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KBc algebra and the gauge invariant overlap in open string field theory

E. Aldo Arroyo*

Centro de Ciências Naturais e Humanas, Universidade Federal do ABC Santo André, 09210-170 São Paulo, SP, Brazil

*E-mail: aldo.arroyo@ufabc.edu.br

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We study in detail the evaluation of the gauge invariant overlap for analytic solutions constructed out of elements in the *KBc* algebra in open string field theory. We compute this gauge invariant observable using analytical and numerical techniques based on the sliver frame $\mathcal{L}_0$ and traditional Virasoro L_0 level expansions of the solutions.

Subject Index B26, B28

1. Introduction

It is well-known that the analytic solutions for tachyon condensation [1–7] in open bosonic string field theory [8] can be formally given in terms of elements in the *KBc* algebra [9,10]. Once a solution Ψ is given, the next step is to evaluate relevant physical gauge invariant quantities, such as the energy and the gauge invariant overlap $\langle I|\mathcal{V}(i)|\Psi\rangle$ discovered in Refs. [11–13]. As argued by Ellwood [14], the gauge invariant overlap represents the shift in the closed string tadpole of the solution relative to the perturbative vacuum. Moreover, using an appropriate zero momentum vertex operator $\mathcal{V}$, defined in Ref. [15], it has been shown that the value of the energy can be obtained from the gauge invariant overlap.

The analytic computation of the gauge invariant overlap for Schnabl's tachyon vacuum solution has been performed in Ref. [13]. Although the evaluation of this gauge invariant appears to be simpler than the energy, the computation presented in Ref. [13] was a bit cumbersome, and the reason for this subtlety was that the authors used a representation of the solution as given in Schnabl's original work [1]. As we will see, the computation of the gauge invariant overlap can be enormously simplified if we express Schnabl's solution in terms of elements in the *KBc* algebra.

Concerning the numerical analysis of the gauge invariant overlap for analytic solutions within the *KBc* algebra, in Ref. [13] the authors evaluated the gauge invariant overlap for the case of Schnabl's original solution using the traditional Virasoro L_0 level truncation scheme. Regarding the case of the so-called Erler–Schnabl's solution, although the analytical computation of the gauge invariant overlap for this solution has been performed in Ref. [2], up to now, using the Virasoro L_0 level truncation scheme, the analysis of the gauge invariant overlap for this type of solution was not performed. Moreover, the analysis of the gauge invariant overlap by means of the curly $\mathcal{L}_0$ level truncation scheme has not been carried out, neither for Schnabl nor for Erler–Schnabl's solution. In the case of the new real tachyon vacuum solution discovered in Ref. [6] (called as Jokel's solution [7]), neither the numerical nor the analytical computation was presented for the gauge invariant overlap.

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Motivated by the above results and open issues, in this work, using analytical and numerical techniques based on the curly $\mathcal{L}_0$ and the traditional Virasoro L_0 level truncation schemes, we show a detailed and pedagogical way of computing the gauge invariant overlap for solutions constructed out of elements in the *KBc* algebra. As explicit examples of our generic results, we present the analytical and numerical computation of the gauge invariant overlap for Schnabl's, Erler–Schnabl's and Jokel's solutions.

By expanding the solution Ψ in the basis of curly $\mathcal{L}_0$ eigenstates, we surprisingly discover that the result for the gauge invariant overlap $\langle I|\mathcal{V}(i)|\Psi\rangle$ turns out to be a finite series. This result is in contrast to the case of the energy, where the series has an infinite number of terms and diverges, though this divergent series can be resummed numerically by means of Padé approximants to give a good approximation to the expected value of the D-brane tension [1,2,16,17].

Regarding the numerical result of the gauge invariant overlap for Erler–Schnabl's and Jokel's solution obtained by means of Virasoro L_0 level truncation computations, we would like to mention that the main reason for performing this numerical computation is to see whether or not higher-level contributions yield to increasingly convergent results which approach the expected answer. We will show that the series that represents the gauge invariant overlap for these solutions turns out to be a non-convergent one, therefore we will be required to use Padé approximants.

This paper is organized as follows. In Sect. 2, we introduce the sliver frame and discuss some conventions and definitions that will be used in the rest of the paper. In Sect. 3, we review the *KBc* algebra. In Sects. 4, 5 and 6, we analytically and numerically evaluate the gauge invariant overlap for solutions expressed in terms of elements in the *KBc* algebra. In Sect. 7, a summary and further directions of exploration are given.

2. The sliver frame: conventions and definitions

Originally, the sliver frame has been defined as the $\tilde{z}$ coordinate obtained by the map [1]

$$\tilde{z} = \arctan z, \tag{1}$$

where z is a point on the upper half-plane (UHP). It is known that the gluing prescription entering into the definition of the star product simplifies if one uses the $\tilde{z}$ coordinate. Under the map (1), the UHP looks like a semi-infinite cylinder of circumference π denoted by C_π.

There is another convention for the definition of the sliver frame which uses the map

$$\tilde{z} = \frac{2}{\pi}\arctan z. \tag{2}$$

This map has been used in Ref. [2], and in this case, the UHP looks like a semi-infinite cylinder of circumference 2 denoted by C_2.

Since the expressions written in terms of elements in the *KBc* algebra which are used in the construction of analytic solutions look different depending on the convention adopted for the $\tilde{z}$ coordinate, it is always useful to mention, from the beginning, which of those conventions will be chosen, i.e., the one given by Eqs. (1) or (2).

In the literature, some authors use the convention Eq. (1) and others Eq. (2); in this work we are going to use a rather generic definition which takes into account both of these conventions. Let us define the $\tilde{z}$ coordinate by the map

$$\tilde{z} = \frac{l}{\pi}\arctan z, \tag{3}$$

so that the UHP looks like a semi-infinite cylinder of circumference l denoted by C_l. Note that the case $l = \pi$ corresponds to the convention Eq. (1) while the case $l = 2$ corresponds to Eq. (2).

Let us define the operators $\hat{\mathcal{L}}$, $\hat{\mathcal{B}}$ and $\tilde{c}_p$, which are very useful in the construction of elements in the *KBc* algebra. These operators are related to the worldsheet energy-momentum tensor T, the b and the c ghosts fields, respectively. Using the map (3), we can write the explicit definition of the operators $\hat{\mathcal{L}}$, $\hat{\mathcal{B}}$ and $\tilde{c}_p$:

$$\hat{\mathcal{L}} \equiv \mathcal{L}_0 + \mathcal{L}_0^\dagger = \oint \frac{dz}{2\pi i} \left(1+z^2\right) (\arctan z + \text{arccot} z)\, T(z)\,, \tag{4}$$

$$\hat{\mathcal{B}} \equiv \mathcal{B}_0 + \mathcal{B}_0^\dagger = \oint \frac{dz}{2\pi i} \left(1+z^2\right) (\arctan z + \text{arccot} z)\, b(z)\,, \tag{5}$$

$$\tilde{c}_p = \left(\frac{l}{\pi}\right)^p \oint \frac{dz}{2\pi i} \frac{1}{(1+z^2)^2} (\arctan z)^{p-2} c(z). \tag{6}$$

In general, if we have a primary field ϕ with conformal weight h, using the map (3) we obtain

$$\tilde{\phi}_p \equiv \oint \frac{d\tilde{z}}{2\pi i} \tilde{z}^{p+h-1} \tilde{\phi}(\tilde{z}) = \left(\frac{l}{\pi}\right)^p \oint \frac{dz}{2\pi i} \frac{1}{(1+z^2)^{1-h}} (\arctan z)^{p+h-1} \phi(z). \tag{7}$$

Using Eq. (7), let us define the operators $\mathcal{L}_{-1}$ and $\mathcal{B}_{-1}$ which are useful in the computation of the star product of string fields involving the operators $\hat{\mathcal{L}}$ and $\hat{\mathcal{B}}$:

$$\mathcal{L}_{-1} \equiv \oint \frac{d\tilde{z}}{2\pi i} \tilde{T}(\tilde{z}) = \frac{\pi}{l} \oint \frac{dz}{2\pi i} \left(1+z^2\right)\, T(z) = \frac{\pi}{l}\left[L_{-1} + L_1\right], \tag{8}$$

$$\mathcal{B}_{-1} \equiv \oint \frac{d\tilde{z}}{2\pi i} \tilde{b}(\tilde{z}) = \frac{\pi}{l} \oint \frac{dz}{2\pi i} \left(1+z^2\right)\, b(z)\ = \frac{\pi}{l}\left[b_{-1} + b_1\right]. \tag{9}$$

