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Note on symmetry operations in quantum mechanics *)

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ABSTRACT

It has been known for long time that symmetry groups on quantum mechanical systems give rise to unitary or antiunitary representations (up to a factor) in Hilbert space. A new and simple proof for this is given.

REMARK

This note does not contain any new result. Only the proof that symmetries of quantum mechanical systems give rise to either unitary or antiunitary representations finds a new and simple form. In order to state the problem, it seemed, however, necessary to discuss the implications of symmetries in some detail.

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1. Introduction.

a) Complete set of observables.

We assume a definite physical system S to be given. The system will be described in a Hilbert space $\mathcal{H}$. Take any observable (hermitian operator) A' and define a class $\mathfrak{A}$ in the following way :

$$A' \in \mathfrak{A}$$

$A \in \mathfrak{A}$ if and only if A commutes with all other $A \in \mathfrak{A}$.
Now if $A \in \mathfrak{A}$; $A' \in \mathfrak{A}$ then

$$A\, A' \in \mathfrak{A}$$

$$aA + bA' \in \mathfrak{A} \quad , \quad (a,\ b \ \text{ real})$$

Hence from the distributive and associative law it follows :

$\mathfrak{A}$ is an abelian ring with real coefficients.

Let $B \notin \mathfrak{A}$. This B defines in the same way an abelian ring $\mathfrak{B}$. In this way we may divide up all observables into such rings $\mathfrak{A}$, $\mathfrak{B}$, $\mathcal{L}$, $\mathfrak{D}$, ... The set of all observables is not a ring since the product of two self-adjoint operators may be not self-adjoint. All operators of any such ring can be brought simultaneously in diagonal form (spectral representation) and hence can be measured without interference. There is a proof by v. Neumann [1] that in such a ring $\mathfrak{A}$ there exists always an element A_o say, such that all elements of the ring are functions of A_o , namely $A_i = f_i(A_o)$. In principle the measurement of A_o alone would determine completely the state of the system.

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[1] J. v. Neumann, Math. Grundlagen der Quantenmechanik, Springer, 1932.

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This might be technically impossible. In fact, normally one takes one convenient operator A_1, say. If now all other elements of $\mathfrak{A}$ cannot be represented as functions of A_1, one takes another one A_2 and so on, till one has as many operators as are necessary in order to generate the whole ring $\mathfrak{A}$. The set of elements $A_1 \ldots . A_n$ is called a complete system of commuting observables. In actual cases, one frequently seems to be able to construct such a system. For every ring $\mathfrak{A}, \mathfrak{B}, \mathfrak{L}, \mathfrak{D} \ldots .$ we assume such a set of generating elements : A_i, $B_i \ldots .$ and call the set of rings $\{ \mathfrak{A}, \mathfrak{B}, \mathfrak{L} \ldots \}$ complete, if every observable belongs to one of the rings.

b) Superselection rules.

Obviously the multiples of the unit operator commute with all our observables. If there exists, apart from that, another operator X' commuting with all observables, then the Hilbert space can be decomposed into superselection subspaces. No observable will have matrix elements between different superselection subspaces (that is the reason for the name).[2)]

Proof :

Together with X' also X'^+ (the hermitian conjugate of X) commutes, hence $X = \frac{1}{2}(X'+X'^+)$ is hermitian and commutes also. We assume now that X has a discrete spectral representation :

$$X = \sum_i \lambda_i X_i \quad ; \quad \text{with}$$

$$X_i^2 = X_i \quad ; \quad X_i X_k = 0 \quad (i \neq k) \quad \text{and} \quad X_i^+ = X_i$$

X_i projects into that subspace $\mathcal{H}_i$ of $\mathcal{H}$, where X has the eigenvalue λ_i.

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2) C.G. Wick, A.S. Wightman, E.P. Wigner, Phys.Rev. 88, 101 (1952).

(Always $\lambda_i \neq \lambda_k$, if λ_j occurs n_j times, $\mathcal{H}_j$ has dimension $n_j \leq \infty$).

