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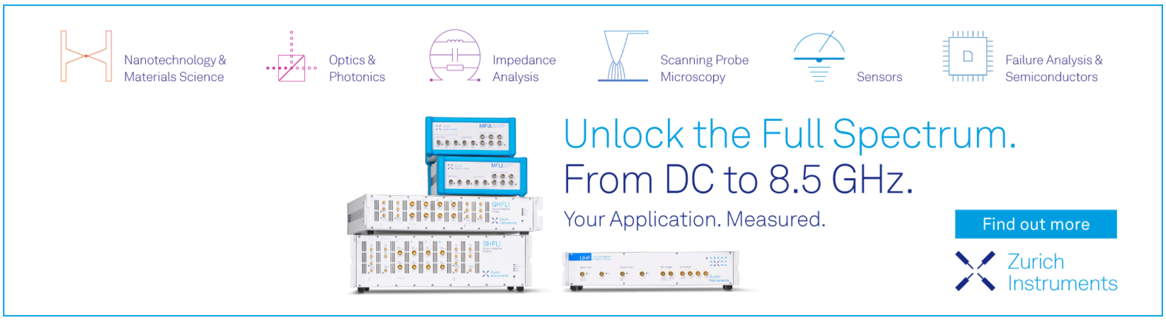

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ABSTRACT

We consider the quantum dynamics of a pair of coupled quantum oscillators coupled to a common correlated dissipative environment. The resulting equations of motion for both the operator moments and covariances can be integrated analytically using the Lyapunov equations. We find that for fully correlated and fully anti-correlated environments, the oscillators relax into a phase-synchronized state that persists for long-times when the two oscillators are nearly resonant and (essentially) forever if the two oscillators are in resonance. We identify an exceptional point that indicates the onset of broken symmetry between an unsynchronized and synchronized dynamical phase of the system as correlations within the environment are increased. We also show that the environmental noise correlation leads to quantum entanglement, and all the correlations between the two oscillators are purely quantum mechanical in origin. This work provides a robust mathematical foundation for understanding how long-lived exciton coherences can be linked to vibronic correlation effects.

I. INTRODUCTION

While recovering from an illness in 1665, Christiaan Huygens observed that two identical pendulum clocks, when suspended from a common substantial beam, quickly synchronized with the same periods and amplitudes.[1–8] He described this intriguing behavior as "an odd sympathy" and concluded that synchronization occurs due to the small vibrations transmitted through the support, which acts as a coupling mechanism between the pendula. Over time, these interactions facilitate an exchange of energy that synchronizes their motion. He presented these results to the Royal Society using a series of complex geometric arguments to explain the synchronization mechanism.[2,8] Surprisingly, the problem of precise time-keeping at sea remains a problem even today. While current laboratory-based atomic clocks reach an accuracy of 1 part in 10^{18} per day, having this standard at sea is made difficult by the spatial and environmental conditions imposed by a marine environment. A recent report by Roslund *et al.* indicates an accuracy of about 300 ps per day over 20 days on a ship-borne device using the optical transitions of I_2 vapor as their time standard. Such precision is needed to achieve nano-second synchronization of clocks for making ultra-precise measurements, sub-meter scale GPS position finding, and long-distance space missions.

The concept of synchronization has wide-ranging implications across various scientific fields, including the coordination of biological rhythms, the operation of power grids, and the development of synchronized communication systems.[9–14] At the microscopic level, synchronization is crucial for developing quantum technologies that are robust against environmental effects, such as decoherence,[15] and for understanding the underlying mechanisms for the transport of quantum excitations in photosynthetic systems in which observed long-lived exciton coherences are linked to vibronic correlation effects. Numerous models and simulations suggest that enhanced excitonic coherence has a profound influence on the enhancement of exciton transport and delocalization.[16–26]

We have recently proposed that synchronization between pairs of qubits could be enhanced by coupling to a noisy ancillary environment with correlated or anti-correlated coupling.[27,28] In these works, we showed that noise correlation can help to preserve the fidelity and purity of a prepared state and determines the degree of phase synchronization such that anti-correlated noise leads to anti-phase synchronization and correlated noise leads to in-phase synchronization. In this work, we consider a pair of coupled quantum oscillators coupled to a common correlated dissipative environment. We show that correlations within the environment act as a shared resource that leads to entanglement in the steady-state solutions.

II. LYAPUNOV EQUATIONS

We begin by writing a suitable Hamiltonian for this system,

$$H/\hbar = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + g(a_1^\dagger a_2 + a_2^\dagger a_1), \tag{1}$$

in which $[a_i, a_j^\dagger] = \delta_{ij}$ are bosonic operators with harmonic frequencies $\{\omega_i\}$ coupled via g, which allows the exchange of quanta from one local mode to the other. For the isolated system, the total number of excitations is conserved since $[a_1^\dagger a_1 + a_2^\dagger a_2, (a_1^\dagger a_2 + a_2^\dagger a_1)] = 0$. Furthermore, we can remove the coupling term by transforming to a normal-mode representation without loss of generality. However, for the purposes of our analysis, we shall continue to work in a local-mode representation. The equations of motion take the general form

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_i D_i(\rho), \tag{2}$$

where $D_i(\rho)$ is the total Lindblad dissipator for oscillator i, which has the general form

$$D_i(\rho) = L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} + L_i' \rho L'^\dagger_i - \frac{1}{2}\{L'^\dagger_i L_i', \rho\}. \tag{3}$$

Here, L_i and L_i' are Lindblad operators describing the local system/bath interactions associated with oscillator i.[29] Following our previous work, and assuming that the coupling to a common environment can be described by correlated Wiener processes, the correlation between the dissipative channels can be introduced by defining a correlation matrix, Ξ, which takes the form

$$\Xi = \begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix}, \tag{4}$$

in which $-1 \le \xi \le 1$ serves as a correlation parameter. Consequently, we use the eigenvalues and corresponding eigenvectors of Ξ to construct new Lindblad operators (now denoted by a tilde),

$$\tilde{L}_\alpha = \sqrt{\lambda_\alpha} \sum_i T_{\alpha i} L_i, \tag{5}$$

where $T^\dagger\ \Xi\ T = \mathrm{diag}\{\lambda_\alpha\}$.