To compute the star product of string fields involving the operators $\hat{\mathcal{L}}$, $\hat{\mathcal{B}}$ and $\tilde{c}_p$, we will need to know the following commutator and anti-commutator relations:

$$[\mathcal{L}_{-1}, \hat{\mathcal{L}}] = [\mathcal{L}_{-1}, \hat{\mathcal{B}}] = 0\,,\ \ [\mathcal{L}_{-1}, \tilde{c}_p] = (2-p)\tilde{c}_{p-1}\,. \tag{10}$$

$$[\hat{\mathcal{B}}, \hat{\mathcal{L}}] = [\mathcal{B}_{-1}, \hat{\mathcal{L}}] = \{\mathcal{B}_{-1}, \hat{\mathcal{B}}\} = 0\,,\ \{\mathcal{B}_{-1}, \tilde{c}_p\} = \delta_{p-1,0}\,. \tag{11}$$

To represent the elements in the *KBc* algebra, we will need to know the operator $U_r^\dagger U_r$. This operator can be written in terms of the operator $\hat{\mathcal{L}}$:

$$U_r^\dagger U_r = \exp\left[\frac{2-r}{2}\hat{\mathcal{L}}\right]. \tag{12}$$

3. Star products and the *KBc* algebra

Before defining the basic elements belonging to the *KBc* algebra, we are going to write the star product of string fields containing the operators $\hat{\mathcal{L}}$ and $\hat{\mathcal{B}}$. Given two string fields ϕ_1 and ϕ_2, we can show that

$$(\hat{\mathcal{B}}\phi_1) * \phi_2 = \hat{\mathcal{B}}(\phi_1 * \phi_2) + (-1)^{\text{gn}(\phi_1)} \frac{l}{2} \phi_1 * \mathcal{B}_{-1}\phi_2, \tag{13}$$

$$\phi_1 * (\hat{\mathcal{B}}\phi_2) = (-1)^{\text{gn}(\phi_1)} \hat{\mathcal{B}}(\phi_1 * \phi_2) - (-1)^{\text{gn}(\phi_1)} \frac{l}{2} (\mathcal{B}_{-1}\phi_1) * \phi_2, \tag{14}$$

$$(\hat{\mathcal{B}}\phi_1) * (\hat{\mathcal{B}}\phi_2) = -(-1)^{\text{gn}(\phi_1)} \frac{l}{2} \hat{\mathcal{B}}\mathcal{B}_{-1}(\phi_1 * \phi_2) + \left(\frac{l}{2}\right)^2 (\mathcal{B}_{-1}\phi_1) * (\mathcal{B}_{-1}\phi_2), \tag{15}$$

$$(\hat{\mathcal{L}}^n\phi_1) * \phi_2 = \sum_{n'=0}^{n} \binom{n}{n'} \left(\frac{l}{2}\right)^{n'} \hat{\mathcal{L}}^{n-n'}(\phi_1 * \mathcal{L}_{-1}^{n'}\phi_2), \tag{16}$$

$$\phi_1 * (\hat{\mathcal{L}}^n\phi_2) = \sum_{n'=0}^{n} \binom{n}{n'} \left(\frac{l}{2}\right)^{n'} (-1)^{n'} \hat{\mathcal{L}}^{n-n'}((\mathcal{L}_{-1}^{n'}\phi_1) * \phi_2), \tag{17}$$

$$(\hat{\mathcal{L}}^m\phi_1) * (\hat{\mathcal{L}}^n\phi_2) = \sum_{m'=0}^{m}\sum_{n'=0}^{n} \binom{m}{m'}\binom{n}{n'} \left(\frac{l}{2}\right)^{m'+n'} (-1)^{n'} \hat{\mathcal{L}}^{m+n-m'-n'}((\mathcal{L}_{-1}^{n'}\phi_1) * (\mathcal{L}_{-1}^{m'}\phi_2)), \tag{18}$$

where $\text{gn}(\phi)$ takes into account the Grassmannality of the string field ϕ. If we set $l = \pi$, the above results match the results given in Ref. [1].

The action of the Becchi-Rouet-Stora-Tyutin (BRST), $\mathcal{L}_{-1}$, and $\mathcal{B}_{-1}$ operators on the star product of two string fields are given by

$$Q(\phi_1 * \phi_2) = (Q\phi_1) * \phi_2 + (-1)^{\text{gn}(\phi_1)}\phi_1 * (Q\phi_2), \tag{19}$$

$$\mathcal{L}_{-1}(\phi_1 * \phi_2) = (\mathcal{L}_{-1}\phi_1) * \phi_2 + \phi_1 * (\mathcal{L}_{-1}\phi_2), \tag{20}$$

$$\mathcal{B}_{-1}(\phi_1 * \phi_2) = (\mathcal{B}_{-1}\phi_1) * \phi_2 + (-1)^{\text{gn}(\phi_1)}\phi_1 * (\mathcal{B}_{-1}\phi_2)\,. \tag{21}$$

Given a operator $\tilde{\phi}(\tilde{z})$ defined in the $\tilde{z}$ coordinate, let us write the wedge state with insertion as

$$U_r^\dagger U_r \tilde{\phi}(\tilde{z})|0\rangle, \tag{22}$$

where $U_r = (2/r)^{\mathcal{L}_0}$ is the scaling operator in the $\tilde{z}$ coordinate. The star product of two states $U_r^\dagger U_r\tilde{\phi}(\tilde{x})|0\rangle$ and $U_s^\dagger U_s\tilde{\psi}(\tilde{y})|0\rangle$ can be derived using the usual gluing prescription,

$$U_r^\dagger U_r\tilde{\phi}(\tilde{x})|0\rangle * U_s^\dagger U_s\tilde{\psi}(\tilde{y})|0\rangle = U_{r+s-1}^\dagger U_{r+s-1}\tilde{\phi}(\tilde{x} + \frac{l}{4}(s-1))\tilde{\psi}(\tilde{y} - \frac{l}{4}(r-1))|0\rangle, \tag{23}$$

where by $\tilde{\phi}(\tilde{z})$ we have denoted the local operator $\phi(z)$ expressed in the sliver frame. For instance, in the case of a primary field with conformal weight h, $\tilde{\phi}(\tilde{z})$ is given by

$$\tilde{\phi}(\tilde{z}) = \left(\frac{dz}{d\tilde{z}}\right)^h \phi(z) = \left(\frac{\pi}{l}\right)^h \cos^{-2h}\left(\frac{\pi\tilde{z}}{l}\right)\phi\left(\tan\left(\frac{\pi\tilde{z}}{l}\right)\right). \tag{24}$$

The elements in the KBc algebra are constructed out of the basic string fields K, B and c. These fields can be represented in terms of operators acting on the identity string field $|I\rangle = U_1^\dagger U_1|0\rangle$:

$$K \equiv \frac{1}{l}\hat{\mathcal{L}}U_1^\dagger U_1|0\rangle, \tag{25}$$

$$B \equiv \frac{1}{l}\hat{\mathcal{B}}U_1^\dagger U_1|0\rangle, \tag{26}$$

$$c \equiv U_1^\dagger U_1\tilde{c}(0)|0\rangle. \tag{27}$$

Let us derive the algebra associated to the set of operators defined by Eqs. (25)–(27). As a pedagogical illustration, we explicitly compute $\{B, c\}$ as

$$\{B, c\} \equiv Bc + cB = \frac{1}{l}\hat{\mathcal{B}}U_1^\dagger U_1|0\rangle * U_1^\dagger U_1\tilde{c}(0)|0\rangle + \frac{1}{l}U_1^\dagger U_1\tilde{c}(0)|0\rangle * \hat{\mathcal{B}}U_1^\dagger U_1|0\rangle; \tag{28}$$

using Eqs. (13), (14) and the anti-commutator (11), we obtain

$$\{B, c\} = U_1^\dagger U_1|0\rangle = |I\rangle, \tag{29}$$

therefore, we have that $\{B, c\} = 1$.

Following the same steps, using Eqs. (13)–(18) and the commutator and anti-commutator relations (10) and (11), we can show that

$$[K, B] = 0, \quad \{B, c\} = 1, \quad \partial c = [K, c], \quad B^2 = 0, \quad c^2 = 0, \tag{30}$$

where the expression ∂c is defined as $\partial c \equiv U_1^\dagger U_1 \partial\tilde{c}(0)|0\rangle$.

The action of the BRST operator Q on the basic string fields K, B and c is given by

$$QK = 0, \quad QB = K, \quad Qc = cKc. \tag{31}$$

Employing the elements in the KBc algebra, we can construct a rather generic solution,

$$\Psi = Fc\frac{KB}{1 - F^2}cF, \tag{32}$$

which formally satisfies the string field equation of motion $Q\Psi + \Psi\Psi = 0$. For this solution to be a well-defined string field, the function $F(K)$ must satisfy some holomorphicity conditions stated in Ref. [3]. From now, we will assume that Ψ belongs to the set of well-defined string fields.

