TN-66-27
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TRANSIENT BEHAVIOR OF BEAM BREAK-UP

In this note we treat the theory of beam break-up (BBU) as observed in a multisection accelerator. The treatment is approximate in several respects. We will use a model of a group of cavities traversed by the beam, each resonant at the same frequency ω_o and having a loss factor Q (Fig. 1). No electromagnetic coupling is assumed.

Let us assume that the rate of build-up of the modulation amplitudes in the beam is small compared to ω_o.

Consider a particle of charge e to cross at a time τ the n^{th} of N cavities at a distance x from the z-axis, taken to be an axis of symmetry. Let L be the distance between cavities, and let the particle velocity be $v \approx c = 1$.

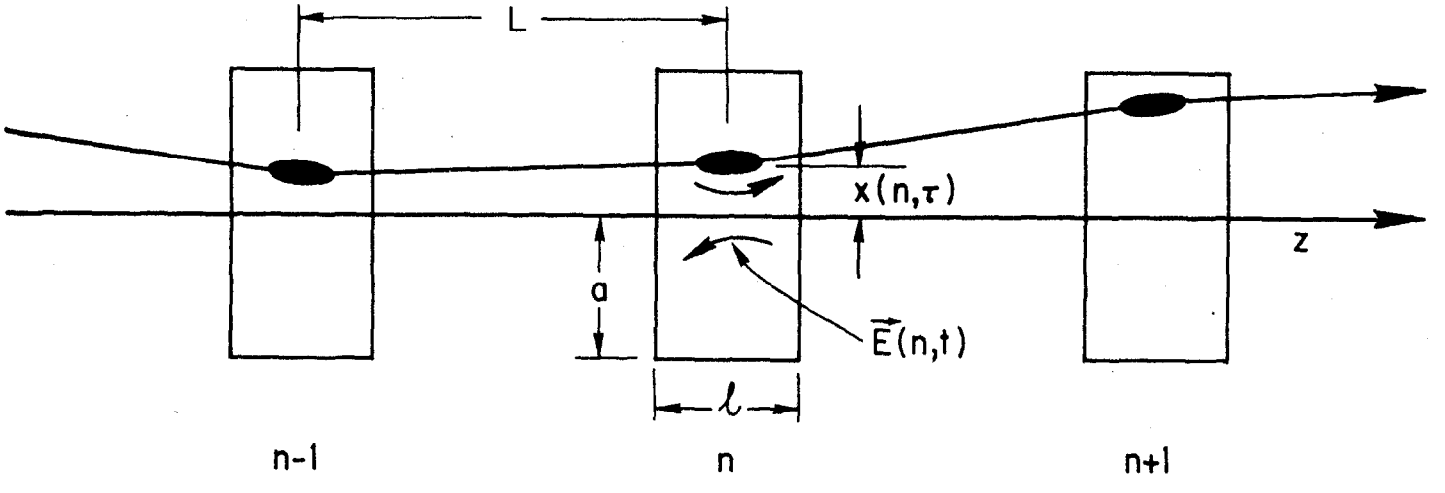

Fig. 1

1. Equation of Motion

Let the electric field $\vec{E}$ in the n^{th} cavity be derivable from a vector potential $\vec{A}$, and let each cavity be excited near a single resonant frequency ω_o. We obtain from the deflection theorem for the change in transverse momentum p_x in the n^{th} cavity:

$$\Delta p_x = e \int \frac{\partial A_z}{\partial x} \, dz = - \frac{ei}{\omega} \int \frac{\partial E_z}{\partial x} \, dz \qquad (1)$$

This transverse momentum Δp_x results in a difference in displacement of $(\Delta p_x/m_o\gamma)\,L$ between the $(n+1)^{th}$ and n^{th} cavity where $m_o\gamma$ is the particle energy. We can thus write a difference equation which can be approximated as a transverse differential equation of motion as follows:

$$\frac{d}{dn}\left(\gamma\,\frac{dx}{dn}\right) = \left(\frac{eL}{im_o\omega}\right) I \tag{2}$$

where I is the field integral

$$I(t) = \int \frac{\partial E_z(t,z)}{\partial x}\,dz \quad . \tag{3}$$

2. Excitation of Cavities

Each cavity excited at a frequency ω near ω_o receives energy U from the current j at the rate $j\int \vec{E}\cdot d\vec{z}$ and loses energy at the rate $\omega U/Q$. The rate of build-up is therefore given by (averaged over many cycles)

$$\frac{\partial U}{\partial t} = \overline{j\int \vec{E}\cdot d\vec{z}} - \frac{\omega U}{Q} \quad . \tag{4}$$

If the field has axial symmetry $\vec{E} = 0$ on the axis and $\int \vec{E}\cdot d\vec{z}$ can be approximated by:

$$\overline{\int \vec{E}\cdot d\vec{z}} \sim x\,\overline{\int \frac{\partial E_z}{\partial x}\,dz} = \overline{xI} \tag{5}$$

where x is the transverse coordinate at which the beam carrying the current j passes from the axis. The cavity excitation is thus proportional to x; on the other hand the cavity, once excited, will affect the motion in x according to Eq. (2). As a result the beam will receive a transverse structure so that x will be modulated at a frequency ω near ω_o.