From

$$[XA] = \sum_i \lambda_i [X_i A] = 0$$

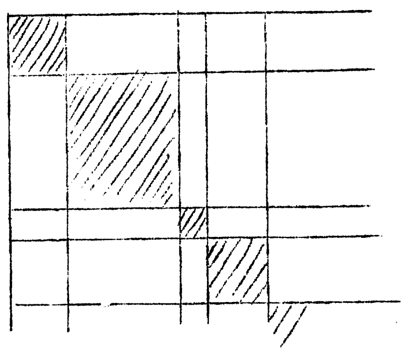
follows by multiplication with X_k from left and right

$$\lambda_k [AX_k] = 0 \quad \text{or} \quad [AX_k] = 0 \quad \text{for} \quad \lambda_k \neq 0.$$

Let be $\varphi_i \in \mathcal{H}_i$, $\varphi_k \in \mathcal{H}_k$ $i \neq k$. Then at least one o λ_i, λ_k is $\neq 0$, say $\lambda_k \neq 0$. Thus

$$\langle \varphi_i, A\varphi_k \rangle = \langle X_i \varphi_i, AX_k \varphi_k \rangle = \langle X_i \varphi_i, X_k A \varphi_k \rangle = \langle X_k X_i \varphi_i, A\varphi_k \rangle = 0$$

Since A can be any observable, the proof is complete. Any observable can then be brought to the following form :

A = 

and the dimensions (in general infinite) of these boxes on the diagonal will be the same for all observables. Examples for such superselection rules are the charge conservation and baryon conservation.

One sees that in the presence of superselection rules arbitrary unitary transformations cannot be admitted, since they would in general mix up the superselection subspaces. Those unitary transformations, however, which transform from a basis $\{\psi_i\}$ belonging to the ring $\mathfrak{A}$ to a basis $\{\varphi_i\}$ belonging to the ring $\mathfrak{L}$, leave the subspace invariant and are admitted. We may even admit those transformations which furthermore induce a permutation of the superselection subspaces.

2. Definition of symmetry.

A transformation group G may be defined in many ways, e.g. :

- Interchanging of particles,
- Charge conjugation (better perhaps: matter conjugation)
- Lorentz transformations
- Gauge transformations

etc.

a) Active and passive interpretation.

There are two ways of interpretation :

i) passive

ii) active

Example : The three dimensional rotation may be interpreted passively as a rotation of the coordinate system or actively as a bodily rotation of the system S under observation.

In what follows we shall adopt throughout the active interpretation, which is always feasible, whereas the passive one looks very strange, if not impossible in some cases.

b) Formal definition of symmetry.

Now we define what we mean by saying that our physical system S is invariant under a group G, which we shall then call a symmetry group of S :

A material system S is said to have a symmetry group G, if for every $g \in G$ there exists

i) another material system $S' = gS$ (symbolically)

ii) a definite function F_g for all observables

such that

α) $A' = F_g(A)$ is again an observable, and

β) the possible results of measuring A' in S' are the same as for measuring A in S and both have the same probability distribution.

In order to put this physical definition in a more mathematical form, we introduce a special notation :

A physical state does not determine an element of $\mathcal{H}$, since together with φ also $\omega\varphi$ with $|\omega| = 1$ represents the same physical state.

For reasons of clarity, we shall denote by :

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(d1)

f, g, h ... elements of $\mathcal{H}$,

φ, ψ, ϕ ... state vectors, i.e. elements with norm 1,

$\hat{\varphi}, \hat{\psi}, \hat{\phi}$ ... unit-rays, representing physical states,

$\hat{f}, \hat{g}, \hat{h}$ ... rays.

A ray $\hat{f}$ is the set of all elements λf with $0 \leq |\lambda| < \infty$ (f fixed). A unit-ray $\hat{\varphi}$ is the set of all state vectors $\omega\varphi$ with $|\omega| = 1$ (φ fixed). Rays are not vectors. Addition, etc. are not defined.