Let us list some solutions of the form (32). As a first example, consider the analytic solution for the tachyon vacuum [1], where $F(K) = e^{-lK/4}$; Schnabl's original solution corresponds to the case where $l = \pi$. Recall that in this work, we are considering the map $\tilde{z} = (l/\pi)\arctan z$, and therefore the Schnabl's solution looks like

$$\Psi_{\text{Sch}} = e^{-lK/4}c\frac{KB}{1 - e^{-lK/2}}ce^{-lK/4}. \tag{33}$$

There is a subtlety with this solution, as shown in Refs. [1,3]; when one performs the expansion of $K/(1 - e^{-lK/2})$ as the sum $\sum_n Ke^{-lKn/2}$, the truncation of this sum produces a remnant which still contributes to certain observables [4]. This is the origin of the phantom term ψ_N. Taking into account the phantom term, the solution (33) can be written as

$$\Psi_{\text{Sch}} = \frac{2}{l}\lim_{N\to\infty}\left[\psi_N - \sum_{n=0}^{N-1}\frac{d\psi_n}{dn}\right], \tag{34}$$

where

$$\psi_n = e^{-lK/4}cBe^{-lKn/2}ce^{-lK/4}. \tag{35}$$

As a second example, let us consider the solution discovered by Erler and Schnabl, namely, the so-called simple tachyon vacuum solution [2]:

$$\Psi_{\text{Er-Sch}} = \frac{1}{\sqrt{1+K}}cB(1+K)c\frac{1}{\sqrt{1+K}}. \tag{36}$$

Note that in this case, $F(K) = 1/\sqrt{1+K}$, and as shown in Refs. [2,3] there is no need for a phantom-like term. It is possible to provide an integral representation of the solution (36); this is given by writing the inverse square root of $1 + K$ as

$$\frac{1}{\sqrt{1+K}} = \frac{1}{\sqrt{\pi}}\int_0^\infty dt\,\frac{1}{\sqrt{t}}e^{-t}\Omega^t, \tag{37}$$

where Ω^t is the wedge state which can be written as [18,19]

$$\Omega^t = e^{-Kt} = U^\dagger_{\frac{2}{t}t+1} U_{\frac{2}{t}t+1}|0\rangle. \tag{38}$$

As a last example, we consider the so-called real tachyon vacuum solution without square roots, or Jokel's real solution for short [6,7]. This solution takes the form

$$\Psi_{\text{Jok}} = \frac{1}{4}\left(\frac{1}{1+K}c + c\frac{1}{1+K} + c\frac{B}{1+K}c + \frac{1}{1+K}c\frac{1}{1+K}\right) + Q\text{-exact terms}, \tag{39}$$

where the Q-exact terms are given by

$$\frac{1}{2}\left[Q(Bc)\frac{1}{1+K} + \frac{1}{1+K}Q(Bc)\right] + \frac{1}{4}\frac{1}{1+K}Q(Bc)\frac{1}{1+K}. \tag{40}$$

Interestingly, the solution does not take the factorized form (32), and is both real and simple, namely, it is without square roots and phantom terms. For this real solution, the corresponding energy has been computed and shown that the value is in agreement with the value predicted by Sen's conjecture.

4. The gauge invariant overlap: analytical computations

In this section, we are going to study the analytic computation of the gauge invariant overlap for solutions given in terms of elements in the *KBc* algebra. This gauge invariant observable has been considered in Refs. [11–13,15,20,21]. For a given solution Ψ of the string field equations of motion, the gauge invariant overlap is defined as the evaluation of the quantity

$$\langle \mathcal{V}|\Psi\rangle = \langle I|\mathcal{V}(i)|\Psi\rangle, \tag{41}$$

where $|I\rangle$ is the identity string field, and the operator $\mathcal{V}(i)$ is an on-shell closed string vertex operator $\mathcal{V} = c\tilde{c}V^{\text{m}}$ which is inserted at the midpoint of the string field Ψ.[1] As argued by Ellwood [14], the gauge invariant overlap represents the shift in the closed string tadpole of the solution relative to the perturbative vacuum.

To evaluate the gauge invariant overlap for solutions given in terms of elements in the *KBc* algebra, the following results are useful:

$$\langle \mathcal{V}|\Omega^{t_1}c\Omega^{t_2}\rangle = (t_1+t_2)\mathcal{C}_{\mathcal{V}}, \tag{42}$$

$$\langle \mathcal{V}|\Omega^{t_1}Bc\Omega^{t_2}c\Omega^{t_3}\rangle = t_2\,\mathcal{C}_{\mathcal{V}}, \tag{43}$$

$$\langle \mathcal{V}|\Omega^{t_1}c\Omega^{t_2}Bc\Omega^{t_3}\rangle = (t_1+t_3)\mathcal{C}_{\mathcal{V}}, \tag{44}$$

where the coefficient $\mathcal{C}_{\mathcal{V}}$ represents the correlator

$$\mathcal{C}_{\mathcal{V}} = \langle \mathcal{V}(i\infty)c(0)\rangle_{C_1}, \tag{45}$$

which is the closed string tadpole evaluated on a cylinder C_1 of unit circumference. The proofs of the above results (42)–(44) are based on usual scaling arguments and can be found in Refs. [2,22].

As an application of Eqs. (42)–(44), we are going to compute the gauge invariant overlap for Schnabl's tachyon vacuum solution. We would like to mention that in Ref. [13], after performing lengthy computations, the authors have evaluated the gauge invariant overlap for Schnabl's solution.

[1] V^{m} is a weight $(1, 1)$ conformal matter primary field.

However, as we will see, this computation can be performed in a few lines if one uses Schnabl's solution expressed in terms of the basic string fields K, B and c,

$$\langle \mathcal{V}|\Psi_{\text{Sch}}\rangle = \frac{2}{l}\lim_{N\to\infty}\Big[\langle \mathcal{V}|\psi_N\rangle - \sum_{n=0}^{N-1}\frac{d\langle \mathcal{V}|\psi_n\rangle}{dn}\Big], \tag{46}$$

therefore, we need to compute $\langle \mathcal{V}|\psi_n\rangle$. Using Eq. (35), we can write

$$\langle \mathcal{V}|\psi_n\rangle = \langle \mathcal{V}|e^{-lK/4}cBe^{-lKn/2}ce^{-lK/4}\rangle = \langle \mathcal{V}|\Omega^{l/4}cB\Omega^{ln/2}c\Omega^{l/4}\rangle. \tag{47}$$

Employing Eq. (44), from Eq. (47) we get

$$\langle \mathcal{V}|\psi_n\rangle = \frac{l}{2}\,\mathcal{C}_{\mathcal{V}}, \tag{48}$$

and plugging this result (48) into Eq. (46) we obtain

$$\langle \mathcal{V}|\Psi_{\text{Sch}}\rangle = \mathcal{C}_{\mathcal{V}} = \langle \mathcal{V}(i\infty)c(0)\rangle_{C_1}. \tag{49}$$

This result coincides with the expected answer of closed string tadpole on the disk [14]. Note that the result (49) does not depend on the parameter l which explicitly appears in the solution (34).

Next we would like to evaluate the gauge invariant overlap for Erler–Schnabl's solution. In fact, using a non-real version of the solution (36), the computation of the gauge invariant overlap has been performed in Ref. [2]. Here we are going to present the computation for the case of the real solution.[2] Let us write the real solution (36) as the following integral representation:

$$\Psi_{\text{Er-Sch}} = \frac{1}{\pi}\Big[(1-\partial_s)\int_0^\infty dt_1dt_2\frac{e^{-t_1-t_2}}{\sqrt{t_1t_2}}\Omega^{t_1}cB\Omega^s c\Omega^{t_2}\Big]\Big|_{s=0}, \tag{50}$$

therefore the gauge invariant overlap for this solution (50) will be given by

$$\langle \mathcal{V}|\Psi_{\text{Er-Sch}}\rangle = \frac{1}{\pi}\Big[(1-\partial_s)\int_0^\infty dt_1dt_2\frac{e^{-t_1-t_2}}{\sqrt{t_1t_2}}\langle \mathcal{V}|\Omega^{t_1}cB\Omega^s c\Omega^{t_2}\rangle\Big]\Big|_{s=0}. \tag{51}$$

Employing Eq. (44), from Eq. (51), we write

$$\begin{aligned}\langle \mathcal{V}|\Psi_{\text{Er-Sch}}\rangle &= \frac{1}{\pi}\Big[(1-\partial_s)\int_0^\infty dt_1dt_2\frac{e^{-t_1-t_2}}{\sqrt{t_1t_2}}(t_1+t_2)\,\mathcal{C}_{\mathcal{V}}\Big]\Big|_{s=0}\\ &= \frac{1}{\pi}\int_0^\infty dt_1dt_2\frac{e^{-t_1-t_2}}{\sqrt{t_1t_2}}(t_1+t_2)\,\mathcal{C}_{\mathcal{V}}\\ &= \mathcal{C}_{\mathcal{V}} = \langle \mathcal{V}(i\infty)c(0)\rangle_{C_1}.\end{aligned} \tag{52}$$

As we can see, this result (52) is exactly the same as the one obtained for Schnabl's solution (49).

