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10HP Project Report

String Compactifications on $\mathrm{AdS}_4 \times \mathrm{S}^6$

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Abstract

The concept of compactifications is used in theoretical physics to perform a dimensional reduction of a theory – often needed in string theory which is formulated in more than four dimensions. This report provides a concise and pedagogical introduction to the idea of compactifications which is accessible to students. We compute the three established massive $\mathrm{AdS}_4 \times S^6$ vacua, discussed recently in e.g. [1]: Starting from the ten-dimensional type IIA supergravity action and employing the $\mathrm{AdS}_4 \times S^6$ ansatz, we perform a dimensional reduction to derive the corresponding four-dimensional effective theory. We analyse two scenarios: Initially, we only consider the field strengths F_0 and F_6; later, we include all possible field strengths, i.e. also H, F_2 and F_4. For both cases, we identify the potential and search for its minima. The comparison with [1] shows that we can reproduce the three $\mathrm{AdS}_4 \times S^6$ minima that they find: two non-supersymmetric and one supersymmetric one. For all minima, we calculate the zero mode of the Kaluza Klein tower and show that they fulfil the BF bound.

Sammanfattning

Begreppet kompaktifiering används inom teoretisk fysik för att göra dimensionella reduktioner av en teori – ofta behövs det inom strängteori som är formulerad i fler än fyra dimensioner. Denna rapport ger en kortfattad och pedagogisk introduktion till idén om kompaktifieringar som är tillgänglig för studenter. Vi beräknar de tre välkända massiva $\mathrm{AdS}_4 \times S^6$ vakuumen: Med utgångspunkt från den tiodimensionella supergravitationsteorin IIA och med $\mathrm{AdS}_4 \times S^6$ ansatzen, utför vi en dimensionell reduktion för att härleda motsvarande fyrdimensionella effektiva teori. Vi analyserar två scenarier: Inledningsvis beaktar vi endast fältstyrkorna F_0 och F_6; senare inkluderar vi alla möjliga fältstyrkor, dvs. även H, F_2 och F_4. I båda fallen identifierar vi potentialen och söker efter dess minima. Jämförelsen med [1] visar att vi kan återskapa de tre $\mathrm{AdS}_4 \times S^6$ minima som de hittar: två icke-supersymmetriska och ett supersymmetriskt. För alla minima beräknar vi nolläget i Kaluza Klein-tornet och visar att de uppfyller BF-gränsen.

Contents

Conventions and Notations

Throughout this report, we will use different indices on different spaces. We will consider spaces of the form $Z_{10} = X_4 \times \mathcal{M}_6$ with a maximally symmetric four-dimensional Lorentzian manifold

X_4 and a compact six-dimensional internal manifold $\mathcal{M}_6$. Capital Latin indices $M, N, \ldots$ run on Z_{10} from 0 to 9, Greek indices $\mu, \nu, \ldots$ run on X_4 from 0 to 3, and lowercase Latin indices run on $\mathcal{M}_6$ from 1 to 6. Further, we denote the coordinates on Z_{10} as z_M, the ones on X_4 as x_μ and the ones on $\mathcal{M}_6$ as y_m:

$$z_M = \begin{cases} x_\mu & \text{for } 0 \leq M \leq 3 \\ y_m & \text{for } 4 \leq M \leq 9. \end{cases}$$

CHAPTER 1

Introduction

In 1915, Einstein's theory of *general relativity* (GR) revolutionised the understanding of gravity – transforming it from a force as described by Newton to a property of the four-dimensional spacetime, the curvature. Later, when *quantum mechanics* (QM) was established as the new theory to describe the microscopic scales of our world, the need arose to describe all known interactions according to the principles of QM. This was indeed possible for almost all interactions – except for gravity: GR's description of gravity breaks down at lengths approaching the Planck scale. The goal of finding a complete quantum description of gravity remains one of the open questions in modern theoretical physics and is called *quantum gravity* (QG). One promising candidate for a QG theory is *string theory*, which postulates that the fundamental particles of nature are not pointlike but excitations of one-dimensional strings. String theory went through several revolutions in the last decades, one of which was based on the introduction of *supersymmetry* – a symmetry between the two types of particles: fermions with half-integer-valued spin and bosons with integer-valued spin. Today, the best understood string theories are supersymmetric. Their low-energy limits describe supergravity – theories that combine GR and supersymmetry. One of these low-energy limits – type IIA supergravity – is the centre of this work.

An essential property of string theory is that it is formulated in more than four dimensions – in the best understood supersymmetric form, it has ten dimensions. Hence, when describing our four-dimensional spacetime in the setting of string theory, one is confronted with a dimensionality problem. One way to approach this is called *compactification*. The idea is – after specifying the shape of the space (in fact, the compact so-called *internal manifold*) on which the additional six dimensions live on – to integrate them out of the ten-dimensional action. One then attempts to make this six-dimensional manifold sufficiently small – so small that we can assume its coordinates are unobservable with our current measurement techniques. Making the internal manifold that small is not always possible – when it is, it is referred to as *scale separation*, a necessary condition for a theory to describe our reality, since we only observe four dimensions.

A common illustration for the concept of scale separation is a straw (with an infinitely thin wall), a two-dimensional object that looks like a one-dimensional object from far enough away:

Figure 1.1: A straw is an example for an object that seems to have less dimensions from far away.

In this example, the two coordinates live on very different length scales – the straw's length is much greater than its radius.

In this work, we choose an ansatz for a six-dimensional manifold, since we want to use a ten-dimensional theory to describe a four-dimensional spacetime. Similar to the straw, where the internal manifold is a circle, we choose the six-sphere S^6 – the generalisation of the one-dimensional circle or the two-dimensional sphere to six dimensions and the most simple and symmetric choice imaginable. In fact, the possibilities for manifolds used in compactifications are much more numerous and complex, but for this work the conceptual idea is central. We go back to our straw to hopefully make the concept tangible. Mathematically, a (now infinite) straw is the product $\mathbb{R} \times S^1$, a line times a circle. With increasing distance from the straw, the dimension living on the circle is not visible anymore; the object seems like a line. In the action, one integrates out the coordinate on the circle to go from the two- to the one-dimensional description. Here, we already note an important property of the internal manifold: It needs to be *compact*, i.e. all its points have to lie within a certain distance (and it has to include all limiting values of its points). This makes sure that we do not get infinite results if we integrate out the internal manifold's coordinates. The coordinate on the circle S^1 lives on the compact interval $[0, 2\pi]$, for example.

Similarly to the infinite straw which mathematically is $\mathbb{R} \times S^1$, for our ten-dimensional theory we choose the ansatz $\mathrm{AdS}_4 \times S^6$ with the four-dimensional anti-de Sitter spacetime AdS_4 and the six-sphere S^6. The analogy to the straw example is illustrated in this table:

	External manifold	Internal manifold
Straw	$\mathbb{R}$	S^1
This work	4d spacetime	S^6

After compactifying the six dimensions on the S^6, one obtains a four-dimensional theory. We then identify the theory's potential and search for its minima – the lowest energy states – since these correspond to the vacuum solutions, the so-called *vacua* that describe empty spacetimes. The vacua can then be examined for stability, the explained scale separation and supersymmetry, for instance.

This work is structured as follows: Chapter 2 will introduce the conceptual framework of general relativity, string theory (in particular, type IIA supergravity) and compactifications. In chapter 3 we will perform the dimensional reduction of the ten-dimensional action of type IIA supergravity. We will then present the found vacua of the effective four-dimensional theory and their properties and eventually interpret our results in chapter 4.

CHAPTER 2

Gravity and Strings

This chapter aims to introduce the key concepts that are essential for this work. We begin with the foundations of general relativity in section 2.1, followed by an introduction to the relevant theory, type IIA supergravity, in section 2.2. We then present the subject of compactifications in section 2.3 and examine vacua and their stability in section 2.4.

2.1 A Brief Review of General Relativity

Even though general relativity (GR) is not the focus of this work, one particular type of solution of the Einstein equations, the Anti-de Sitter (AdS) space, will occur frequently. Hence, we will spend a little time to present a few basic ideas of GR in order to introduce the AdS space. We will loosely follow David Tong's lecture notes [2].

Curvature of Spacetime and Geodesics

In GR, the motion of relativistic particles is described by the *geodesic equation*

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \tag{2.1}$$

with a scalar parameter τ, e.g. the proper time, and the objects $\Gamma^\mu_{\nu\rho}$, called *Christoffel symbols* and defined as

$$\Gamma^\mu_{\nu\rho} := \frac{1}{2}g^{\mu\gamma}\left(\partial_\rho g_{\gamma\nu} + \partial_\nu g_{\rho\gamma} - \partial_\gamma g_{\nu\rho}\right) \tag{2.2}$$

with the metric tensor g. They obviously satisfy

$$\Gamma^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\rho\nu}. \tag{2.3}$$

A solution of the geodesic equation is called a *geodesic*, the generalisation of a straight line in curved space: Freely falling or moving particles always follow geodesics.

Geometric Invariants

We define the *covariant derivative*, a generalisation of the partial derivative that transforms covariantly under Lorentz transformations:

$$\nabla_{\mu}v^{\nu} \equiv (\nabla_{\mu}v)^{\nu} := \partial_{\mu}v^{\nu} + \Gamma^{\nu}_{\mu\rho}v^{\rho}; \qquad \nabla_{\mu}v_{\nu} \equiv (\nabla_{\mu}v)_{\nu} := \partial_{\mu}v_{\nu} - \Gamma^{\rho}_{\mu\nu}v_{\rho}, \tag{2.4}$$

where we introduced the rather sloppy yet widely used bracketless notation. Note that $\nabla_{\mu}v^{\nu}$ really is the ν-th component of $\nabla_{\mu}v$, and not a differentiation of v's ν-th component. In contrast to partial derivatives, covariant derivatives do not commute anymore. In fact, the *Riemann tensor* encodes the non-commutativity of the covariant derivative. Its components are defined as

$$R^{\sigma}_{\rho\mu\nu} := \partial_{\mu}\Gamma^{\sigma}_{\nu\rho} - \partial_{\nu}\Gamma^{\sigma}_{\mu\rho} + \Gamma^{\lambda}_{\nu\rho}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\lambda}_{\mu\rho}\Gamma^{\sigma}_{\nu\lambda} \tag{2.5}$$

which by definition are antisymmetric in their last two indices,

$$R^{\sigma}_{\rho\mu\nu} := -R^{\sigma}_{\rho\nu\mu}. \tag{2.6}$$

Contracting the first with the third index yields the *Ricci tensor*

$$R_{\mu\nu} := R^{\rho}_{\mu\rho\nu} \tag{2.7}$$

whose trace is the *Ricci scalar*

$$R := g^{\mu\nu}R_{\mu\nu}. \tag{2.8}$$

Finally, we define the *Einstein tensor* as

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{2.9}$$

The Vacuum Einstein Equations

The goal is to understand the dynamics of the gravitational field which is described by the Einstein field equations. In the absence of matter, varying the *Einstein-Hilbert action*, defined as

$$S = \int d^4x\sqrt{-|g_{\mu\nu}|}R \tag{2.10}$$

with the Ricci scalar R, yields the *vacuum Einstein field equations*:

$$G_{\mu\nu} = 0. \tag{2.11}$$

They actually simplify to

$$R_{\mu\nu} = 0 \tag{2.12}$$

which means that the metric is Ricci flat.

We can then extend this action by adding a potential term,

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-|g_{\mu\nu}|}(R - 2\Lambda), \tag{2.13}$$

where the prefactor has dimensional reasons. Λ is called *cosmological constant* and has the interpretation of a potential energy. The corresponding equations of motion are given by

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{2.14}$$

and simplify to

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{2.15}$$

These are the vacuum Einstein equations in the presence of the cosmological constant. Depending on if Λ is positive, negative or zero, the solutions will take different forms. Solutions with $\Lambda = 0$ are called *Minkowski*, solutions with $\Lambda > 0$ *de-Sitter*, and the ones with $\Lambda < 0$ *Anti-de-Sitter* (AdS).

The metric of AdS space is takes the form

$$ds^2 = -\left(1 + \frac{r^2}{R^2}\right) dt^2 + \left(1 + \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.16}$$

where $R^2 = -\frac{3}{\Lambda}$. Another common notation for the AdS metric is, after introducing $r = R\sinh\rho$,

$$ds^2 = -\cosh^2\rho dt^2 + R^2 d\rho^2 + R^2 \sinh^2\rho(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.17}$$

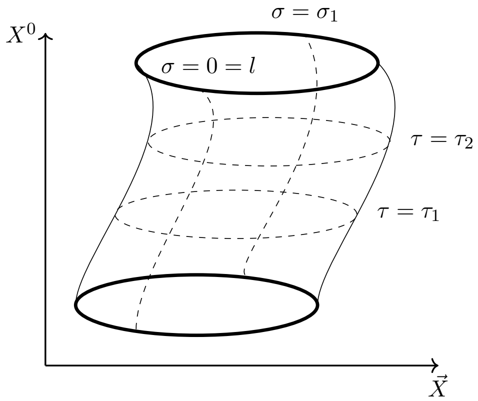
AdS spacetime pushes massive particles towards the origin $r = 0$ while it acts as a homogeneous space for massless particles. Moreover, the isometry group of AdS is the ten-dimensional group $\mathrm{SO}(2,3)$.

2.2 Type IIA Supergravity

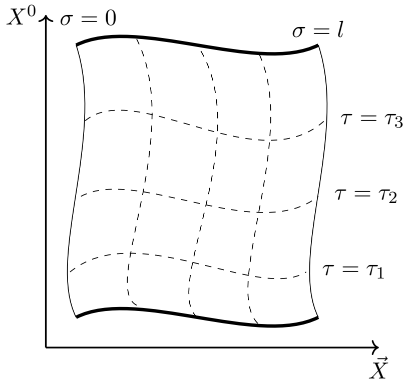
We present the theory that forms the basis of this work – type IIA supergravity. It is the low-energy limit of type IIA string theory, one of the five consistent supersymmetric string theories. This section will first briefly introduce the idea of string theory's framework before discussing the ten-dimensional action of type IIA supergravity.