In the subsequent material we will denote by the quantities Δp_x, x, and I the complex amplitudes of the respective variables oscillating at the frequency ω. These amplitudes will generally not be in phase; in fact, according to Eq. (1),

Δp_x and I must be in quadrature. The averaging process in Eqs. (4) and (5) results in the "build up" equation for the energy in the cavity:

$$\frac{\partial U}{\partial t} + \frac{\omega}{Q} U = j\,\overline{xI} = j\,|x|\,|I|\,(\cos\,\theta)/2 \ . \qquad (6)$$

Note that the driving term varies as $\cos\,\theta$ where θ is the phase angle between x and I. On the other hand, according to Eq. (2) the second derivative controlling the growth of x is proportional to $\sin\,\theta$, and therefore maximum growth will occur if x is 45° out of phase between I and Δp_x (Fig. 2).

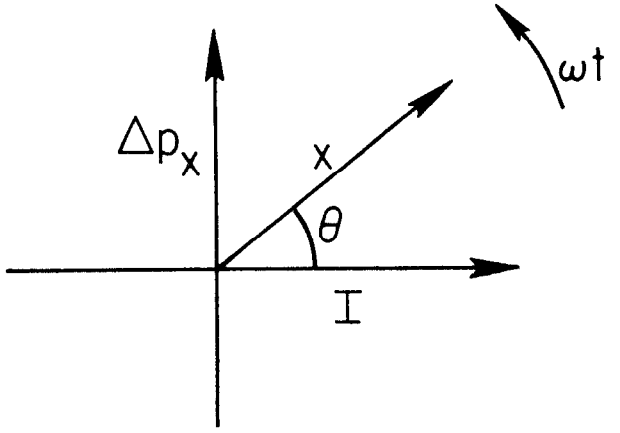

Fig. 2--Phase relations between the field integral I, the transverse momentum Δp_x and the transverse deflection x. Note that I leads Δp_x in time.

We can understand these relationships by noting the electric field configuration if $E_z = 0$ on the z axis: the fields near the axis will be shaped as shown in Fig. 3.

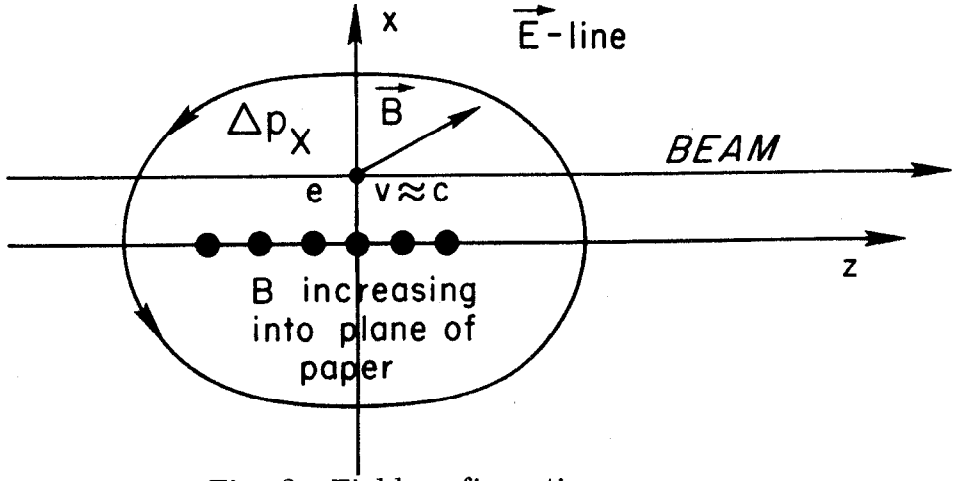

Fig. 3--Field configuration

In order for the particle (assumed of positive charge) to do work on the field (and therefore build it up), the fields must be in the direction as shown in Fig. 3, averaged over the time of passage of the particle. The transverse deflection is essentially due to the magnetic field $\vec{B}$ which is increasing into the paper. A magnetic field in that direction would produce a force along the positive direction of x. Therefore build-up of the deflection involves a time of passage of the particle later than the crest of the electric field configuration in Fig. 3. The time sequence is shown in Fig. 4.