As the last example of analytical calculation, let us evaluate the gauge invariant overlap for Jokel's real solution. Since BRST exact terms do not contribute to the evaluation of the gauge invariant

[2] The reality condition of a string field is defined as $\Psi^\ddagger = \Psi$, where the operation $\ddagger$ means the composition of BPZ and Hermitian conjugation. Since the basic string fields K, B and c are real string fields in this sense, the reality condition requires that the string field reads the same way from the left as from the right.

overlap, we just need to consider the non-BRST exact terms of the solution. These terms are given on the right-hand side of Eq. (39) and they can be written as

$$\hat{\Psi}_{\text{Jok}} \equiv \frac{1}{4}\left(\frac{1}{1+K}c + c\frac{1}{1+K} + c\frac{B}{1+K}c + \frac{1}{1+K}c\frac{1}{1+K}\right)$$
$$= \frac{1}{4}\int_0^\infty dt\, e^{-t}\left(\Omega^t c + c\Omega^t + c\Omega^t Bc\right) + \frac{1}{4}\int_0^\infty dt_1 dt_2\, e^{-t_1-t_2}\,\Omega^{t_1} c\Omega^{t_2}. \quad (53)$$

Therefore the gauge invariant overlap for Jokel's real solution is given by

$$\langle\mathcal{V}|\hat{\Psi}_{\text{Jok}}\rangle = \frac{1}{4}\int_0^\infty dt\, e^{-t}\,\langle\mathcal{V}|\Omega^t c + c\Omega^t + c\Omega^t Bc\rangle + \frac{1}{4}\int_0^\infty dt_1 dt_2\, e^{-t_1-t_2}\,\langle\mathcal{V}|\Omega^{t_1} c\Omega^{t_2}\rangle. \quad (54)$$

Using Eqs. (42) and (44), from Eq. (54) we obtain

$$\langle\mathcal{V}|\hat{\Psi}_{\text{Jok}}\rangle = \left[\frac{1}{2}\int_0^\infty dt\; te^{-t} + \frac{1}{4}\int_0^\infty dt_1 dt_2\,(t_1+t_2)e^{-t_1-t_2}\right]\mathcal{C}_{\mathcal{V}}$$
$$= \mathcal{C}_{\mathcal{V}} = \langle\mathcal{V}(i\infty)c(0)\rangle_{C_1}. \quad (55)$$

Note that this result (55) is the same as the ones obtained in the case of Schnabl's (49) and Erler–Schnabl's solutions (52).

It should be nice to obtain the above analytic results by numerical means. For instance, using the traditional Virasoro L_0 level truncation scheme, in Ref. [13], the authors have evaluated the gauge invariant overlap for Schnabl's solution. However, up to now, using the Virasoro L_0 level truncation scheme, the analysis of the gauge invariant overlap for Erler–Schnabl's and Jokel's real solution was not performed. Moreover, the analysis of the gauge invariant overlap by means of the curly $\mathcal{L}_0$ level truncation scheme has not been carried out for neither Schnabl's, Erler–Schnabl's nor Jokel's real solution.

In the next two sections, using the curly $\mathcal{L}_0$ and the Virasoro L_0 level truncation scheme, we are going to present the evaluation of the gauge invariant overlap for solutions constructed out of elements in the *KBc* algebra.

5. The gauge invariant overlap: $\mathcal{L}_0$ level truncation computations

Since from the beginning we do not know if the result for the gauge invariant overlap obtained by analytical computations will match the result obtained by numerical means (either by using the $\mathcal{L}_0$ or the L_0 level truncation scheme), it is important for the consistency of the solutions to check explicitly if these different schemes provide the same answer. In this section, using the $\mathcal{L}_0$ level expansion of a rather generic solution Ψ, we will present the evaluation of the gauge invariant overlap.

As we know, the solution is given in terms of elements in the *KBc* algebra (which involves the operators $\hat{\mathcal{L}}$, $\hat{\mathcal{B}}$ and $\tilde{c}$); in general, we can write the following $\mathcal{L}_0$ level expansion

$$\Psi = \sum_{n,p} f_{n,p}\hat{\mathcal{L}}^n\tilde{c}_p|0\rangle + \sum_{n,p,q} f_{n,p,q}\hat{\mathcal{L}}^n\hat{\mathcal{B}}\tilde{c}_p\tilde{c}_q|0\rangle, \quad (56)$$

where $n = 0, 1, 2, \cdots$, and $p, q = 1, 0, -1, -2, \cdots$. The coefficients of the expansion $f_{n,p}$ and $f_{n,p,q}$ can be regarded as generic ones, and obviously these coefficients depend on the solution we choose. For instance, for the case of Schnabl's solution (33), these coefficients are given by

$$f_{n,p} = \frac{1-(-1)^p}{2}\,\frac{l^{-p}}{2^{n-2p+1}}\,\frac{1}{n!}(-1)^n B_{n-p+1}, \quad (57)$$

$$f_{n,p,q} = \frac{1-(-1)^{p+q}}{2}\frac{l^{-p-q}}{2^{n-2(p+q)+3}}\frac{1}{n!}(-1)^{n-q}B_{n-p-q+2}, \tag{58}$$

where B_m are the Bernoulli's numbers.

To compute the gauge invariant overlap for solutions expanded in terms of $\mathcal{L}_0$ eigenstates, we start by replacing the string field Ψ with $z^{\mathcal{L}_0}\Psi$, so that states in the $\mathcal{L}_0$ level expansion will acquire different integer powers of z at different levels. As usual, at the end, we will simply set $z = 1$.

Let us start with the evaluation of the gauge invariant overlap as a formal power series expansion in z. Plugging the expansion (56) into the definition of the gauge invariant overlap (41), we obtain

$$\langle\mathcal{V}|z^{\mathcal{L}_0}\Psi\rangle = \sum_{n,p} z^{n-p} f_{n,p}\langle\mathcal{V}|\hat{\mathcal{L}}^n\tilde{c}_p|0\rangle + \sum_{n,p,q} z^{n+1-p-q} f_{n,p,q}\langle\mathcal{V}|\hat{\mathcal{L}}^n\hat{\mathcal{B}}\tilde{c}_p\tilde{c}_q|0\rangle. \tag{59}$$

As we can see, we need to compute $\langle\mathcal{V}|\hat{\mathcal{L}}^n\tilde{c}_p|0\rangle$ and $\langle\mathcal{V}|\hat{\mathcal{L}}^n\hat{\mathcal{B}}\tilde{c}_p\tilde{c}_q|0\rangle$. To evaluate these quantities, we need to express $\hat{\mathcal{L}}^n\tilde{c}_p|0\rangle$ and $\hat{\mathcal{L}}^n\hat{\mathcal{B}}\tilde{c}_p\tilde{c}_q|0\rangle$ in terms of elements in the *KBc* algebra; for this purpose, the following relations will be useful

$$\Omega^{s_1}c\Omega^{s_2} = e^{u\hat{\mathcal{L}}}\tilde{c}(x)|0\rangle, \tag{60}$$

$$B\Omega^{t_1}c\Omega^{t_2}c\Omega^{t_3} - \frac{1}{2}\Omega^{t_1+t_2}c\Omega^{t_3} + \frac{1}{2}\Omega^{t_1}c\Omega^{t_2+t_3} = \frac{1}{l}\hat{\mathcal{B}}e^{u\hat{\mathcal{L}}}\tilde{c}(x)\tilde{c}(y)|0\rangle, \tag{61}$$

where

$$s_1 = \frac{l}{4} - \frac{lu}{2} - x, \quad s_2 = \frac{l}{4} - \frac{lu}{2} + x, \tag{62}$$

$$t_1 = \frac{l}{4} - \frac{lu}{2} - x, \quad t_2 = x - y, \quad t_3 = \frac{l}{4} - \frac{lu}{2} + y. \tag{63}$$

Employing the above relations, we can write $\hat{\mathcal{L}}^n\tilde{c}_p|0\rangle$ and $\hat{\mathcal{L}}^n\hat{\mathcal{B}}\tilde{c}_p\tilde{c}_q|0\rangle$ in terms of elements in the *KBc* algebra

$$\hat{\mathcal{L}}^n\tilde{c}_p|0\rangle = n!\oint\frac{du}{2\pi i}\frac{dx}{2\pi i}u^{-n-1}x^{p-2}\Omega^{s_1}c\Omega^{s_2}, \tag{64}$$

$$\hat{\mathcal{L}}^n\hat{\mathcal{B}}\tilde{c}_p\tilde{c}_q|0\rangle = n!\oint\frac{du}{2\pi i}\frac{dx}{2\pi i}\frac{dy}{2\pi i}u^{-n-1}x^{p-2}y^{q-2}\left[lB\Omega^{t_1}c\Omega^{t_2}c\Omega^{t_3} - \frac{l}{2}\Omega^{t_1+t_2}c\Omega^{t_3} + \frac{l}{2}\Omega^{t_1}c\Omega^{t_2+t_3}\right]. \tag{65}$$