String Theory

String theory postulates a completely new understanding of particles: The fundamental objects are not zero-dimensional points anymore but vibration modes of one-dimensional strings. One distinguishes between *open* and *closed* strings, where the former are topologically equivalent to a line interval and the latter to a circle. When describing the propagation of a string in spacetime, the one-dimensional worldline that describes the propagation of zero-dimensional particles is replaced by the two-dimensional worldsheet Σ, see figure 2.1. Σ is parameterised by two parameters – the time and the spatial coordinates τ and σ, where it is $0 \leq \sigma \leq l$ with the string length l.



(a) closed string



(b) open string

Figure 2.1: Worldsheets for (a) closed and (b) open strings

String theory has only one free parameter, α', that defines the string length, the string mass scale and the string tension:

$$l_s = \sqrt{\alpha'} \qquad \text{string length} \tag{2.18}$$

$$M_s = \frac{1}{\sqrt{\alpha'}} \qquad \text{string mass scale} \tag{2.19}$$

$$T = \frac{1}{2\pi\alpha'} \qquad \text{string tension.} \tag{2.20}$$

Supersymmetric String Theory

After the original version of string theory – *bosonic string theory*, developed in the late 1960s and formulated in 26 dimensions – failed in several ways (for instance in predicting fermions), the discovery of *supersymmetry* in the context of string theory in the 1970s led to *superstring theory*. Supersymmetry is a symmetry based on the assumption that for each boson there exists a fermion and vice versa, and that these pairs of particles share some properties such as their masses. As far as we know, our world does not obey supersymmetry. However, because of a phenomenon called *symmetry breaking* this does not make the study of supersymmetry obsolete. In fact, it is possible that supersymmetry is a symmetry of our world but a broken one and therefore not observable at low energies.

There are five consistent supersymmetric string theories, all formulated in ten dimensions and assumed to be different limits of one single theory called *M-theory*. Their low-energy limits describe supergravity, theories that combine GR and supersymmetry. As mentioned before, we will study type IIA supergravity in this work.

The Action of Type IIA Supergravity

The ten-dimensional action of type IIA supergravity in Einstein frame is given as follows:

$$\begin{aligned} S_{10} = \frac{1}{2\kappa_{10}^2} \int d^{10}z \sqrt{-G'_{10}} \left(R'_{10} - \frac{1}{2}(\partial\phi)'^2 - \frac{1}{2\cdot 3!} e^{-\phi}|H'|^2 - \frac{1}{2} \sum_{n \text{ even}} e^{\frac{5-n}{2}\phi}|F'_n|^2 \right) \\ + \frac{1}{2\kappa_{10}^2} \int B \wedge F_4 \wedge F_4. \end{aligned} \tag{2.21}$$

Note that the prime indicates that quantities are contracted with the primed metric, since we will also have an unprimed metric in the following. Typically, the action is given either in Einstein frame or in String frame where the two are related via a metric rescaling, $G'_{MN} \equiv G^E_{MN} = e^{-\frac{\phi}{2}} G^S_{MN}$. The Einstein frame is named as such because its first term takes the form of the Einstein-Hilbert action from GR (compare (2.10)). The Einstein-Hilbert term is followed by the kinetic term of the dilaton ϕ, a scalar field appearing in theories with extra dimensions. The *Neveu-Schwarz-Neveu-Schwarz* (NSNS) field strength H and the *Ramond-Ramond* (RR) field strengths F_n's are exterior derivatives of gauge fields, similar to electrodynamics. Note that we sum over even n's – in type IIB supergravity one sums over odd n's. Since the F_n's are differential forms and we work in ten dimensions, there are no F_n's with $n > 10$, and further F_6, F_8 and F_{10} are related to F_4, F_2 and F_0, respectively, via Hodge duality,

$$F_n = (-1)^{\frac{(n-1)(n-2)}{2}} * F_{10-n}. \tag{2.22}$$

We work with the four RR field strengths: F_0, F_2, F_4 and F_6. The NSNS and the RR field strengths have to satisfy the Bianchi-identities

$$\mathrm{d}F_n = H \wedge F_{n-3} + \delta_{\mathrm{D}(8-n)/\mathrm{O}(8-n)} \tag{2.23}$$

where the second term will not be present in the setup we work with. Fields H or F_n's satisfying the Bianchi-identites are often called fluxes. Finally, the last term is the Chern-Simons term which will also not be present in our setup.

We will later fix specific choices for the field strengths in order to transform them.

2.3 Compactifications

The term *compactification* refers to a dimensional reduction of a theory. Regarding the terminology, we refer to the manifold of the dimensions we want to retain (in our case, the four-dimensional spacetime) as the *external* manifold. Conversely, the manifold containing the dimensions we aim to compactify (in our case, the six-sphere) is called the *internal* manifold. In the following, we will give a quick historical review and introduce the idea behind compactifications.

A Historical Perspective: Kaluza-Klein Reduction

The attempt to unify descriptions of different fundamental forces by adding spacetime dimensions is not a new one and did not arise in string theory for the first time: Independently from each other, Nordström in 1914 and Kaluza in 1921 developed a five-dimensional theory that yielded the field equations of both gravity and electromagnetism. The naturally arising question why we do not observe a fifth dimension was answered by both of them with the assumption that our reality is located on a four-dimensional hypersurface in a five-dimensional universe [3]. Klein further developed Kaluza's ansatz in 1926 by choosing the topology of the fifth dimension to be a tiny circle. The fields, depending on the circle periodically, could then be Fourier-expanded, with, if choosing the circle to be small enough, Fourier modes with energies so high that they are effectively unobservable. Based on their work, the general procedure of expanding the fields in terms of the eigenfunctions of the compact space is called Kaluza-Klein reduction. It leads to an infinite tower of modes appearing in the lower dimensional theory, a so-called *Kaluza-Klein tower.*

This brings us to a very important constraint for a compactification to be realistic: Scale separation.

Scale separation

The common explanation for why we do not observe specific dimensions in higher dimensional theories is that these dimensions are too small to be detected. This concept, known as *scale separation*, suggests that the length scales of the external and the internal manifold differ by several orders of magnitudes. However, this condition is not always achievable. After choosing the internal manifold, performing the dimensional reduction and obtaining an effective lower dimensional theory, it can be determined if the internal manifold's length scale can indeed be made sufficiently small.

Moduli Stabilisation and Fluxes

When considering theories with extra dimensions, one main issue is the appearance of *modulus fields*, unobserved massless scalar fields associated with the geometry of the internal manifold [4]. They lack a potential, meaning their vacuum expectation values (vevs), known as moduli, are not restricted to any values. Among other problems, this undermines the predictive power of the theory. To get rid of the ambiguity, the moduli need to be stabilised by some mechanism. A common approach is to introduce *fluxes*, background values of field strengths, to stabilise the moduli by giving them fixed vevs – which means that they become massive. In our setup, the moduli are the dilaton ϕ, a volume field φ and the two axion fields ζ and b, while the fluxes are the field strengths H and F_n.

The Problem with de-Sitter Compactifications

Since we observe a positive cosmological constant we would like to focus on compactifications where the external manifold is a de Sitter space. Unfortunately, these solutions are hard to obtain. As explained in [5], theories with an Einstein-Hilbert term satisfying the *strong energy condition* – the often required condition that gravity is always attractive in GR [6] – have no de Sitter compactifications and only marginally allow Minkowski solutions. Hence, one usually deals with external Anti-de Sitter spaces.

2.4 Stability of Vacua

In order to find vacuum solutions, so-called *vacua*, of a theory, one searches for minima of the potential. One distinguishes between local potential minima, corresponding to so-called *false* *vacua*, and global minima, corresponding to *true vacua*. False vacua are metastable and can decay into stable vacua at a lower energy in various ways, events that are called *false vacuum decay*. We call the potential minima the *critical points*: If the potential V depends on fields $\phi_1, \dots, \phi_n$, the critical points $(\phi_1^*, \dots, \phi_n^*)$ fulfil

$$\partial_{\phi_1} V(\phi_1, \dots, \phi_n)|_{(\phi_1^*, \dots, \phi_n^*)} = 0; \dots ; \partial_{\phi_n} V(\phi_1, \dots, \phi_n)|_{(\phi_1^*, \dots, \phi_n^*)} = 0 \tag{2.24}$$

and describe the vacuum energies of the solutions. Note that at the minimum, $V(\phi_1^*, \dots, \phi_n^*)$ acts as a cosmological constant Λ in the d-dimensional theory. In four dimensions and for $\Lambda < 0$, we can identify [7]

$$V(\phi_1^*, \dots, \phi_n^*) = M_{\mathrm{Pl}}^2 \cdot \Lambda. \tag{2.25}$$

The cosmological constant is further related to the length scale of the external AdS space as

$$\Lambda = -\frac{3}{L_{\mathrm{AdS}}^2}. \tag{2.26}$$

As mentioned, there are several types of false vacuum decays where metastable vacua decay into other ones at a lower energy. In the following, we will discuss some of these decays.

Bubble Nucleation

In the process of *bubble nucleation*, a bubble of true vacuum materialises inside a false vacuum and expands with a speed that asymptotically reaches c while converting the false to true vacuum.

A bubble of true vacuum inside a false vacuum has a positive surface tension and a negative volume term, since the true vacuum has a lower energy compared to the false one. There is a critical radius R_c at which the bubble has total energy zero – which is required due to energy conservation. However, through a quantum tunnelling event a bubble can overcome R_c which then leads to the bubble's expansion. In 1977, this phenomenon was introduced by Coleman [8]; and in 1980, Coleman and Luccia expanded the idea by taking the effects of gravitation into account [9]. They found that gravitation can both favour and unfavour a decay, depending on the energies of the initial and final vacuum. They restricted themselves to the *thin-wall approximation*, i.e. the assumption that the energy density difference between the true and the false vacuum is small. In the absence of gravity, a bubble with total energy zero can always be achieved: The smaller the energy difference between the false and the true vacuum, the larger the radius has to be. This is no longer true when including the effects of gravity. The negative energy density inside the bubble diminishes the volume/surface ratio with the result that for sufficiently small energy difference ϵ there is no bubble with a big enough volume/surface ratio [9]. Supersymmetric AdS-vacua are protected against this type of tunnel effect, while for non-supersymmetric vacua several examples have been found where this vacuum decay indeed happens [5].

The Bubble of Nothing

The *bubble of nothing* is another vacuum decay which is similar to the already discussed bubble decay. It also describes the nucleation of a bubble that then expands rapidly – yet for this effect it is literally nothing inside the bubble, where nothing means the absence of spacetime [5]. For AdS, *nothing* should be thought of as the limit of AdS space in which the curvature length approaches zero [10].

Breitenlohner Freedman Bound

The *Breitenlohner-Freedman* (BF) bound [11] is a bound for perturbative stability and describes a lower bar for the mass below which the AdS space becomes unstable.

In flat space, fields with negative squared masses – so-called *tachyons* – signal an instability: The solution of the Klein-Gordon equation $\partial_\mu \partial^\mu \phi = m^2 \phi$ is a plane wave $e^{i(\omega t + k_i x^i)}$. ω is a function of m, and for $m^2 < 0$, then ω becomes imaginary which then leads to the solution becoming an exponentially growing function. However, in AdS, stable solutions with negative squared masses are actually possible: If the squared mass is negative but sufficiently small, the Compton wavelength m^{-1} can become so large that it eliminates the instability [5]. We find that the frequency ω is real and thus the solution stable if and only if the mass satisfies the BF bound,

$$m^2 L_{AdS}^2 \geq -\frac{(d-1)^2}{4}. \tag{2.27}$$

In our convention, the masses are given by the eigenvalues of the Hessian $H_{ij} = \partial_i \partial_j V|_{(\phi_1^*,\dots,\phi_n^*)}$ when they are canonically normalised.

While supersymmetric vacua always have masses above the BF bound and therefore avoid this instability, non-supersymmetric vacua usually have masses below the bound [5].

CHAPTER 3

Compactification on $\mathbf{S^6}$

We present our results in this chapter. Starting with an ansatz for the ten-dimensional metric and the introduction of the necessary Weyl rescalings, we perform the general dimensional reduction from the ten- to the four-dimensional action. We then further investigate two different setups: In a first one, we only include the two field strengths F_0 and F_6, and in a second, general one we also include F_2, F_4 and H. For both setups we identify the potential of the theory, search for its minima and investigate their properties.

3.1 General Dimensional Reduction

As introduced in (2.21), the ten-dimensional action of type IIA string theory in Einstein frame is given by

$$\begin{aligned} S_{10} = \frac{1}{2\kappa_{10}^2} \int d^{10}z \sqrt{-G'_{10}} \bigg(R'_{10} - \frac{1}{2}(\partial\phi)'^2 - \frac{1}{2\cdot 3!} e^{-\phi} |H'|^2 - \frac{1}{2} \sum_{n \text{ even}} e^{\frac{5-n}{2}\phi} |F'_n|^2 \bigg) \\ + \frac{1}{2\kappa_{10}^2} \int B \wedge F_4 \wedge F_4. \end{aligned} \tag{3.1}$$

For the ten-dimensional spacetime we choose an ansatz of the form $X_4 \times \mathcal{M}_6$ where X_4 is a four-dimensional maximally symmetric Lorentzian manifold (either Minkowski, de Sitter or Anti-de Sitter space) and $\mathcal{M}_6$ is a six-dimensional compact manifold. We will find that for the compactifications we consider in this work, X_4 is actually an Anti-de-Sitter space, $X_4 = \mathrm{AdS}_4$, but we want to underline that it is not an assumption we make. We set

$$ds_{10}^2 = G'_{MN}dz^M dz^N = e^{2\alpha\varphi(x)}g_{\mu\nu}dx^\mu dx^\nu + e^{2\beta\varphi(x)}g_{mn}dy^m dy^n \tag{3.2}$$

where μ, ν run from 0 to 3 and m, n from 1 to 6, and where x are the coordinates on X_4. The factor $e^{2\beta\varphi(x)}$ describes the dependence of $\mathcal{M}_6$ on the X_4-coordinates – in particular, it controls the volume of $\mathcal{M}_6$ – and the factor $e^{2\alpha\varphi(x)}$ is chosen for convenience. The parameters α and β are initially arbitrary, yet we will fix them later to attain a four-dimensional effective action with a standard Einstein-Hilbert term. From now on we write $\varphi = \varphi(x)$. The aim is to express all terms in the action (3.1) in terms of the metric $g_{\mu\nu}$. In particular, in order to integrate out the six y-coordinates, we must decompose the Ricci scalar R'_{10} into $R'_{10} = R'_4 + R'_6$. This can only be achieved if the two spaces X_4 and M_6 are independent of each other, requiring Weyl rescalings for that purpose.

