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Asymptotes in $SU(2)$ Recoupling Theory]Asymptotes in $SU(2)$ Recoupling Theory:
Wigner Matrices, $3j$ Symbols, and Character Localization

J. Ben Geloun]Joseph Ben Geloun
R. Gurau]Razvan Gurau

Abstract In this paper, we employ a technique combining the Euler Maclaurin formula with the saddle point approximation method to obtain the asymptotic behavior (in the limit of large representation index J) of generic Wigner matrix elements $D^J_{MM'}(g)$. We use this result to derive asymptotic formulae for the character $\chi^J(g)$ of an $SU(2)$ group element and for Wigner's $3j$ symbol. Surprisingly, given that we perform five successive layers of approximations, the asymptotic formula we obtain for $\chi^J(g)$ is in fact *exact*. The result hints at a "Duistermaat-Heckman like" localization property for discrete sums.

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1 Introduction

The saddle point approximation (SPA) is a classical algorithm to determine asymptotic behavior of a large class of integrals in some large parameter limit (1). One uses it when exact calculations are either too complex or not very relevant. Recently, SPA has been used in conjunction with the Euler Maclaurin (EM) formula to derive asymptotic behavior of discrete sums (2; 3). In the combined EM SPA scheme corrections to the leading behavior come from two sources: the derivative terms in the EM formula and sub-leading terms in the SPA estimate.

It is worthwhile emphasizing that similar approximation methods can be traced back for years (4) and such methods have led to more or less accurate results depending on the oscillatory character of the summand. As pointed out in (4) (see from page 358 for a review), one of the best way to convert discrete sums to integrals in semiclassical cases, is the Poisson summation formula. For instance, Braun et al. (5) discussed the semiclassical approximation of the Floquet operator (which is a composition of a rotation and then a torsion around the z axis) in a stroboscopic period-to-period dynamics that in return possesses an application in the asymptotic of the small Wigner d-matrix element. They were able to prove also using the Poisson summation formula that SPA asymptotes of the $SU(2)$ character turns out to be exact.

The semiclassical analysis of a Wigner matrix element has been performed in many different ways (see (6; 7; 8) and also (9) for a recent review and the geometric perspective attached to it). One the first contribution on this analysis is may be the work by Brussaard et al. (6). Therein, relations of Clebsch-Gordan

Joseph Ben Geloun · Razvan Gurau Perimeter Institute for Theoretical Physics 31, Caroline N. Waterloo, ON N2L 2Y5, Canada jbengeloun@perimeterinstitute.ca; rgurau@perimeterinsitute.ca Joseph Ben Geloun International Chair in Mathematical Physics and Applications (ICMPA–UNESCO Chair) University of Abomey-Calavi 072 B.P. 50, Cotonou, Republic of Benin

and Racah coefficients for large angular momenta are derived. Classical analogues of the square of Clebsch-Gordan coefficients and the square of the little Wigner matrix element are suggested based on their geometrical meaning. In (8), again a combined Poisson sum formula and the stationary phase method have been used to perform the semiclassical approximation for a reduced rotation matrix element expressed in terms of a classical generating function.

In this paper, we use a slightly different EM SPA (using deformation contours exploring the complex plane) method to derive the asymptotic behavior of Wigner rotation matrix elements. We subsequently use this asymptotic formula to derive the asymptotic behavior of the character of an $SU(2)$ group element. Although our estimate is obtained after using twice the EM SPA approximation and once the Stirling approximation for Euler's Gamma functions it turns out to be the exact result. We then proceed to obtain the asymptotic expression for Wigner's $3j$ symbol, recovering with this method the results of (10).

Both our results and method are relevant for computing topological (Turaev Viro like (11)) invariants and in connection to the volume conjecture (12). From a theoretical physics perspective, they are of consequence for spin foam models (13), group field theory (14; 15), discretized BF theory and lattice gravity (16; 17; 18). Continuous SPA has been extensively used in this context to derive asymptotic behaviors of spin foam amplitudes (19; 20; 21) and (22; 23; 24).

In the recoupling theory of $SU(2)$, the EM SPA method has already been used to obtain in a particularly simple way the Ponzano-Regge asymptotic of the $6j$ symbol (3; 25). The main strength of this approach is the following: most relevant quantities in the recoupling theory of $SU(2)$ are expressed in Fourier space by discrete sums. In particular, the Wigner matrix elements admit a single sum representation (26). However, generically, the sums are alternated, hence it is difficult to handle. Our EM SPA method deals very efficiently with alternating signs: generically such signs lead to complex saddle points situated outside the initial summation interval. After exchanging the original sums (via the EM formula) for integrals, only one deforms the integration contour in the complex plane to pass trough the saddle points in a completely standard manner. This feature is the crucial strength of our method, and allows rapid access to explicit results. The EM SPA method should allow one to prove for instance the asymptotic behavior (27) of the $9j$ symbol.