Now we are in a position to evaluate the quantities $\langle\mathcal{V}|\hat{\mathcal{L}}^n\tilde{c}_p|0\rangle$ and $\langle\mathcal{V}|\hat{\mathcal{L}}^n\hat{\mathcal{B}}\tilde{c}_p\tilde{c}_q|0\rangle$. For instance, using Eqs. (42) and (64), let us compute

$$\begin{aligned}\langle\mathcal{V}|\hat{\mathcal{L}}^n\tilde{c}_p|0\rangle &= n!\oint\frac{du}{2\pi i}\frac{dx}{2\pi i}u^{-n-1}x^{p-2}\langle\mathcal{V}|\Omega^{s_1}c\Omega^{s_2}\rangle \\ &= n!\oint\frac{du}{2\pi i}\frac{dx}{2\pi i}u^{-n-1}x^{p-2}\left(\frac{l}{2} - lu\right)\mathcal{C}_{\mathcal{V}} \\ &= l\left(\frac{\delta_{p,1}\delta_{n,0}}{2} - \delta_{p,1}\delta_{n,1}\right)\mathcal{C}_{\mathcal{V}}.\end{aligned} \tag{66}$$

Performing similar calculations as above, using Eqs. (42), (43) and (65), we obtain

$$\langle\mathcal{V}|\hat{\mathcal{B}}\hat{\mathcal{L}}^n\tilde{c}_p\tilde{c}_q|0\rangle = l\left(\delta_{n,0}\delta_{p,0}\delta_{q,1} - \delta_{n,0}\delta_{q,0}\delta_{p,1}\right)\mathcal{C}_{\mathcal{V}}. \tag{67}$$

Finally, plugging the results (66) and (67) into the definition of the gauge invariant overlap (59), and setting $z = 1$, we get

$$\langle \mathcal{V}|\Psi\rangle = l\left(\frac{f_{0,1}}{2} - f_{1,1} - 2f_{0,1,0}\right)\mathcal{C}_{\mathcal{V}}. \tag{68}$$

To compute the gauge invariant overlap for a solution expanded in terms of $\mathcal{L}_0$ eigenstates (56), we only need to know the value of the first three coefficients appearing at levels z^{-1} and z^0. Remarkably, this result (68) is simpler than the one obtained for the case of the energy. Evaluating the energy in the $\mathcal{L}_0$ level expansion gives a very complicated non-convergent series, though the series can be resummed numerically by means of the so-called Padé approximants to give a good approximation to the brane tension [1,2,16].

Let us apply the general result (68) for some particular solutions such as the Schnabl's solution Ψ_{Sch}. Using the explicit expressions of the coefficients (57) and (58)

$$f_{0,1} = \frac{2}{l}, \quad f_{1,1} = \frac{1}{2l}, \quad f_{0,1,0} = -\frac{1}{4l}, \tag{69}$$

which appear in the $\mathcal{L}_0$ level expansion of Schnabl's solution, from Eq. (68) we obtain

$$\langle \mathcal{V}|\Psi_{\text{Sch}}\rangle = \mathcal{C}_{\mathcal{V}}. \tag{70}$$

This result does not depend on the parameter l and is the same result as the one obtained from analytic computations.

In the case of Erler–Schnabl's solution $\Psi_{\text{Er-Sch}}$, using its integral representation (50), we can compute the first three coefficients appearing in the $\mathcal{L}_0$ level expansion of the solution

$$f_{0,1} = 1, \quad f_{1,1} = \frac{1}{2}, \quad f_{0,1,0} = -\frac{1}{2l}. \tag{71}$$

Therefore, plugging these results (71) into Eq. (68), we obtain

$$\langle \mathcal{V}|\Psi_{\text{Er-Sch}}\rangle = \mathcal{C}_{\mathcal{V}}. \tag{72}$$

This result also does not depend on the parameter l and is the same result as the one obtained for the case of Schnabl's solution.

In the case of Jokel's real solution, we can also calculate the curly $\mathcal{L}_0$ level expansion of the non-BRST exact terms of the solution (53). The first three coefficients of this $\mathcal{L}_0$ level expansion are given by

$$f_{0,1} = \frac{2}{l}, \quad f_{1,1} = -\frac{1}{4l}, \quad f_{0,1,0} = \frac{1}{8l}. \tag{73}$$

Substituting these results (73) into Eq. (68), we get

$$\langle \mathcal{V}|\hat{\Psi}_{\text{Jok}}\rangle = \mathcal{C}_{\mathcal{V}}. \tag{74}$$

As we can see, the result (74) is the same as the ones obtained in the case of Schnabl's and Erler–Schnabl's solutions.

So far, we have computed the gauge invariant overlap by two means: analytically and using the curly $\mathcal{L}_0$ level expansion of the solutions. In what follows, we are going to evaluate the gauge invariant overlap by a third method, namely, using the traditional Virasoro L_0 level expansion of the solutions.

6. The gauge invariant overlap: Virasoro L_0 level truncation computations

In this section, using the L_0 level truncation scheme, the evaluation of the gauge invariant overlap will be shown. Since the solution Ψ involves the operators $\hat{\mathcal{L}}$, $\hat{\mathcal{B}}$ and $\tilde{c}$, we can write its L_0 level expansion as follows:

$$\Psi = \sum g_{n_1 n_2 \cdots n_i p} L_{n_1} L_{n_2} \cdots L_{n_i} c_p |0\rangle + \sum g_{m_1 m_2 \cdots m_j s p q} L_{m_1} L_{m_2} \cdots L_{m_j} b_s c_p c_q |0\rangle, \tag{75}$$

where $n_i, m_j, s \leq -2$ and $p, q = 1, 0, -1, -2, \cdots$. The L_n terms are the ordinary Virasoro generators with zero central charge $c = 0$ of the total (i.e. matter and ghost) conformal field theory. For instance, Schnabl's solution (34), with $l = \pi$, expanded up to level two states is given by

$$\Psi_{\text{Sch}} = 0.553465\, c_1|0\rangle + 0.043671\, c_{-1}|0\rangle + 0.137646\, L_{-2}c_1|0\rangle + 0.131082\, b_{-2}c_0c_1|0\rangle. \tag{76}$$

To compute the gauge invariant overlap by means of the L_0 level truncation scheme, it is clear that if we insert the expansion (75) into the definition of the gauge invariant overlap (41), we will need to evaluate the quantities

$$\langle \mathcal{V}|L_{n_1} L_{n_2} \cdots L_{n_i} c_p|0\rangle, \quad \langle \mathcal{V}|L_{m_1} L_{m_2} \cdots L_{m_j} b_s c_p c_q|0\rangle. \tag{77}$$

We are going to calculate these quantities by means of a recursive method based on the evaluation of the following commutation and anti-commutation relations

$$[L_m, L_n] = (m - n)L_{m+n}, \tag{78}$$

$$[L_m, b_n] = (m - n)b_{m+n}, \tag{79}$$

$$[L_n, c_p] = (-2n - p)c_{n+p}, \tag{80}$$

$$\{b_m, c_n\} = \delta_{m+n,0}. \tag{81}$$

As an illustration, suppose we need to calculate $\langle \mathcal{V}|L_n c_p|0\rangle$. Since for $n \leq -2$ the operator L_n does not annihilate the vacuum $|0\rangle$, and in order to apply the commutator (80), we must first express the operator L_n in terms of annihilation operators. This can be achieved if we use the fact that the on-shell closed string state $\mathcal{V} = c\tilde{c}V^{\text{m}}$ is invariant by the transformation generated by $K_n = L_n - (-1)^n L_{-n}$, namely, we have [13]

$$\langle \mathcal{V}|L_n = \langle \mathcal{V}|(-1)^n L_{-n}. \tag{82}$$

Now, since $L_{-n}|0\rangle = 0$ for $n \leq -2$, we are able to compute $\langle \mathcal{V}|L_n c_p|0\rangle$ using the commutator (80)

$$\langle \mathcal{V}|L_n c_p|0\rangle = (-1)^n \langle \mathcal{V}|[L_{-n}, c_p]|0\rangle = (-1)^n (2n - p)\langle \mathcal{V}|c_{p-n}|0\rangle. \tag{83}$$