In general, the correlation matrix, Ξ, describes the topological adjacency between correlated Wiener processes such that $dW_i(t) dW_j(t') = \Xi_{ij}\delta(t - t')dt$ and may represent a connected graph belonging to a particular permutation group. Consequently, the eigenvectors of Ξ can be classified according to the irreducible representations of that group. However, we do need to require the correlation matrix, Ξ, to be positive semi-definite so that the eigenvalues $\lambda_\alpha \ge 0$. For the special case of a k-regular graph (that is, a graph with each node having k edges, for which we can write $\Xi = I + \xi A$, where A is the adjacency matrix for an undirected graph and I is the identity matrix), one can show that positivity is obtained for $\xi \in [-1/k, 1/k]$.

For example, for a cyclic two-regular graph with N nodes, the eigenvalues are given by

$$\lambda_\alpha = 1 + 2\xi \cos\left(\frac{2\pi}{N}\alpha\right), \tag{6}$$

where $\alpha = 0, 1, \ldots, \lfloor N/2 \rfloor$. The eigenvectors with $1 \le \alpha \le \lceil N/2 \rceil - 1$ are doubly degenerate according to the Coulson–Rushbrooke pairing theorem.[30,31] For even N, the $\alpha = 0$ and $\alpha = N/2$ eigenvalues are 0 and 2ξ, respectively, with the corresponding eigenvectors $(1, 1, \ldots, 1)$ and $(1, -1, 1, -1, \ldots)$. However, if $\xi = 1/2$, the $\alpha = 0$ eigenvector is the totally anti-symmetric eigenvector and, if $\xi = -1/2$, then the $\alpha = 0$ eigenvector is totally symmetric. Consequently, the symmetry of the 0-mode implies that corresponding irreducible representations of the system will be totally *decoupled* from the environment.

In our calculations that follow, we use the following set of local Lindblad operators:

$$L_i = \sqrt{\gamma(\overline{n}_i + 1)}\, a_i, \tag{7}$$

$$L_i' = \sqrt{\gamma \overline{n}_i}\, a_i^\dagger, \tag{8}$$

where L_i describes the relaxation and L_i' describes the thermal excitations. The relaxation rate is specified by γ, and $\overline{n}_i$ is the equilibrium thermal population of oscillator i. We use the eigenvalues and corresponding eigenvectors of Ξ to construct new Lindblad operators corresponding to the correlated and anti-correlated components

$$\tilde{L}_\pm = \sqrt{\frac{1 \pm \xi}{2}}(L_1 \pm L_2), \tag{9}$$

$$\tilde{L}'_\pm = \sqrt{\frac{1 \pm \xi}{2}}(L_1' \pm L_2'). \tag{10}$$

The total dissipator now includes cross terms involving L_1 and L_2 and similarly L_1' and L_2' arising from correlation between the two local environments. When $\xi = 0$, the cross components vanish and dissipation channels are independent of each other. When $\xi = 1$, the anti-symmetric contribution vanishes, leaving only symmetric terms $\tilde{L}_+$ and $\tilde{L}'_+$. Likewise, in case of anti-correlated noise ($\xi = -1$), symmetric terms vanish, leaving only the anti-symmetric terms $\tilde{L}_-$ and $\tilde{L}'_-$.

In principle, one can also write H in terms of the normal modes of the oscillators themselves. For the two-oscillator system, we have a symmetric and an anti-symmetric normal mode. Consequently, when $\xi > 0$ (more correlated environment), the symmetric normal mode will be strongly damped compared to the anti-symmetric mode such that, in the extreme case of $\xi = +1$, the anti-symmetric mode is completely undamped and vice versa for $\xi < 0$.

Since our equations of motion generate a Gaussian map, we will instead work with the operator expectation values and covariances rather than with the density matrix itself. To establish our notation, we write $\mathbf{x} = \{a_1, a_1^\dagger, a_2, a_2^\dagger\}$ as a vector of system operators and the covariances as

$$\Theta_{ij} = \frac{1}{2}\langle\{\delta x_i^\dagger, \delta x_j\}\rangle = \frac{1}{2}\langle\{x_i^\dagger, x_j\}\rangle - \langle x_i^\dagger\rangle\langle x_j\rangle. \tag{11}$$

The covariance matrix, Θ, must be positive semi-definite and should satisfy the uncertainty principle at all times. This can be stated compactly as

$$\Theta + i\Omega \ge 0, \tag{12}$$

where Ω is the symplectic form,

$$\Omega = \oplus_{n=1}^{N}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{13}$$

For the case at hand, we can see that eigenvalues of Θ are always positive semi-definite. Likewise, the symplectic form remains invariant under any symplectic transformation, including the one that diagonalizes Θ (Williamson's theorem).[32] Consequently, Θ represents a *bona fide* covariance matrix for a physical Gaussian state.[33]

The covariance matrix and moments can be used to construct the Wigner function in multi-variable form,

$$\mathcal{W}(\alpha) = \frac{1}{(2\pi)^N\sqrt{|\Theta|}}\exp\left[-\frac{1}{2}(\alpha - \mathbf{x})^\dagger \cdot \Theta^{-1}\cdot(\alpha - \mathbf{x})\right], \tag{14}$$

where N is the number of oscillators and α is a continuous multi-dimensional complex variable whose real and imaginary components are related to the position and momentum conjugate variables. Note that the Husimi-Q function and Glauber–Sudarshan P functions are similarly constructed; however, the Q-function has a covariance $\Theta_Q = \Theta + I$, while the P-function uses $\Theta_P = \Theta - I$.