We now give the formal definition of symmetry :

(D)

If we denote the state vectors of S by φ, ψ, ... and those of S' by φ', ψ', ..., then there is a one-to-one correspondence of the physical states (represented by unit-rays)

$$\hat{\varphi} \xleftrightarrow[g]{} \hat{\varphi}'$$

such that the probabilities are conserved :

$$|\langle \varphi', \psi' \rangle| = |\langle \varphi, \psi \rangle|$$

if $\varphi' \in \hat{\varphi}'$, $\psi' \in \hat{\psi}'$ and $\varphi \in \hat{\varphi}$, $\psi \in \hat{\psi}$.

We shall quote this definition henceforward as (D).

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3. Consequences of the existence of a symmetry group.

a) Existence of a unit-ray transformation.

It follows from (D) that the existence of a symmetry group G implies only the existence of a group t of unit-ray transformations $\hat{\Theta}$, which maps the set of all unit-rays onto itself

$$\hat{\varphi} \longleftrightarrow \hat{\varphi}' = \hat{\Theta}\hat{\varphi}$$

That this transformation group is uniquely defined by G and is isomorphic to G, follows from physical experience. In this form, however, it is useless for quantum mechanics, since the superposition principle is true for state vectors rather than for unit-rays. The problem is therefore to find a group of transformations Θ defined for elements of the whole Hilbert space, which is homomorphic, or even possibly isomorphic to the group $\hat{\Theta}$, and as simple as possible.

b) Class of transformations Θ leading to the same $\hat{\Theta}$.

Let t be a group of unit-ray transformations. If $\hat{\Theta} \in t$, we xonsider all transformations Θ of $\mathcal{H}$ which correspond to $\hat{\Theta}$. Correspondence is defined by

(d2)
$$\Theta \sim \hat{\Theta} \quad \text{if}$$
$$\Theta\varphi \in \hat{\Theta}\hat{\varphi} \quad \text{for all} \quad \varphi \in \hat{\varphi}$$

Consider the set of all Θ, which correspond to all $\hat{\Theta} \in t$. These Θ form a group T, which is homomorphic to the group t. This is a very large homomorphism, since many Θ correspond to the same $\hat{\Theta}$; for instance Θ_1 and Θ_2 may be defined completely different for all elements with norm $\neq 1$ and yet correspond to the same $\hat{\Theta}$. Consider the unit element of t, $\hat{\Theta} = E$. The set of transformations $\Theta \sim E$ forms an invariant sub-group T_0 of T (the kernel of the homomorphism). The factor group T/T_0 is then isomorphic to t. If $\Theta \sim \hat{\Theta}$, then the coset $\Theta T_0 = T_0\Theta$ contains all transformations which correspond to $\hat{\Theta}$, we write $\Theta T_0 \sim \hat{\Theta}$. The transformations $\Theta \in T_0$ have the property to multiply state vectors φ only with unimodular complex numbers (leaving their "length" and "direction" constant). On other elements with norm $\neq 1$ they may be defined arbitrarily.

Any coset of T_0 may be represented by one of its elements, but if we select one element of each coset, then in general these elements will not form a group.

It will be shown that elements Θ can be selected from the cosets such that they are either unitary or antiunitary and form a representation up to a factor.

c) Conservation of the norm.

First we postulate

$$\Theta f = \|f\| \cdot \Theta \frac{f}{\|f\|} \tag{3.1}$$

By this Θ is defined in the whole of $\mathcal{H}$ once it is defined for state vectors. These Θ still form a group. The invariant sub-group T_0 contains now all those Θ_0 which merely multiply each element of $\mathcal{H}$ with a unimodular complex factor such that $\Theta_0 f = \omega_{f/\|f\|} \cdot f$. The factor depends only on $\frac{f}{\|f\|}$. We shall call henceforward this invariant sub-group Ω . This is in fact a very strong restriction. Its effect is that two transformations Θ and Θ', which both correspond to one single $\hat{\Theta}$, differ at most by a $\Theta_0 \in \Omega$.

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Obviously the restriction is not too strong, since it affects not the transformation of state vectors.

d) Complete orthonormal systems.

Theorem 1 :

Complete orthonormal systems are transformed into complete orthonormal systems.