Let us comment that for the case of the operator b_n, which corresponds to the modes of the ghost field b, we have a similar result as the one given by Eq. (82) [11–13,23]:

$$\langle \mathcal{V}|b_n = \langle \mathcal{V}|(-1)^n b_{-n}. \tag{84}$$

As we have seen, after the use of the commutation and anti-commutation relations (78)–(81), we can express the quantities (77) as linear combinations of terms like

$$\langle \mathcal{V}|c_p|0\rangle. \tag{85}$$

To evaluate (85), first let us express the mode c_p in the $\tilde{z}$-coordinate. Using the conformal transformation of the $c(z)$ ghost, under the map (3), we get

$$c_p = \oint \frac{dz}{2\pi i} z^{p-2} c(z) = \left(\frac{\pi}{l}\right)^2 \oint \frac{d\tilde{z}}{2\pi i} \sec^4\left(\frac{\pi \tilde{z}}{l}\right) \tan^{p-2}\left(\frac{\pi \tilde{z}}{l}\right) \tilde{c}(\tilde{z}). \tag{86}$$

If we substitute Eq. (86) into Eq. (85), it is clear that we will need to evaluate the quantity $\langle \mathcal{V}|\tilde{c}(\tilde{z})|0\rangle$. Using Eqs. (42) and (60), we can compute this quantity

$$\langle \mathcal{V}|\tilde{c}(\tilde{z})|0\rangle = \left\langle \mathcal{V}|\Omega^{-\tilde{z}+l/4} c\, \Omega^{\tilde{z}+l/4}\right\rangle = \frac{l}{2}\,\mathcal{C}_{\mathcal{V}}. \tag{87}$$

Therefore, employing Eqs. (86) and (87), we obtain

$$\langle \mathcal{V}|c_p|0\rangle = \left(\frac{\pi}{l}\right)^2 \left(\frac{l}{2}\right) \mathcal{C}_{\mathcal{V}} \oint \frac{d\tilde{z}}{2\pi i} \sec^4\left(\frac{\pi \tilde{z}}{l}\right) \tan^{p-2}\left(\frac{\pi \tilde{z}}{l}\right) = \frac{\pi}{2}\left(\delta_{p,-1} + \delta_{p,1}\right) \mathcal{C}_{\mathcal{V}}. \tag{88}$$

As a first example, let us compute the gauge invariant overlap for Schnabl's solution expanded up to level two states:

$$\Psi_{\text{Sch}} = t'\, c_1|0\rangle + u'\, c_{-1}|0\rangle + v'\, L_{-2}c_1|0\rangle + w'\, b_{-2}c_0c_1|0\rangle, \tag{89}$$

where the values of the coefficients t', u', v' and w' are given in Eq. (76). Using the property that $\langle \mathcal{V}|L_{-2} = \langle \mathcal{V}|L_2$ and $\langle \mathcal{V}|b_{-2} = \langle \mathcal{V}|b_2$, the evaluation of the gauge invariant overlap reads as

$$\begin{aligned}\langle \mathcal{V}|\Psi_{\text{Sch}}\rangle &= t'\langle \mathcal{V}|c_1|0\rangle + u'\langle \mathcal{V}|c_{-1}|0\rangle + v'\langle \mathcal{V}|[L_2, c_1]|0\rangle + w'\langle \mathcal{V}|[b_2, c_0c_1]|0\rangle \\ &= t'\langle \mathcal{V}|c_1|0\rangle + u'\langle \mathcal{V}|c_{-1}|0\rangle - 5v'\langle \mathcal{V}|c_3|0\rangle = \frac{\pi}{2}\left(t' + u'\right)\mathcal{C}_{\mathcal{V}}.\end{aligned} \tag{90}$$

We would like to compare this result (90) with the one obtained in Ref. [13], where Schnabl's solution has been expanded from a slightly different basis. Instead of considering the Virasoro generators L_n with zero central charge, the authors have used the α_n's oscillators; for instance, up to level two states, they have written the expansion

$$\Psi_{\text{Sch}} = t\, c_1|0\rangle + u\, c_{-1}|0\rangle + v\,(\alpha_{-1}\cdot\alpha_{-1})c_1|0\rangle + w\, b_{-2}c_0c_1|0\rangle, \tag{91}$$

where the coefficients have the following values[3]

$$t = 0.553465, \quad u = 0.456611, \quad v = 0.068823, \quad w = -0.144210. \tag{92}$$

Then, by using an explicit oscillator representation for the on-shell closed string state which can be found in Refs. [13,24], the gauge invariant overlap for the expanded Schnabl's solution (91) turns out to be [13]

$$\langle \mathcal{V}|\Psi_{\text{Sch}}\rangle = \frac{1}{4}t - \frac{3}{2}v + \frac{1}{4}u = 0.149284. \tag{93}$$

[3] We have noted that if we use Eq. (3.36) of reference [13], the value of the coefficient v turns out to be twice the value presented here (92). This means that if the authors want to use the definition of v as given in their Eqs. (3.31) and (3.32), their Eq. (3.36) should be replaced by a half of it. We have communicated this issue to one of the authors, and he has confirmed this little mistake which nevertheless does not change the main result presented in Ref. [13].

Let us compare this result (93) with the one obtained by us (90). To get the same answer, we should choose the normalization where $\mathcal{C}_{\mathcal{V}} = 1/(2\pi)$, and in fact with this normalization from Eq. (90), we obtain

$$\langle \mathcal{V} | \Psi_{\text{Sch}} \rangle = \frac{1}{4}\big(t' + u'\big) = 0.149284. \tag{94}$$

Taking into account higher-level states, we have performed the computation of the gauge invariant overlap for Schnabl's solution, and the results we have obtained with the normalization $\mathcal{C}_{\mathcal{V}} = 1/(2\pi)$ are in agreement with the ones presented in Ref. [13]. We can consider this agreement as a test for the method of computing the gauge invariant overlap based on the use of Eqs. (82), (84) and the commutation and anti-commutation relations (78)–(81).

The advantage of this method compared to the one presented in Ref. [13] is that we do not need to use an explicit oscillator representation for the on-shell closed string state. The implication of this observation will be reflected in the simplification of the evaluation of the gauge invariant overlap. Recall that the L_0 level expansion of analytic solutions constructed out of elements in the KBc algebra, as presented in (75), is naively given in terms of the total (matter+ghost) Virasoro generators L_n and the b_n and c_p modes, and since we do not need to use an explicit oscillator representation for the on-shell closed string state, using the expansion (75) we can directly evaluate the gauge invariant overlap without the necessity of re-expressing the expansion in terms of the α_n's oscillators (which will require an additional work).

Before studying the numerical evaluation of the gauge invariant overlap for the case of Erler–Schnabl's and Jokel's solutions, we would like to mention some motivations for doing this computation. First, using the L_0 level truncation scheme, the numerical analysis of the gauge invariant overlap for Erler–Schnabl's and Jokel's solutions has not been carried out. This analysis should be crucial if we want to confirm the analytic result. However, the main motivation for performing such numerical computations is to see whether or not higher-level contributions yield to increasingly convergent results which approach to the expected answer. In the case of Schnabl's solution, it has been shown that every time we increase the level of the truncated solution, the gauge invariant overlap converges to the expected analytical result without the necessity of using any regularization scheme such as Padé approximants [13].

Let us start with the L_0 level truncation analysis of the gauge invariant overlap for Erler–Schnabl's solution. To simplify the computations, it will be useful to write the solution (36) in the following way:

$$\Psi_{\text{Er-Sch}} = \frac{1}{\sqrt{1+K}} c \frac{1}{\sqrt{1+K}} + Q\Big\{\frac{1}{\sqrt{1+K}} Bc \frac{1}{\sqrt{1+K}}\Big\}. \tag{95}$$

Inserting the solution (95) into the definition of the gauge invariant overlap, the BRST exact term does not contribute, and so we only need to consider the first term appearing on the right-hand side of Eq. (95); let us denote this term as

$$\Psi^{(1)} \equiv \frac{1}{\sqrt{1+K}} c \frac{1}{\sqrt{1+K}}. \tag{96}$$

To compare the L_0 level expansion of the string field (96) with the one presented in Ref. [2], we choose the value of the parameter l, which appears in the definition of the map (3), as $l = 2$. The L_0

level expansion of the string field (96) can be obtained from the following result [2,25]

$$\Psi^{(1)} = \frac{1}{2\pi^2} \int_0^\infty dsdt \, \frac{1}{\sqrt{st}} e^{-s-t} r^2 \cos^2\left(\frac{\pi x}{r}\right) \widetilde{U}_r c \left(\frac{2\tan(\pi x/r)}{r}\right) |0\rangle, \tag{97}$$

where r and x are given by

$$r = s + t + 1, \quad x = \frac{s-t}{2}. \tag{98}$$

The operator $\widetilde{U}_r$ is defined as

$$\widetilde{U}_r \equiv \cdots e^{u_{10,r}L_{-10}} e^{u_{8,r}L_{-8}} e^{u_{6,r}L_{-6}} e^{u_{4,r}L_{-4}} e^{u_{2,r}L_{-2}}. \tag{99}$$