The time-evolution of any observable, $\hat{O}$, is given by the master equation

$$\frac{d\langle\hat{O}\rangle}{dt} = i\langle[H,\hat{O}]\rangle + \langle\overline{D}_L(\hat{O})\rangle. \tag{15}$$

Since these are equations of motion for the expectation values rather than the operators themselves, we use the adjoint dissipator

$$\overline{D}_L(\hat{O}) = \langle\hat{O}D_L(\rho)\rangle = \mathrm{tr}\left[\hat{O}\left(L\rho L^\dagger - \frac{1}{2}\{L^\dagger L,\rho\}\right)\right]$$
$$= \frac{1}{2}\langle L[\hat{O},L^\dagger] + [L,\hat{O}]L^\dagger\rangle \tag{16}$$

in deriving these terms. Note that, in Eq. (15), we sum over all the Lindblad operators.

It is straightforward to show that the moments, $\mathbf{x}$, and covariances, Θ, evolve according to the following set of equations of motion:

$$\frac{d\mathbf{x}}{dt} = W\cdot\mathbf{x} + f \tag{17}$$

where W is the dynamical matrix and f includes any source or sink terms. Similarly, the covariances evolve according to

$$\frac{d\Theta}{dt} = W\cdot\Theta + \Theta\cdot W^\dagger + F, \tag{18}$$

where the matrix F depends only on the dissipative terms in the Liouville equation. These are referred to as the Lyapunov equations and are useful for understanding the stability and dynamics of chaotic systems.[34]

For the coupled oscillators, we find the dynamical matrix given by

$$W = \begin{pmatrix} -i\omega_1 & 0 & -ig - \xi\gamma_{12}/2 & 0 \\ 0 & +i\omega_1 & 0 & ig - \xi\gamma_{12}/2 \\ -ig - \xi\gamma_{12}/2 & 0 & -i\omega_2 & 0 \\ 0 & ig - \xi\gamma_{12}/2 & 0 & +i\omega_2 \end{pmatrix} - \frac{\gamma}{2}I_4, \tag{19}$$

where I_4 is the 4×4 identity matrix and

$$\gamma_{12} = \gamma\left(\sqrt{(\bar{n}_1+1)(\bar{n}_2+1)} - \sqrt{\bar{n}_1\bar{n}_2}\right) \tag{20}$$

is a temperature dependent relaxation rate. Similarly,

$$F = -\begin{pmatrix} \frac{\gamma}{2}+\gamma\bar{n}_1 & 0 & \xi\gamma\sqrt{\bar{n}_1\bar{n}_2} & 0 \\ 0 & \frac{\gamma}{2}+\gamma\bar{n}_1 & 0 & \xi\gamma\sqrt{\bar{n}_1\bar{n}_2} \\ \xi\gamma\sqrt{\bar{n}_1\bar{n}_2} & 0 & \frac{\gamma}{2}+\gamma\bar{n}_2 & 0 \\ 0 & \xi\gamma\sqrt{\bar{n}_1\bar{n}_2} & 0 & \frac{\gamma}{2}+\gamma\bar{n}_2 \end{pmatrix}. \tag{21}$$

The derivation of these equations is straightforward and was facilitated by a Mathematica package we developed for doing multi-boson operator algebra. This package is publicly available from the URL given in the Data Availability section.

Both W and F can be rearranged into block-diagonal form, and from the Lyapunov equations, we obtain the equations of motion for the moments and covariances.

The dynamical matrix, W, is non-Hermitian and has complex eigenvalues,

$$\lambda_\pm = \left(\frac{i}{2}(\omega_1+\omega_2) - \gamma\right) \pm \frac{i}{2}\sqrt{(2g - i\xi\gamma_{12})^2 + (\omega_1-\omega_2)^2}. \tag{22}$$

If we take the case where the two oscillators are decoupled $(g = 0)$, synchronization becomes spontaneous when $\xi > (\omega_1 - \omega_2)/\gamma_{12}$. This defines the critical value of ξ necessary for synchronization to occur. As $\xi \to \pm 1$, one of the two normal modes will become completely decoupled from the dissipative environment, leading to the formation of undamped oscillations in which the two oscillators are completely in-phase or out-of-phase with each other. This is the synchronized (or anti-synchronized) regimes.