Proof :

Let $\{\psi_i\}$ be a complete orthonormal system, and

$$\theta \sim \hat{\theta}$$

$$\theta' \sim \hat{\theta}^{-1}$$

then $\theta\theta' \sim E$. $\psi_i \rightarrow \psi_i' = \theta\psi_i$ and from (D)

$$|\langle\theta\psi_i, \theta\psi_k\rangle| = |\langle\psi_i, \psi_k\rangle| = \delta_{ik} \tag{3.2}$$

It remains to show that $\{\psi_i'\}$ is complete. Assume it is not, then there exists a ψ_o' orthogonal on all ψ_i'. Then $\theta'\psi_o' = \psi_o$ were orthogonal on all ψ_i against the assumption.

e) The restricted distribution law. Linearly independent elements.

Theorem 2 :

i) If f, g, h ... are lineraly independent, then Θf, Θg, Θh ... are also.

ii) $\Theta(f+g) = \omega(f,g)\Theta f + \omega(g,f)\Theta g$ with $|\omega(f,g)| = 1$ for any pair of elements f, g .

Proof :

i) Let $f_1 \dots f_n$ be linearly independent. For that it is necessary and sufficient that

det $F > 0$, where F is the matrix with elements

$F_{ik} = F^*_{ki} = \langle f_i, f_k \rangle$.

Assume $\Theta f_1 \dots \Theta f_n$ were linearly dependent, then

det $F' = 0$, where, according to (D)

$F'_{ik} = F'^*_{ki} = \langle \Theta f_i, \Theta f_k \rangle = \omega_{ik} \langle f_i f_k \rangle$ with $|\omega_{ik}| = 1$.

Decomposing in an arbitrary way $\omega_{ik} = \omega_i \omega_k$ such that $|\omega_i| = |\omega_k| = 1$, one can write

$F' = \omega F \omega^*$, where

$$\omega = \begin{pmatrix} \omega_1 & & & 0 \\ & \omega_2 & & \\ & & \omega_3 & \\ 0 & & & \ddots \; \omega_n \end{pmatrix}$$

Then det $F' = (\det\omega)(\det\omega^*)$ det F = det $F = 0$ against the presupposition.

ii) For $f = \lambda g$ it is trivial. Let f, g be linearly independent. They span a sub-space M. For any $h \in M^{\perp}$

$$|\langle \Theta h, \Theta(f+g)\rangle| = |\langle h, f+g\rangle| = 0$$

Hence $\Theta(f+g)$ belongs to the subspace spanned by Θf and Θg :

$$\Theta(f+g) = \lambda(f,g)\Theta f + \mu(f,g)\Theta g.$$

Define $k(f,g) = f - \dfrac{\langle g,f\rangle}{\|g\|^2} g$; $\langle k,g\rangle = 0$

Then

$$|\langle \Theta k, \Theta(f+g)\rangle| = |\langle k, f+g\rangle| = |\langle k,f\rangle| \quad \text{but also}$$

$$|\langle \Theta k, \Theta(f+g)\rangle| = |\langle \Theta k, \lambda(f,g)\Theta f\rangle| = |\lambda(f,g)| \cdot |\langle k,f\rangle|$$

Hence $|\lambda(f,g)| = 1$. From interchanging f and g it follows that $\lambda(f,g) = \mu(g,f) = \omega(f,g)$, q.e.d.

In particular :

$$\Theta \sum c_i \psi_i = \sum c_i' \Theta \psi_i \quad \text{with} \quad |c_i'| = |c_i| \qquad (3.3)$$

f) The operator function.

We can now write down explicitly the operator function

$$A' = F_g(A) .$$

Let $\Theta \sim \hat{\Theta}(g)$. If

$$A = \sum a_i \; |\psi_i\rangle\langle\psi_i| ,$$

then

$$F_g(A) = A' = \sum a_i \, |\Theta\psi_i\rangle\langle\Theta\psi_i| = \sum a_i \, |\psi_i'\rangle\langle\psi_i'| \qquad (3.4)$$

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We have to show that this is unique. Thus any $\Theta \sim \hat{\Theta}(g)$ must give the same. Take $\Theta' = \Theta_0 \Theta$; $\Theta_0 \in \Omega$:

$$A'' = \sum a_i |\Theta'\psi_i\rangle\langle\Theta'\psi_i| = \sum a_i |\omega_i \psi_i'\rangle\langle\omega_i\psi_i'| = \sum a_i \omega_i \omega_i^* |\psi_i'\rangle\langle\psi_i'| = A' .$$

It follows that A' has the same eigenvalues as A and the transformed state vectors ψ_i' are the new eigenvectors.

