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Planck Length Emerging as the Invariant Quantum Minimum Effective Length Determined by the Heisenberg Uncertainty Principle in Manifestly Covariant Quantum Gravity Theory

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Abstract: The meaning of the quantum minimum effective length that should distinguish the quantum nature of a gravitational field is investigated in the context of manifestly covariant quantum gravity theory (CQG-theory). In such a framework, the possible occurrence of a non-vanishing minimum length requires one to identify it necessarily with a 4-scalar proper length s.It is shown that the latter must be treated in a statistical way and associated with a lower bound in the error measurement of distance, namely to be identified with a standard deviation. In this reference, the existence of a minimum length is proven based on a canonical form of Heisenberg inequality that is peculiar to CQG-theory in predicting massive quantum gravitons with finite path-length trajectories. As a notable outcome, it is found that, apart from a numerical factor of $O(1)$, the invariant minimum length is realized by the Planck length, which, therefore, arises as a constitutive element of quantum gravity phenomenology. This theoretical result permits one to establish the intrinsic minimum-length character of CQG-theory, which emerges consistently with manifest covariance as one of its foundational properties and is rooted both on the mathematical structure of canonical Hamiltonian quantization, as well as on the logic underlying the Heisenberg uncertainty principle.

Keywords: quantum gravity; invariant minimum length; Planck length; Heisenberg uncertainty principle; Heisenberg inequality; Hamiltonian quantization; stochastic graviton trajectories

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1. Introduction

A physically acceptable model of quantum gravity should be characterized by the occurrence of a minimum length, whose existence should be regarded as a ground-level requisite of consistency for the validity of any theory of this kind [1]. In this reference, the possibility of introducing a notion of minimum length for the quantum representation of gravitational space–time can be motivated by first principles. This refers to the characteristic intrinsically discrete nature of quantum physics per se, as well as quantum field dynamics with respect to the corresponding classical counterparts, as it occurs, for example, in the case of quantum mechanics, electromagnetic fields and radiation theories. According to this reasoning, the minimum length should be interpreted as a quantum of the geometry of space–time, namely the minimum length of discretization that composes its microscopic structure and admits the continuum space–time of General Relativity (GR) as a macroscopic limit [2,3].

Nevertheless, beyond these general remarks, at least two crucial questions remain to be addressed. The first one concerns the very existence of a minimum length to be intended both from a physical and a mathematical perspective. This issue pertains to the identification and subsequent establishment of a unique character of minimum length, which relies on the theory of quantum gravity implemented and/or the physical context introduced. In fact, a priori, one cannot exclude the possibility of multiple admissible definitions depending on the type of quantum-gravity phenomena and scenarios treated [4–6]. On the other hand, in the case of uniqueness, the minimum length should represent a constitutive intrinsic element of a given quantum-gravity theory with a universal character that overcomes any specific realization.

In the last few decades, the literature on minimum-length theories has been abundant and apparently fertile [7–16]. The concept of minimum length arises often in association with quantum-gravity phenomenological models focused on modification of the Heisenberg indeterminacy principle in the framework of so-called Generalized Uncertainty Principle (GUP) theories [17–23]. The target of these approaches is hopefully that of predicting the manifestation of some kind of quantum phenomena for particle physics [24–27], or for semi-classical GR solutions in cosmological scenarios [28–31] and in strong gravitational field regimes, like in the case of Hawking radiation near black holes [32–37]. These kinds of studies attract interest from research communities in both scientific and philosophical fields, and it is partly related to the possibility of representing valuable theoretical ways for making concrete progress on the formulation of a comprehensive theory of quantum gravitational fields [38]. However, it is clear that only rigorous approaches that are formulated within canonical quantization schemes and established conceptual frameworks may be susceptible to bringing concrete theoretical advances on the matter.

The second question to ascertain pertains instead to the relationship that exists between the effective minimum length of quantum gravity and the Planck length (ℓ_P), namely the characteristic unit of length that arises in the system of Planck units. The latter is defined as an invariant quantity in terms of the reduced Planck constant ($\hbar$), the gravitational constant (G) and the speed of light in vacuum (c) as

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}}. \tag{1}$$

In SI units, where lengths are expressed in meters (m), the value of the Planck length is of the order of 1.6×10^{-35} m, which represents, by far, a much smaller length than any known size in atomic or subatomic physics. Because of its dependence on both the Planck and the gravitational constants, it has been speculated that the invariant Planck length can acquire a physical meaning in the framework of quantum gravity.

More precisely, it is believed that ℓ_P should express the invariant scale at which the laws of classical gravity theory, in its currently accepted formulation due to Einstein, fail and the structure of continuum space–time predicted by the same GR theory breaks down [39]. This leads to the conjecture that, at Planck-length scales, quantum-gravity phenomena should appear or become relevant, thus demanding the adoption of a theory of quantum gravity for their correct understanding [40–42]. For example, in quantum-gravity approaches that rely on suitable slices of space–time into sub-dimensional spaces [43,44], like the quantum Geometrodynamics, to be distinguished into its two main variants by the Wheeler–DeWitt equation [45–48] and Loop Quantum Gravity theory [49–53], the Planck length is claimed to be the minimum length of the discretization of the underlying geometric space–time grid. Hence, a crucial question to be addressed is about the nature of ℓ_P and its meaning as a minimum length for quantum gravity. This must be ascertained based on first-principle mathematical and physical approaches in order to disclose whether the Planck length can truly represent an intrinsic element of the conceptual framework of quantum gravity.

Given these premises, the purpose of the present research is to investigate the meaning of a quantum minimum effective length that should distinguish the quantum nature of gravitational field in contrast to its classical deterministic description. To this aim, in the following, we shall adopt the framework of manifestly covariant quantum gravity theory (CQG-theory), whose fundamentals have been proposed in a series of recent works [54,55].

The main peculiar features of CQG-theory that distinguish it from alternative literature approaches to quantum gravity are as follows:

- It satisfies the principle of manifest covariance, which states that all quantum operators, dynamical variables and physical observables must be represented exclusively in 4-tensor form and be endowed with tensor properties with respect to a suitable group of local point-coordinate transformations [56,57] (see extended definition given below).
- It relies on the canonical quantization of the continuum space–time field theory provided by classical General Relativity, such that it possesses abstract quantum Hamiltonian and Hilbert space structures that are formally analogous and ontologically equivalent to Quantum Mechanics.
- It admits a 4-dimensional continuous quantum space–time that is expressed by the unconstrained and independent quantum field variables $g(r) \equiv \{g_{\mu\nu}(r)\} = \{g^{\mu\nu}(r)\}$. These must be distinguished from the background metric tensor $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} = \{\widehat{g}^{\mu\nu}(r)\}$, which determines the geometric properties of space–time and raises and lowers tensor indices with respect to which covariance and tensor transformation laws are defined [58].
- It obeys the same quantum logic of Quantum Mechanics, as was recently pointed out in [59], and it relies on the stochastic character of quantum canonical theory and the related validity of Heisenberg inequalities in conventional canonical invariant form.

As a consequence, it must be noticed that the possible occurrence of a non-vanishing invariant quantum minimum length for CQG-theory requires one necessarily to identify it with a 4-scalar structure, and it is denoted with the symbol σ. This must be defined with respect to the continuous background space–time $\widehat{g}(r)$ in order to warrant its objective, i.e., unique, character.

In agreement with these criteria, the target of the present research is twofold. First, it is proven that a consistent realization of the invariant quantum minimum length σ is obtained by expressing it as a proper arc-length. In order to be non-vanishing, the latter must be related to the path-length of quantum particles endowed with finite masses and having corresponding sub-luminal finite-length geodesic trajectories on the background space–time $\widehat{g}(r)$. In the framework of CQG-theory, the natural choice for such a particle endowed with a gravitational-field nature is to identify it with the massive graviton, whose existence is predicted by the validity of manifestly covariant Hamiltonian structure and the canonical quantization at the basis of CQG-theory. It must be noted that the only key assumption is that of non-null geodesics. This is a mandatory requisite for proper-time parametrization to exist both at the classical and quantum levels for gravitational field dynamics. Instead, the existence of massive gravitons with finite masses can be interpreted as a fundamental consequence. In this regard, the existence of a minimum and non-vanishing mass for gravitons was proven in ref. [55].

