Quark-quark-gluon vertex for heavy quarks up to order $1/m^5$

Guojun Huang and Pengfei Zhuang
Physics Department, Tsinghua University, Beijing 100084, China

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Instead of the often used Foldy-Wouthuysen-Tani (FWT) transformation in nonrelativistic quantum chromodynamics (NRQCD), we take a more general relation between the relativistic and nonrelativistic on-shell spinors to recalculate the quark-quark-gluon vertex for heavy quarks. In comparison with the previous result using FWT, we obtain the high-order coefficients in the NRQCD Lagrangian up to $1/m^5$, where m is the heavy quark mass.

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Nonrelativistic quantum chromodynamics (NRQCD) is often used to describe the dynamics of heavy quarks at low energy [1–3]. The usual understanding of the total heavy quark momentum is the summation of the quark mass m times its velocity v and the residual momentum k, $p = mv + k$. Luke and Manohar constructed the transformation for spinor fields in analogy to a vector transformation, which yields the relation between the reparametrized spinor field Ψ_v and the conventional heavy quark field ψ_v [4],

$$\Psi_v = \frac{m + i\partial\cdot v + i\not{\partial}_\perp + \sqrt{(m + i\partial\cdot v)^2 + (i\partial_\perp)^2}}{[2\sqrt{(m + i\partial\cdot v)^2 + (i\partial_\perp)^2}(m + i\partial\cdot v + \sqrt{(m + i\partial\cdot v)^2 + (i\partial_\perp)^2})]^{1/2}}\psi_v \tag{1}$$

with the definition of a transverse vector $a^\mu_\perp = a^\mu - v^\mu a\cdot v$. On the other hand, the often used Foldy-Wouthuysen-Tani (FWT) transformation yields a simple representation [5],

$$\Psi_v = e^{\frac{i\not{\partial}_\perp}{2m}}\psi_v, \tag{2}$$

which can be expanded in terms of the inverse heavy quark mass,

$$\Psi_v = \left(1 + \frac{\not{k}_\perp}{2m} + \frac{k_\perp^2}{8m^2} + \cdots\right)\psi_v \tag{3}$$

in momentum representation. To the order $1/m^2$, the two results (1) and (2) are consistent with each other, under the condition of $i\partial\cdot v\psi_v \to k\cdot v\psi_v = 0$, which is, however, not easy to understand for $p = mv + k$. Attempts to obtain exact FWT transformation in some cases can be seen, for instance, in Ref. [6].

In this paper, we recalculate the quark-quark-gluon vertex, using a more general relation for relativistic and nonrelativistic on-shell fields, and fix the coefficients of the corresponding NRQCD Lagrangian by matching the result with the perturbative QCD calculation. We also make a comparison with the previous calculation using the FWT transformation to see the difference.

Under the assumption of the on-shell condition $p_0 = E = \sqrt{m^2 + \mathbf{p}^2}$, which is valid for heavy quarks, the relation between the relativistic spinors u and v corresponding to positive and negative energies and their nonrelativistic limits u_{NR} and v_{NR} can be expressed as [7]

$$\begin{aligned} u(p) &= \frac{1}{\sqrt{2E(E+m)}}(m + \not{p})u_{NR}(p) = \frac{m + E + \not{p}_\perp}{\sqrt{2E(E+m)}}u_{NR}(p), \\ v(p) &= \frac{1}{\sqrt{2E(E+m)}}(m - \not{p})v_{NR}(p) = \frac{m + E - \not{p}_\perp}{\sqrt{2E(E+m)}}v_{NR}(p) \end{aligned} \tag{4}$$

in the local rest frame with quark velocity $v_\mu = (1, 0, 0, 0)$. The $1/m$ expansion of the relation for heavy quarks to the order $1/m^2$ is just the normal FWT transformation (2), when we take $k_\perp$ to be $p_\perp$. In this sense, the relation (4) can be considered a generalized FWT transformation. Under the assumption $i\partial \cdot v\psi_v \to k \cdot v\psi_v = 0$ and $i\partial^\mu \psi_v \Rightarrow k^\mu \psi_v = p_\perp^\mu \psi_v$ in the momentum representation, the reparameterization (1) and the relation (4) are also consistent with each other.

