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Generalization of the Schrödinger Equation for Open Systems Based on the Quantum-Statistical Approach

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Abstract: Within the framework of the quantum-statistical approach, utilizing both non-Hermitian Hamiltonian and Lindblad's jump operators, one can derive various generalizations of the von Neumann equation for reduced density operators, also known as hybrid master equations. If one considers the evolution of pure states only, i.e., disregarding the coherence between states and spontaneous transitions from pure to mixed states, then one can resort to quantum-mechanical equations of the Schrödinger type. We derive them from the hybrid master equations and study their main properties, which indicate that our equations have a larger range of applicability compared to other generalized Schrödinger equations proposed hitherto. Among other features, they can describe not only systems which remain in the stationary eigenstates of the Hamiltonian as time passes, but also those which evolve from those eigenstates. As an example, we consider a simple but important model, a quantum harmonic oscillator driven by both Hamiltonian and non-Hamiltonian terms, and derive its classical limit, which turns out to be the damped harmonic oscillator. Using this model, we demonstrate that the effects of dissipative environments of different types can cancel each other, thus resulting in an effectively dissipation-free classical system. Another discussed phenomenon is whether a non-trivial quantum system can reduce to a classical system in free motion, i.e., without experiencing any classical Newtonian forces. This uncovers a large class of quantum-mechanical non-Hamiltonian systems whose dynamics are not determined by conventional mechanics' potentials and forces, but rather come about through quantum statistical effects caused by the system's environment.

Keywords: open quantum systems; quantum dissipative phenomena; generalized Schrödinger equation; master equation; density operator; non-Hermitian Hamiltonian

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1. Introduction

Any realistic quantum system can be influenced not only by its natural environment, but can also be affected by the process of measurement, i.e., the interaction with an external measuring apparatus. Therefore, any observable quantum system is, or eventually becomes, open. The history of the theory of open quantum systems (OQS) goes back a few decades, counting from the early studies of dissipative systems based on quantizing classical models with damping, viscosity, and other phenomenologically introduced characteristics of dissipation [1].

It soon became clear that these initial attempts were quite limited in terms of their physical relevance because the role, complexity, and diversity of dissipative phenomena drastically increase, both qualitatively and quantitatively, in quantum systems, compared to classical ones. Therefore, one cannot take them all into account by simply starting from classical dissipative models and quantizing them in a "textbook" manner.

Moreover, quantum systems and experiments exist whose complete description cannot be achieved using a single-state vector because of the additional randomness or uncertainty regarding precisely which of their possible quantum states is prepared. These additional uncertainties could be caused by imperfectness of the devices used in experiments, which

inevitably induce the classical randomness, or they could occur as a result of correlations of states due to quantum entanglement.

Thus, a consistent description of open quantum systems requires extensive usage of quantum statistical approaches [2,3], due to the occurrence of mixed states, i.e., statistical ensembles of pure states, as soon as a system "opens up". This requires the OQS formalism to be upgraded from state vectors to density operators because the latter contain information not only about quantum-mechanical states, but also about the classical statistical probabilities thereof and correlations between them.

In conventional quantum mechanics, dealing with conservative systems, the transition from the state-vector to density-matrix descriptions is straightforward, and leads to the von Neumann equation, but, in the case of open systems, the situation is obviously more complicated, as a result of the above-mentioned diversity in dissipative processes and environments. This results in various generalizations of the von Neumann equation, usually aggregated into the OQS approach commonly referred to as the master equations for reduced density operators.

The Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) master equation, or, simply, the Lindblad equation, became popular first (chronologically) [4,5], due to the relative simplicity of its foundations, which are based on the Markovian and rotating-wave approximations [3]. Despite the achieved substantial technical simplifications in open systems' descriptions, it is obvious that these approximations significantly reduce the range of applicability of the GKSL approach.

An alternative framework and a range of models, which do not use the above-mentioned approximation, are based on non-Hermitian Hamiltonians (NH). Probably the earliest example can be found in the works of Feshbach, who showed that Hamiltonian operators can acquire anti-Hermitian terms as a result of partition between a subsystem and its environment [6,7].

Although master equations with non-Hermitian Hamiltonians have long been used—for example, in quantum optics [2]—the modern version of the NH approach was proposed relatively recently: when the post-selecting normalization procedure of a reduced density matrix was introduced [8,9] and transitions from pure to mixed states were considered [10]. This immediately resulted in a vast number of applications in nearly all branches of quantum physics; for examples from only the very recent literature, see [11–33].

In this paper, we derive various generalizations of the Schrödinger equation for open systems while using the quantum-statistical approach as a starting point and motivation. In Section 2, we give a brief introduction to the modern quantum-statistical approach based on unification of the NH- and GKSL-type master equations. In Section 3, we perform the dephasing and rank reduction procedures on density matrices, thus resorting from the density-matrix to the state-vector description. We discuss three types of generalized Schrödinger equations which occur as a result of dephasing, determine their equivalence class, and discuss special cases: NH-driven versus GKSL-driven evolution. Their basic properties are discussed in Section 4, where we focus on the evolution of stationary states, invariances, and relations between mean values. An instructive example is considered in Section 5, where we consider OQS models that reduce to the damped oscillator upon dephasing and averaging in the coherent states basis. Conclusions are drawn in Section 6.

2. Generalized von Neumann Equations

Here, we adopt the quantum-statistical approach that deals exclusively with the degrees of freedom of a subsystem, wherein all degrees of freedom of the environment have been averaged ("traced out"). We thus regard the (reduced) density operator of a subsystem as a primary operator for quantum-statistical description: the mean value of a quantum observable defined on a Hilbert space associated with the subsystem's degrees of freedom is computed as the trace of the product of its operator with the density operator.

If density operator is normalized, i.e., its trace's value is constant (can thus be set to 1) and preserved during the system's time evolution, then we denote it by $\hat{\rho}$; otherwise, we denote it by $\hat{\varrho}$. The relation between these two density matrices is straightforward:

$$\hat{\rho} = \hat{\varrho}/\operatorname{Tr}\hat{\varrho}, \tag{1}$$

therefore, definitions for statistical mean values are

$$\langle\hat{O}\rangle \equiv \operatorname{Tr}(\hat{\rho}\,\hat{O}), \;\; \langle\hat{O}\rangle_{\varrho} \equiv \operatorname{Tr}(\hat{\varrho}\,\hat{O}) = \langle\hat{O}\rangle \operatorname{Tr}\hat{\varrho}, \tag{2}$$

where we use the subscript ϱ to differentiate the notations.

Furthermore, we assume that the dissipative environment's influence upon a subsystem can be encoded in two different, but not mutually exclusive, ways.

The first approach is to use the anti-Hermitian terms of the subsystem's Hamiltonian. Any non-Hermitian Hamiltonian operator can always be written as the sum of self- and skew-adjoint components:

$$\hat{\mathcal{H}} = \hat{H}_+ + \hat{H}_- = \hat{H}_+ - i\,\hat{\Gamma}, \tag{3}$$

where we denote $\hat{H}_\pm = \frac{1}{2}(\hat{\mathcal{H}} \pm \hat{\mathcal{H}}^\dagger) = \pm\hat{H}_\pm^\dagger$ and introduce a self-adjoint operator $\hat{\Gamma} \equiv i\hat{H}_-$, which is commonly referred to as the decay rate operator. It should be noted that, strictly speaking, the operator $\hat{\mathcal{H}}$ is not a true Hamiltonian, but rather an analytical way of writing down a set of two physically relevant operators $\hat{\mathcal{H}} \mapsto (\hat{H}_+, \hat{\Gamma})$, as we shall demonstrate below.

A second type of dissipation can be described using the non-Hamiltonian (Liouvillian) term, dubbed the Lindblad's dissipator:

$$\hat{\mathcal{D}}\hat{\rho} = \sum_k \gamma_k \left(\hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2}\{\hat{\rho},\, \hat{L}_k^\dagger \hat{L}_k\} \right) = \frac{1}{2}\sum_k \gamma_k \Big([\hat{L}_k,\, \hat{\rho}\hat{L}_k^\dagger] + [\hat{L}_k \hat{\rho},\, \hat{L}_k^\dagger] \Big), \tag{4}$$

where $\hat{L}_k$ is called the Lindblad, collapse, or jump operator; γs are non-negative couplings which can be derived from the correlation functions of the environment (usually playing the role of relaxation rates for different decay modes); and the summation index runs $k = 1, \ldots, N^2 - 1$, where N is the dimension of the system [4,5]. Thus, $\hat{\mathcal{D}}$ is a Liouvillian-type operator in the quantum dynamical semigroup form, which is traceless, linear with respect to the density matrix, and quadratic with respect to the jump operators.