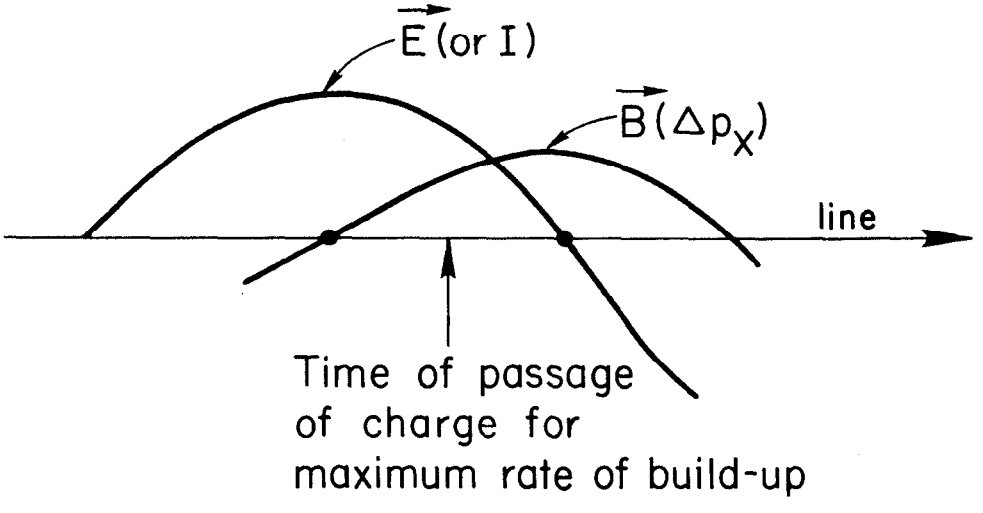

Fig. 4--Time sequence of electric and magnetic fields in the deflecting mode.

Note that this implies that the cavities will not be driven at the resonant frequency ω_o but at a frequency displaced to give the correct phase relationship as shown between the driving current and the fields.

In general, the energy storage U and the field integral I are related quadratically. If ℓ is the interaction length in each cavity and a its radius, we can write $U = KI^2$ and estimate K by putting

$$U = \frac{1}{8\pi}\int E^2 dv \sim \frac{1}{8\pi}\,\frac{I^2 a^2}{\ell^2}\cdot \pi a^2 \ell = \left(\frac{a^4}{8\ell}\right) I^2 \qquad (7)^*$$

* E^2 is the crest value of the square of the field; since the field energy is stored either in the electric or magnetic fields, the average energy is correctly given by Eq. (7) without dividing by two.

In practice the exact value of $K \sim a^4/8\ell$ depends of course on the particular mode configuration. Hence from Eq. (6):

$$|I| \frac{\partial |I|}{\partial t} + \frac{\omega}{2Q} |I| = \frac{j|x||I|}{4K} \cos\theta \qquad (8)$$

or expressed as complex amplitudes:

$$\frac{\partial I}{\partial t} + \frac{\omega}{2Q} I = \frac{jx}{4K} \quad . \qquad (9)$$

Hence the field integral $I(t)$ in the n^{th} cavity will build up as,

$$I(t) = \frac{j}{4K} \int_{-\infty}^{t} x(t') \exp\left[-(\omega/2Q)(t-t')\right] dt' \qquad (10)$$

under the influence of the current j which passed previously off-axis at a distance $x(t')$ for a time $0 < t' < t$.

3. Blow-Up Equation

Combining Eq. (10) with the equation of motion (2), we obtain the integral-differential equation for $x(n,t)$

$$\frac{d}{dn}\left(\gamma \frac{dx}{dn}\right) = \frac{e^2}{4im_o LK} J \int^{t} x(t') \exp\left[-\beta(t-t')\right] dt' \qquad (11)$$

where $\beta = \omega/2Q$ and $J = (j/e)$ is the number of electrons/unit length.
If we put

$$\frac{e^2 L}{4m_o \omega K} \sim \frac{1}{\pi} \frac{e^2}{m_o} \frac{\ell \lambda L}{a^4} = \frac{r_o \ell \lambda L}{\pi a^4} = C \qquad (12)$$

as a dimensionless number in terms of the classical electron radius $r_o = e^2/m_o$, then

$$\frac{d}{dn}\left(\gamma \frac{dx}{dn}\right) = \frac{CJ}{i} \int_{-\infty}^{\tau} x(t') \exp\left[-\beta(\tau-t')\right] dt' \qquad (13)$$

or as a partial differential equation:

$$\left(\frac{\partial}{\partial\tau} + \beta\right) \frac{\partial}{\partial n} \left(\gamma \frac{\partial x}{\partial n}\right) = \frac{CJ}{i} x \quad . \tag{14}$$ *