Weyl Rescalings

We will perform two Weyl rescalings of the metric in order to express all terms in the action (3.1) in terms of the $g_{\mu\nu}$ and the g_{mn} metric. We already defined

$$G'_{MN} = \left(\begin{array}{c|c} e^{2\alpha\varphi}g_{\mu\nu} & 0 \\ \hline 0 & e^{2\beta\varphi}g_{mn} \end{array}\right), \tag{3.3}$$

and now set

$$\tilde{G}_{MN} := e^{-2\beta\varphi}G'_{MN} = \left(\begin{array}{c|c} e^{2(\alpha-\beta)\varphi}g_{\mu\nu} & 0 \\ \hline 0 & g_{mn} \end{array}\right). \tag{3.4}$$

This is the first Weyl rescaling in which we successfully removed the x-dependence from the g_{mn}-metric. We further define a second Weyl rescaling, now only for the four-dimensional metric:

$$\hat{g}_{\mu\nu} := e^{2(\alpha-\beta)\varphi}g_{\mu\nu}. \tag{3.5}$$

With this, we can write

$$\tilde{G}_{MN} = \left(\begin{array}{c|c} \hat{g}_{\mu\nu} & 0 \\ \hline 0 & g_{mn} \end{array}\right). \tag{3.6}$$

Transformation of the Metric Determinant

With definition (3.4) from above, we can rewrite the metric determinant as

$$\begin{aligned} \sqrt{-G'_{10}} &= \sqrt{-\det(e^{2\beta\varphi}\tilde{G}_{10})} = e^{10\beta\varphi}\sqrt{-\tilde{G}_{10}} = e^{10\beta\varphi}\sqrt{-\det(e^{2(\alpha-\beta)\varphi}g_4)}\sqrt{g_6} \\ &= e^{10\beta\varphi}e^{4(\alpha-\beta)\varphi}\sqrt{-g_4}\sqrt{g_6} = e^{(4\alpha+6\beta)\varphi}\sqrt{-g_4}\sqrt{g_6} \end{aligned} \tag{3.7}$$

where we used that $\det(aA) = a^n \det(A)$ and that we can write the determinant of $\tilde{G}$ as the product of the determinants of the single blocks as it is in block-diagonal form.

Transformation of the Ricci Scalar

In appendix A, we calculate how the Ricci scalar transforms under Weyl rescalings. Under the metric rescaling (3.4), $\tilde{G}_{MN} = e^{-2\beta\varphi} G'_{MN}$, R_{10} transforms as

$$R'_{10} = e^{-2\beta\varphi}(\tilde{R}_{10} - 18\beta\tilde{\nabla}^2\varphi - 72\beta^2(\tilde{\partial}\varphi)^2). \tag{3.8}$$

Note that the second and third term are effectively only contracted with the four-dimensional metric since φ is a function of the x-coordinates on X_4 and thus, derivatives of φ w.r.t. the y-coordinates on M_6 vanish. This means that we can write $\tilde{\nabla}^2\varphi = \hat{\nabla}^2\varphi$ and $(\tilde{\partial}\varphi)^2 = (\hat{\partial}\varphi)^2$. Furthermore, since the M_6 part lost its x-dependence in $\tilde{G}_{MN}$, we can now split the Ricci scalar, i.e.

$$\tilde{G}_{MN} = \left(\begin{array}{c|c} \hat{g}_{\mu\nu} & 0 \\ \hline 0 & g_{mn} \end{array}\right) \quad \Longrightarrow \quad \tilde{R}_{10} = \hat{R}_4 + R_6. \tag{3.9}$$

This then yields

$$R'_{10} = e^{-2\beta\varphi}(\hat{R}_4 + R_6 - 18\beta\hat{\nabla}^2\varphi - 72\beta^2(\hat{\partial}\varphi)^2). \tag{3.10}$$

We can rewrite $\hat{R}_4$ using the found formula for transformed Ricci scalars (now for $n = 4$) and the transformation $g_{\mu\nu} = e^{-2(\alpha-\beta)\varphi}\hat{g}_{\mu\nu}$:

$$\hat{R}_4 = e^{-2(\alpha-\beta)\varphi}(R_4 - 6(\alpha-\beta)\nabla^2\varphi - 6(\alpha-\beta)^2(\partial\varphi)^2). \tag{3.11}$$

The expressions $\hat{\nabla}^2\varphi$ and $(\hat{\partial}\varphi)^2$ in (3.10) still need to be expressed in terms of the metric $g_{\mu\nu}$. It is

$$(\hat{\partial}\varphi)^2 = \hat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = e^{-2(\alpha-\beta)\varphi}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = e^{-2(\alpha-\beta)\varphi}(\partial\varphi)^2 \tag{3.12}$$

and

$$\begin{aligned} \hat{\nabla}^2\varphi &= \hat{\nabla}_\mu\hat{\nabla}^\mu\varphi = \hat{g}^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu\varphi = \hat{g}^{\mu\nu}\hat{\nabla}_\mu\partial_\nu\varphi = \hat{g}^{\mu\nu}(\partial_\mu\partial_\nu\varphi - \hat{\Gamma}^\rho_{\mu\nu}\partial_\rho\varphi) \\ &= e^{-2(\alpha-\beta)\varphi}g^{\mu\nu}(\partial_\mu\partial_\nu\varphi - \hat{\Gamma}^\rho_{\mu\nu}\partial_\rho\varphi) = e^{-2(\alpha-\beta)\varphi}g^{\mu\nu}(\partial_\mu\partial_\nu\varphi - (\Gamma^\rho_{\mu\nu} + a^\rho_{\mu\nu})\partial_\rho\varphi) \\ &= e^{-2(\alpha-\beta)\varphi}\underbrace{g^{\mu\nu}(\partial_\mu\partial_\nu\varphi - \Gamma^\rho_{\mu\nu}\partial_\rho\varphi)}_{=\nabla^2\varphi} - e^{-2(\alpha-\beta)\varphi}g^{\mu\nu}a^\rho_{\mu\nu}\partial_\rho\varphi \end{aligned} \tag{3.13}$$

where we used the expression for transformed Christoffel symbols we derive in appendix A. Hence, using

$$a^\rho_{\mu\nu} = (\alpha-\beta)(\partial_\nu\varphi\delta^\rho_\mu + \partial_\mu\varphi\delta^\rho_\nu - \partial_\gamma\varphi g^{\rho\gamma}g_{\mu\nu}) \tag{3.14}$$

we can write

$$\hat{\nabla}^2\varphi = e^{-2(\alpha-\beta)\varphi}\nabla^2\varphi - e^{-2(\alpha-\beta)\varphi}(\alpha-\beta)\, \underbrace{g^{\mu\nu}\,\underbrace{(\partial_\nu\varphi\delta^\rho_\mu + \partial_\mu\varphi\delta^\rho_\nu - \partial_\gamma\varphi g^{\rho\gamma}g_{\mu\nu})\partial_\rho\varphi}_{=\partial_\nu\varphi\partial_\mu\varphi+\partial_\mu\varphi\partial_\nu\varphi-\partial_\gamma\varphi\partial^\gamma\varphi g_{\mu\nu}}}_{=\partial^\mu\varphi\partial_\mu\varphi+\partial_\mu\varphi\partial^\mu\varphi-4\partial_\gamma\varphi\partial^\gamma\varphi=-2(\partial\varphi)^2}$$
$$= e^{-2(\alpha-\beta)\varphi}\nabla^2\varphi + 2e^{-2(\alpha-\beta)\varphi}(\alpha-\beta)(\partial\varphi)^2. \tag{3.15}$$

Now, we finally have all the terms to rewrite the ten-dimensional Ricci scalar:

$$\begin{aligned} R'_{10} &= e^{-2\beta\varphi}\Big(e^{-2(\alpha-\beta)\varphi}(R_4 - 6(\alpha-\beta)\nabla^2\varphi - 6(\alpha-\beta)^2(\partial\varphi)^2) + R_6 \\ &\quad - 18\beta(e^{-2(\alpha-\beta)\varphi}\nabla^2\varphi + 2e^{-2(\alpha-\beta)\varphi}(\alpha-\beta)(\partial\varphi)^2) - 72\beta^2 e^{-2(\alpha-\beta)\varphi}(\partial\varphi)^2\Big) \\ &= e^{-2\beta\varphi}R_6 + e^{-2\alpha\varphi}\Big(R_4 - (6(\alpha-\beta)+18\beta)\nabla^2\varphi - (6(\alpha-\beta)^2 \\ &\quad + 36\beta(\alpha-\beta) + 72\beta^2)(\partial\varphi)^2\Big). \end{aligned} \tag{3.16}$$

Fixing the Relation between α and β

To fix the relation between the two parameters α and β we consider the action

$$S_R = \frac{1}{2\kappa_{10}^2}\int d^{10}z\sqrt{-G'_{10}}R'_{10}. \tag{3.17}$$

In the previous subsections, we saw how both $\sqrt{-G'_{10}}$ and R'_{10} transform under the discussed Weyl rescalings. Plugging in the found results yields

$$\begin{aligned} S_R = \frac{1}{2\kappa_{10}^2}\int d^4x\ d^6y\ e^{(4\alpha+6\beta)\varphi}\sqrt{-g_4}\sqrt{g_6}\Big[e^{-2\beta\varphi}R_6 + e^{-2\alpha\varphi}\Big(R_4 - (6(\alpha-\beta)+18\beta)\nabla^2\varphi \\ - (6(\alpha-\beta)^2 + 36\beta(\alpha-\beta) + 72\beta^2)(\partial\varphi)^2\Big)\Big]. \end{aligned} \tag{3.18}$$

Note that we can now clearly separate the terms depending on x and the ones depending on y. In particular, as we discussed earlier, for the last two terms we only sum over the x-coordinates. Therefore, we can split the integral as follows:

$$\begin{aligned} S_R &= \frac{1}{2\kappa_{10}^2}\int d^4x\ d^6y\ e^{(4\alpha+6\beta)\varphi}\sqrt{-g_4}\sqrt{g_6}e^{-2\alpha\varphi}\Big(R_4 - (6(\alpha-\beta)+18\beta)\nabla^2\varphi - (6(\alpha-\beta)^2 \\ &\quad + 36\beta(\alpha-\beta) + 72\beta^2)(\partial\varphi)^2\Big) + \frac{1}{2\kappa_{10}^2}\int d^4x\ d^6y\ e^{(4\alpha+6\beta)\varphi}\sqrt{-g_4}\sqrt{g_6}e^{-2\beta\varphi}R_6 \\ &= \Big(\frac{1}{2\kappa_{10}^2}\int d^4x\ e^{(2\alpha+6\beta)\varphi}\sqrt{-g_4}\Big(R_4 - (6(\alpha-\beta)+18\beta)\nabla^2\varphi - (6(\alpha-\beta)^2 + 36\beta(\alpha-\beta) \\ &\quad + 72\beta^2)(\partial\varphi)^2\Big)\Big)\Big(\int d^6y\sqrt{g_6}\Big) + \Big(\frac{1}{2\kappa_{10}^2}\int d^4x\ e^{4(\alpha+\beta)\varphi}\sqrt{-g_4}\Big)\Big(\int d^6y\sqrt{g_6}R_6\Big). \end{aligned} \tag{3.19}$$

Since all the information about the volume of $\mathcal{M}_6$ is stored in the term $e^{2\beta\varphi}$, we need to impose that the integral just gives the unit volume of $\mathcal{M}_6$:

$$\int d^6y\sqrt{g_6} = 1. \tag{3.20}$$

Additionally, since R_6 is constant for the choice of M_6 we will consider, the integral

$$\int d^6y\sqrt{g_6}R_6 = R_6 \tag{3.21}$$

just gives a constant as well.

When performing the dimensional reduction, in principle the κ_{10}-factor is replaced by a κ_4-factor, related via

$$\frac{1}{\kappa_4^2} = \frac{e^{-2\alpha\varphi_0}}{\kappa_{10}^2} \tag{3.22}$$

where φ_0 is a constant one can fix, for example to be the value of φ in the potential minimum. However, we set

$$\kappa_4 \stackrel{!}{=} 1 \tag{3.23}$$

which means that in four dimensions, we work in Planck units.

Using that, we can write down the dimensional reduced action:

$$\begin{aligned} S_R = \Bigg(\int d^4x \; e^{(2\alpha+6\beta)\varphi}\sqrt{-g_4}\Big(R_4 - (6(\alpha-\beta)+18\beta)\nabla^2\varphi - (6(\alpha-\beta)^2 + 36\beta(\alpha-\beta) \\ + 72\beta^2)(\partial\varphi)^2\Big)\Bigg) + R_6\Big(\int d^4x \; e^{4(\alpha+\beta)\varphi}\sqrt{-g_4}\Big). \end{aligned} \tag{3.24}$$

We then fix the relation between α and β by demanding the prefactor in front of R_4 to be 1, i.e. $2\alpha + 6\beta \stackrel{!}{=} 0 \implies \alpha = -3\beta$. Plugging in this relation gives us

$$S_R = \Bigg(\int d^4x\sqrt{-g_4}\Big(R_4 + 6\beta\nabla^2\varphi - 24\beta^2(\partial\varphi)^2\Big)\Bigg) + R_6\Big(\int d^4x \; e^{-8\beta\varphi}\sqrt{-g_4}\Big). \tag{3.25}$$

Fixing β

The parameter β remains undetermined, allowing us to set the prefactor in front of the kinetic term of φ to $-\frac{1}{2}$, a common convention. In order to do so, first note that the integral $\int dx^\mu\sqrt{-g_4}\nabla^2\varphi$ vanishes: We can write the Laplace-Beltrami operator as [12]

$$\nabla^2\varphi = \frac{1}{\sqrt{-g_4}}\partial_\mu\left(\sqrt{-g_4}g^{\mu\nu}\partial_\nu\varphi\right), \tag{3.26}$$

and thus

$$\begin{aligned}\int_{X_4} d^4x\sqrt{-g_4}\nabla^2\varphi &= \int_{X_4} d^4x\sqrt{-g_4}\frac{1}{\sqrt{-g_4}}\partial_\mu\left(\sqrt{-g_4}g^{\mu\nu}\partial_\nu\varphi\right)\\ &= \int_{X_4} d^4x\ \partial_\mu\left(\sqrt{-g_4}\partial^\mu\varphi\right)\\ &= \int_{\partial X_4} d\sigma\sqrt{-g_4}\partial^\mu\varphi = 0\end{aligned} \tag{3.27}$$

where we used Stokes' theorem in the second last step, and for the last step we assumed that the derivative of the field φ tends to zero at infinity. With this part of the action being zero, we are left with

$$S_R = \left(\int d^4x\sqrt{-g_4}\left(R_4 - 24\beta^2(\partial\varphi)^2\right)\right) + R_6\left(\int d^4x\ e^{-8\beta\varphi}\sqrt{-g_4}\right). \tag{3.28}$$

Imposing the condition for β explained above and choosing the positive solution, it is $\beta = \frac{1}{\sqrt{48}}$.