Furthermore, the general equation of motion for the density matrix was proposed in [9]. For the normalized density matrix, it reads as

$$\begin{aligned} \frac{d}{dt}\hat{\rho} &= -\frac{i}{\hbar}\left(\hat{\mathcal{H}}\hat{\rho} - \hat{\rho}\,\hat{\mathcal{H}}^\dagger\right) + \frac{i}{\hbar}\langle\hat{\mathcal{H}} - \hat{\mathcal{H}}^\dagger\rangle\hat{\rho} + \sum_k \gamma_k \left(\hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2}\{\hat{\rho},\, \hat{L}_k^\dagger \hat{L}_k\} \right) \\ &= -\frac{i}{\hbar}[\hat{H}_+, \hat{\rho}] - \frac{1}{\hbar}\{\hat{\Gamma}, \hat{\rho}\} + \frac{2}{\hbar}\langle\hat{\Gamma}\rangle\hat{\rho} + \sum_k \gamma_k \left(\hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2}\{\hat{\rho},\, \hat{L}_k^\dagger \hat{L}_k\} \right), \end{aligned} \tag{5}$$

where square and curly brackets denote, respectively, the commutator and anticommutator, $\langle\hat{\Gamma}\rangle = \operatorname{tr}(\hat{\rho}\,\hat{\Gamma})$, according to our notations for mean values in (2).

Furthermore, the right-hand side of Equation (5) is essentially a sum of the following three terms:

$$-\frac{i}{\hbar}[\hat{H}_+, \hat{\rho}], \;\; -\frac{1}{\hbar}\{\hat{\Gamma}, \hat{\rho}\} + \frac{2}{\hbar}\langle\hat{\Gamma}\rangle, \;\; \sum_k \gamma_k \left(\hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2}\{\hat{\rho},\, \hat{L}_k^\dagger \hat{L}_k\} \right), \tag{6}$$

which means that it is the generalization of the quantum von Neumann equation (represented by the first term) based on the unification of the quantum-statistical equation for non-Hermitian Hamiltonians (second term) with the Lindblad master equation approach (third term). As discussed in Section 1, the Lindblad's dissipator and non-Hermitian Hamiltonian parts have different physical origins and, thus, should describe different types of

dissipation. Comparing the structure of Equation (5) to the conventional von Neumann equation, one can see that it is the operator $\hat{H}_+$ that plays the role of a Hamiltonian of the system, whereas the anti-Hermitian part of $\hat{\mathcal{H}}$ contributes, together with the Lindblad's dissipator, to the non-Hamiltonian (Liouvillian) part. This reaffirms our earlier remark after Equation (3) about the physical meaning of $\hat{\mathcal{H}}$.

Each of the three terms in (6) independently has a vanishing trace, which means that the trace of $\hat{\rho}$ is conserved at all times, as it is necessary for the probabilistic interpretation. It should also be noted that the Lindblad term cannot be entirely absorbed into the non-Hermitian Hamiltonian term, in general, because it contains not only the anticommutator $\{\hat{\rho}, \hat{L}_k^\dagger \hat{L}_k\}$, but also the term $\hat{L}_k \hat{\rho} \hat{L}_k^\dagger$, which is required by the semigroup structure and tracelessness of the Lindblad's dissipator.

Since the density operator is capable of describing not only pure but also mixed states, it is an important feature when dealing with dissipative systems. However, the case of pure states is still fairly special: using the master equation in (5), one can show that the evolution equation for the quantum purity reads as

$$\frac{d}{dt}\mathcal{P} = \mathcal{R} = \frac{4}{\hbar}\Big[\langle \hat{\Gamma}\rangle \mathcal{P} - \mathrm{Tr}(\hat{\rho}^2\,\hat{\Gamma})\Big] + 2\sum_k \gamma_k \Big[\mathrm{Tr}(\hat{\rho}\hat{L}_k^\dagger \hat{\rho}\hat{L}_k) - \mathrm{Tr}(\hat{\rho}^2 \hat{L}_k^\dagger \hat{L}_k)\Big], \tag{7}$$

where $\mathcal{P} = \mathrm{tr}(\hat{\rho}^2)$ is the purity. This equation indicates that the purity evolution rate $\mathcal{R}$ is determined by the NH terms and Lindblad terms independently of each other and in different ways. If evolution starts from an initially pure state, for which $\hat{\rho}^2(t_0) = \hat{\rho}(t_0) = |\Psi_0\rangle\langle\Psi_0|$, then one obtains

$$\mathcal{R}|_{t=t_0} = 2\sum_k \gamma_k \Big(\langle \hat{L}_k^\dagger\rangle_{\Psi_0}\langle \hat{L}_k\rangle_{\Psi_0} - \langle \hat{L}_k^\dagger \hat{L}_k\rangle_{\Psi_0}\Big), \tag{8}$$

where we used the notation $\langle\,.\,\rangle_\Psi = \langle\Psi|\,.\,|\Psi\rangle$ for mean values in the Hilbert space of $|\Psi\rangle$s. This formula indicates that initially pure states continuously evolve into mixed ones due to the Lindblad term in the master equation, while its non-Hermitian Hamiltonian part does not affect this process (this, however, does not preclude non-Hermitian Hamiltonians alone from triggering spontaneous transitions from pure to mixed states in the presence of quantum fluctuations, cf. [10]).

Let us now consider the non-normalized density matrix. Its evolution equation can be derived similarly:

$$\begin{aligned}\frac{d}{dt}\hat{\varrho} &= -\frac{i}{\hbar}\Big(\hat{\mathcal{H}}\hat{\varrho} - \hat{\varrho}\,\hat{\mathcal{H}}^\dagger\Big) + \sum_k \gamma_k \Big(\hat{L}_k \hat{\varrho} \hat{L}_k^\dagger - \frac{1}{2}\{\hat{\varrho}, \hat{L}_k^\dagger \hat{L}_k\}\Big) \\ &= -\frac{i}{\hbar}[\hat{H}_+, \hat{\varrho}] - \frac{1}{\hbar}\{\hat{\Gamma}, \hat{\varrho}\} + \sum_k \gamma_k \Big(\hat{L}_k \hat{\varrho} \hat{L}_k^\dagger - \frac{1}{2}\{\hat{\varrho}, \hat{L}_k^\dagger \hat{L}_k\}\Big),\end{aligned} \tag{9}$$

where the trace conservation breaks due to the absence of the compensating term $\frac{2}{\hbar}\langle\hat{\Gamma}\rangle\hat{\varrho}$, cf. Equation (6). The dynamical value of the density matrix's trace is no longer an identity but a solution to the first-order differential equation

$$\frac{d}{dt}\mathrm{Tr}\,\hat{\varrho} = \frac{2}{i\hbar}\mathrm{Tr}\big(\hat{\varrho}\hat{H}_-\big) = -\frac{2}{\hbar}\langle\hat{\Gamma}\rangle_\varrho\,, \tag{10}$$

where, in the last step, we assume the notations in (2).

The existence of two types of density operators, normalized and non-normalized, can be useful in certain applications, for example, when a system can evolve in either a sustainable or a non-sustainable way [34]. Then, the non-normalized density operator describes the system which inevitably decays (or diverges), whereas the normalized density operator indicates that gain and loss somehow balance each other out; therefore, the system sustains itself.

3. Generalized Schrödinger Equations

In this section, we derive relations between quantum-statistical master equations from the previous section and the quantum-mechanical Schrödinger-type equations for state vectors.

3.1. Dephasing and State-Vector Reduction

If one considers the evolution of only the state vectors, then one narrows the physical picture by neglecting various quantum-statistical phenomena, such as coherence, because they are predominantly described by the off-diagonal components of a density matrix. In other words, a forced dephasing can be performed, which turns a system into a plain statistical mixture, thus making its description closer to the classical. If this is acceptable, if an initial state can be prepared to be pure, and also if "mixing" quantum fluctuations can be neglected, then one can indeed resort to a state-vector description, i.e., consider, for each state, the time evolution of only one diagonal element of a density matrix. The rank of the latter thus effectively reduces to one because its other elements are assumed to be zeros.

