcal{V}_{1,n}$ for $n \geq 1$ and $\mathcal{V}_{g,n}$ for $g, n \geq 0$. However, it makes sense to introduce additional vertices. First, we remark that all $\mathcal{V}_{g,n}$ introduced so far are truncated Green functions: but, so is the free action (2.75) which motivates the notation $\mathcal{V}_{0,2}$ introduced in (2.67). Second, it is also necessary to introduce a 1-loop vertex $\mathcal{V}_{1,0}$ (1-loop cosmological constant) for background independence [174]. Third, the remaining vertices $\mathcal{V}_{0,0}$ (1-loop cosmological constant) and $\mathcal{V}_{0,1}$ (classical source) appears when SFT is formulated around a non-conformal background. As a consequence, the general form of the SFT action reads:

$$S = \sum_{g,n\geq 0} \frac{\hbar^{2g} g_s^{-\chi_{g,n}}}{n!} \mathcal{V}_{g,n}(\Psi^n), \tag{2.90}$$

where we have introduced the appropriate power of $\hbar$.

The string vertices can be seen as function $\mathcal{V}_{g,n} : \mathcal{H}^{\otimes n} \to \mathbb{C}$ linear in each argument. Since the string inner products (2.69) are non-degenerate, we can introduce string products $\ell_{g,n} : \mathcal{H}^{\otimes n} \to \mathcal{H}$ such that

$$\mathcal{V}_{g,n+1}(\Psi_0, \Psi_1, \dots, \Psi_n) =: \langle A_0, \ell_{g,n}(A_1, \dots, A_n)\rangle. \tag{2.91}$$

This allows to rewrite the action as

$$S = \sum_{g,n\geq 0} \frac{\hbar^{2g} g_s^{-\chi_{g,n}}}{n!} \langle \Psi, \ell_{g,n-1}(\Psi^{n-1})\rangle. \tag{2.92}$$

In order to restore the gauge invariance, it would be necessary to use the BV formalism. To avoid this technical complication, it is simpler to work with the 1PI action:

$$S_{\text{1PI}} = \sum_{n\geq 0} \frac{g_s^{n-2}}{n!} \mathcal{V}_n^{\text{1PI}}(\Psi^n) = \sum_{n\geq 0} \frac{g_s^{n-2}}{n!} \langle \Psi, \ell_{n-1}^{\text{1PI}}(\Psi^{n-1})\rangle + \frac{1}{g_s^2} \mathcal{V}_0^{\text{1PI}}. \tag{2.93}$$

In terms of Riemann surfaces, the regions V_n^{1PI} are defined by decomposing the moduli spaces like for $\mathcal{V}_{g,n}$, but keeping only 1PI surfaces (which do not fall in two pieces if one cut any single propagator). Like in usual QFT, the 1PI vertices include all quantum corrections. The 1PI action shares the same properties as the classical action: the difference is that the latter has $\mathcal{V}_0 = \mathcal{V}_1 = 0$. For this reason and to simplify the notations, we will omit the index 1PI. We will also set $\mathcal{V}_0 = 0$ since it will not appear anywhere in this thesis.

To obtain the gauge invariant action, it is sufficient to release the Siegel gauge fixing condition (2.74). The only difference is the definition of $\mathcal{V}_2$ where $c_0 L_0$ and $c_0^+ L_0^+$ are replaced by the BRST charge Q [5]. The action is invariant under the gauge transformation:

$$\delta_\Lambda |\Psi\rangle = Q\Lambda + \sum_{n\geq 1} \frac{1}{n!} \ell_{n+1}(\Psi^n, \Lambda) \tag{2.94}$$

if the string products satisfy the L_∞ relations:

$$0 = \sum_{\substack{\{i_k, j_\ell\}\subset\{1,\dots,n\} \\ k+\ell=n}} \sigma(i_k, j_\ell)\, \ell_{k+1}\big(A_{i_1}, \dots, A_{j_k}, \ell_\ell(A_{j_1}, \dots, A_{j_\ell})\big), \tag{2.95}$$

for generic states $A_1, \dots, A_n \in \mathcal{H}$, where by convention $\ell_0(\cdots) = 0$, and $\sigma(i_k, j_\ell)$ is the sign obtained by the rearrangement:

$$Q, A_1, \dots, A_n \longrightarrow A_{i_1}, \dots, A_{j_k}, Q, A_{j_1}, \dots, A_{j_\ell}. \tag{2.96}$$

The first four relations are:

$$0 = Q\ell_0, \tag{2.97a}$$

$$0 = Q^2 A_1 + (-1)^{|A_1|}\,\ell_2(A_1, \ell_0), \tag{2.97b}$$

$$\begin{aligned} 0 = {} & Q\ell_2(A_1, A_2) + \ell_2(QA_1, A_2) + (-1)^{|A_1|}\,\ell_2(A_1, QA_2) \\ & + (-1)^{|A_1|+|A_2|}\,\ell_3(A_1, A_2, \ell_0), \end{aligned} \tag{2.97c}$$

$$\begin{aligned} 0 = {} & Q\ell_3(A_1, A_2, A_3) + \ell_3(QA_1, A_2, A_3) + (-1)^{|A_1|}\,\ell_3(A_1, QA_2, A_3) \\ & + (-1)^{|A_1|+|A_2|}\,\ell_3(A_1, A_2, QA_3) + (-1)^{|A_1|}\,\ell_2\big(A_1, \ell_2(A_2, A_3)\big) \\ & + (-1)^{|A_2|(1+|A_1|)}\,\ell_2\big(A_2, \ell_2(A_1, A_3)\big) + (-1)^{|A_3|(1+|A_1|+|A_2|)}\,\ell_2\big(A_3, \ell_2(A_1, A_2)\big) \\ & + (-1)^{|A_1|+|A_2|+|A_3|}\,\ell_4(A_1, A_2, A_3, \ell_0). \end{aligned} \tag{2.97d}$$

where $|A_i|$ is the Grassmann parity of the state A_i. The first identity says that $\ell_0 \in \mathcal{H}$ is fixed and must be annihilated by Q. The second means that Q would be nilpotent if $\ell_0 = 0$. The third indicates that Q would be a derivative of ℓ_2 if $\ell_0 = 0$. Finally, the fourth says that the failure of Q to be a derivative of ℓ_3 is related to the failure of the Jacobi identity plus terms depending on ℓ_0. Hence, a L_∞ algebra is a generalization of a differential Lie algebra (where only ℓ_1 and ℓ_2 non-zero).

As reviewed at the end of Section 2.2.1, open string amplitudes are naturally given in terms of worldsheet expressions which are only invariant under cyclic permutations of external states. Following the same procedure as for the closed string, one obtains string vertices and products m_n with the same properties such that the action reads:

$$S = \frac{1}{2}\,\langle \Psi, Q\Psi \rangle + \sum_{n\geq 2} \frac{1}{n+1}\,\langle \Psi, m_n(\Psi^n) \rangle. \tag{2.98}$$

This action is gauge invariant under the transformation

$$\delta_\Lambda\,|\Psi\rangle = Q\,|\Lambda\rangle + \sum_{n\geq 1}\sum_{m=0}^{n} m_{n+1}(\Psi^m, \Lambda, \Psi^{n-m}) \tag{2.99}$$

if the m_n products satisfy the A_∞ (called weak or curved if $m_0 \neq 0$) conditions. The first four relations are [19]:

$$0 = Qm_0, \tag{2.100a}$$

$$0 = Q^2 A - m_2(m_0, A) + m_2(A, m_0), \tag{2.100b}$$

$$\begin{aligned} 0 = {} & Qm_2(A_1, A_2) - m_2(QA_1, A_2) - (-1)^{|A_1|}\,m_2(A_1, QA_2) \\ & + m_3(m_0, A_1, A_2) - m_3(A_1, m_0, A_2) + m_3(A_1, A_2, m_0), \end{aligned} \tag{2.100c}$$

$$\begin{aligned} 0 = {} & Qm_3(A_1, A_2, A_3) + m_3(QA_1, A_2, A_3) + (-1)^{|A_1|}\,m_3(A_1, QA_2, A_3) \\ & + (-1)^{|A_1|+|A_2|}\,m_3(A_1, A_2, QA_3) - m_2\big(m_2(A_1, A_2), A_3\big) \\ & + m_2\big(A_1, m_2(A_2, A_3)\big) + m_4(m_0, A_1, A_2, A_3) \\ & - m_4(A_1, m_0, A_2, A_3) + m_4(A_1, A_2, m_0, A_3) - m_4(A_1, A_2, A_3, m_0), \end{aligned} \tag{2.100d}$$

for generic states $A_1, \dots, A_n \in \mathcal{H}$,. Since all arguments of the string products in the action are identical, it is possible to anti-symmetrize (the string field is Grassmann-odd) the action explicitly. In the same way that a Lie algebra can be constructed from a matrix algebra, a L_∞ algebra can be built from a A_∞ algebra [175]. For example, a graded-symmetric 2-product ℓ_2 is induced from the m_2 product:

$$\ell_2(A_1, A_2) := m_2(A_1, A_2) \pm (-1)^{|A_1||A_2|} m_2(A_2, A_1). \tag{2.101}$$

Which sign to choose to use ℓ_2 in the action? This depends on the parity of the string field: since the open string field is odd, the product does not vanish if one takes the minus sign. Indeed, for $A_1 = A_2 = A$, one has:

$$\ell_2(A^2) = \pm(-1)^{|A|}\, \ell_2(A^2). \tag{2.102}$$

This means that open string products are graded-symmetric for odd arguments, while for the closed string it was for even arguments. To avoid this difference between the two theories, a new parity called the degree is often defined (as the Grassmann parity plus one, modulo two) such that the open string field has degree 0. If m_2 is a non-commutative product (like the star product from Witten's theory [6]), then ℓ_2 is a graded commutator.[6] This is the same thing as Yang–Mills theory: the gauge fields can be viewed as matrices or as elements of the Lie algebra. In the first case, one can write $\operatorname{tr} A^a A^b A^c$, which is only cyclically symmetric since the matrix product is defined. In the second case, one must write $\operatorname{tr} A^a [A^b, A^c]$ which is completely symmetric.

2.5 Momentum space string field theory

In this section, we describe the general properties of SFT actions in the momentum space. This allows to make SFT more intuitive, but also to use standard QFT methods to prove various properties of string theory (Chapter 4). We explain how the Wick rotation is generalized for theories with vertices diverging at infinite real energies (Lorentzian signature). I conclude this section by describing the properties which can be proven in this way.

2.5.1 General form

Since the explicit expressions of the string vertices are not known, it is not possible to write explicitly the SFT action. However, the general properties of the vertices are known: then, one can write a general QFT which contains SFT as a subcase. This is sufficient to already extract a lot of informations. The other advantage is that the QFT language is more familiar and intuitive in many situations. Hence, one can use this general form to built intuition before translating the results in a more stringy language. In a nutshell, SFT is a QFT (see below for more details):

- with an infinite number of fields (of all spins);
- with an infinite number of interactions;
- with non-local interactions $\propto \mathrm{e}^{-\# k^2}$;
- which reproduces the worldsheet amplitudes (whenever the latter are well-defined).

The non-locality of the interactions is the most salient property of SFT, beyond the infinite number of fields. This has a number of consequences:

- the Wick rotation is ill-defined;
- the position representation cannot be used, nor any property relying on it (micro-causality, largest time equation...);
- standard assumptions from local QFT (in particular, from the constructive S-matrix program, such as micro-causality) break down.

[6] This explains why it is often denoted as $[\cdot,\cdot]$. Then, it is logical to use the same bracket notation for higher-order products: $\ell_n(A_1, \dots, A_n) = [A_1, \dots, A_n]$.

Together, these points imply that the usual arguments from QFTs must be improved. This has been an active topic in the recent years and the results will be summarized in Section 2.5.2.

We expand the string field in Fourier space using a basis $\{\phi_\alpha(k)\}$ as:

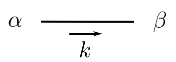
$$|\Psi\rangle = \sum_j \int \frac{\mathrm{d}^D k}{(2\pi)^D}\, \psi_\alpha(k)\, |\phi_\alpha(k)\rangle\,, \tag{2.103}$$

where k is the D-dimensional momentum and α the discrete indices (Lorentz indices, group representation, KK modes...) of the spacetime fields $\psi_\alpha(k)$. The action in momentum space takes the form (in Lorentzian signature):

$$\begin{aligned} S = &- \int \mathrm{d}^D k\, \psi_\alpha(k) K_{\alpha\beta}(k) \psi_\beta(-k) \\ &- \sum_{n\geq 0} \int \mathrm{d}^D k_1 \cdots \mathrm{d}^D k_n\, V^{(n)}_{\alpha_1\cdots\alpha_n}(k_1,\ldots,k_n)\, \psi_{\alpha_1}(k_1)\cdots\psi_{\alpha_n}(k_n). \end{aligned} \tag{2.104}$$

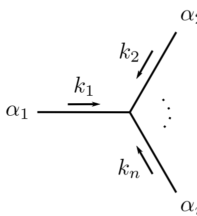
The kinetic matrix $K_{\alpha\beta}$ is usually quadratic in the momentum.

From the action, we can write the Feynman rules (for the path integral weight $\mathrm{e}^{\mathrm{i}S}$ and S-matrix $S = 1 + \mathrm{i}T$). The propagator reads:



$$= K_{\alpha\beta}(k)^{-1} = \frac{-\mathrm{i}\, M_{\alpha\beta}}{k^2 + m_\alpha^2}\, Q_\alpha(k), \tag{2.105}$$

where $M_{\alpha\beta}$ is mixing matrix for states of equal mass and Q_α a polynomial in k (there is no sum over α). The interactions are obtained by plugging the basis states $\{\phi_\alpha\}$ inside the vertices $\mathcal{V}_n$ (2.89):



$$\begin{aligned} &= \mathrm{i}\, V^{(n)}_{\alpha_1\cdots\alpha_n}(k_1,\ldots,k_n) := \mathrm{i}\, \mathcal{V}_n\big(\phi_{\alpha_1}(k_1),\ldots,\phi_{\alpha_n}(k_n)\big) \\ &= \mathrm{i} \int \mathrm{d}t\, \mathrm{e}^{-g_{ij}^{\{\alpha_k\}}(t)\, k_i\cdot k_j - \lambda \sum\limits_\alpha m_\alpha^2}\, P_{\alpha_1,\ldots,\alpha_n}(k_1,\ldots,k_n;t), \end{aligned} \tag{2.106}$$

where t denotes collectively the moduli parameters, $P_{\{\alpha_i\}}$ is a polynomial in k, g_{ij} is a positive-definite matrix, $\lambda > 0$ is a number. There is an implicit sum over the momentum indices.

The terms quadratic in the momenta inside the exponential arise from two sources:

- The correlation functions of the vertex operators $\langle \prod_i \mathrm{e}^{\mathrm{i}k_i\cdot X(z_i)}\rangle$ is proportional to $\mathrm{e}^{-k_i\cdot k_j G(z_i,z_j)}$, where G is the Green function. Additional factors like ∂X contribute to the polynomial $P_{\alpha_1,\ldots,\alpha_n}$.
- It is possible to add stubs to the vertices. The effect is to multiply each leg by a factor $\mathrm{e}^{-\lambda(k_i^2+m_i^2)}$ with $\lambda > 0$ (we take λ to be the same for all vertices for simplicity). The first term of the exponential contributes to the diagonal of the matrix g_{ij}. By taking λ sufficiently large, one can enforce that all eigenvalues are positive.

Finally, the exponential term with the masses m_α^2 ensures that the sum over all intermediate states converge despite an infinite number of states. Indeed, the number of states of mass

m_α grows as $\mathrm{e}^{c m_\alpha}$, which is dominated by $\mathrm{e}^{-\lambda m_\alpha^2}$ for sufficiently large λ. Hence, the addition of stubs make explicit the absence of divergences in SFT.[7]

The vertices have no singularity for $k_i \in \mathbb{C}$ finite. As the energy becomes infinite $|k_i^0| \to \infty$, they behave as:

$$\lim_{k^0 \to \pm \mathrm{i}\infty} V^{(n)} = 0, \qquad \lim_{k^0 \to \pm\infty} V^{(n)} = \infty. \tag{2.107}$$

The first condition is responsible for the soft UV behavior of string theory in Euclidean signature. The second prevents from performing the Wick rotation (indeed, the pole at infinity implies that the arcs closing the contour contribute).

The g-loop n-point amputated Green functions are sums of Feynman diagrams, each of the form:

$$\begin{aligned} F_{g,n}(p_1, \ldots, p_n) \sim \int \mathrm{d}T \prod_s \mathrm{d}^D \ell_s \, \mathrm{e}^{-G_{rs}(T)\, \ell_r \cdot \ell_s - 2H_{ri}(T)\, \ell_r \cdot p_i - F_{ij}(T)\, p_i \cdot p_j} \\ \times \prod_a \frac{1}{k_a^2 + m_a^2} \, \mathcal{P}(p_i, \ell_r; T), \end{aligned} \tag{2.108}$$

where $\{p_i\}$ are the external momenta, $\{\ell_r\}$ the loop momenta and $\{k_i\}$ the internal momenta, with the latter given by a linear combination of the others. Moreover, T denotes the dependence in the moduli parameters of all the internal vertices, and $\mathcal{P}$ is a polynomial in (p_i, ℓ_r). The matrix G_{rs} is positive definite, which implies that:

- integrations over spatial loop momenta $\boldsymbol{\ell}_r$ converge;
- integrations over loop energies ℓ_r^0 diverge.

As a consequence, the Feynman diagrams in Lorentzian signature are ill-defined: we will explain in the next section how to fix this problem.

2.5.2 Generalized Wick rotation

We have seen that loop integrals in Lorentzian signature are divergent because of the large energy behaviour of the interactions. But, this is not different from the usual QFT, where the loop integrals are also ill-defined in Lorentzian signature. Indeed, poles of the propagators sit on the real axis and also give divergent loop integrals (note that the same problem arise also here). In that case, the strategy is to define the Feynman diagrams in Euclidean space and to perform a Wick rotation: the latter matches the expressions in Lorentzian signature up to the iε-prescription. The goal of the latter is to move slightly the poles away from the real axis.

As an example, consider a scalar field of mass m with a quartic interaction. The 1-loop 4-point Feynman diagram is given in Figure 2.1. The external momenta are p_i, $i = 1, \ldots, 4$. There are one loop momentum ℓ and two internal momenta $k_1 = \ell$ and $k_2 = p - \ell$, where $p = p_1 + p_2$. The poles in the loop energy ℓ^0 are located at:

$$p_\pm = \pm\sqrt{\boldsymbol{\ell}^2 + m^2}, \qquad q_\pm = p^0 \pm \sqrt{(\boldsymbol{p} - \boldsymbol{\ell})^2 + m^2}. \tag{2.109}$$

The graph is first defined in Euclidean signature, where the external and loop energies are pure imaginary, $p_i^0, \ell^0 \in \mathrm{i}\mathbb{R}$. The poles are shown in Figure 2.2. Then, the external momenta are analytically continued to real values, $p_i^0 \in \mathbb{R}$. At the same time, the integration contour is also analytically continued thanks to the Wick rotation (Figure 2.3). The contour is closed with arcs, but they don't contribute since there is no poles in the upper-right and lower-left

[7] Remember that λ is not a physical parameter and disappears on-shell. This means that the cancellation of the divergences is independent of λ and must always happen on-shell.

quadrants, and no poles at infinity. However, one cannot continue the contour such that $\ell^0 \in \mathbb{R}$ because of the poles on the real axis. The Wick rotation is possible for ℓ^0 in the upper-right quadrant, $\operatorname{Re} \ell^0 \geq 0, \operatorname{Im} \ell^0 > 0$, which leads to the i$\varepsilon$-prescription $\ell^0 \in \mathbb{R} + \mathrm{i}\varepsilon$.

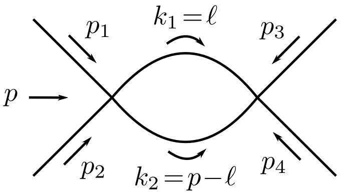


Figure 2.1 – 1-loop 4-point function for a scalar field theory.

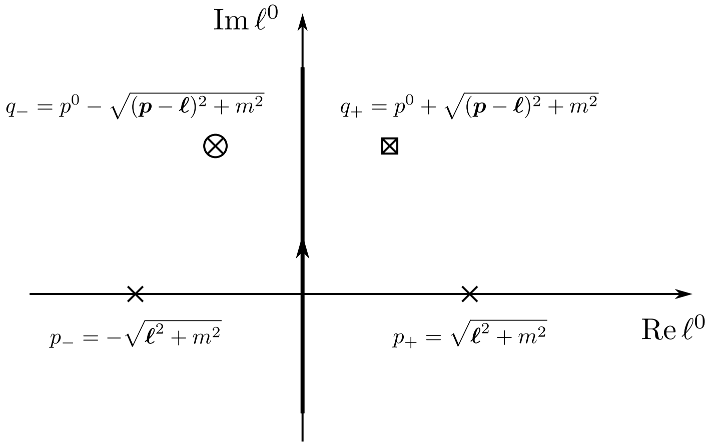


Figure 2.2 – Integration contour for external Euclidean momenta.

Since the Feynman diagram (2.108) is not defined in Lorentzian signature because of the poles at $\ell^0_r \to \pm\infty$, it is also necessary to start with Euclidean momenta. However, the same behaviour at infinity prevents from using the Wick rotation since the contribution from the arcs does not vanish. It is then necessary to find another prescription for defining the Feynman diagrams in SFT starting from the Euclidean Green functions. This is given by the following *generalized Wick rotation* (Pius–Sen [46]):

1. Define the Green functions for Euclidean internal and external momenta.
2. Perform an analytic continuation of the external energies and of the integration contour such that:
    - keep poles on the same side;
    - keep the contour ends fixed at $\pm\mathrm{i}\infty$.

One can show [46] that the Green functions are analytic in the upper-right quadrant $\operatorname{Im} p_a^0 > 0, \operatorname{Re} p_a^0 \geq 0$, for $\boldsymbol{p}_a \in \mathbb{R}$, p_a^0. Moreover, the result is independent of the contour chosen as long as it satisfies the conditions described above. In fact, this generalized Wick rotation is valid even for normal QFT, which raises interesting questions. For example, it seems that the internal and external sets of states have no intersection, which can be puzzling when trying to interpret the Cutkosky rules. Nonetheless, everything works as expected.

The fact that the amplitude is analytic only when the imaginary parts of the momenta are not zero, $\operatorname{Im} p_a^0 > 0$, is equivalent to the usual iε-prescription for QFT. Moreover, it has

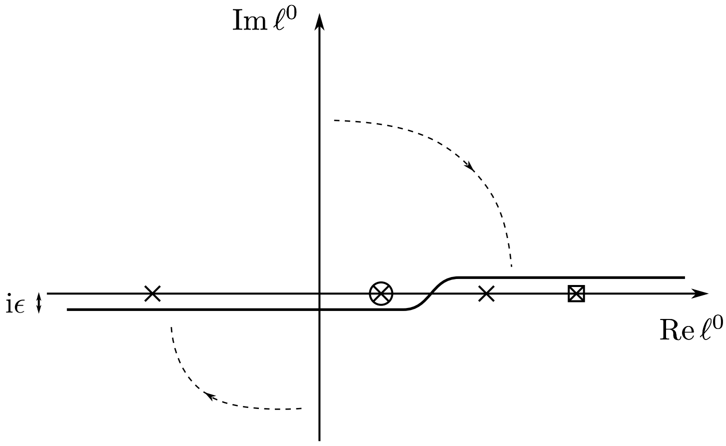


Figure 2.3 – Integration contour for external Lorentzian momenta after Wick rotation (regular vertices).

been shown [49] to be equivalent to the moduli space iε-prescription from [176]. Then, it has also been used to prove several important properties of string theory shared by local QFTs: Cutkosky rules [46, 48], unitarity [47, 177], analyticity in a subset of the primitive domain and crossing symmetry [151, 178] (Chapter 4). Finally, general soft theorems for string theory (and, in fact, any theory of quantum gravity) have been proven in [50–54]. All together, these properties establish string theory as a very strong candidate for a consistent theory of everything.