Transformation of the Dilaton

There are other terms in the action (3.1) that are affected by the transformation of the metric as well. First, consider the dilaton contribution $(\partial\phi)^2$ in (3.1):

$$S_\phi = \frac{1}{2\kappa_{10}^2}\int d^{10}z\sqrt{-G'_{10}}(-\frac{1}{2}(\partial\phi)'^2). \tag{3.29}$$

Under the discussed metric transformations it is

$$(\partial\phi)'^2 = G'^{MN}\partial_M\phi\partial_N\phi = e^{-2\beta\varphi}\tilde{G}^{MN}\partial_M\phi\partial_N\phi = e^{-2\beta\varphi}(\tilde{\partial}\phi)^2. \tag{3.30}$$

The dilaton only depends on the X_4-coordinates and thus it is $(\tilde{\partial}\phi)^2 = (\hat{\partial}\phi)^2$, and with the second rescaling we find

$$(\partial\phi)'^2 = e^{-2\beta\varphi}(\hat{\partial}\phi)^2 = e^{-2\beta\varphi}e^{-2(\alpha-\beta)\varphi}(\partial\phi)^2 = e^{-2\alpha\varphi}(\partial\phi)^2. \tag{3.31}$$

With $\sqrt{-G'_{10}} = e^{(4\alpha+6\beta)\varphi}\sqrt{-g_4}\sqrt{g_6}$ it is

$$\begin{aligned}S_\phi &= -\frac{1}{4\kappa_{10}^2}\int d^4x\ d^6y\ e^{(4\alpha+6\beta)\varphi}\sqrt{-g_4}\sqrt{g_6}e^{-2\alpha\varphi}(\partial\phi)^2\\ &= -\frac{1}{2}\int d^4x\sqrt{-g_4}e^{(2\alpha+6\beta)\varphi}(\partial\phi)^2\\ &= -\frac{1}{2}\int d^4x\sqrt{-g_4}(\partial\phi)^2\end{aligned} \tag{3.32}$$

where we identified again the unit volume $\int dy^m\sqrt{g_6}$ and used the found relation $\alpha = -3\beta$. Hence, the kinetic terms of both scalar fields ϕ and φ come with a prefactor of $\frac{1}{2}$ in the four-dimensional action, which is convenient.

To be able to transform the other terms in the action (3.1) we need to choose specific ansatzes which we will do in the next sections.

3.2 Setup 1: Turning on F_0 and F_6

We choose the six-dimensional internal manifold to be the six-sphere which is a compact nearly Kähler-manifold. For now, we assume the presence of only F_0 and F_6. Since F_4 vanishes in this scenario, $B \wedge F_4 \wedge F_4$ and $H = dB$ do that as well. The $|F'_p|^2$ all only depend on the $\mathcal{M}_6$-coordinates and are therefore only affected by the first Weyl rescaling:

$$\begin{aligned} |F'_p|^2 &= \frac{1}{p!} G'^{M_1 N_1} \dots G'^{M_p N_p} F_{M_1 \dots M_p} F_{N_1 \dots N_p} \\ &= \frac{1}{p!} e^{-2p\beta\varphi} \tilde{G}^{M_1 N_1} \dots \tilde{G}^{M_p N_p} F_{M_1 \dots M_p} F_{N_1 \dots N_p} \\ &= \frac{1}{p!} e^{-2p\beta\varphi} g^{m_1 n_1} \dots g^{m_p n_p} F_{m_1 \dots m_p} F_{n_1 \dots n_p} \\ &= e^{-2p\beta\varphi} |F_p|^2 \end{aligned} \tag{3.33}$$

where we used that we only have to sum over the indices on $\mathcal{M}_6$. We further choose an ansatz for F'_p of the form [7]

$$F'_p = Q_p \tilde{\epsilon}_p \tag{3.34}$$

with a constant Q_p and the Levi-Civita tensor $\tilde{\epsilon}_p$ satisfying $\int_{M_6} \tilde{\epsilon}_p = 1$. It is then

$$|F'_p|^2 = e^{-2p\beta\varphi} Q_p^2 (\tilde{\epsilon}_p)^2 = e^{-2p\beta\varphi} Q_p^2. \tag{3.35}$$

Thus,

$$\begin{aligned} S_{F_p} &= \frac{1}{2\kappa_{10}^2} \int d^{10}z \sqrt{-G'_{10}} \Big(-\frac{1}{2} \sum_{p \text{ even}} e^{\frac{5-p}{2}\phi} |F'_p|^2\Big) \\ &= -\frac{1}{4\kappa_{10}^2} \int d^4x \; d^6y \sqrt{-g_4} \sqrt{g_6} e^{(4\alpha+6\beta)\varphi} \sum_{p \text{ even}} e^{\frac{5-p}{2}\phi} e^{-2p\beta\varphi} Q_p^2 \\ &= -\frac{1}{2} Q_p^2 \int d^4x \sqrt{-g_4} e^{-6\beta\varphi} \sum_{p \text{ even}} e^{-2p\beta\varphi} e^{\frac{5-p}{2}\phi}. \end{aligned} \tag{3.36}$$

In particular, we find

$$S_{F_0} = -\frac{1}{2} Q_0^2 \int d^4x \sqrt{-g_4} e^{-6\beta\varphi} e^{\frac{5}{2}\phi} \tag{3.37}$$

with Q_0 being the Romans mass and

$$S_{F_6} = -\frac{1}{2} Q_6^2 \int d^4x \sqrt{-g_4} e^{-18\beta\varphi} e^{-\frac{1}{2}\phi}. \tag{3.38}$$

As before, we choose to work in Planck units in the four-dimensional setup where it is $\kappa_4 = 1$.

We can then put everything together to get the four-dimensional action of the theory. Hence, we consider the action

$$\begin{aligned} S = \frac{1}{2} \int d^4x \sqrt{-g_4} \bigg[&\Big(R_4 - \frac{1}{2}((\partial\varphi)^2 + (\partial\phi)^2) \Big) + e^{-8\beta\varphi} R_6 \\ &- \frac{Q_0^2}{2} e^{-6\beta\varphi} e^{\frac{5}{2}\phi} - \frac{Q_6^2}{2} e^{-18\beta\varphi} e^{-\frac{1}{2}\phi} \bigg]. \end{aligned} \tag{3.39}$$

We identify the kinetic and the potential terms in this action and find the potential to be

$$V = \frac{1}{2}\Big(- e^{-8\beta\varphi} R_6 + \frac{Q_0^2}{2} e^{-6\beta\varphi} e^{\frac{5}{2}\phi} + \frac{Q_6^2}{2} e^{-18\beta\varphi} e^{-\frac{1}{2}\phi}\Big). \tag{3.40}$$

From the action above we can extract three equations of motion, for R_4, φ and ϕ. The one for R_4 will yield the Einstein equations, but for now we are more interested in the potentials.

Find Minima of the Potential

The potential depends on two fields, the dilaton ϕ and the volume φ. As discussed in 2.4, the critical points (ϕ^*, φ^*) with

$$\partial_\phi V|_{(\phi^*,\varphi^*)} = 0; \quad \partial_\varphi V|_{(\phi^*,\varphi^*)} = 0, \tag{3.41}$$

determine the vacuum energy of the solution. Recall also that at the minimum, $V(\phi^*, \varphi^*)$ acts as a cosmological constant Λ. We start setting the partial derivatives of (3.40) to zero:

$$\partial_\phi V|_{(\phi^*,\varphi^*)} \stackrel{!}{=} 0 \implies \phi^*(\varphi) \stackrel{!}{=} -4\beta\varphi + \frac{1}{3}\ln\left(\frac{1}{5}\left(\frac{Q_6}{Q_0}\right)^2\right). \tag{3.42}$$

Further, setting $\partial_\varphi V|_{(\phi^*,\varphi^*)} \stackrel{!}{=} 0$ and using the found relation for $\phi^*(\varphi)$ yields

$$\varphi^* = -\frac{1}{8\beta}\ln\left(\frac{5^{\frac{5}{6}}\left(\frac{Q_6}{Q_0}\right)^{\frac{1}{3}} R_6}{6Q_6^2}\right). \tag{3.43}$$

Thus, we found the critical point

$$(e^{\phi^*}, e^{\varphi^*}) = \left(\left(\frac{Q_6^2}{5Q_0^2}\right)^{\frac{1}{3}}\left(\frac{5^{\frac{5}{6}}\left(\frac{Q_6}{Q_0}\right)^{\frac{1}{3}} R_6}{6Q_6^2}\right)^{\frac{1}{2}}, \left(\frac{6Q_6^2}{5^{\frac{5}{6}}\left(\frac{Q_6}{Q_0}\right)^{\frac{1}{3}} R_6}\right)^{\frac{1}{8\beta}}\right), \tag{3.44}$$

and plugging this into the potential V gives

$$V(\phi^*, \varphi^*) = -\frac{5^{\frac{5}{6}}(\frac{Q_6}{Q_0})^{\frac{1}{3}} R_6^2}{24\kappa_{10}^2 Q_6^2}. \tag{3.45}$$

Note that the cosmological constant relates to the potential at the minimum, $V(\phi^*, \varphi^*) = M_{\text{Pl}}^2 \cdot \Lambda$ – that we just find to be negative. That means, that the four-dimensional maximally symmetric Lorentzian manifold is indeed an Anti-de Sitter space.

The BF bound

Recall the BF bound from section 2.4 that vacua need to fulfil to be perturbatively stable: The masses, i.e. the eigenvalues of the Hessian, need to satisfy

$$m^2 \geq -\frac{(d-1)^2}{4L_{\text{AdS}}^2}. \tag{3.46}$$

We plug (ϕ^*, φ^*) into the Hessian matrix and get

$$H(\phi^*, \varphi^*) = \begin{pmatrix} \partial_\phi^2 V & \partial_\phi \partial_\varphi V \\ \partial_\varphi \partial_\phi V & \partial_\varphi^2 V \end{pmatrix}\Bigg|_{(\phi^*,\varphi^*)} = -\frac{5^{\frac{5}{6}} R_6^2 (\frac{Q_6}{Q_0})^{\frac{1}{3}}}{\kappa_{10}^2 Q_6^2} \begin{pmatrix} \frac{5}{96} & \frac{5}{\sqrt{3}\cdot 96} \\ \frac{5}{\sqrt{3}\cdot 96} & \frac{37}{288} \end{pmatrix}. \tag{3.47}$$

The eigenvalues are given by

$$\begin{aligned} m^2 &= \frac{R_6^2 \left(\frac{Q_6}{Q_0}\right)^{\frac{1}{3}} 5^{\frac{5}{6}}}{12\kappa_{10}^2 Q_6^2} \mu_{1,2} \\ \text{with} \quad \mu_1 &= \frac{1}{2}, \quad \mu_2 = \frac{5}{3}. \end{aligned} \tag{3.48}$$

Using $L_{\text{AdS}}^2 = -\frac{6}{V(\phi^*,\varphi^*)}$ we can rewrite these as

$$\text{eig}(H) = m^2 = \left\{ \frac{6}{L_{\text{AdS}}^2}, \frac{20}{L_{\text{AdS}}^2} \right\}. \tag{3.49}$$

Comparing this with the BF-bound (3.46), both eigenvalues fulfil this stability condition and therefore describe stable solutions:

$$m^2 = \left\{ \frac{6}{L_{\text{AdS}}^2}, \frac{20}{L_{\text{AdS}}^2} \right\} \geq -\frac{9}{4L_{\text{AdS}}^2} = -\frac{(d-1)^2}{4L_{\text{AdS}}^2}. \tag{3.50}$$

Scale Separation

As discussed in section 2.3, a compactification can only describe our physical reality if scale separation holds, i.e.

$$L_{\text{AdS}} >> L_{S^6}. \tag{3.51}$$

Hence, we compare the length scales of both the AdS_4-space and the S^6-sphere. We have to be careful with the dimensions. As we defined the potential, it has mass dimension 4 while the cosmological constant has mass dimension 2. They are related via

$$V(\phi^*, \varphi^*) = M_{\text{Pl}}^2 \Lambda = \frac{e^{2\alpha\varphi^*}}{\kappa_{10}^2} \Lambda. \tag{3.52}$$

The length scale of the external space is given by

$$L_{AdS_4}^2 = -\frac{3}{\Lambda} = -\frac{e^{2\alpha\varphi^*}}{\kappa_{10}^2} \frac{3}{V(\phi^*, \varphi^*)} \tag{3.53}$$

while the length scale of the sphere is given by

$$L_{S^6}^2 = e^{2\beta\varphi^*}. \tag{3.54}$$

We find

$$\frac{L_{AdS_4}^2}{L_{S^6}^2} = \frac{12}{R_6} = \frac{2}{5} \tag{3.55}$$

where we used that the Ricci scalar of the S^6 is given by $R_6 = 30$. Hence, scale separation does not hold; the internal and the external manifold have length scales in the same order.