We thus consider the special case of a statistical operator:

$$\hat{\rho} \to |\Phi\rangle\langle\Phi|, \;\; \hat{\varrho} \to |\Omega\rangle\langle\Omega|, \tag{11}$$

for which Equation (5) reduces to a Hermitian conjugate pair of equations of a Schrödinger type:

$$\begin{aligned} i\hbar\,\partial_t\,|\Phi\rangle &= \left[\hat{\mathcal{H}} + \frac{i}{2}\hbar\sum_k \gamma_k\left(\frac{\langle\hat{L}_k^\dagger\rangle_\Phi}{\langle\Phi|\Phi\rangle} - \hat{L}_k^\dagger\right)\hat{L}_k\right]|\Phi\rangle - \frac{1}{2}\langle\hat{\mathcal{H}} - \hat{\mathcal{H}}^\dagger\rangle_\Phi|\Phi\rangle \\ &= \left[\hat{H}_+ + \frac{i}{4}\hbar\sum_k \frac{\gamma_k}{\langle\Phi|\Phi\rangle}\Big(\langle\hat{L}_k^\dagger\rangle_\Phi\hat{L}_k - \hat{L}_k^\dagger\langle\hat{L}_k\rangle_\Phi\Big)\right]|\Phi\rangle \\ &\quad -i\left[\hat{\Gamma} - \langle\hat{\Gamma}\rangle_\Phi - \frac{1}{4}\hbar\sum_k \gamma_k\left(\frac{\langle\hat{L}_k^\dagger\rangle_\Phi\hat{L}_k + \hat{L}_k^\dagger\langle\hat{L}_k\rangle_\Phi}{\langle\Phi|\Phi\rangle} - 2\hat{L}_k^\dagger\hat{L}_k\right)\right]|\Phi\rangle, \end{aligned} \tag{12}$$

and, similarly, for the non-normalized density matrix evolution (9):

$$\begin{aligned} i\hbar\,\partial_t\,|\Omega\rangle &= \left[\hat{\mathcal{H}} + \frac{i}{2}\hbar\sum_k \gamma_k\left(\frac{\langle\hat{L}_k^\dagger\rangle_\Omega}{\langle\Omega|\Omega\rangle} - \hat{L}_k^\dagger\right)\hat{L}_k\right]|\Omega\rangle \\ &= \left[\hat{H}_+ + \frac{i}{4}\hbar\sum_k \frac{\gamma_k}{\langle\Omega|\Omega\rangle}\Big(\langle\hat{L}_k^\dagger\rangle_\Omega\hat{L}_k - \hat{L}_k^\dagger\langle\hat{L}_k\rangle_\Omega\Big)\right]|\Omega\rangle \\ &\quad -i\left[\hat{\Gamma} + \frac{1}{4}\hbar\sum_k \gamma_k\left(2\hat{L}_k^\dagger\hat{L}_k - \frac{\langle\hat{L}_k^\dagger\rangle_\Omega\hat{L}_k + \hat{L}_k^\dagger\langle\hat{L}_k\rangle_\Omega}{\langle\Omega|\Omega\rangle}\right)\right]|\Omega\rangle, \end{aligned} \tag{13}$$

where, in the second and subsequent lines of either Equation (12) or (13), we explicitly separate Hermitian and anti-Hermitian components of the Lindblad part:

$$\hat{\mathcal{H}}_L^{(\Phi)} = \frac{i}{2}\hbar\sum_k \gamma_k\left(\frac{\langle\hat{L}_k^\dagger\rangle_\Phi}{\langle\Phi|\Phi\rangle} - \hat{L}_k^\dagger\right)\hat{L}_k, \;\; \hat{\mathcal{H}}_L^{(\Omega)} = \frac{i}{2}\hbar\sum_k \gamma_k\left(\frac{\langle\hat{L}_k^\dagger\rangle_\Omega}{\langle\Omega|\Omega\rangle} - \hat{L}_k^\dagger\right)\hat{L}_k, \tag{14}$$

$$\hat{H}_{L\pm} = \frac{1}{2}(\hat{\mathcal{H}}_L \pm \hat{\mathcal{H}}_L^\dagger) = \pm\hat{H}_{L\pm}^\dagger, \; \hat{\Gamma}_L = i\hat{H}_{L-}\,, \tag{15}$$

assuming our usual notations.

One can see that the Lindblad term can be now viewed as a part of the non-Hermitian Hamiltonian in Equation (12) or Equation (13), which was not possible in the underlying master equations (due to the presence of the term containing a density matrix "sandwiched" between two Lindblad operators). This is a side effect of the imposed dephasing procedure in (11), which turns some of the Lindblad operators into their averages.

Another side effect of the dephasing and rank-truncating of the original density matrices is that the norms of the corresponding state vectors $|\Phi\rangle$ and $|\Omega\rangle$ are not conserved in general (which is not surprising for $|\Omega\rangle$ but is for $|\Phi\rangle$). From Equations (12) and (13), one obtains the evolution equation for the norms:

$$\partial_t \langle\Phi|\Phi\rangle = -\frac{2}{\hbar}\langle\hat{\Gamma}\rangle_\Phi(1-\langle\Phi|\Phi\rangle)+\sum_k \gamma_k\left(\frac{\langle\hat{L}_k^\dagger\rangle_\Phi\langle\hat{L}_k\rangle_\Phi}{\langle\Phi|\Phi\rangle}-\langle\hat{L}_k^\dagger\hat{L}_k\rangle_\Phi\right), \tag{16}$$

$$\partial_t \langle\Omega|\Omega\rangle = -\frac{2}{\hbar}\langle\hat{\Gamma}\rangle_\Omega+\sum_k \gamma_k\left(\frac{\langle\hat{L}_k^\dagger\rangle_\Omega\langle\hat{L}_k\rangle_\Omega}{\langle\Omega|\Omega\rangle}-\langle\hat{L}_k^\dagger\hat{L}_k\rangle_\Omega\right), \tag{17}$$

which can be viewed as first-order differential equations for, respectively, $\langle\Phi|\Phi\rangle$ and $\langle\Omega|\Omega\rangle$ as functions of time. Each of these equations is in fact a generalization of the normalization condition of a state vector in terms of an inner product. They also indicate that the GKSL-type dissipation dynamically breaks the normalization of both wavefunctions Φ and Ω.

The norm (non-)conservation can be used in certain applications, for example, in establishing the classical limit of a dissipative system. According to Equation (16), the classical regime is necessary to stop the ultimate decay of the system because the Lindblad operators tend to become uncorrelated in the classical limit; therefore, the right-hand side of Equation (16) tends to zero, and $\langle\Phi|\Phi\rangle = 1$ becomes a solution. This feature can be useful in decoherence models and in theories of mesoscopic systems and quantum chaos [35,36].

For other applications, it is more robust to use models with a state vector whose norm is conserved at all times, cf. the generalized Schrödinger equations proposed in [37–39]. These models also cover situations when the measuring process is restarted or calibrated to compensate for the "leakage" of probability.

Therefore, let us introduce the normalized state vector:

$$|\Psi\rangle \equiv \langle\Omega|\Omega\rangle^{-1/2}\,|\Omega\rangle, \tag{18}$$

by analogy with the normalization procedure for density matrices (1). Its dynamics are governed by the following equation:

$$\begin{aligned} i\hbar\,\partial_t|\Psi\rangle &= \left[\hat{\mathcal{H}}+\frac{i}{2}\hbar\sum_k\gamma_k\Big(\big(\langle\hat{L}_k^\dagger\rangle_\Psi-\hat{L}_k^\dagger\big)\hat{L}_k-\langle\hat{L}_k^\dagger\rangle_\Psi\langle\hat{L}_k\rangle_\Psi+\langle\hat{L}_k^\dagger\,\hat{L}_k\rangle_\Psi\Big)\right]|\Psi\rangle-\frac{1}{2}\langle\hat{\mathcal{H}}-\hat{\mathcal{H}}^\dagger\rangle_\Psi|\Psi\rangle \\ &= \left[\hat{H}_+ +\frac{i}{4}\hbar\sum_k\gamma_k\Big(\langle\hat{L}_k^\dagger\rangle_\Psi\hat{L}_k-\hat{L}_k^\dagger\langle\hat{L}_k\rangle_\Psi\Big)\right]|\Psi\rangle \\ &\quad -i\left[\hat{\Gamma}-\langle\hat{\Gamma}\rangle_\Psi-\frac{1}{4}\hbar\sum_k\gamma_k\Big(\langle\hat{L}_k^\dagger\rangle_\Psi\hat{L}_k+\hat{L}_k^\dagger\langle\hat{L}_k\rangle_\Psi-2\hat{L}_k^\dagger\hat{L}_k-2\langle\hat{L}_k^\dagger\rangle_\Psi\langle\hat{L}_k\rangle_\Psi+2\langle\hat{L}_k^\dagger\,\hat{L}_k\rangle_\Psi\Big)\right]|\Psi\rangle, \end{aligned} \tag{19}$$

where we remember the notation in (3). One can directly check that the norm of a state vector is conserved at all times: $\partial_t\langle\Psi|\Psi\rangle = 0$. Although the corresponding density matrix $|\Psi\rangle\langle\Psi|$ would no longer be a solution to either of the original master equations—(5) and (9)—this equation belongs to the same equivalence class as Equations (12) and (13).