In addition, the space–time trajectories of gravitons are expected to be non-deterministic, i.e., stochastic, in character. This means that the phenomenon of quantum minimum length must also be closely related to fundamental physical principles that may limit the precision of quantum position measurements, giving rise to an effective "delocalization" of quantum particles and quantum trajectories. Indeed, CQG-theory involves, just like in the case of particle dynamics in QM [60], the treatment of the stochastic space–time trajectories associated with massive quantum gravitons [61]. Departing from the customary GUP literature, the physical basis is provided by a new form of generalized Heisenberg uncertainty principle in tensor form. The latter is peculiar to CQG-theory as it involves the Riemann distance, i.e., the arc length along the background geodetics of gravitons, together with its canonically conjugate quantum momentum. The goal is to show that the quantum minimum length σ must be treated in a statistical way and associated with a lower bound in the error measurement of distance, namely to be identified with a standard deviation constrained by a suitable generalized Heisenberg inequality.

In this regard, as a remarkable feature, it must be stressed that such a statistical and intrinsically quantum definition for the minimum length σ is reached by uniquely implementing a canonical form of the Heisenberg indeterminacy principle, which is analogous to the one that is held in quantum mechanics. Therefore, the result is established without any phenomenological modification of the same Heisenberg principle (like it occurs instead in GUP models), warranting the full consistency of the treatment with the conceptual scheme of canonical Hamiltonian field quantization. Finally, it must be also noted that the existence for CQG-theory of the arc length σ is not at variance nor in contradiction with the postulates of manifest covariance. In fact, the minimum length σ is not pertinent to any kind of discretization model or symmetry breaking of the continuum space–time coordinate systems, which remain associated to the (possibly quantum-modified) background metric tensor $\widehat{g}(r)$.

The second target deals instead with the establishment of the relationship existing between the minimum length σ and the Planck length ℓ_P. This represents a continuation of the theoretical investigation proposed recently in ref. [62] on the meaning of Planck length in the Hamiltonian formulation of classical GR and its quantum version expressed by CQG-theory. In particular, it was shown that the requirements for a realization of a manifestly covariant classical GR Hamiltonian theory for the Einstein field equations with non-vanishing canonical momenta place stringent constraints on the admissible Planck length-dependent contributions that can appear in the corresponding Lagrangian function. In fact, it was proved that, in such a framework and beyond the apparent freedom characterizing the formulation of Lagrangian theory, only terms independent of ℓ_P were ultimately admitted at the classical and quantum variational level for the Hamiltonian theory to hold. This suggests the conjecture that the Planck length ℓ_P should not be introduced from first principles as a coupling constant at the level of the variational treatment of classical GR but, rather, should emerge consistently a posteriori from the canonical quantization procedure as part of the logical structure of CQG-theory [63].

Indeed, the target of the present research includes the proof that, apart from a numerical factor of $O(1)$, the Heisenberg inequality implies a realization of the minimum length σ as an extremal standard deviation in terms of the Planck length, namely—under a suitable ordering assumption on the effective Planck mass (see the related discussion in Section 6 below)—the following notable outcome:

$$\sigma \sim \ell_P. \tag{2}$$

Therefore, ℓ_P arises as a constitutive element of quantum gravity phenomenology characterizing the stochastic quantum dynamics of massive gravitons. This theoretical result is unique among quantum gravity theories and appears peculiar to CQG-theory due to its invariant manifestly covariant representation. In fact, without invoking an implausible hypothesis for the ad hoc modifications of the ontological principles of quantum mechanics, it permits one to establish the intrinsic minimum-length character of CQG-theory. This is found to emerge consistently as a foundational property from the combined validity of both the mathematical structure of canonical Hamiltonian quantization, as well as that of the logic underlying the Heisenberg uncertainty principle.

In summary, the scheme of this paper is as follows. In Section 2, the theoretical setting underlying the manifestly covariant quantum gravity theory is recalled. In Section 3, the validity of the Heisenberg indeterminacy principle holding for the same quantum gravity theory is established. Based on this result, Section 4 deals with the proof of validity of the proper-time Heisenberg inequality, which is peculiar to the manifestly covariant quantum gravity theory and applies to a novel set of conjugate canonical variables. Given this mathematical framework, in Section 5, the proof of existence of the quantum minimum length is reached. Then, the relationship holding between the representation of the quantum minimum length and the Planck length is elucidated in Section 6. Final remarks and concluding discussion are reported in Section 7.

2. Theoretical Setting

In this section, we summarize the main features of the theoretical setting adopted in the following treatment, which is represented by the theory of manifestly covariant quantum gravity (CQG-theory). These notions are propedeutical for the mathematical proof of existence and realization of quantum minimum length in such a framework. In this regard, the problem treated below demands the introduction of the following requirements:

1. The principle of manifest covariance. This principle states that it should always be possible to cast the physical laws of relativistic field theories in a coordinate-independent form, namely in 4-tensor form with respect to the group of local-point transformations (LPT-group), which leaves invariant the differential manifold of the background space–time [64]. It is assumed that the space–time is represented by a Lorentzian differential manifold of the type $\{\mathbf{Q}^4, \widehat{g}(r)\}$, with $\mathbf{Q}^4$ being the 4-dimensional real vector space $\mathbb{R}^4$ representing space–time and $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} \equiv \{\widehat{g}^{\mu\nu}(r)\}$ being a real and symmetric metric tensor represented in terms of a coordinate system (or GR-frame) $r \equiv \{r^\mu\} \in \mathbf{Q}^4$. The same coordinates are parametrized in terms of the arc length s defined along a suitable family of geodetics. Then, it follows that the coordinate system itself is $r \equiv \{r^\mu\}$ and can also be parametrized with respect to s. As a consequence, LPTs are necessarily realized by diffeomorphisms (i.e., differentiable bijections) of the form
$$r(s) \to r'(s) = r'(r(s)), \tag{3}$$
which are globally defined for all $s \in I \subseteq \mathbb{R}$ and are referred to as the LPT-group, with the inverse
$$r'(s) \to r(s) = r(r'(s)) \tag{4}$$
being characterized by a non-singular Jacobian matrix $M \equiv \left\{M^k_\mu(r)\right\} \equiv \left\{\frac{\partial r^k(r)}{\partial r'^\mu}\right\}$. Thus, $r(s) \equiv \{r^\mu(s)\}$ and $r'(s) \equiv \{r'^\mu(s)\}$ denote arbitrary points along suitable geodetics that belong to the initial and transformed space–time structures $\{\mathbf{Q}^4, \widehat{g}(r)\}$ and $\{\mathbf{Q}'^4, \widehat{g}'(r')\}$, respectively. The space–time structure is then preserved by construction under the action of LPT-group (3) and (4) so that $\{\mathbf{Q}^4, \widehat{g}(r)\} \equiv \{\mathbf{Q}'^4, \widehat{g}'(r')\}$ while the Riemann distance s is realized by means of the 4-scalar $ds^2 = \widehat{g}_{\mu\nu}(r)dr^\mu dr^\nu = \widehat{g}'_{\mu\nu}(r')dr'^\mu dr'^\nu$, which is manifestly coordinate-independent (i.e., invariant with respect to the globally defined LPT-group). Instead, any other 4-tensor, including the Ricci and Riemann tensors, necessarily transforms in accordance with appropriate 4-tensor transformation laws [56].
2. The formulation of Classical Hamiltonian theory of GR consistent with the principle of manifest covariance [65]. This is realized by identifying an invariant (i.e., 4-scalar) evolution parameter s, which is denoted as proper-time, and by introducing the Classical Hamiltonian structure represented by the set $\{x, H\}$, which is formed by the canonical state, expressed in 4-tensor form, $x(s) \equiv (g(s), \pi(s))$ and a suitable classical 4-scalar Hamiltonian density H. Here, $g(s) = \{g_{\mu\nu}(s)\}$ and $\pi(s) = \{\pi^{\mu\nu}(s)\}$ denote, respectively, the variational tensor field and the conjugate tensor canonical momentum. The canonical state $x(s)$ must therefore fulfill a corresponding set of continuum 4-tensor Hamilton equations
$$\begin{cases} \frac{dg_{\mu\nu}}{ds} = [g_{\mu\nu}, H] = \frac{\partial H}{\partial \pi^{\mu\nu}}, \\ \frac{d\pi^{\mu\nu}}{ds} = [\pi^{\mu\nu}, H] = -\frac{\partial H}{\partial g_{\mu\nu}}, \end{cases} \tag{5}$$
which are subject to the initial-value condition
$$x(s_o) \equiv (g_{\mu\nu}(s_o), \pi^{\mu\nu}(s_o)). \tag{6}$$