Figures 1(a) and 1(b) show the real and imaginary eigenvalues of the dynamical matrix for a system at $T = 0K$ with detuning $\delta = 0.2$ and relaxation rate $\gamma = 0.5$. Here, we set $\omega_1 = 1$ and $\omega_2 = \omega_1 + \delta$ to set the time scale of our system. Both the real and imaginary components of the eigenvalues are shifted as indicated on the plots. Here, the eigenvalues of W appear to exhibit an exceptional point at $\xi_{crit} = (\delta/\gamma_{12})$, where the two frequencies coalesce $[\mathrm{Im}(\lambda_+) = \mathrm{Im}(\lambda_-)]$ and one mode becomes less damped while the other becomes more strongly damped. An exceptional point signals a breaking of parity–time $(\mathcal{PT})$ symmetry.[35–41] For the case at hand for an open-quantum system, parity corresponds to the permutation the two sites $(\mathcal{P}a_1 \mapsto a_2)$ and time-reversal corresponds to $\mathcal{T}a_i \mapsto a_i^\dagger$.[42] In the normal regime, the system exhibits a balanced distribution of gain and loss. For the case at hand, the thermodynamic ground state of the system is where the oscillator displacements vanish and the covariance matrix is diagonal. The fluctuation–dissipation theorem (FTD) imposes that the influx of thermal energy from the environment (i.e., fluctuations) is balanced by the efflux of energy back to the environment for a mode of

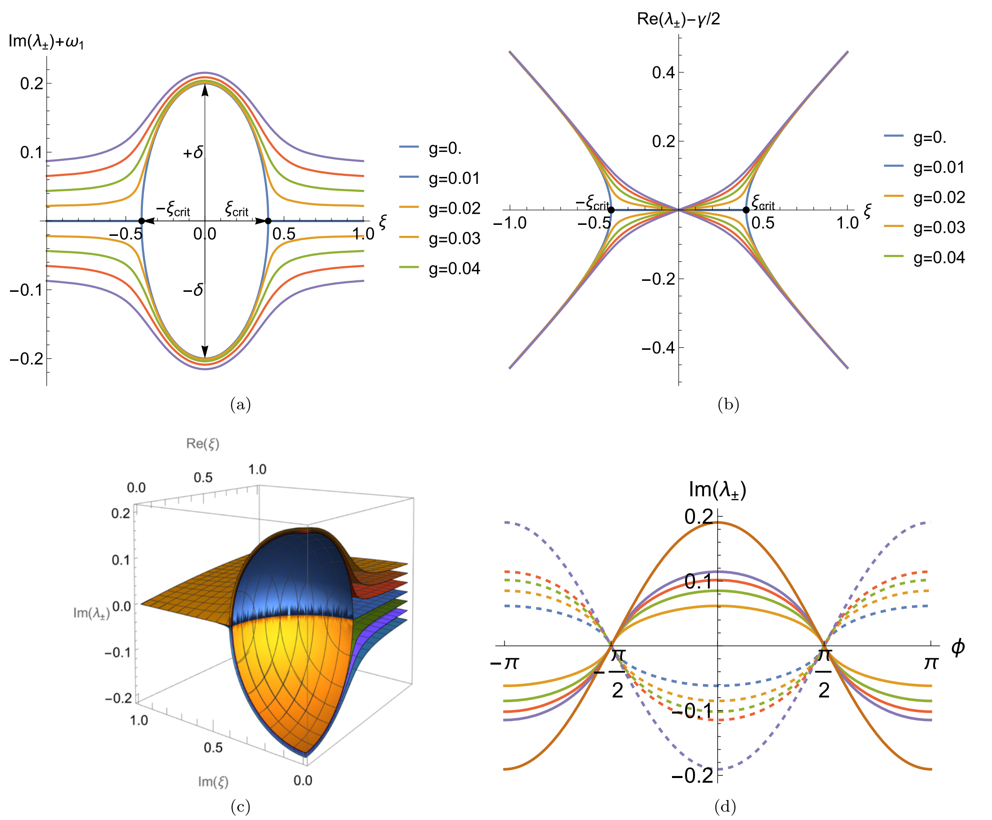

FIG. 1. Imaginary (a) and Real (b) components of the eigenvalues of the non-Hermitian dynamical matrix in Eq. (19) for various parametric values: $\delta = \omega_1 - \omega_2 = 0.2$, $\gamma = 0.5$, and increasing values of exchange coupling $g = 0$ to $g = 0.04$. The system enters the spontaneous synchronized domain above the critical coupling $\xi_{crit} = \delta/\gamma$. (c) is analytic continuation of (b) onto the complex ξ-plane for showing the presence of a branch cut starting at $\pm i\xi_{crit}$. (d) Slice through (c) along $|\xi_{crit}|$ passing through the exceptional points at $\phi = \pm\pi/2$ for $g > 0$. Solid: Im(λ_+). Dashed: Im(λ_-).

the system. When this balance is disrupted, the system can transition into a phase where the $\mathcal{PT}$ symmetry is broken. However, in the $\mathcal{PT}$ broken symmetry regime, excitations of the system do not decay back to the thermodynamic ground state of the system, but rather to a phase-locked non-equilibrium steady state. The nature of that phase-locking depends upon the correlation or anti-correlation between the local environments acting on the oscillators.

Setting $g \neq 0$ lifts the degeneracy between the two oscillators, and there is no longer an exceptional point along the real ξ-axis. However, one can find the exceptional point by performing an analytical continuation of $\xi \to |\xi|e^{i\phi}$ in W. This places the exceptional point along the imaginary ξ-axis ($\phi = \pi/2$) for all values of g. Furthermore, this is the start of a seam where $\mathrm{Im}(\lambda_+) = \mathrm{Im}(\lambda_-)$ along the imaginary axis staring at $\pm i\xi_{crit}$. This is illustrated in Fig. 1(c), where we show the analytical continuation of $\mathrm{Im}(\lambda_\pm)$ on the complex ξ-plane. Consequently, one can adiabatically transition between synchronized and anti-synchronized regimes by transforming the system along $|\xi|e^{i\phi}$. While this may not be feasible in a material system, it may be possible to construct light–matter or fully optical systems that exhibit this behavior.[43] Finally, we note that $g < 0$ swaps the two regimes such that $\xi < 0$ leads to anti-synchronized dynamics and $\xi > 0$ produces synchronized dynamics.