Indeed, three state vectors are related through the transformations

$$\begin{aligned} |\Psi\rangle &= \langle\Omega|\Omega\rangle^{-1/2}\,|\Omega\rangle = \langle\Phi|\Phi\rangle^{-1/2}\,|\Phi\rangle, \\ |\Omega\rangle &= \exp\left[-\int\left(\frac{1}{\hbar}\langle\hat{\Gamma}\rangle_\Psi+\frac{1}{2}\sum_k\gamma_k\Big(\langle\hat{L}_k^\dagger\hat{L}_k\rangle_\Psi-\langle\hat{L}_k^\dagger\rangle_\Psi\langle\hat{L}_k\rangle_\Psi\Big)\right)dt\right]|\Psi\rangle \\ &= \exp\left(-\frac{1}{\hbar}\int\langle\hat{\Gamma}\rangle_\Phi\,dt\right)|\Phi\rangle, \end{aligned} \tag{20}$$

defined up to a standard unitary transformation. These relations mean that the three generalized Schrödinger equations we have derived, (12), (13), and (19), belong to the same equivalence class as defined above.

To summarize, by performing dephasing and rank reduction of the quantum-statistical equations, we derived three forms of the generalized Schrödinger equation: (12), (13), and (19). The former two were derived from, respectively, Equations (5) and (9), whereas the latter is the one with a conserved norm. These equations belong to the same equivalence class as (20); therefore, the choice between them for a specific model is a matter of physical relevance, cf. the remarks at the end of Section 2.

3.2. Special Cases of a Normalized State-Vector Equation

For reasons given in the previous section, we choose Equation (19) as the main object of our studies. For brevity, we omit the Lindblad operators' summation and related indices from now on, i.e.,

$$\sum_k \gamma_k \left(\hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2} \{ \hat{\rho},\, \hat{L}_k^\dagger \hat{L}_k \} \right) \mapsto \gamma \left(\hat{L} \hat{\rho} \hat{L}^\dagger - \frac{1}{2} \{ \hat{\rho},\, \hat{L}^\dagger \hat{L} \} \right), \tag{21}$$

while implying the summation whenever the Lindblad operators occur in a quadratic combination.

Since the NH and GKSL parts of Equation (19) have quite different physical origins and applications, it is convenient to consider them separately:

- *Generic NH-driven evolution*. In this case, we neglect the Lindblad part; therefore, Equation (19), as well as Equations (12) and (13), reduce to

$$\begin{aligned} i\hbar\, \partial_t \, |\Psi\rangle &= \hat{\mathcal{H}} |\Psi\rangle - \frac{1}{2} \langle \hat{\mathcal{H}} - \hat{\mathcal{H}}^\dagger \rangle_\Psi |\Psi\rangle \\ &= \hat{H}_+ |\Psi\rangle - i \big(\hat{\Gamma} - \langle \hat{\Gamma} \rangle_\Psi \big) |\Psi\rangle, \end{aligned} \tag{22}$$

  where the anti-Hermitian part $\hat{H}_- = -i\hat{\Gamma}$ is assumed to account for all dissipative effects except those coming from the Lindblad's jump operators;
- *GKSL-driven evolution*. In this case, we neglect the non-Lindblad dissipation part; therefore, Equation (19) reduces to

$$\begin{aligned} i\hbar\, \partial_t \, |\Psi\rangle &= \left[\hat{H}_+ + \frac{i}{2} \hbar \gamma \Big(\Big(\langle \hat{L}^\dagger \rangle_\Psi - \hat{L}^\dagger \Big) \hat{L} + \langle \hat{L}^\dagger\, \hat{L} \rangle_\Psi - \langle \hat{L}^\dagger \rangle_\Psi \langle \hat{L} \rangle_\Psi \Big) \right] |\Psi\rangle \\ &= (\hat{H}_+ + \hat{H}_{L+}) |\Psi\rangle - i\, (\hat{\Gamma}_L - \langle \hat{\Gamma}_L \rangle) |\Psi\rangle, \\ \hat{H}_{L+} &\equiv \frac{i}{4} \hbar \gamma \Big(\langle \hat{L}^\dagger \rangle_\Psi \hat{L} - \langle \hat{L} \rangle_\Psi \hat{L}^\dagger \Big), \\ \hat{\Gamma}_L &\equiv -\frac{1}{4} \hbar \gamma \Big(\langle \hat{L}^\dagger \rangle_\Psi \hat{L} + \langle \hat{L} \rangle_\Psi \hat{L}^\dagger - 2 \hat{L}^\dagger \hat{L} \Big), \end{aligned} \tag{23}$$

  where $\hat{H}_+$ is a Hermitian part of the original subsystem's Hamiltonian, and we assume the notations in (21).
  This equation confirms that the GKSL-driven evolution of a state vector belongs to a class of non-Hermitian Hamiltonian evolutions with a conserved norm; see also the remark after Equation (15).

In the next section, we study these special cases separately, while keeping OQS applications in mind.

4. Basic Properties

Let us study the main properties of the generalized Schrödinger equation in (19) while focusing on its limit cases, as discussed in Section 3.2 above.

4.1. Generic NH-Driven Evolution

In this special case, evolution is governed by Equation (22). It conserves the norm of the state vector because

$$\partial_t \langle \Psi | \Psi \rangle = 0 \tag{24}$$

at all times. The evolution equation is invariant under a complex-valued shift of the zero of the operator $\hat{\mathcal{H}}$:

$$\hat{\mathcal{H}} \mapsto \hat{\mathcal{H}} + (\varepsilon_0 - i\gamma_0)\hat{I}, \;\; |\Psi\rangle \mapsto \mathrm{e}^{-i\varepsilon_0 t/\hbar}\,|\Psi\rangle, \tag{25}$$

with $\hat{I}$ being the identity operator, as per usual. Remember that the conventional Schrödinger equation for conservative systems is invariant only under a real-valued shift in the zero of the Hamiltonian operator; therefore, the invariance in (25) is something which is more general. This seems plausible from the phenomenological point of view because observables should be independent not only from the real-valued shifts in the energy's zero but also from the imaginary-valued shifts that could possibly occur due to quantum fluctuations and virtual particles.

Norm conservation and shift invariance are the features that simultaneously occur in the master equation in (5).

Furthermore, if $|\Psi\rangle$ is an eigenvector of the decay rate operator $\hat{\Gamma}$, then our equation reduces to the ordinary Schrödinger equation with the Hermitian Hamiltonian $\hat{H}_+$.

If $\hat{\mathcal{H}}$ is time-independent, then mean values of its components evolve according to the following formulae:

$$\frac{d}{dt}\langle \hat{H}_+ \rangle_\Psi = \frac{2}{\hbar}\left(\langle \hat{H}_+ \rangle_\Psi \langle \hat{\Gamma} \rangle_\Psi - \frac{1}{2}\langle \{\hat{H}_+, \hat{\Gamma}\} \rangle_\Psi \right), \tag{26}$$

$$\frac{d}{dt}\langle \hat{\Gamma} \rangle_\Psi = \frac{2}{\hbar}\left(\langle \hat{\Gamma} \rangle_\Psi^2 - \langle \hat{\Gamma}^2 \rangle_\Psi + \frac{i}{2}\langle \big[\hat{H}_+, \hat{\Gamma}\big] \rangle_\Psi \right) = \frac{i}{\hbar}\langle \big[\hat{H}_+, \hat{\Gamma}\big] \rangle_\Psi - \frac{2}{\hbar}\mathrm{Var}(\Gamma), \tag{27}$$

where $\mathrm{Var}(X) \equiv \langle X^2 \rangle_\Psi - \langle X \rangle_\Psi^2$ is a variance of a physical variable described by the operator $\hat{X}$. Therefore, if $|\Psi\rangle$ is an eigenvector of $\hat{H}_+$, then the average energy is conserved, and, if it is an eigenvector of the decay operator $\hat{\Gamma}$, then the mean values of both energy and decay rate are conserved.