4. Solution By Laplace Transform

Let us try the Laplace transform solution, using an appropriate contour C,

$$x = \int_C w(\mu)\, f(n,\mu)\, e^{\mu\tau}\, d\mu \tag{16}$$

in Eqs. (13) and (14). The function $f(n,\mu)$ satisfies

$$(\mu + \beta)(\gamma f')' + iCJf = 0 \tag{17}$$

where $'$ denotes differentiation with respect to n. We can integrate this equation by assuming adiabatic variation of γ with n (WKB approximation). The result is:**

$$f(n,\mu) \sim \sqrt[4]{\frac{(\mu+\beta)}{\gamma}}\ \exp\left\{\left[(1-i)/\sqrt{2}\right](CJ)^{\frac{1}{2}}(\mu+\beta)^{-\frac{1}{2}}\int_{n_o}^{n}\gamma^{-\frac{1}{2}}\,dn\right\} . \tag{18}$$

The i term in the exponent is carried from Eq. (1) and it governs the phase of the space harmonic in x at the frequency ω relative to the phase of the electric field in each cavity, as discussed previously. As far as the amplitude of excitation is concerned, the term can be neglected. The $1/\sqrt{2}$ term remaining in the exponent is a consequence of the 45^{o} phase relation between I and x. Hence the general solution is

$$x(n,\tau) = \frac{1}{\sqrt[4]{\gamma}} \int_C w(\mu) \exp\left[\mu\tau + \left|CJ/2(\mu+\beta)\right|^{\frac{1}{2}} \int_{n_o}^{n} \gamma^{-\frac{1}{2}}\,dn\right] d\mu \tag{20}$$

where in general $w(\mu)$ depends on the starting conditions; we have absorbed the term $\sqrt[4]{\mu+\beta}$ into $w(\mu)$.

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* A differential equation of form equal to (14) can be deduced for $I(n,t)$ by eliminating x between Eqs. (2) and (9) and by putting

$$\frac{d}{dn} = L\frac{\partial}{\partial t} + \frac{\partial}{\partial n} \tag{15}$$

in Eq. (2), and carrying out extensive algebra.

** The WKB solution of the equation

$$(Af')' + Bf = 0 \text{ is : } f \sim \frac{1}{\sqrt[4]{AB}} \exp\left[i\int\sqrt{B/A}\ dn\right] \tag{19}$$

5. Solution By "Steepest Descent"

Evaluation of the function $w(\mu)$ in terms of specific starting condition, such as a unit disturbance in x occurring at $n = n_o$ at a fixed time, is not possible in closed analytical form. M. Bander, in a separate note (TN-66-28), has derived such a Green's function as a rapidly converging power series in the variable

$$CJ\left(\int_{n_o}^{n} \gamma^{-\frac{1}{2}}\, dn\right)^2 ,$$

and R. Helm has developed a program for numerical integration of the basic differential equation (14). However the question of largest practical interest is the dependence of x on the various physical parameters (current, length of current pulse, number of cavities) once a "blow-up" of x, sufficiently large to lead to beam loss, has occurred. Such loss requires a large (of the order of 10^{10}) amplification. For this purpose an asymptotic solution is adequate which can be generated by the method of steepest descent.

The appropriate contour C for the integration of Eq. (20) is from $\epsilon - i\infty$ to $\epsilon + i\infty$, where ϵ is some positive real number:

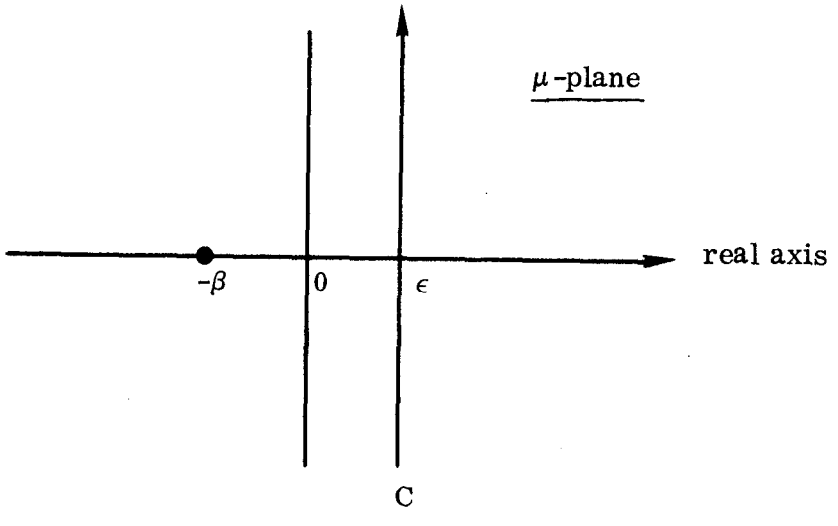

Fig. 5

An approximation to a solution of Eq. (17) can be found by choosing ϵ such that $f(n,\mu)$ falls off steepest in both directions along C away from the real axis; this will occur if ϵ is at a saddle point, i.e., if $f(n,\mu)$ has a minimum along the real axis and a maximum along C. This leads to a useful approximation if the exponent is large, i.e., if "blow-up" has largely progressed. As will be shown, we can then ignore initially the variation of $w(\mu)$ with μ.