We now calculate the quark-quark-gluon vertex for heavy quarks. Taking the on-shell condition, the vertex can be expressed in terms of the form factors $F_1(q^2/m^2)$ and $F_2(q^2/m^2)$ [4],

$$-ig\bar{u}(p')T^a\left(\gamma^\mu F_1(q^2/m^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2/m^2)\right)u(p), \tag{5}$$

with coupling constant g, Gell-Mann matrices T_a, and momentum transfer $q = p' - p$ between the initial and final momenta. Taking the relations $\gamma_0 = \sigma_3 \otimes I_2$ and $\boldsymbol{\gamma} = i\sigma_2 \otimes \boldsymbol{\sigma}$, the vertex can be explicitly written as

$$\begin{aligned} &\frac{-igT^a}{\sqrt{4E'(E'+m)E(E+m)}}\psi^\dagger\{(F_1(q^2/m^2) + F_2(q^2/m^2)) \\ &\times [\delta^\mu{}_0((m+E')(m+E) + \mathbf{p}'\cdot\mathbf{p} + i\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})) + \delta^\mu{}_j((m+E')\sigma^j\mathbf{p}\cdot\boldsymbol{\sigma} + (m+E)\mathbf{p}'\cdot\boldsymbol{\sigma}\sigma^j)] \\ &- \frac{F_2(q^2/m^2)}{2m}(p'^\mu + p^\mu)[(m+E')(m+E) - \mathbf{p}'\cdot\mathbf{p} - i\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})]\}\psi \end{aligned} \tag{6}$$

with the final-state energy $E' = \sqrt{m^2 + \mathbf{p}'^2}$.

In terms of the small variable q^2/m^2 for heavy quarks, the form factors F_i $(i = 1, 2)$ can be expanded as

$$\begin{aligned} F_i(q^2/m^2) &= \sum_{n=0}^{+\infty}\frac{1}{n!}\frac{d^nF_i(q^2/m^2)}{d(q^2/m^2)^n}\bigg|_{q^2/m^2=0}(q^2/m^2)^n \\ &= F_i(0) - \frac{\mathbf{q}^2}{m^2}F_i'(0) + \frac{1}{4m^4}[(\mathbf{p}^2 - \mathbf{p}'^2)^2F_i'(0) + 2\mathbf{q}^4F_i''(0)] + \cdots. \end{aligned} \tag{7}$$

To simplify the notation, we take $F_i = F_i(0), F_i' = F_i'(0)$ and $F_i'' = F_i''(0)$ in the following.

Taking into account the transformation (4) between relativistic and nonrelativistic quark fields, the expansion (7) for the form factors, and the relations $u_{NR}^\dagger = (\psi^\dagger, 0)$ and $\gamma^0 u_{NR} = u_{NR}$, the vertex can be expressed in terms of the current j_μ,

$$-igT^a u_{NR}^\dagger j_\mu A_a^\mu u_{NR} \tag{8}$$

with

$$\begin{aligned} j_0 = F_1 &- \frac{1}{4m^2}\left[\left(\frac{1}{2}F_1 + F_2 + 4F_1'\right)\mathbf{q}^2 - i(F_1 + 2F_2)\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})\right] \\ &+ \frac{1}{8m^4}\left[\left(\frac{5}{16}F_1 + \frac{1}{4}F_2 + 2F_1'\right)(\mathbf{p}^2 - \mathbf{p}'^2)^2 + (F_1' + 2F_2' + 4F_1'')\mathbf{q}^4 + \left(\frac{3}{8}F_1 + \frac{1}{2}F_2\right)(\mathbf{p}'^2 + \mathbf{p}^2)\mathbf{q}^2\right. \\ &\left.- i(2F_1' + 4F_2')\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})\mathbf{q}^2 - i\left(\frac{3}{4}F_1 + F_2\right)\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})(\mathbf{p}'^2 + \mathbf{p}^2)\right] + \mathcal{O}(1/m^6) \end{aligned} \tag{9}$$