Furthermore, the evolution Equation (22) allows the usual stationary states, i.e, the eigenvectors of the Hamiltonian $\hat{H}_+$, only if the latter commutes with the decay operator (which corresponds to $\hat{\mathcal{H}}$ being a normal operator).

This means that our model, even in its truncated form (without the Lindblad part), can describe two types of dissipative systems: those with balanced gain and loss, for which $\big[\hat{H}_+, \hat{\Gamma}\big] = -(i/2)\big[\hat{\mathcal{H}}, \hat{\mathcal{H}}^\dagger\big] = 0$ (hence, the usual stationary states can exist [13]), and those with strong dissipation, for which $\big[\hat{H}_+, \hat{\Gamma}\big] \neq 0$ (hence, no usual stationary states can exist in general). This feature significantly expands the range of applicability of our generalized Schrödinger equation approach compared to some other generalizations, cf. [37], and places the proverbial pyramid on its base.

4.2. GKSL-Driven Evolution of L-Eigenvector

Let us consider the special case of purely GKSL-driven evolution in (23), with two additional assumptions: we restrict ourselves to only one species of Lindblad jump operators (hence, the index k takes only one value), and we assume that our state vector is an eigenvector of the jump operator:

$$|\Psi\rangle = |L\rangle : \quad \hat{L}|L\rangle = L|L\rangle, \;\; \langle L|L\rangle = 1, \tag{28}$$

where an eigenvalue $L = \langle L|\hat{L}|L\rangle$ is a complex-valued function of time in general.

Under these assumptions, Equation (23) yields the following evolution equation for the state vector $|L\rangle$:

$$\begin{aligned} i\hbar\,\partial_t\,|L\rangle &= \left[\hat{H}_+ - \frac{i}{2}\hbar\gamma_1\Big(\hat{L}^\dagger - L^*\hat{I}\Big)\hat{L}\right]|L\rangle \\ &= (\hat{H}_+ + \hat{H}_{L+})|\Psi\rangle - i\,(\hat{\Gamma}_L - \langle\hat{\Gamma}_L\rangle_L)|\Psi\rangle, \\ \hat{H}_{L+} &\equiv \frac{i}{4}\hbar\gamma_1\Big(L^*\hat{L} - L\hat{L}^\dagger\Big) \mapsto \frac{i}{4}\hbar\gamma_1\Big(|L|^2 - L\hat{L}^\dagger\Big), \\ \hat{\Gamma}_L &\equiv -\frac{1}{4}\hbar\gamma_1\Big(L^*\hat{L} + L\hat{L}^\dagger - 2\hat{L}^\dagger\hat{L}\Big) \mapsto -\frac{1}{4}\hbar\gamma_1\Big(|L|^2 - L\hat{L}^\dagger\Big), \end{aligned} \tag{29}$$

where the notation '$\mapsto$' indicates that the projection of the jump operator on the eigen(ket)vector is implied. Notice also that, for this model,

$$\langle\hat{H}_{L+}\rangle_L = 0, \;\; \langle\hat{\Gamma}_L\rangle_L = 0, \tag{30}$$

which can be directly verified.

Otherwise, this model belongs to a class of NH models discussed in Section 4.1, as one can clearly see from the form of its evolution equation. This means that the properties in (24) and (25), as well as other generic properties of NH-driven evolution equations, would be valid for this case too.

If operators $\hat{H}_+$ and $\hat{L}$ are themselves time-independent, then their mean values evolve according to the following equations:

$$\begin{aligned} \frac{d}{dt}\langle\hat{H}_+\rangle_L &= \frac{\gamma_1}{2}|L|^2\Big(2\langle\hat{H}_+\rangle_L - \frac{1}{L}\langle\hat{L}\hat{H}_+\rangle_L - \frac{1}{L^*}\langle\hat{H}_+\hat{L}^\dagger\rangle_L\Big) \\ &= \frac{\gamma_1}{2}|L|^2\langle\hat{C}_{L+} + \hat{C}_{L+}^\dagger\rangle_L = \frac{\gamma_1}{2}|L|^2\langle\hat{C}_{\Delta+}\rangle_L, \end{aligned} \tag{31}$$

$$\begin{aligned} \frac{1}{L}\frac{d}{dt}L &= -\frac{i}{\hbar}\Big(\frac{1}{L}\langle\hat{L}\hat{H}_+\rangle_L - \langle\hat{H}_+\rangle_L\Big) + \frac{\gamma_1}{2}\Big(|L|^2 - \langle\hat{L}\hat{L}^\dagger\rangle_L\Big) \\ &= \frac{i}{\hbar}\langle\hat{C}_{L+}\rangle_L - \frac{\gamma_1}{2}\langle\hat{C}_L\rangle_L, \end{aligned} \tag{32}$$

$$\begin{aligned} \frac{1}{L^*}\frac{d}{dt}L^* &= \frac{i}{\hbar}\Big(\frac{1}{L^*}\langle\hat{H}_+\hat{L}^\dagger\rangle_L - \langle\hat{H}_+\rangle_L\Big) + \frac{\gamma_1}{2}\Big(|L|^2 - \langle\hat{L}\hat{L}^\dagger\rangle_L\Big) \\ &= -\frac{i}{\hbar}\langle\hat{C}_{L+}^\dagger\rangle_L - \frac{\gamma_1}{2}\langle\hat{C}_L\rangle_L, \end{aligned} \tag{33}$$

where we denoted $\hat{C}_L \equiv [\hat{L}, \hat{L}^\dagger] = \hat{C}_L^\dagger$, $\hat{C}_{L+} \equiv [\hat{H}_+, \hat{L}/L]$, $\hat{C}_{\Delta+} \equiv [\hat{H}_+, \hat{\Delta}_L]$ and $\hat{\Delta}_L \equiv \hat{L}/L - \hat{L}^\dagger/L^* = -\hat{\Delta}_L^\dagger$.

From the first of these equations, one can deduce that the original subsystem's average energy is conserved if operators $\hat{H}_+$ and $i\,\hat{\Delta}_L$ commute, i.e., they have common eigenvectors.

Furthermore, by rearranging those equations one can obtain the relation

$$\frac{d}{dt}\langle\hat{H}_+\rangle_L = \gamma_1|L|^2\langle\hat{H}_+\rangle_L - \frac{i\hbar\gamma_1}{2}\Big(\frac{1}{L}\frac{dL}{dt} - \frac{1}{L^*}\frac{dL^*}{dt}\Big), \tag{34}$$

which can be regarded as a first-order differential equation for $\langle\hat{H}_+\rangle_L$ as a function of time. Solving it, we obtain the relation between mean values

$$\begin{aligned} \langle\hat{H}_+\rangle_L &= \langle\hat{H}_+\rangle_0\,\mathrm{e}^{\gamma_1\int_{t_0}^t|L|^2dt'} - \frac{i\hbar\gamma_1}{2}\ln(L/L^*) \\ &\quad - \frac{i\hbar\gamma_1^2}{2}\,\mathrm{e}^{\gamma_1\int_{t_0}^t|L|^2dt'}\int_{t_0}^t \mathrm{e}^{-\gamma_1\int_{t_0}^{t'}|L|^2dt''}\ln(L/L^*)|L|^2dt', \end{aligned} \tag{35}$$

where $\langle \hat{H}_+ \rangle_0 = \langle \hat{H}_+ \rangle_L \big|_{t=t_0}$ is an initial value of average energy as a function of time.

Furthermore, let us study the eigenvectors of the Hamiltonian $\hat{H}_+$, which are solutions to the evolution equation (Equation (29)). For simplicity, let us consider a special case when $\hat{H}_+$ commutes with the Lindblad operator:

$$\left[\hat{H}_+, \hat{L}\right] = 0, \tag{36}$$

thus, having common eigenvectors, $\hat{H}_+|L_\varepsilon\rangle = \varepsilon|L_\varepsilon\rangle$. In this case, mean value of $\hat{H}_+$ is conserved in time, which corresponds to an open system with balanced gain and loss.