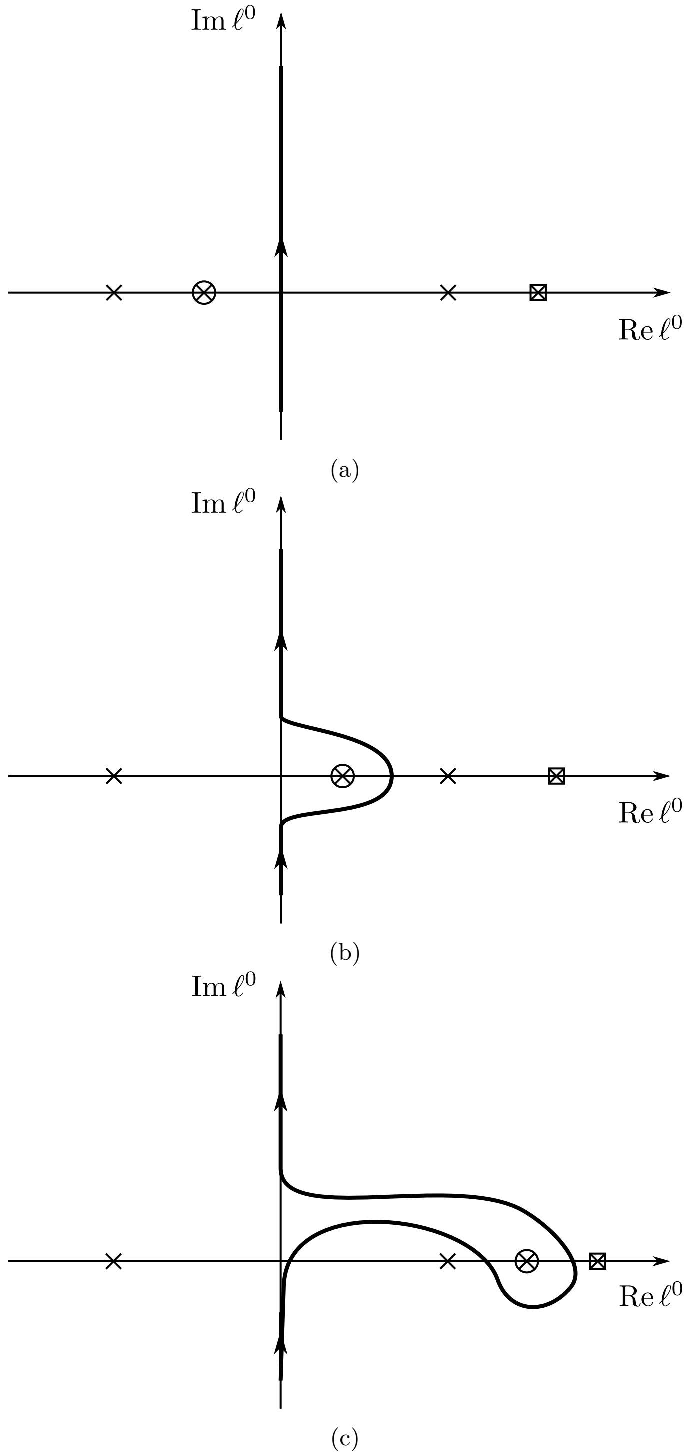


Figure 2.4 – Integration contour after analytic continuation to external Lorentzian momenta. Depending on the values of the external momenta, different cases can happen.

Chapter 3

Two-point amplitude

In Section 2.2.1, we have restricted the computation of tree-level amplitudes to the case with $n \geq 3$ external states, leaving open the case of the 2-point amplitude. It has long been believed that it vanishes following two arguments: there are not sufficiently many vertex operators 1) to fix completely the $\mathrm{SL}(2, \mathbb{C})$ invariance or 2) to saturate the number of c-ghost zero-modes. This would mean that the worldsheet path integral computes truncated Green functions and not amplitudes. However, both arguments are incorrect and the 2-point function does not vanish. Instead, it matches the universal form required by general QFT properties.

We first provide a refresher on amplitudes and Green functions in QFT. In particular, we show why the on-shell 2-point truncated Green function vanishes and why the amplitude has a universal form. Then, we review the incorrect arguments from string theory, before performing a simple computation of the amplitude by regularizing the divergences and proving that they cancel. Finally, we present a new gauge fixing procedure such that no divergence even appears. This chapter is based on the paper [150] but focuses on the closed string.[1] An alternative approach has been presented in [179].

3.1 Two-point amplitude and Green functions

The S-matrix can be split as:

$$S = 1 + \mathrm{i}T, \tag{3.1}$$

where 1 denotes the contribution where all particles propagate without interaction. The connected components of S and T are denoted by S^c and T^c. The n-point connected scattering amplitudes $A_n := T_n$ for $n \geq 3$ are computed from the Green functions G_n through the LSZ prescription (amputation of the external propagators):

$$A_n(k_1, \ldots, k_n) := G_n(k_1, \ldots, k_n) \prod_{i=1}^{n} (k_i^2 + m_i^2). \tag{3.2}$$

The path integral computes the Green functions G_n; perturbatively, they are obtained from the Feynman rules. They include a D-dimensional delta function

$$G_n(k_1, \ldots, k_n) \propto \delta^{(D)}(k_1 + \cdots + k_n). \tag{3.3}$$

[1] The paper focuses mostly on the open string [150], details on the closed string were discussed privately with Nathan Berkovits, Juan Maldacena and Dimitri Skliros.

The 2-point amputated Green function T_2 computed from the LSZ prescription vanishes on-shell. For example, considering a scalar field at tree-level, one finds:

$$T_2(k, k') := G_2(k, k')\,(k^2 + m^2)^2 \sim (k^2 + m^2)\,\delta^{(D)}(k + k') \xrightarrow[k^2 \to -m^2]{} 0 \tag{3.4}$$

since

$$G_2(k, k') = \frac{\delta^{(D)}(k + k')}{k^2 + m^2}. \tag{3.5}$$

Hence, $T_2 = 0$ and the S-matrix (3.1) reduces to the identity component $A_2 = S_2^c = 1_2$ (which is a connected process). There are several way to understand this last relation:

1. The recursive definition of the connected S-matrix S^c from the cluster decomposition principle requires a non-vanishing 2-point amplitude [180, sec. 5.1.5, 181, sec. 4.3, 182, sec. 6.1, 183, sec. 2.2].

2. The 2-point amplitude corresponds to the normalization of the 1-particle states (overlap of a particle state with itself, which is non-trivial) [184, eq. 4.1.4, 185, chap. 5].

3. A single particle in the far past propagating to the far future without interacting is a connected and physical process [182, p. 133].

4. Unitarity of the 2-point amplitude is compatible only with $A_2 = 0$ or $A_2 = 1_2$ [150].

These points indicate that the 2-point amplitude is proportional to the identity in the momentum representation [180, p. 212, 184, eq. 4.3.3 and 4.1.5]:

$$A_2(k, k') = 2k^0\,(2\pi)^{D-1}\delta^{(D-1)}(\boldsymbol{k} - \boldsymbol{k}'). \tag{3.6}$$

The absence of interactions implies that the spatial momentum does not change (the on-shell condition implies that energy is also conserved). This relation is consistent with the commutation relation of the operators with the Lorentz invariant measure[2]

$$[a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')] = 2k^0\,(2\pi)^{D-1}\delta^{(D-1)}(\boldsymbol{k} - \boldsymbol{k}'). \tag{3.7}$$

That this holds for all particles at all loops can be proven using the Källen–Lehman representation [180, p. 212].

On the other hand, the identity part in (3.1) is absent for $n \geq 3$ for connected amplitudes: $S_n^c = T_n^c$ for $n \geq 3$. This shows that the Feynman rules and the LSZ prescription compute only the interacting part T of the on-shell scattering amplitudes. The reason is that the derivation of the LSZ formula assumes that the incoming and outgoing states have no overlap, which is not the case for the 2-point function. A complete derivation of the S-matrix from the path integral is more involved [180, sec. 5.1.5, 186, sec. 6.7, 187] (see also [188]). The main idea is to consider a superposition of momentum states (here, in the holomorphic representation [186, sec. 5.1, 6.4])

$$\phi(\alpha) = \int \mathrm{d}^{D-1}\boldsymbol{k}\,\alpha(\boldsymbol{k})^* a^\dagger(\boldsymbol{k}). \tag{3.8}$$

They contribute a quadratic piece to the connected S-matrix and, setting them to delta functions, one recovers the above result.

[2] If the modes are defined as $\tilde{a}(k) = a(k)/\sqrt{2k^0}$ such that $[\tilde{a}(\boldsymbol{k}), \tilde{a}^\dagger(\boldsymbol{k}')] = (2\pi)^{D-1}\delta^{(D-1)}(\boldsymbol{k} - \boldsymbol{k}')$, then one finds $\tilde{A}_2(k, k') = (2\pi)^{D-1}\delta^{(D-1)}(\boldsymbol{k} - \boldsymbol{k}')$.

3.2 Statement of the problem

The main question is whether the worldsheet path integral (2.38) computes truncated Green functions or amplitudes. In this section, we review the arguments which have lead to the conclusion that it computes the former. We will briefly explain why it is necessary to be more careful, providing more rigorous computations in the other sections of this chapter.

For simplicity, we consider a flat background metric $\hat{g} = \delta$ and an orthonormal basis of CKV. The two weight-$(1,1)$ matter vertex operators are denoted as $V_k(z,\bar{z})$ and $V_{k'}(z',\bar{z}')$ such that the 2-point correlation function on the sphere reads (see Appendix A.1 for more details):

$$\langle V_k(z,\bar{z}) V_{k'}(z',\bar{z}') \rangle_{S^2} = \frac{\mathrm{i}\,(2\pi)^D \delta^{(D)}(k+k')}{|z-z'|^4}. \tag{3.9}$$

The numerator comes from the zero-modes $\mathrm{e}^{\mathrm{i}(k+k')\cdot x}$ for a target spacetime with a Lorentzian signature [189, p. 866, 172] (required to make use of the on-shell condition).

Gauge fixing only the metric but not the CKV in the tree-level amplitude (2.38) for $n = 2$ gives:

$$A_{0,2}(k,k') = \frac{C_2}{\operatorname{Vol}\mathcal{K}_{0,0}} \int \mathrm{d}^2 z\, \mathrm{d}^2 z' \; \langle V_k(z,\bar{z}) V_{k'}(z',\bar{z}') \rangle_{S^2}\,, \tag{3.10}$$

where $\mathcal{K}_{0,n}$ is the CKV group of the sphere with n punctures. With have reintroduced the normalization of the amplitude $C_2 = 8\pi\alpha'^{-1}$ for $g_s = 1$ [172, 190]. Since there are two insertions, the CKV group can be partially gauge fixed by fixing the positions of the two punctures to $z = 0$ and $z' = \infty$. In this case, the amplitude (3.10) becomes:

$$A_{0,2}(k,k') = \frac{C_2}{\operatorname{Vol}\mathcal{K}_{0,2}} \; \langle V_k(\infty,\infty) V_{k'}(0,0) \rangle_{S^2}\,, \tag{3.11}$$

where $\mathcal{K}_{0,2} = \mathbb{R}_+^* \times \mathrm{U}(1)$ is the CKV group of the 2-punctured sphere – containing dilatations and rotations.[3] Since the volume of this group is infinite $\operatorname{Vol}\mathcal{K}_{0,2} = \infty$, it looks like $A_{0,2} = 0$. However, this forgets that the 2-point correlation function (3.9) contains a D-dimensional delta function. The on-shell condition implies that the conservation of the momentum $k + k' = 0$ is automatic for one component, such that the numerator in (3.11) contains a divergent factor $\delta(0)$:

$$A_{0,2}(k,k') = (2\pi)^{D-1} \delta^{(D-1)}(\boldsymbol{k} + \boldsymbol{k}')\, \frac{C_2\, 2\pi\mathrm{i}\, \delta(0)}{\operatorname{Vol}\mathcal{K}_{0,2}}. \tag{3.12}$$

Hence, (3.11) is of the form $A_{0,2} = \infty/\infty$ and one should be careful when evaluating it.

The second argument relies on a loophole in the understanding of the gauge fixed amplitudes (2.55). The result (2.48) is often summarized by saying that one can go from the amplitude without ghosts to (2.48) by replacing K_g^c integrated vertices $\int V$ by unintegrated vertices $c\bar{c}V$ in order to saturate the ghost zero-modes and to obtain a non-zero result. For $g = 0$, this requires 3 unintegrated vertices. But, since there are only two operators in (3.10), this is impossible and the result must be zero. However, this is also incorrect because it is always possible to insert 6 c zero-modes [**Erbin:2020:StringFieldTheory**]. Indeed, they are part of how the path integral measure is defined and do not care of the matter operators. The question is whether they can be attached to vertex operators (for aesthetic reasons or more pragmatically to get natural states of the BRST cohomology). We will discuss below how ghosts can be introduced properly in the 2-point amplitude.

[3] The subgroup and the associated measure depend on the locations of the two punctures.

3.3 Simple approach

In this section, we compute the 2-point amplitude from (3.11):

$$A_{0,2}(k,k') = \frac{C_2}{\operatorname{Vol}\mathcal{K}_{0,2}}\,\langle V_k(\infty,\infty)V_{k'}(0,0)\rangle_{S^2}\,. \tag{3.13}$$

The volume of $\mathcal{K}_{0,2}$ reads (by writing a measure invariant under rotations and dilatations, but not translations nor special conformal transformations) [191, 192]:

$$\operatorname{Vol}\mathcal{K}_{0,2} = \int \frac{\mathrm{d}^2 z}{|z|^2} = 2\int_0^{2\pi}\mathrm{d}\sigma\int_0^\infty \frac{\mathrm{d}r}{r}, \tag{3.14}$$

where the second equality follows from the change of variables $z = r\,\mathrm{e}^{\mathrm{i}\sigma}$. Since the volume is infinite, it must be regularized. A first possibility is to cut-off a small circle of radius ϵ around $r = 0$ and $r = \infty$ (corresponding to removing the two punctures at $z = 0, \infty$). A second possibility consists in performing the change of variables $r = \mathrm{e}^\tau$ and to add an imaginary exponential:

$$\operatorname{Vol}\mathcal{K}_{0,2} = 4\pi\int_0^\infty \frac{\mathrm{d}r}{r} = 4\pi\int_{-\infty}^\infty \mathrm{d}\tau = 4\pi\lim_{\varepsilon\to 0}\int_{-\infty}^\infty \mathrm{d}\tau\,\mathrm{e}^{\mathrm{i}\varepsilon\tau} = 4\pi\times 2\pi\lim_{\varepsilon\to 0}\delta(\varepsilon), \tag{3.15}$$

such that the regularized volume reads

$$\operatorname{Vol}_\varepsilon\mathcal{K}_{0,2} = 8\pi^2\,\delta(\varepsilon). \tag{3.16}$$

In fact, τ can be interpreted as the Euclidean worldsheet time on the cylinder since r corresponds to the radial direction of the complex plane.

Since the worldsheet is an embedding into the target spacetime, both must have the same signature. As a consequence, for the worldsheet to be also Lorentzian, the formula (3.15) must be analytically continued as $\varepsilon = -\mathrm{i}E$ and $\tau = \mathrm{i}t$ such that

$$\operatorname{Vol}_{M,E}\mathcal{K}_{0,2} = 8\pi^2\mathrm{i}\,\delta(E), \tag{3.17}$$

where the subscript M reminds that one considers the Lorentzian signature. Inserting this expression in (3.12) and taking the limit $E \to 0$, it looks like the two $\delta(0)$ will cancel. However, we need to be careful about the dimensions. Indeed, the worldsheet time τ and energy E are dimensionless, while the spacetime time and energy are not. Thus, it is not quite correct to cancel directly both $\delta(0)$ since they don't have the same dimensions. In order to find the correct relation between the integrals in (3.15) and of the zero-mode in (3.9), we can look at the mode expansion for the scalar field (removing the useless oscillators):

$$X^0(z,\bar z) = x^0 + \frac{\mathrm{i}}{2}\,\alpha' k^0\,\ln|z|^2 = x^0 + \mathrm{i}\alpha' k^0\tau, \tag{3.18}$$

where the second equality follows by setting $z = \mathrm{e}^\tau$. After analytic continuation $k^0 = -\mathrm{i}k^0_M$, $X^0 = \mathrm{i}X^0_M$, $x^0 = \mathrm{i}x^0_M$ and $\tau = \mathrm{i}t$, we find [193, p. 186]:

$$X^0_M = x^0_M + \alpha' k^0_M t. \tag{3.19}$$

This indicates that the measure of the worldsheet time in (3.17) must be rescaled by $1/\alpha' k^0_M$ such that:

$$\operatorname{Vol}_M\mathcal{K}_{0,2} \longrightarrow \frac{8\pi^2\,\delta(0)}{\alpha' k^0_M} = \frac{C_2\,2\pi\,\delta(0)}{2k^0_M}. \tag{3.20}$$

This is equivalent to rescale E by $\alpha' k^0$ and to use $\delta(ax) = a^{-1}\delta(x)$.

Ultimately, the 2-point amplitude becomes (removing the subscript on k^0):

$$A_{0,2}(k,k') = 2k^0(2\pi)^{D-1}\delta^{(D-1)}(\boldsymbol{k}+\boldsymbol{k}') \tag{3.21}$$

and matches the QFT formula (3.6). We see that taking into account the scale of the coordinates is important to reproduce this result.

There are different ways to rewrite the 2-point amplitude in terms of ghosts. A first approach is to insert $1 = \int \mathrm{d}^2 z\, \delta^{(2)}(z)$ inside (3.10) to mimic the presence of a third operator. This is equivalent to use the identity

$$\langle 0|\, c_{-1}\bar{c}_{-1}c_0\bar{c}_0 c_1\bar{c}_1\, |0\rangle = 1 \tag{3.22}$$

inside (3.11), leading to:

$$A_{0,2}(k,k') = \frac{C_2}{\operatorname{Vol}\mathcal{K}_{0,2}}\, \langle \mathscr{V}_k(\infty,\infty)c_0\bar{c}_0\, \mathscr{V}_{k'}(0,0)\rangle_{S^2}\,, \tag{3.23}$$

where $\mathscr{V}_k(z,\bar{z}) = c\bar{c}V_k(z,\bar{z})$. This shows that (3.6) can also be recovered using the correct insertions of ghosts. The presence of $c_0\bar{c}_0$ can be expected from string field theory since they appear in the kinetic term (2.75). The disadvantage of this formula is that it still contains the infinite volume of the dilatation group.

The regularization of the volume from this section presents some ambiguities because the cut-off can always be rescaled. However, this ambiguity can be fixed from unitarity of the scattering amplitudes. A more general version of the Faddeev–Popov gauge fixing free of ambiguities and which avoids dealing altogether with infinities is introduced in the next section.

3.4 Improved gauge fixing

Under an infinitesimal $\mathrm{SL}(2,\mathbb{C})$ transformation, the coordinate and field transform as:

$$\delta z = \beta + \alpha z + \gamma z^2, \qquad \delta X(z,\bar{z}) = \delta z\, \partial X(z,\bar{z}) + \delta\bar{z}\, \bar{\partial} X(z,\bar{z}). \tag{3.24}$$

For $n \geq 3$, the positions of three vertices are fixed such that the gauge fixing conditions reads:

$$f_i(z_i^0) = z_i - z_i^0, \qquad \bar{f}_i(\bar{z}_i^0) = \bar{z}_i - \bar{z}_i^0. \tag{3.25}$$

However, any other objects which transforms under the CKV group can be used for gauge fixing. We have seen that the divergent part of the CKV volume is related to the worldsheet time, which in turns is related to the target time. Thus, for $n = 2$ where fixing the vertex positions provide only 4 conditions, it makes sense to use $X^0(z,\bar{z})$ to write the missing gauge fixing condition.

First, we continue to fix the positions of the two vertices:

$$\begin{aligned} f_1 &= z - z^0, & f_2 &= z', \\ \bar{f}_1 &= \bar{z} - \bar{z}^0, & \bar{f}_2 &= \bar{z}', \end{aligned} \tag{3.26}$$

where the limit $z^0 \to \infty$ is understood in all expressions.[4] Only the dilatation subgroup of $\mathcal{K}_{0,2}$ has an infinite volume, it is really necessary to add a single gauge fixing condition. However, for avoiding supplementary factors, we will add a sixth condition. A natural choice is to enforce the level-matching condition $L_0^- = 0$ since it removes global rotations.

[4] Putting z_1 and z_2 at other positions would lead to more complicated expressions. For example, the ghost insertions below would not be $c_0^\pm$ but more general combinations.

To write the additional conditions, it is better to introduce coordinates on the cylinder as $z = r\,\mathrm{e}^{\mathrm{i}\sigma}$, where $r = \mathrm{e}^{\tau}$. In this coordinate system, an infinitesimal rotation of angle θ and dilatation by λ acts as

$$r \to \lambda r, \qquad \delta\sigma = \theta. \tag{3.27}$$

These parameters are related to α in (3.24) as $\alpha = \lambda + \mathrm{i}\theta$, and the normalization of the integration measure gives:

$$\mathrm{d}^2\alpha = 2\,\mathrm{d}\lambda\mathrm{d}\theta. \tag{3.28}$$

The corresponding variations of the field are (with θ infinitesimal):

$$\delta X^\mu = \lambda\, r\partial_r X^\mu + \theta\, \partial_\sigma X^\mu. \tag{3.29}$$

The level-matching condition amounts to fix the origin of the angular coordinate. Since there is no more coordinate available in the 2-point function, we introduce a new coordinate (r', σ') by inserting the identity

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma' \tag{3.30}$$

in the path integral. Then, rotations are gauge fixed with the condition:

$$f_3 = \sigma'. \tag{3.31}$$

Finally, it remains to put a condition on X^0. It must be invariant under rotation, which leads to integrate X^0 over the angular direction for the last condition:

$$f_4 = \frac{1}{2\pi}\int \mathrm{d}\sigma\, X^0(r,\sigma). \tag{3.32}$$

We can now perform the Faddeev–Popov trick. The conditions f_1, f_2, $\bar{f}_1$ and $\bar{f}_2$ give:

$$1 = \Delta_1(z^0)\int \mathrm{d}^2\beta \mathrm{d}^2\gamma\, \delta^{(2)}(z - z^0)\delta^{(2)}(z), \qquad \Delta_1(z^0) := |z^0|^4. \tag{3.33}$$

For f_5, we get:

$$1 = \int \mathrm{d}\theta\, \delta(\sigma'), \tag{3.34}$$

and, finally, for f_6:

$$1 = \Delta_2(X^0)\int \mathrm{d}\lambda\, \delta^{(2)}(z - z^0), \qquad \Delta_2(X^0) := \frac{1}{2\pi}\int \mathrm{d}\sigma\, r\partial_r X^0. \tag{3.35}$$

The determinant Δ_2 can be simplified using the mode expansion of X^0 on the cylinder. The integral over σ removes all non-zero modes since the contain an oscillatory $\mathrm{e}^{\mathrm{i}n\sigma}$, which leaves only (3.18). The derivative with respect to r kills the first term, which leaves:

$$\Delta_2(X^0) := \alpha' p^0, \tag{3.36}$$

where p^0 is the momentum operator (Lorentzian signature).

Inserting these identities in the 2-point amplitude (3.10) gives:

$$A_{0,2}(k,k') = \frac{C_2}{4\pi}\,\Delta_1(z^0)\, \left\langle \delta(x^0)\Delta_2(X^0) V_k(z^0,\bar{z}^0) V_{k'}(0,0)\right\rangle_{S^2}, \tag{3.37}$$

after canceling the integration over the gauge parameters with the volume of $\mathcal{K}_{0,0}$ (note the factor of 2 in changing the measure for rotations and dilatations) and integrating over z, z' and ϕ' to remove the delta functions. We have put Δ_2 inside the correlation function since it contains X^0. Taking the limit $z^0 \to \infty$, performing the OPE of Δ_2 with the vertex operators and evaluating the 2-point function give:

$$A_{0,2}(k, k') = 2k^0 (2\pi)^{D-1} \delta^{(D-1)}(\boldsymbol{k} + \boldsymbol{k}'). \tag{3.38}$$

Note that p^0 acts only on one operator as can be seen by mapping the contour integral to the complex plane.