Here, $g_{\mu\nu}(s_o)$ and $\pi^{\mu\nu}(s_o)$ denote two initial conjugate 4-tensor fields, where s_o is the initial proper-time and $\frac{d}{ds}$ is a suitable covariant $s-$derivative operator defined in ref. [54]. Then, the solution of the initial-value problem (5) and (6) generates the Hamiltonian flow

$$x(s_o) \to x(s), \tag{7}$$

which is associated with the Hamiltonian structure $\{x, H\}$. For the validity of the theory, the canonical Equation (5) must correctly recover the form of the Einstein field equations among their extremal solutions so that $\{x, H\}$ effectively identifies the Hamiltonian structure of GR [66]. In particular, we require the extremal field equations to take the customary 4-tensor form

$$\widehat{R}_{\mu\nu} - \frac{1}{2}\widehat{R}\widehat{g}_{\mu\nu} + \Lambda\widehat{g}_{\mu\nu} = \kappa\widehat{T}_{\mu\nu}, \tag{8}$$

where $\widehat{R}_{\mu\nu} = R_{\mu\nu}(\widehat{g}(r))$ and $\widehat{R} = \widehat{g}^{\mu\nu}(r)\widehat{R}_{\mu\nu} \equiv R(\widehat{g}(r))$ denote, respectively, the background Ricci 4-tensor and Ricci 4-scalar, Λ is the cosmological constant, $\widehat{T}_{\mu\nu} = T_{\mu\nu}(\widehat{g}(r))$ is the background stress-energy tensor associated with the external source fields and κ denotes the dimensional constant $\kappa = 8\pi G/c^4$. In addition, the background metric tensor $\widehat{g}_{\mu\nu}$ is subject to the so-called metric compatibility condition $\widehat{\nabla}^{\mu}\widehat{g}_{\mu\nu} = 0$ with respect to the background covariant derivative operator $\widehat{\nabla}^{\mu}$, which, in turn, yields the representation of the Christoffel symbols.

3. The validity of a quantum Hamiltonian theory of GR [67]. Given the classical GR–Hamiltonian structure $\{x, H\}$, this is represented by the set $\left\{x^{(q)}, H^{(q)}\right\}$ realized by the formal map

$$\begin{cases} g_{\mu\nu} \to g^{(q)}_{\mu\nu} \equiv g_{\mu\nu}, \\ \pi_{\mu\nu} \to \pi^{(q)}_{\mu\nu} \equiv -i\hbar\frac{\partial}{\partial g^{\mu\nu}}, \\ H \to H^{(q)}, \end{cases} \tag{9}$$

where the dimensional 4-scalar $H^{(q)}$ identifies the quantum Hamiltonian operator, where $x^{(q)} \equiv \left\{g^{(q)}_{\mu\nu}, \pi^{(q)}_{\mu\nu}\right\}$ is the quantum canonical state and $\pi^{(q)}_{\mu\nu}$ is the quantum momentum operator prescribed so that the commutator $\left[g^{(q)}_{\mu\nu}, \pi^{(q)\alpha\beta}\right] = i\hbar\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}$ applies. The actual realization of the mapping (9) also requires the prescription of dimensional constants to warrant the correct physical dimensions of the quantum Hamiltonian function and quantum operators. This quantum correspondence must then imply the realization of a manifestly covariant Schroedinger-like quantum-wave equation for a 4-scalar quantum-wave function ψ of the form

$$i\hbar\frac{\partial}{\partial s}\psi(s) = \left[H^{(q)}, \psi(s)\right] \equiv H^{(q)}\psi(s), \tag{10}$$

with $\frac{\partial}{\partial s}$ denoting a covariant s-derivative, $H^{(q)}$ being the quantum Hamiltonian operator and $[A, B]$ being the quantum commutator defined in general as $[A, B] \equiv AB - BA$, which is therefore specified here such that $\left[H^{(q)}, \psi(s)\right] \equiv H^{(q)}\psi(s)$. The complex quantum-wave function ψ must admit a Madelung representation as follows:

$$\psi(s) = \sqrt{\rho(s)}\exp\left\{\frac{i}{\hbar}S^{(q)}(s)\right\}, \tag{11}$$

where the real quantum fluid fields $\left\{\rho(s), S^{(q)}(s)\right\}$ identify the quantum probability density function (PDF) and the quantum phase function. As a consequence, consistent with the unitarity principle, the quantum-wave Equation (10) is equivalent to a set of two real PDEs, which are referred to as the quantum continuity and quantum Hamilton–Jacobi equations advancing in proper-time $\rho(s)$ and $S^{(q)}(s)$, respectively.

The acceptable quantum gravity theory must warrant the probabilistic character of quantum mechanics and, particularly, the validity of the Born rule, such that the 4-scalar $\rho(s) \equiv |\psi(s)|^2$ is a probability density on 4-dimensional space–time.

The appropriate setup that realizes the scheme proposed above is provided by the manifestly covariant classical and quantum theories of the gravitational field, which are denoted, respectively, as the CCG-theory and CQG-theory. In this regard, the explicit representation of the classical and quantum Hamiltonian functions H and $H^{(q)}$ can be found in refs. [54,55]. However, it should be noted that, for the subsequent development, their mathematical definitions is not actually required. For this reason, and without missing clarity, we omit to report them here. In fact, the remarkable aspect to underline is the Hamiltonian character that distinguishes the formalisms of the CCG-theory and CQG-theory with respect to other non-Hamiltonian formulations of classical and quantum gravity theories, including the validity of canonical quantization rules and the conceptual framework of quantum mechanics.

This follows precisely because CCG-theory and CQG-theory are rooted on a variational formulation that is based on synchronous Lagrangian and Hamiltonian variational principles [68], which realize a De Donder–Weyl manifestly covariant variational representation of classical and quantum GR [69–76]. Characteristic features of the synchronous approach with respect to the customary literature asynchronous principles (which include, for example, the original Hilbert–Einstein approach) are the space–time representation in terms of the superabundant and unconstrained field variables $g_{\mu\nu}$, and adopting the background metric tensor formalism in terms of a suitably-prescribed background metric tensor $\widehat{g}_{\mu\nu}$. Accordingly, manifest covariance is defined with respect to a continuum background metric tensor $\widehat{g} \equiv \{\widehat{g}_{\mu\nu}\}$, which satisfies both the orthogonality condition $\widehat{g}_{\mu\nu}\widehat{g}^{\mu k} = \delta^k_\nu$, so that it raises/lowers tensor indices, as well as the metric compatibility condition $\widehat{\nabla}_\alpha \widehat{g}_{\mu\nu} = 0$ so that it defines the standard Christoffel connections and curvature tensors of space–time, namely in defining the geometric properties of space–time. The background metric tensor $\widehat{g}$ is then determined self-consistently as a solution of the quantum-modified Einstein field equations implied by the same CQG quantum-wave equation. In contrast, the variational theory applies to the tensor field $g \equiv \{g_{\mu\nu}\}$, which is such that $g_{\mu\nu}g^{\mu k} \neq \delta^k_\nu$, and, generally, has a non-vanishing covariant derivative in the action functional so that $\widehat{\nabla}_\alpha g^{\mu\nu} \neq 0$, where $\widehat{\nabla}_\alpha$ is the covariant derivative whose connections are expressed in terms of $\widehat{g}_{\mu\nu}$. The field $g_{\mu\nu}$ must be interpreted as the quantum gravitational field of the quantum Hamiltonian theory. In this sense, the same quantum field $g_{\mu\nu}$ is allowed to evolve dynamically on the background space–time defined by $\widehat{g}_{\mu\nu}$ according to the quantum-wave equation (CQG-wave equation), a feature that implies that it can correspondingly carry a non-vanishing quantum momentum $\pi_{\mu\nu}$.