to order $1/m^4$ and

$$
\begin{aligned}
\boldsymbol{j} = {} & \frac{1}{2m}[F_1(\mathbf{p}+\mathbf{p}') + i(F_1+F_2)\boldsymbol{\sigma}\times\mathbf{q}] \\
& - \frac{1}{8m^3}\left\{\left[F_1(\mathbf{p}'^2+\mathbf{p}^2) + \left(\frac{1}{2}F_2 + 4F_1'\right)\mathbf{q}^2\right](\mathbf{p}+\mathbf{p}') + i[(F_1+F_2)(\mathbf{p}'^2+\mathbf{p}^2) + 4(F_1'+F_2')\mathbf{q}^2]\boldsymbol{\sigma}\times\mathbf{q}\right. \\
& \left. + \frac{1}{2}(F_1+F_2)(\mathbf{p}'^2-\mathbf{p}^2)\mathbf{q} + \frac{i}{2}(F_1+F_2)(\mathbf{p}'^2-\mathbf{p}^2)\boldsymbol{\sigma}\times(\mathbf{p}+\mathbf{p}') - iF_2\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})(\mathbf{p}+\mathbf{p}')\right\} \\
& + \frac{1}{8m^5}\left\{\left[-(F_1'+F_2')(\mathbf{p}'^2-\mathbf{p}^2)\mathbf{p}'\cdot\mathbf{p} + \left(\frac{1}{2}(F_1'+F_2') + \frac{5}{16}(F_1+F_2)\right)(\mathbf{p}'^4-\mathbf{p}^4)\right]i\boldsymbol{\sigma}\times(\mathbf{p}'+\mathbf{p})\right. \\
& + \left[8(F_1''+F_2'')(\mathbf{p}'\cdot\mathbf{p})^2 - 2(F_1'+F_2'+4(F_1''+F_2''))(\mathbf{p}'^2+\mathbf{p}^2)\mathbf{p}'\cdot\mathbf{p}\right. \\
& \left. + \left(\frac{3}{16}(F_1+F_2) + 4(F_1''+F_2'')\right)\mathbf{p}^2\mathbf{p}'^2 + \left(\frac{21}{32}(F_1+F_2) + 2(F_1'+F_2'+F_1''+F_2'')\right)(\mathbf{p}'^4+\mathbf{p}^4)\right]i\boldsymbol{\sigma}\times\mathbf{q} \\
& - \left[\left(F_2' + \frac{3}{8}F_2\right)(\mathbf{p}'^2+\mathbf{p}^2) - 2F_2'\mathbf{p}'\cdot\mathbf{p}\right]i\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})(\mathbf{p}'+\mathbf{p}) \\
& + \left[\frac{1}{2}(F_1'+F_2')(\mathbf{p}'^2-\mathbf{p}^2)\mathbf{q}^2 + \frac{5}{16}(F_1+F_2)(\mathbf{p}'^4-\mathbf{p}^4)\right]\mathbf{q} \\
& + \left[(2F_2'+8F_1'')(\mathbf{p}'\cdot\mathbf{p})^2 - \left(\frac{3}{8}F_2 + 2F_1' + 2F_2' + 8F_1''\right)(\mathbf{p}'^2+\mathbf{p}^2)\mathbf{p}'\cdot\mathbf{p}\right. \\
& \left.\left. + \left(\frac{3}{16}F_1 + \frac{1}{8}F_2 + F_2' + 4F_1''\right)\mathbf{p}^2\mathbf{p}'^2 + \left(\frac{21}{32}F_1 + \frac{5}{16}F_2 + 2F_1' + \frac{1}{2}F_2' + 2F_1''\right)(\mathbf{p}'^4+\mathbf{p}^4)\right](\mathbf{p}'+\mathbf{p})\right\} \\
& + \mathcal{O}(1/m^7) \qquad (10)
\end{aligned}
$$

to order $1/m^5$.