The proofs of our three main results (Theorems 1, 2 and 3) are straightforward, but the shear amount of computations performed renders this a somewhat technical paper. In Sect. 2, we give a quick review of iterated saddle point approximations. In Sect. 3, we establish Theorem 1 and use it in Sect. 4 to derive the character formula (Theorem 2). Section 5 proves the asymptotic formulae of the $3j$ symbol (Theorem 3). Section 6 draws the conclusion of our work and roughly discusses a possible connection between our result for the character and the Duistermaat Heckman theorem. The (very detailed) Appendices present explicit computations and detail the EM derivative terms.

2 Successive Saddle Point Approximations

We briefly review the iterated SPA approximations. The result of this section justifies the use of our asymptote of the Wigner matrices to derive the asymptotic behavior of $SU(2)$ characters and Wigner $3j$ symbols.

Consider a function f of two real variables. We are interested in evaluating the asymptotic behavior of the integral

$$I = \int \mathrm{d}u\mathrm{d}x\, e^{Jf(u,x)}, \tag{1}$$

for large J. One can chose to either evaluate I via an SPA in both variables at the same time or via two successive SPAs, one for each variable. The question is if the two estimates coincide. This problem is addressed in full detail in (1) and the answer to the above question is yes (for sufficiently smooth functions), with known estimates. Let us give a quick flavor of the origin of this result.

Remark 1 Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a function with an unique critical point (u_c, x_c) and non degenerate Hessian at (u_c, x_c) such that $I = \int \mathrm{d}u\mathrm{d}x e^{Jf(u,x)}$ admits a SPA at large J. Assume that the equation $\partial_u f(u,x) = 0$ admits an unique solution $u_c = h(x)$, such that $[\partial_u^2 f](h(x),x) \neq 0$. Then, the SPA of $\int \mathrm{d}u\mathrm{d}x e^{Jf(u,x)}$ in both variables (u,x) gives the same estimate as two successive SPAs, the first one in u and the second one in x.

Proof The simultaneous SPA in u and x yields the estimate

$$I \approx \frac{2\pi}{J\sqrt{[\partial_u^2 f \partial_x^2 f - (\partial_u \partial_x f)^2] \mid_{(u_c,x_c)}}} e^{J f(u_c,x_c)}. \tag{2}$$

The saddle point equation for $u, [\partial_u f](u,x) = 0$, is solved by $u_c = h(x)$. Thus, a first SPA in u gives

$$I \approx \sqrt{\frac{2\pi}{J}} \int \mathrm{d}x \frac{1}{\sqrt{-\partial_u^2 f \mid_{(h(x),x)}}} e^{J f(h(x),x)}. \tag{3}$$

We evaluate Eq. (3) by a second SPA, in the x variable. The saddle point equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f\left(h(x),x\right)\right) = [\partial_u f] \mid_{(h(x),x)} \frac{\mathrm{d}h}{\mathrm{d}x} + [\partial_x f] \mid_{(h(x),x)}, \tag{4}$$

and, as $[\partial_u f](h(x),x) = 0$, the first term above vanishes. The critical point x_c is therefore a solution of $[\partial_x f] \mid_{(h(x),x)} = 0$. The second derivative of $f(h(x),x)$ computes to

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(f(h(x),x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left([\partial_x f]\Big|_{(h(x),x)}\right) \\ &= [\partial_u \partial_x f]\Big|_{(h(x),x)} \frac{\mathrm{d}h}{\mathrm{d}x} + [\partial_x^2 f]\Big|_{(h(x),x)},\end{aligned} \tag{5}$$

and noting that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\partial_u f]\Big|_{(h(x),x)} = 0 \Rightarrow [\partial_u^2 f]\Big|_{(h(x),x)} \frac{\mathrm{d}h}{\mathrm{d}x} + \partial_x[\partial_u f]\Big|_{(h(x),x)} = 0 \Rightarrow \frac{\mathrm{d}h}{\mathrm{d}x} = -\frac{[\partial_x \partial_u f]}{[\partial_u^2 f]}\Big|_{(h(x),x)}, \tag{6}$$

the estimate obtained by two successive SPAs is

$$I \approx \frac{2\pi}{J\sqrt{\partial_u^2 f|_{(h(x_c),x_c)} \left(-\frac{[\partial_u \partial_x f]^2}{[\partial_u^2 f]} + \partial_x^2 f\right)|_{(h(x_c),x_c)}}} e^{J f(u_c,x_c)}, \tag{7}$$

identical with Eq. (2).