The next step is to rewrite the 2-point amplitude in terms of ghosts. The conditions f_1, f_2, $\bar{f}_1$ and $\bar{f}_2$ lead to the usual $c\bar{c}(z^0, z^0) = c_{-1}\bar{c}_{-1}$ (in the limit $z^0 \to \infty$) and $c\bar{c}(0,0) = c_1\bar{c}_1$ insertions. The other contributions are found by looking at the form of the BRST transformations: the ghost insertion is given by the variation of the gauge fixing condition under a BRST transformation (without the factor of i) and evaluated at the gauge fixing condition. The BRST transformation of $\mathcal{O}$ is obtained from the variation $\delta_z \mathcal{O}(z, \bar{z})$ as

$$\delta_B \mathcal{O}(z, \bar{z}) = c(z)\delta_z \mathcal{O}(z) + \bar{c}(\bar{z})\delta_{\bar{z}} \mathcal{O}(z, \bar{z}) \tag{3.39}$$

(setting all parameters to 1). For example, we have $\delta_B f_1 = c(z)$, which comes with $\delta(z - z^0)$ such that the insertion is $c(z^0)$. A global rotation $\delta\sigma = \theta$ corresponds to $\delta_z \sigma = 1$ and $\delta_{\bar{z}}\sigma = -1$ (the sign comes from conjugating $\alpha = \lambda + \mathrm{i}\theta$) such that the corresponding BRST transformation reads:

$$\delta_B f_3 = \delta_B \sigma' = c(\sigma') - \bar{c}(\sigma'). \tag{3.40}$$

Evaluating the ghosts at $\sigma' = 0$ picks the zero-mode (because c is written in the coordinates on the cylinder), giving $2c_0^-$. Following the same procedure for a dilatation $\delta r = \lambda r$ gives $\delta_z r = \delta_{\bar{z}} r = r$ such that

$$\delta f_4 = \frac{1}{2\pi} \int \mathrm{d}\sigma \, \big(c(\sigma) + \bar{c}(\sigma)\big) r \partial_r X^0 = \alpha' p^0 \, c_0^+. \tag{3.41}$$

Plugging everything in the amplitude gives:

$$A_{0,2}(k, k') = 4k^0 \, \frac{\alpha' C_2}{8\pi} \, \big\langle \delta(x^0) \mathscr{V}_k(z^0, \bar{z}^0) c_0^- c_0^+ \mathscr{V}_{k'}(0,0) \big\rangle_{S^2} . \tag{3.42}$$

Since $2c_0^- c_0^+ = c_0\bar{c}_0$, this expression correctly matches (3.37) after using the normalization $\langle c_{-1}\bar{c}_{-1}c_0\bar{c}_0c_1\bar{c}_1 \rangle = 1$. This form is expected from the SFT kinetic term (2.75).

Chapter 4

Analyticity and crossing symmetry

String theory looks very different from usual point particle QFT, and it is natural to ask how much they differ. Moreover, we have seen that string theory is non-local, so an important question is how this impacts other properties such as causality. Consistency of a QFT is usually assessed by studying properties of the S-matrix since it is the most important observable. Ideally, one would like to prove directly the different consistency conditions. However, when this is not possible, we can look for indirect tests, such as proving consequences of these properties by another way. While this does not establish them, this still helps understand better string theory and how it differs (or not) from usual local QFTs.

In this chapter, we are interested in analyticity and crossing symmetry of n-point superstring amplitudes at all loops. This provides an indirect test of locality in string theory. In local QFTs, these properties are consequences of locality and causality [194–206]. Since momenta must be off-shell, SFT is the natural framework to address these questions. The idea is to first prove analyticity in the *primitive domain* [194–198]. The original proof uses micro-causality (fields commute at spacelike separation) of the complete S-matrix written in position representation. Then, an analytic extension of a subdomain allows to prove crossing symmetry of 4- and 5-point amplitudes [201–203]. As reviewed in Section 2.5, string theory is non-local and the position representation is not well-defined. As a consequence, we will investigate analyticity of the perturbative S-matrix by studying singularities of Feynman diagrams in momentum space. We are able to prove analyticity in a restricted subset of the primitive domain, however, it is sufficient to imply crossing symmetry. Other properties can also be derived from this analyticity, for example analyticity in the Jost–Lehman–Dyson domain [207, 208], which implies analyticity of the elastic forward scattering amplitude $t = 0$ in the full complex s-plane [180]. Recently, analyticity of the 4-point amplitude have been extended to the full primitive domain, and for a large part of it for the 5-point amplitude [178]. This chapter is based on the paper [151].

4.1 Analyticity and crossing symmetry in QFT

Crossing symmetry is a set of relations between amplitudes with exchange of particles/antiparticles in initial/final states. It is often assumed (for example, in the scattering amplitude program) or just observed for amplitudes with a small number of external particles and at tree-level. Why is it interesting to get a general proof? First, it ensures that the observed examples are not accidents because the corresponding amplitudes are simple.[1] Second, it also teaches something about fundamental properties of QFT. In local QFT, crossing symmetry has been proved only for 4- and 5-point amplitudes. We will focus on the former.

Let's consider a non-physical 4-point process where the external states have momenta $p_a \in \mathbb{C}$ (taken to be incoming) with $a = 1, \ldots, 4$ and such that momentum is conserved:

$$p_1 + p_2 + p_3 + p_4 = 0. \tag{4.1}$$

The most interesting object is the on-shell amplitude $A(p_1, p_2, p_3, p_4)$. It is obtained from the truncated Green function $\tilde{G}(p_1, \ldots, p_4)$ by taking the on-shell limit for the external states:

$$A(p_1, p_2, p_3, p_4) = \lim_{p_a^2 \to -m_a^2} \tilde{G}(p_1, p_2, p_3, p_4). \tag{4.2}$$

In turn, the truncated Green function is itself found from the Green function $G(p_1, p_2, p_3, p_4)$ through the LSZ prescription by amputating external propagators (see also Section 3.1):

$$\tilde{G}(p_1, p_2, p_3, p_4) = G(p_1, p_2, p_3, p_4) \prod_{a=1}^{4} (p_a^2 + m_a^2). \tag{4.3}$$

In QFT with an action, the Green function is given by a sum of Feynman diagrams.

The mass-shell condition is not sufficient to make the amplitude physical because of kinematical constraints. To characterize the physical regions, the best is to use the Mandelstam variables:

$$s = -(p_1 + p_2)^2, \qquad t = -(p_1 + p_3)^2, \qquad u = -(p_1 + p_4)^2. \tag{4.4}$$

On-shell, the sum of the Mandelstam variables equals the sum of the squared masses:

$$s + t + u = \sum_a m_a^2. \tag{4.5}$$

The three physical regions are (Figure 4.1):

$$\begin{aligned}
&S \ \ (s\text{-channel}): \qquad && s \geq \sum_a m_a^2, \qquad && t, u \leq 0 \\
&T \ \ (t\text{-channel}): \qquad && t \geq \sum_a m_a^2, \qquad && s, u \leq 0 \\
&U \ \ (u\text{-channel}): \qquad && u \geq \sum_a m_a^2, \qquad && s, t \leq 0
\end{aligned} \tag{4.6}$$

Then, the three physical amplitudes A_S, A_T and A_U are obtained after taking the momenta in the corresponding regions:

$$A_{S,T,U}(p_1, \ldots, p_4) = \lim_{p_a \in S,T,U} A(p_1, \ldots, p_4). \tag{4.7}$$

Then, crossing symmetry is the statement that the processes

$$\begin{aligned}
S: &\quad 1 + 2 \to 3 + 4, \\
T: &\quad 1 + \bar{3} \to \bar{2} + 4, \\
U: &\quad 1 + \bar{4} \to 3 + \bar{2}
\end{aligned} \tag{4.8}$$

(and CPT conjugates) are related by analytic continuation on the complex mass-shell:

$$A_S(s, t) = A_T(t, s), \qquad A_S(s, u) = A_U(u, s). \tag{4.9}$$

[1] One case where crossing symmetry is violated is in $3d$ Chern–Simons theories [209, 210].

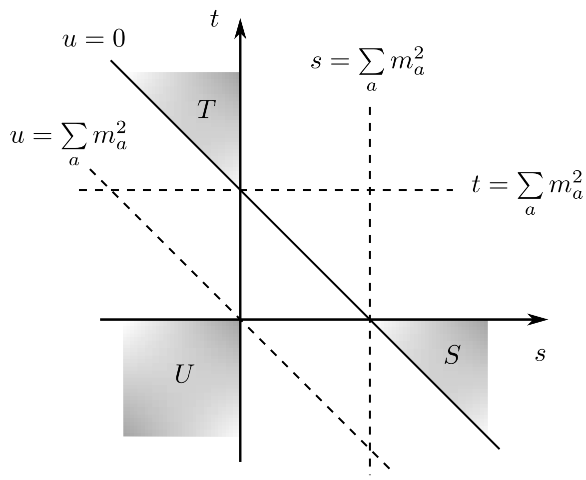


Figure 4.1 – Mandelstam plane for on-shell real momenta. Physical regions are shaded in gray.

At tree-level, crossing symmetry is obvious by direct investigation of the functional form of the amplitudes. More generally, it looks natural from the LSZ prescription since the three amplitudes $A_{S,T,U}$ all come from a single function A. However, it is not guaranteed that A is analytic in a domain with paths between S, T, U as required by the definition of crossing symmetry.

Let's outline the proof [201, 202]. The first step is to prove analyticity in the primitive domain. This can be derived directly for any n-point amplitudes. Then, crossing symmetry of the 4-point amplitude follows by analytic extension of a 2-dimensional subset of the primitive domain.

We need to assume that there is a mass gap $m_a^2 > 0$, and that the asymptotic states are stable particles. The former condition is expected since amplitudes with massless states are ill-defined due to infrared divergences (the two main solutions include working with the inclusive cross-section or with the Kulish–Faddeev S-matrix [211]). Next, let's define a sequence of *primitive domains* for n-point amplitudes:

$$\begin{aligned}\Delta_k = \bigcap_{A_\alpha} \Big[& \big\{ \operatorname{Im} P_{(\alpha)} \neq 0, (\operatorname{Im} P_{(\alpha)})^2 \leq 0 \big\} \cup \big\{ \operatorname{Im} P_{(\alpha)} = 0, -P_{(\alpha)}^2 < M_\alpha^2 \big\} \\ & \cap \big\{ \operatorname{Im} p_a^i = 0, i = k, \ldots, D-1 \big\} \Big],\end{aligned} \tag{4.10}$$

where $A_\alpha \subset \{1, \ldots, n\}$, M_α is the production threshold for the channel A_α and $P_{(\alpha)}$ is the sum over the momenta in the channel A_α:

$$P_{(\alpha)} = \sum_{a \in A_\alpha} p_a. \tag{4.11}$$

In words, momenta p_a in Δ_k have at most k complex components (the energy p_a^0 and the $(k-1)$ first spatial components) and $D - k$ real components such that $P_{(\alpha)}$ satisfies one of the following two conditions:

1. $P_{(\alpha)}$ has non-zero imaginary timelike part;

2. $P_{(\alpha)}$ is real but its norm is below the multi-particle threshold in channel A_α.

Analyticity of the full S-matrix inside Δ_D has been proven from micro-causality [194–198]. The problem is that this domain has no intersection with the mass-shell

$$\Delta_D \cap \text{mass-shell} = \emptyset, \tag{4.12}$$

and thus is not sufficient to obtain crossing symmetry. Indeed, consider the complex mass-shell conditions:

$$\operatorname{Re} p_a \cdot \operatorname{Im} p_a = 0, \qquad (\operatorname{Re} p_a)^2 - (\operatorname{Im} p_a)^2 + m_a^2 = 0. \tag{4.13}$$

If $\operatorname{Im} p_a$ is timelike – first condition in (4.10) –, $(\operatorname{Im} p_a)^2 \leq 0$, then $\operatorname{Re} p_a$ must also be timelike, $(\operatorname{Re} p_a)^2 < 0$, to satisfy the second equation, but, this violates the first one. If $\operatorname{Im} p_a = 0$ timelike – second condition in (4.10) –, then $-P_{(\alpha)}^2 \geq M_\alpha^2$ which is not possible.

As a consequence, the next step is to compute the *envelope of holomorphy* $\mathcal{H}(\Delta_2)$ of Δ_2 (it is not necessary to consider larger domains). Then, one can show that the intersection of $\mathcal{H}(\Delta_2)$ with the mass-shell is not empty

$$\mathcal{H}(\Delta_2) \cap \text{mass-shell} \neq \emptyset \tag{4.14}$$

and contains a path between all pairs of iϵ-neighbourhoods of physical regions (which is expected since amplitudes are well-defined only up to the iϵ-prescription) [201, 202].

The envelope of holomorphy $\mathcal{H}(\Delta)$ of a domain Δ provides an analytic continuation of any function analytic in the domain Δ. The reason is that analyticity of a function $f(z_1, \ldots, z_n)$ of several complex variables is so constraining that the shape of its analyticity domain is not arbitrary. Hence, given a domain Δ, it is possible to look for an analytic extension $\mathcal{H}(\Delta)$ independently of the function f, such that any function analytic in Δ is also analytic in $\mathcal{H}(\Delta)$.

This implies that any statement following from the analyticity in Δ_k can be made for any function analytic in Δ_k. Thus, it is sufficient to prove analyticity in Δ_2 in SFT to prove crossing symmetry of superstring amplitudes.

4.2 Analyticity of superstring amplitudes

As discussed in the introduction of this chapter and in Section 2.5, the non-locality of string theory prevents using the position representation and assuming usual properties of local QFT (like micro-causality). An alternative procedure is to prove analyticity perturbatively by studying the singularities of Green functions from the corresponding Feynman diagrams in momentum space [212].

In our work [151], we have proven that n-point superstring Green functions are analytic in Δ_2 at all loop orders. This implies crossing symmetry for $n = 4$. Moreover, this domain has been later extended to Δ_D [178], which shows that 4-point amplitudes have the same analyticity properties in string theory and local QFT. We outline and comment the main steps of the proof.

We consider a Feynman diagram (2.108) with n external states with momenta p_a (see Section 2.5). Loop momenta are written as ℓ_s, where s runs over the number of loops. Momenta of internal states are denoted by k_i, where i runs over the number of internal propagators, and correspond to linear combinations of p_a and ℓ_s.

As superstring theory contains massless states, we have to be careful since amplitudes have infrared singularities which spoil analyticity. There are two approaches: 1) regularize the massless states by introducing a small mass, 2) consider the part of the amplitude where all internal particles are massive. In the first case, the regulated amplitude has no IR problems and removing the cut-off produces the real amplitude when $D > 4$. In the second case, we consider the projector over massless states (see also Chapter 6)

$$\widehat{P}_0 = \mathrm{e}^{-\infty \widehat{L}_0^+} \tag{4.15}$$

and decompose all internal propagators $\Delta(k_i)$ as:

$$\Delta(k_i) = \Delta(k_i)\widehat{P}_0 + \Delta(k_i)(1 - \widehat{P}_0). \tag{4.16}$$

Then, we can analyze the analyticity of the part where all propagators are accompanied with the projector $(1 - \widehat{P}_0)$ on massive states. The analyticity of the remaining contributions has to be analyzed separately. To make contact with Chapter 6, the projected Green functions with only massless external states correspond to the vertices of the Wilsonian effective action obtained after integrating out the massive fields. The generating functional of the vertices satisfies a BV master equation which encodes the gauge invariance of the full theory even if the projected Green function itself is not gauge invariant. In both cases, this does not allow to establish the crossing symmetry of the full amplitude. But, we stress that it is a problem which is encountered in any QFT with massless fields and which has nothing to do with string theory.[2]

We can now proceed to the proof of analyticity. The method for studying the singularities of a Feynman diagram is as follow [213]:

1. start with some $p_a = p_a^{(1)}$, $\ell_r^0 \in \mathrm{i}\mathbb{R}$, $\boldsymbol{\ell}_r \in \mathbb{R}$ such that there is no singularity (according to Section 2.5.2);

2. find a path in momentum space from $p_a = p_a^{(1)}$ to the desired $p_a = p_a^{(2)}$;

3. deform the contours of ℓ_s integrals as the poles move;

4. assume that there is a singularity (translated by *pinching* of the contour: two poles collide from opposite sides which prevents deforming further the contour), corresponding to at least one internal propagator going on-shell;

5. analyze the reduced diagram and display an inconsistency.

In a reduced graph, all non-singular propagators are contracted to points such that the total momentum entering reduced vertices corresponds to one of the $P_{(\alpha)}$ from (4.11). Following this procedure, we prove analyticity in Δ_2 in two steps, first going from $p_a = 0$ to $p_a \in \Delta_1$, and then to $p_a \in \Delta_2$. Since a Green function is a sum over a finite number of Feynman diagrams, no new singularity can appear from summing all processes.

Analyticity in Δ_1 We start at the point $p_a = 0$ where the amplitude is analytic according to Section 2.5.2. Then, we consider a point where $p_a^0 \in \mathbb{C}$ and $\boldsymbol{p}_a \in \mathbb{R}$ (keeping $\operatorname{Im} \boldsymbol{p}_a = 0$) such that $p_a \in \Delta_1$.

We assume that there is a singularity. The associated diagram is given in Figure 4.2, where a vertical cut intersects m propagators and where the arrows give the signs of k_i^0. Since the k_i are linear combinations of p_a and ℓ_s, we have:

$$\boldsymbol{p}_a, \boldsymbol{\ell}_r \in \mathbb{R} \quad \Longrightarrow \quad \boldsymbol{k}_i \in \mathbb{R}, \tag{4.17}$$

and the mass-shell conditions of the internal propagators imply:

$$k_i^2 = -m_i^2 \quad \Longrightarrow \quad k_i \in \mathbb{R}. \tag{4.18}$$

[2]Crossing symmetry seems to hold also for massless particles but has never been proven rigorously because the amplitudes are not analytic.

Next, it is possible to show that the arrows in Figure 4.2 cannot make a closed loop, such that all arrows point towards the right or, equivalently, $\forall i : k_i^0 > 0$. Then, using the mass-shell condition for the external states, we deduce that the total momentum $P_{(\alpha)}$ passing through this part of the diagram satisfies:

$$P_{(\alpha)} = \sum_{i=1}^{m} k_i \in \mathbb{R}, \qquad p_a^2 = -m_a^2 \quad \Longrightarrow \quad -P_{(\alpha)}^2 \geq M_\alpha^2. \tag{4.19}$$

This gives a contradiction since $p_a \in \Delta_1$ requires $-P_{(\alpha)}^2 < M_\alpha^2$, see (4.10). Thus, the Feynman diagram is analytic in Δ_1.

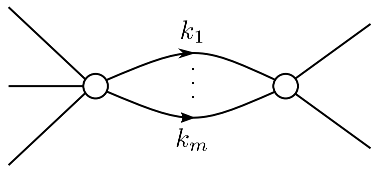


Figure 4.2 – Reduced diagram where all propagators are on-shell. The blob represents the rest of the diagram which does not matter for the argument.

Analyticity in Δ_2 Starting from $p_a \in \Delta_1$, we continue the momenta towards $p_a \in \Delta_2$ by going from $\operatorname{Im} p_a^1 = 0$ to the desired value of $\operatorname{Im} p_a^1$. Let's define:

$$p_a^{\parallel} := (p_a^0, p_a^1) \in \mathbb{C}, \qquad p_a^{\perp} := (p_a^2, \dots, p_a^{D-1}) \in \mathbb{R}. \tag{4.20}$$

Assuming a singularity during the momentum continuation leads to a diagram similar to Figure 4.2. However, this time, the arrows define the signs of $\operatorname{Im} k_i^1$. It is again possible to show that there is no closed loop and that all arrows point towards the right, i.e. $\forall i$: $\operatorname{Im} k_i^1 > 0$. Then, using the mass-shell condition for internal propagators, we find:

$$k_i^2 = -m_i^2 \quad \Longrightarrow \quad \operatorname{Im} k_i^{\parallel} \in W^+ \quad \Longrightarrow \quad \operatorname{Im} P_{(\alpha)} = \sum_i \operatorname{Im} k_i \in W^+ \tag{4.21}$$

where $W^\pm$ are the two-dimensional spacelike regions in Minkowski space (Figure 4.3). This yields a contradiction with the definition (4.10) of the primitive domain Δ_2 since $\operatorname{Im} P_{(\alpha)}$ must be timelike. This proves the analyticity in Δ_2.

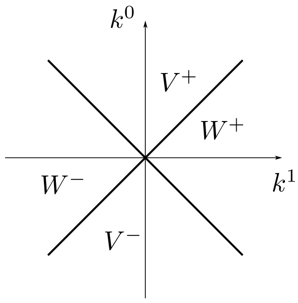


Figure 4.3 – Two-dimensional Minkowski space. $V^\pm$ denote the forward and backward light-cones, $W^\pm$ denote the disconnected spacelike regions.

Analyticity in Δ_k with $k > 2$ could not be proven similarly because the spacelike regions become connected in more than two dimensions. This is a problem because the general sum of spacelike vectors is not necessarily spacelike. However, even if analyticity could not be proven from Feynman diagrams, it has been shown that Δ_2 can be analytically extended to Δ_D (resp. a large subset of it) for the 4-point (resp. 5-point) amplitude [178].

In any case, analyticity in Δ_2 is sufficient to run the arguments of [201, 202] and to derive crossing symmetry of n-point superstring amplitudes. Moreover, we note that our proof provides an alternative derivation of analyticity valid for more general QFTs than discussed in [201, 202].

Extensions of this work includes finding analyticity in Δ_D and $\mathcal{H}(\Delta_2)$ from Feynman diagrams only, in order to get a better physical picture of the result. It would also be interesting to transform the amplitudes in position space to study the (non-)locality of string amplitudes perturbatively.

Chapter 5

Timelike Liouville theory

Liouville theory arises as the universal gravitational action (Wess–Zumino action) of matter described by a conformal field theory (CFT), coupled to two-dimensional gravity in the conformal gauge [121] (see Section 1.2.2 for some motivations). The Liouville action is characterized by two parameters: the sign of the kinetic term $\epsilon = \pm 1$, and the background charge $Q \in \mathbb{C}$. The central charge depends on these two parameters and is related to the central charge c_m of the matter as:

$$c_L := 26 - c_m := 1 + 6\,\epsilon\, Q^2. \tag{5.1}$$

The theory is called spacelike when $\epsilon = 1$, and timelike when $\epsilon = -1$. Since Liouville theory is a two-dimensional CFT, it can also be studied using only CFT techniques. It is within this framework that Liouville theory is defined for $Q \in \mathbb{C}$ without ambiguities.

In this chapter, our goal is to provide a definition of the timelike Liouville theory with $c_L < 1$ not as a standalone CFT, but as a theory of quantum gravity.

To achieve this goal, in Section 5.3, we compute the BRST cohomology of Liouville theory for generic values of the parameters $\epsilon = \pm 1$ and $c_L \in \mathbb{R}$, coupled to a spacelike Coulomb gas and a generic transverse CFT. This provides the most general discussion of the BRST cohomology in $2d$ gravity, but also for non-critical string theory. In particular, it generalizes previous studies for the spacelike Liouville theories with $c_L \geq 25$ [214] (see also [215–226]) and $c_L \in (1, 25)$ [227]. We prove a no-ghost theorem for the Hermitian sector in the timelike theory and for some spacelike models. The computation directly extends methods originally introduced to describe physical states in string theory (Section 2.2.2).

However, the corresponding spectrum cannot be used as internal states in correlation functions. This situation is well-known from QFT: the solution is to define the correlation functions by analytic continuation of the external and internal states. A naive analytic continuation encounters singularities: the proper definition needs the generalized Wick rotation originally introduced in SFT [46, 47, 151] (Section 2.5.2). In Section 5.4, we describe the 2- and 3-point functions. We then explain how to implement the Wick rotation for the 4-point function and obtain a result which is crossing symmetric.

This chapter is based on the papers [152, 153]. With respect to Appendix A.1 and the other chapters of this thesis, we rescale the momenta of the vertex operatos by a factor 2 to agree with the usual conventions for Liouville theory: $k \to 2k$, and $a \to 2a$ (and similarly for the associated operators).

5.1 Context

The most studied and well-known regimes are the ones where the background charge Q is real. This implies that the central charge is real, and that the (quadratic part of the) action is also real. Like for the Coulomb gas, spacelike Liouville ($\epsilon = 1$) corresponds then to a central charge $c_L \geq 1$. For the exponential term in the action to be real, it is further necessary for b to be real. This implies $Q \geq 2$, that is $c_L \geq 25$, which is the regime typically referred to as "*the* spacelike Liouville theory". Giving up on the reality of the exponential gives the range $b \in \mathrm{i}\mathbb{R}$ or $c_L \in (1, 25)$. The regime typically referred to as "*the* timelike Liouville theory" corresponds to a timelike kinetic term $\epsilon = -1$ and real Q, hence with $c_L \leq 1$. It can also be obtained after an analytic continuation of the field and all parameters (except the cosmological constant) in the action. We emphasize that another two regimes compatible with a real central charge exist: that of spacelike Liouville with $Q \in \mathrm{i}\mathbb{R}$ hence $c_L \leq 1$ (considered in [228]), and that of timelike Liouville also with $Q \in \mathrm{i}\mathbb{R}$ and hence $c_L > 1$. All different regimes compatible with real central charge are summarized in Table 5.1. One aim of our papers was to elucidate the differences between the different regimes, especially within each pair with the same central charge.

c	$Q \in \mathbb{R}$		$Q \in \mathrm{i}\mathbb{R}$
	$Q \in [0, 2)$	$Q \in [2, \infty)$	
spacelike: $\epsilon = +1$	$c \in [1, 25)$	$c \geq 25$	$c < 1$
timelike: $\epsilon = -1$	$c < 1$		$c > 1$

Table 5.1 – Range of real values of the Liouville (or Coulomb gas) central charge depending on the parameters Q and ϵ. For the spacelike case $\epsilon = 1$, the two different ranges $Q \in [0, 2)$ and $Q \geq 2$ correspond to $b \in e^{\mathrm{i}\mathbb{R}}$ and $b \in \mathbb{R}$ respectively. What is typically known as the spacelike Liouville theory corresponds to the regime $\epsilon = 1$ and $Q \geq 2$, and the range typically known as the timelike Liouville theory corresponds to $\epsilon = -1$ and $Q \in \mathbb{R}$.