We now turn to the NRQCD calculation. The corresponding NRQCD Lagrangian density can be fixed by matching the above calculated quark-quark-gluon vertex. Integrating out the unphysical field ψ_v' which is defined as $\psi(x) = e^{-imv\cdot x}(h_v(x) + \psi_v'(x))$ and satisfies the relation $\not{v}\psi_v' = -\psi_v'$, the familiar Lagrangian density

$$
\mathcal{L} = \bar{h}_v(iD\cdot v)h_v + \bar{h}_v\left(i\not{D}_\perp \frac{1}{2m + iD\cdot v} i\not{D}_\perp\right)h_v \qquad (11)
$$

can be expanded in terms of $1/m$,

$$
\begin{aligned}
\mathcal{L} = {} & \bar{h}_v(iD_0)h_v + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2m)^{n+1}}\bar{h}_v\{(iD_0)^n, \mathbf{D}^2 + g\boldsymbol{\sigma}\cdot\mathbf{B}\}h_v \\
& + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2m)^{n+2}}\sum_{l=0}^{n}\bar{h}_v(iD_0)^{n-l}g([\mathbf{D},\mathbf{E}]_{\cdot} + i\boldsymbol{\sigma}\cdot[\mathbf{D},\mathbf{E}]_\times)(iD_0)^l h_v \\
& + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2m)^{n+2}}\sum_{l=0}^{n-1}\sum_{l'=0}^{n-l-1}\bar{h}_v(iD_0)^{n-1-l-l'}g^2(\mathbf{E}\cdot(iD_0)^{l'}\mathbf{E} + i\boldsymbol{\sigma}\cdot(\mathbf{E}\times(iD_0)^{l'}\mathbf{E}))(iD_0)^l h_v \qquad (12)
\end{aligned}
$$

in the local rest frame with the derivatives $D_0 = \partial_t + igZA_0$ and $\mathbf{D} = \nabla - igZ\mathbf{A}_a T^a$, the color field strengths $E_i = -G_{i0}$ and $B_i = -\epsilon_{ijk}G^{jk}/2$, and the definitions of $[\mathbf{a},\mathbf{b}]_{\cdot} = \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a}$, $[\mathbf{a},\mathbf{b}]_\times = \mathbf{a}\times\mathbf{b} - \mathbf{b}\times\mathbf{a}$, $\{\mathbf{a},\mathbf{b}\}_{\cdot} = \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a}$, and $\{\mathbf{a},\mathbf{b}\}_\times = \mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{a}$, where Z is the renormalization constant. By redefining the field

$$
h_v \to \left[1 + \frac{\mathbf{D}^2 + g\boldsymbol{\sigma}\cdot\mathbf{B}}{8m^2} + \frac{(iD_0)^3 + g([\mathbf{D},\mathbf{E}]_{\cdot} + i\boldsymbol{\sigma}\cdot[\mathbf{D},\mathbf{E}]_\times)}{16m^3} + \cdots\right]h_v, \qquad (13)
$$

dropping the terms unrelated to the quark-quark-gluon vertex, and taking the process used in [8], we have, to order $1/m^5$,