This remark can be generalized (1), for sufficiently smooth functions of more variables with non-degenerate critical points. In the sequel, we will express the Wigner matrix elements $D^J_{MM'}$ (up to corrections coming from the EM formula) as integrals which we approximate by a first SPA. To compute more involved sums or integrals of products of such matrix elements (the character of an $SU(2)$ group element and the $3j$ symbol) we will substitute the SPA approximation for each $D^J_{MM'}$ and evaluate the resulting expressions by subsequent SPAs.

3 Asymptotic Formula of a Wigner Matrix Element

In this section, we prove an asymptotic formula for a Wigner matrix element. Before proceeding, let us mention that many of our results are expressed in terms of angles. We will always denote them as $\imath\phi = \ln w$ for some complex number w with $|w| = 1$ without mentioning at this formal level which of the logarithm branches is used. For numerical evaluations, one could take the principal branch of the logarithm function.

Our starting point is the classical expression of $D^J_{MM'}$ in terms of Euler angles (α,β,γ) in $z\,y\,z$ order (see (26))

$$D^J_{MM'}(\alpha,\beta,\gamma) = e^{-\imath\alpha M} e^{-\imath\gamma M'} \sum_t (-)^t \frac{\sqrt{(J+M)!(J-M)!(J+M')!(J-M')!}}{(J+M-t)!(J-M'-t)!t!(t-M+M')!} \times \xi^{2J+M-M'-2t} \eta^{2t-M+M'}, \tag{8}$$

with $\xi = \cos(\beta/2)$, $\eta = \sin(\beta/2)$. The sum is taken over all t such that all factorials have positive argument (hence it has $1+\min\{J+M, J-M, J+M', J-M'\}$ terms). We call a Wigner matrix generic if its second Euler angle $\beta \notin \mathbb{Z}\pi$ (that is $0 < \xi^2 < 1$). We define the reduced variables $x = \frac{J}{M}$ and $y = \frac{J}{M'}$. A priori the asymptotic behavior we derive below holds in certain region of the parameters x, y and ξ detailed in Appendices E and C.

Theorem 1 *A generic Wigner matrix element in the spin J representation of an SU(2) group element has in the large J limit the asymptotic behavior*

$$
\begin{aligned}
&D^J_{xJ,yJ}(\alpha,\beta,\gamma)\\
&\quad\approx e^{-\imath J\alpha x-\imath J\gamma y}\left(\frac{1}{\pi J\sqrt{\Delta}}\right)^{\frac{1}{2}}\cos\left[\left(J+\frac{1}{2}\right)\phi + xJ\psi - yJ\omega - \frac{\pi}{4}\right],
\end{aligned}
\tag{9}
$$

with

$$
\Delta = (1-\xi^2)(\xi^2 - xy) - \frac{(x-y)^2}{4} \geq 0,
\tag{10}
$$

with ϕ, ψ and ω the three angles

$$
\begin{aligned}
\imath\phi = \ln\frac{2\xi^2 - 1 - xy + 2\imath\sqrt{\Delta}}{\sqrt{(1-x^2)(1-y^2)}}\,, \quad \imath\psi = \ln\frac{\frac{x+y}{2} - x\xi^2 + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-x^2)}}\,,\\
\imath\omega = \ln\frac{-\frac{x+y}{2} + y\xi^2 + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-y^2)}}.
\end{aligned}
\tag{11}
$$

Proof The proof of Theorem 1 is divided into two steps: first the approximation of Eq. (8) by an integral via the EM formula, and second the evaluation of the latter by an SPA.