To find the coefficients $u_{n,r}$ appearing in the exponentials, we use

$$\begin{aligned} \frac{r}{2}\tan\left(\frac{2}{r}\arctan z\right) &= \lim_{N\to\infty}\left[f_{2,u_{2,r}} \circ f_{4,u_{4,r}} \circ f_{6,u_{6,r}} \circ f_{8,u_{8,r}} \circ f_{10,u_{10,r}} \circ \cdots \circ f_{N,u_{N,r}}(z)\right] \\ &= \lim_{N\to\infty}\left[f_{2,u_{2,r}}(f_{4,u_{4,r}}(f_{6,u_{6,r}}(f_{8,u_{8,r}}(f_{10,u_{10,r}}(\cdots(f_{N,u_{N,r}}(z))\ldots)))))\right], \end{aligned} \tag{100}$$

where the function $f_{n,u_{n,r}}(z)$ is given by

$$f_{n,u_{n,r}}(z) = \frac{z}{(1-u_{n,r}nz^n)^{1/n}}. \tag{101}$$

By performing the change of variables

$$s \to \frac{1}{2}(u - u\eta), \quad t \to \frac{1}{2}(u + u\eta), \quad dsdt \to \frac{u}{2} du d\eta, \tag{102}$$

where $u \in [0, \infty)$ and $\eta \in (-1, 1)$, we are going to evaluate the double integrals coming from Eq. (97) numerically.

Employing the above results, let us write the string field (96), expanded up to level four states:

$$\begin{aligned} \Psi^{(1)} = &+ 0.509038\, c_1|0\rangle + 0.13231\, c_{-1}|0\rangle - 0.001576\, L_{-2}c_1|0\rangle + 0.0893356\, c_{-3}|0\rangle \\ &- 0.0135795\, L_{-4}c_1|0\rangle - 0.00694698\, L_{-2}c_{-1}|0\rangle + 0.0231579\, L_{-2}L_{-2}c_1|0\rangle. \end{aligned} \tag{103}$$

To evaluate the gauge invariant overlap using the L_0 level truncation scheme, first we perform the replacement $\Psi^{(1)} \to z^{L_0}\Psi^{(1)}$ and then, using the resulting string field $z^{L_0}\Psi^{(1)}$, we define

$$\langle \mathcal{V}|\Psi^{(1)}\rangle(z) \equiv \langle \mathcal{V}|z^{L_0}\Psi^{(1)}\rangle. \tag{104}$$

The value of the gauge invariant overlap is obtained just by setting $z = 1$. As we can see, our problem has been reduced to the computation of quantities like $\langle \mathcal{V}|L_{n_1}L_{n_2}\cdots L_{n_i}c_p|0\rangle$, which can be evaluated using Eqs. (78)–(81), (82) and (88).

As an example, plugging the level expansion (103) into the definition (104), we obtain

$$\langle \mathcal{V}|\Psi^{(1)}\rangle(z) = \left[\frac{0.79959514}{z} + 0.20783242z - 0.11276868z^3\right]\mathcal{C}_{\mathcal{V}}. \tag{105}$$

If we set $z = 1$, from Eq. (105) we get about 89% of the expected result for the gauge invariant overlap (52). This result may appear good; however, considering the string field (96) expanded up to level twenty-four, we obtain about 116% of the expected result. This behavior is in contrast with the case of Schnabl's solution, where it has been shown that every time we increase the level of the truncated solution, the gauge invariant overlap converges to the expected analytical result [13]. Therefore, as we suspect, for the case of Erler–Schnabl's solution, by naively setting $z = 1$, we are obtaining a non-convergent result. Recall that in numerical L_0 level truncation computations a regularization procedure based on Padé approximants produces desired results for gauge invariant quantities like the energy [2]. Let us see if, after applying Padé approximants, we can obtain the expected answer for the case of the gauge invariant overlap.

For the numerical evaluation, we have considered the string field $\Psi^{(1)}$ expanded up to level twenty-four, so that we obtain a series expansion for (104) truncated up to the order z^{23}. The explicit expression for the gauge invariant overlap, truncated up to this order, is given by

$$\begin{aligned}\langle \mathcal{V}|\Psi^{(1)}\rangle(z) = \Big[&\frac{0.79959514}{z} + 0.20783242z - 0.11276868z^3 + 0.03183002z^5\\ &+ 0.1105491863z^7 + 0.003197445654z^9 - 0.14509620056z^{11}\\ &+ 0.0040708415z^{13} + 0.1939886423z^{15} + 0.002321956902z^{17}\\ &- 0.2468785966z^{19} + 0.0009635172z^{21} + 0.313942988469z^{23}\Big]\mathcal{C}_{\mathcal{V}}. \end{aligned} \quad (106)$$

As an illustration of the numerical method based on Padé approximants, let us compute the value of the gauge invariant overlap using a Padé approximant of order $P^{4+1}_{4+2}(z)$. First, we express $\langle \mathcal{V}|\Psi^{(1)}\rangle(z)$ as the rational function $P^{4+1}_{4+2}(z)$

$$\langle \mathcal{V}|\Psi^{(1)}\rangle(z) = P^{4+1}_{4+2}(z) = \frac{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5}{z + b_2 z^2 + b_3 z^3 + b_4 z^4 + b_5 z^5 + b_6 z^6}\mathcal{C}_{\mathcal{V}}\,. \quad (107)$$

Expanding the right-hand side of Eq. (107) around $z = 0$ up to the seventh order in z and equating the coefficients of z^{-1}, z^0, z^1, z^2, z^3, z^4, z^5, z^6 and z^7 with the expansion (106), we get a system of algebraic equations for the unknown coefficients a_0, a_1, a_2, a_3, a_4, a_5, b_2, b_3, b_4, b_5 and b_6. Solving those equations, we get

$$a_0 = 0.799595, \quad a_1 = 0, \quad a_2 = 3.68919, \quad a_3 = 0, \quad a_4 = 2.55861, \quad a_5 = 0, \quad (108)$$

$$b_2 = 0, \quad b_3 = 4.35389, \quad b_4 = 0, \quad b_5 = 2.20925, \quad b_6 = 0. \quad (109)$$

Replacing the value of these coefficients inside the definition of $P^{4+1}_{4+2}(z)$ (107), and evaluating this at $z = 1$, we get the following value of the gauge invariant overlap:

$$P^{4+1}_{4+2}(z = 1) = 0.931807965\, \mathcal{C}_{\mathcal{V}}. \quad (110)$$

The results of our calculations are summarized in table 1. As we can see from the last column, the naive evaluation of the gauge invariant overlap produces non-convergent values that oscillate around the expected analytic result; nevertheless, the value of the gauge invariant overlap evaluated using Padé approximants confirms the expected analytic result (52). Although the convergence to the expected answer gets quite slow, by considering higher-level contributions, we will eventually reach the correct value of the gauge invariant overlap $\langle \mathcal{V}|\Psi^{(1)}\rangle \to 1\,\mathcal{C}_{\mathcal{V}}$.

Finally, let us show the L_0 level truncation analysis of the gauge invariant overlap for Jokel's solution. In order to expand the string field (53) in the state space of Virasoro L_0 eigenstates, we

Table 1. The Padé approximation for the value of the gauge invariant overlap $\langle \mathcal{V}|z^{L_0}\Psi^{(1)}\rangle$ divided by $\mathcal{C}_{\mathcal{V}}$ and evaluated at $z = 1$. The third column shows the P^{n+1}_{n+2} Padé approximation. In the last column, P^{2n}_{1} represents a trivial approximation, a naively summed series. At each line, we have considered the string field expanded up to level $2n$ states. The value P^{5+1}_{5+2} appears to be a bit anomalous due to an accidental position of a zero and a pole of the Padé approximant close to the value $z = 1$.