$$
\begin{aligned}
\mathcal{L} = h_v^\dagger\Bigg[& iD_0 + c_2\frac{\mathbf{D}^2}{2m} + c_4\frac{\mathbf{D}^4}{8m^3} + c_6\frac{\mathbf{D}^6}{16m^5} + g\frac{c_F\boldsymbol{\sigma}\cdot\mathbf{B}}{2m} + g\frac{c_D[\mathbf{D},\mathbf{E}]_. + ic_S\boldsymbol{\sigma}\cdot[\mathbf{D},\mathbf{E}]_\times}{8m^2} \\
&+ g\frac{c_{W1}\{\mathbf{D}^2,\boldsymbol{\sigma}\cdot\mathbf{B}\} - 2c_{W2}\mathbf{D}\cdot(\boldsymbol{\sigma}\cdot\mathbf{B})\mathbf{D} + c_{p'p}((\boldsymbol{\sigma}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{D}\cdot\mathbf{B})(\boldsymbol{\sigma}\cdot\mathbf{D})) + ic_M\{\mathbf{D},\mathbf{B}\times\mathbf{D}\}_.}{8m^3} \\
&+ g\frac{c_{X1}[\mathbf{D}^2,\{\mathbf{D},\mathbf{E}\}_.] + c_{X2}\{\mathbf{D}^2,[\mathbf{D},\mathbf{E}]_.\} + c_{X3}[D_i,[D_i,[\mathbf{D},\mathbf{E}]_.]]}{m^4} \\
&+ g\frac{ic_{X5}D_i\boldsymbol{\sigma}\cdot[\mathbf{D},\mathbf{E}]_\times D_i + ic_{X6}\epsilon_{ijk}\sigma_i D_j[\mathbf{D},\mathbf{E}]_. D_k}{m^4} \\
&+ g\frac{c_{Y1}\{\mathbf{D}^4,\boldsymbol{\sigma}\cdot\mathbf{B}\} + c_{Y2}\mathbf{D}^2(\boldsymbol{\sigma}\cdot\mathbf{B})\mathbf{D}^2 + c_{Y3}\{\mathbf{D}^2,D_i(\boldsymbol{\sigma}\cdot\mathbf{B})D_i\} + c_{Y4}D_iD_j(\boldsymbol{\sigma}\cdot\mathbf{B})D_jD_i}{m^5} \\
&+ g\frac{c_{Y5}\{\mathbf{D}^2,(\boldsymbol{\sigma}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{D}\cdot\mathbf{B})(\boldsymbol{\sigma}\cdot\mathbf{D})\} + c_{Y6}D_i((\boldsymbol{\sigma}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D}) + (\mathbf{D}\cdot\mathbf{B})(\boldsymbol{\sigma}\cdot\mathbf{D}))D_i}{m^5} \\
&+ g\frac{ic_{Y7}\{\mathbf{D}^2,\mathbf{D}\cdot(\mathbf{B}\times\mathbf{D}) + (\mathbf{D}\times\mathbf{B})\cdot\mathbf{D}\} + ic_{Y8}D_i(\mathbf{D}\cdot(\mathbf{B}\times\mathbf{D}) + (\mathbf{D}\times\mathbf{B})\cdot\mathbf{D})D_i}{m^5} \\
&+ g\frac{c_{Y9}[\mathbf{D}^2,[\boldsymbol{\sigma}\cdot\mathbf{D},\mathbf{B}\cdot\mathbf{D}+\mathbf{D}\cdot\mathbf{B}]]}{m^5} + \cdots\Bigg]h_v
\end{aligned}
\tag{14}
$$

with the definition of $\{\mathbf{a},\mathbf{b}\times\mathbf{c}\}_. = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{b})\cdot\mathbf{a}$, where c_2, c_4, c_6, c_F, c_D, c_S, c_{W1}, c_{W2}, $c_{p'p}$, c_M, c_{X1}, c_{X2}, c_{X3}, c_{X5}, c_{X6}, c_{Y1}, c_{Y2}, c_{Y3}, c_{Y4}, c_{Y5}, c_{Y6}, c_{Y7}, c_{Y8}, and c_{Y9} are the coefficients to be determined. The first four terms in the Lagrangian determine the quark propagator in the gauge field A_μ^a, and the other terms control the quark-quark-gluon vertex [9,10].