Step 1: In the large J limit, the leading behavior of the Wigner matrix element Eq. (8) is

$$
D^J_{xJ,yJ}(\alpha,\beta,\gamma) \approx \frac{1}{2\pi}\int \mathrm{d}u\,\sqrt{K(x,y,u)}\,e^{Jf(x,y,u)},
\tag{12}
$$

where

$$
\begin{aligned}
f(x,y,u) = {}& -\imath\alpha x - \imath\gamma y + \imath\pi u + (2+x-y-2u)\ln\xi + (2u-x+y)\ln\eta\\
&+\frac{1}{2}(1-x)\ln(1-x) + \frac{1}{2}(1+x)\ln(1+x)\\
&+\frac{1}{2}(1-y)\ln(1-y) + \frac{1}{2}(1+y)\ln(1+y)\\
&-(1+x-u)\ln(1+x-u) - (1-y-u)\ln(1-y-u)\\
&-u\ln u - (u-x+y)\ln(u-x+y),
\end{aligned}
\tag{13}
$$

and

$$
K(x,y,u) = \frac{\sqrt{(1-x)(1+x)(1-y)(1+y)}}{(1+x-u)(1-y-u)(u)(u-x+y)}.
\tag{14}
$$

To prove this we rewrite Eq. (8) in terms of Gamma functions

$$
\begin{aligned}
D^J_{MM'}(\alpha,\beta,\gamma) &= \sum_t F(J,M,M',t), \\
F(J,M,M',t) &= e^{\imath\pi t} e^{-\imath\alpha M} e^{-\imath\gamma M'} \xi^{2J+M-M'-2t} \eta^{2t-M+M'} \\
&\quad \times \frac{\sqrt{\Gamma(J+M+1)\Gamma(J-M+1)\Gamma(J+M'+1)\Gamma(J-M'+1)}}{\Gamma(J+M-t+1)\Gamma(J-M'-t+1)\Gamma(t+1)\Gamma(t-M+M'+1)},
\end{aligned}
\tag{15}
$$

and use the Euler–Maclaurin formula

$$
\begin{aligned}
\sum_{t_{\min}}^{t_{\max}} h(t) &= \int_{t_{\min}}^{t_{\max}} h(t)\mathrm{d}t - B_1\left[h(t_{\max}) + h(t_{\min})\right] \\
&\quad + \sum_k \frac{B_{2k}}{(2k)!}\left[h^{(2k-1)}(t_{\max}) - h^{(2k-1)}(t_{\min})\right],
\end{aligned}
\tag{16}
$$

where B_1, B_{2k} are the Bernoulli numbers.[1] To derive our asymptote, we only take into consideration the integral approximation of Eq. (15) (the boundary terms are discussed in Appendix E), hence

$$
D^J_{MM'}(\alpha,\beta,\gamma) \approx \int \mathrm{d}t\; F(J,M,M',t). \tag{17}
$$

We define $u = \frac{t}{J}$ hence $\mathrm{d}u = \frac{1}{J}\mathrm{d}t$ and using the Stirling formula for the Gamma functions (see Appendix A) we get Eq. (12).

Step 2: We now proceed to evaluate the integral (12) by an SPA. Some of the computations relevant for this proof are included in Appendix B. Denoting the set of saddle points by $\mathscr{C}$, the leading asymptotic behavior of a generic Wigner matrix element can be written

$$
D^J_{xJ,yJ}(\alpha,\beta,\gamma) \approx \frac{1}{\sqrt{2\pi J}} \sum_{u_*\in\mathscr{C}} \frac{\sqrt{K|_{x,y,u_*}}}{\sqrt{(-\partial_u^2 f)|_{x,y,u_*}}}\, e^{Jf(x,y,u_*)}. \tag{18}
$$

Our task is to identify $\mathscr{C}$ and to calculate $K|_{x,y,u_*}, (-\partial_u^2 f)|_{x,y,u_*}$ and $f(x,y,u_*)$.

The set $\mathscr{C}$. The derivative of f with respect to u is

$$
\partial_u f = \imath\pi - 2\ln\xi + 2\ln\eta + \ln(1+x-u) + \ln(1-y-u) - \ln u - \ln(u-x+y). \tag{19}
$$

A straightforward computation shows that the saddle points are the solutions of

$$
(1+x-u)(1-y-u)\tfrac{(1-\xi^2)}{\xi^2} + u(u-x+y) = 0 \tag{20}
$$

$$
\Leftrightarrow u^2 - u[2(1-\xi^2)+x-y] + (1-\xi^2)(1+x)(1-y) = 0. \tag{21}
$$

The region of parameters x, y, ξ for which the discriminant of Eq. (21) is positive gives exponentially suppressed matrix elements, while the region for which it is zero gives an Airy function estimate. Both cases are detailed in Appendix C.

[1] Equation (16) holds for all C^∞ functions $h(t)$, such that the sum over k converges.