While the spacelike theory is well understood for $c_L \geq 25$ from different points of view [122–125, 229], it is not the case for the other parameter ranges. The most interesting case is the timelike theory with $c_L \leq 1$, studied extensively in [230–241]. Indeed, this theory serves as a toy model for four-dimensional quantum gravity since the kinetic term of the conformal factor is negative definite in $d = 4$ [242] (see also [243–247]). Earlier works on the $c_L \leq 1$ timelike Liouville theory [230–234, 239] approached the question using the minisuperspace approximation, but there are subtelties: the Hamiltonian is not Hermitian, the spectrum does not match the one of the conformal bootstrap and displays strange discrete states, and the 3-point function does not match the $c_L \leq 1$ structure constant.

To solve this problem, we propose to use BRST quantization as the fundamental guiding principle to determine the spectrum, since it encodes the constraints from the diffeomorphism invariance, the gauge symmetry of gravity.

It was recently proven numerically that the spacelike Liouville theory ($\epsilon = +1$) is a consistent CFT for all $c_L \in \mathbb{C}$ as it solves the conformal bootstrap constraints [123, 228] (see also [248–250] for a connection to statistical loop models). The 3-point function which solves the conformal bootstrap constraints depends only on the value of the central charge: it is given by the DOZZ formula everywhere in the complex c_L-plane except for the real interval $c_L \leq 1$, where it is instead given by the "$c_L \leq 1$ structure constant" (also called timelike DOZZ formula) [233, 236–238]. Convergence of the 4-point function also determines the internal spectrum, defined as the set of states on which the correlation functions factorize. The internal spectrum is unique for $c_L \in \mathbb{C}$ and made of states with real momenta. With these ingredients, the 4-point function is crossing symmetric.

In [228], the spectrum of the theory is defined by the internal states on which the correlation functions factorize. However, they are different from the states found in the BRST cohomology. For this reason, we conclude that this definition is too restrictive in quantum gravity and that the internal spectrum and the external (or physical) spectrum must distinguished, in the same way that one differentiates between off-shell and on-shell states in QFT. The internal spectrum is constrained by the convergence of the 4-point function, so it cannot be changed arbitrarily. This explains why we need to introduce a generalized Wick rotation.

5.2 Liouville CFT

Liouville theory corresponds to a Coulomb gas deformed by an exponential interaction representing the coupling to the cosmological constant μ:

$$S_L = \frac{\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{g}\, \left(g^{\mu\nu}\partial_\mu\phi\, \partial_\nu\phi + QR\phi + 4\pi\epsilon\, \mu\, \mathrm{e}^{2b\phi}\right). \tag{5.2}$$

Thanks to the relation (A.8) between b and Q, this is a marginal deformation.

5.2.1 Fock space

The presence of the exponential potential in the Liouville action entails a time-dependence of the field Fourier modes. For this reason, the mode expansion of the Liouville field ϕ is complicated. However, within the conformal bootstrap program (see [122, 123] for reviews), the theory can be completely defined in terms of the current J and of the vertex operators V_a, which satisfy the same properties as the ones of a Coulomb gas. Hence, the central charge (A.10), the definition of Q (A.8) in terms of b, the relation between the plane and cylinder momenta (A.14) and the conformal weights (A.22) are the same as for a Coulomb gas:

$$c = 1 + 6\,\epsilon Q^2, \qquad Q = \frac{1}{b} + \epsilon b. \tag{5.3}$$

The intuition is that, in the regime of small parameter b, the exponential wall is only relevant for high values of the field. For most of the field range, the theory is effectively a Coulomb gas. One can then understand the effect of the cosmological constant as a reflecting wall which mixes the positive- and negative-frequency modes. As a consequence, Liouville vertex operators $\mathcal{V}_k$ are linear combinations of $V_{\pm k}$ such that

$$\mathcal{V}_k := V_k + R(k)V_{-k} = R(k)\mathcal{V}_{-k}, \tag{5.4}$$

where $R(k)$ is the reflection coefficient and satisfies $R(k)R(-k) = 1$. It can be shown [123] that $R(k)$ is proportional to the cosmological constant μ and vanishes when the latter is set to zero.

5.2.2 Analytic continuation

The theories ($\epsilon = 1, Q \in \mathbb{R}$) and ($\epsilon = -1, Q \in \mathrm{i}\mathbb{R}$) are related by the following analytic continuation:

$$\phi = \mathrm{i}\chi, \qquad Q = \mathrm{i}q, \qquad b = -\mathrm{i}\beta, \qquad a = -\mathrm{i}\alpha, \qquad p = -\mathrm{i}E. \tag{5.5}$$

Indeed, starting from (5.2) with $\epsilon = 1$, this analytic continuation yields the timelike Liouville action:

$$S_{tL} = \frac{1}{4\pi} \int \mathrm{d}^2\sigma \sqrt{g}\, \left(-g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - qR\chi + 4\pi\, \mu\, \mathrm{e}^{2\beta\chi}\right). \tag{5.6}$$

This is usually how one gets "*the* timelike Liouville theory" from "*the* spacelike Liouville theory" at the classical level, as presented in the common literature. However, this is only a simple way to translate certain classical expressions from one theory to the other (in the usual case, both Q and q are taken to be real).

Most quantities (like 3-point correlation functions and higher), though, are not analytic in the central charge, such that this analytic continuation cannot be used to derive the properties of the timelike theory from those of the spacelike one [123, 228, 233–235, 239, 240].

5.2.3 Correlation functions

Two-point function

The vertex operators V_α can be normalized so that the two-point function takes the form [123, 240]

$$C_2(z_1,\alpha_1;z_2,\alpha_2) = \frac{1}{|z_1-z_2|^{4\Delta_{\alpha_1}}}\big(\delta(q+\alpha_1+\alpha_2) + R(\alpha_1)\,\delta(\alpha_1-\alpha_2)\big)\,. \tag{5.7}$$

The dependence in z_1 and z_2 is completely fixed by global conformal invariance. The presence of the reflection coefficient $R(\alpha)$ in (5.7) is due to the reflection-invariance of the two-point function. The reflection coefficient can be computed by considering the four-point function of two degenerate fields [123, 240] and reads

$$R(\alpha) = -\left(\frac{e^{i\pi}}{-\pi\mu\,\gamma(-\beta^2)}\right)^{\frac{q+2\alpha}{\beta}} \frac{\Gamma\big(\beta(-q-2\alpha)\big)\Gamma\big(\beta^{-1}(q+2\alpha)\big)}{\Gamma\big(\beta(q+2\alpha)\big)\Gamma\big(\beta^{-1}(-q-2\alpha)\big)}\,, \tag{5.8}$$

with $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. The two-point function has a well-defined analytic continuation in β and in α. In particular, the normalization chosen here is an analytic continuation of the normalization for spacelike Liouville.

Three-point function

Conformal invariance fixes the form of the three-point function to be

$$C_3(z_1,\alpha_1;z_2,\alpha_2;z_3,\alpha_3) = \frac{\widehat{C}_{\alpha_1,\alpha_2,\alpha_3}}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}|z_{23}|^{2(\Delta_2+\Delta_3-\Delta_1)}|z_{13}|^{2(\Delta_3+\Delta_1-\Delta_2)}}\,, \tag{5.9}$$

where $z_{ij} := z_i - z_j$ and $\widehat{C}_{\alpha_1,\alpha_2,\alpha_3} := \widehat{C}(\alpha_1,\alpha_2,\alpha_3)$ are the structure constants of the theory. They are given by[1]

$$\widehat{C}_{\alpha_1,\alpha_2,\alpha_3} = \left(\frac{e^{i\pi}}{-\beta^{2+2\beta^2}\,\pi\mu\,\gamma(-\beta^2)}\right)^{\frac{q+\alpha}{\beta}} \frac{\Upsilon_\beta(\beta-q-\alpha)}{\Upsilon_\beta(\beta)} \prod_{i=1}^{3} \frac{\Upsilon_\beta(\beta+2\alpha_i-\alpha)}{\Upsilon_\beta(\beta-2\alpha_i)}\,, \tag{5.10}$$

where $\alpha = \alpha_1+\alpha_2+\alpha_3$. The Upsilon function $\Upsilon_\beta(x)$ is defined in Appendix A.2. This formula is valid for all $\alpha_i \in \mathbb{C}$. This structure constant was found by Zamolodchikov, and independently by Kostov and Petkova [235–238]. The expression at $c_L = 1$ already appeared in [233]. More insights on this formula from the path integral perspective can be found in [240, 241].

As already mentioned, the structure constant $\widehat{C}_{\alpha_1,\alpha_2,\alpha_3}$ is the unique solution to the degenerate crossing relations when $c_L \leq 1$, and even if the degenerate relations admit a continuation to all $c_L \in \mathbb{C}$, $\widehat{C}$ can only be analytically continued to $c_L \notin (25,\infty)$.

[1] The hat on the $\widehat{C}_{\alpha_1,\alpha_2,\alpha_3}$ is added to distinguish these structure constants in the timelike regime from the ones in the spacelike regime, for which C_{a_1,a_2,a_3} is used in most of the literature. The hat reminds of the fact that these are two different functions of the Liouville momenta.

Four-point function

Higher-point correlation functions can be constructed from the structure constants by using the OPEs. Concretely, the s-channel decomposition of the four-point function reads

$$C_4(z_i, \alpha_i) = \int_{\mathcal{S}_{\text{int}}} \mathrm{d}\alpha_s \, \widehat{C}_{\alpha_1,\alpha_2,\alpha_s} \, \widehat{C}_{-q-\alpha_s,\alpha_3,\alpha_4} \left|\mathcal{F}^{(s)}_{\alpha_s}(z_i, \alpha_i)\right|^2 \tag{5.11}$$

where $\mathcal{F}^{(s)}_{\alpha_s}$ are the s-channel conformal blocks (see [123] for a complete characterization). The integral runs over the internal states $\mathcal{S}_{\text{int}}$ (taken to be continuous by assumption) to be determined. The t- and u-channels are obtained similarly by considering different OPEs.

Requiring convergence of the decomposition (5.11) imposes a restriction on the contour of integration (see Section 5.4 for more details), i.e. on the internal spectrum. Indeed, the integrand behaves as $|\mathfrak{q}|^{2\Delta_s}$ for large $|\Delta_s|$, where the elliptic nome $\mathfrak{q}$ is defined by

$$\mathfrak{q}(x) = \exp\left(-\pi \, \frac{K'(x)}{K(x)}\right) \tag{5.12}$$

in terms of the complete elliptic integral of the first kind $K(x)$, with x being the cross-ratio $x := z_{12}z_{34}/z_{13}z_{24}$. One has $|\mathfrak{q}| < 1$ for all $x \in \mathbb{C}$. Hence, the integral diverges when the real part of the conformal dimensions of the operators appearing in $\mathcal{S}_{\text{int}}$ is unbounded from below. For both the spacelike and timelike regimes, the respective conformal dimensions $\Delta_p = Q^2/4 + p^2$ or $\Delta_E = -q^2/4 - E^2$ are bounded from below when $E := \mathrm{i}p \in \mathrm{i}\mathbb{R}$. Therefore, this continuous family of states can be identified to be the internal spectrum in both regimes (i.e. the family with $p \in \mathbb{R}$ in the spacelike case and the family with $E \in \mathrm{i}\mathbb{R}$ in the timelike case). By continuity, this internal spectrum is also used for any $c_L \in \mathbb{C}$.

However, there is a small caveat in the timelike regime: in this case, the so-identified internal spectrum includes states with dimensions equal to those of the degenerate states, which happen to correspond to the poles of the conformal blocks. This is another indication that the case $c_L \leq 1$ is subtler than the $c_L \geq 25$, and cannot be obtained by an analytic continuation from the latter. The remedy to avoid the poles is to slightly shift the contour of integration by a small real number as [123, 228]

$$\mathcal{S}_{\text{int}} = \left\{\alpha = -\frac{q}{2} + \mathrm{i}E, \, E \in \mathrm{i}\mathbb{R} + \epsilon\right\}. \tag{5.13}$$

The poles and integration contour are described in Figure 5.1. This prescription is equivalent to shifting the momentum on the cylinder E by $\mathrm{i}\epsilon$, which can be interpreted as the standard $\mathrm{i}\epsilon$ prescription of QFT. It is also consistent with the fact that a continuous internal spectrum can be deformed in the complex plane when no singularity is encountered. It has been checked numerically that the result does not depend on ϵ, and that the spectrum (5.13) together with the structure constant C and $\widehat{C}$ respectively for $c_L \notin (-\infty, 1]$ and $c_L \leq 1$, lead to a crossing-symmetric four-point function for all $c_L \in \mathbb{C}$ [228].

Depending on the central charge, two different 3-point functions are compatible with the degenerate crossing equations of the Liouville theory: the first, for $c_L \notin (-\infty, 1)$, is given by the DOZZ formula [192, 228, 251], while the second is valid for $c_L \notin (25, \infty)$ [228, 235–238]. However, the latter range gets restricted to $c_L \in (-\infty, 1]$ when considering the full set of crossing equations [228]. Note that these two ranges are specified by the value of the central charge, regardless of the value of $\epsilon = \pm 1$ because the bootstrap is insensitive to the sign of the current–current OPE. Hence, the choice of the 3-point function is uniquely fixed by the central charge, not by the regime.

Finally, the conformal bootstrap selects a specific internal spectrum to ensure that the 4-point function is well defined (convergence of the integration over the internal states given by the OPE). It is characterized by $p \in \mathbb{R}$ for $\epsilon = 1$, $p \in \mathrm{i}\mathbb{R}$ for $\epsilon = -1$, but since the associated

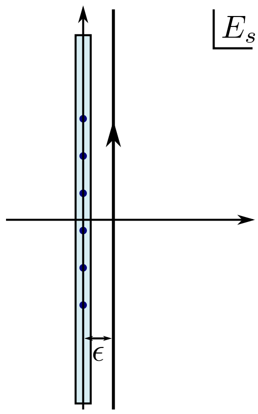


Figure 5.1 – Poles of the four-point function integrand and integration contour for the $c_L \leq 1$ Liouville theory. The poles in the E_s-plane are located on the imaginary axis (shaded area, only few poles are displayed) and depend only on β, *not* on the external momenta $\alpha_1, \alpha_2, \alpha_3$ and α_4. The iϵ prescription shifts the contour away from the poles.

operators have the same conformal weights (and the same lower bound $(c_L - 1)/24$) for identical central charges, they can be identified (remember that the definition of p is ϵ-dependent). However, this does not prevent to consider different spectra – which may be required by other consistency conditions – if the theory can be consistently defined by analytic continuation [152].

5.3 BRST quantization and physical states

The BRST cohomology is computed by generalizing the derivation of Bouwknegt, McCarthy and Pilch [214], where the Liouville theory is represented by a Coulomb gas. This generalizes to all regimes (ϵ, Q) the results from [214, 227] for spacelike Liouville. The main result is that the Hermitian sector of the BRST cohomology associated to $2d$ diffeomorphisms is free from negative-norm states. We provide an argument to match the cohomology of the Liouville and Coulomb gas when the Fock space of the latter contains no degenerate states for Hermitian momenta. This proves the *no-ghost theorem* in full generality for the timelike Liouville theory, and for some of the spacelike Liouville cases. Interestingly, in timelike case, this holds even when the Liouville field is not Hermitian, only the matter. Examples of applications can be found in [153].

Another conclusion of our analysis is that two theories with the same value of the central charge can have different BRST spectra. Indeed, consider the Liouville theories with $(\epsilon_\phi = -1, Q_\phi \in \mathbb{R})$ and with $(\epsilon_\phi = +1, Q_\phi \in i\mathbb{R})$, both coupled to the same spacelike Coulomb gas with $Q_X \in i\mathbb{R}$ and the same transverse CFT. The two have the same central charge $c_L \leq 1$, yet the first one only has continuous states in the spectrum, while the second one will generically also contain discrete states (the spectrum given by the conformal bootstrap is not compatible with Hermitian Virasoro operators). This should not come as a surprise: as it is well known in string theory, the cohomology is empty in Euclidean signature but not in Lorentzian signature.

5.3.1 Setup

Consider a general CFT with a Coulomb gas scalar field X and a Liouville field ϕ, with background charges Q_X, Q_ϕ, and a generic transverse CFT. The field X is taken to be spacelike, while the field ϕ can be spacelike or timelike, i.e. formulas depend on ϵ_ϕ. We consider all values of $Q_\phi \in \mathbb{R}, i\mathbb{R}$, for which the central charge of ϕ is real and for which the BRST operator can be Hermitian. The quantities associated to each scalar are distinguished by an index ϕ or X, placed as a superscript for the modes, as a subscript otherwise. Compared to string theory, we have $\alpha' = 1$.

Together, the two scalar fields and the ghosts form the *longitudinal* sector. The *transverse* CFT is unitary and is only characterized by its energy–momentum tensor $T^\perp$. The Hilbert space $\mathcal{H}$ of the theory is

$$\mathcal{H} = \mathcal{H}_\parallel \otimes \mathcal{H}_\perp, \qquad \mathcal{H}_\parallel := \mathcal{F}_\phi(k_\phi) \otimes \mathcal{F}_X(k_X) \otimes \mathcal{H}_{\text{gh}}, \tag{5.14}$$

where k_ϕ and k_X correspond to the momenta of the scalar vacua and $\mathcal{H}_\perp$ is the Hilbert space of the transverse CFT. The total Virasoro zero-mode operator L_0 is

$$L_0 = L_0^\perp - m^2 - 1 + \widehat{L}_0^\parallel \tag{5.15}$$

where $\widehat{L}_0^\parallel$ is the total longitudinal level operator

$$\widehat{L}_0^\parallel = N^\phi + N^X + N^b + N^c, \tag{5.16}$$

and the (longitudinal) mass m^2 corresponds to the vacuum energy of the scalars

$$-m^2 = \frac{1}{4}\left(Q_X^2 + \epsilon_\phi Q_\phi^2\right) + k_X^2 + \epsilon_\phi k_\phi^2. \tag{5.17}$$

The total central charge must vanish as required by gauge invariance of the two-dimensional gravitational theory, which leads to the relation:

$$Q_X^2 + \epsilon_\phi Q_\phi^2 = 4 - \frac{c_\perp}{6}. \tag{5.18}$$

The Hamiltonian can be written more simply as:

$$L_0 = L_0^\perp - \frac{c_\perp}{24} + k_X^2 + \epsilon_\phi k_\phi^2 + \widehat{L}_0^\parallel. \tag{5.19}$$

5.3.2 Relative cohomology

Physical states are the states in the Hilbert space which belong to the BRST cohomology $\mathcal{H}_{\text{abs}}$, i.e. which are Q-closed but non-exact:

$$\mathcal{H}_{\text{abs}}(Q, \mathcal{H}) := \Big\{ |\psi\rangle \in \mathcal{H} \;\Big|\; Q\,|\psi\rangle = 0, \nexists\, |\chi\rangle \in \mathcal{H} : |\psi\rangle = Q\,|\chi\rangle \Big\}. \tag{5.20}$$

The subscript refers to the absolute cohomology, as opposed to the relative cohomology which will be defined shortly.

The general method to construct the absolute cohomology follows [214]. Other works and reviews include [172, 218, 227, 252–256]. The strategy is to find a sequence of isomorphisms between cohomologies of simpler BRST operators. This is achieved by finding a "contracting homotopy" operator which inverts the BRST operator in some subspace. Then, one can restrict the BRST operator in the orthogonal subspace to compute the cohomology, because a BRST closed state with a definite eigenvalue of the contracting homotopy operator is necessarily exact. Restricting to this subspace is what defines the relative cohomology, in which the BRST operator takes a simpler form. Introducing a light-cone parametrization and iterating the procedure allows to construct the states explicitly. Finally, one needs to map them to the original space, which is an easy task when the states have no ghosts beyond the one of the vacuum.

A necessary condition for a state $|\psi\rangle$ to be an element of the BRST cohomology is to be on-shell $L_0 |\psi\rangle = 0$. It is convenient to first consider the subspace $\mathcal{H}_0$ of on-shell states that further satisfy the Siegel gauge (2.74) $b_0 |\psi\rangle = 0$

$$\mathcal{H}_0 := \mathcal{H} \cap \ker b_0. \tag{5.21}$$

since this additional condition is sufficient to ensure on-shellness of BRST closed states. The relative cohomology is defined as the restriction of the absolute cohomology on this subspace:

$$\mathcal{H}_{\text{rel}}(Q, \mathcal{H}) := \mathcal{H}_{\text{abs}}(Q, \mathcal{H}) \cap \ker b_0 \simeq \mathcal{H}(\widehat{Q}, \mathcal{H}_0), \tag{5.22}$$

where $\mathcal{H}(\widehat{Q}, \mathcal{H}_0)$ is the cohomology of $\widehat{Q}$ in $\mathcal{H}_0$. The last equivalence follows from the decomposition (A.47) where $\widehat{Q}$ corresponds to the part of Q which does not contain ghost zero-modes. Since the operator $\widehat{Q}$ is nilpotent in $\mathcal{H}_0$, it is then sufficient to work out the BRST cohomology for the $\widehat{Q}$ operator.

Then, it is possible to show that the absolute cohomology is given by $\mathcal{H}_{\text{abs}} = \mathcal{H}_{\text{rel}} \oplus c_0 \mathcal{H}_{\text{rel}}$ [214]. However, the additional states in $c_0 \mathcal{H}_{\text{rel}}$ are not physical [172, 257]. For this reason, we focus on the relative cohomology.

The next step is to introduce a light-cone parametrization of both Coulomb gas:

$$\alpha_n^\pm = \frac{1}{\sqrt{2}} \left(\alpha_n^\phi \pm \frac{\mathrm{i}}{\sqrt{\epsilon_\phi}} \alpha_n^X \right), \qquad \alpha^\pm = \frac{1}{\sqrt{2}} \left(\epsilon_\phi \, \alpha_\phi \pm \frac{\mathrm{i}}{\sqrt{\epsilon_\phi}} \alpha_X \right), \tag{5.23a}$$

$$x^\pm = \frac{1}{\sqrt{2}} \left(\epsilon_\phi \, x_\phi \pm \frac{\mathrm{i}}{\sqrt{\epsilon_\phi}} x_X \right), \qquad p^\pm = \frac{1}{\sqrt{2}} \left(\epsilon_\phi \, p_\phi \pm \frac{\mathrm{i}}{\sqrt{\epsilon_\phi}} p_X \right), \tag{5.23b}$$

$$Q^\pm = \frac{1}{\sqrt{2}} \left(Q_\phi \pm \frac{\mathrm{i}}{\sqrt{\epsilon_\phi}} Q_X \right). \tag{5.23c}$$

Defining the momenta

$$P_n^\pm = p^\pm + \frac{\mathrm{i}}{2} Q^\pm n, \tag{5.24}$$

the $\widehat{Q}$ operator is decomposed as:

$$\widehat{Q} = Q_0 + Q_1 + Q_2, \tag{5.25}$$

where

$$\begin{aligned} Q_0 &= \epsilon_\phi \sqrt{2} \sum_{m \neq 0} P_m^+ \, c_{-m} \alpha_m^-, \qquad Q_2 = \epsilon_\phi \sqrt{2} \sum_{m \neq 0} P_m^- \, c_{-m} \alpha_m^+, \\ Q_1 &= \sum_{n \neq 0} c_{-n} L_n^\perp + \sum_{\substack{m,n \neq 0 \\ m+n \neq 0}} c_{-m} \left(\epsilon_\phi \, \alpha_{-n}^+ \alpha_{m+n}^- - \frac{1}{2} (m-n) \, c_{-n} b_{m+n} \right). \end{aligned} \tag{5.26}$$

The subscripts $0, 1, 2$ refer to the *degree* of the operator defined as $N^+ - N^- + N^c - N^b$, with $N^\pm$ being the light-cone level operators defined as in (A.17). Nilpotency of $\widehat{Q}$ gives the following conditions:

$$Q_0^2 = Q_2^2 = 0, \qquad \{Q_0, Q_1\} = \{Q_1, Q_2\} = 0, \qquad Q_1^2 + \{Q_0, Q_2\} = 0. \tag{5.27}$$

Therefore, Q_0 and Q_2 are both nilpotent and define a cohomology. The whole point of this decomposition and of the light-cone parametrization is that the cohomologies of $\widehat{Q}$ and Q_0 are isomorphic [214, theorem A.3]:[2]

$$\mathcal{H}(\widehat{Q}, \mathcal{H}_0) \simeq \mathcal{H}(Q_0, \mathcal{H}_0) \tag{5.28}$$

[2] The theorem states that the Q_0- and $\widehat{Q}$-cohomologies are isomorphic if states at fixed ghost number have all the same degree.