Theorem 3 :

The expectation values are conserved.

Proof :

$$\varphi = \sum c_i \psi_i \quad ; \quad \Theta\varphi = \sum c_i' \,\Theta\psi_i$$

then

$$\langle \Theta\varphi, A'\Theta\varphi \rangle = \sum c_j'^* a_i c_l' \langle\Theta\psi_j|\Theta\psi_i\rangle\langle\Theta\psi_i|\Theta\psi_l\rangle = \sum a_i |c_i'|^2$$
$$\langle \varphi, A\varphi \rangle = \sum a_i |c_i|^2$$

These are equal on account of (3.3). That means : Corresponding measurements in corresponding states will always yield the same results. Hence the whole physics is unchanged. That is the meaning of our definition of symmetry.

g) Superselection subspaces.

Theorem 4 :

All $\Theta \in T$ will, apart from a mapping of superselection subspaces onto themselves, at most induce a permutation of these spaces.

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Proof :

i) From theorem 1 it follows that orthogonal subspaces are transformed into orthogonal subspaces.

ii) From theorem 3 it follows that the transformed observables have no matrix elements between the transformed superselection subspaces.

iii) The transformation of the observables is a mapping of the set of all observables onto itself. Th s set defines the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \dots$ up to the order in which the $\mathcal{H}_i$ are counted, thus a certain $\Theta \in T$ induces a well-defined permutation of the $\mathcal{H}_i$ (which might be the unit-permutation).

h) Continuity.

There are two types of continuity :

i) By symmetry operations on the physical system neighbouring physical states are transformed into neighbouring physical states. This is a physical experience. It is contained in our definition of symmetry (D) :

$$|\langle \psi', \varphi' \rangle| = |\langle \psi, \varphi \rangle|$$

Let $\varphi = \varphi(t)$ follow a continuous path, then both sides have to be continuous functions of t. This does not imply that $\varphi'(t)$ is also continuous, since it may contain any discontinuous phase factor $\omega(t)$.

ii) The symmetry group G may be continuous or partly continuous : There may exist continuous paths $g(t)$.

We shall presently discuss the first type of continuity. For this we define distance and neighbourhood of unit-rays :

(d3) The distance of two unit-rays $\hat{\varphi}$ and $\hat{\psi}$ is $\hat{d} = \min_{\omega} \| \varphi - \omega \psi \|$ where $\varphi \in \hat{\varphi}$ and $\psi \in \hat{\psi}$ is an otherwise arbitrary pair of state vectors.

(d4) A ε-neighbourhood of $\hat{\varphi}$ is the set of all unit-rays $\hat{\psi}$ whose distance from $\hat{\varphi}$ is $< \varepsilon$.

With these definitions the first type of continuity can be stated as

Theorem 5 :

The transformations $\hat{\theta}$, which correspond to elements $g \in G$ are continuous.

This is obvious since neighbouring rays are transformed in neighbouring rays.

Theorem 6 :

In the set of all $\theta \sim \hat{\theta}$ there is at least one which is continuous in the whole considered superselection subspace.

Sketch of a proof :

By construction. Take an arbitrary state vector φ_o. It belongs to $\hat{\varphi}$. Take now an arbitrary state vector $\varphi_o' \in \hat{\varphi}'$ where $\hat{\Theta}\hat{\varphi} = \hat{\varphi}'$. Define $\Theta\lambda\varphi_o = \lambda\varphi_o'$ for all λ $(0 \leq |\lambda| < \infty)$. Hereby Θ is defined for multiples of φ_o. We now define it for any element g :

Let ψ be any state vector and $\hat{\psi}$ its unit-ray. Find ω such that

$$\|\varphi_o - \omega\psi\| = \min \; ; \quad (\psi_o = \omega\psi \text{ is nearest to } \varphi_o)$$

Find further ω' such that, with $\psi' \in \hat{\Theta}\hat{\psi}$

$$\|\varphi_o' - \omega'\psi'\| = \min \; ; \quad (\psi_o' = \omega'\psi' \text{ is nearest to } \varphi_o')$$

Define $\Theta\mu\psi_o = \mu\psi_o'$ for any μ $(0 \leq |\mu| < \infty)$.