In the rest of this proof, we treat the region in which the discriminant of Eq. (21) is negative. We denote by Δ minus the reduced discriminant, that is

$$\Delta = (1-\xi^2)(\xi^2 - xy) - \frac{(x-y)^2}{4} > 0, \tag{22}$$

and the two saddle points, solutions of Eq. (21), can be written as

$$u_\pm = (1-\xi^2) + \frac{x-y}{2} \pm \imath\sqrt{\Delta}, \tag{23}$$

thus the set of saddle points is $\mathscr{C} = \{u_+, u_-\}$.

Evaluation of $f(x,y,u_\pm)$. We rearrange the terms in Eq. (13) and then write

$$\begin{aligned} f(x,y,u) &= -\imath\alpha x - \imath\gamma y + (2+x-y)\ln\xi + (-x+y)\ln\eta \\ &+\frac{1}{2}(1-x)\ln(1-x) + \frac{1}{2}(1+x)\ln(1+x) \\ &+\frac{1}{2}(1-y)\ln(1-y) + \frac{1}{2}(1+y)\ln(1+y) \\ &-(1+x)\ln(1+x-u) - (1-y)\ln(1-y-u) - (-x+y)\ln(u-x+y) \\ &+u\ln\left[(-)\frac{1-\xi^2}{\xi^2}\,\frac{(1+x-u)(1-y-u)}{u(u-x+y)}\right]. \end{aligned} \tag{24}$$

Note that by the saddle point equations the last line in Eq. (24) is zero for $u_\pm$. The rest of Eq. (24) can be worked out to (see Appendix B.1 for details)

$$f(x,y,u_\pm) = -\imath\alpha x - \imath\gamma y \pm \imath\,(\phi + x\psi - y\omega), \tag{25}$$

with

$$\imath\phi = \ln\frac{2\xi^2 - 1 - xy + 2\imath\sqrt{\Delta}}{\sqrt{(1-x^2)(1-y^2)}}, \quad \imath\psi = \ln\frac{\frac{x+y}{2} - x\xi^2 + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-x^2)}},$$
$$\imath\omega = \ln\frac{-\frac{x+y}{2} + y\xi^2 + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-y^2)}}. \tag{26}$$

Second derivative. The derivative of Eq. (19) is

$$-\partial_u^2 f(x,y,u) = \frac{1}{1+x-u} + \frac{1}{1-y-u} + \frac{1}{u} + \frac{1}{u-x+y}. \tag{27}$$

At the saddle points, a straightforward computation shows that (see Appendix B.2)

$$(-\partial_u^2 f)\Big|_{x,y,u_\pm} = \frac{1}{(1-x^2)(1-y^2)\xi^2(1-\xi^2)}\Big(4\Delta \pm \imath 2\sqrt{\Delta}\left[1+xy-2\xi^2\right]\Big). \tag{28}$$

The prefactor K. The prefactor $K(x,y,u)$ is given by

$$K = \frac{\sqrt{(1-x^2)(1-y^2)}}{u(1+x-u)(1-y-u)(u-x+y)}, \tag{29}$$

which can be calculated at the saddle points to (see Appendix B.3)

$$K\,|_{x,y,u_\pm} = \frac{-\sqrt{(1-x^2)(1-y^2)}\left(2\xi^2-1-xy\pm 2\imath\sqrt{\Delta}\right)^2}{\xi^2(1-\xi^2)(1-x^2)^2(1-y^2)^2}. \tag{30}$$

Final evaluation. Before collecting all our previous results we first evaluate, using Eqs. (28) and (30)

$$\begin{aligned}\frac{K|_{x,y,u_\pm}}{(-\partial_u^2 f)|_{x,y,u_\pm}} &= -\frac{\left(2\xi^2-1-xy\pm 2\imath\sqrt{\Delta}\right)^2}{\sqrt{(1-x^2)(1-y^2)}\left(4\Delta \pm \imath 2\sqrt{\Delta}\,[1+xy-2\xi^2]\right)}\\ &= \frac{1}{\sqrt{(1-x^2)(1-y^2)}(\pm 2\imath\sqrt{\Delta})}\left(2\xi^2-1-xy\pm 2\imath\sqrt{\Delta}\right)\\ &= \frac{1}{\pm\imath 2\sqrt{\Delta}}\frac{\left(2\xi^2-1-xy\pm 2\imath\sqrt{\Delta}\right)}{\sqrt{(1-x^2)(1-y^2)}}.\end{aligned} \tag{31}$$