In Fig. 2, we show two representative trajectories for $\langle a_1(t)\rangle$ and $\langle a_2(t)\rangle$ for the case of zero detuning ($\delta = 0$) and with various degrees of environmental correlations. Here, we simply chose random initial conditions for the $\langle a_1\rangle$ and $\langle a_2\rangle$ moments. We see clearly that anti-correlated noise ($\xi = -1$) leads to in-phase synchronization, while fully correlated noise ($\xi = +1$) leads to anti-phase synchronization. Furthermore, at long time, the synchronized states are fully decoupled from the environment and do not decay as compared to the fully uncorrelated case that relaxes to zero on the time scale dictated by γ_{12}. Similar regimes have been suggested by a series of recent papers concerning harmonic chains coupled by dissipative nearest neighbor terms.[44–47]

We can quantify phase synchronization by examining the phase-locking value (PLV), taken as the average phase difference between $\langle a_1(t)\rangle$ and $\langle a_2(t)\rangle$,

$$PLV = \lim_{t_2\to\infty} \frac{1}{t_2 - t_1} \mathrm{Re}\left\langle \int_{t_1}^{t_2} e^{i(\phi_1(t)-\phi_2(t))} dt \right\rangle, \tag{23}$$

where $\phi_i(t) = \arg(\langle a_i(t)\rangle)$ is the phase-angle associated with the complex-valued moment. For this, we first integrate the equation of motion for some sufficiently long time, $t_1 \gg 1/\gamma$. We then sampled over 5000 initial phase differences, varying g to accumulate the data shown in Fig. 3(a). For $g = 0$, synchronization does not occur for $-\xi_{crit} \le \xi \le \xi_{crit}$. Outside this regime, the PLV goes to +1 for $-\xi_{crit} \ge -1$ and to −1 for $\xi_{crit} \le +1$, indicating the presence of synchronized or anti-synchronized states. For $g \neq 0$, phase synchronization or anti-synchronization readily occurs, except when $\xi = 0$. For $-\xi_{crit} \le \xi \le +\xi_{crit}$ and $g = 0$, the two oscillators are asymptotically unsynchronized. The oscillations in the computed PLVs are

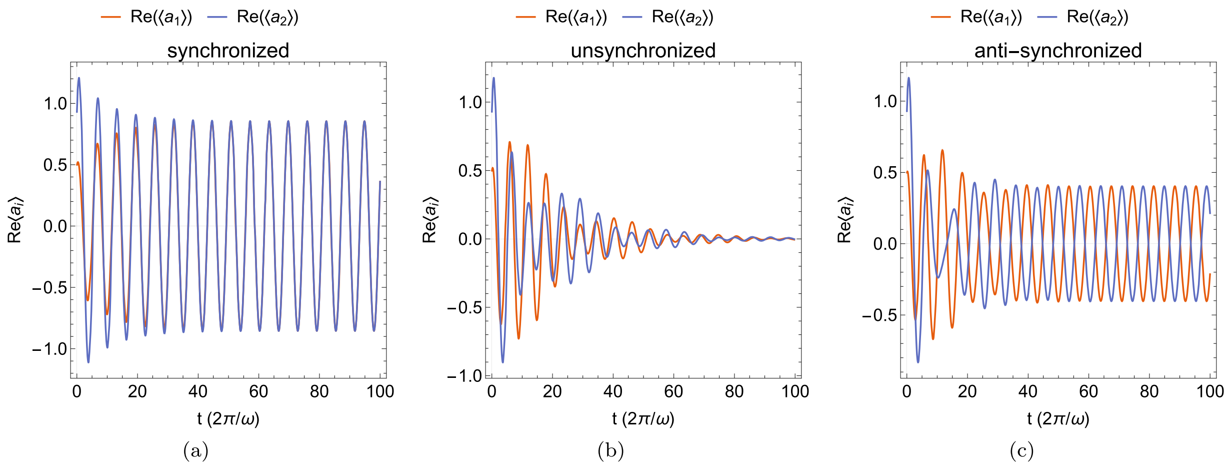

FIG. 2. Representative moment trajectories of $\mathrm{Re}(\langle a_i(t)\rangle)$ (equivalent to the displacement) for pairs of equivalent oscillators driven by (a) anti-correlated ($\xi = -1$), (b) uncorrelated ($\xi = 0$), and (c) correlated ($\xi = +1$) noise, and each starting from an identical starting condition (parameters: $\delta = 0$, $g = 0.1$, and $\gamma = 0.1$).

due to numerical sampling of the initial conditions and finite time integration of the PLV integral. These can be attributed to the competition between gain and loss mechanisms while the system is in the non-synchronized regime.

When the coupling $g \neq 0$, the PLVs show obvious dips close to the critical value of the correlation parameter, indicating the "competition" between relaxation and synchronization. In Fig. 3(b), we set $\delta = 0$ and vary the relaxation rate γ to illustrate how the switch between synchronized to anti-synchronized behavior depends on the coupling to the environment, again averaging over initial conditions of the moments. Weakening the system/bath coupling γ weakens the synchronization effect.