Plot the Potential

We want to plot the potential $V(\phi^*, \varphi^*)$ as a function of both ϕ^* and φ^*. We would like to have the same Q_0, Q_6-dependence in the last two terms since we do not know their values. For that purpose, we redefine the fields ϕ and φ. We write the potential (3.40) as

$$V = \frac{1}{2} e^{-8\beta\varphi} \Big(R_6 - \underbrace{\frac{Q_0^2}{2} e^{2\beta\varphi} e^{\frac{5}{2}\phi}}_{(1)} - \underbrace{\frac{Q_6^2}{2} e^{-10\beta\varphi} e^{-\frac{1}{2}\phi}}_{(2)} \Big) \tag{3.56}$$

We now make an ansatz for $e^{\beta\varphi}$ and $e^{\frac{\phi}{2}}$, namely

$$\begin{aligned} e^{\beta\varphi} &\sim Q_6^{\frac{5}{24}} Q_0^{\frac{1}{24}} R_6^{-\frac{1}{8}} =: Q_6^{\frac{5}{24}} Q_0^{\frac{1}{24}} R_6^{-\frac{1}{8}} L, \\ e^{\phi} &\sim Q_6^{-\frac{1}{6}} Q_0^{-\frac{5}{6}} R_6^{\frac{1}{2}} =: Q_6^{-\frac{1}{6}} Q_0^{-\frac{5}{6}} R_6^{\frac{1}{2}} \delta. \end{aligned} \tag{3.57}$$

where we fixed the exponents by requiring that both terms (1) and (2) go with $Q_0^0 Q_6^0 R_6^1$. Note that the L corresponds to the radius of the six-sphere, and the δ corresponds to the string coupling constant g_s. With this ansatz, we find

$$V = -A \cdot L^{-8} (1 - \frac{1}{2}(L^2 \delta^5 + L^{-10} \delta^{-1})). \tag{3.58}$$

with $A = \frac{1}{2\kappa_{10}^2} Q_6^{-\frac{5}{3}} Q_0^{-\frac{1}{3}}$. We plot V/A as a function of L and δ, i.e. of the (rescaled) radius of S^6 and of the (rescaled) string coupling constant, see figure 3.1. The potential has its minimum in

$$(L^*, \delta^*) = \left(\frac{6^{\frac{1}{8}}}{5^{\frac{5}{48}}}, \frac{5^{\frac{1}{24}}}{6^{\frac{1}{4}}} \right). \tag{3.59}$$

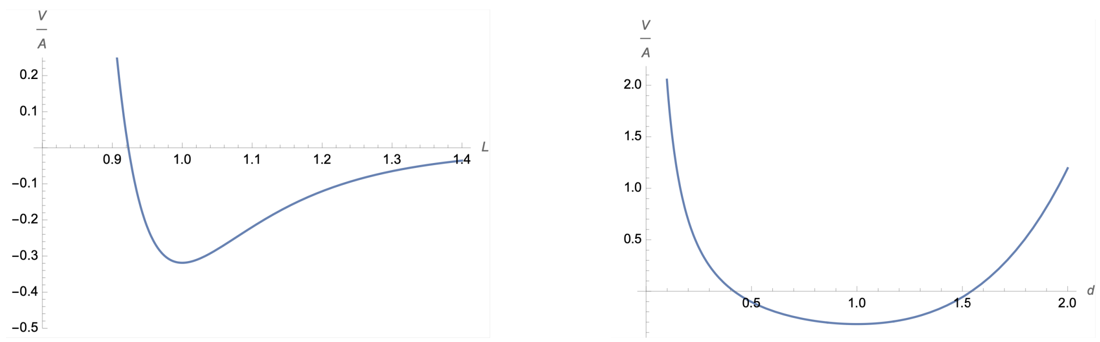


Figure 3.1: Potential V as a function of L (left) and d (right).

3.3 Setup 2: Turning on all Field Strengths

So far, we considered the action with only the two RR field strengths, F_0 and F_6, and found a first vacuum of the theory. However, that is not the most general action. In fact, we can turn on axion fields as well – scalar fields with a continuous shift symmetry.

The Setting

We will turn on the other two RR field strengths F_2 and F_4 and also the NSNS field strength H. The RR and NSNS field strengths are not completely free to choose, in particular, they have to fulfil the Bianchi identities

$$\mathrm{d}F_n = H \wedge F_{n-3} + \delta_{\mathrm{D}(8-n)/\mathrm{O}(8-n)} \tag{3.60}$$

where the second δ-term vanishes in the setting we work with. One can check that the following choice for B indeed fulfils the Bianchi identities:

$$B_{mn} = b(x)J_{mn}, \quad H = \mathrm{d}B \implies H_{\mu mn} = \partial_\mu b(x)J_{mn}, \quad H_{mnl} = 2b(x)\tilde{m}(\mathrm{Re}\Omega)_{mnl}. \tag{3.61}$$

Here, $b(x)$ is a four-dimensional axion, $\tilde{m}$ is a parameter, and J is a 2-form that satisfies

$$\frac{1}{2!}J_{mn}J_{kl}g^{mk}g^{nl} = 3 \tag{3.62}$$

and

$$\frac{1}{6}J \wedge J \wedge J = \mathrm{vol}_6. \tag{3.63}$$

It encodes the volume of 2-cycles or 2d submanifolds. Similarly, the complex 3-form Ω encodes the volume of 3-cycles or three-dimensional submanifolds, and satisfies

$$\begin{aligned} \frac{1}{3!}(\mathrm{Re}\Omega)_{mnk}(\mathrm{Re}\Omega)_{uvw}g^{mn}g^{nv}g^{kw} = \frac{1}{3!}(\mathrm{Im}\Omega)_{mnk}(\mathrm{Im}\Omega)_{uvw}g^{mn}g^{nv}g^{kw} = 4 \\ (\mathrm{Re}\Omega)_{mnk}(\mathrm{Im}\Omega)^{mnk} = 0 \\ \Omega_{mnl}J^{mn} = 0. \end{aligned} \tag{3.64}$$

The field strength F_2 and its components are given by

$$F_2 = Q_0 B \quad \implies \quad F_{mn} = Q_0 b(x)J_{mn}. \tag{3.65}$$

Moreover, we set

$$F_4 = \frac{1}{2}B \wedge BQ_0 + \mathrm{d}(\zeta(x)\mathrm{Im}\Omega) = \frac{1}{2}B \wedge BQ_0 + \mathrm{d}\zeta(x) \wedge \mathrm{Im}\Omega - \frac{4\tilde{m}}{3}\zeta(x)J \wedge J \tag{3.66}$$

with components

$$F_{\mu mnl} = \partial_\mu \zeta(\mathrm{Im}\Omega)_{mnl}, \; F_{klmn} = (Q_0 b^2 - \frac{8\tilde{m}}{3}\zeta)\epsilon_{klmnop}J^{op}. \tag{3.67}$$

The ζ that appears here is another axion. Finally, also F_6 gains additional terms:

$$F_6 = Q_6 \mathrm{vol}_6 + \frac{1}{6}Q_0 B \wedge B \wedge B + B \wedge \mathrm{d}C_3 = (Q_6 - b^3 Q_0 + 8\tilde{m}\zeta b)\mathrm{vol}_6 \tag{3.68}$$

with components

$$F_{klmnop} = (Q_6 - b^3 Q_0 + 8\tilde{m}\zeta b)\epsilon_{klmnop}. \tag{3.69}$$

Transformation of the Field Strengths

We compute how the field strengths $H,\ F_2,\ F_4$ and F_6 transform under the Weyl rescalings. The computations can be found in appendix B. It is

$$\begin{aligned} H' &= 6e^{(2\alpha-8\beta)\varphi}(\partial b)^2 + 96e^{-6\beta\varphi}\tilde{m}^2 b^2, \\ |F_2'|^2 &= 3e^{-4\beta\varphi}Q_0^2(b(x))^2, \\ |F_4'|^2 &= e^{(2\alpha-10\beta)\varphi}(\partial\zeta)^2 + 12e^{-8\beta\varphi}\left(Q_0 b^2 - \frac{8\tilde{m}}{3}\zeta\right)^2, \\ |F_6'|^2 &= e^{-12\beta\varphi}\left(Q_6 - b^3 Q_0 + 8\tilde{m}\zeta b\right)^2. \end{aligned} \tag{3.70}$$

Action

Collecting all terms together yields the four-dimensional action, as before we now work in Planck units with $\kappa_4 = 1$. The derivation can be found in more detail in appendix B. We find the action to be

$$\begin{aligned} S = \frac{1}{2}\int d^4x\sqrt{-g_4}\Big(R_4 &- \frac{1}{2}((\partial\varphi)^2 + (\partial\phi)^2) - \frac{1}{2}e^{-\phi}e^{-20\beta\varphi}(\partial b)^2 \\ &- \frac{1}{2}e^{\frac{\phi}{2}}e^{-22\beta\varphi}(\partial\zeta)^2 - V\Big) \end{aligned} \tag{3.71}$$

with the potential

$$\begin{aligned} V = -e^{-8\beta\varphi}R_6 &+ 8e^{-\phi}e^{-12\beta\varphi}\tilde{m}^2 b^2 + \frac{Q_0^2}{2}e^{\frac{5}{2}\phi}e^{-6\beta\varphi} + \frac{3}{2}e^{\frac{3}{2}\phi}e^{-10\beta\varphi}Q_0^2 b^2 \\ &+ \frac{3}{2}e^{\frac{\phi}{2}}e^{-14\beta\varphi}(Q_0 b^2 - \frac{8\tilde{m}}{3}\zeta)^2 + \frac{1}{2}e^{-\frac{\phi}{2}}e^{-18\beta\varphi}(Q_6 - b^3 Q_0 + 8\tilde{m}\zeta b)^2. \end{aligned} \tag{3.72}$$

Similar to the first setup, we rescale the four moduli fields to achieve an overall dependence of Q_0, Q_6 and R_6:

$$\begin{aligned} e^{\beta\varphi} &= Q_6^{\frac{5}{24}}Q_0^{\frac{1}{24}}R_6^{-\frac{1}{8}}L, \\ e^{\frac{\phi}{2}} &= Q_6^{-\frac{1}{12}}Q_0^{-\frac{5}{12}}R_6^{\frac{1}{4}}\delta, \\ b &= Q_6^{\frac{1}{3}}Q_0^{-\frac{1}{3}}\tilde{b}, \\ \zeta &= \frac{\sqrt{30}}{4}Q_6^{\frac{2}{3}}Q_0^{\frac{1}{3}}R_6^{-\frac{1}{2}}\tilde{\zeta}. \end{aligned} \tag{3.73}$$

We can then write the potential as

$$\begin{aligned} V = e^{-8\beta\varphi}R_6\Bigg(&-1 + \frac{3}{5}\delta^{-2}L^{-4}\tilde{b}^2 + \frac{1}{2}\delta^5 L^2 + \frac{3}{2}\delta^3 L^{-2}\tilde{b}^2 \\ &+ \frac{3}{2}\left(\tilde{b}^2 - \tilde{\zeta}\right)^2\delta L^{-6} + \frac{1}{2}\delta^{-1}L^{-10}(1 - \tilde{b}^3 + 3\tilde{\zeta}\tilde{b})^2\Bigg) \end{aligned} \tag{3.74}$$

where we used $R_6 = \frac{15}{2}\left(\frac{4}{3}\tilde{m}\right)^2 \Leftrightarrow \tilde{m}^2 = \frac{3}{40}R_6$.

Searching for minima of V, now dependent on four fields, yields three solutions, see table 3.1. Note that the first solution is the one we already found in the first setup.

	δ^*	L^*	$\tilde{b}^*$	$\tilde{\zeta}^*$
Solution 1	$\frac{5^{1/24}}{6^{1/4}}$	$\frac{6^{1/8}}{5^{5/48}}$	0	0
Solution 2	$\frac{1}{2^{1/12}3^{1/8}5^{1/6}}$	$\frac{3^{5/16}}{2^{7/24}5^{1/12}}$	$-\frac{1}{2^{2/3}5^{1/3}}$	$\left(\frac{2}{5}\right)^{\frac{2}{3}}$
Solution 3	$\frac{1}{2^{1/3}3^{1/8}5^{1/24}}$	$\frac{3^{5/16}5^{5/48}}{2^{2/3}}$	$\frac{1}{2 \cdot 2^{2/3}5^{1/3}}$	$-\frac{1}{2^{1/3}5^{2/3}}$

Table 3.1: Three found minima

We compare the three found minima with the ones found in [1]. Indeed, the results only differ by overall prefactors which do not change the results due to freedom of convention. Multiplying our AdS length scale L_{AdS^2} with $\frac{1}{9}\left(\frac{2}{5}\right)^{1/3}$ yields the AdS length scale in [1], multiplying our ζ with $-\left(\frac{5}{2}\right)^{1/3}$ and our b with $-\left(\frac{5}{2}\right)^{2/3}$ yields the axions ζ and b in [1]. Finally, multiplying our L^8 with $\frac{1}{3}\left(\frac{5}{2}\right)^{2/3}$ and our δ^4 with $3\cdot\left(\frac{5}{2}\right)^{2/3}$ gives the dilaton and volume fields in [1]. Hence, we could reproduce the results in [1]. From their discussions, we can also conclude that – while solutions 1 and 2 are not supersymmetric – the third solution is actually a supersymmetric one.