When comparing Eqs. (31) with (11), it can be inferred that

$$\frac{K|_{x,y,u_\pm}}{(-\partial_u^2 f)|_{x,y,u_\pm}} = \frac{1}{\pm\imath 2\sqrt{\Delta}} e^{\pm\imath\phi}. \tag{32}$$

Substituting Eqs. (32) and (25) into Eq. (18), we obtain

$$\begin{aligned}D^J_{xJ,yJ}(\alpha,\beta,\gamma) \approx \frac{1}{\sqrt{2\pi J}}\left(\frac{1}{2\sqrt{\Delta}}\right)^{\frac{1}{2}} e^{-\imath J\alpha x-\imath J\gamma y}\\ \times\left(\sqrt{\frac{1}{\imath}e^{\imath\phi}}\; e^{\imath J(\phi+x\psi-y\omega)}+\sqrt{\frac{1}{-\imath}e^{-\imath\phi}}\; e^{-\imath J(\phi+x\psi-y\omega)}\right),\end{aligned} \tag{33}$$

and a straightforward computation proves Theorem 1.

4 Characters

In this section, we use Theorem 1 to derive an asymptotic formula for the character of an $SU(2)$ group element.

Theorem 2 *The leading asymptotic behavior of the character of an $SU(2)$ group element (with Euler angles (α,β,γ)) in the J representation, $\chi^J(\alpha,\beta,\gamma)$ is*

$$\chi^J(\alpha,\beta,\gamma) \approx \frac{\sin\left[\left(J+\frac{1}{2}\right)\theta\right]}{\sin\frac{\theta}{2}}, \tag{34}$$

with θ defined by

$$\cos\frac{\theta}{2} = \cos\frac{\beta}{2}\cos\frac{(\alpha+\gamma)}{2}. \tag{35}$$

Let us emphasize that up to this point we already performed three different approximations: first the EM approximation, second the Stirling approximation and third the SPA approximation. To prove Theorem 2, we will use a second EM approximation and a second SPA approximation. However, formula (35) is exactly the classical relation between the Euler angle parametrization and the $\theta, \mathbf{n}$ parametrization of an $SU(2)$ group element, thus the leading behavior we find (after five levels of approximation) is in fact the exact formula of the character! We will discuss this rather surprising result in Sect. 6.

Proof of Theorem 2 To establish Theorem 2, we follow again the EM SPA recipe. The character χ^J of a group element can be written

$$\chi^J(\alpha,\beta,\gamma) = \sum_{M=-J}^{J} D^J_{MM}(\alpha,\beta,\gamma) = \sum_{x=-1}^{1} D^J_{xJ,xJ}(\alpha,\beta,\gamma), \tag{36}$$

with $x = \frac{M}{J}$ the re-scaled variable. Note that the step in the second sum is $\mathrm{d}x = \frac{1}{J}$. The leading EM approximation (see end of Appendix E) for the character is, therefore, the continuous integral (dropping henceforth the arguments (α,β,γ))

$$\chi^J \approx J \int_{-1}^{1} \mathrm{d}x\, D^J_{xJ,xJ}. \tag{37}$$

We now use Theorem 1 (more precisely Eq. (33)) and write a diagonal Wigner matrix element as

$$D^J_{xJ,xJ} \approx \left[\frac{1}{4\pi J\sqrt{\Delta}}\right]^{\frac{1}{2}} \left[\sqrt{\frac{e^{\imath\phi}}{\imath}}\, e^{Jf(x,x,u_+)} + \sqrt{\frac{e^{-\imath\phi}}{-\imath}}\, e^{Jf(x,x,u_-)}\right]. \tag{38}$$

Note that for diagonal matrix elements the exponents can be further simplified such that

$$f(x,x,u_\pm) = -\imath(\alpha+\gamma)x \pm \imath\,(\phi + x(\psi-\omega)), \tag{39}$$

while the discriminant Δ and angles ϕ, ψ and ω from Eq. (11) become

$$\imath\phi = \ln\frac{2\xi^2 - 1 - x^2 + 2\imath\sqrt{\Delta}}{(1-x^2)}, \quad \imath\psi = \ln\frac{x(1-\xi^2) + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-x^2)}}, \tag{40}$$

$$\imath\omega = \ln\frac{-x(1-\xi^2) + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-x^2)}}\,, \quad \Delta = (1-\xi^2)(\xi^2 - x^2). \tag{41}$$

We follow the same steps as in the proof of Theorem 1.