We can directly take out the quark-quark-gluon vertex from the NRQCD Lagrangian density (14). By matching it with the current (8) controlled by the form factors, we extract the NRQCD coefficients to order $1/m^5$,

$$
\begin{aligned}
c_2 &= c_4 = c_6 = 1, \\
c_F &= F_1 + F_2, \\
c_D &= F_1 + 2F_2 + 8F_1', \\
c_S &= F_1 + 2F_2, \\
c_{W1} &= F_1 + F_2/2 + 4F_1' + 4F_2', \\
c_{W2} &= F_2/2 + 4F_1' + 4F_2', \\
c_{p'p} &= F_2, \\
c_M &= F_2/2 + 4F_1', \\
c_{X1} &= 5F_1/128 + F_2/32 + F_1'/4, \\
c_{X2} &= 3F_1/64 + F_2/16, \\
c_{X3} &= F_1'/8 + F_2'/4 + F_1''/2, \\
c_{X5} &= 3F_1/32 + F_2/8, \\
c_{X6} &= -3F_1/32 - F_2/8 - F_1'/4 - F_2'/2, \\
c_{Y1} &= 27F_1/256 + 23F_2/256 + 5F_1'/16 \\
&\quad + 5F_2'/16 + F_1''/4 + F_2''/4, \\
c_{Y2} &= -3F_1/128 - 11F_2/128 - F_1'/8 - 3F_2'/8 \\
&\quad + F_1''/2 + F_2''/2, \\
c_{Y3} &= -3F_2/64 - F_1'/4 - F_2'/4 - F_1'' - F_2'', \\
c_{Y4} &= F_2'/4 + F_1'' + F_2'', \\
c_{Y5} &= 3F_2/64 + F_2'/8, \\
c_{Y6} &= -F_2'/4, \\
c_{Y7} &= 3F_2/128 + F_1'/8 + F_2'/16 + F_1''/4, \\
c_{Y8} &= -F_2'/8 - F_1''/2, \\
c_{Y9} &= -3F_1/128 - F_2/32 - F_1'/16 - F_2'/8.
\end{aligned}
\tag{15}
$$

We now compare our result with the previous calculations [9,11–14]. Instead of the general transformation (4) between relativistic and nonrelativistic on-shell spinors, which is employed in the above calculation, the usual FWT transformation (2) was used in the previous calculations. To order $1/m^3$, the obtained coefficients here are consistent with the recent ones, but c_M differs by a sign of the previous one [9]. It is clear that the above calculation can also be applied to heavy fermions in nonrelativistic quantum electrodynamics (NRQED). In comparison with the NRQED calculation in the frame of the variational method [8], we obtain exactly the same coefficients up to order $1/m^4$. In particular, we have the same relation $2c_M = c_D - c_F$ among the coefficients at different orders [8,11].

Note that the previous calculations, including NRQCD and NRQED, are up to order $1/m^4$; the calculation here up to order $1/m^5$ results in the new coefficients $c_{X1}, c_{X2}, c_{X3}, c_{X5}$, c_{X6}, c_{Y1}, c_{Y2}, c_{Y3}, c_{Y4}, c_{Y5}, c_{Y6}, c_{Y7}, c_{Y8}, and c_{Y9}.

The final step to numerically determine the coefficients is to calculate the form factors $F_1(q^2/m^2)$ and $F_2(q^2/m^2)$ in QCD at some specific level [15]. For instance, computing the Feynman diagrams to the one-loop correction and taking into account the related renormalization, we have

$$F_1\left(\frac{q^2}{m^2}\right) = 1 + \frac{\alpha_s}{144\pi}\frac{q^2}{m^2}\left[\left(-51 + 154\ln\frac{m}{\mu}\right) + \frac{1}{10}\frac{q^2}{m^2}\left(131 + 888\ln\frac{m}{\mu}\right)\right],$$
$$F_2\left(\frac{q^2}{m^2}\right) = \frac{\alpha_s}{6\pi}\left\{\left(13 - 9\ln\frac{m}{\mu}\right) + \frac{q^2}{m^2}\left[\frac{1}{6}\left(13 - 54\ln\frac{m}{\mu}\right) - \frac{3}{4}\frac{q^2}{m^2}\left(1 + 6\ln\frac{m}{\mu}\right)\right]\right\} \tag{16}$$

with the redefined coupling constant $\alpha_s = g^2/(4\pi)$ and the cutoff μ in dimensional renormalization.