For these minima, we calculate the critical masses using the Hessian. In contrast to setup 1, not all kinetic terms in the action (3.71) have a prefactor $\frac{1}{2}$, i.e. the matrix g in

$$T = \frac{1}{2}\,(\partial\Phi)^T \cdot g \cdot (\partial\Phi) \tag{3.75}$$

with kinetic energy T and $\Phi = (\phi, \varphi, b, \zeta)^T$ is not canonical. The introduction of a metric E such that $E^T \cdot E = g$ and a rescaled moduli vector $\Xi := E \cdot \Phi$ solves the problem: We can write the kinetic term in a canonic form, $T = \frac{1}{2}(\partial\Xi)^T \cdot (\partial\Xi)$, and the mass matrix is then given by $(E^{-1})^T \cdot H \cdot E^{-1}$ with the Hessian matrix H. We find the following critical masses (in $\left[\frac{1}{L^2_{\text{AdS}}}\right]$ with $L^2_{\text{AdS}} = -\frac{6}{V(\delta^*, L^*, \tilde{b}^*, \tilde{\zeta}^*)}$):

Solution 1	20	20	6	6
Solution 2	20	$\frac{64}{5}$	6	$-\frac{6}{5}$
Solution 3	$\frac{47}{3}+\sqrt{\frac{53}{3}}$	$\frac{47}{3}-\sqrt{\frac{53}{3}}$	$4+\sqrt{6}$	$4-\sqrt{6}$

Table 3.2: Masses of the found minima

They all satisfy the BF bound

$$m^2 \geq -\frac{9}{4L^2_{\text{AdS}}} = -\frac{(d-1)^2}{4L^2_{\text{AdS}}}. \tag{3.76}$$

CHAPTER 4

Conclusion

In this work, we looked at string compactifications on $\mathrm{AdS}_4 \times S^6$ in type IIA supergravity. After choosing an ansatz for the ten-dimensional metric, we performed the dimensional reduction of the ten-dimensional action to obtain the four-dimensional effective theory. We then examined this theory for its vacua, and investigated certain properties of those. In a first setup, where we only included the two field strengths F_0 and F_6, we were able to identify one vacuum. In a second, general setup, we included all field strengths, namely $F_0,\ F_2,\ F_4,\ F_6$ and H. Besides the vacuum already found in the first setup, we found two additional vacua. We were able to identify the correspondence between our results and the three well-known solutions for $\mathrm{AdS}_4 \times S^6$, first found in [13], [14] and [15], and for example presented in [1]: Two non-supersymmetric vacua and one supersymmetric one. For all three vacua, we calculated the corresponding masses of the axion fields and found that for each vacuum and each axion, the first mode of the tower of masses fulfils the BF-bound.

Widely discussed is the stability of non-supersymmetric AdS vacua as it could enhance our understanding of time-dependent de Sitter solutions and the use of holography in realistic systems, such as those found in condensed matter physics or QCD [16]. The (non-) perturbative stability of the two non-supersymmetric $\mathrm{AdS}_4 \times S^6$ vacua is the subject matter of several publications.

In [17], Ooguri and Vafa sharpen the famous weak gravity conjecture, which then implies that any non-supersymmetric AdS vacuum supported by fluxes must be unstable. This conjecture applies for the two found vacua with $\mathcal{N} = 0$: While the perturbative instability of the non-supersymmetric, SO(7) preserving $\mathrm{AdS}_4 \times S^6$ vacuum was proven in [18], [1] presents a vacuum decay for the G_2 invariant vacuum with $\mathcal{N} = 0$, a decay via expanding D-branes going through a bubble of nothing regime. This instability does not only apply in this scenario, but more generally for related $\mathrm{AdS}_4 \times \mathcal{M}_6$ solutions with $\mathcal{M}_6$ being a nearly-Kähler manifold [1].

As discussed, the supersymmetry of the three $\mathrm{AdS}_4 \times S^6$ vacua is well-known. However, this was not examined in this work and offers possibilities for further investigations. Moreover, in a next step one could investigate a slightly more complex ten-dimensional spacetime, such as $\mathrm{AdS}_4 \times \mathbb{C}\mathrm{P}^3$.

APPENDIX A

The Transformation of the Ricci Scalar under Weyl Rescalings

We compute how the Ricci scalar transforms under a Weyl transformation of the n-dimensional metric,

$$g'_{\mu\nu} = e^{2c\omega(x)} g_{\mu\nu}; \qquad g'^{\mu\nu} = e^{-2c\omega(x)} g^{\mu\nu}, \tag{A.1}$$

with c being a constant and ω a smooth function of the coordinates. Note that the following two useful relations hold:

$$\begin{aligned} \Gamma_{jik} + \Gamma_{kji} &= g_{lj}\Gamma^l_{ik} + g_{km}\Gamma^m_{ji} \\ &= \frac{1}{2} g_{lj} g^{l\gamma} \left(\partial_i g_{\gamma k} + \partial_k g_{i\gamma} - \partial_\gamma g_{ik}\right) + \frac{1}{2} g_{km} g^{m\mu} \left(\partial_j g_{\mu i} + \partial_i g_{j\mu} - \partial_\mu g_{ij}\right) \\ &= \frac{1}{2}\left(\partial_i g_{jk} + \partial_k g_{ij} - \partial_j g_{ik} + \partial_j g_{ki} + \partial_i g_{jk} - \partial_k g_{ij}\right) \\ &= \partial_i g_{jk} \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} \partial_i g^{jk} = \partial_i g^{jm} \delta^k_m = \underbrace{\partial_i (g^{jm}) g_{ml}}_{=-g^{jm}\partial_i(g_{ml})} g^{lk} = -g^{lk} g^{jm} \partial_i g_{ml} = -g^{lk} g^{jm} (\Gamma_{mil} + \Gamma_{lmi}) \\ = -(g^{lk}\Gamma^j_{il} + g^{jm}\Gamma^k_{mi}). \end{aligned} \tag{A.3}$$

We first consider the transformed connection:

$$\begin{aligned}
\Gamma'^i_{kl} &= \frac{1}{2} g'^{i\gamma}(\partial_l g'_{\gamma k} + \partial_k g'_{\gamma l} - \partial_\gamma g'_{kl}) \\
&= \frac{1}{2} e^{-2c\omega} g^{i\gamma}[\partial_l(e^{2c\omega} g_{\gamma k}) + \partial_k(e^{2c\omega} g_{\gamma l}) - \partial_\gamma(e^{2c\omega} g_{kl})] \\
&= \frac{1}{2} e^{-2c\omega} g^{i\gamma}[2c\partial_l(\omega) e^{2c\omega} g_{\gamma k} + e^{2c\omega}\partial_l(g_{\gamma k}) + 2c\partial_k(\omega) e^{2c\omega} g_{\gamma l} + e^{2c\omega}\partial_k(g_{\gamma l}) \\
&\quad - 2c\partial_\gamma(\omega) e^{2c\omega} g_{kl} - e^{2c\omega}\partial_\gamma(g_{kl})] \\
&= \frac{1}{2} g^{i\gamma}(\partial_l g_{\gamma k} + \partial_k g_{\gamma l} - \partial_\gamma g_{kl}) + c g^{i\gamma}[\partial_l(\omega) g_{\gamma k} + \partial_k(\omega) g_{\gamma l} - \partial_\gamma(\omega) g_{kl}] \\
&= \Gamma^i_{kl} + a^i_{kl}
\end{aligned} \tag{A.4}$$

with

$$\begin{aligned}
a^i_{kl} &= c g^{i\gamma}[\partial_l(\omega) g_{\gamma k} + \partial_k(\omega) g_{l\gamma} - \partial_\gamma(\omega) g_{kl}] \\
&= c[\partial_l(\omega)\delta^i_k + \partial_k(\omega)\delta^i_l - \partial_\gamma(\omega) g^{i\gamma} g_{kl}].
\end{aligned} \tag{A.5}$$

The transformed Ricci scalar is given by

$$\begin{aligned}
R' = g'^{ij} R'_{ij} = g'^{ij} R'^k_{ikj} &= g'^{ij}\left(\partial_k \Gamma'^k_{ji} - \partial_j \Gamma'^k_{ki} + \Gamma'^k_{k\gamma}\Gamma'^\gamma_{ji} - \Gamma'^k_{j\gamma}\Gamma'^\gamma_{ki}\right) \\
&= g'^{ij}\left(R^k_{ikj} + \partial_k a^k_{ji} - \partial_j a^k_{ki} + \Gamma^k_{k\gamma} a^\gamma_{ji} + a^k_{k\gamma}\Gamma^\gamma_{ji} + a^k_{k\gamma} a^\gamma_{ji} - \Gamma^k_{j\gamma} a^\gamma_{ki} - a^k_{j\gamma}\Gamma^\gamma_{ki} - a^k_{j\gamma} a^\gamma_{ki}\right) \\
&= g'^{ij}(R^k_{ikj} + \underbrace{\partial_k a^k_{ji} - \partial_j a^k_{ki}}_{(1)} + \underbrace{a^k_{k\gamma}\Gamma^\gamma_{ji} - a^k_{j\gamma}\Gamma^\gamma_{ki}}_{(2)} + \underbrace{\Gamma^k_{k\gamma} a^\gamma_{ji} - \Gamma^k_{j\gamma} a^\gamma_{ki}}_{(3)} + \underbrace{a^k_{k\gamma} a^\gamma_{ji} - a^k_{j\gamma} a^\gamma_{ki}}_{(4)}).
\end{aligned} \tag{A.6}$$

We compute the single terms, starting with (1): It is

$$\begin{aligned}
\partial_k a^k_{ji} &= \partial_k\left(c\left(\partial_j(\omega)\delta^k_i + \partial_i(\omega)\delta^k_j - \partial_\gamma(\omega) g^{k\gamma} g_{ij}\right)\right) \\
&= c(\partial_k(\partial_j(\omega))\delta^k_i + \partial_k(\partial_i(\omega))\delta^k_j - \partial_k(\partial_\gamma(\omega)) g^{k\gamma} g_{ij} + \partial_j(\omega)\underbrace{\partial_k \delta^k_i}_{=0} \\
&\quad + \partial_i(\omega)\underbrace{\partial_k \delta^k_j}_{=0} - \partial_\gamma(\omega)\partial_k(g^{k\gamma} g_{ij})) \\
&= c(\partial_i(\partial_j(\omega)) + \partial_j(\partial_i(\omega)) - \partial^\gamma(\partial_\gamma(\omega)) g_{ij} - \partial_\gamma(\omega)\partial_k(g^{k\gamma} g_{ij})) \\
&= c(2\partial_i(\partial_j(\omega)) - \partial^\gamma(\partial_\gamma(\omega)) g_{ij} - \partial_\gamma(\omega)\partial_k(g^{k\gamma} g_{ij}))
\end{aligned} \tag{A.7}$$

where we used that we can interchange the derivatives because of Schwarz, and

$$\begin{aligned}
-\partial_j a^k_{ki} &= -\partial_j(c(\partial_i(\omega)\underbrace{\delta^k_k}_{=n} + \partial_k(\omega)\delta^k_i - \partial_\gamma(\omega)\delta^\gamma_i)) \\
&= c(-n\partial_j(\partial_i(\omega)) - \partial_j(\partial_k(\omega))\delta^k_i + \partial_j(\partial_\gamma(\omega))\delta^\gamma_i \\
&\quad - \partial_i(\omega)\underbrace{\partial_j(n)}_{=0} - \partial_k(\omega)\underbrace{\partial_j(\delta^k_i)}_{=0} + \partial_\gamma(\omega)\underbrace{\partial_j(\delta^\gamma_i)}_{=0}) \\
&= c(-n\partial_j(\partial_i(\omega)) - \partial_j(\partial_i(\omega)) + \partial_j(\partial_i(\omega))) \\
&= -nc\partial_j(\partial_i(\omega)).
\end{aligned} \tag{A.8}$$

Thus,

$$\begin{aligned} g'^{ij}(\partial_k a^k_{ji} - \partial_j a^k_{ki}) &= g'^{ij}\left(2\partial_i(\partial_j(\omega)) - \partial^\gamma(\partial_\gamma(\omega))g_{ij} - \partial_\gamma(\omega)\partial_k(g^{k\gamma}g_{ij}) - n\ \partial_j(\partial_i(\omega))\right) \\ &= e^{-2c\omega}c\left(2\partial^j(\partial_j(\omega)) - n\ \partial^\gamma(\partial_\gamma(\omega)) - \partial_\gamma(\omega)g^{ij}\partial_k(g^{k\gamma}g_{ij}) - n\ \partial^i(\partial_i(\omega))\right) \\ &= e^{-2c\omega}c\left(-\partial_\gamma(\omega)g^{ij}\partial_k(g^{k\gamma}g_{ij}) + (2-2n)\ \partial^i(\partial_i(\omega))\right). \end{aligned} \tag{A.9}$$

Further, (2) is given by

$$\begin{aligned} a^k_{k\gamma}\cdot\Gamma^\gamma_{ji} - a^k_{j\gamma}\cdot\Gamma^\gamma_{ki} &= c\underbrace{\left(\partial_\gamma(\omega)\delta^k_k + \partial_k(\omega)\delta^k_\gamma - \partial_\mu(\omega)\delta^\mu_\gamma\right)}_{=n\partial_\gamma(\omega)}\cdot\Gamma^\gamma_{ji} \\ &\quad - c\left(\partial_\gamma(\omega)\delta^k_j + \partial_j(\omega)\delta^k_\gamma - g^{k\mu}\partial_m u(\omega)g_{j\gamma}\right)\cdot\Gamma^\gamma_{ki} \\ &= c(n\ \partial_\gamma(\omega)\Gamma^\gamma_{ji} - \partial_\gamma(\omega)\Gamma^\gamma_{ji} - \partial_j(\omega)\Gamma^\gamma_{\gamma i} + \partial^k(\omega)g_{j\gamma}\Gamma^\gamma_{ki}) \\ &= c((n-1)\ \partial_\gamma(\omega)\Gamma^\gamma_{ji} - \partial_j(\omega)\Gamma^\gamma_{\gamma i} + \partial^k(\omega)g_{j\gamma}\Gamma^\gamma_{ki}). \end{aligned} \tag{A.10}$$

Having that, it is

$$\begin{aligned} g'^{ij}(a^k_{k\gamma}\cdot\Gamma^\gamma_{ji} - a^k_{j\gamma}\cdot\Gamma^\gamma_{ki}) &= e^{-2c\omega}c((n-1)\ \partial_\gamma(\omega)g^{ij}\Gamma^\gamma_{ji}\underbrace{-\partial^i(\omega)\Gamma^\gamma_{\gamma i} + \partial^k(\omega)\delta^i_\gamma\Gamma^\gamma_{ki}}_{=0}) \\ &= (n-1)e^{-2c\omega}c\partial_\gamma(\omega)g^{ij}\Gamma^\gamma_{ji}. \end{aligned} \tag{A.11}$$