Critical set $\mathscr{C}_\chi$. The derivatives of the exponents for each of the two terms in Eq. (38) are

$$\partial_x f(x,x,u_\pm) = -\imath(\alpha+\gamma) \pm \imath(\psi-\omega) \pm \imath\partial_x\phi \pm \imath x\partial_x(\psi-\omega). \tag{42}$$

The derivative of ϕ is given by

$$\imath\partial_x\phi = \partial_x\left[\ln(\sqrt{\xi^2-x^2}+\imath\sqrt{1-\xi^2})^2-\ln(1-x^2)\right] = \imath\frac{2x\sqrt{1-\xi^2}}{(1-x^2)\sqrt{\xi^2-x^2}}. \tag{43}$$

The difference $\psi-\omega$ can be recast as

$$\imath(\psi-\omega) = \ln\frac{x(1-\xi^2)+\imath\sqrt{\Delta}}{-x(1-\xi^2)+\imath\sqrt{\Delta}} = \ln\frac{\left(\sqrt{\xi^2-x^2}-\imath x\sqrt{1-\xi^2}\right)^2}{\xi^2(1-x^2)}, \tag{44}$$

so that its derivative is expressed as follows:

$$\imath\partial_x(\psi-\omega) = 2\frac{\frac{-x}{\sqrt{\xi^2-x^2}}-\imath\sqrt{1-\xi^2}}{\sqrt{\xi^2-x^2}-\imath x\sqrt{1-\xi^2}}-\frac{-2x}{1-x^2} = \imath\frac{-2\sqrt{1-\xi^2}}{(1-x^2)\sqrt{\xi^2-x^2}}. \tag{45}$$

When combining Eqs. (43) and (45), we have

$$\partial_x\phi + x\partial_x(\psi-\omega) = 0, \tag{46}$$

and therefore simplify the saddle point Eq. (42) as

$$\psi-\omega = \pm(\alpha+\gamma). \tag{47}$$

Dividing by 2 and exponentiating, the following holds:

$$\frac{\sqrt{\xi^2-x^2}-\imath x\sqrt{1-\xi^2}}{\sqrt{\xi^2(1-x^2)}} = e^{\pm\imath\frac{\alpha+\gamma}{2}} \Rightarrow \frac{x\sqrt{1-\xi^2}}{\sqrt{\xi^2-x^2}} = \mp\tan\frac{\alpha+\gamma}{2}. \tag{48}$$

Hence, the saddle points are solutions of the quadratic equation

$$x^2(1-\xi^2) = (\xi^2-x^2)\tan^2\frac{\alpha+\gamma}{2} \Rightarrow x^2 = \frac{\xi^2\sin^2\frac{\alpha+\gamma}{2}}{1-\xi^2\cos^2\frac{\alpha+\gamma}{2}}. \tag{49}$$

Defining a new variable θ via the relation $\cos\frac{\theta}{2} = \xi\cos\frac{\alpha+\gamma}{2}$, the saddle points can be rewritten

$$x^2 = \frac{\xi^2\sin^2\frac{\alpha+\gamma}{2}}{\sin^2\frac{\theta}{2}}. \tag{50}$$

Taking into consideration Eq. (48), one identifies an unique saddle point (x_1) for $f(x,x,u_+)$ and an unique saddle point (x_2) for $f(x,x,u_-)$ with x_1 and x_2 given by

$$x_1 = -\frac{\xi\sin\frac{\alpha+\gamma}{2}}{\sin\frac{\theta}{2}}, \quad x_2 = \frac{\xi\sin\frac{\alpha+\gamma}{2}}{\sin\frac{\theta}{2}}. \tag{51}$$

Evaluation of the functions and Hessian on $\mathscr{C}_\chi$. Straightforward computations lead to

$$\begin{aligned} \xi^2 - x_{1,2}^2 &= (1-\xi^2)\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}, \\ \Delta|_{x_{1,2}} &= (1-\xi^2)^2\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} \geq 0, \\ 1-x_{1,2}^2 &= \frac{(1-\xi^2)}{\sin^2\frac{\theta}{2}}. \end{aligned} \tag{52}$$