Let us also perform the decomposition into the projected and orthogonal parts with respect to the vector $|L_\varepsilon\rangle$:

$$\hat{L}^\dagger|L_\varepsilon\rangle = L^*|L_\varepsilon\rangle + L_\perp|\perp\rangle, \tag{37}$$

where $\langle L_\varepsilon|\perp\rangle = 0$ and $L_\perp = \langle\perp|\hat{L}^\dagger|L_\varepsilon\rangle / \langle\perp|\perp\rangle$. Then, Equation (29) reduces to the differential equation

$$\partial_t |L_\varepsilon\rangle \;\; = \;\; -\frac{i\varepsilon}{\hbar}|L_\varepsilon\rangle - \frac{\gamma_1}{2} L L_\perp|\perp\rangle, \tag{38}$$

whose formal solution can be written in the form

$$|L_\varepsilon\rangle = \mathrm{e}^{-i\varepsilon(t-t_0)/\hbar}\left(|L\rangle_0 - \frac{\gamma_1}{2}\int\limits_{t_0}^{t} dt'\, \mathrm{e}^{i\varepsilon t'/\hbar} L L_\perp|\perp\rangle\right), \tag{39}$$

where $|L\rangle_0 = |L_\varepsilon\rangle\big|_{t=t_0}$ is an initial value of $|L_\varepsilon\rangle$ as a vector function of time.

Equation (38) and the solution in (39) demonstrate that the eigenstates of the Hamiltonian $\hat{H}_+$, which are solutions of Equation (29), are not stationary in general, even in the balanced case with conserved average energy. This reaffirms that the evolution in (29) is indeed of a dissipative type, with the time-dependent source/sink vector $LL_\perp|\perp\rangle$.

5. Example: Dissipative Harmonic Oscillator

In the conventional quantum mechanics of conservative systems, the harmonic oscillator is a benchmark model with a clearly derivable classical limit and plenitude of applications. Its Hamiltonian can be written in the standard form:

$$\hat{H}_{\mathrm{HO}} = \hat{p}^2/2 + m\omega^2\hat{q}^2/2 = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2), \tag{40}$$

where $\hat{a} = (\sqrt{m\omega/\hbar}\,\hat{q} + i\hat{p}/\sqrt{m\omega\hbar})/\sqrt{2}$ and $\hat{a}^\dagger$ are ladder operators whose commutator $[\hat{a}, \hat{a}^\dagger] = 1$, as per usual.

By taking quantum averages in the eigenbasis of the operator $\hat{a}$, usually referred to as coherent states, one can obtain a set of ordinary differential equations:

$$\frac{d}{dt}\alpha = -i\omega\alpha, \tag{41}$$

$$\frac{d}{dt}\begin{pmatrix}\langle\hat{p}\rangle_\alpha \\ \langle\hat{q}\rangle_\alpha\end{pmatrix} = \begin{pmatrix}0 & -m\omega^2 \\ 1/m & 0\end{pmatrix}\begin{pmatrix}\langle\hat{p}\rangle_\alpha \\ \langle\hat{q}\rangle_\alpha\end{pmatrix}, \tag{42}$$

where $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. From these equations, a second-order differential equation immediately follows:

$$\frac{d^2q}{dt^2} + \omega^2 q = 0, \tag{43}$$

where $q = \langle\hat{q}\rangle_\alpha$, which describes the classical harmonic oscillator.

On the other hand, let us recall the classical damped oscillator

$$\frac{d^2q}{dt^2} + 2\delta\frac{dq}{dt} + \omega_0^2 q = 0, \tag{44}$$

where $\delta = \mu/2\tilde{m}$, $\omega_0 = \sqrt{\kappa/\tilde{m}}$, and $\tilde{m}$, κ, and μ are, respectively, mass, spring constant, and friction. This model was extensively used in the early quantization attempts of dissipative systems.

The straightforward quantization of the system in (44) via the Lagrangian approach has been shown to be somewhat inconsistent [1,40]. The currently prevailing point of view is that the damped oscillator can be a classical limit of a number of quantum models; one example is the model based on Gisin's generalized Schrödinger equation.

In our case, the question arises: what model, as defined by means of both Hamiltonian and non-Hamiltonian (Liouvillian) terms, recovers Equation (44) in the classical limit?

Similarly to the previous section, let us consider cases of generic NH-driven and GKSL-driven evolutions separately first; then, we consider the unified NH–GKSL evolution.

5.1. Generic NH-Driven Evolution

Let us generalize the Hamiltonian (40) by moving its overall coupling constant into the complex domain. From the quantum-mechanical point of view, it is a natural generalization because coupling constants are not observables *per se* and, therefore, do not have to be real numbers, especially in the presence of quantum fluctuations and virtual particles. We thus assume that

$$\hat{\mathcal{H}} = (1 - i\eta/\omega)\hat{H}_{\mathrm{HO}} = \hbar(\omega - i\eta)(\hat{a}^\dagger\hat{a} + 1/2), \tag{45}$$

while the notations in Equation (40) remain intact. Correspondingly, the evolution of this system is described by Equation (22), with the following associations:

$$\hat{H}_+ = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2),\ \hat{\Gamma} = \hbar\eta\,\hat{a}^\dagger\hat{a}, \tag{46}$$

while mean values are assumed to be computed in the eigenbasis $|\alpha\rangle$ of the operator $\hat{a}$. Note that solutions to other generalized Schrödinger Equations (12) and (13), in this basis, are related to $|\Psi\rangle$ via the transformations

$$\exp\left(\eta\int|\alpha|^2\,dt\right)|\alpha(\Omega)\rangle = |\alpha(\Psi)\rangle = |\alpha(\Phi)\rangle, \tag{47}$$

which can be easily derived from Equation (20); it is, thus, sufficient to study a case of a normalized state vector $|\Psi\rangle$, as mentioned before.

Upon averaging in the basis $|\alpha\rangle$, we obtain the analogues of Equations (41) and (42) for the model in (45):

$$\frac{d}{dt}\alpha = -(\eta + i\omega)\alpha, \tag{48}$$

$$\frac{d}{dt}\begin{pmatrix}\langle\hat{p}\rangle_\alpha\\ \langle\hat{q}\rangle_\alpha\end{pmatrix} = \begin{pmatrix}-\eta & -m\omega^2\\ 1/m & -\eta\end{pmatrix}\begin{pmatrix}\langle\hat{p}\rangle_\alpha\\ \langle\hat{q}\rangle_\alpha\end{pmatrix}, \tag{49}$$

from which a second-order differential equation immediately follows:

$$\frac{d^2q}{dt^2} + 2\eta\frac{dq}{dt} + (\omega^2 + \eta^2)q = 0, \tag{50}$$

where $q = \langle\hat{q}\rangle_\alpha$. Thus, the classical limit of the model (45) is the damped oscillator (44), with

$$\delta \mapsto \eta,\ \omega_0^2 \mapsto \omega^2 + \eta^2, \tag{51}$$

which means that the NH coupling induces friction in the classical limit. Solving this, we reveal the following cases:

- If $\omega \neq 0$ and $\eta > 0$, then the damping is subcritical, which is an exponential decay with oscillations with the frequency ω:

$$q(t) = q_0 \exp(-\eta t)\left[\cos(\omega t) + \frac{\eta + \dot{q}_0/q_0}{\omega}\sin(\omega t)\right], \tag{52}$$

  where q_0 and $\dot{q}_0$ are, respectively, oscillator's position and velocity at $t = 0$;
- If $\omega = 0$ and $\eta > 0$, then the damping is critical:

$$\begin{aligned} q(t) &= \lim_{\omega\to 0} q_0 \exp(-\eta t)\left[\cos(\omega t) + \frac{\eta + \dot{q}_0/q_0}{\omega}\sin(\omega t)\right] \\ &= q_0 \exp(-\eta t)[1 + (\eta + \dot{q}_0/q_0)t], \end{aligned} \tag{53}$$

  which is rather surprising because the "Hamiltonian" (45) is completely skew-adjoint in this case, yet it still yields a realistic classical system upon averaging.

Thus, in the classical limit, the model described by Equations (45) and (22) reduces to either an underdamped or a critically damped harmonic oscillator, depending on the value of frequency.