With the known form factors, it is straightforward to represent the coefficients with m and μ,

$$c_F = 1 + \frac{\alpha_s}{6\pi}\left(13 - 9\ln\frac{m}{\mu}\right),$$
$$c_D = 1 + \frac{\alpha_s}{18\pi}\left(27 + 100\ln\frac{m}{\mu}\right),$$
$$c_S = 1 + \frac{\alpha_s}{3\pi}\left(13 - 9\ln\frac{m}{\mu}\right),$$
$$c_{W1} = 1 + \frac{\alpha_s}{36\pi}\left(40 - 89\ln\frac{m}{\mu}\right),$$
$$c_{W2} = \frac{\alpha_s}{36\pi}\left(40 - 89\ln\frac{m}{\mu}\right),$$
$$c_{p'p} = \frac{\alpha_s}{6\pi}\left(13 - 9\ln\frac{m}{\mu}\right),$$
$$c_M = \frac{\alpha_s}{36\pi}\left(-12 + 127\ln\frac{m}{\mu}\right),$$
$$c_{X1} = \frac{5}{128} + \frac{\alpha_s}{576\pi}\left(-12 + 127\ln\frac{m}{\mu}\right),$$
$$c_{X2} = \frac{3}{64} + \frac{\alpha_s}{96\pi}\left(13 - 9\ln\frac{m}{\mu}\right),$$
$$c_{X3} = \frac{\alpha_s}{5760\pi}\left(789 + 2162\ln\frac{m}{\mu}\right),$$
$$c_{X5} = \frac{3}{32} + \frac{\alpha_s}{48\pi}\left(13 - 9\ln\frac{m}{\mu}\right),$$
$$c_{X6} = -\frac{3}{32} + \frac{\alpha_s}{576\pi}\left(-209 + 386\ln\frac{m}{\mu}\right),$$
$$c_{Y1} = \frac{27}{256} + \frac{\alpha_s}{23040\pi}\left(4143 - 7741\ln\frac{m}{\mu}\right),$$
$$c_{Y2} = -\frac{3}{128} + \frac{\alpha_s}{11520\pi}\left(-3587 + 4889\ln\frac{m}{\mu}\right),$$
$$c_{Y3} = \frac{\alpha_s}{5760\pi}\left(-203 + 2561\ln\frac{m}{\mu}\right),$$
$$c_{Y4} = \frac{\alpha_s}{360\pi}\left(8 - 231\ln\frac{m}{\mu}\right),$$
$$c_{Y5} = \frac{\alpha_s}{1152\pi}\left(169 - 297\ln\frac{m}{\mu}\right),$$
$$c_{Y6} = \frac{\alpha_s}{144\pi}\left(-13 + 54\ln\frac{m}{\mu}\right),$$
$$c_{Y7} = \frac{\alpha_s}{11520\pi}\left(859 + 3607\ln\frac{m}{\mu}\right),$$
$$c_{Y8} = \frac{\alpha_s}{720\pi}\left(-98 - 309\ln\frac{m}{\mu}\right),$$
$$c_{Y9} = -\frac{3}{128} + \frac{\alpha_s}{2304\pi}\left(-209 + 386\ln\frac{m}{\mu}\right). \tag{17}$$

In summary, we recalculated the quark-quark-gluon vertex for heavy quarks. Instead of the usual FWT transformation which is often used in previous calculations, we employed a more general relation between relativistic and nonrelativistic on-shell spinors. By matching our calculation with the standard NRQCD calculation, the coefficients in the NRQCD Lagrangian are fixed. The result to order $1/m^3$ is almost the same as the previous calculation using the FWT transformation, and the new coefficients at orders $1/m^4$ and $1/m^5$ are derived in the current calculation and may have applications in high precision heavy hadron physics.

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