Differentiating the exponent of Eq. (20) with respect to μ, we obtain the equation for ϵ:

$$\tau - \frac{1}{2}\left[\frac{CJ}{2(\epsilon+\beta)^3}\right]^{1/2} \int_{n_o}^{n} \gamma^{-1/2}\, dn = 0$$

$$\epsilon = -\beta + \left(\frac{CJ}{2}\right)^{1/3} \left(\frac{1}{2\tau} \int_{n_o}^{n} \gamma^{-1/2}\, dn\right)^{2/3} \tag{21}$$

giving an asymptotic variation:

$$x(n,\tau) \sim \exp\left[-\beta\tau + (3/2)(CJ)^{1/3}\, \tau^{1/3}\left(\int_{n_o}^{n} \gamma^{-1/2} dn\right)^{2/3} - \frac{1}{4}\log(\gamma/\gamma_o)\right] \tag{22}$$

We can write this out for two cases: (a) "Coasting" at energy γ_c or (b) constant acceleration at a rate $d\gamma/dn = \gamma'$. We have:

$$x(n,\tau) \sim \exp\left[-\beta\tau + (3/2)(CJ)^{1/3}\, \tau^{1/3}\, \gamma_c^{-1/3}\, (n-n_o)^{2/3}\right] \tag{23a}$$

$$x(n,\tau) \sim \exp\left[-\beta\tau + (3/2)(4CJ)^{1/3}\, \tau^{1/3} (\gamma')^{-1/3} (n^{1/2} - n_o^{1/2})^{2/3} - \frac{1}{4}\log(\gamma/\gamma_o)\right] \tag{23b}$$

in the two cases. Intermediate cases of mixed coasting and acceleration can of course be written down.

6. Scaling Laws and Numerical Examples

It is difficult to determine quantitatively the range of validity of Eqs. (23a) and (23b). Nevertheless they are apt to be reliable to determine the scaling laws among the variables γ_c or γ', τ, n and current to determine the "blow-up" condition, since this probably requires an e-folding of 20 to 30 from initial noise. Clearly the solution will be invalid unless the damping term $\beta\tau$ is relatively small in the exponent.

Let F be the e-folding factor required. The scaling laws for Blow-up are then

(a) Coasting:

$$J\tau\,(n-n_o)^2/\gamma_c = \left[F + \beta\tau\right]^3/(27C/8) \tag{24a}$$

(b) Constant energy gain:

$$J\tau\left(n^{1/2} - n_o^{1/2}\right)^2 \Big/ \gamma' = \left[F + \beta\tau + 1/4\log(\gamma/\gamma_o)\right]^3/(27C/2). \tag{24b}$$

For a numerical example let us use in Eq. (12):

$$a = 2\text{cm} \qquad \ell = 20\text{cm} \qquad \lambda = 7\text{cm} \qquad L = 300\text{cm}$$

$$C = \frac{r_o \ell \lambda L}{\pi a^4} = 2.4 \times 10^{-10}$$

$$\omega = 2.6 \times 10^{10}\ \text{sec}^{-1} \qquad Q = 10^4 \qquad \beta = 1.3 \times 10^6\ \text{sec}^{-1}\ .$$

Since we are interested in values of F between 20-30, $\beta\tau \sim 2$ and can be ignored in the scaling. For the higest energy $\gamma/\gamma_o = 4 \times 10^4$; $\frac{1}{4}\log(\gamma/\gamma_o) \sim 2.7$; hence cavity losses and adiabatic damping combined reduce the e-folding by at most 5; since the threshold current raises as the cube of the e-folding, these terms are important, but difficult to use since the e-folding factor is not known precisely.

For numerical purposes let us take F = 20. We have no numerical data on a pure coasting case, so we will compute only a constant energy and a "mixed" coasting and acceleration case.

(a) Acceleration to 20 GeV in 1000 Sections.

The scaling equation is ($\gamma' = 40$)

$$J\tau\left(n^{1/2}-n_o^{1/2}\right)^2/\gamma' \sim J\tau n/\gamma' = 4.8 \times 10^{12} \qquad (25)$$

$$J\tau n = 1.9 \times 10^{14} \qquad J\tau = 1.9 \times 10^{11} \text{ particles/pulse} .$$

The predicted value from experiment under these conditions is

$$J\tau = 2.4 \times 10^{11} \text{ particles/pulse (25 ma peak; 1.5 } \mu\text{sec)} .$$

This agreement can be considered fortuitous, considering the somewhat arbitrarily chosen constants F and C.