In particular, this holds automatically if there are no ghosts and no light-cone oscillators. Note that Q_0 commutes with L_0 and b_0: to compute the cohomology of $\widehat{Q}$, one can compute the cohomology of Q_0 for the Fock space $\mathcal{H}$ and restrict it at the end to $\mathcal{H}_0$.

The computation of the Q_0- and Q_2-cohomologies requires to invert one of the momenta $P_n^\pm$. A subtlety therefore arises if both $P_n^\pm$ vanish for some integers. In this case, some oscillators are not present in the expression of the contracting homotopy operator, and they can be used to write more states. For this reason, two cases must be considered: 1) $P_n^+ \neq 0$ for all $n \neq 0$ (P_n^- can vanish for some value) and 2) $P_r^+ = P_s^- = 0$ for some $r, s \neq 0$. The case $P_n^- \neq 0$ for all $n \neq 0$ (P_n^+ can vanish for some value) is analogous to the first case by reversing the definition of the degree.

In the first case (continuous state), we find that $\widehat{L}_0^{\parallel} = 0$ which implies:

$$N^+ = N^- = N^b = N^c = 0, \qquad L_0 = L_0^\perp - \frac{c_\perp}{24} + k_X^2 + \epsilon_\phi\, k_\phi^2 = 0. \tag{5.29}$$

As a consequence, states in the cohomology do not contain any $\alpha_n^\pm$, b or c excitations: they don't contain ghosts and correspond to the ground state of the Fock space $\mathcal{F}_\phi \otimes \mathcal{F}_X$ with momenta constrained by the above condition. Since they all have the same degree, they are also elements of the $\widehat{Q}$-cohomology.

In the second case (discrete states), there exist two non-zero integers r and s such that the operators $P_n^\pm$ vanish

$$\exists \quad r, s \in \mathbb{Z}^* : \qquad P_r^+ = P_s^- = 0. \tag{5.30}$$

This sets all number operators in $L_0^{\parallel}$ to zero except for the modes r and s. As a consequence, the corresponding states are of the form:

$$r > 0 : \qquad (\alpha_{-r}^+)^u\,(c_{-r})^v\,|k_\phi, k_X, \downarrow\rangle \otimes |\psi_\perp\rangle\,, \tag{5.31a}$$

$$r < 0 : \qquad (\alpha_r^-)^u\,(b_r)^v\,|k_\phi, k_X, \downarrow\rangle \otimes |\psi_\perp\rangle\,, \tag{5.31b}$$

where u and v are some positive integers to be determined in each case by consistency with the other conditions. The allowed values for v are 0 and 1: the cohomology will hence contain states with ghost number $N_{\text{gh}} = 0, 1, 2$. The on-shell condition reduces to:

$$L_0 = L_0^\perp + (1 - rs)\left(1 - \frac{c_\perp}{24}\right) - 1 + |r|(u + v) = 0. \tag{5.32}$$

The momenta k_ϕ and k_X are fixed in terms of r and s by (5.30). It is not possible to apply theorem A.3 from [214] to show that the discrete states (5.31) are also elements of the $\widehat{Q}$-cohomology. Indeed, there can be states of different degrees with $N_{\text{gh}} = 1$ satisfying the on-shell condition (5.32). One such example is the case where $Q_X = 0$ and the transverse CFT is made of $D - 1$ scalars [227]. Discrete states contain ghost excitations and can have negative-norm states. For these reasons, it is useful to project them out, which can be achieved by imposing Hermiticity of the BRST charge.

Hermitian sector and no-ghost theorem. Hermiticity of the BRST charge follows from the Hermiticity of the Virasoro operators of each sector. For the Coulomb gas, the standard Hermiticity conditions (A.27) require $p^\dagger = p$ if $Q \in \mathbb{R}$, and $p^\dagger = -p$ if $Q \in \mathrm{i}\mathbb{R}$.

For continuous states, this simply restricts the range of the momenta k_X and k_ϕ. Note that this is independent of $\epsilon_\phi = \pm 1$.

To study the discrete states, we first rewrite the momenta for the scalar fields when $P_r^+ = P_s^- = 0$ holds:

$$p_\phi = -\frac{\mathrm{i}\epsilon_\phi}{4}(r + s)Q_\phi + \frac{\sqrt{\epsilon_\phi}}{4}(r - s)Q_X, \tag{5.33a}$$

$$p_X = -\frac{\sqrt{\epsilon_\phi}}{4}(r - s)Q_\phi - \frac{\mathrm{i}}{4}(r + s)Q_X, \tag{5.33b}$$

using (5.24) and (5.23b), (5.23c). As a consequence, restricting the BRST cohomology to its Hermitian sector imposes constraints on the possible values of r and s. Tables 5.2 and 5.3 display the conditions for which X is Hermitian for the timelike and spacelike Liouville cases respectively:

Timelike case We find that the only allowed possibility is for $r = s = 0$ when both fields are Hermitian, and hence there are no discrete states.

Spacelike case When both fields are Hermitian, the only solution is $r = s = 0$ if one charge is real and the other imaginary. On the other hand, there are solutions with $r = -s$ when the charges are either both real or both imaginary. For example, if there is no transverse CFT, then $r = -s$ does not solve the on-shell condition (5.32) since $rs < 0$.

$\epsilon_\phi = -1$	$Q_X \in \mathbb{R}$	$Q_X \in i\mathbb{R}$
$Q_\phi \in \mathbb{R}$	$r = s = 0$	$r + s = 0$
$Q_\phi \in i\mathbb{R}$	$r + s = 0$	$r = s = 0$

Table 5.2 – Conditions on the integers r and s for a *timelike* ϕ following from the Hermiticity of X. Imposing further ϕ to be Hermitian reduces all conditions to $r = s = 0$, in which case there are no solutions.

$\epsilon_\phi = +1$	$Q_X \in \mathbb{R}$	$Q_X \in i\mathbb{R}$
$Q_\phi \in \mathbb{R}$	$r + s = 0$	$r = s = 0$
$Q_\phi \in i\mathbb{R}$	$r = s = 0$	$r + s = 0$

Table 5.3 – Conditions on the integers r and s for a *spacelike* ϕ following from the Hermiticity of X. Hermiticity of ϕ does not impose further conditions.

In conclusion, there are no discrete states, hence no ghosts, in the Hermitian subsector of the BRST cohomology in most cases. Indeed, in the case where the Liouville field (or one of the two Coulomb gases) is timelike, only continuous states remain. In that case, it is possible to apply theorem A.3 from [214] which implies that the Q_0-cohomology is isomorphic to the relative cohomology.

Cosmological constant. Finally, we outline a simple argument to argue that restoring the cosmological constant does not change the cohomology beyond identifying states with $\pm k$. This relies on the isomorphism between the Fock basis $\{\alpha_{-n}\}$ and the Virasoro basis $\{L_{-n}\}$, which holds when the momentum of the vacuum $|k\rangle$ (A.25) is not equal to the momentum of a degenerate state [258–261].[3] This is mostly the case since, as discussed below (A.27), there are no degenerate states for Hermitian momenta when $c_L \notin (1, 25)$ for both $\epsilon_\phi = \pm 1$. The argument is as follows:

1. The cohomology is computed for the Coulomb gas in the oscillator basis $\{\alpha_{-n}\}$, that is, as a subspace of the Fock space built on all primaries $V_{\pm p}$.
2. The isomorphism is used to rewrite *all states* in the Virasoro basis $\{L_{-n}\}$, which is possible when there are no degenerate states. In particular, we map the states from the cohomology.

[3] A degenerate state is a primary state which has a null state among its descendants. A null state is a state which is both primary and descendant.

3. The Liouville states are written as linear combinations (5.4) of Coulomb gas states in the Virasoro basis. This can be done because Liouville primaries are given by

$$\mathcal{V}_p = V_p + R(p)V_{-p}. \tag{5.34}$$

Therefore their descendents are generated by the Virasoro basis $\{L_{-n}\}$ acting on these. Since they are written as linear combinations of states $V_{\pm p}$, which have identical conformal weights h_p, they also have a well-defined conformal weight (and similarly for the descendents).

4. The BRST charge Q is completely determined by the Virasoro algebra, and its action depends only on the conformal weight of the states and the central charge c_L. Since the central charges and conformal weights of the Liouville theory and of the corresponding Coulomb gas are the same, the Liouville cohomology follows from the one of the Coulomb gas.

5.4 Generalized Wick rotation for correlation functions

In this section, we describe how to define the timelike theory at a given c_L as the analytic continuation of the spacelike theory *with the same* c_L. Thus, this analytic continuation is not performed on the complex plane of the central charge, but rather on the energies of external and internal states through a generalized Wick rotation [3, 46, 48, 151].

5.4.1 Two- and three-point functions

The 2-point function is taken to be (5.7) since this formula is well defined for any external complex momentum and there is no internal momentum. Moreover, it matches the saddle-point computations in the semi-classical limit for the timelike action (5.6) [240].

The 3-point function is taken to be (5.10) for the same reasons. Indeed, the path integral and Coulomb gas computations from [240, 241] convincingly show that this is the correct 3-point function for the timelike theory. It is also analytic in the external momenta and has no internal momentum. Moreover, solving the full crossing equations shows that the 3-point function is unique for a given central charge [123, 228].

5.4.2 Divergence in the four-point function

The large conformal dimension behaviour of the conformal block is [228, 240, 251, 262]

$$\mathcal{F}^{(s)}_{\alpha_s}(z_i, \alpha_i) \sim_{|\Delta_{\alpha_s}| \to \infty} \big(16\mathfrak{q}(x)\big)^{\Delta_{\alpha_s}}, \tag{5.35}$$

where $x := z_{12}z_{34}/z_{13}z_{24}$ is the cross-ratio. The elliptic nome $\mathfrak{q}$ is defined in (5.12) Writing $\alpha_s = -\frac{q}{2} + \mathrm{i}E_s$ and combining together (5.35) and (A.61), the large E_s behaviour of the integrand of the four-point function (5.11) is found to be

$$\widehat{C}_{\alpha_1,\alpha_2,-q-\alpha_s}\,\widehat{C}_{\alpha_s,\alpha_3,\alpha_4}\big|\mathcal{F}^{(s)}_{\alpha_s}(z_i,\alpha_i)\big|^2 \sim_{|E_s|\to\infty} |\mathfrak{q}|^{-2E_s^2} \propto |\mathfrak{q}|^{L_0+\bar{L}_0}. \tag{5.36}$$

This behavior implies that the four-point function is finite for $\alpha_s \in \mathbb{R}$ or equivalently $E_s \in \mathrm{i}\mathbb{R}$ up to the iϵ prescription. It diverges for $\alpha_s \in -\frac{q}{2} + \mathrm{i}\mathbb{R}$ or $E_s \in \mathbb{R}$ [228]. Note that the factor $|\mathfrak{q}|^{L_0+\bar{L}_0}$ is formally similar to the one which appears in the closed string propagator (2.76) using Schwinger parametrization (2.82) after setting $\mathfrak{q} = \mathrm{e}^{-s}$. It was shown numerically in [228] that as long as the internal states belong to $\mathcal{S}_{\text{int}}$ then the four-point function is well defined and consistent with the full crossing symmetry constraints for external states with any complex momenta. However, this is not sufficient for our purposes because we would like to define the timelike Liouville theory as an analytic continuation of the theory in [228] in the same way Lorentzian QFTs are defined by an analytic continuation of Euclidean QFTs. That is, the correlation functions with external states in the BRST cohomology must be reached by an appropriate analytic continuation, including the integration contour.

5.4.3 Definition of the four-point function

We restrict ourselves to the task of defining the unintegrated correlation functions of the Liouville sector, because the other (matter and ghosts) sectors are decoupled. The goal is to define a well-defined crossing-symmetric four-point function for all allowed external states in the BRST cohomology. Following SFT prescription (Section 2.5.2), this can be achieved as follows:

1. Start with (5.11) such that all external states with $E_i \in \mathrm{i}\mathbb{R}$, i.e. in $\mathcal{S}_{\text{int}}$. With the choice of the contour along the imaginary E_s axis, it is a well-defined crossing-symmetric integral.

2. As the external energies are analytically continued to $E_i \in \mathbb{R}$, examine if any poles of the integrand cross the contour along the imaginary axis.

3. Deform the contour, if necessary, to avoid any of the above poles while holding the ends fixed at $\pm\mathrm{i}\infty$.

The main idea is that for an analytic continuation it is not necessary to analytically *rotate* the entire contour as one usually does with the Wick rotation; it is sufficient if one can analytically *deform* the contour while avoiding all poles to reach the physical regime of interest.

To implement step (2) we need to investigate the poles of the integrand (Figure 5.1). The poles of the conformal block are located at the values of α_s such that the conformal dimension Δ_s equals the one of a degenerate state, but they are otherwise independent of the external momenta [123]. Thus these poles do not move as one analytically continues the external energies. Next, one needs to investigate the poles of the three-point functions. From the expression (5.10) and the formulas in Appendix A.2, one finds that the poles are located at

$$\Upsilon_\beta(\beta - 2\alpha_s) = 0, \tag{5.37}$$

which are again independent of external energies and hence do not move.

In conclusion, the prescription for timelike gravity is in fact even simpler than the analogous prescription in string field theory. The relevant poles of the integrand do not move at all as the external energies are analytically continued and it is not even necessary to change the integration contour (which stays the same as in Figure 5.1). The same formula (5.11) should be used with external states belonging to $\mathcal{S}_{\text{ext}}$. This implies that the code written for [228] is directly usable for the timelike Liouville theory without the need to deform the contour in any way. We have explicitly checked numerically that the four-point functions are indeed convergent for all our physical states and crossing symmetric.

Higher-order correlation functions can be similarly defined by factorization, following for example [263], and analytic continuation from $E \in \mathrm{i}\mathbb{R}$ to $E \in \mathbb{R}$. Together with a concrete identification of the spectrum, this provides a complete definition of the timelike Liouville theory which satisfies several basic states such as crossing symmetry of the 4-point function. We focused on the interpretation within quantum gravity, but it is clear that this section does not use the fact that the Liouville theory is coupled to some matter. Hence, our prescription can be applied for more general external spectrum than the one given by the BRST cohomology.

Chapter 6

Effective string field theory

Low-energy effective actions provide one of the most direct avenue for studying phenomenological stringy effects. Indeed, the masses of the string states are proportional to $\sqrt{\alpha'}$, itself related to the string scale, which is expected to be very large. Thus, only the lightest (in particular, massless) states are relevant for making contact with possible experiments. For this reason, it is interesting to integrate out massive fields. This is a huge simplifications since it allows to work with a finite number of fields of low spins instead of the full infinite tower of string states.

The goal of this chapter is first to investigate the algebraic structure of the effective theory (IR), in particular, to understand which properties are dictated from the ones of the microscopic theory (UV). The two most important aspects are 1) the symmetries and 2) the observables. There are two approaches: the first, perturbative, is useful to understand the general ideas but is quickly cumbersome. The second, based on coalgebra, is more compact and allows to make statement at all orders but is more mathematically involved. Using the homological perturbation lemma, this language provides an efficient way to combine multiple projections and perturbations of a free theory. Moreover, we clarify the role of the auxiliary Nakanishi–Lautrup fields. We illustrate some of these ideas with the open bosonic SFT, keeping only massless states.

Next, we study the effective action using the WZW formulation of heterotic SFT [7, 23, 24]. The latter does not fit in the previous framework because it has no L_∞ structure. However, it allows to show in an elegant manner that cancellations happen thanks to supersymmetry, which makes it possible to compute the quartic potential (at zero momentum) *without* knowing the interactions of the UV theory. This is particularly useful since, as explained in Chapter 1, the interactions are not known beyond the first orders.

We focus on the open classical SFT. Formulas for the closed string can be recovered easily by adding factors of 2 and plus on operators.

Other works on algebraic aspects of SFT and effective theories are [44, 264–270]. Exact computations have been provided in [148, 149, 271] while level truncated effective actions of gauge fields have been computed in [42, 272–275]. Finally, related papers are [62, 276, 277].

This chapter is based on the papers [154, 155].

6.1 Algebraic structure

The effective action can be computed in the same way for both the open and closed SFT (and both for A_∞ and L_∞). For this reason, we introduced unified notations for the action and equations of motion (Chapter 2):

$$S := \frac{1}{2}\langle \Psi, Q\Psi \rangle + \langle \Psi, V(\Psi) \rangle, \tag{6.1a}$$

$$\mathcal{E}(\Psi) := Q\Psi + V'(\Psi), \tag{6.1b}$$

where $\Psi \in \mathcal{H}$ is the string field and

$$V(\Psi) := \int_0^1 \mathrm{d}t \, V'(t\Psi). \tag{6.2}$$

The inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ is defined in (2.69) and $Q := \ell_1$ is the BRST charge. The potential for A_∞ and L_∞ are respectively (Section 2.4.2):

$$\begin{aligned} A_\infty: \quad V(\Psi) &= \sum_{n\geq 2} \frac{1}{n+1}\, m_n(\Psi^n), \\ L_\infty: \quad V(\Psi) &= \sum_{n\geq 2} \frac{1}{(n+1)!}\, \ell_n(\Psi^n). \end{aligned} \tag{6.3}$$

Let's introduce a projector P and its conjugate $\bar{P} = 1 - P$ such that:

$$\begin{aligned} &[L_0, P] = [Q, P] = [b_0, P] = 0, \\ &P^\dagger = P, \qquad \ker \widehat{L}_0 \in \operatorname{Im} P \end{aligned} \tag{6.4}$$

The last condition means that the projector kernel contains the massless states of the theory. For this reason, we call:

- light the states in $P\mathcal{H}$;
- heavy the states in $(1 - P)\mathcal{H}$.

For example, the projector $\widehat{P}_0$ on massless states (which already appeared in Section 4.2) reads:

$$\widehat{P}_0 := \mathrm{e}^{-\infty\, \widehat{L}_0}, \tag{6.5}$$

where we recall that $L_0 = \alpha' k^2 + \widehat{L}_0$ and $\widehat{L}_0 =: \alpha' m^2$ for the open string. Another example is the projector P_0 on on-shell states:

$$P_0 := \mathrm{e}^{-\infty\, L_0}, \tag{6.6}$$

which forces $k^2 = -m^2$. At zero-momentum $k = 0$, this projects on massless states.

The procedure for integrating out the heavy fields to get an effective action for the light fields is the following:

1. Siegel gauge fixing heavy fields;
2. integrate out heavy fields;
3. check out-of-Siegel gauge constraints;
4. integrate out light auxiliary fields.

6.1.1 Perturbative description

Heavy fields are integrated out by solving their equations of motion. However, the kinetic term can be inverted only if the gauge invariance is gauge fixed. For this reason, we impose the Siegel gauge condition. An important point is that gauge fixing a symmetry leads to constraints, here called out-of-Siegel gauge. To derive them, we need to study the gauge fixing of the equations of motion and not of the action.

Gauge fixing The first step is to introduce a projector Π_s on Siegel gauge and its conjugate $\bar{\Pi}_s$:

$$\Pi_s := b_0 c_0, \qquad \bar{\Pi}_s := c_0 b_0, \tag{6.7}$$

in order to decompose the field as:

$$\begin{gathered}\Psi = \varphi + R_\downarrow + R_\uparrow, \\ \varphi := P\Psi, \qquad R_\downarrow := \Pi_s \bar{P}\Psi, \qquad R_\uparrow := \bar{\Pi}_s \bar{P}\Psi,\end{gathered} \tag{6.8}$$

where ϕ denotes the light fields, and $R_\downarrow$ and $R_\uparrow$ are the heavy fields in the Siegel and out-of-Siegel gauge subspaces.

Next, we apply the projectors P and Π_s and their conjugates on the equation of motion (6.1b) to obtain the equations of motion for each of the field components. Using the decomposition (A.47), we find:

$$\mathcal{E}_\varphi(\Psi) := P\mathcal{E}(\Psi) = Q\varphi + PV'(\Psi), \tag{6.9a}$$
$$\mathcal{E}_{R_\downarrow}(\Psi) := \bar{P}\bar{\Pi}_s\mathcal{E}(\Psi) = c_0 L_0 R_\downarrow + \widehat{Q}R_\uparrow + \bar{P}\bar{\Pi}_s V'(\Psi), \tag{6.9b}$$
$$\mathcal{E}_{R_\uparrow}(\Psi) := \bar{P}\Pi_s\mathcal{E}(\Psi) = \widehat{Q}R_\downarrow - b_0 M_+ R_\uparrow + \bar{P}\Pi_s V'(\Psi). \tag{6.9c}$$

Note that the equation of motion for $R_\downarrow$ (resp. $R_\uparrow$) is obtained by acting with $\bar{\Pi}_s$ (resp. Π_s): the reason is that the Siegel projector behaves as $\Pi_s^\dagger = \bar{\Pi}_s$ for the inner products (2.69).

Afterwards, the Siegel gauge is imposed on the heavy field:

$$b_0(\bar{P}\Psi) = 0 \quad \Longrightarrow \quad R_\uparrow = 0. \tag{6.10}$$

Then, the gauge fixed equations of motion are:

$$\mathcal{E}_{\text{gf},\varphi}(\Psi_{\text{gf}}) = Q\varphi + PV'(\Psi_{\text{gf}}), \tag{6.11a}$$
$$\mathcal{E}_{\text{gf},R_\downarrow}(\Psi_{\text{gf}}) = c_0 L_0 R_\downarrow + \bar{P}\bar{\Pi}_s V'(\Psi_{\text{gf}}). \tag{6.11b}$$

Effective action The propagator (also called contracting homotopy operator) in Siegel gauge was given in (2.76):

$$\Delta := \frac{b_0}{L_0}, \qquad \{Q, \Delta\} = \bar{P}_0, \tag{6.12}$$

where $\bar{P}_0 = 1 - P_0$ and P_0 is the projector on on-shell states (6.6). It can be used to invert the linear term in the equation of motion for $R_\downarrow$:

$$\mathcal{E}_{\text{gf},R_\downarrow}(\Psi_{\text{gf}}) = 0 \quad \Longrightarrow \quad R_\downarrow = -\frac{b_0}{L_0}\,\bar{P}V'(\varphi + R_\downarrow). \tag{6.13}$$

This equation must be solved to obtain $R_\downarrow = R_\downarrow(\varphi)$. Plugging the solution inside the gauge-fixed action gives:

$$S_{\text{eff}} = \frac{1}{2}\,\langle\varphi, Q\varphi\rangle + \langle\varphi, PV(\Psi_{\text{eff}})\rangle + \left\langle \bar{\Pi}_s V'(\Psi_{\text{eff}}), \frac{b_0}{L_0}\,\bar{P}\left(\frac{V'(\Psi_{\text{eff}})}{2} - V(\Psi_{\text{eff}})\right)\right\rangle \tag{6.14}$$

where $\Psi_{\text{eff}} := \varphi + R_\downarrow(\varphi)$.

There is one point to be careful of: L_0 can be singular in $\bar{P}\mathcal{H}$. But, this has to be expected for an effective theory which is always accompanied with a momentum cut-off corresponding to the mass of the lightest state which has been integrated out:

$$\alpha' p^2 \ll \min_{\bar{P}\mathcal{H}} \widehat{L}_0. \tag{6.15}$$

Then, the action (6.14) can be expanded in powers of α' or equivalently k^2 (higher-derivative expansion) using:

$$\frac{1}{L_0} = \frac{1}{\alpha' p^2 + \widehat{L}_0} = \sum_{n\geq 0} (-1)^n \alpha'^n \, \frac{p^{2n}}{\widehat{L}_0^{n+1}}. \tag{6.16}$$

We use the momentum operator to show that it acts on any state on the right of L_0^{-1}.