It must be noted that the distinction between g and $\widehat{g}$ holds only at the variational level of virtual curves, while, for the extremal fields yielding Einstein field equations, the identity $g = \widehat{g}$ is restored. In the synchronous setting, hatted quantities depend on the background metric tensor $\widehat{g}$ and do not contribute to the variational calculus. Thus, denoting, in particular, the synchronous volume element as $d\widehat{\Omega} = d^4 r\sqrt{-|\widehat{g}|}$, its variation vanishes by construction so that $\delta d\widehat{\Omega} = 0$, where $|\widehat{g}|$ denotes the determinant of $\widehat{g}$. This volume-preserving property under the action of the variational operator δ justifies the name given to this approach as the synchronous variational principle. Indeed, this feature provides a point of connection between the synchronous setting and literature approaches that go under the name of non-metric volume forms, or modified measures, which were defined, for example, in refs. [77,78], or the so-called non-Riemannian space–time volume elements [79,80]. Contrary to the customary assumption of the Hilbert–Einstein theory, these works also propose variational approaches to the GR equations in which the volume elements of integration in the action principles are metric-independent and are determined dynamically through additional degrees of freedom. In practice, this is met by the inclusion of new scalar fields. Therefore, as a common point of contact, both synchronous and non-metric approaches treat the 4-dimensional volume element of integration as a non-variational quantity, namely to be independent of the variational metric tensor. However, the synchronous setting distinguishes itself because it does not invoke nor does it demand any additional field of unknown nature but only the use of superabundant field variables that ultimately coincide with the unique observable space–time metric tensor in extremal Einstein equations.

3. The Heisenberg Indeterminacy Principle

A peculiar property of CQG-theory, which follows from the principle of manifest covariance and the Hamiltonian structure of its formulation, is the possibility of proving the validity of the Heisenberg indeterminacy principle for the CQG-wave Equation (10). This outcome is essential for the subsequent establishment of the quantum minimum-length character of the same CQG-theory. The extended mathematical formulation of the Heisenberg uncertainty theory can be found in refs. [81,82]. Based on these works, in the following section, we summarize the main conceptual steps that are needed for the present research, providing here the formal extension of the theory that includes the case of stochastic graviton trajectories.

First, we require that the quantum-wave function $\psi(s)$ must span a Hilbert space, namely a functional linear space on which a suitable definition of the scalar product $\langle\langle\psi_a(s)|\,\psi_b(s)\rangle\rangle$ can be given. This prescription is needed to warrant the validity of the principle of quantum unitarity, namely the condition for the arbitrary $s \in I \equiv \mathbb{R}$ of the normalization constraint

$$\langle\langle\psi(s)|\,\psi(s)\rangle\rangle = 1. \tag{12}$$

This, in turn, requires the Hamiltonian operator $H^{(q)}$ in Equation (10) to be a Hermitian operator, i.e., to also satisfy, in terms of the same scalar product, the identity

$$\left\langle\left\langle\psi(s)\middle|H_R^{(q)}\psi(s)\right\rangle\right\rangle = \left\langle\left\langle H_R^{(q)}\psi(s)\middle|\psi(s)\right\rangle\right\rangle. \tag{13}$$

As shown in ref. [82], the appropriate prescription of the scalar product is introduced according to the following non-local definition:

$$\langle\langle\psi_a|\psi_b\rangle\rangle \equiv \frac{1}{\int\limits_{s_0}^{s_1} ds} \int\limits_{s_0}^{s_1} ds\langle\psi_a|\psi_b\rangle_L, \tag{14}$$

where a proper-time integration on the interval $[s_0, s_1]$ (with $s_0, s_1 \in I \subseteq \mathbb{R}$) is included. Here, $\langle\psi_a|\psi_b\rangle_L$ denotes the customary local scalar product

$$\langle\psi_a|\psi_b\rangle_L \equiv \int\limits_{U_g} d(g)\psi_a^*\psi_b, \tag{15}$$

where $\psi_a \equiv \psi_a(g, \widehat{g}(r,s), r(s), s)$ and $\psi_b \equiv \psi_b(g, \widehat{g}(r,s), r(s), s)$ are two arbitrary elements of the Hilbert space. We notice that $\langle\psi_a|\psi_b\rangle_L$ is still a function of the arguments $\widehat{g}(r)$, $r(s)$ and s. Thus, while, generally, for an arbitrary Hermitian quantum operator A, one has that $\langle\langle\psi_a|A\psi_b\rangle\rangle \neq \langle\psi_a|A\psi_b\rangle_L$ thanks to Equation (12), it follows that

$$\begin{aligned} \langle\langle\psi(s)|\,\psi(s)\rangle\rangle \equiv & \int\limits_{U_g} d(g)|\psi(g, \widehat{g}(r,s), r(s), s)|^2 \\ \equiv & \langle\psi(s)|\,\psi(s)\rangle_L. \end{aligned} \tag{16}$$

This equation was obtained by setting $\psi_a^* \equiv \psi^*(s)$ and $\psi_b \equiv \psi(s)$, with $\psi(s) \equiv \psi(g, \widehat{g}(r,s), r(s), s)$ denoting the arbitrary element of the same Hilbert space, such as a particular solution of the CQG-wave Equation (10). Notice that, furthermore, the equality of the rhs is warranted by the fact that, in the unitary case, $\langle\psi(s)|\,\psi(s)\rangle_L$ is independent of the proper-time s.

Furthermore, as is appropriate for the theory of stochastic CQG-theory [61] and in view of subsequent development, we generalized the notion of a non-local scalar product and introduced to this aim the stochastic-averaged scalar product, which is defined as

$$\langle\langle\langle\psi_a|\psi_b\rangle\rangle\rangle_\alpha = \int_{I_\alpha} d\alpha g_\varepsilon(\alpha)\langle\langle\psi_a|\psi_b\rangle\rangle, \tag{17}$$

where $\langle\bullet\rangle_\alpha$ denotes the stochastic α-average

$$\langle\bullet\rangle_\alpha \equiv \int_{I_\alpha} d\alpha \bullet g_\varepsilon(\alpha). \tag{18}$$

Here, α denotes a dimensionless stochastic independent 4-scalar parameter (hidden variable), while $r(s,\alpha)$ is a space–time stochastic curve that is required to satisfy the condition that the displacement $r(s,\alpha) - r(s)$ is suitably small with respect to the characteristic scale length of the geodesics $r(s)$. For definiteness, α is assumed here to belong to the finite set (stochasticity domain)

$$I_\alpha = [-a,a] - \{-\varepsilon,\varepsilon\}, \tag{19}$$

where either $a = 1,\infty$, and the value $\alpha = 0$ is assumed to be forbidden, while α is assumed to be endowed with a stochastic PDF. The form of such a PDF remains arbitrary so that possible examples include the following: (a) a binomial PDF with α taking only the values ± 1, and (b) a Gaussian PDF of the form

$$g_\alpha(\varepsilon) = N\exp\left\{-\alpha^2/\varepsilon^2\right\}. \tag{20}$$