	Level	P^{n+1}_{n+2}	P^{2n}_{1}
$n = 0$	0	0.7995951404	0.7995951404
$n = 1$	2	1.0074275622	1.0074275622
$n = 2$	4	0.9343242915	0.8946588687
$n = 3$	6	0.9234761474	0.9264888970
$n = 4$	8	0.9318079653	1.0370380866
$n = 5$	10	1.2045451257	1.0402355322
$n = 6$	12	0.9644587833	0.8951393273
$n = 7$	14	0.9673353296	0.8992101689
$n = 8$	16	0.9815354429	1.0931988113
$n = 9$	18	0.9814639822	1.0955207682
$n = 10$	20	0.9728969059	0.8486421716
$n = 11$	22	0.9730134315	0.8496056888
$n = 12$	24	0.9757472737	1.1635486772

need to write this string field as follows [7]:

$$\hat{\Psi}_{\text{Jok}} = \int_0^\infty dt \frac{e^{-t} r \sin^2\left(\frac{\pi}{2r}\right)\left(2\pi r - r\sin\left(\frac{\pi}{r}\right) + \pi\right)}{16\pi^2} \widetilde{U}_r \left(c\left(-\frac{2\tan\left(\frac{\pi t}{2r}\right)}{r}\right) + c\left(\frac{2\tan\left(\frac{\pi t}{2r}\right)}{r}\right)\right)$$
$$+ \int_0^\infty dt \sum_{k=1}^{\infty} \frac{e^{-t}(-1)^{k+1} 2^{2k-3}\left(\frac{1}{r}\right)^{2k-3}\sin^4\left(\frac{\pi}{2r}\right)}{\pi^2\left(4k^2-1\right)} \widetilde{U}_r b_{-2k} c\left(-\frac{2\tan\left(\frac{\pi t}{2r}\right)}{r}\right) c\left(\frac{2\tan\left(\frac{\pi t}{2r}\right)}{r}\right)$$
$$+ \int_0^\infty dt_1 \int_0^\infty dt_2 \frac{e^{-t_1-t_2}(1+t_1+t_2)^2\cos^2\left(\frac{\pi(t_2-t_1)}{2(1+t_1+t_2)}\right)}{8\pi} \widetilde{U}_{1+t_1+t_2} c\left(\frac{2\tan\left(\frac{\pi(t_2-t_1)}{2(1+t_1+t_2)}\right)}{1+t_1+t_2}\right), \quad (111)$$

where $r = 1 + t$.

By writing the c ghost in terms of its modes $c(z) = \sum_m c_m/z^{m-1}$ and employing Eqs. (99) and (111), the string field $\hat{\Psi}_{\text{Jok}}$ can be readily expanded and the individual coefficients can be numerically integrated. For instance, let us write the expansion of $\hat{\Psi}_{\text{Jok}}$ up to level four states

$$\begin{aligned}\hat{\Psi}_{\text{Jok}} = &+ 0.45457753 c_1|0\rangle + 0.17214438 c_{-1}|0\rangle - 0.03070678 L_{-2}c_{-1}|0\rangle - 0.01400692 b_{-2}c_0c_1|0\rangle \\ &- 0.00605891 L_{-4}c_1|0\rangle + 0.02033379 L_{-2}L_{-2}c_1|0\rangle + 0.16194599 c_{-3}|0\rangle \\ &- 0.00976204 b_{-2}c_{-2}c_1|0\rangle - 0.01053192 L_{-2}c_{-1}|0\rangle + 0.00976204 b_{-2}c_{-1}c_0|0\rangle \\ &+ 0.00465417 b_{-4}c_0c_1|0\rangle - 0.00308797 L_{-2}b_{-2}c_0c_1|0\rangle. \end{aligned} \quad (112)$$

In order to compute the gauge invariant overlap using the L_0 level truncation scheme, we perform the replacement $\hat{\Psi}_{\text{Jok}} \to z^{L_0}\hat{\Psi}_{\text{Jok}}$ and then, using the resulting string field $z^{L_0}\hat{\Psi}_{\text{Jok}}$, we define

$$\langle \mathcal{V}|\hat{\Psi}_{\text{Jok}}\rangle(z) \equiv \langle \mathcal{V}|z^{L_0}\hat{\Psi}_{\text{Jok}}\rangle. \quad (113)$$

It turns out that if we naively set $z = 1$ in (113), we obtain a non-convergent result, therefore in the case of Jokel's solution we are also required to use Padé approximants.

Table 2. The Padé approximation for the value of the gauge invariant overlap $\langle \mathcal{V}|z^{L_0}\hat{\Psi}_{\text{Jok}}\rangle$ divided by $\mathcal{C}_{\mathcal{V}}$ and evaluated at $z = 1$. The third column shows the P^{n+1}_{n+2} Padé approximation. In the last column, P^{2n}_{1} represents a trivial approximation, a naively summed series. At each line, we have considered the string field expanded up to level $2n$ states. The value P^{5+1}_{5+2} appears to be a bit anomalous due to an accidental position of a zero and a pole of the Padé approximant close to the value $z = 1$.

	Level	P^{n+1}_{n+2}	P^{2n}_{1}
$n = 0$	0	0.7140487176	0.7140487176
$n = 1$	2	0.9844524899	0.9844524899
$n = 2$	4	0.9048229924	0.8715855076
$n = 3$	6	0.9010464675	0.9057032252
$n = 4$	8	0.9042818456	1.0387366169
$n = 5$	10	1.2106757561	1.0438779002
$n = 6$	12	0.9506363141	0.8658994735
$n = 7$	14	0.9547341261	0.8689244179
$n = 8$	16	0.9699784236	1.1105628220
$n = 9$	18	0.9698405361	1.1162899369
$n = 10$	20	0.9642533690	0.8089606332
$n = 11$	22	0.9646503269	0.8080979232
$n = 12$	24	0.9715134811	1.1962242655

We have considered the string field $\hat{\Psi}_{\text{Jok}}$ expanded up to level twenty-four, so that we obtain a series expansion for (113) truncated up to the order z^{23}. The explicit expression for the gauge invariant overlap, truncated up to this order, is given by

$$\begin{aligned}\langle \mathcal{V}|\hat{\Psi}_{\text{Jok}}\rangle(z) = \Bigg[&\frac{0.71404871}{z} + 0.27040377z - 0.11286698z^3 + 0.03411771z^5 \\ &+ 0.133033393978z^7 + 0.0051412823z^9 - 0.17797842572z^{11} \\ &+ 0.00302494385z^{13} + 0.24163840461z^{15} + 0.0057271144z^{17} \\ &- 0.30732930326z^{19} - 0.00086271048z^{21} + 0.3881263427z^{23}\Bigg]\mathcal{C}_{\mathcal{V}}. \end{aligned} \quad (114)$$

Starting from this expression (114), we have computed the value of the gauge invariant overlap using Padé approximants of order $P^{n+1}_{n+2}(z)$. Since these computations are similar to the ones developed in the case of Erler–Schnabl's solution, at this point we only present the results which are shown in table 2. We observed that the value of the gauge invariant overlap evaluated using Padé approximants confirms the expected analytic result.

7. Summary and discussion

Through analytical and numerical techniques, we have evaluated the gauge invariant overlap for solutions within the *KBc* algebra. In order to analyze the gauge invariant overlap numerically, we have used two types of expansions for the truncated solutions, namely, the curly $\mathcal{L}_0$ and the Virasoro L_0 level expansions.

We have shown that when we expand a solution Ψ in the basis of curly $\mathcal{L}_0$ eigenstates, the resulting expression for the gauge invariant overlap $\langle I|\mathcal{V}(i)|\Psi\rangle$ is given in terms of a finite series and so the use of Padé approximants was not necessary. This is quite a generic result provided that the solution belongs to the state space constructed out of elements in the *KBc* algebra. As explicit examples, we have presented the results for the case of Schnabl's, Erler–Schnabl's and Jokel's solutions.

Regarding the Virasoro L_0 level truncation analysis of the gauge invariant overlap for Erler–Schnabl's and Jokel's solutions, we have shown that the resulting expressions are given in terms of non-convergent series which nevertheless can be numerically evaluated using Padé approximants. These results are in contrast to the case of Schnabl's original solution, where the expression of the gauge invariant overlap obtained from Virasoro L_0 level truncation computations becomes a convergent series; therefore, in that case [13], there was no need for using Padé approximants.

Our original motivation for studying the level truncation analysis of the gauge invariant overlap has been to prepare a numerical background to analyze more cumbersome solutions, such as the multibrane solutions [22]; however, there are problems that can arise when using the *KBc* algebra to construct such solutions, for instance, depending on the regularization used to define the solutions, the analytic computation of the energy and the gauge invariant overlap becomes ambiguous [26, 27]. Moreover, these solutions are not well defined when expanded in the basis of Virasoro L_0 eigenstates [28].

With the hope of constructing well-behaved solutions other than the tachyon vacuum, the *KBc* algebra has recently been extended to a larger algebra given as a string field representation of the Virasoro algebra [29]. Since the evaluation of the gauge invariant overlap is simpler than the energy, it should be nice to extend the results presented in our work in order to compute the gauge invariant overlap for solutions constructed within the proposed Mertes–Schnabl's algebra.

Finally, we would like to comment that the evaluation of the gauge invariant overlap can be generalized for solutions in the context of superstring field theory [30–32]. For instance, we should analyze the gauge invariant overlap for solutions constructed out of elements in the so-called $GKBc\gamma$ algebra introduced in Refs. [33–37]. The analytic computation of this gauge invariant quantity has already been presented for some particular solutions [38–40]; however, it remains to carry out the numerical analysis.

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