The out-of-Siegel gauge constraints ($\sim$ Gauss constraints)

$$\mathcal{E}_{\text{gf},R_\uparrow}(\Psi_{\text{eff}}) = \widehat{Q} R_\downarrow + \bar{P} \Pi_s V'(\Psi_{\text{eff}}) \tag{6.17}$$

cannot be derived from effective action (6.14) and must be enforced on the side. However, it can be shown using the constraints from the A_∞ or L_∞ algebra that it holds identically if the light fields satisfy their equations of motion:

$$\mathcal{E}_{\text{gf},\varphi} = 0 \quad \Longrightarrow \quad \mathcal{E}_{\text{gf},R_\uparrow} = 0. \tag{6.18}$$

A solution for $R_\downarrow$ can be found perturbatively. Expanding the fields and the potential as:

$$\varphi = \sum_{n\geq 1} \mu^n \varphi_n, \qquad R_\downarrow = \sum_{n\geq 1} \mu^n R_n, \qquad V' = \sum_{n\geq 2} \mu^n V'_n \tag{6.19}$$

where $\mu \ll 1$ is the perturbative parameter, we can solve order by order in μ. After resummation, we have (taking an L_∞ structure to fix the ideas):

$$\begin{aligned} R_\downarrow(\varphi) &= -\frac{1}{2} \frac{b_0}{L_0} \bar{P} \ell_2(\varphi^2) + \frac{1}{2} \frac{b_0}{L_0} \bar{P} \ell_2 \left(\varphi, \frac{b_0}{L_0} \bar{P} \ell_2(\varphi^2) \right) \\ &\quad - \frac{1}{3!} \frac{b_0}{L_0} \bar{P} \ell_3(\varphi^3) + O(\varphi^4), \\ \varphi &= \mu \varphi_1 + \mu^2 \varphi_2 + O(\mu^3). \end{aligned} \tag{6.20}$$

Plugging in the effective action (6.14) gives:

$$\begin{aligned} S_{\text{eff}} = \frac{1}{2} \langle \varphi, Q\varphi \rangle &+ \frac{1}{3!} \langle \varphi, P\ell_2(\varphi^2) \rangle + \frac{1}{4!} \langle \varphi, P\ell_3(\varphi^3) \rangle \\ &- \frac{1}{8} \left\langle \bar{\Pi}_s \ell_2(\varphi^2), \frac{b_0}{L_0} \bar{P} \ell_2(\varphi^2) \right\rangle + O(\varphi^5). \end{aligned} \tag{6.21}$$

In full generality, the action and equation of motion

$$S_{\text{eff}} := \frac{1}{2} \langle \varphi, Q\varphi \rangle + \langle \varphi, \tilde{V}(\varphi) \rangle, \tag{6.22a}$$

$$\mathcal{E}_{\text{eff}} := Q\varphi + \tilde{V}'(\varphi), \tag{6.22b}$$

are written in terms of an effective potential $\tilde{V}$ defined in terms of new string products $\tilde{m}_n$ or $\tilde{\ell}_n : P\mathcal{H}^{\otimes n} \to P\mathcal{H}$:

$$\begin{aligned} A_\infty: &\quad \tilde{V}(\Psi) = \sum_{n\geq 2} \frac{1}{n+1} \, \tilde{m}_n(\Psi^n), \\ L_\infty: &\quad \tilde{V}(\Psi) = \sum_{n\geq 2} \frac{1}{(n+1)!} \, \tilde{\ell}_n(\Psi^n). \end{aligned} \tag{6.23}$$

In the previous example, the effective products are:

$$\begin{aligned}
\tilde{\ell}_1(A_1) = PQA_1, \qquad \tilde{\ell}_2(A_1, A_2) = P\ell_2(A_1, A_2),& \\
\tilde{\ell}_3(A_1, A_2, A_3) = P\ell_3(A_1, A_2, A_3) - P\ell_2\left(A_1, \frac{b_0}{L_0}\bar{P}\ell_2(A_2, A_3)\right)& \\
- (-1)^{A_1(A_2+A_3)}\, P\ell_2\left(A_2, \frac{b_0}{L_0}\bar{P}\ell_2(A_3, A_1)\right)& \\
- (-1)^{A_3(A_1+A_2)}\, P\ell_2\left(A_3, \frac{b_0}{L_0}\bar{P}\ell_2(A_1, A_2)\right).&
\end{aligned} \tag{6.24}$$

It can be checked that they solve the first three L_∞ conditions (2.95). This implies the invariance of the action under the effective gauge transformation:

$$\delta_\lambda\varphi = Q\lambda + \tilde{\ell}_2(\varphi, \lambda) + \frac{1}{2}\,\tilde{\ell}_3(\varphi^2, \lambda) + O(\varphi^3), \qquad \bar{P}\lambda = 0 \tag{6.25}$$

Hence, it is natural to conjecture that the effective action also enjoys an effective A_∞ or L_∞ structure. This can be proven using the coalgebra language introduced below (Section 6.1.2).

Auxiliary fields The light string field φ contains both physical and auxiliary spacetime fields. To obtain an effective action only for the physical fields, one can either gauge fix (sets the fields to zero) or integrate them out. The advantage of the second approach is to preserve the gauge invariance of the action. To simplify the discussion, we consider the projector (6.5) on the massless states in the open bosonic string.

The massless fields of the open string are the gauge field A_μ and the Nakanishi–Lautrup (NL) field B associated to the following vertex operators:

$$\varphi_A := \frac{\sqrt{2}}{\alpha'}\, A_\mu(k)\, c\, \mathrm{i}\partial X^\mu \mathrm{e}^{\mathrm{i}k\cdot X} \in \Pi_s P\mathcal{H}, \qquad \varphi_B := \frac{B(k)}{\sqrt{2}}\, \partial c\, \mathrm{e}^{\mathrm{i}k\cdot X} \in \bar{\Pi}_s P\mathcal{H}. \tag{6.26}$$

The field φ_A is primary only if A is transverse, $k \cdot A = 0$, while the field φ_B is never primary. The on-shell condition for both fields is $k^2 = 0$. The free action is easily evaluated by inserting $\varphi = \varphi_A + \varphi_B$ in (2.67):

$$S_{\text{free}} = \frac{1}{2}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\Big[A_\mu(k)k^2 A_\mu(-k) - B(k)B(-k) + 2k\cdot A(k)B(-k)\Big]. \tag{6.27}$$

The Siegel gauge condition plus the associated constraint read:

$$\begin{cases} b_0\varphi_B = 0, \\ \widehat{Q}\varphi_A = 0, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} B(k) = 0, \\ k\cdot A(k) = 0. \end{cases} \tag{6.28}$$

The action (6.27) becomes the gauge-fixed Maxwell equation. In order to keep the gauge invariance, it is better to integrate out the NL field.

The NL field can be integrated out by solving its equation of motion:

$$\Pi_s \mathcal{E}_{\text{eff}} = -b_0 M_+ \varphi_B + \widehat{Q}\varphi_A + PV'(\varphi_A + \varphi_B) = 0, \tag{6.29}$$

where the first equality follows from the decomposition (A.47). Using the properties (A.50) of the $\mathrm{SU}(1,1)$ algebra, the inverse of the kinetic operator $b_0 M_+$ for φ_B is:

$$g := -c_0 M_- \widehat{P}_0, \tag{6.30}$$

and we note that

$$g = \frac{1}{2}\, c_0 b_1 b_{-1}\, \widehat{P}_0. \tag{6.31}$$

This propagator is algebraic, which makes sense since the kinetic term is also algebraic. For this reason, it is not necessary to fix a gauge contrary to what we did for the heavy fields. Moreover, the operator for the NL field does not depend on the matter (except for the exponential), which is reflected in the absence of matter operators in the propagator. Using this propagator, the equation can be rewritten as:

$$\varphi_B = -c_0 M_- \Big(\widehat{Q}\varphi_A + PV'(\varphi_A + \varphi_B)\Big). \tag{6.32}$$

This solution is linear in the gauge field: $\varphi_B = O(\varphi_A)$.

However, there is a better approach: it is not convenient to work with a non-primary operator for the gauge field. Making a field redefinition of the NL field

$$\beta(k) := B(k) - k \cdot A(k), \tag{6.33}$$

gives the following operators:

$$\tilde{\varphi}_A := \frac{A_\mu(k)}{\sqrt{2}} \left(\frac{2}{\alpha'}\, c\, \mathrm{i}\partial X^\mu + k^\mu\, \partial c\right) \mathrm{e}^{\mathrm{i}k\cdot X}, \qquad \varphi_\beta := \frac{\beta(k)}{\sqrt{2}}\, \partial c\, \mathrm{e}^{\mathrm{i}k\cdot X}, \tag{6.34}$$

Another motivation for this field redefinition is that β is gauge invariant under linearized gauge transformations.

In fact, this field redefinition can be implemented by introducing a new projector Π_{NL}: we have seen that the anti-commutator of the kinetic operator and of the propagator gives a projector. Making this computation for $\tilde{m}_1 := \widehat{P}_0 Q$ and g gives:

$$\{\tilde{m}_1, g\} = 1 - \Pi_{\mathrm{NL}}, \qquad \Pi_{\mathrm{NL}} := \Pi_s - c_0 W = (b_0 + W)c_0. \tag{6.35}$$

The Siegel gauge projector is modified by an extra term W given by:

$$W := [\widehat{Q}, M_-] = -\frac{1}{2}\, b_{-1} L^{\mathrm{m}}_1 + \cdots = -\sqrt{\frac{\alpha'}{2}}\, k\cdot\alpha_1 b_{-1} + \cdots \tag{6.36}$$

where the dots indicate terms which does not contribute when acting on φ. We have as expected:

$$\tilde{\varphi}_A = \Pi_{\mathrm{NL}}\varphi, \qquad \varphi_\beta = \bar{\Pi}_{\mathrm{NL}}\varphi. \tag{6.37}$$

The free action with these new fields is:

$$S_{\mathrm{free}} = \frac{1}{2}\int \frac{\mathrm{d}^D k}{(2\pi)^D} \Big[A_\mu(k)(k^2\,\eta^{\mu\nu} - k^\mu k^\nu)A_\nu(-k) - \beta(k)\beta(-k)\Big]. \tag{6.38}$$

One advantage of this form is to recover a covariant form. The equation of motion for β starts quadratically in $\tilde{\varphi}_A$:

$$\varphi_\beta = O(\tilde{\varphi}_A^2). \tag{6.39}$$

This simplifies the computation of the action after integrating out the NL field. However, like when integrating out heavy fields, it is difficult to include interactions. The latter are more easily describe using the coalgebra language introduced in the next subsection.

6.1.2 Coalgebra description

Coalgebras and coderivations The coalgebra language has several advantages:

- no need for explicit field decomposition;
- optimal projector clearly characterized;
- packaged perturbative expansion and effective interactions;
- manifest effective L_∞ structure;
- perform both projections at the same time ("horizontal composition");
- deformation (e.g. Ellwood invariant) combined directly with projection ("vertical composition").

However, this language is much more mathematical and introducing everything would take more space than can be dedicated in this thesis. I will outline only the key points and the interested reader is referred to the literature for more details, for example [83, 267–270, 278].

Let's introduce the tensor product Hilbert space $T\mathcal{H}$

$$T\mathcal{H} := \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} + \cdots , \tag{6.40}$$

and projections π_k from this space to the kth tensor product:

$$\pi_k : T\mathcal{H} \to \mathcal{H}^{\otimes k}. \tag{6.41}$$

A *coderivation* embeds the A_∞ products $m_n : \mathcal{H}^{\otimes n} \to \mathcal{H}$ inside $T\mathcal{H}$:

$$\begin{gathered} \boldsymbol{m}_n : T\mathcal{H} \to T\mathcal{H}, \qquad \boldsymbol{m} := \sum_{n\geq 1} \boldsymbol{m}_n \\ \boldsymbol{m}_n \pi_N = \sum_{k=0}^{N-n} 1_{\mathcal{H}}^{\otimes (N-n-k)} \otimes m_n \otimes 1_{\mathcal{H}}^{\otimes k}. \end{gathered} \tag{6.42}$$

In fact, coderivations $\boldsymbol{X}$ and $\boldsymbol{A}$ can be introduced for any operator $X : \mathcal{H} \to \mathcal{H}$ and any state $A \in \mathcal{H}$ such that:

$$\pi_1 \boldsymbol{X} \pi_1 := X, \qquad \pi_1 \boldsymbol{A} \pi_0 := A. \tag{6.43}$$

Then, the A_∞ relations take the very simple form:

$$[\boldsymbol{m}, \boldsymbol{m}] = 0. \tag{6.44}$$

A *group-like element* is an element such that:

$$\frac{1}{1-A} = 1_{T\mathcal{H}} + A + A^{\otimes 2} + \cdots \tag{6.45}$$

where $A \in \mathcal{H}$.

Introducing a symplectic form

$$\omega : \mathcal{H}^{\otimes 2} \to \mathbb{C}, \tag{6.46}$$

the action can be written as:

$$S = \int_0^1 \mathrm{d}t\, \omega \left(\pi_1 \boldsymbol{\partial}_t \frac{1}{1-\Psi(t)} \otimes \pi_1 \boldsymbol{m} \frac{1}{1-\Psi(t)} \right) \tag{6.47}$$

where $\boldsymbol{\partial}_t$ is the coderivation associated to ∂_t and $\Psi(t)$ is an interpolating fields such that:

$$\Psi(1) = \Psi, \qquad \Psi(0) = 0. \tag{6.48}$$

The equation of motion is:

$$\pi_1 \boldsymbol{m} \frac{1}{1 - \Psi(t)} = 0, \tag{6.49}$$

while the gauge transformation reads:

$$\delta_\Lambda \frac{1}{1 - \Psi} = [\boldsymbol{m}, \boldsymbol{\Lambda}] \frac{1}{1 - \Psi} n \tag{6.50}$$

where $\boldsymbol{\Lambda}$ is the coderivation for Λ.

Homological perturbation lemma Given an interacting theory described as a perturbation of a free theory, the *homological perturbation lemma* [279] describes the theory obtained after peforming a projection.

The free theory is encoded as strong deformation retract which is defined by:

- a vector space $T\mathcal{H}$;
- a differential corresponding to the BRST charge $\boldsymbol{Q} = \boldsymbol{m}_1$;
- a contracting operator corresponding to the free propagator $\boldsymbol{\Delta}$;
- a projector $\boldsymbol{P}$.

Then, the interactions correspond to a perturbation of the free theory such that:

- $\delta\boldsymbol{m}$ is the perturbation, giving interactions $\boldsymbol{m}_n$ for $n \geq 2$;
- $\boldsymbol{m} = \boldsymbol{m}_1 + \delta\boldsymbol{m}$ is the full differential;
- $\boldsymbol{\eta}$ is the full contracting operator;
- $\boldsymbol{\Pi}$ is the full projector.

Consistency conditions on $\boldsymbol{Q}$ and $\boldsymbol{m}$ imply:

$$[\boldsymbol{P}, \boldsymbol{Q}] = 0, \qquad \boldsymbol{Q}^2 = 0, \qquad [\boldsymbol{\Pi}, \boldsymbol{m}] = 0, \qquad \boldsymbol{m}^2 = 0. \tag{6.51}$$

The gauge-fixing conditions and Hodge–Kodaira decomposition correspond to the statement that the field is annihilated by an appropriate projector (hidden in the propagator) and that the propagator inverts the kinetic operator outside the kernel of the projector:

$$\begin{aligned} \boldsymbol{\Delta}\Psi &= 0, & [\boldsymbol{Q}, \boldsymbol{\Delta}] &= 1 - \boldsymbol{P}, & \boldsymbol{P}\boldsymbol{\Delta} &= \boldsymbol{\Delta}\boldsymbol{P} = \boldsymbol{\Delta}^2 = 0 \\ \boldsymbol{\eta}\Psi &= 0, & [\boldsymbol{m}, \boldsymbol{\eta}] &= 1 - \boldsymbol{\Pi}, & \boldsymbol{\Pi}\boldsymbol{\eta} &= \boldsymbol{\eta}\boldsymbol{\Pi} = \boldsymbol{\eta}^2 = 0. \end{aligned} \tag{6.52}$$

The projection does not modify $\boldsymbol{Q}$ since they commute, however, it modifies the interactions (as seen earlier, this corresponds to effective interactions). The deformation corresponding to the new interactions is denoted by $\delta\tilde{\boldsymbol{m}}$ such that $\tilde{\boldsymbol{m}} = \boldsymbol{Q} + \delta\tilde{\boldsymbol{m}}$. These different relations are summarized as:

$$\begin{tikzcd} \boldsymbol{\Delta} \circlearrowright (T\mathcal{H}, \boldsymbol{Q}) \arrow[r, "\boldsymbol{P}"] \arrow[d, "\delta\boldsymbol{m}"'] & (TP\mathcal{H}, \boldsymbol{Q}) \arrow[d, "\delta\tilde{\boldsymbol{m}}"'] \\ \boldsymbol{\eta} \circlearrowright (T\mathcal{H}, \boldsymbol{m}) \arrow[r, "\boldsymbol{\Pi}"] & (T\Pi\mathcal{H}, \tilde{\boldsymbol{m}}) \end{tikzcd} \tag{6.53}$$

The homological perturbation lemma allows to write the full propagator $\boldsymbol{\eta}$, the perturbation after projection $\delta\tilde{\boldsymbol{m}}$, and the projector after perturbation Π in terms of the rest of the data:

$$\begin{gathered}\delta\tilde{\boldsymbol{m}} = \boldsymbol{P}\delta\boldsymbol{m}\,\frac{1}{1+\boldsymbol{\Delta}\delta\boldsymbol{m}}, \qquad \boldsymbol{\eta} = \boldsymbol{\Delta} - \boldsymbol{\Delta}\delta\tilde{\boldsymbol{m}}\boldsymbol{\Delta}, \\ \boldsymbol{\Pi} = \frac{1}{1+\boldsymbol{\Delta}\delta\boldsymbol{m}}\,\boldsymbol{P}\,\frac{1}{1+\delta\boldsymbol{m}\boldsymbol{\Delta}}.\end{gathered} \tag{6.54}$$

This can be applied directly to the situation from Section 6.1.1, for integrating out heavy fields. The procedure can be summarized by the following diagram:

$$\begin{tikzcd} \text{free theory} \arrow[r, "\boldsymbol{P}"] \arrow[d, "\delta\boldsymbol{m}"'] & \text{free effective theory} \arrow[d, "\delta\tilde{\boldsymbol{m}}"'] \\ \text{interacting theory} \arrow[r, "\boldsymbol{\Pi}"] & \text{interacting effective theory} \end{tikzcd} \tag{6.55}$$

The effective interactions are read by expanding $\delta\tilde{\boldsymbol{m}}$ given in (6.54).

Horizontal composition We wish to perform two successive projections P_1 and P_2 which is described by the diagram:

$$\begin{tikzcd} \boldsymbol{\Delta}_1 \circlearrowright (T\mathcal{H},\, \boldsymbol{Q}) \arrow[r, "\boldsymbol{P}_1"] \arrow[d, "\delta\boldsymbol{m}"'] & \boldsymbol{\Delta}_2 \circlearrowright (TP_1\mathcal{H},\, \boldsymbol{Q}) \arrow[r, "\boldsymbol{P}_2"] \arrow[d, "\delta\tilde{\boldsymbol{m}}"'] & (TP_2P_1\mathcal{H},\, \boldsymbol{Q}) \arrow[d, "\delta\boldsymbol{m}'"'] \\ \boldsymbol{\eta}_1 \circlearrowright (T\mathcal{H},\, \boldsymbol{m}) \arrow[r, "\boldsymbol{\Pi}_1"] & \boldsymbol{\eta}_2 \circlearrowright (T\Pi_1\mathcal{H},\, \tilde{\boldsymbol{m}}) \arrow[r, "\boldsymbol{\Pi}_2"] & (T\Pi_2\Pi_1\mathcal{H},\, \boldsymbol{m}') \end{tikzcd} \tag{6.56}$$

Using the homological perturbation lemma twice, it is possible to prove that both projections are equivalent to a single projection:

$$\begin{tikzcd} \boldsymbol{\Delta}_{12} \circlearrowright (T\mathcal{H},\, \boldsymbol{Q}) \arrow[r, "\boldsymbol{P}_{12}"] \arrow[d, "\delta\boldsymbol{m}"'] & (TP_{12}\mathcal{H},\, \boldsymbol{Q}) \arrow[d, "\delta\tilde{\boldsymbol{m}}"'] \\ \boldsymbol{\eta}_{12} \circlearrowright (T\mathcal{H},\, \boldsymbol{m}) \arrow[r, "\boldsymbol{\Pi}_{12}"] & (T\Pi_{12}\mathcal{H},\, \tilde{\boldsymbol{m}}) \end{tikzcd} \tag{6.57}$$

where

$$\begin{aligned} P_{12} &= P_2P_1, & \Pi_{12} &= \Pi_2\Pi_1, \\ \Delta_{12} &= \Delta_1 + \Delta_2P_1, & \eta_{12} &= \eta_1 + \eta_2\Pi_1. \end{aligned} \tag{6.58}$$

The simplest application is for integrating out the NL field (Section 6.1.1). In this case, $P_1 = \widehat{P}_0$ and $P_2 = \Pi_{\text{NL}}$ project out heavy and NL fields respectively. The propagator and lowest vertices for the physical field $\tilde{\varphi}_A$ are:

$$\begin{aligned} \Delta_{\text{eff}} &:= \frac{b_0}{L_0} + c_0M^-\widehat{P}_0, \\ m_1'(A_2) &:= \Pi_{\text{NL}}\widehat{P}_0\,QA_1, \\ m_2'(A_1, A_2) &:= \Pi_{\text{NL}}\widehat{P}_0\,m_2(A_1, A_2), \\ m_3'(A_1, A_2, A_3) &:= \Pi_{\text{NL}}\widehat{P}_0\,m_3(A_1, A_2, A_3) + \Pi_{\text{NL}}\widehat{P}_0\,m_2\Big(\Delta_{\text{eff}}m_2(A_1, A_2), A_3\Big) + \cdots \end{aligned} \tag{6.59}$$

We note that, at zero momentum, these formulas provide an explicit realization of the minimal model since $m_1' = 0$ ($\Pi_{\text{NL}}P$ maps to the zero-momentum cohomology).

Vertical composition We are now interested in performing two successive deformations $\delta\boldsymbol{m}_1$ and $\delta\boldsymbol{m}_2$:

$$
\begin{tikzcd}
\boldsymbol{\Delta} \circlearrowright (T\mathcal{H},\, \boldsymbol{Q}) \arrow[r, "\boldsymbol{P}"] \arrow[d, "\delta\boldsymbol{m}_1"'] & (TP\mathcal{H},\, \boldsymbol{Q}) \arrow[d, "\delta\tilde{\boldsymbol{m}}_1"'] \\
\boldsymbol{\eta}_1 \circlearrowright (T\mathcal{H},\, \boldsymbol{m}) \arrow[r, "\boldsymbol{\Pi}_1"] \arrow[d, "\delta\boldsymbol{m}_2"'] & (T\Pi_1\mathcal{H},\, \tilde{\boldsymbol{m}}) \arrow[d, "\delta\tilde{\boldsymbol{m}}_2"'] \\
\boldsymbol{\eta}_2 \circlearrowright (T\mathcal{H},\, \boldsymbol{M}) \arrow[r, "\boldsymbol{\Pi}_2"] & (T\Pi_2\mathcal{H},\, \tilde{\boldsymbol{M}})
\end{tikzcd}
\tag{6.60}
$$

Using again the homological perturbation lemma, this is equivalent to a single deformation

$$
\begin{tikzcd}
\boldsymbol{\Delta} \circlearrowright (T\mathcal{H},\boldsymbol{Q}) \arrow[r, "\boldsymbol{P}"] \arrow[d, "\delta\boldsymbol{m}_{12}"'] & (TP\mathcal{H},\boldsymbol{Q}) \arrow[d, "\delta\tilde{\boldsymbol{m}}_{12}"'] \\
\boldsymbol{\eta}_2 \circlearrowright (T\mathcal{H},\boldsymbol{M}) \arrow[r, "\boldsymbol{\Pi}_2"] & (T\Pi_2\mathcal{H},\tilde{\boldsymbol{M}})
\end{tikzcd}
\tag{6.61}
$$

where

$$\delta\boldsymbol{m}_{12} = \delta\boldsymbol{m}_1 + \delta\boldsymbol{m}_2, \qquad \delta\tilde{\boldsymbol{m}}_{12} = \delta\tilde{\boldsymbol{m}}_1 + \delta\tilde{\boldsymbol{m}}_2. \tag{6.62}$$

The natural application is deforming Witten's SFT by the Ellwood invariant [280–282].[1] We consider only the zero-momentum case.