Here, $\varepsilon > 0$ is a dimensionless parameter such that $\varepsilon \ll 1$, and N is a normalization constant defined so that

$$\langle 1\rangle_\alpha \equiv \int_{I_\alpha} d\alpha g_\varepsilon(\alpha) = 1. \tag{21}$$

Given these definitions, it is possible to prove that, as a consequence of the strict positivity and smoothness of the quantum PDF $\rho(s)$, the wave function $\psi(s)$ of quantum gravitational field satisfies suitable generalized Heisenberg inequalities, which are related to the fluctuations (and corresponding standard deviations) of the Lagrangian variables and of conjugate quantum canonical momenta. Denoting, for brevity, $\psi_a = \psi_a(s)$ and $\psi_b = \psi_b(s)$, we first notice that the definitions of the scalar product introduced above satisfy the standard properties of the Cauchy–Schwartz inequality, namely

$$\langle\langle\psi_a|\,\psi_a\rangle\rangle\langle\langle\psi_b|\,\psi_b\rangle\rangle \geq |\langle\langle\psi_a|\,\psi_b\rangle\rangle|^2. \tag{22}$$

On the other hand, it can also be proven that

$$\left|(\langle\langle\psi_a|\,\psi_b\rangle\rangle)^2\right| \geq \left|\frac{1}{2i}(\langle\langle\psi_a|\,\psi_b\rangle\rangle - \langle\langle\psi_b|\,\psi_a\rangle\rangle)\right|^2, \tag{23}$$

and, by construction, that

$$\langle\langle\psi_a|\,\psi_b\rangle\rangle = \langle\langle\psi_b|\,\psi_a\rangle\rangle^*. \tag{24}$$

Let us consider now two Hermitian operators A and B, which are denoted as

$$|\psi_a\rangle\rangle = |(A-a)\psi\rangle\rangle, \tag{25}$$

$$|\psi_b\rangle\rangle = |(B-b)\psi\rangle\rangle, \tag{26}$$

with $a = \langle\langle\psi\,|A\psi\rangle\rangle$ and $b = \langle\langle\psi\,|B\psi\rangle\rangle$ being the respective expectation values. Furthermore, let us assume that the quantum-wave function ψ is not an eigenfunction of either operators, namely let us identically exclude that either $(A-a)\psi \equiv 0$ or $(B-b)\psi \equiv 0$. Then, it follows that

$$\langle\langle \psi_a | \psi_a \rangle\rangle = \sigma_A^2 \equiv \langle \psi \left| (A-a)^2 \psi \right\rangle > 0, \tag{27}$$

$$\langle\langle \psi_b | \psi_b \rangle\rangle = \sigma_B^2 \equiv \langle \psi \left| (B-b)^2 \psi \right\rangle > 0, \tag{28}$$

where σ_A and σ_B denote the standard deviations of the quantum operators A and B. Furthermore, one can show that, under the same assumption of the operators A and B being Hermitian, it follows that

$$\langle\langle \psi_a | \psi_b \rangle\rangle - \langle\langle \psi_b | \psi_a \rangle\rangle = \left| \frac{1}{2i} \langle\langle \psi | [A,B] \psi \rangle\rangle \right|^2 . \tag{29}$$

We also conclude that, therefore, in the context of CQG-theory, and in the analogy with foundations of standard quantum mechanics, the formal inequality

$$\sigma_A^2 \sigma_B^2 \geq \left| \frac{1}{2i} \langle\langle \psi | [A,B] \psi \rangle\rangle \right|^2 \tag{30}$$

holds, namely

$$\sigma_A \sigma_B \geq \left| \frac{1}{2i} \langle\langle \psi | [A,B] \psi \rangle\rangle \right| . \tag{31}$$

This identifies the so-called Robertson uncertainty relation, and it actually proves the validity of the Heisenberg indeterminacy principle that holds for the CQG-theory and is expressed in tensorial form, namely it is consistent with the principle of manifest covariance.

For completeness, it must be noticed that analogous inequalities can also be determined in terms of the local scalar product, thus yielding

$$\sigma_A \sigma_B \geq \left| \frac{1}{2i} \langle \psi | [A,B] \psi \rangle \right| , \tag{32}$$

where the standard deviations are again prescribed in terms of the local scalar product defined above. The appropriate choice of the inequalities depends on the Hermitian properties of the operators A and B, i.e., whether they are fulfilled or not by the local scalar product rather than by the non-local one. However, in both cases, the physical meaning of the inequalities is analogous. It is stated that the product of the standard deviations of two non-commuting quantum observables A and B is necessarily bounded from below by one half of the modulus of the expectation value of the commutator $[A,B]$.

As an example of application of the theory appropriate for the CQG-theory, one can consider the conjugate quantum canonical variables that are expressed by the set

$$(A,B) = \left(g_{\mu\nu}, \pi_{\mu\nu}^{(q)} \equiv -i\hbar \frac{\partial}{\partial g^{\mu\nu}} \right), \tag{33}$$

for which the commutator is given by

$$\left[g_{(\mu)(\nu)}, \pi_{\mu\nu}^{(q)} \right] = i\hbar . \tag{34}$$

The inequality for the corresponding standard deviations is then

$$\sigma_{g_{(\mu)(\nu)}} \sigma_{\pi^{(q)}_{(\mu)(\nu)}} \geq \frac{\hbar}{2}, \tag{35}$$

which realizes the generalized Heisenberg inequality for this set of conjugate canonical variables (33). The corresponding squared standard deviations are then given in terms of the local scalar product by

$$\begin{aligned} \sigma^2_{g_{(\mu)(\nu)}} &\equiv \left\langle \left(g_{(\mu)(\nu)} - \widetilde{g}_{(\mu)(\nu)} \right)^2 \right\rangle \\ &\equiv \left\langle \left\langle \psi | \left(g_{(\mu)(\nu)} - \widetilde{g}_{(\mu)(\nu)} \right)^2 \psi \right\rangle \right\rangle, \end{aligned} \tag{36}$$

$$\sigma^2_{\pi^{(q)}_{(\mu)(\nu)}} \equiv \left\langle \left(\Delta\pi^{(q)}_{(\mu)(\nu)} \right)^2 \right\rangle \equiv \left\langle \left\langle \psi | \left(\Delta\pi^{(q)}_{(\mu)(\nu)} \right)^2 \psi \right\rangle \right\rangle, \tag{37}$$

where $\Delta\pi^{(q)}_{(\mu)(\nu)} = \pi^{(q)}_{(\mu)(\nu)} - \widetilde{\pi}^{(q)}_{(\mu)(\nu)}$, while $\widetilde{g}_{(\mu)(\nu)}$ and $\widetilde{\pi}^{(q)}_{(\mu)(\nu)}$ are the quantum expectation values.

4. Proper-Time Heisenberg Inequality

The Heisenberg indeterminacy principle expressed in terms of the non-local scalar product (14) brings a unique result that is peculiar to CQG-theory and its stochastic extension [61]. This concerns the validity of the Heisenberg inequality for additional sets of conjugate canonical variables besides the fundamental ones provided by Equation (33). This feature permits one to establish the existence of a minimum length and its relationship with the Planck length in terms of a statistical notion related to the lower bound on error measurements of invariant lengths in the CQG-theory. As a result, it is shown that the quantum minimum length σ can actually be related to the small-scale stochastic character of quantum graviton trajectories.