We continue with the third term (3). It is

$$\begin{aligned} \Gamma^k_{k\gamma}\cdot a^\gamma_{ji} &= \Gamma^k_{k\gamma}\cdot c\left(\partial_i(\omega)\delta^\gamma_j + \partial_j(\omega)\delta^\gamma_i - g^{\gamma\nu}\partial_\nu(\omega)g_{ji}\right) \\ &= c(\partial_i(\omega)\Gamma^k_{kj} + \partial_j(\omega)\Gamma^k_{ki} - \partial^\gamma(\omega)\Gamma^k_{k\gamma}g_{ji}) \end{aligned} \tag{A.12}$$

and

$$\begin{aligned} -\Gamma^k_{j\gamma}\cdot a^\gamma_{ki} &= -\Gamma^k_{j\gamma}\cdot c\left(\partial_i(\omega)\delta^\gamma_k + \partial_k(\omega)\delta^\gamma_i - g^{\gamma\nu}\partial_\nu(\omega)g_{ki}\right) \\ &= c(-\partial_i(\omega)\Gamma^k_{jk} - \partial_k(\omega)\Gamma^k_{ji} + \partial^\gamma(\omega)\Gamma^k_{j\gamma}g_{ki}), \end{aligned} \tag{A.13}$$

thus

$$\Gamma^k_{k\gamma}\cdot a^\gamma_{ji} - \Gamma^k_{j\gamma}\cdot a^\gamma_{ki} = c\Big(\partial_j(\omega)\Gamma^k_{ki} - \partial_k(\omega)\Gamma^k_{ji} + \partial^\gamma(\omega)\left(\Gamma^k_{j\gamma}g_{ki} - \Gamma^k_{k\gamma}g_{ji}\right)\Big) \tag{A.14}$$

and further

$$\begin{aligned} g'^{ij}(\Gamma^k_{k\gamma}\cdot a^\gamma_{ji} - \Gamma^k_{j\gamma}\cdot a^\gamma_{ki}) &= e^{-2c\omega}c\left(\partial^i(\omega)\Gamma^k_{ki} - \partial_k(\omega)g^{ij}\Gamma^k_{ji} + \partial^\gamma(\omega)\left(g^{ij}\Gamma^k_{j\gamma}g_{ki} - g^{ij}\Gamma^k_{k\gamma}g_{ji}\right)\right) \\ &= e^{-2c\omega}c\left(\partial^i(\omega)\Gamma^k_{ki} - \partial_k(\omega)g^{ij}\Gamma^k_{ji} + \partial^\gamma(\omega)(\Gamma^k_{k\gamma} - n\Gamma^k_{k\gamma})\right) \\ &= e^{-2c\omega}c\left((2-n)\partial^i(\omega)\Gamma^k_{ki} - \partial_k(\omega)g^{ij}\Gamma^k_{ji}\right). \end{aligned} \tag{A.15}$$

Finally, we have to look at the last term (4):

$$a^k_{k\gamma}a^\gamma_{ji} - a^k_{j\gamma}a^\gamma_{ki} = c^2\left(\partial_\gamma(\omega)\delta^k_k + \partial_k(\omega)\delta^k_\gamma - g^{k\mu}\partial_\mu(\omega)g_{k\gamma}\right)\left(\partial_i(\omega)\delta^\gamma_j + \partial_j(\omega)\delta^\gamma_i - g^{\gamma\nu}\partial_\nu(\omega)g_{ji}\right)$$

$$
\begin{aligned}
&- c^2 \left(\partial_\gamma(\omega)\delta^k_j + \partial_j(\omega)\delta^k_\gamma - g^{k\mu}\partial_\mu g_{j\gamma}\right) \left(\partial_i(\omega)\delta^\gamma_k + \partial_k(\omega)\delta^\gamma_i - g^{\gamma\nu}\partial_\nu(\omega)g_{ki}\right)\\
= c^2 \Bigg(&\underbrace{\partial_\gamma(\omega)\partial_i(\omega)(n\delta^\gamma_j - \delta^\gamma_j)}_{=(n-1)\partial_j(\omega)\partial_i(\omega)} \underbrace{+\partial_\gamma(\omega)\partial_j(\omega)(\delta^k_k\delta^\gamma_i)}_{=+n\ \partial_i(\omega)\partial_j(\omega)} \underbrace{-\partial_\gamma(\omega)\partial_\nu(\omega)(ng^{\gamma\nu}g_{ji} - g^{\gamma\nu}g_{ji})}_{=-(n-1)\partial^\nu(\omega)\partial_\nu(\omega)g_{ji}}\\
&\underbrace{+\partial_k(\omega)\partial_i(\omega)\delta^k_j}_{=+\partial_j(\omega)\partial_i(\omega)} + \partial_k(\omega)\partial_j(\omega)\underbrace{(\delta^k_\gamma\delta^\gamma_i - \delta^k_\gamma\delta^\gamma_i)}_{=0} \underbrace{-\partial_k(\omega)\partial_\nu(\omega)(\delta^k_\gamma g^{\gamma\nu}g_{ji})}_{=-\partial^\nu(\omega)\partial_\nu(\omega)g_{ji}}\\
&- \partial_\mu(\omega)\partial_i(\omega)\underbrace{(\delta^\mu_j - \delta^\mu_j)}_{=0} \underbrace{-\partial_\mu(\omega)\partial_j(\omega)\delta^\mu_i}_{=-\partial_i(\omega)\partial_j(\omega)}\\
&\underbrace{+\partial_\mu(\omega)\partial_\nu(\omega)\underbrace{(g^{k\mu}g_{k\gamma}g^{\gamma\nu}g_{ji} - g^{k\mu}g_{j\gamma}g^{\gamma\nu}g_{ki})}_{=g^{\mu\nu}g_{ji}-\delta^\nu_j\delta^\mu_i}}_{=+\partial^\nu(\omega)\partial_\nu(\omega)g_{ji}-\partial_i(\omega)\partial_j(\omega)} \underbrace{-\partial_\gamma(\omega)\partial_k(\omega)(\delta^k_j\delta^\gamma_i)}_{=-\partial_i(\omega)\partial_j(\omega)}\\
&\underbrace{-\partial_j(\omega)\partial_i(\omega)\delta^k_\gamma\delta^\gamma_k}_{=-n\ \partial_j(\omega)\partial_i(\omega)} \underbrace{+\partial_j(\omega)\partial_\nu(\omega)\underbrace{(\delta^k_\gamma g^{\gamma\nu}g_{ki})}_{=\delta^\nu_i}}_{=+\partial_j(\omega)\partial_i(\omega)} \underbrace{+\partial_\mu(\omega)\partial_k(\omega)(g^{k\mu}g_{j\gamma}\delta^\gamma_i)}_{=+\partial^k(\omega)\partial_k(\omega)g_{ji}} \Bigg)\\
= c^2\Bigg(&(n-1)\ \partial_j(\omega)\partial_i(\omega) + n\ \partial_i(\omega)\partial_j(\omega) - (n-1)\ \partial^\nu(\omega)\partial_\nu(\omega)g_{ji} + \partial_j(\omega)\partial_i(\omega)\\
&- \partial^\nu(\omega)\partial_\nu(\omega)g_{ji} - \partial_i(\omega)\partial_j(\omega) + \partial^\nu(\omega)\partial_\nu(\omega)g_{ji} - \partial_i(\omega)\partial_j(\omega) - \partial_i(\omega)\partial_j(\omega)\\
&- n\ \partial_j(\omega)\partial_i(\omega) + \partial_j(\omega)\partial_i(\omega) + \partial^k(\omega)\partial_k(\omega)g_{ji}\Bigg)\\
= c^2&((n-2)\ \partial_i(\omega)\partial_j(\omega) - (n-2)\ \partial^k(\omega)\partial_k(\omega)g_{ji})
\end{aligned}
$$

and hence,

$$
\begin{aligned}
g'^{ij}(a^k_{k\gamma}a^\gamma_{ji} - a^k_{j\gamma}a^\gamma_{ki}) &= g'^{ij}c^2((n-2)\partial_i(\omega)\partial_j(\omega) - (n-2)\partial^k(\omega)\partial_k(\omega)g_{ji})\\
&= e^{-2c\omega}c^2((n-2)\partial^j(\omega)\partial_j(\omega) - n(n-2)\partial^k(\omega)\partial_k(\omega))\\
&= -(n-1)(n-2)c^2e^{-2c\omega}\partial^j(\omega)\partial_j(\omega).
\end{aligned}
\tag{A.16}
$$

Collecting all computed terms together yields

$$
\begin{aligned}
R' = e^{-2c\omega}(&R - c\partial_\gamma(\omega)g^{ij}\partial_k(g^{k\gamma}g_{ij}) - 2(n-1)c\partial^i(\partial_i(\omega)) + (n-1)c\partial_\gamma(\omega)g^{ij}\Gamma^\gamma_{ji}\\
&- (n-2)c\partial^i(\omega)\Gamma^k_{ki} - c\partial_k(\omega)g^{ij}\Gamma^k_{ji} - (n-1)(n-2)c^2\partial^j(\omega)\partial_j(\omega))\\
= e^{-2c\omega}(&R - 2(n-1)c\partial^i(\partial_i(\omega)) - (n-1)(n-2)c^2\partial^j(\omega)\partial_j(\omega)\\
&\underbrace{-c\partial_\gamma(\omega)g^{ij}\partial_k(g^{k\gamma}g_{ij}) + (n-2)c\partial_\gamma(\omega)g^{ij}\Gamma^\gamma_{ji} - (n-2)c\partial^i(\omega)\Gamma^k_{ki}}_{(=:A)}).
\end{aligned}
\tag{A.17}
$$

We want to rewrite the term A. For that, first consider its first summand:

$$\begin{aligned} c\partial_\gamma(\omega)g^{ij}\partial_k(g^{k\gamma}g_{ij}) &= c(\partial_\gamma(\omega)(g^{ij}\partial_k(g^{k\gamma})g_{ij} + g^{ij}g^{k\gamma}\partial_k(g_{ij}))) \\ &\overset{A.2,A.3}{=} c(\partial_\gamma(\omega)\left(-n\left(g^{l\gamma}\Gamma^k_{kl} + g^{km}\Gamma_{mk}\right) + g^{ij}g^{k\gamma}\left(\Gamma_{ikj} + \Gamma_{jik}\right)\right)) \\ &= c(\partial_\gamma(\omega)\left(-n\left(g^{l\gamma}\Gamma^k_{kl} + g^{km}\Gamma^\gamma_{mk}\right) + g^{k\gamma}\Gamma^j_{kj} + g^{k\gamma}\Gamma^i_{ik}\right)) \\ &= c(\partial_\gamma(\omega)\left((1-n)g^{l\gamma}\Gamma^k_{kl} - ng^{km}\Gamma^\gamma_{mk} + g^{k\gamma}\Gamma^j_{kj}\right)) \\ &= c((1-n)\partial^l(\omega)\Gamma^k_{kl} - n\partial_\gamma(\omega)g^{km}\Gamma^\gamma_{mk} + \partial^k(\omega)\Gamma^j_{kj}) \\ &= c((2-n)\partial^l(\omega)\Gamma^k_{kl} - n\partial_\gamma(\omega)g^{km}\Gamma^\gamma_{mk}). \end{aligned} \tag{A.18}$$

With this, we can rewrite A as follows:

$$\begin{aligned} &c(- \partial_\gamma(\omega)g^{ij}\partial_k(g^{k\gamma}g_{ij}) + (n-2)\partial_\gamma(\omega)g^{ij}\Gamma^\gamma_{ji} - (n-2)\partial^i(\omega)\Gamma^k_{ki}) \\ &= c((n-2)\partial^l(\omega)\Gamma^k_{kl} + n\partial_\gamma(\omega)g^{km}\Gamma^\gamma_{mk} + (n-2)\partial_\gamma(\omega)g^{ij}\Gamma^\gamma_{ji} - (n-2)\partial^i(\omega)\Gamma^k_{ki}) \\ &= c(2n-2)\partial_\gamma(\omega)g^{ij}\Gamma^\gamma_{ji}. \end{aligned} \tag{A.19}$$

We plug this back into R':

$$\begin{aligned} R' &= e^{-2c\omega}(R - 2(n-1)c\partial^i(\partial_i(\omega)) - (n-1)(n-2)c^2\partial^j(\omega)\partial_j(\omega) + c(2n-2)\partial_\gamma(\omega)g^{ij}\Gamma^\gamma_{ji}) \\ &= e^{-2c\omega}\left(R - 2(n-1)c\underbrace{\left(\partial^i(\partial_i(\omega)) - \partial_\gamma(\omega)g^{ij}\Gamma^\gamma_{ji}\right)}_{=g^{ij}(\nabla_j\nabla_i\omega)=\nabla^2\omega} - (n-1)(n-2)c^2(\partial\omega)^2\right) \end{aligned} \tag{A.20}$$

where we identified the covariant derivative $\nabla_\mu v_\nu = \partial_\mu v_\nu - \Gamma^\rho_{\mu\nu}v_\rho$ and introduced the rather sloppy notation $(\partial\omega)^2 := \partial^j(\omega)\partial_j(\omega)$. Thus, our final result for the transformed Ricci scalar under a Weyl transformation $g'_{\mu\nu} = e^{2c\omega(x)}g_{\mu\nu}$ is given by

$$R' = e^{-2c\omega}\left(R - 2(n-1)c\nabla^2\omega - (n-1)(n-2)c^2(\partial\omega)^2\right). \tag{A.21}$$

APPENDIX B

Computations

We calculate how the field strengths transform in section 3.3.