A. Steady state solutions

It is useful to examine the long-time or stationary state solutions (denoted by "tilde") in which

$$W \cdot \tilde{\Theta} + \tilde{\Theta} \cdot W^{\dagger} = -F. \tag{24}$$

The Lyapunov equation has a closed form solution for $\tilde{\Theta}$ when $\omega_1 = \omega_2$,[48]

$$\tilde{\Theta}_{11} = \langle a_1^{\dagger} a_1 \rangle_{ss} + 1/2 = \left(\overline{n}_1 + \frac{1}{2}\right) + \frac{2g^2}{4g^2 + \gamma^2}(\overline{n}_2 - \overline{n}_1) + \xi^2 \gamma_{12} \Delta, \tag{25}$$

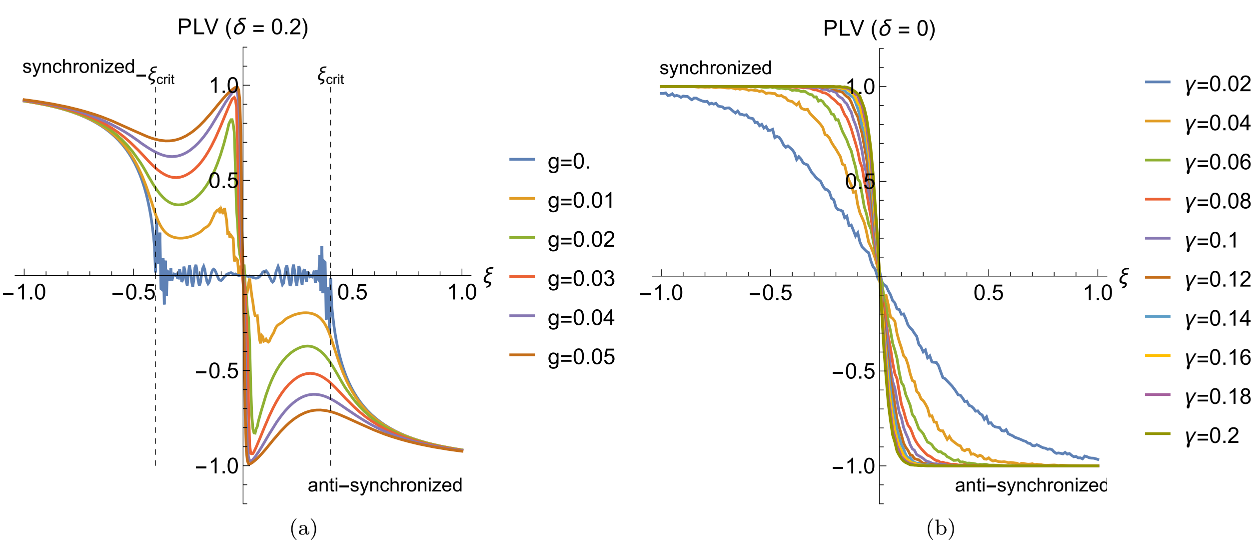

FIG. 3. (a) Average phase-locking value (PLV) vs ξ for $\delta = 0.2$ and off-diagonal coupling $0 \leq g \leq 0.05$. (b) Average PLV for $\delta = 0$ and $g = 0$, varying the relaxation constant γ.

$$\tilde{\Theta}_{22} = \langle a_2^\dagger a_2 \rangle_{ss} + 1/2 = \left(\overline{n}_2 + \frac{1}{2}\right) - \frac{2g^2}{4g^2 + \gamma^2}(\overline{n}_2 - \overline{n}_1) + \xi^2 \gamma_{12}\Delta, \tag{26}$$

$$\tilde{\Theta}_{12} = \langle a_1^\dagger a_2 \rangle_{ss} = i\frac{g\gamma}{4g^2 + \gamma^2}(\overline{n}_2 - \overline{n}_1) + \gamma\xi\Delta, \tag{27}$$

where Δ is a collection of variables given by

$$\Delta = \frac{\left(\gamma_{12}(\overline{n}_1 + \overline{n}_2 + 1) + 2\gamma\sqrt{\overline{n}_1 \overline{n}_2}\right)}{2\left(\gamma^2 - \gamma_{12}^2 \xi^2\right)}. \tag{28}$$

This term imposes an additional constraint on the range of ξ since it becomes singular when $\overline{n}_1 \neq \overline{n_2}$,

$$\xi = \pm\sqrt{\frac{1}{\overline{n}_1 + \overline{n}_2 + 1}}. \tag{29}$$

Otherwise, $-1 \le \xi \le 1$. In what follows, we focus our attention on the special case where the two oscillators are at the same frequency but may be held at different temperatures.

B. Correlation under thermal bias

The off-diagonal $\tilde{\Theta}_{12}$ terms corresponds to the steady-state "flow" of quanta between the two oscillators. Interestingly, if there is a temperature bias between the two oscillators, the system develops correlation since $\tilde{\Theta}_{12} \neq 0$. However, a temperature bias does not create synchronization (or anti-synchronization) since the sign of γ_{12} in the equations of motion for the moments does not change if $\bar{n}_1 \neq \bar{n}_2$. In fact, if we consider the time evolution of the population of oscillators 1 and 2 at different temperatures,

$$\frac{dn_1(t)}{dt} = \gamma(\overline{n}_1 - n_1(t)) - ig(\langle a_1^\dagger a_2 \rangle + \langle a_2^\dagger a_1 \rangle), \tag{30}$$

$$\frac{dn_2(t)}{dt} = \gamma(\overline{n}_2 - n_2(t)) + ig(\langle a_1^\dagger a_2 \rangle + \langle a_2^\dagger a_1 \rangle). \tag{31}$$

For the stationary state (in which populations do not evolve with time), the two time-derivatives must be equal and opposite in sign, and we can define a flux

$$J = \frac{2g\gamma}{4g^2 + \gamma^2}(\overline{n}_2 - \overline{n}_1) \tag{32}$$

describing the flow of quanta from the warmer to cooler baths. This does not depend upon correlations within the environment and only depends upon the exchange coupling term within the Hamiltonian and the local system/bath coupling described by γ.