5.2. GKSL-Driven Evolution

Let us consider the conventional harmonic oscillator in (40) and assume that its evolution is affected by the dissipative effects described by two jump operators:

$$\hat{L}_1 = \hat{a},\ \hat{L}_2 = \hat{a}^\dagger, \tag{54}$$

with the non-negative relaxation rates γ_1 and γ_2, respectively. Then, Equation (23) takes the form

$$\begin{aligned} i\hbar\,\partial_t|\Psi\rangle &= \hat{H}_{\rm HO}|\Psi\rangle + \frac{i}{2}\hbar\Big[\gamma_1\Big((\langle\hat{a}^\dagger\rangle_\Psi - \hat{a}^\dagger)\hat{a} + \langle\hat{a}^\dagger\hat{a}\rangle_\Psi - \langle\hat{a}^\dagger\rangle_\Psi\langle\hat{a}\rangle_\Psi\Big) + \gamma_2\Big(\hat{a}\leftrightarrow\hat{a}^\dagger\Big)\Big]|\Psi\rangle \\ &= (\hat{H}_{\rm HO} + \hat{H}_{L+})|\Psi\rangle - i\,(\hat{\Gamma}_L - \langle\hat{\Gamma}_L\rangle)|\Psi\rangle, \\ \hat{H}_{L+} &\equiv \frac{i}{4}\hbar\Big[\gamma_1\Big(\langle\hat{a}^\dagger\rangle_\Psi\hat{a} - \langle\hat{a}\rangle_\Psi\hat{a}^\dagger\Big) + \gamma_2\Big(\langle\hat{a}\rangle_\Psi\hat{a}^\dagger - \langle\hat{a}^\dagger\rangle_\Psi\hat{a}\Big)\Big], \\ \hat{\Gamma}_L &\equiv -\frac{1}{4}\hbar\Big[\gamma_1\Big(\langle\hat{a}^\dagger\rangle_\Psi\hat{a} + \langle\hat{a}\rangle_\Psi\hat{a}^\dagger - 2\hat{a}^\dagger\hat{a}\Big) + \gamma_2\Big(\langle\hat{a}\rangle_\Psi\hat{a}^\dagger + \langle\hat{a}^\dagger\rangle_\Psi\hat{a} - 2\hat{a}\hat{a}^\dagger\Big)\Big], \end{aligned} \tag{55}$$

where $\hat{H}_{\rm HO}$ is borrowed from Equation (40), and a shorthand notation $\leftrightarrow$ means an interchange of symbols.

In the eigenbasis of the operator $\hat{a}$, solutions to other generalized Schrödinger Equations (12) and (13) are related via the transformations

$$|\alpha(\Omega)\rangle = \exp\left(-\frac{1}{2}\gamma_2 t\right)|\alpha(\Psi)\rangle = |\alpha(\Phi)\rangle, \tag{56}$$

which can be easily derived from Equation (20); it is, thus, sufficient to study a case of a normalized state vector $|\Psi\rangle$ only.

Upon averaging in the basis $|\alpha\rangle$, Equation (55) yields the following analogues of Equations (41) and (42):

$$\frac{d}{dt}\alpha = -\left(\frac{1}{2}\gamma_1 + i\omega\right)\alpha, \tag{57}$$

$$\frac{d}{dt}\begin{pmatrix}\langle\hat{p}\rangle_\alpha \\ \langle\hat{q}\rangle_\alpha\end{pmatrix} = \begin{pmatrix}-\gamma_1/2 & -m\omega^2 \\ 1/m & -\gamma_1/2\end{pmatrix}\begin{pmatrix}\langle\hat{p}\rangle_\alpha \\ \langle\hat{q}\rangle_\alpha\end{pmatrix}, \tag{58}$$

from which a second-order differential equation immediately follows:

$$\frac{d^2q}{dt^2} + \gamma_1 \frac{dq}{dt} + \left(\omega^2 + \frac{1}{4}\gamma_1^2\right)q = 0, \tag{59}$$

where $q = \langle \hat{q} \rangle_\alpha$, as per usual.

Thus, the classical limit of the model in (55) is the damped oscillator (44) with

$$\delta \mapsto \frac{1}{2}\gamma_1,\ \omega_0^2 \mapsto \omega^2 + \frac{1}{4}\gamma_1^2, \tag{60}$$

which means that the Lindblad evolution also induces friction in the classical limit.

It is worth noticing that the γ_2-coupled term, $\gamma_2\big((\langle \hat{a} \rangle_\Psi - \hat{a})\hat{a}^\dagger + \langle \hat{a}\hat{a}^\dagger \rangle_\Psi - \langle \hat{a} \rangle_\Psi \langle \hat{a}^\dagger \rangle_\Psi\big)$, does not affect the evolution of coherent-state mean values based on Equation (19). This indicates that this term represents higher-order corrections that can be taken into account when returning to the more general (density operator) formalism.

Furthermore, by solving Equation (59), we reveal the following special cases:

- If $\omega \neq 0$, then the damping is subcritical, which is an exponential decay with oscillations with the frequency ω:

$$q(t) = q_0 \exp\left(-\gamma_1 t/2\right)\left[\cos\left(\omega t\right) + \frac{\gamma_1/2 + \dot{q}_0/q_0}{\omega}\sin\left(\omega t\right)\right], \tag{61}$$

  where q_0 and $\dot{q}_0$ are, respectively, oscillator's position and velocity at $t = 0$;
- If $\omega \to 0$, then the damping turns critical:

$$\begin{aligned} q(t) &= \lim_{\omega \to 0} q_0 \exp\left(-\gamma_1 t/2\right)\left[\cos\left(\omega t\right) + \frac{\gamma_1/2 + \dot{q}_0/q_0}{\omega}\sin\left(\omega t\right)\right] \\ &= q_0 \exp\left(-\gamma_1 t/2\right)[1 + (\gamma_1/2 + \dot{q}_0/q_0)t], \end{aligned} \tag{62}$$

  which is also surprising because the Hamiltonian can be neglected. In this case, system's evolution is driven solely by the non-Hamiltonian (Liouvillian) term, yet we obtain a physically relevant classical system upon averaging.

5.3. NH–GKSL Evolution and Cancellation of Dissipations

Now let us unify the models described in Sections 5.1 and 5.2: we assume that the dissipative effects are encoded both in the non-Hermitian Hamiltonian (45) and in the Lindblad term with two jump operators (54).

In this case, Equation (19) takes the form

$$\begin{aligned} i\hbar\,\partial_t |\Psi\rangle &= \left(1 - i\frac{\eta}{\omega}\right)\hat{H}_{\rm HO}|\Psi\rangle + i\frac{\eta}{\omega}\langle \hat{H}_{\rm HO} \rangle_\Psi |\Psi\rangle + \\ &\quad \frac{i}{2}\hbar\Big[\gamma_1\Big((\langle \hat{a}^\dagger \rangle_\Psi - \hat{a}^\dagger)\hat{a} + \langle \hat{a}^\dagger \hat{a} \rangle_\Psi - \langle \hat{a}^\dagger \rangle_\Psi \langle \hat{a} \rangle_\Psi\Big) + \gamma_2\Big(\hat{a} \leftrightarrow \hat{a}^\dagger\Big)\Big]|\Psi\rangle \\ &= \big(\hat{H}_{\rm HO} + \hat{H}_{L+}\big)|\Psi\rangle - i\big(\hat{\Gamma}_L - \langle \hat{\Gamma}_L \rangle_\Psi + \hat{\Gamma}_\eta - \langle \hat{\Gamma}_\eta \rangle_\Psi\big)|\Psi\rangle, \\ \hat{H}_{L+} &\equiv \frac{i}{4}\hbar\Big[\gamma_1\Big(\langle \hat{a}^\dagger \rangle_\Psi \hat{a} - \langle \hat{a} \rangle_\Psi \hat{a}^\dagger\Big) + \gamma_2\Big(\langle \hat{a} \rangle_\Psi \hat{a}^\dagger - \langle \hat{a}^\dagger \rangle_\Psi \hat{a}\Big)\Big], \\ \hat{\Gamma}_L &\equiv -\frac{1}{4}\hbar\Big[\gamma_1\Big(\langle \hat{a}^\dagger \rangle_\Psi \hat{a} + \langle \hat{a} \rangle_\Psi \hat{a}^\dagger - 2\hat{a}^\dagger\hat{a}\Big) + \gamma_2\Big(\langle \hat{a} \rangle_\Psi \hat{a}^\dagger + \langle \hat{a}^\dagger \rangle_\Psi \hat{a} - 2\hat{a}\hat{a}^\dagger\Big)\Big], \\ \hat{\Gamma}_\eta &\equiv \hbar\eta\,\hat{a}^\dagger\hat{a}, \end{aligned} \tag{63}$$

where $\hat{H}_{\rm HO}$ has been defined in Equation (40), and shorthand notation '$\leftrightarrow$' means an interchange of symbols.