The Ellwood invariant corresponds to an open string 1-point function

$$E[\Psi] := \langle \mathcal{V}(i) f \circ \Psi \rangle_{\text{UHP}} := \langle \Psi, e_0 \rangle \tag{6.63}$$

where $\mathcal{V}$ is an on-shell closed string state inserted at the mid-point of the string. It defines a 0-product e_0. After integrating out the massive fields, one obtains an effective invariant:

$$
\begin{aligned}
\tilde{E}[\varphi] &= \sum_{n\geq 0} \frac{1}{n+1}\, \omega\big(\varphi, \tilde{e}_n(\varphi^n)\big) = E\big[\Psi_{\text{eff}}(\varphi)\big], \\
\tilde{e}_n(\varphi^n) &= -\tilde{m}_{n+1}(-\Delta e_0, \varphi^n) + \text{perms}.
\end{aligned}
\tag{6.64}
$$

However, the UV action deformed by the Ellwood invariant does *not* become the effective action deformed by the effective invariant after integrating out the massive fields:

$$S_{\text{ell}}[\Psi] = S[\Psi] + \lambda\, E[\Psi], \qquad S_{\text{eff,ell}}[\varphi] = S_{\text{eff}}[\varphi] + \lambda\, \tilde{E}[\varphi] + O(\lambda^2). \tag{6.65}$$

The reason is that the combined effective interactions and invariant do not satisfy the A_∞ relations.

The 0-product gives a classical tadpole, implying that the field must be shifted [58]. Interestingly, the obstruction to the vacuum shift of full SFT is given by the massless equation of motion of the effective SFT. In simple cases, it reduces to the tadpole in $S_{\text{eff,ell}}$.

Defining the two perturbations to be:

$$\delta\boldsymbol{m}_1 = \boldsymbol{m}_2 + \cdots, \qquad \delta\boldsymbol{m}_2 = \lambda\boldsymbol{e}, \tag{6.66}$$

computing the effective vertices in two times or using vertical composition give two formulas:

$$\tilde{\boldsymbol{M}} = \boldsymbol{P}(\boldsymbol{m} + \lambda\boldsymbol{e})\, \frac{1}{1 + \boldsymbol{\Delta}(\boldsymbol{m} + \lambda\boldsymbol{e} - \boldsymbol{Q})} = \tilde{\boldsymbol{m}} + \lambda\, \boldsymbol{\Pi}_1 \boldsymbol{e}\, \frac{1}{1 + \lambda\boldsymbol{\eta}_1\boldsymbol{e}}. \tag{6.67}$$

[1] Similar results have been presented in [19].

It implements automatically the vacuum shift of the products. In components, we have:

$$\tilde{M}_n(A_1, \ldots, A_n) = \sum_{k \geq 0} \tilde{m}_{n+k}\Big((-\Delta e_0)^k, A_1, \ldots, A_n\Big) + \text{perms} \tag{6.68}$$

6.2 Localization and Yang–Mills effective action

In this section, we compute the quartic potential of massless fields at zero-momentum in heterotic SFT. This potential was originally computed in [271] for non-Abelian gauge fields. The commutator-squared term from Yang–Mills theory is recovered thanks to a cancellation between the 4-point fundamental interaction and the effective interaction obtained by integrating out the massive fields. In [148, 149] (see also [277]), an elegant rephrasing of this result was obtained by showing how the effective 4-point interaction localizes on the boundary of the moduli space, corresponding to a single contribution given by the exchange of massive fields with an infinitely long propagator. Our goal is to show how this generalizes to the case of the heterotic SFT.

We will introduce only the most important notations and refer the reader to the literature for more details on superstrings (for example [3]).

6.2.1 Large Hilbert space SFT

The heterotic superstring worldsheet theory is described by two inequivalent holomorphic and anti-holomorphic sectors: the first sector has $N = 1$ supersymmetry while the second is bosonic. As a consequence, there is an additional pair of commuting holomorphic ghosts $\beta(z)$ and $\gamma(z)$ associated to local supersymmetry. They are bosonized into a set of 3 fields $\xi(z)$, $\eta(z)$ and $\phi(z)$ as:

$$\gamma = \eta\, \mathrm{e}^{\phi}, \qquad \beta = \partial\xi\, \mathrm{e}^{-\phi}. \tag{6.69}$$

(ϕ should not be confused with the Liouville field from Chapter 5). The fields ξ and η are anti-commuting ghosts with weights 0 and 1, while ϕ is a Coulomb gas (Appendix A.1.2). Since the bosonization uses only $\partial\xi$ and not ξ, the zero-mode

$$\xi_0 = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\xi(z)}{z} \tag{6.70}$$

is not a good operator to write states in the Hilbert space – called the *small Hilbert space* – defined by $\beta(z)$ and $\gamma(z)$. Including the zero-mode leads to the large Hilbert space.

In the WZW formulation of SFT, the string field Φ lives in the large Hilbert space. This description is due to Berkovits for the open superstring [7] and was later extended to the heterotic superstring [23, 24]. It provides a construction of the superstring SFT by dressing the bosonic products with insertions of Q and η_0:

$$\eta_0 = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\, \eta(z). \tag{6.71}$$

The action up to fourth order in the field reads:

$$\begin{aligned} S = \frac{1}{2}\langle\eta_0\Phi, Q\Phi\rangle + \frac{1}{3!}\langle\eta_0\Phi, \ell_2(\Phi, Q\Phi)\rangle + \frac{1}{4!}\big\langle\eta_0\Phi, \ell_3(\Phi, (Q\Phi)^2)\big\rangle \\ + \frac{1}{4!}\big\langle\eta_0\Phi, \ell_2\big(\Phi, \ell_2(\Phi, Q\Phi)\big)\big\rangle + O(\Phi^5), \end{aligned} \tag{6.72}$$

where ℓ_2 and ℓ_3 are the products from the closed bosonic string and the inner product is defined as

$$\langle A, B\rangle := \langle A|\, \xi_0 c_0^- \,|B\rangle\,. \tag{6.73}$$

Note that the gauge invariance of this action does not have an L_∞ structure.

6.2.2 Quartic action

Following the approach from Section 6.1.1, we find the following effective action for the massless field φ (in the large Hilbert space):

$$S_{\text{eff}} = \frac{1}{2}\,\langle \eta_0\varphi, Q\varphi\rangle + \frac{1}{3!}\,\langle \eta_0\varphi, \ell_2(\varphi, Q\varphi)\rangle + S^{(4)}_{\text{eff}} + O(\Phi^5), \tag{6.74}$$

where the quartic interaction reads:

$$\begin{aligned} S^{(4)}_{\text{eff}} = \frac{1}{4!}\,\big\langle \eta_0\varphi, \ell_3(\varphi, (Q\varphi)^2)\big\rangle + \frac{1}{4!}\,\big\langle \eta_0\varphi, \ell_2\big(\varphi, \ell_2(\varphi, Q\varphi)\big)\big\rangle \\ - \frac{1}{8}\,\Big\langle \ell_2(\eta_0\varphi, Q\varphi), \xi_0\frac{b_0^+}{L_0^+}\,\bar{P}_0\ell_2(\eta_0\varphi, Q\varphi)\Big\rangle . \end{aligned} \tag{6.75}$$

The gauge fixing is more complicated because the gauge symmetry is extended in the large Hilbert space, but one can show that everything works like in Section 6.1.1.[2]

In order to proceed, we restrict to zero momentum and we assume that there is a global $N = 2$ worldsheet supersymmetry such that a $N = 1$ super-primary can be written as the sum of two short $N = 2$ super-primaries charged under R-symmetry:

$$\varphi := \varphi^+ + \varphi^-. \tag{6.76}$$

Zero-momentum fields automatically satisfy the free equation of motion $\eta_0 Q\varphi = 0$ such that there is no quadratic term. Moreover, it can also be shown that the $N = 2$ supersymmetry implies the vanishing of the cubic term, leaving only the quartic term:

$$S_{\text{eff}}(\varphi) = S^{(4)}_{\text{eff}}(\varphi) + O(\varphi^5). \tag{6.77}$$

Using ghost number and R-symmetry conservation, it reduces to:

$$\begin{aligned} S^{(4)}_{\text{eff}} &= -\,\frac{1}{8}\,\big\langle \ell_2(\varphi^-, \eta_0\varphi^-), P_0\ell_2(\varphi^+, Q\varphi^+)\big\rangle + (+ \leftrightarrow -) \\ &\quad -\,\frac{1}{8}\,\big\langle \ell_2(\varphi^-, \varphi^+), P_0\ell_2(\eta_0\varphi^-, Q\varphi^+)\big\rangle + (+ \leftrightarrow -). \end{aligned} \tag{6.78}$$

The projector $P_0 = \mathrm{e}^{-\infty L_0}$ corresponds to an infinitely long propagator connecting the two cubic vertices given by ℓ_2. From the point of view of Riemann surfaces, it corresponds to the boundary of the moduli space.

It is very interesting that the 3-product ℓ_3 disappeared completely from the effective action. Moreover, the presence of P_0 in the correlation function implies that the precise form of ℓ_2 is irrelevant because only the leading term from the OPE survives the projection:

$$P_0\ell_2(A, B) = b_0^-\,\{A_1 A_2\}_{0,0}(0,0)\,|0\rangle\,, \tag{6.79}$$

where the notation $\{A_1 A_2\}_{n,\bar{n}}$ means that one keeps the coefficient of $z^n\bar{z}^{\bar{n}}$ in the OPE. As argued before, the main obstruction to SFT computations is the difficulty to characterize explicitly the string products. Hence, localization simplifies tremendously the computations by removing unnecessary off-shell data.

Replacing the 2-product by the OPE leading term means that the quartic effective action reduces to a sum of 2-point functions. Writing the string field as

$$\varphi = c\bar{c}\,V_{\frac{1}{2},1}\,\xi\mathrm{e}^{-\phi}, \tag{6.80}$$

[2] There is a subtelty related to the ghost-dilaton [283–285] which we ignore.

where $V_{\frac{1}{2},1}$ is a weight $(\frac{1}{2},1)$-primary matter operator, the quartic effective action reads:

$$s_{\text{eff}}^{(4)} = \frac{1}{4}\,\langle \mathbb{H}_{1,1}^{+}, \mathbb{H}_{1,1}^{-}\rangle + \frac{1}{4}\,\langle \mathbb{H}_{0,1}, \mathbb{H}_{0,1}\rangle, \tag{6.81}$$

where the auxiliary fields $\mathbb{H}_{1,1}^{\pm}$ and $\mathbb{H}_{0,1}$ are defined as:

$$\begin{aligned}
\mathbb{H}_{1,1}^{\pm}(z,\bar z) &:= \lim_{\epsilon,\bar\epsilon\to 0} 2\bar\epsilon\, V_{\frac{1}{2},1}^{\pm}(z+\epsilon,\bar z+\bar\epsilon)V_{\frac{1}{2},1}^{\pm}(z-\epsilon,\bar z-\bar\epsilon),\\
\mathbb{H}_{0,1}(z,\bar z) &:= \lim_{\epsilon,\bar\epsilon\to 0} |2\epsilon|^2\, V_{\frac{1}{2},1}^{\pm}(z+\epsilon,\bar z+\bar\epsilon)V_{\frac{1}{2},1}^{\pm}(z-\epsilon,\bar z-\bar\epsilon).
\end{aligned} \tag{6.82}$$

The equations of motion (yielding flat directions of the potential) correspond to ADHM-like constraints:

$$\mathbb{H}_{1,1}^{\pm} = 0, \qquad \mathbb{H}_{0,1}^{\pm} = 0. \tag{6.83}$$

As an example, we can consider a flat background. Then, the metric and the B-field do not appear in the auxiliary fields $\mathbb{H}_{1,1}^{\pm}$ ad $\mathbb{H}_{0,1}$ because there is no first order pole in the OPE of the corresponding operators. Only the non-Abelian gauge fields remain. Then, working out the auxiliary fields and evaluating the effective action, one finds the quartic Yang–Mills potential with the correct normalization:

$$S_{\text{eff}}^{(4)} = -\frac{1}{16C}\,\mathrm{tr}[A_\mu, A_\nu]^2, \tag{6.84}$$

where C is the Dynkin index of the representation.

Chapter 7

Other results

In this chapter, I outline the results obtained in fields other than SFT (Section 1.2).

7.1 Machine learning

ML for lattice QFT In a first paper [157], we have trained a neural network to compute the Casimir energy for a scalar field in 3 dimensions with general Dirichlet boundary conditions. Training and predictions are much faster than Monte Carlo (MC): 10 minutes are required for training, and only 5 ms for predicting one sample, where MC needs 3 hours. In most cases, the neural network gives prediction with an error of the same order as the MC error. For general boundary conditions, the problem is intractable analytically and MC requires a long time, thus, ML provides an interesting alternative.

In a second paper [158], we have studied the confinement phase transition of $3d$ QED. This system provides a good toy model for QCD as it displays similar properties: confinement, generation of a mass gap and temperature phase transition. The main question is how ML can be used to extrapolate properties at different lattice sizes. We trained a neural network to predict different quantities (phase, Polyakov loop, monopole density...) from the monopole distribution for a lattice of size $(4, 16, 16)$. Then, we used the neural network to predict the same quantities for lattices of other sizes, from which we extracted the critical temperature of the phase transition. We found that it matched the MC results within a few percents.

Creativity for generating EFT Another interesting use of ML is to explore the space of QFTs. We have implemented in [156] a toy model to study the "creativity" of ML for generating effective field theories. Considering a single-field $N = 1$ superpotential, we found that a generative adversarial network (GAN) can create superpotentials with properties not found in the original training set. Noteworthy, we did not impose the holomorphicity of the superpotential ourselves but the network encoded itself this property. This shows that neural networks can be used to propose new models, while also understanding consistency requirements.

7.2 Two-dimensional gravity

Spectrum of the Mabuchi theory In [160, 161], we identified the physical states associated with the Mabuchi action through a minisuperspace analysis. We found that both the Mabuchi and Liouville theories share the same spectrum since the Hamiltonians coincide in this approximation. This result is essential because knowing the spectrum is the starting point for computing correlation functions. It is also surprising since both theories are different at the full quantum level (they have different string susceptibilities, Liouville is conformal while Mabuchi is not).

Degrees of freedom in $2d$ gravity While this is a simple exercise, the degrees of freedom of classical $2d$ gravity both with conformal and non-conformal matter have never been completely analyzed. We have shown in [162] that it displays some interesting and uncommon properties. In particular, theories where the matter action is Weyl invariant has generically more degrees of freedom than actions without the invariance because the equation of motion of the metric are always Weyl invariant. Moreover, we found that, in models with scalar fields, solutions to the equations of motion are typically trivial or inconsistent.

7.3 Tensor and SYK models

Conformal invariance of SYK at NLO In [163], we investigated the near-conformal invariance of the SYK and tensor models at the next-to-leading order (NLO) in the $1/N$-expansion. We could show that the NLO 2-point function is proportional to the leading order 2-point function, and thus that the near-conformal invariance is preserved. Our work was the first to study the explicit properties of subleading corrections.

Constructive analysis of quartic vector models We used the loop vertex expansion in [**Erbin:2019:ConstructiveExpansionQuartic**] to prove in a unified manner the analyticity and Borel summability of general quartic vector models (corresponding to models from Section 2.1 with $V^{(3)} = 0$) – bosonic and fermionic, relativistic and non-relativistic – in low dimensions. As a by-product, we proved these properties for the SYK model (large N and quenched disorder). One of the motivations is to study toy models of SFT, following the general ideas described in Section 2.5 and chapter 4, and trying to study more and more complex models.

7.4 Black holes and supergravity

Domain walls in CSO-gauged supergravity In [164], we proved the completeness of the existing classification of domain walls in CSO-gauged supergravities (including a large class of various maximal and half-maximal supergravities in different dimensions).

Universality of the Hawking radiation We have established in [165] the universality of the Hawking temperature within the Hamilton–Jacobi tunnelling method,[286] in agreement with the other known methods. Indeed, it requires to choose a background and a specific particle to compute the Hawking radiation, and there is no hint of background and spin independence. It has been checked case by case in many papers, but no general proof had been given before.

Chapter 8

Perspectives

8.1 Consistency of string theory

My objective is to bring string theory to the same level as point particle QFTs with respect to the study of fundamental properties, in order to establish that it is a completely consistent theory of unification and quantum gravity. As described in Chapter 4, analyticity plays an important role in this respect. This first set of projects amounts to push further the study of analyticity and to study its consequences, in particular, concerning locality and the CPT theorem. Moreover, we have seen that massless particles fall outside the scope of most methods, and the question is how to accommodate them. A second line of research consists in continuing the study of background independence of the open-closed super-SFT.

Analyticity and locality In Chapter 4 and [151], we have proved analyticity of n-point superstring amplitudes in a subset of the primitive domain. It has been extended to the full primitive domain Δ_D in (4.10) for 4-point amplitudes, and partially for 5-point amplitudes [178].

The first step is to extend the analyticity of all n-point amplitudes from Δ_2 to Δ_D. This could be done using the theory of functions of several complex variables, however, it would also be interesting to find a proof using Feynman diagrams directly. Next, one would need to find the full envelope of holomorphy $\mathcal{H}(\Delta_D)$ for n-point amplitudes. An intermediate step would be to find the domain which allows to prove crossing symmetry. However, analytic extensions of the primitive domain and crossing symmetry for n-point amplitudes are still open problems in local QFT [203], which means that it is likely a difficult and long-term project.

In any case, the current knowledge of analyticity may already be used for studying locality of n-point amplitudes. Indeed, analyticity in momentum space implies certain behavior in position space, which can be used to assess locality [205, app. 2.B]. Since we have obtained analyticity conditions perturbatively, this would only say something about locality at the perturbative level. Hence, even if this analysis showed – counter-intuitively – that string theory is local, non-perturbative effects could break locality.

CPT theorem The CPT theorem [287–292] is one of the most celebrated result in QFT (see [293, 294] for reviews). It asserts that theories are invariant under charge conjugation, parity transformations (including spatial reflections) and time reversal under very general conditions. Moreover, it is related to the spin-statistics theorem which says that the spin and the bosonic or fermionic character of a particle are not independent.

Since it is one of the most general result in QFT, it would be interesting to prove it for string theory. While it has already been studied in the worldsheet formalism [295–299], I aim at giving a QFT proof using SFT (the only attempt using SFT is [300] which focuses on the open bosonic SFT). The reason is that the worldsheet theory is ill-defined as described in Chapter 1.

Following Chapter 4, I am planning to use QFT techniques to extend the proof of the CPT theorem to a general QFT which includes SFT as a particular case. The standard proof of the CPT theorem is done in position space and uses the counterpart of the primitive domain of analyticity in this space. The first step is then to translate the proof in momentum space and to assess which subspace of the primitive domain in momentum space is necessary to establish the theorem. Preliminary investigations indicate that it may correspond to the same subdomain as the one used for proving crossing symmetry.

Kulish–Faddeev method Several techniques have been designed in QFT to handle massless particles (see for example [184, 301, 302]), and have recently been again the focus of interest [303]. The usual solution is to consider inclusive cross-section, defined by summing over all amplitudes where additional soft massless particles are inserted and whose energies are below some threshold. But, this is not very satisfactory since consistency properties are usually assessed using the S-matrix (Section 2.5.2 and chapter 4). The main insight for an alternative solution is that asymptotic states cannot be free if there are massless particles because the interaction does not fall off sufficiently fast at infinity. Insisting on using eigenstates of the free theory to write amplitudes, in this context, explains the emergence of divergences, which are just saying that such processes are not physical [211, 304, 305]. The solution is to modify the asymptotic states by dressing free states with a coherent state of massless particles. Interestingly for our purpose, this method has been recently extended to gravity [306, 307]. Instead of modifying the asymptotic states, it is possible to modify the S-matrix (by changing the basis of the Hilbert space). This approach culminates in [308, 309] which describes the construction of a finite S-matrix for theory with massless particles, giving concrete examples with QED, QCD and $N = 4$ Yang–Mills.

I would like to extend this analysis to SFT. This involves dressing the string field which a coherent states of all massless states (which can be described using projectors as in Chapter 6), building on the coherent string state studied in [310–312]. The next question is how to write amplitudes, possibly modifying the S-matrix as in [308, 309]. Beyond providing a full definition of the string theory S-matrix, this may help in studying analyticity properties without removing massless states as was done in Chapter 6.

Before attacking the full SFT problem, I am planning to investigate simpler toy models. In particular, systems with both gauge fields and gravity have not been investigated. Thus, Einstein–Maxwell with a massive scalar field provides an example to study the interplay between the dressing in terms of both photons and gravitons. Then, the model can be complexified to get closer to SFT, or ideally reformulated in terms of string fields.

Ghost-dilaton theorem in superstring field theory The ghost-dilaton theorem [283–285] states that shifting the ghost-dilaton state is equivalent to a shift of the string coupling. I would like to extend this result to closed super-SFT [21].

An intermediate step has been achieved in [313] where it is shown that amplitudes with insertion of a ghost-dilaton equals the same amplitude without it but multiplied by the Euler characteristics. The next goal is to use this result to prove the theorem for the SFT action. This uses different geometrical subspaces introduced in [314] for proving background independence of closed super-SFT. However, it is not possible to follow directly the proof from [283]: the latter works with the BV quantum action and the path integral, while [314] works at the level of the 1PI action. For this reason, this involved some adaptations of the proof.

The natural continuation of this work would be to prove background independence for open-closed super-SFT. This is a longer-term project since it has not even been done for the bosonic SFT.

8.2 Computing with string field theory

Beside proving that string theory is consistent, it is also primordial to develop techniques for explicit computational with SFT.

Tree-level partition function An interesting question is whether the techniques described in Chapter 3 can be extended to compute the tree-level 1- and 0-point amplitudes on the sphere.

In most cases, the 1-point amplitude is expected to vanish since 1-point correlation functions of primary operators other than the identity vanish in unitary CFTs. The integral over the zero-mode gives a factor $\delta^{(D)}(k)$ which implies $k = 0$. At zero momentum, the time scalar X^0 is a unitary CFT. However, there can be some subtleties when considering marginal operators.

The 0-point function corresponds to the sphere partition function: the saddle point approximation to leading order relates it to the spacetime action evaluated on the classical solution ϕ_0, $Z_0 \sim \mathrm{e}^{-S[\phi_0]/\hbar}$. Since the normalization is not known and because $S[\phi_0]$ is expected to be infinite, only comparison between two spacetimes is meaningful (*à la* Gibbons–Hawking–York [315, sec. 4.1]). In particular, for Minkowski spacetime, we find naively:

$$Z_0 \sim \frac{\delta^{(D)}(0)}{\operatorname{Vol} \mathcal{K}_0}, \tag{8.1}$$

which is not well-defined. I am interested in generalizing the new gauge fixing procedure from Chapter 3 to this case. Of primary interest is the computation of the 0-point function for a background with a black hole, to read the on-shell action with all stringy corrections.

QFT with stubs Stubs, discussed below (2.89), are an important feature of string theory. But, in fact, they can also be introduced in QFT (work in progress with Victor Godet). I am currently studying the simple model from Section 2.1 with $V^{(4)} = 0$ in part, for pedagogical reasons, in part, to study some questions (such as making SFT cubic, see below) in a simpler setup. One interesting aspect is to give an explicit example demonstrating the result from [44]: SFT is a Wilsonian effective action, where the stub parameter is identified with the UV cut-off (see also [316]).

Quartic $\mathrm{SL}(2,\mathbb{C})$ string vertex The quartic vertex has been constructed numerically [8] using local coordinates induced from the minimal area prescription. With Ted Erler and Suvajit Majumder, we have started to investigate vertices constructed from local coordinates described by $\mathrm{SL}(2,\mathbb{C})$ functions instead. They are interesting because it allows to push analytic continuation much further. In particular, the boundary of the vertex region in $\mathcal{M}_{0,4}$ are parametrized by one ellipse and two limaçons of Pascal. It remains only to find local coordinates for the 4-point vertex. Since formulas can be written in closed-form, this raises the hope of characterizing analytically higher-order vertices.

Cubic closed string field theory The huge success of Witten's bosonic string field theory [6] is due to its simple form: it does not have any interaction beyond the cubic vertex, which is in turn easily constructed. A major endeavor is to obtain a cubic action for the bosonic closed SFT, but a no-go theorem [317] seemed to doom any such attempt. There is, however, a loophole in the argument: introducing auxiliary fields thanks to a Hubbard–Stratonovich transformation.