C. Information theory for Gaussian states

The total entropy of a bi-partite system is strongly sub-additive in that $S_A + S_B \ge S_{AB}$. This holds true for the von Neumann entropy but is not true in general for the various orders of the Rényi entropies. However, strong sub-additivity for Rényi order 2 does hold for Gaussian states and is simply related to the purity,[33]

$$S_2 = -\ln \mathrm{tr}(\rho^2) \tag{33}$$

$$= -\ln \int d\alpha \mathcal{W}(\alpha)^2. \tag{34}$$

One can easily show that for the 2-mode case,

$$S_2(\Theta) = \frac{1}{2}\ln \det\Theta + N \ln 2. \tag{35}$$

Furthermore, since the joint covariance matrix can be written in block form as

$$\Theta = \begin{bmatrix} \Theta_A & \Gamma \\ \Gamma^\dagger & \Theta_B \end{bmatrix} \tag{36}$$

such that the covariance within subspace A or B is given by the two diagonal blocks and the correlations between the two are the off-diagonal blocks (Γ). The mutual information is then simply

$$I_2(A:B) = S_A + S_B - S_{AB} = \frac{1}{2}\ln\frac{\det\Theta_A \det\Theta_B}{\det\Theta} \tag{37}$$

in terms of the Rényi-2 entropy. This mutual information is a measure of the total correlation between the two oscillators.

Figure 4(a) shows steady-state mutual information, $I_2(A:B)$, for various cases where there is a thermal bias between the two oscillators. $I_2(A:B)$ diverges as ξ approaches the limit imposed by Eq. (29). Since the fully synchronized (or anti-synchronized) steady-state is only achieved when $\xi = \pm 1$, a thermal bias will inhibit the formation of this state. Figure 4(b) illustrates the case where the two oscillators are at a common temperature. Here, $I_2(A:B)$ diverges at $\xi = \pm 1$. This means that in the fully synchronized limit, a measurement on subsystem A gives you full access to the information in subsystem B.

1. Discord

For the given bi-partite system and the specific form of the covariance matrix, it is possible to filter out the classical part of the total mutual information as we showed in our previous work.[28] Following the work by Adesso *et al.* and Adesso and Datta,[33,49] we can define the measurement-dependent classical part of the total mutual information by $\mathcal{J}_2(\rho_{A|B}) \ge 0$. This leaves us with a lack of information about subsystem A when a Gaussian measurement is performed on subsystem B. Such measurements preserve the Gaussian nature of states. We have

$$\mathcal{J}_2(\rho_{A|B}) = \sup_{\Sigma_B^\Pi} \frac{1}{2}\ln\left(\frac{\det\Theta_A}{\det\tilde{\Theta}_A^\Pi}\right), \tag{38}$$

where

$$\tilde{\Theta}_A^\Pi = \Theta_A - \Gamma_{AB}\left(\Theta_B + \Sigma_B^\Pi\right)^{-1}\Gamma_{AB}^T \tag{39}$$

is the Schur complement of the Θ_A block and Σ_B^Π is the seed of a Gaussian measurement on B. Naturally then the *discord*[50] or the degree of quantumness can be defined as the difference between the total correlation and the purely classical contributions,[50]

$$\mathcal{D}_2 = I_2(A:B) - \mathcal{J}_2(A|B) = \inf_{\Sigma_B^\Pi}\frac{1}{2}\ln\left(\frac{\det\Theta_B \det\tilde{\Theta}_A^\Pi}{\det\Theta}\right). \tag{40}$$