(b) Mixed Acceleration and Coasting.

Let us accelerate from n_o to n_1 at a uniform energy gain γ' and coast from there to section n_2. We obtain:

$$g = \int_{n_o}^{n_2} \gamma^{-1/2} dn = \frac{2}{\gamma'^{1/2}} \left(n_1^{1/2} - n_o^{1/2}\right) + \frac{1}{(n_1\gamma')^{1/2}} (n_2 - n_1) .$$

If we neglect n_o this becomes

$$g = \frac{1}{\gamma'^{1/2}} \left[n_1^{1/2} + n_2/n_1^{1/2} \right] .$$

Hence, the scaling equation is:

$$\frac{J\tau}{\gamma'} \left(n_1 + 2n_2 + \frac{n^2}{n_1} \right) \sim (F + \beta\tau)^3 \Big/ (27C/8) . \qquad (26)$$

The data reported in Fig. 3, SLAC Pub 197 taken by R. Miller on May 17, 1966, were taken by accelerating the beam through $n_1 = 32$ sections to 600 MeV ($\gamma' = 38$) and then letting the beam coast to n_2 sections as given in the following table.

n_2	$n_1+2n_2+n_2^2/n_1 = N$	$J\tau$ Product observed at $\tau = 10^{-6}$ sec	$\frac{J\tau N}{\gamma'}$
140	0.92×10^3	18.8×10^{10}	0.460×10^{13}
179	1.39	15.0	0.550
222	2.02	11.9	0.635
270	2.84	10.0	0.750
334	4.20	7.25	0.800
420	6.38	5.6	0.920
508	9.10	4.5	1.08
610	12.85	3.8	1.28

The variation of the threshold current for beam break-up with drift distance is somewhat less rapid than predicted; this may partially be due to the presence of quadrupole focusing which is not included in the analysis leading to Eq. (26). As we shall see later (Section 8) the beam loss with distance is reduced by focusing. The theoretical value for the "constant" in the last column from Eq. (26) is $(8/27)\,(F+\beta\tau)^3/C = 1.3\times10^{13}$, in excellent agreement.

(c) BBU of a Function of Energy.

Data were taken by R. Miller on June 1, 1966, in which the beam was accelerated for n_1 sections at $\gamma' = 40$ to various energy values and then permitting it to drift to $n_2 = 960$ sections.

The data are given in the following table:

γ	$J\tau$	n_1	$N = 2n_2+n_1+n_2^2/n_1$	$NJ\tau/\gamma'$
7.1×10^3	5.8×10^{10}	178	7258	1.05×10^{13}
12.6×10^3	8.9	315	5175	1.15×10^{13}
17.0×10^3	10.4	425	4515	1.18×10^{13}
24.5×10^3	11.4	614	4030	1.15×10^{13}

The theoretical value of the constant in the last column is again 1.3×10^{13}. The agreement is excellent, in particular in regards to the constancy of the entries in the last column.

Another set of data was taken on June 3, 1966 in which γ' was held constant for $n_o < n < n_1 = 32$ at a value $\gamma'_1 = 18.5$ (300 MeV in 32 sections) and then γ' was raised to give a range of energies at $n_2 = 960$ sections. The data are given in the following table:

Final energy (GeV)	γ'	$\frac{n_2^{\frac{1}{2}}-n_1^{\frac{1}{2}}}{\gamma'^{\frac{1}{2}}} = A$	$\frac{n_1^{\frac{1}{2}}-n_o^{\frac{1}{2}}}{\gamma'^{\frac{1}{2}}_1} = B$	$(A+B)^2$	$J\tau$ Observed	$J\tau(A+B)^2$
2.3	4.2	12.7	1.33	196	3.2×10^{10}	6.3×10^{12}
4.7	9.3	8.6	1.33	98	3.8	3.8
6.8	13.7	7.05	1.33	70	4.6	3.2
9.1	18.5	6.05	1.33	54.4	5.0	2.7
11.35	23.9	5.32	1.33	44.1	6.4	2.8
13.60	28.0	4.94	1.33	39.2	7.6	3.0
15.70	32.5	4.59	1.33	35.0	8.1	2.8
18.20	37.6	4.26	1.33	31.2	9.4	3.0

The nominal value of the last column is 4.8×10^{12}. With the exception of the first point, the last column is reasonably constant.