We have to show that this Θ is continuous. Given f, we must find a neighbourhood N_f of f such that for all $g \in N_f$ $\|\Theta f - \Theta g\| < \varepsilon$.

Put $f = \lambda\psi_o$, $g = \mu\phi_o$, where ψ_o and ϕ_o are nearest to φ_o in the above sense. Then

$$\|\Theta f - \Theta g\| = \|\lambda\psi_o' - \mu\phi_o'\| \leq |\lambda| \cdot \|\psi_o' - \phi_o'\| + |\lambda - \mu| \; .$$

To make this smaller than ε , choose

$$|\lambda - \mu| < \varepsilon/2$$

$$\|\phi_o' - \psi_o'\| < \frac{\varepsilon}{2|\lambda|} \; .$$

To achieve the last inequality, take the set of all ϕ_o' fulfilling it. To each of them belongs a unit-ray $\hat{\phi}'$. Call the set of these unit-rays $\{\hat{\phi}'\}_\delta$. Transform each of these unit-rays with $\hat{\Theta}^{-1}$. This gives a set $\{\hat{\phi}\}_\delta$. Find the nearest element ϕ_o to φ_o of each $\hat{\phi} \in \{\hat{\phi}\}_\delta$. These ϕ_o

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form an open set containing ψ_o and defining a neighbourhood of ψ_o. Any $g = \mu\phi_o$ for which ϕ_o lies in this neighbourhood and for which $|\lambda-\mu| < \varepsilon/2$, fulfils the condition.

i) Reduction to unitary and antiunitary transformations Θ.

Theorem 7 :

Given any $\hat{\Theta}$, we can choose such a $\Theta \sim \hat{\Theta}$ that Θ is either unitary or antiunitary in the whole considered superselection subspace and moreover continuous.

This theorem has first been proved by E.P. Wigner [3] without taking into account the antiunitary operators. This has been done explicitly also by E.P. Wigner in his Lorentz-chair lectures, University of Leyden 1957 (not published). His proof is rather complicated and the following simple proof is the only essentially new thing in this paper.

Proof :

The theorem means that given a $\hat{\Theta}$ and a $\Theta' \sim \hat{\Theta}$, we can find such a $\Theta_o \in \Omega$, that $\Theta_o\Theta' = \Theta$ is either unitary or antiunitary. We construct Θ_o.

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[3] E.P. Wigner, Gruppentheorie und ihre Anwendungen an die Quantenmechanik, (1931), p.251.

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α) Take any continuous $\Theta' \sim \hat{\Theta}$. From theorem 2 it follows that for linearly independent elements f, g, h

$$\Theta'(f+h+g) = \omega(f+h,g)\Big[\omega(f,h)\Theta'f + \omega(h,f)\Theta'h\Big] + \omega(g,f+h)\Theta'g$$

Interchanging g and h and comparing the coefficients leads to

$$\omega(f+g,h)\,\omega(f,g) = \omega(f+h,g)\,\omega(f,h) \qquad \text{coefficient of } \Theta'f$$

$$\omega(f+g,h)\,\omega(g,f) = \omega(g,f+h) \qquad \text{coefficient of } \Theta'g$$

$$\omega(f+h,g)\,\omega(h,f) = \omega(h,f+g) \qquad \text{coefficient of } \Theta'h$$

This gives, by dividing the first two and eliminating $\omega(f+h,g)$ with the help of the third equation

$$\omega(f,g) = \frac{\omega(h,f+g)}{\omega(h,f)} \cdot \omega(f,h) \cdot \frac{\omega(g,f)}{\omega(g,f+h)}$$

From $\Theta'(f+0) = \omega(f,0)\Theta'f = \Theta'f$ follows $\omega(f,0) = 1$.
We put $\omega(0,f) = \frac{1}{u(f)}$. Letting now $h \to 0$ one obtains

$$\omega(f,g) = \frac{u(f)}{u(f+g)} \; .$$

$u(f)$ is unimodular by definition. That it is continuous and independent of the norm $\|f\|$, follows from the corresponding properties of Θ'.