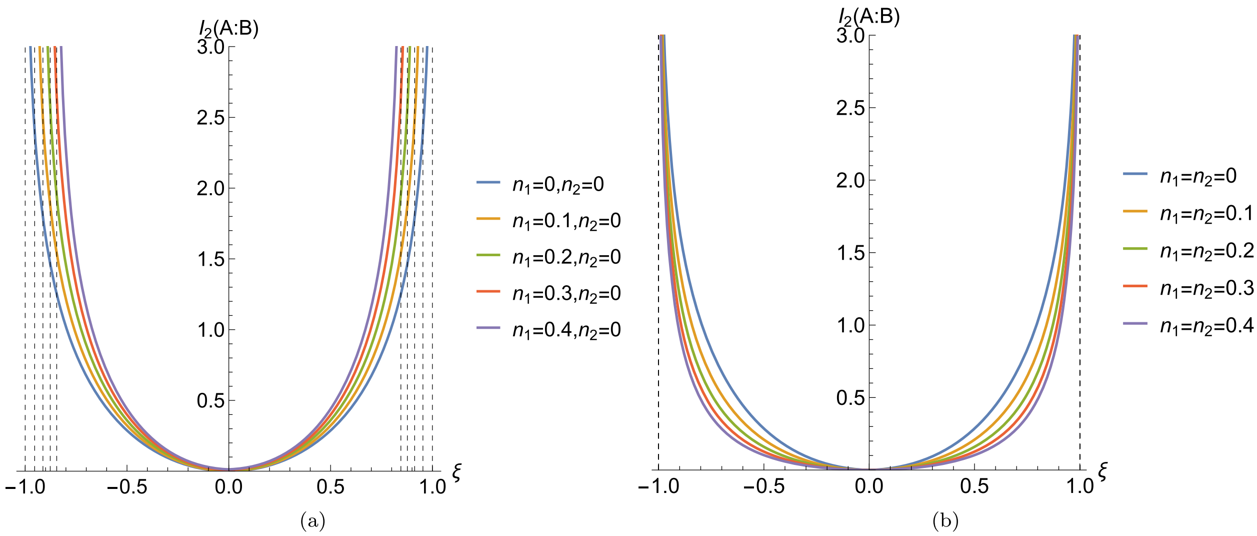

FIG. 4. (a) Mutual information shared by two oscillators in the presence of a temperature bias as specified by the average occupations $\bar{n}_1$ and $\bar{n}_2$. (b) Mutual information shared by two oscillators at a common temperature. In each case, we set $\omega_1 = \omega_2 = 1$, $g = 1$, and $\gamma = 0.1$. The vertical dashed lines correspond to the limiting conditions imposed by Eq. (29) when $\bar{n}_1 \neq \bar{n}_2$.

In Ref. 28, we showed that one can obtain a tight lower bound on the amount of quantum correlation between subsystems by simply removing the coherences between the two subspaces within the total density matrix and computing the mutual information between the two remaining diagonal blocks. For Gaussian systems, this follows from Eq. (38) since $\mathcal{J}_2(\rho_{A|B})$ is extremized under this condition and Eq. (40) becomes equal to the total correlation.

In our previous work on coupled qubit dimers, we found that depending upon the correlation within the environment as specified by ξ, by and large, coupling to a common reservoir introduces *quantum* correlations between the spins.[27,28] For the case at hand, we have two possible sources of correlation in our model. First, the direct coupling introduces correlation when the two oscillators are at different temperatures. Second, even when the direct coupling is turned off or the two oscillators are at the same temperature, correlations within the common environmental mode introduce correlations between the two oscillators. Our results are consistent with the formal results in Ref. 33, which show that the only two-mode Gaussian states with zero Gaussian quantum discord are product states, i.e., states with no correlations at all, that constitute a zero measure set. Consequently, all correlated two-mode Gaussian states have non-classical correlations certified by a nonzero quantum discord.

III. DISCUSSION

We demonstrate here that synchronization and entanglement are universal phenomena that can arise via *indirect* coupling to a correlated dissipative environmental mode. Environmental correlations impose a symmetry on the system by selectively increasing the relaxation rates for modes that have the same symmetry as the environmental correlations, leaving the remaining modes *effectively decoupled* from the environment. This selectivity will persist in multi-mode systems, especially in systems with a high degree of topological symmetry. One can use this to engineer dissipation and decoherence-free domains within the total Hilbert space of the system. It is worth mentioning that such ideas were being explored by Braun *et al.*[51,52] and others,[53–56] particularly on creating entanglement between qubits as they interact with a common heat bath. This body of work suggests that entanglement can be created for a very short duration upon interaction with the heat bath and may persist for longer times depending on the system, coupling, and the properties of the heat bath. Our work sheds light on the underlying mechanism and provides insights into achieving this. A striking result is that the environmental correlation parameter ξ, which gives us more control on the mode-selection process, also serves as an *order parameter* breaking $\mathcal{PT}$ symmetry of the system. For k-regular graphs, we also found that $\xi \in [-1/k, 1/k]$, which means that this order parameter is *emergent* in finite dimensional systems and vanishes in the $k \to \infty$ limit corresponding to zero-dimensional systems. This leads us to the conjecture that environmental correlations are destroyed in an all-to-all connected system in which $k \to \infty$. This is currently being explored and will be part of a forthcoming report.

Our results also have implications in terms of understanding the underlying mechanisms for the transport of quantum excitations in photosynthetic systems in which observed long-lived exciton coherences are linked to vibronic correlation effects. The enhanced excitonic coherence has a profound influence on the enhancement of exciton transport and delocalization.[16–21] Another example of enhanced collective behavior is the recent report of quantum collective motion in a superconducting circuit optomechanical platform composed of six-equivalent oscillators in a hexagonal arrangement.[15] While the authors do not consider the influence of the environmental noise, the experiments do suggest that similar devices could be fabricated that could have controllable degrees of environmental correlations.

One of the biggest challenges in quantum computing is decoherence—the loss of quantum coherence due to interactions with the environment. Quantum states, particularly superpositions and entanglement, are very fragile and prone to disruption due to environmental fluctuations. This work suggests that by carefully engineering the local environment about each qubit and correlations between qubits, one can construct symmetry-adapted baths that force the system into phase-locked state. For instance, synchronized quantum oscillators may continue to operate in coherence for longer durations, increasing the reliability of quantum computations and preserving the integrity of qubits. By actively mitigating decoherence, facilitating error correction, and improving qubit control, synchronization could address several of the most pressing challenges in quantum computing. In addition, it enables more efficient quantum information transfer, helps to stabilize entanglement by preserving fidelity,[27] and supports the development of precise quantum clocks. These advancements will be pivotal in realizing robust and scalable quantum computers.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Both authors discussed the results and contributed to the final manuscript.

Eric R. Bittner: Conceptualization (lead); Data curation (lead); Formal analysis (lead); Funding acquisition (lead); Investigation (lead); Methodology (lead); Project administration (lead); Resources (lead); Software (lead); Supervision (lead); Validation (equal); Visualization (equal); Writing – original draft (lead); Writing – review & editing (equal). **Bhavay Tyagi**: Formal analysis (equal); Software (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data files and source codes used in this work are freely and publicly available on the Wolfram Cloud at https://www.wolframcloud.com/obj/a7bd6168-46c4-47e6-9f01-1755e79553f6.

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