To illustrate the issue, let us consider the set of canonical variables $\left(s, p_s^{(q)}\right)$. More precisely, the invariant path-length (s) along the suitable space–time trajectories of graviton particles is introduced, whereby graviton quantum particles are assumed to move freely along one of the infinite set of possible space–time stochastic geodesics that they may belong to. Introducing the quantum momentum

$$p_s^{(q)} \equiv -i\hbar \frac{\partial}{\partial s}, \tag{38}$$

and denoting by $\hbar$ the Planck constant, it is assumed that, within the framework of the CQG-theory, the ensemble of 4-scalar operators

$$(A, B) = \left(s, p_s^{(q)} \equiv -i\hbar \frac{\partial}{\partial s} \right) \tag{39}$$

identifies a set of quantum canonical variables. Here, s is the proper time that parameterizes the local-field geodetic $\{r(s) | s \in [s_0, s_1]\}$, and $p_s^{(q)}$ is the corresponding extended canonical momentum. We notice that both $\left(s, p_s^{(q)}\right)$ are Hermitian operators. In particular, setting in Equation (14) $\psi_a^* \equiv \psi^*(s)$, $\psi_b \equiv p_s^{(q)} \psi(s)$, where $\psi(s) \equiv \psi(g, \widehat{g}(r,s), r(s), s)$ denotes an arbitrary element of the Hilbert space to be identified with a particular solution of the CQG-wave Equation (10), then, by construction, it follows that

$$\begin{aligned}\left\langle\left\langle\psi|p_s^{(q)}\psi\right\rangle\right\rangle &\equiv F\left\{\int_{U_g} d(g)\psi^*(s)p_s^{(q)}\psi(s)\right\} \\ &= F\left\{\int_{U_g} d(g)\left(p_s^{(q)}\psi^*(s)\right)\psi(s)\right\} \\ &= \left\langle\left\langle p_s^{(q)}\psi|\psi\right\rangle\right\rangle.\end{aligned} \tag{40}$$

Here, according to Equation (14), F is the linear operator $F = \frac{1}{\Delta s}\int_{s_0}^{s_0+\Delta s} ds$, where $\Delta s \equiv \int_{s_0}^{s_1} ds$. The same relationship also follows by partial integration with respect to the proper-time s. In fact, $\left\langle\left\langle\psi|p_s^{(q)}\psi\right\rangle\right\rangle \equiv -i\hbar\left\langle\left\langle\psi|\frac{\partial}{\partial s}\psi\right\rangle\right\rangle$; hence, integrating by parts $\left\langle\left\langle\psi|p_s^{(q)}\psi\right\rangle\right\rangle = i\hbar\left\langle\left\langle\frac{\partial}{\partial s}\psi|\psi\right\rangle\right\rangle = \left\langle\left\langle p_s^{(q)}\psi|\psi\right\rangle\right\rangle$. However, the products of Hermitian operators need not be Hermitian. In particular, direct evaluation shows that $\left\langle\left\langle\psi|sp_s^{(q)}\psi\right\rangle\right\rangle \neq \left\langle\left\langle sp_s^{(q)}\psi|\psi\right\rangle\right\rangle$. On the other hand, the commutator of $(s, p_s^{(q)})$ is evidently

$$\left[s, p_s^{(q)}\right] = i\hbar. \tag{41}$$

Assuming, for greater generality, the case of stochastic quantum graviton trajectories, the quantum expectation values of the variables $\left(s, p_s^{(q)}\right)$ are evaluated by means of the quantum averages

$$\widetilde{s} \equiv \langle\langle\psi|s\psi\rangle\rangle_\alpha, \tag{42}$$

$$\widetilde{p_s^{(q)}} \equiv \left\langle\left\langle\psi\middle|p_s^{(q)}\psi\right\rangle\right\rangle_\alpha, \tag{43}$$

with $\langle\langle\bullet|\bullet\rangle\rangle_\alpha$ denoting the stochastic-averaged scalar product prescribed in Section 3. Similarly, the corresponding quantum standard deviations (i.e., quantum fluctuations) σ_s and $\sigma_{p_s^{(q)}}$ were prescribed by the quantum averages of the form

$$\sigma_s^2 \equiv \left\langle\langle\psi\middle|(s-\widetilde{s})^2\psi\right\rangle\rangle_\alpha, \tag{44}$$

$$\sigma_{p_s^{(q)}}^2 \equiv \left\langle\left\langle\psi\middle|\left(p_s^{(q)} - \widetilde{p_s^{(q)}}\right)^2\psi\right\rangle\right\rangle_\alpha. \tag{45}$$

In particular, in the case of a unitary solution, one can show that $\widetilde{p_s^{(q)}} = -\widetilde{H}^{(q)}$, namely $\widetilde{p_s^{(q)}}$ is related to the expectation value of the quantum Hamiltonian. Then, provided σ_s and $\sigma_{p_s^{(q)}}$ are both non-vanishing, from the Robertson inequality (31), it therefore follows that

$$\sigma_s\sigma_{p_s^{(q)}} \geq \frac{\hbar}{2}, \tag{46}$$

which realizes the proper-time-extended canonical momentum Heisenberg inequality for the canonical variables $\left(s, p_s^{(q)}\right)$. Therefore, the simultaneous quantum measurement of proper-time s and its conjugate quantum-extended momentum $p_s^{(q)}$ during the proper-time interval $(s_o, s_1 = s_0 + \Delta s)$ involves the evaluation of the expectation values $\left(\widetilde{s}, \widetilde{p_s^{(q)}}\right)$ together with the related standard deviations σ_s and $\sigma_{p_s^{(q)}}$. The corresponding quantum fluctuations (i.e., the squares of the standard deviations σ_s and $\sigma_{p_s^{(q)}}$) are therefore given by Equations (44) and (45).

5. Quantum Minimum Length

Let us now prove the existence of the quantum minimum length σ for the CQG-theory based on the theoretical framework provided by the proper-time Heisenberg indeterminacy principle.

First, let us consider the prescription of the proper-time extrema, which enter the stochastic non-local scalar product introduced in Equation (17) and affects the prescription of the fluctuations in Equations (44) and (45). In fact, under validity of the unitarity principle, let

$$s_1 = s_0 + \Delta s \tag{47}$$

and let both s_0 and Δs remain in principle arbitrary, with Δs being interpreted as the amplitude of proper-time intervals during which a quantum measurement is performed, and which is associated with a massive graviton along its stochastic trajectory on the background space–time. As such, it follows that the amplitude Δs cannot be arbitrarily small.

Then, formally setting $s_0 = 0$ in the time averages, one finds out that the expectation value and standard deviation of s are, respectively, given by

$$\widetilde{s} = \int_{I_\alpha} d\alpha g_\varepsilon(\alpha) \frac{1}{\int_{s_0}^{s_1} ds} \int_{s_0}^{s_1} dss = \frac{\langle \Delta s \rangle_\alpha}{2}, \tag{48}$$

$$\sigma_s \equiv \sqrt{\langle\langle \psi | (s - \widetilde{s})^2 \psi \rangle\rangle_\alpha} = \frac{\sqrt{\langle \Delta s^2 \rangle_\alpha}}{2\sqrt{3}}. \tag{49}$$

Then, the generalized Heisenberg inequality, written in terms of the invariant length Δs and the standard deviation $\sigma_{p_s^{(q)}}$, requires that

$$\sigma_s \sigma_{p_s^{(q)}} = \frac{\sqrt{\langle \Delta s^2 \rangle_\alpha}}{2\sqrt{3}} \sigma_{p_s^{(q)}} \geq \frac{\hbar}{2}, \tag{50}$$

which, therefore, reduces to

$$\sqrt{\langle \Delta s^2 \rangle_\alpha} \sigma_{p_s^{(q)}} \geq \hbar\sqrt{3}. \tag{51}$$

Let us briefly analyze the physical implications of this result in the context of the unitary representation of the CQG-theory. The first one follows from the validity of manifest covariance. The consistency with such a principle warrants that Inequality (51) holds in arbitrary GR-frames, which belong to the same space–time $\{\mathbf{Q}^4, \widehat{g}\}$, i.e., are defined with respect to arbitrary coordinate systems that are mutually related by means of the local-point transformations $r' = r'(r)$. In fact, the physical meaning of inequality (50) is that of expressing a lower bound for the length σ_s, i.e., the proper-time amplitude of the quantum measurement. In turn, this naturally gives rise to a minimal length that is determined by the same inequality, i.e., in terms of the standard deviation of the conjugate quantum momentum operator $\sigma_{p_s^{(q)}}$. As a remarkable consequence, such a minimal length is necessarily an invariant, i.e., it is a 4-scalar with respect to the background space–time of CQG-theory that is expressed by the set $\{\mathbf{Q}^4, \widehat{g}\}$. This means that σ_s itself has an objective character (which is, in a proper sense, a mandatory requirement for a quantum observable); therefore, it has the same value when expressed again in arbitrary GR-frames belonging to $\{\mathbf{Q}^4, \widehat{g}\}$. The generalized Heisenberg Inequality (50) implies that, by construction, σ_s, is strictly positive and therefore that the limit $\sigma_s \to 0$ is physically meaningless.