7. Estimate of Correction to Steepest Descent Calculation

In Eq. (23) we simply evaluated the exponent at $\epsilon = \mu$. Let us now instead expand the exponent along the contour C near the point ϵ, i.e., let

$$\mu = \epsilon + i\nu \quad . \tag{26}$$

The terms in the exponent of Eq. (20) become, assuming ϵ, to be given by (21),

and carrying terms in ν^2 only ($g = \int_{n_o}^{n} \gamma^{-1/2}\, dn$):

$$E = -\beta\tau + \left(\frac{CJ}{2}\right)^{1/3}\left(\frac{g}{2\tau}\right)^{2/3}$$

$$+\left(\frac{CJ}{2}\right)^{1/2} g \quad \frac{1}{\sqrt{\epsilon+\beta}}\left\{1 - \frac{3}{8} \quad \frac{\nu^2}{(\epsilon+\beta)^2}\right\}$$

$$= -\beta\tau + \frac{3}{2}\ (CJ)^{1/3}\left(\frac{g}{\tau}\right)^{2/3} - \frac{3}{2}\ \nu^2\,(CJ)^{-1/3}\, g^{-2/3}\ \tau^{5/3}\ .$$

The extra correction factor to Eq. (19) is

$$\int_{-\infty}^{\infty} e^{-\alpha\nu^2}\, d\nu = \sqrt{\frac{\pi}{\alpha}}$$

where $\alpha = \frac{3}{2}\ (CJ)^{1/3}\ \tau^{5/3}\left(\int_{n_o}^{n} \gamma^{-1/2}\, dn\right)^{-2/3}$. (27)

If we ignore constant factors, the exponent is to be corrected by the addition of:

$$\frac{1}{6}\log\left[J^{-1}\,\tau^{-5}\left(\int_{n_o}^{n} \gamma^{-1/2}\ dn\right)^2\right]. \tag{28}$$

Over the range of interest for the variables J and τ (say a factor of 10) this contributes an additive uncertainty of about 2 in the exponent. The range of the integral

$$g = \int_{n_o}^{n} \gamma^{-1/2}\, dn$$

might be quite large if we are interested in tracing the growth of the instability from the beginning of the accelerator. If we wish to restrict the uncertainty in the exponent to 2 (corresponding according to Eq. (27) to a 30% uncertainty in

the prediction for the current to give an e-folding of 20) then the range in the integral g is restricted to 400; for the uniform acceleration case this limit is in practice unrestrictive; for the coasting case this means that this analysis is not valid if applied to coasting from the first few sections. At any rate, the analysis is inapplicable if behavior of the build-up in a small range of e- folding is considered; the quantitative extent can be estimated by comparing Eq. (27) with the exponent in Eq. (22).

8. The Effect of Radial Focusing

The preceding discussions ignored the effect of the quadrupole triplet lenses located between every 32 sections of the accelerator. Let us represent their effect by a continuous focusing term which would produce cross-overs spaced a distance (half-wavelength) π/k apart. Such a system would add a term $+\gamma k^2 x$ to the left hand side of Eq. (13), giving

$$\frac{d}{dn}\left(\gamma \frac{dx}{dn}\right) + \gamma k^2 x = \frac{CJ}{i} \int_{-\infty}^{\tau} x(t')\exp\left[-\beta(\tau - t')\right] dt' \ . \tag{29}$$

Using the Laplace transform (16), the equation for $f(n,\mu)$ becomes

$$(\gamma f')' + \left(k^2\gamma + \frac{iCJ}{\mu+\beta}\right) f = 0 \ . \tag{30}$$

Ignoring a phase factor, and omitting the slowly varying amplitude factor, the adiabatic solution (20) becomes:

$$x(n,\tau) = \int w(\mu)\exp\left[\mu\tau + (1/\sqrt{2})\int_{n_o}^{n}\left\{\left[k^4 + (CJ)^2/(\mu+\beta)^2\gamma^2\right]^{1/2} - k^2\right\}^{1/2} dn\right] d\mu \tag{31}$$

where in general k and γ could be given functions of n .

To use the method of steepest descent we would like to obtain the minimum of the exponent in Eq. (31) as a function of μ. This is difficult in general; an

algebraic solution is possible only if we are interested in a relatively small corrective effect of focusing, i.e., if we let

$$k^2 \ll (CJ)/(\mu + \beta)\,\gamma \tag{32}$$

near $\mu = \epsilon$, where ϵ is given by Eq. (21).