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We have therefore a unimodular function $u(f)$ such that

$$u(f+g)\ \Theta'(f+g) = u(f)\ \Theta' f + u(g)\ \Theta' g.$$

$u(f)$ is an element $\Theta_o \in \Omega$. Now $\Theta_o \Theta' = \Theta$ is distributive : $\Theta(f+g) = \Theta f + \Theta g$, and Θ is again continuous.

β)[4] $\langle \Theta(f+g), \Theta(f+g) \rangle = \langle f+g,\ f+g \rangle = \| f \| + \| g \| + 2\mathrm{Re}\langle f,g \rangle$

$$\langle \Theta(f+g) . \Theta(f+g) \rangle = \| f \| + \| g \| + 2\mathrm{Re}\ \langle \Theta f, \Theta g \rangle$$

Hence

$$\mathrm{Re}\ \langle \Theta f,\ \Theta g \rangle = \mathrm{Re}\ \langle f,g \rangle \quad \text{and}$$

$$|\langle \Theta f,\ \Theta g \rangle| = |\langle f,g \rangle|$$

Only two solutions : $\mathrm{Im}\ \langle \Theta f,\ \Theta g \rangle = \pm\ \mathrm{Im}\ \langle fg \rangle$
thus :

Either $\langle \Theta f,\ \Theta g \rangle = \langle f,g \rangle$ $\quad\Theta$ is unitary
or $\langle \Theta f,\ \Theta g \rangle = \langle g,f \rangle$ $\quad\Theta$ is antiunitary.

Since Θ is continuous, it is either in the whole subspace unitary or in the whole subspace antiunitary. Note that the same cannot be concluded for the whole of $\mathcal{H}$.

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4) The following argument is due to Dr. J.M. Jauch (private communication).

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j) Dependence on the group element g. Representation up to a factor.

We have now unitary or antiunitary Θ's.. For this we have paid much. Beginning with a given Θ', we found the unimodular u(f) uniquely defined up to a unimodular factor $\omega(\Theta)$, which does no longer depend on f.

Assume we had started with a Θ'', we would have found a u''(f) represented by the group element $\Theta_0'' \in \Omega$. Now $\Theta_0\Theta'$ and $\Theta_0''\Theta''$ can differ only by another $\Theta_0''' \in \Omega$. But this Θ_0''' can be only a constant unimodular ω, since otherwise Θ_0 and Θ_0'' were <u>not</u> uniquely defined (apart from a constant factor).

Therefore we can postulate that Θ shall be unitary or antiunitary (we cannot choose between these two) but we pay for this by loosing all freedom but for the freedom of multiplying Θ by a unimodular $\omega(\Theta)$. Apart from that, Θ is then a uniquely defined function of $\hat{\Theta}$ and therefore of the group element g.

With a choice of the remaining free function $\omega(\Theta)$ everything is fixed : We have then chosen exactly one transformation $\Theta(g)$ from every coset. We cannot expect these transformations to form a group, but we have always

$$\Theta(g_1)\ \Theta(g_2) = \omega(g_1 g_2)\ \Theta(g_1 g_2)$$

This is called a "representation up to a factor" or a "ray representation".

It should be clear that if we admit arbitrary phase factors, then the Θ form a group. Its invariant sub-group corresponding to $\hat{\Theta} = E$ is now the group ω of complex unimodular numbers, and the factor group T/ω still is isomorphic to the group G. It is by fixing these phase factors that the group property is lost. It may happen, however, that these phase factors can be chosen such that the Θ's still form a group. Under which circumstances this is possible, is discussed in full generality by V. Bargmann, Ann. of Math. 59, 1 (1954).

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