In the eigenbasis of the operator $\hat{a}$, solutions to the other generalized Schrödinger Equations (12) and (13) are related via the transformations in (20):

$$\exp\left(\eta\int|\alpha|^2\,dt\right)|\alpha(\Omega)\rangle=\exp\left(-\frac{1}{2}\gamma_2 t\right)|\alpha(\Psi)\rangle=|\alpha(\Phi)\rangle, \tag{64}$$

which supports our decision to study the normalized state-vector case (see the end of Section 3.1).

Furthermore, upon averaging in the basis $|\alpha\rangle$, Equation (63) yields the following analogues of Equations (41) and (42):

$$\frac{d}{dt}\alpha=-\left(\frac{1}{2}\gamma_1+\eta+i\omega\right)\alpha, \tag{65}$$

$$\frac{d}{dt}\begin{pmatrix}\langle\hat{p}\rangle_\alpha\\ \langle\hat{q}\rangle_\alpha\end{pmatrix}=\begin{pmatrix}-\eta-\gamma_1/2 & -m\omega^2\\ 1/m & -\eta-\gamma_1/2\end{pmatrix}\begin{pmatrix}\langle\hat{p}\rangle_\alpha\\ \langle\hat{q}\rangle_\alpha\end{pmatrix}, \tag{66}$$

from which a second-order differential equation immediately follows:

$$\frac{d^2q}{dt^2}+2(\eta+\gamma_1/2)\frac{dq}{dt}+\left(\omega^2+(\eta+\gamma_1/2)^2\right)q=0, \tag{67}$$

where $q=\langle\hat{q}\rangle_\alpha$, as per usual.

Thus, the classical limit of the model in (55) is the damped oscillator in (44) with

$$\delta\mapsto\eta+\gamma_1/2,\ \omega_0^2\mapsto\omega^2+(\eta+\gamma_1/2)^2, \tag{68}$$

which means that the NH–GKSL evolution also induces friction in the classical limit. Its solutions reveal the following special cases:

- If $\omega\neq0$ and $\eta=-\gamma_1/2$ (the latter is possible because η can be both positive and negative, unlike the strictly non-negative Lindblad relaxation constants), then the damping term disappears in Equation (67). In this case, evolution becomes a plain oscillation with the frequency ω:

$$q(t)=q_0\left(\cos(\omega t)+\frac{\dot{q}_0}{q_0\omega}\sin(\omega t)\right), \tag{69}$$

  where q_0 and $\dot{q}_0$ are, respectively, the oscillator's position and velocity at $t=0$. This indicates that the dissipative effects associated with NH and GKSL terms can act against each other, which leads to their mutual cancellation;
- If $\omega\neq0$ and $\eta\neq-\gamma_1/2$, then the damping is subcritical, which is an exponential decay with oscillations at the frequency ω:

$$q(t)=q_0\exp\left(-(\eta+\gamma_1/2)t\right)\left[\cos(\omega t)+\frac{\eta+\gamma_1/2+\dot{q}_0/q_0}{\omega}\sin(\omega t)\right], \tag{70}$$

  where q_0 and $\dot{q}_0$ are, respectively, oscillator's position and velocity at $t=0$;
- If $\omega\to0$ and $\eta+\gamma_1/2>0$, then the damping becomes critical:

$$\begin{aligned}q(t)&=\lim_{\omega\to0}q_0\exp\left(-(\eta+\gamma_1/2)t\right)\left[\cos(\omega t)+\frac{\eta+\gamma_1/2+\dot{q}_0/q_0}{\omega}\sin(\omega t)\right]\\&=q_0\exp\left(-(\eta+\gamma_1/2)t\right)[1+(\eta+\gamma_1/2+\dot{q}_0/q_0)t],\end{aligned} \tag{71}$$

  which is somewhat surprising because the Hamiltonian is neglected in this case; therefore, the evolution becomes dominated by the Liouvillian term, yet we obtain a physically realistic classical system upon averaging;

- If $\omega \to 0$ and $\eta + \gamma_1/2 = 0$, then the damping disappears, while the average value of $\langle \hat{q} \rangle_\alpha$ grows linearly in time:

$$q(t) \quad = \quad \lim_{\omega \to 0} \, q_0 \left(\cos(\omega t) + \frac{\dot{q}_0}{q_0 \omega} \sin(\omega t) \right) = q_0 + \dot{q}_0 t, \tag{72}$$

  which corresponds to a system in free motion. Notice that this classically free system descends from a quantum system that does not have a Hamiltonian, hence it experiences no forces in a sense of Newton's first law of motion, yet its underlying dynamics are non-trivial (due to the presence of non-Hamiltonian terms).

6. Conclusions

In the present study, we started out with the quantum-statistical approach, using both the non-Hermitian Hamiltonian and Lindblad's jump operators: we presented various generalizations of the von Neumann equation for density operator, which did not assume the Markovian approximation in a general case. We demonstrated that, if one performs the dephasing procedure by neglecting off-diagonal components of the density matrix that are responsible for coherence and spontaneous transitions from pure to mixed states, followed by neglecting all but one diagonal component thereof, then one reduces quantum-statistical equations to quantum-mechanical Schrödinger-type equations for state vectors. We thus restricted ourselves to dealing with pure states only, but it can be a robust approximation in a number of cases where one has the full classical statistical information about a system (which does not imply the full knowledge of the possible outcomes of quantum measurement).

We first obtained generalized Schrödinger Equations (12) and (13), which directly originated from the master equations, i.e., Equations (5) and (9), for normalized and non-normalized density operators, respectively. We showed that both evolutions, (12) and (13), do not preserve the norm of a state vector, due to the fact that the dephasing and rank reduction procedures replace some jump operators with their mean values. Therefore, in addition to those equations, we derived the generalized Schrödinger Equation (19), which directly descends from neither of the original master equations, but describes the time evolution of the state vector with a conserved norm. In a sense, Equation (19) can be viewed as a state-vector analogue of Equation (5) because it is also derived using the post-selecting procedure of normalization.

Furthermore, we demonstrated that all three Equations, (12), (13), and (19), belong to the same equivalence class. For this reason, we focused our studies on the evolution in (19), while keeping in mind applications to open quantum systems. We demonstrated that equations from the equivalence class in (20) can describe not only systems which can stay in the stationary eigenstates of the Hamiltonian, but also those which can evolve from those eigenstates. This indicates that our generalizations have a broader range of applicability than some of the generalized Schrödinger equations proposed hitherto. Another advantage is that they do not contain any stochastic terms or other random functions.

From a mathematical viewpoint, each of these equations in general is both nonlinear (with respect to wavefunction) and non-local because various mean values inside their structure contain wavefunction and integration over the whole configuration space. These two features indicate that our equations form a modeling framework that not only has a rich mathematical structure, but is also *a priori* more natural for describing diverse quantum phenomena and systems, including those involving measuring apparati.

Finally, as an instructive example, we considered the model of the quantum harmonic oscillator in the presence of dissipative effects described by non-Hermitian Hamiltonian and Lindblad's terms. Using coherent states' basis, we derived its classical limit, which turned out to be the damped harmonic oscillator, though with different regimes. In particular, we considered the case when non-Hermitian Hamiltonian and Lindblad's dissipative effects cancel each other out, thus resulting in the decay-free evolution of an oscillating system, at least in the leading-order approximation after dephasing and averaging.

Yet another interesting example shown is the classically free system which descends from a non-trivial (not free from interactions) quantum system. A seeming contradiction to Newton's first law of motion is resolved due to the fact that the quantum system in question does not have a Hamiltonian, therefore no Newtonian forces are present therein; instead, its quantum dynamics are governed by NH and GKSL terms, which are both non-Hamiltonian by nature. This uncovers a way to formulate a large class of quantum-mechanical non-Hamiltonian models whose dynamics are not determined by conventional potentials and forces, but which come about through the quantum statistical effects of the system's environment.

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Abbreviations

The following abbreviations are used in this manuscript:

GKSL	Gorini–Kossakowski–Sudarshan–Lindblad
HO	Harmonic oscillator (quantum)
NH	Non-Hermitian Hamiltonian
OQS	Open quantum system

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