Let us consider, in more detail, Equation (50). First, we noticed that, by construction, both σ_s and $mc \equiv \sigma_{p_s^{(q)}}$ may generally depend on the neighborhood of space–time in which the quantum averages are evaluated. On the other hand, in Equation (50) the case of equality effectively determines a threshold condition. Under such an occurrence, the standard deviations σ_s and $\sigma_{p_s^{(q)}}$ are assumed to take extremal values expressed as

$$\sigma_s \rightarrow \sigma_s|_{extr}, \tag{52}$$

$$\sigma_{p_s^{(q)}} \rightarrow \sigma_{p_s^{(q)}}\Big|_{extr} \equiv mc, \tag{53}$$

where mc can be also equivalently referred to as the (minimum) mass–momentum particle. As a consequence, the extremal form of Equation (50) becomes

$$\sigma_s|_{extr}\,\sigma_{p_s^{(q)}}\Big|_{extr} = \frac{\hbar}{2}, \tag{54}$$

where $\sigma_s|_{extr}$ and $\sigma_{p_s^{(q)}}\Big|_{extr}$ are universal constants, namely, by assumption, they depend only on the universal constants G, $\hbar$ and c. The problem of the determination of $\sigma_s|_{extr}$ is referred to here as the quantum minimum-length problem. This ultimately permits one to establish that

$$\sigma \equiv \sigma_s|_{extr}, \tag{55}$$

which clearly identifies the quantum minimum length and proves its existence within the conceptual framework of the CQG-theory.

In particular, such an evaluation is conveniently carried out in terms of Planck units, namely introducing the Planck length and momentum that are, respectively, defined as $\ell_P = \sqrt{\frac{\hbar G}{c^3}}$ and $P_P = cm_P$, where $m_P = \sqrt{\frac{\hbar c}{G}}$ is the corresponding Planck mass. Thus, once Inequality (50) is set in terms of the Planck units ℓ_P and P_P, it follows that

$$\left[\frac{\sqrt{\langle \Delta s^2 \rangle_\alpha}}{\ell_P}\frac{\sigma_{p_s^{(q)}}}{P_P}\right]\ell_P P_P \geq \hbar\sqrt{3}, \tag{56}$$

namely

$$\overline{\Delta s}\,\overline{\sigma_{p_s^{(q)}}} \geq \sqrt{3}, \tag{57}$$

where $\overline{\Delta s} = \frac{\sqrt{\langle \Delta s^2 \rangle_\alpha}}{\ell_P}$ and $\overline{\sigma_{p_s^{(q)}}} = \frac{\sigma_{p_s^{(q)}}}{P_P}$ denote the normalized, stochastic-averaged proper length and the normalized stochastic-averaged conjugate momentum uncertainties, respectively.

These conclusions provide the basis for inferring an important physical clue on the concept of minimal length. Specifically, we refer here to the debate that usually arises in the framework of phenomenological Generalized Uncertainty Principle (GUP) theories [83,84]. GUP models have been investigated as alternative quantum approaches in the study of the quantum properties of classic black-hole solutions and related Hawking evaporation phenomena [85–87]. First of all, we wish to stress that the present Heisenberg indeterminacy principle originates from a canonical quantization of the gravitational field satisfying manifestly covariant representation. In contrast, in the literature, GUP proposals usually arise as heuristic non-canonical quantum theories, although similar generalized uncertainty relations can be inferred by non-commutative geometry [88–92] and string theory [93–95]. As an immediate consequence, the minimal length predicted by GUP models might not be generally a 4-scalar, a fact which implies the violation of the principle of manifest covariance. The notable aspect of the present prescription is that it suggests a possible way out to the problem since the formalism of CQG-theory has a tensorial character and is therefore manifestly covariant. Nevertheless, as a matter of consistency, the relation with the Heisenberg principle determined above is intrinsically distinguished from GUP models. This follows precisely because of the canonical Hamiltonian structure of the same CQG-theory in comparison with non-canonical and non-Hamiltonian approaches. In particular, the theory proposed here leads to a relation that does not contain any ad hoc minimum length parameter to be introduced as a modification of the rhs of the Heisenberg inequality [96]. Hence, on the conceptual side, thanks to the adoption of the concept of stochastic trajectories in the context of CQG-theory, the GUP is no longer needed.

Because of these features, given the existence of a quantum minimum-effective length, at the same time, CQG-theory can be interpreted as a minimum-length theory. The latter phenomenon arises as a lower bound in the error measurement of physical distances in a curved space–time for massive gravitons that are generally characterized by stochastic quantum dynamics. Hence, the intuitive reason behind the existence of such a minimum length can be related to the stochastic behavior that may/might appear to characterize the small-scale behavior of quantum gravity or, equivalently, the quanta of gravitational fields.

6. Minimum-Length Representation: The Planck Length

In this section, we establish the relationship that exists between the quantum minimum length σ determined in Section 5 and its representation in terms of the Planck length. This permits one to explicitly draw the physical meaning of the Planck length in CQG-theory as an invariant scale length that emerges consistently with the fundamental principles of quantum field theory and quantum mechanics, namely from the theory of stochastic error measurements of the conjugate set of observables determined by the Heisenberg indeterminacy principle.

In detail, the generalized Heisenberg Inequality (50), or its normalized form (56), permit us to draw definite conclusions concerning the extremal values of σ_s and $\sigma_{p_s^{(q)}}$. In fact, the validity of Equation (54) leads one to constrain the value of the minimum length σ. One can show that this implies that

$$\sigma_s\big|_{extr} = \sqrt{\beta}\sqrt{\frac{\hbar G}{c^3}}, \tag{58}$$

$$\sigma_{p_s^{(q)}}\Big|_{extr} = \frac{c}{2\sqrt{\beta}}\sqrt{\frac{\hbar c}{G}}, \tag{59}$$

where $m_P = \sqrt{\frac{\hbar c}{G}}$ and $\ell_P = \sqrt{\frac{\hbar G}{c^3}}$ denote, respectively, the Planck mass and length, while

$$\beta = \beta(r/L_C) \tag{60}$$

represents some suitable dimensionless and non-vanishing function that will be discussed below and is expressed in terms of the ratio r/L_C, with L_C being the the Compton length associated with massive quantum gravitons. The freedom in the prescription of function β is a characteristic property of the theory that may be seen as a gauge property or as an arbitrariness that is analogous to the (arbitrary) choice of the unit system.