To order linear in $k^2/CJ/(\mu + \beta)\,\gamma$ the exponential in Eq. (31) becomes

$$\exp\left[\mu\tau + 1/\sqrt{2}\int_{n_o}^{n}\left\{\left[CJ/(\mu+\beta)\gamma\right]^{1/2} - (k^2/2)\left[(\mu+\beta)\gamma/CJ\right]^{1/2}\right\}dn\right]. \tag{33}$$

To the same order the saddle point $\mu = \epsilon$ as given in Eq. (21) will not be shifted; we obtain therefore the exponential solution

$$x(n,\tau) \sim \exp\left[-\beta\tau + (3/2)(CJ)^{1/3}\,\tau^{1/3}\,g^{2/3} - (1/4)\,(CJ)^{-1/3}\,\tau^{-1/3}\,g^{1/3}\int_{n_o}^{n}\gamma^{1/2}\,k^2\,dn\right]. \tag{34}$$

Hence the dominant exponent is reduced by a factor

$$1 - (CJ)^{-2/3}\,\tau^{-2/3}\,g^{-1/3}\int_{n_o}^{n}\gamma^{1/2}\,k^2\,dn/6 \tag{35}$$

which can be written in the surprisingly simple form:

$$1 - \frac{3F^{-2}}{8}\left(\int_{n_o}^{n}\gamma^{1/2}\,k^2\,dn\right)\left(\int_{n_o}^{n}\gamma^{-1/2}\,dn\right) \tag{36}$$

where F is the e-folding before reduction due to focusing.

Let us examine the required focal strength k to improve the beam current by a factor f above the value at which break-up would be observed without

focusing. Then

$$f^{1/3}\ F\left[1-\frac{3f^{-2/3}F^{-2}}{8}\left(\int_{n_o}^{n}\gamma^{1/2}k^2\,dn\right)\left(\int_{n_o}^{n}\gamma^{-1/2}\,dn\right)\right]=F\ . \tag{37}$$

If k^2 is assumed to be constant,* this becomes

$$k^2=\frac{8F^2}{3}\ f^{1/3}\left(f^{1/3}-1\right)\Big/\left(\int_{n_o}^{n}\gamma^{1/2}\,dn\right)\int_{n_o}^{n}\left(\gamma^{-1/2}\,dn\right) \tag{38}$$

or:

$$k=(8/3)^{1/2}\,F\,f^{1/6}\left(f^{1/3}-1\right)^{1/2}/\,(n-n_o) \tag{39}$$

in the coasting situation, independent of energy and:

$$k=2^{1/2}\,F\,f^{1/6}\left(f^{1/3}-1\right)^{1/2}\,(n-n_o) \tag{40}$$

in the case of uniform acceleration, independent of gradient.

Numerically let us consider a desired improvement factor $f=2$ at constant e-folding $F=20$. We obtain $k=18/(n-n_o)$ in the coasting case and $k=16/(n-n_o)$ in the case of uniform energy gain.

For the full length, uniform energy gain machine this would require $k=1.7\times10^{-2}$ or a distance between cross-overs of $\pi/(1.7\times10^{-2})=190$ sections. Hence to double the current before beam break-up, roughly 1 cross-over per 6 sectors of 32 sections is required. This leaves a margin by a factor of 6 above the stop-band; for the machine operating at 20 GeV and 50 ma, an interlaced beam

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*This assumption is of course not required but provided optimum separation from stop-bands if the lenses are located periodically along the structure.

down to about 5 GeV should be possible; the lower energy interlaced beam would have a correspondingly larger improvement factor over beam break-up; such larger factors cannot be calculated reliably in this approximation.

Some experimental data on the increase in BBU threshold current were taken by R. Miller on June 1, 1966. The data are given in the following table:

Threshold current for BBU at $\tau = 1.5$ μsec (ma)	k	$f^{\frac{1}{3}}(f^{\frac{1}{3}}-1)$	$f^{\frac{1}{3}}(f^{\frac{1}{3}}-1)/k$
3 (extrapolated)	0	0	-
4.3	0.008	0.147	18
5.0	0.015	0.225	15
7.0	0.02	0.44	22
11.6	0.033	0.90	27

Data were taken by accelerating to 3.7 GeV in $n_1 = 224$ sections and then coasting to $n_2 = 960$ sections. The theoretical value of the last column, assuming a constant value of k is thus:

$$\left(\frac{3}{8}\right)^{1/2} F^{-1} \left[\left(n_1^{1/2}\, n_2 - \frac{1}{3} n_1^{3/2}\right)\left(n_1^{1/2} + n_2/n_1^{1/2}\right)\right]^{1/2} = 630\, F^{-1} \quad . \tag{41}$$

Hence an e-folding value of $F \sim 25$ gives resonable agreement with the data.

9. Conclusion

The theory described here gives at least semi-quantitative agreement with the data on BBU obtained thus far. The asymptotic solution given is applicable only for large e-folding; therefore it can be used for scaling the various experimental parameters (current, pulse-length, focusing field, accelerator length, energy gradient) at BBU threshold. The theory contains only one free parameter ($F/C^{1/3}$); we have estimated this quantity but without being able to study the build-up from

known initial disturbances and without exact knowledge of the cavity fields (in particular of the interaction length ℓ) the quantitative agreement is actually better than could be expected.

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