Preliminary results seem to indicate that the quartic vertex can be rewritten in terms of a cubic interaction thanks to an auxiliary field living in $\mathcal{H}^{\otimes 2}$. This last element points towards an homotopy algebra where the string field lives in $T\mathcal{H}$ defined in (6.40): the corresponding algebra is called IBL_∞ [83, 318–321]. Importantly, this gives the first formulation of a SFT with auxiliary fields within an algebraic structure (allowing for example a consistent BV quantization). It seems that the Hubbard–Stratonovich action can be written by gauge fixing the IBL_∞ action. Playing with this structure may allow to find a cubic form of the action or, at least, to find a simpler parametrization.

The Hubbard–Stratonovich representation has also several nice properties. First, there are consistent truncations of part of the auxiliary fields [322], which could be interesting to solve the equations of motion to find analytic solutions. Second, the auxiliary field for a quartic model is identified with the order parameter of the system. Hence, the saddle point corresponds to the mean field approximation, and fluctuations can be consistently studied as an expansion using the Landau–Ginzburg method. This may help investigating thermal effects in SFT.

Machine learning for level-truncated solutions We have started to investigate how machine learning (ML) techniques can be used for level-truncated solutions of SFT (in progress, with Riccardo Finotello, Matej Matěj and Martin Schnabl). The first question is whether this can help to improve extrapolations from results at finite level to infinite levels. There are preliminary encouraging results, even if algorithms seem unable to generalize on different backgrounds (in the sense that it is necessary to train from scratch the algorithm for each different background). Other possibilities are to use ML techniques to manipulate the huge matrices which appear when solving the equations of motion (such techniques are also interesting in lattice simulations [323]) or to explore the space of solutions using reinforcement learning.

Numerical string vertices If string vertices cannot be constructed analytically at all orders, one can try, at least, to approximate them numerically to the desired accuracy. A practical approach has recently been proposed in [40, 41], providing an efficient optimization algorithms to obtain the vertices in the minimal area representation [5].

More generally, building the string vertex $\mathcal{V}_{0,n}$ requires to find 1) a set of local coordinates (holomorphic functions) for each n which satisfy various constraints, 2) a subspace of the moduli space $\mathcal{M}_{0,n}$. Both are typical tasks for ML.

Appendix A

Formulas

A.1 Two-dimensional conformal field theory

In this section, we gather the relevant expressions for the conformal field theories (CFTs) involved in the thesis: the Coulomb gas, the Liouville theory, and the *bc* ghosts [**Erbin:2020:StringFieldTheory**, 152]. Normal ordering is implicit. We work with a flat Euclidean background metric $g_{ab} = \delta_{ab}$ and complex coordinates $(z, \bar z)$.

We focus on the holomorphic sector of the closed string, the anti-holomorphic formulas following by adding a bar. The corresponding expressions for the open string are found by letting $k \to 2k$.

A.1.1 Two-dimensional CFTs

The energy–momentum tensor has two components, respectively holomorphic and anti-holomorphic: $T(z) := T_{zz}(z)$ and $\bar T(\bar z) := T_{\bar z \bar z}(\bar z)$. The mode expansions of read:

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}, \qquad \bar T(\bar z) = \sum_{n\in\mathbb{Z}} \frac{\bar L_n}{\bar z^{n+2}}. \tag{A.1}$$

We define:

$$L_n^\pm := L_n \pm \bar L_n. \tag{A.2}$$

A primary operator transforms as:

$$f \circ \phi(z) = \big(f'(z)\big)^{h_\phi}\, \phi\big(f(z)\big) \tag{A.3}$$

under holomorphic changes of coordinates.

The BPZ conjugation is defined as

$$\phi^t(z) := I^\pm \circ \phi(z), \qquad I^\pm(z) := \begin{cases} +1 & \text{closed,} \\ -1 & \text{open.} \end{cases} \tag{A.4}$$

As a consequence, the modes transform as:

$$\phi_n^t := \begin{cases} (-1)^h \phi_{-n} & \text{closed,} \\ (-1)^{h+n} \phi_{-n} & \text{open.} \end{cases} \tag{A.5}$$

A.1.2 Coulomb gas and scalar field

The Coulomb gas CFT [324] consists of a free scalar field X in the presence of a background charge Q

$$S = \frac{\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{g}\, \left(g^{\mu\nu} \partial_\mu X \partial_\nu X + QRX\right) \tag{A.6}$$

where R is the Ricci scalar. It is necessary to consider a curved background in the action to display the coupling of the scalar field to the curvature. The field X can be spacelike or timelike (in reference to the signature of the target space) depending on the sign of the kinetic term:

$$\epsilon = \begin{cases} +1 & \text{spacelike,} \\ -1 & \text{timelike,} \end{cases} \qquad \sqrt{-1} = \mathrm{i}. \tag{A.7}$$

The charge Q is parametrized in terms of another parameter b as

$$Q = \frac{1}{b} + \epsilon b\,, \tag{A.8}$$

which is defined such that the conformal weight (A.22) of the vertex operator $V_b = \mathrm{e}^{2bX}$ is $h_b = 1$.

The energy–momentum tensor T on flat space reads:

$$T = -\epsilon\, (\partial X)^2 + \epsilon Q\, \partial^2 X, \tag{A.9}$$

and the associated central charge is

$$c = 1 + 6\, \epsilon Q^2. \tag{A.10}$$

The real values of the central charge in terms of Q and ϵ are summarised in Table 5.1. The action (A.6) changes by a constant term under a constant shift of X which leads to the conserved current

$$J = \mathrm{i}\epsilon\, \partial X. \tag{A.11}$$

This current is anomalous at the quantum level if $Q \neq 0$.

String theory in the critical dimension contains D such scalar fields X^μ with $Q = 0$. In Lorentzian signature, the field X^0 has $\epsilon = -1$ while the others have $\epsilon = 1$. In Euclidean signature, they all have $\epsilon = 1$.

Mode expansions. The Fourier expansion of the Coulomb gas field is

$$X = \frac{x}{2} - \epsilon\, \alpha \ln z + \frac{\mathrm{i}}{\sqrt{2}} \sum_{n \neq 0} \frac{\alpha_n}{n}\, z^{-n}. \tag{A.12}$$

where we use the rescaled zero-mode $\alpha := \mathrm{i}\epsilon\, \alpha_0/\sqrt{2}$. The zero-mode α is related to the charge of the conserved current as[1]

$$\alpha = \frac{1}{2\pi} \oint \mathrm{d}z\, J. \tag{A.13}$$

It corresponds to the momentum on the plane and, because of the current's anomaly, it is related to the momentum p on the cylinder by a shift:

$$\alpha = \epsilon Q + \mathrm{i}p. \tag{A.14}$$

[1] This is the appropriate normalization for the closed string, i.e. when both holomorphic and anti-holomorphic sectors are included. The open string momentum is twice smaller than the closed string one.

The Virasoro operators are

$$L_m = \frac{\epsilon}{2} \sum_{n \neq 0} \alpha_n \alpha_{m-n} + \frac{\mathrm{i}}{\sqrt{2}} \left(\epsilon\, Q(m+1) - \frac{\alpha}{2} \right) \alpha_m \,. \tag{A.15}$$

The expression of the zero-mode can be simplified to

$$L_0 = N + \frac{\alpha}{2} \left(Q - \epsilon \, \frac{\alpha}{2} \right) = N + \frac{\epsilon}{4}(Q^2 + p^2), \tag{A.16}$$

where the level operator N is defined in terms of the number operators N_n at level $n > 0$ as

$$N = \sum_{n>0} n\, N_n \,, \qquad N_n = \frac{\epsilon}{n}\, \alpha_{-n} \alpha_n \,. \tag{A.17}$$

The canonical commutation relations are

$$[x, \alpha] = -\epsilon \,, \qquad [x, p] = \mathrm{i}\epsilon \,, \qquad [\alpha_m, \alpha_n] = \epsilon\, m\, \delta_{m+n,0}. \tag{A.18}$$

The commutator of the modes with the Virasoro operators is

$$[L_m, \alpha_n] = -n\, \alpha_{m+n} + \frac{\mathrm{i}Q}{\sqrt{2}}\, m(m+1)\, \delta_{m+n,0}. \tag{A.19}$$

The commutator of the creation modes ($n > 0$) with the number operators are

$$[N_m, \alpha_{-n}] = \alpha_{-m} \delta_{m,n}. \tag{A.20}$$

Fock space. The operator ∂X is primary with conformal weight $h = 1$ only if $Q = 0$. For any Q, the vertex operators

$$V_a = \mathrm{e}^{aX} \tag{A.21}$$

are primaries with conformal weights

$$h_a = \frac{a}{2} \left(Q - \epsilon \, \frac{a}{2} \right). \tag{A.22}$$

and eigenstates of the zero-mode α with eigenvalues $a \in \mathbb{C}$. According to the anomalous shift (A.14), they correspond to the operators

$$V_k = \mathrm{e}^{\mathrm{i}kX}, \qquad h_k = \frac{\epsilon}{4}(Q^2 + k^2) \tag{A.23}$$

on the cylinder, with eigenvalue $p = k$ such that the eigenvalues are related by

$$a = \epsilon Q + \mathrm{i}k. \tag{A.24}$$

We will use V_k or V_a indistinguishably. The operators V_k and V_{-k} have the same weights. Note that $h_a, h_k \in \mathbb{R}$ only if $Q, p \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$, which also implies $c \in \mathbb{R}$. For this reason, we restrict our focus to these values of the parameters.

A set of Fock vacua $|k\rangle$ are obtained by acting with the vertex operators on the $\mathrm{SL}(2, \mathbb{C})$ vacuum $|0\rangle$

$$|k\rangle = V_k \,|0\rangle \,. \tag{A.25}$$

The Fock space $\mathcal{F}(k)$ of the theory is generated by all the states obtained by applying creation operators α_{-n} with $n > 0$ on the vacuum

$$|\psi\rangle = \prod_{n \geq 1} \frac{(\alpha_{-n})^{N_n}}{\sqrt{n^{N_n} N_n!}} \,|k\rangle \,, \qquad N_n \in \mathbb{N}. \tag{A.26}$$

The state-operator correspondence maps this to operators built from $\partial^n X$ and $\mathrm{e}^{\mathrm{i}kX}$.

Hermiticity conditions. The Virasoro modes are Hermitian $(L_n)^\dagger = L_{-n}$ if the Coulomb gas modes satisfy (signs are correlated across) [123, 252, 325, 326]:

$$Q^* = \pm Q, \qquad \alpha_n^\dagger = \pm\alpha_{-n}, \qquad \alpha^\dagger = \pm(2\epsilon Q - \alpha), \qquad k^\dagger = \pm k, \qquad x^\dagger = \pm x. \tag{A.27}$$

This implies that $Q \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$ and $k \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$. The first condition gives $c \in \mathbb{R}$ while both together give $h_k \in \mathbb{R}$. The Hermiticity condition is chosen such that $Q \in \mathbb{R} \to 0$ reproduces the standard results for the free scalar CFT (in particular, that its momentum is Hermitian).

A.1.3 Ghosts

In two-dimensional gravity, the gauge fixing of the metric in the conformal gauge introduces b and c ghosts with action

$$S_{\text{gh}} = \frac{1}{4\pi}\int \mathrm{d}^2\sigma \sqrt{g}\, b_{\mu\nu}\big(\nabla^\mu c^\nu + \nabla^\nu c^\mu - g^{\mu\nu}\nabla_\rho c^\rho\big). \tag{A.28}$$

The energy–momentum tensor on the plane reads

$$T^{\text{gh}} = -\partial(bc) - b\partial c \tag{A.29}$$

from which it is straightforward to compute the central charge and conformal weights of the ghosts

$$c_{\text{gh}} = -26, \qquad h_b = 2, \qquad h_c = -1. \tag{A.30}$$

The ghost action is invariant under an anomalous $\mathrm{U}(1)$ global symmetry with current $j = -bc$. The associated charge is called the ghost number N_{gh} and is normalized such that

$$N^{\text{gh}}(b) = -1, \qquad N^{\text{gh}}(c) = 1 \tag{A.31}$$

on the plane.

Mode expansions. The mode expansions of the ghosts are

$$b(z) = \sum_n \frac{b_n}{z^{n+2}}, \qquad c(z) = \sum_n \frac{c_n}{z^{n-1}}, \qquad \bar b(\bar z) = \sum_n \frac{\bar b_n}{\bar z^{n+2}}, \qquad \bar c(\bar z) = \sum_n \frac{\bar c_n}{\bar z^{n-1}}, \tag{A.32}$$

and we introduce the combinations

$$b_n^\pm := b_n \pm \bar b_n, \qquad c_n^\pm := \frac{1}{2}(c_n \pm \bar c_n). \tag{A.33}$$

The Virasoro operators are

$$L_m^{\text{gh}} = \sum_n (m-n)\, b_{m+n} c_{-n}\,, \qquad L_0^{\text{gh}} = N^b + N^c - 1, \tag{A.34}$$

where the zero-mode is written in terms of the ghost level and number operators

$$N^b = \sum_{n>0} n\, N_n^b\,, \quad N_n^b = b_{-n} c_n\,, \qquad N^c = \sum_{n>0} n\, N_n^c\,, \quad N_n^c = c_{-n} b_n\,. \tag{A.35}$$

The anticommutation relations between the ghosts are

$$\{b_m, c_n\} = \delta_{m+n,0}\,, \tag{A.36}$$

which imply that b_n and c_n with $n > 0$ are respectively annihilation operators for c_{-n} and b_{-n}. The commutation relations with the Virasoro and number operators are

$$[L_m^{\text{gh}}, b_n] = (m-n)\, b_{m+n}, \qquad [L_m^{\text{gh}}, c_n] = -(2m+n)\, c_{m+n}, \tag{A.37a}$$

$$[N_m^b, b_{-n}] = b_{-m}\delta_{m,n}, \qquad [N_m^c, c_{-n}] = c_{-m}\delta_{m,n}. \tag{A.37b}$$

Fock space. The $\mathrm{SL}(2,\mathbb{C})$-invariant vacuum $|0\rangle$ is defined by

$$\forall n \geq -1: \quad b_n |0\rangle = 0, \qquad \forall n \geq 2: \quad c_n |0\rangle = 0. \tag{A.38}$$

However, there exists a 2-fold degenerate state with a lower energy since $|0\rangle$ is not annihilated by c_1. The degeneracy arises because b_0 and c_0 commute with the Hamiltonian. The two ground states are given by

$$|\downarrow\rangle = c_1 |0\rangle, \qquad |\uparrow\rangle = c_0 c_1 |0\rangle. \tag{A.39}$$

They are annihilated by all positive frequency modes b_n, c_n with $n > 0$, and are related as

$$b_0 |\downarrow\rangle = 0, \qquad c_0 |\downarrow\rangle = |\uparrow\rangle, \qquad c_0 |\uparrow\rangle = 0, \qquad b_0 |\uparrow\rangle = |\downarrow\rangle. \tag{A.40}$$

The two ghost ground states have a vanishing norm and their inner product is normalised to one

$$\langle\downarrow|\downarrow\rangle = \langle\uparrow|\uparrow\rangle = 0, \qquad \langle\downarrow|\uparrow\rangle = \langle 0| c_{-1} c_0 c_1 |0\rangle = 1. \tag{A.41}$$

By analogy with the critical string, we take $|\downarrow\rangle$ to be the physical vacuum and we use it to build the Fock space $\mathcal{H}_{\mathrm{gh}}$ by acting with the creation and annihilation operators

$$|\psi\rangle = c_0^{N_0^c} \prod_{n\geq 1} (b_{-n})^{N_n^b} (c_{-n})^{N_n^c} |\downarrow\rangle, \qquad N_n^b, N_n^c = 0, 1. \tag{A.42}$$

Taking into account the anti-holomorphic sector doubles the structure of the Fock space. For example, there are four vacua $|\downarrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$ and $|\uparrow\uparrow\rangle$ from which states are built by applying b_{-n}, c_{-n}, $\bar{b}_{-n}$ and $\bar{c}_{-n}$ for $n \geq 1$.

Hermiticity conditions. The Virasoro modes are Hermitician $(L_n^{\mathrm{gh}})^\dagger = L_{-n}^{\mathrm{gh}}$ if

$$b_n^\dagger = b_{-n}, \qquad c_n^\dagger = c_{-n}. \tag{A.43}$$

A.1.4 BRST quantization

In the following, we use the superscripts "gh" and "m" to denote the ghost and matter sectors. The BRST current is given by:

$$j_B(z) = c(z) T^{\mathrm{m}}(z) + \frac{1}{2} c(z) T^{\mathrm{gh}}(z) + \frac{3}{2} \partial^2 c(z). \tag{A.44}$$

It is primary only if $c_m = 26$. The mode expansion of the associated conserved charge Q_B reads

$$Q = \sum_n c_n \left(L_{-n}^{\mathrm{m}} + \frac{1}{2} L_{-n}^{\mathrm{gh}} \right) = \sum_n c_n L_{-n}^{\mathrm{m}} + \frac{1}{2} \sum_{m,n} (n-m)\, c_{-m} c_{-n} b_{m+n} - c_0. \tag{A.45}$$

Its ghost number is $N_{\mathrm{gh}}(Q_B) = 1$ and it is nilpotent $Q_B^2 = 0$ if $c_m = 26$. Importantly, the Virasoro operators are BRST exact:

$$L_n = \{Q_B, b_n\}, \qquad [Q_B, L_m] = 0. \tag{A.46}$$

The BRST operator (A.45) can be decomposed in terms of the ghost zero-modes as:

$$Q = c_0 L_0 - b_0 M_+ + \widehat{Q} \tag{A.47}$$

where

$$\widehat{Q} = \sum_{n\neq 0} c_{-n} L_n^{\mathrm{m}} - \frac{1}{2} \sum_{\substack{m,n\neq 0\\ m+n\neq 0}} (m-n)\, c_{-m} c_{-n} b_{m+n}, \qquad M_+ = \sum_{n\neq 0} n\, c_{-n} c_n. \tag{A.48}$$

Nilpotency of the BRST operator implies the relations:

$$[L_0, M] = [\widehat{Q}, M_+] = [\widehat{Q}, L_0] = 0, \qquad \widehat{Q}^2 = L_0 M_+. \tag{A.49}$$

The operator M_+ is part of an $\mathrm{SU}(1,1)$ algebra

$$[M_+, M_-] = \widehat{N}_{\mathrm{gh}}, \qquad [\widehat{N}_{\mathrm{gh}}, M_\pm] = \pm 2 M_\pm \tag{A.50}$$

together with the ghost number without zero-mode $\widehat{N}_{\mathrm{gh}}$ and the operator M_- defined as:

$$M_- := \frac{1}{2} \sum_{n>0} \frac{1}{n}\, b_{-n} b_n, \qquad \widehat{N}_{\mathrm{gh}} := \sum_{n>0} (c_{-n} b_n - b_{-n} c_n). \tag{A.51}$$

A.1.5 String theory CFT

We have:

$$T = T^{\mathrm{m}} + T^{\mathrm{gh}}, \tag{A.52a}$$

$$T^{\mathrm{gh}} = -2 b \partial c - \partial b\, c, \tag{A.52b}$$

$$T^{\mathrm{m}} = -\frac{1}{\alpha'}\, \eta_{\mu\nu} \partial X^\mu \partial X^\nu. \tag{A.52c}$$

Useful OPE are:

$$T(z)\phi(w) \sim \frac{h_\phi\, \phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}, \tag{A.53a}$$

$$\partial X^\mu(z) \partial X^\nu(w) \sim -\frac{\alpha'}{2} \frac{\eta^{\mu\nu}}{(z-w)^2}, \tag{A.53b}$$

$$\mathrm{i}\partial X^\mu(z)\, V_k(w) \sim \frac{\alpha' k^\mu}{2} \frac{V_k(w)}{z-w}, \tag{A.53c}$$

$$T(z) V_k(w) \sim \frac{\alpha' k^2}{4} \frac{V_k(w)}{(z-w)^2} + \frac{\partial V_k(w)}{z-w}, \tag{A.53d}$$

$$V_k(z) V_{k'}(w) \sim \frac{V_{k+k'}(w)}{(z-w)^{-\alpha' k\cdot k'/2}}. \tag{A.53e}$$

The modes and fields are related as:

$$\begin{gathered} \alpha^\mu_{-n} = \sqrt{\frac{2}{\alpha'}}\, \frac{\mathrm{i}}{(n-1)!}\, \partial^n X^\mu(0), \\ b_{-n} = \frac{1}{(n-2)!}\, \partial^{n-2} b(0), \qquad c_{-n} = \frac{1}{(n+1)!}\, \partial^{n+1} c(0). \end{gathered} \tag{A.54}$$

Useful correlation functions are:

$$\left\langle \prod_{i=1}^n \mathrm{e}^{\mathrm{i}(k_i\cdot X + \rho_i\cdot\partial X)}(z_i) \right\rangle = \exp\left(\frac{\alpha'}{2} \sum_{i<j} \frac{\rho_i\cdot\rho_j}{z_{ij}^2} + \frac{\alpha'}{2} \sum_{i\neq j} \frac{\rho_i\cdot k_j}{z_{ij}} \right) \prod_{i<j} (z_{ij})^{\frac{\alpha'}{2} k_i\cdot k_j}, \tag{A.55a}$$

$$\langle c(z_1) c(z_2) c(z_3) \rangle = z_{12} z_{13} z_{23}, \tag{A.55b}$$

where $z_{ij} := z_i - z_j$.

A.2 Upsilon function

The Υ-function $\Upsilon_\beta(x)$ appearing in the three-point correlation functions [123, 327] has an integral definition for $\mathrm{Re}(x) \in (0, \mathrm{Re}(\hat{q}))$:

$$\ln \Upsilon_\beta(x) = \int_0^\infty \frac{\mathrm{d}t}{t} \left[\left(\frac{\hat{q}}{2} - x \right)^2 \mathrm{e}^{-2t} - \frac{\sinh^2 \left(\left(\frac{\hat{q}}{2} - x \right) t \right)}{\sinh(\beta t) \sinh\left(\frac{t}{\beta} \right)} \right] \tag{A.56}$$

where

$$\hat{q} = \frac{1}{\beta} + \beta. \tag{A.57}$$

This formula admits an analytic continuation to $x \in \mathbb{C}$ and can be represented by an infinite product

$$\Upsilon_\beta(x) = \lambda_\beta^{\left(\frac{\hat{q}}{2} - x\right)^2} \prod_{m,n \in \mathbb{N}} f\left(\frac{\frac{\hat{q}}{2} - x}{\frac{\hat{q}}{2} + m\beta + n\beta^{-1}} \right), \qquad f(x) = (1 - x^2)\, \mathrm{e}^{x^2}, \tag{A.58}$$

where λ_β is a constant. This formula indicates that $\Upsilon_\beta(x)$ has no poles and an infinite number of zeros located at (Figure A.1)

$$(-\beta\mathbb{N} - \beta^{-1}\mathbb{N}) \cup (\hat{q} + \beta\mathbb{N} + \beta^{-1}\mathbb{N}). \tag{A.59}$$

The function has also a reflection property

$$\Upsilon_\beta(\hat{q} - x) = \Upsilon_\beta(x). \tag{A.60}$$

A useful limit of this function (to analyse the behaviour of the four-point function integrand) is:

$$\ln \Upsilon_\beta \left(\frac{\hat{q}}{2} + \mathrm{i}E \right) \sim_{E \to \infty} -E^2 \ln |E| + \frac{3}{2} E^2. \tag{A.61}$$

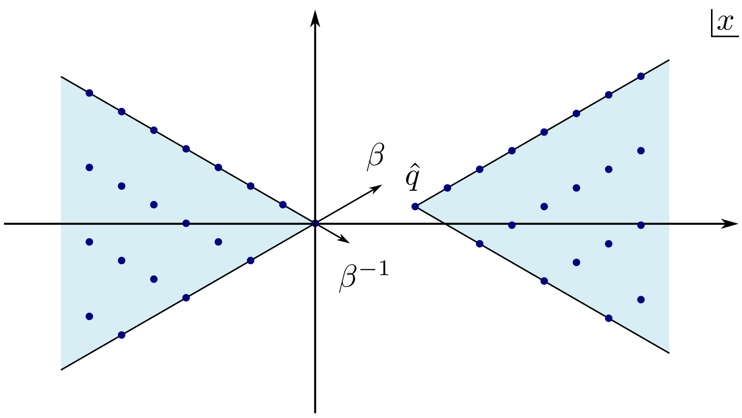


Figure A.1 – Zeros of the function $\Upsilon_\beta(x)$ for $\beta \in \mathbb{C}$. If $\beta \in \mathbb{R}$ (resp. $\beta \in \mathrm{i}\mathbb{R}$), the zeros become all real (resp. pure imaginary).

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