The proof is as follows. First, one notices that dimensional analysis requires

$$\sigma_s\big|_{extr} = \sigma, \tag{61}$$

$$\sigma_{p_s^{(q)}}\Big|_{extr} = mc, \tag{62}$$

where σ and m are, respectively, a length and a rest mass that are both to be determined. Then, Equation (54) requires that r and m are related via

$$2\sigma = \frac{\hbar}{mc} \equiv L_C(m), \tag{63}$$

where $L_C(m)$ denotes the Compton length of a particle of rest mass m. On the other hand, given a graviton particle of mass m, it is always possible to also introduce the corresponding Schwarzschild radius $r_{S(m)} \equiv \frac{2Gm}{c^2}$ and to require that σalso takes the form

$$\sigma = \beta \frac{2Gm}{c^2} \equiv \beta r_S(m), \tag{64}$$

namely it coincides, up to a factor β, with the same characteristic graviton Schwarzschild radius $r_{S(m)}$. Notice that, in general, here, β is a real non-vanishing function of the form $\beta = \beta(\sigma/L_C)$, which remains, in principle, arbitrary. Let us now require and assume that σ and m only depend on the universal constants G, $\hbar$ and c. This demands that, therefore, m and σ take the general form

$$m = M_P^{(\beta)} \equiv \frac{1}{2\sqrt{\beta}}\sqrt{\frac{\hbar c}{G}} = \frac{1}{2\sqrt{\beta}} m_P, \tag{65}$$

$$\begin{aligned} \sigma &= r_P^{(\beta)} = \beta \frac{2GM_P^{(\beta)}}{c^2} = \sqrt{\beta} r_S(m_P) \\ &\equiv \beta r_S(M_P^{(\beta)}), \end{aligned} \tag{66}$$

where $r_S(m_P)$ and $r_S(M_P^{(\beta)}) \equiv \frac{2GM_P^{(\beta)}}{c^2}$ denote, respectively, the Schwarzschild radius for the Planck mass and the effective Planck mass $M_P^{(\beta)}$. Again, we notice that the function β remains arbitrary and non-vanishing in order to allow for a generality of the formalism since, a priori, one should not assume Planck units to identically hold. Furthermore, as a matter of consistency, invoking Equation (63), it follows that

$$\begin{aligned} \frac{r_P^{(\beta)}}{L_C\left(M_P^{(\beta)}\right)} &= \frac{\beta \frac{2GM_P^{(\beta)}}{c^2}}{\frac{\hbar}{M_P^{(\beta)} c}} = \beta \frac{2GM_P^{(\beta)2}}{\hbar c} \\ &= \beta \frac{2G}{\hbar c} \frac{1}{4\beta} \frac{\hbar c}{G} = \frac{1}{2}. \end{aligned} \tag{67}$$

Hence, provided β is non-vanishing, Equation (54) is identically satisfied since

$$r_P^{(\beta)} M_P^{(\beta)} c \equiv \sqrt{\beta}\sqrt{\frac{\hbar G}{c^3}} \frac{c}{2\sqrt{\beta}} \sqrt{\frac{\hbar c}{G}} = \frac{\hbar}{2}. \tag{68}$$

Thus, when comparing Equation (68) with Equation (63), it follows that the extremal values σ_s and $\sigma_{p_s^{(q)}}$ must coincide with Equations (58) and (59), respectively. Notice that, in principle, nothing prevents one to assume that the arbitrary factor β appearing in the previous Equations (65) and (66) should be different from $O(1)$ (and possibly even either $\ll 1$ or $\gg 1$). A definite answer on the issue requires the explicit evaluation of the extremal standard deviation $\sigma_s|_{extr}$, i.e., one that is based on the prescription of the quantum Hamiltonian operator $H^{(q)}$, which appears in the quantum-wave Equation (10)—a task that shall be reported elsewhere. However, the question is under what conditions, if any, it is warranted that—consistently with Claim (2), which was stated at the beginning—the ordering $\beta \sim O(1)$ actually holds. The answer is related to the possible independent estimates of either the effective Schwarzschild or the Compton lengths, and it occurs provided the assumptions

$$r_S(M_P^{(\beta)}) \simeq \ell_P, \tag{69}$$

or equivalently

$$L_C\left(M_P^{(\beta)}\right) \simeq L_C(m_P), \tag{70}$$

hold. However, both orderings apply if the effective Planck mass $M_P^{(\beta)}$ becomes of the same order of the Planck mass m_P. Therefore, under this condition, one concludes that

$$\sigma \equiv \sigma_s|_{extr} \simeq \sqrt{\frac{\hbar G}{c^3}} \equiv \ell_P, \tag{71}$$

$$\sigma_{p_s^{(q)}}\Big|_{extr} \simeq \sqrt{\frac{\hbar c^3}{G}} \equiv m_P c, \tag{72}$$

namely, as anticipated, the extremal values σ_s and $\sigma_{p_s^{(q)}}$ coincide up to factors of the order $O(1)$, respectively, with the Planck length and Planck particle mass momentum. As a consequence, invoking the validity of Equation (55) implies that the quantum minimum length σ, whose existence is proven on the basis of the Heisenberg indeterminacy principle up to a factor of order $O(1)$, is also identified at the same time with the Planck length. This proves the validity of Equation (2).

Let us briefly analyze the physical meaning of the results expressed by Equations (71) and (72):

- Since the extremal value of σ_s coincides with the quantum minimum-length σ, the obvious consequence is that the CQG-theory can be viewed as an invariant minimum-length theory, where the same length is not, however, associated with the possible discreteness of space–time, nor does it modify, in any way, the form of the Heisenberg indeterminacy principle.
- The existence of the quantum minimum length σ is not at variance with the validity of the CQG-theory nor, in particular, its manifest covariance.
- The mathematical proof established above determines the connection between the quantum minimum length σ and the Planck length ℓ_P. The relevant outcome is that they coincide up to a factor of order $O(1)$; thus, an emerging character as a foundational element of CQG-theory and its logical framework is assigned to the Planck length ℓ_P.

7. Conclusions

The rationale at the basis of the present research is that, in quantum gravity theory, a quantum minimum effective length can correspond to the standard deviation in the path length (s) defined along the non-zero geodesic curves that identify the possible stochastic space–time trajectories of massive graviton particles. In order that the same arc length and the standard deviation make sense at all (i.e., actually identify observable dynamical variables), however, requires placing well-defined constraints on the requisite admissible quantum gravity theory in which the notion of minimum length is determined. Thus, in order to satisfy the principle of general covariance, the arc length and the standard deviation must necessarily preserve their form in arbitrary GR-frames, i.e., they should be expressed in 4-tensor form with respect to the group of local-point transformations (diffeomorphisms).

In particular, this means that the following apply: (1) gravitons must be realized by massive particles with quantum stochastic trajectories; (2) the physical space–time in which the quantum gravity theory is developed must be identified with a 4-dimensional background space–time, which is endowed with a well-defined Lorentzian time-oriented and curved structure; and (3) quantum gravity theory must be manifestly covariant and satisfy a generalized Heisenberg uncertainty principle in 4-tensor form, which involves—in particular—the standard deviation of the path length s and that of its canonically conjugate quantum momentum.

A possible model of quantum gravity theory satisfying these requirements is realized by the theory of manifestly covariant quantum gravity (CQG-theory) that was recently achieved. This is rooted on the canonical quantization of the classical Hamiltonian structure underlying General Relativity and is displayed by means of a synchronous variational principle. As shown here, the theory permits one to reach well-defined conclusions regarding the proof of existence of a quantum minimum length and its realization in terms of Planck length. The analysis is based on the implementation of the stochastic quantization underlying CQG-theory, particularly the validity of a generalized Heisenberg inequality for the set of conjugate canonical variables that are represented by $\left(s, p_s^{(q)} \equiv -i\hbar\frac{\partial}{\partial s}\right)$. As a result, it has been shown that, ultimately, the existence of a quantum minimum length arises due to the stochastic character of quantum graviton trajectories. The minimum length in such a case remains associated to the lower bound of the error measurement of graviton arc lengths. Remarkably, apart from a numerical factor of order unity, such a limit is found to be expressed by the Planck length.

A relevant aspect of the treatment proposed here is that the existence of a minimum length follows from the validity of the Heisenberg indeterminacy principle for the CQG-theory, which is expressed in canonical form according to the algebraic prescription of quantum mechanics. In addition, the same feature permits one to prove that the existence of the minimum length obtained in this way is not at variance with the principle of manifest covariance on which the CQG-theory is rooted. In fact, in such a framework, the minimum length appears as a property of the geodesic trajectory of massive gravitons and does not rely on any assumption of the discreteness of quantum space–time.

In conclusion, what emerges is that the stochastic CQG-theory appears to meet the physical requirements set by the quantum minimum-length problem. Nevertheless, further investigations are demanded to understand the full implications of the present theory, with particular reference to the role of the stochastic properties of quantum gravitational fields at the Planck scale.

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