Transformation of H

It is

$$H = \mathrm{d}B \quad \Longrightarrow \quad H_{\mu mn} = \partial_\mu b(x) J_{mn}, \; H_{mnl} = b(x) 2\tilde{m} (\mathrm{Re}\Omega)_{mnl}. \tag{B.1}$$

With that, we calculate how H transforms under the Weyl rescalings:

$$\begin{aligned}
H' &= G'^{M_1N_1} G'^{M_2N_2} G'^{M_3N_3} H_{M_1M_2M_3} H_{N_1N_2N_3} \\
&\overset{(1)}{=} e^{-6\beta\varphi} \tilde{G}^{M_1N_1} \tilde{G}^{M_2N_2} \tilde{G}^{M_3N_3} H_{M_1M_2M_3} H_{N_1N_2N_3} \\
&\overset{(2)}{=} e^{-6\beta\varphi} \tilde{G}^{M_1N_1} g^{m_2n_2} g^{m_3n_3} H_{M_1m_2m_3} H_{N_1n_2n_3} \\
&\overset{(3)}{=} e^{-6\beta\varphi} e^{2(\alpha-\beta)\varphi} g^{\mu\nu} g^{m_2n_2} g^{m_3n_3} H_{\mu m_2m_3} H_{\nu n_2n_3} + e^{-6\beta\varphi} g^{m_1n_1} g^{m_2n_2} g^{m_3n_3} H_{m_1m_2m_3} H_{n_1n_2n_3} \\
&\overset{(4)}{=} e^{(2\alpha-8\beta)\varphi} g^{\mu\nu} g^{m_2n_2} g^{m_3n_3} \partial_\mu b(x) J_{m_2m_3} \partial_\nu b(x) J_{n_2n_3} \\
&\quad + e^{-6\beta\varphi} g^{m_1n_1} g^{m_2n_2} g^{m_3n_3} (b(x))^2 (2\tilde{m})^2 (\mathrm{Re}\Omega)_{m_1m_2m_3} (\mathrm{Re}\Omega)_{n_1n_2n_3} \\
&\overset{(5)}{=} 2!3 e^{(2\alpha-8\beta)\varphi} g^{\mu\nu} \partial_\mu b(x) \partial_\nu b(x) + 3!4 e^{-6\beta\varphi} (b(x))^2 (2\tilde{m})^2 \\
&= 6 e^{(2\alpha-8\beta)\varphi} (\partial b)^2 + 96 e^{-6\beta\varphi} \tilde{m}^2 b^2.
\end{aligned} \tag{B.2}$$

Here, we used for (1) that $\tilde{G}^{MN} = e^{2\beta\varphi} G'^{MN}$, for (2) that the last two indices of H are internal ones, for (3) we divided the expression in two parts for M, N either internal or external, for (4) we used (B.1), and finally for (5) we used the relations (3.64).

Transformation of F_2

In (3.65) we set

$$F_2 = Q_0 B \quad \Longrightarrow \quad F_{mn} = Q_0 b(x) J_{mn}. \tag{B.3}$$

With that we can transform as follows:

$$\begin{aligned}
|F_2'|^2 &= \frac{1}{2!} G'^{M_1N_1} G'^{M_2N_2} F_{M_1M_2} F_{N_1N_2} \\
&\overset{(1)}{=} \frac{1}{2!} e^{-4\beta\varphi} \tilde{G}^{M_1N_1} \tilde{G}^{M_2N_2} F_{M_1M_2} F_{N_1N_2} \\
&\overset{(2)}{=} \frac{1}{2!} e^{-4\beta\varphi} g^{m_1n_1} g^{m_2n_2} F_{m_1m_2} F_{n_1n_2} \\
&\overset{(3)}{=} \frac{1}{2!} e^{-4\beta\varphi} g^{m_1n_1} g^{m_2n_2} Q_0^2 (b(x))^2 J_{m_1m_2} J_{n_1n_2} \\
&\overset{(4)}{=} 3e^{-4\beta\varphi} Q_0^2 (b(x))^2
\end{aligned} \tag{B.4}$$

where we used for (1) that $\tilde{G}^{MN} = e^{2\beta\varphi} G'^{MN}$, for (2) that F_2 only depends on the internal coordinates, for (3) the definition (3.65) for the components of F_2, and for (4) the contraction (3.62) of J.

Transformation of F_4

The components of F_4 were defined as

$$F_{\mu mnl} = \partial_\mu \zeta (\mathrm{Im}\Omega)_{mnl}, \; F_{klmn} = (Q_0 b^2 - \frac{8\tilde{m}}{3}\zeta) \epsilon_{klmnop} J^{op} \tag{B.5}$$

(compare (3.67)). It follows

$$\begin{aligned}
|F_4'|^2 &= \frac{1}{4!} G'^{M_1N_1} G'^{M_2N_2} G'^{M_3N_3} G'^{M_4N_4} F_{M_1M_2M_3M_4} F_{N_1N_2N_3N_4} \\
&\overset{(1)}{=} \frac{1}{4!} e^{-8\beta\varphi} \tilde{G}^{M_1N_1} \tilde{G}^{M_2N_2} \tilde{G}^{M_3N_3} \tilde{G}^{M_4N_4} F_{M_1M_2M_3M_4} F_{N_1N_2N_3N_4} \\
&\overset{(2)}{=} \frac{1}{4!} e^{-8\beta\varphi} \tilde{G}^{M_1N_1} g^{m_2n_2} g^{m_3n_3} g^{m_4n_4} F_{M_1m_2m_3m_4} F_{N_1n_2n_3n_4} \\
&\overset{(3)}{=} \frac{1}{4!} e^{-8\beta\varphi} e^{2(\alpha-\beta)\varphi} g^{\mu\nu} g^{m_2n_2} g^{m_3n_3} g^{m_4n_4} F_{\mu m_2m_3m_4} F_{\nu n_2n_3n_4} \\
&\quad + \frac{1}{4!} e^{-8\beta\varphi} g^{m_1n_1} g^{m_2n_2} g^{m_3n_3} g^{m_4n_4} F_{m_1m_2m_3m_4} F_{n_1n_2n_3n_4} \\
&\overset{(4)}{=} \frac{1}{4!} e^{(2\alpha-10\beta)\varphi} g^{\mu\nu} g^{m_2n_2} g^{m_3n_3} g^{m_4n_4} \partial_\mu \zeta (\mathrm{Im}\Omega)_{m_2m_3m_4} \partial_\nu \zeta (\mathrm{Im}\Omega)_{n_2n_3n_4} \\
&\quad + \frac{1}{4!} e^{-8\beta\varphi} g^{m_1n_1} g^{m_2n_2} g^{m_3n_3} g^{m_4n_4} (Q_0 b^2 - \frac{8\tilde{m}}{3}\zeta)^2 \epsilon_{m_1m_2m_3m_4o_1p_1} J^{o_1p_1} \epsilon_{n_1n_2n_3n_4o_2p_2} J^{o_2p_2} \\
&\overset{(5)}{=} e^{(2\alpha-10\beta)\varphi} g^{\mu\nu} \partial_\mu \zeta \partial_\nu \zeta + \frac{1}{4!} e^{-8\beta\varphi} (Q_0 b^2 - \frac{8\tilde{m}}{3}\zeta)^2 \underbrace{\epsilon^{n_1n_2n_3n_4q_1r_1} \epsilon_{n_1n_2n_3n_4o_2p_2}}_{=4!(\delta^{q_1}_{o_2}\delta^{r_1}_{p_2} - \delta^{q_1}_{p_2}\delta^{r_1}_{o_2})} g_{o_1q_1} g_{p_1r_1} \\
&\quad J_{s_1t_1} g^{o_1s_1} g^{p_1t_1} J_{s_2t_2} g^{o_2s_2} g^{p_2t_2}
\end{aligned}$$

$$= e^{(2\alpha-10\beta)\varphi}(\partial\zeta)^2 + e^{-8\beta\varphi}(Q_0 b^2 - \frac{8\tilde{m}}{3}\zeta)^2(\delta^{q_1}_{o_2}\delta^{r_1}_{p_2} - \delta^{q_1}_{p_2}\delta^{r_1}_{o_2})\delta^{s_1}_{q_1}\delta^{t_1}_{r_1}g^{o_2 s_2}g^{p_2 t_2}J_{s_1 t_1}J_{s_2 t_2}$$
$$= e^{(2\alpha-10\beta)\varphi}(\partial\zeta)^2 + e^{-8\beta\varphi}(Q_0 b^2 - \frac{8\tilde{m}}{3}\zeta)^2(g^{q_1 s_2}g^{r_1 t_1} - g^{r_1 s_2}g^{q_1 t_2})J_{q_1 r_1}J_{s_2 t_2}$$
$$\overset{(6)}{=} e^{(2\alpha-10\beta)\varphi}(\partial\zeta)^2 + 12e^{-8\beta\varphi}(Q_0 b^2 - \frac{8\tilde{m}}{3}\zeta)^2$$

where we used $\tilde{G}^{MN} = e^{2\beta\varphi}G'^{MN}$ for (1), for (2) that the last three indices of F_4 are internal ones, for (3) that $\tilde{G}^{MN} = e^{2(\alpha-\beta)\varphi}g^{\mu\nu} + g^{mn}$, for (4) we used (3.67), for (5) we used (3.64) and rewrote the coloured expressions. Finally, for (6) we used (3.62) and the fact that J is antisymmetric. Using $\alpha = -3\beta$, we find

$$|F'_4|^2 = e^{-16\beta\varphi}(\partial\zeta)^2 + 12e^{-8\beta\varphi}(Q_0 b^2 - \frac{8\tilde{m}}{3}\zeta)^2 \tag{B.6}$$

Transformation of F_6

F_6's components are given by

$$F_{klmnop} = (Q_6 - b^3 Q_0 + 8\tilde{m}\zeta b)\epsilon_{klmnop} \tag{B.7}$$

(compare (3.69)). Then

$$\begin{aligned}
|F'_6|^2 &= \frac{1}{6!}G'^{M_1 N_1}\dots G'^{M_6 N_6}F_{M_1\dots M_6}F_{N_1\dots N_6} \\
&\overset{(1)}{=} \frac{1}{6!}e^{-12\beta\varphi}\tilde{G}^{M_1 N_1}\dots\tilde{G}^{M_6 N_6}F_{M_1\dots M_6}F_{N_1\dots N_6} \\
&\overset{(2)}{=} \frac{1}{6!}e^{-12\beta\varphi}g^{m_1 n_1}\dots g^{m_6 n_6}F_{m_1\dots m_6}F_{n_1\dots n_6} \\
&\overset{(3)}{=} \frac{1}{6!}e^{-12\beta\varphi}g^{m_1 n_1}\dots g^{m_6 n_6}(Q_6 - b^3 Q_0 + 8\tilde{m}\zeta b)^2\epsilon_{m_1\dots m_6}\epsilon_{n_1\dots n_6} \\
&= \frac{1}{6!}e^{-12\beta\varphi}(Q_6 - b^3 Q_0 + 8\tilde{m}\zeta b)^2 \underbrace{\epsilon^{n_1\dots n_6}\epsilon_{n_1\dots n_6}}_{=6!} \\
&= e^{-12\beta\varphi}(Q_6 - b^3 Q_0 + 8\tilde{m}\zeta b)^2.
\end{aligned} \tag{B.8}$$

We again used $\tilde{G}^{MN} = e^{2\beta\varphi}G'^{MN}$ for (1), we used for (2) the fact that F_6 only depends on the internal coordinates, and we used (3.69) for (3).

Action

Putting all terms together and using

$$\begin{aligned}
S_F &= -\frac{1}{2\kappa_{10}^2}\int d^{10}z\sqrt{-G_{10}}\frac{1}{2}e^{\frac{5-n}{2}\phi}|F'_n|^2 \\
&= -\frac{1}{4}\int d^4x\sqrt{-g_4}e^{(4\alpha-6\beta)\varphi}e^{\frac{5-n}{2}\phi}|F'_n|^2
\end{aligned} \tag{B.9}$$

where we work in Planck units in the four-dimensional theory, i.e. $\kappa_4 = 1$, we get the full action:

$$\begin{aligned}
S &= \frac{1}{2\kappa_{10}^2}\Bigg\{\int d^{10}z\sqrt{-G'_{10}}\Bigg(R'_{10} - \frac{1}{2}(\partial\phi)'^2 - \frac{1}{2\cdot 3!}e^{-\phi}|H'|^2 - \frac{1}{2}\sum_{n\text{ even}} e^{\frac{5-n}{2}\phi}|F'_n|^2\Bigg) \\
&\quad + \frac{1}{2\kappa_{10}^2}\int B\wedge F_4\wedge F_4\Bigg\} \\
&= \frac{1}{2\kappa_{10}^2}\Bigg\{\int d^{10}z\sqrt{-G'_{10}}\Bigg(R'_{10} - \frac{1}{2}(\partial\phi)'^2 - \frac{1}{2\cdot 3!}e^{-\phi}|H'|^2 - \frac{1}{2}e^{\frac{5}{2}\phi}|F'_0|^2 - \frac{1}{2}e^{\frac{3}{2}\phi}|F'_2|^2 \\
&\quad - \frac{1}{2}e^{\frac{\phi}{2}}|F'_4|^2 - \frac{1}{2}e^{-\frac{\phi}{2}}|F'_6|^2\Bigg) + \frac{1}{2\kappa_{10}^2}\int B\wedge F_4\wedge F_4\Bigg\} \\
&= \frac{1}{2}\int d^4x\sqrt{-g_4}\Big(R_4 - \frac{1}{2}((\partial\varphi)^2 + (\partial\phi)^2) + e^{-8\beta\varphi}R_6 - \frac{1}{12}e^{-\phi}(6e^{-20\beta\varphi}(\partial b(x))^2 \\
&\quad + 96e^{-12\beta\varphi}\tilde{m}^2(b(x))^2) - \frac{Q_0^2}{2}e^{-6\beta\varphi}e^{\frac{5}{2}\phi} - \frac{3}{2}e^{\frac{3}{2}\phi}e^{-10\beta\varphi}Q_0^2(b(x))^2 - \frac{1}{2}e^{\frac{\phi}{2}}\Big(e^{-22\beta\varphi}(\partial\zeta)^2 \\
&\quad + 12e^{-14\beta\varphi}(Q_0b^2 - \frac{8\tilde{m}}{3}\zeta)^2\Big) - \frac{1}{2}e^{-\frac{\phi}{2}}e^{-18\beta\varphi}(Q_6 - b^3Q_0 + 8\tilde{m}\zeta b)^2\Big) \\
&= \frac{1}{2}\int d^4x\sqrt{-g_4}\Big(R_4 - \frac{1}{2}((\partial\varphi)^2 + (\partial\phi)^2) - \frac{1}{2}e^{-\phi}e^{-20\beta\varphi}(\partial b(x))^2 \\
&\quad - \frac{1}{2}e^{\frac{\phi}{2}}e^{-22\beta\varphi}(\partial\zeta)^2 - V\Big).
\end{aligned} \tag{B.10}$$

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