Also note that at the saddle points, the angle ϕ can be simplified further to

$$\begin{aligned} \imath\phi &= \ln\frac{2\xi^2 - 1 - x_{1,2}^2 + 2\imath\sqrt{\Delta|_{x_{1,2}}}}{(1-x_{1,2}^2)} \\ &= \ln\frac{(1-\xi^2)\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} - (1-\xi^2) + 2\imath(1-\xi^2)\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}}{\frac{(1-\xi^2)}{\sin^2\frac{\theta}{2}}} \\ &= \ln\left[\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} + \imath 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right] = \ln e^{i\theta} = i\theta. \end{aligned} \tag{53}$$

Substituting the saddle point Eqs. (47) into (39), we see that, at the saddles

$$f(x_1,x_1,u_+) = \imath\phi = \imath\theta, \quad f(x_2,x_2,u_-) = -\imath\phi = -\imath\theta. \tag{54}$$

To evaluate the Hessian at the saddle, we first simplify Eqs. (42) using (46) hence

$$\partial_x^2 f(x,x,u_\pm) = \pm\imath\partial_x(\psi-\omega) = \mp 2\imath\frac{\sqrt{1-\xi^2}}{(1-x^2)\sqrt{\xi^2-x^2}} \tag{55}$$

which becomes at the saddle points

$$\mp 2\imath\frac{\sqrt{1-\xi^2}}{\frac{(1-\xi^2)}{\sin^2\frac{\theta}{2}}\sqrt{(1-\xi^2)}\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}} = \mp 2\imath\frac{1}{1-\xi^2}\frac{\sin^3\frac{\theta}{2}}{\cos\frac{\theta}{2}}. \tag{56}$$

Final evaluation. Using Eqs. (54) and (56), the SPA of the character Eq. (37) is

$$\chi^J \approx \frac{1}{\sqrt{2(1-\xi^2)\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}}}\left(\sqrt{\frac{e^{\imath\theta}}{\imath}}\frac{e^{\imath J\theta}}{\sqrt{\imath\frac{2}{1-\xi^2}\frac{\sin^3\frac{\theta}{2}}{\cos\frac{\theta}{2}}}} + \sqrt{\frac{e^{-\imath\theta}}{-\imath}}\frac{e^{-\imath J\theta}}{\sqrt{-\imath\frac{2}{1-\xi^2}\frac{\sin^3\frac{\theta}{2}}{\cos\frac{\theta}{2}}}}\right), \tag{57}$$

which is

$$\chi^J \approx \frac{1}{2\sin\frac{\theta}{2}}\left(\frac{1}{\imath}e^{\imath(J+\frac{1}{2})\theta} + \frac{1}{-\imath}e^{-\imath(J+\frac{1}{2})\theta}\right) = \frac{\sin\left[(J+\frac{1}{2})\theta\right]}{\sin\frac{\theta}{2}}. \tag{58}$$

5 Asymptotes of $3j$ Symbols

In this section, we employ the asymptotic formula for the Wigner matrices to obtain an asymptotic formula for Wigner's $3j$ symbol. Note that one can use directly the EM SPA method to derive this asymptotic starting from the single sum representation of the $3j$ symbol (26). We take here the alternative route of using the results of Theorem 1 and the representation of $3j$ symbols in terms of Wigner matrices

$$\int \mathrm{d}g\, D^{J_1}_{M_1M_1'}(g) D^{J_1}_{M_2M_2'}(g) D^{J_3}_{M_3M_3'}(g) \\ = \begin{pmatrix} J_1 & J_2 & J_3 \\ M_1 & M_2 & M_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3 \\ M_1' & M_2' & M_3' \end{pmatrix}, \tag{59}$$

where the integral is taken over $SU(2)$ with the normalized Haar measure

$$\int \mathrm{d}g := \frac{1}{8\pi^2} \int_0^{2\pi} \mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\gamma \int_0^{\pi} d\beta \sin\beta. \tag{60}$$

Substituting the asymptote (9) for each matrix element $D^{J_i}_{M_iM_i'}(g)$ $(i = 1,2,3)$, the main contribution to the integral (59) is

$$\int \mathrm{d}g \left(\frac{1}{4\pi J_1\sqrt{\Delta_1}}\right)^{1/2} \left(\frac{1}{4\pi J_2\sqrt{\Delta_2}}\right)^{1/2} \left(\frac{1}{4\pi J_3\sqrt{\Delta_3}}\right)^{1/2} \\ \times \prod_{i=1}^{3} \sum_{s_i=\pm 1} e^{-\imath J_i(\alpha+\gamma)} \frac{1}{\sqrt{s_i \imath}} e^{\imath s_i\left(\frac{\phi_i}{2} + J_i(\phi_i + x_i\psi_i - y_i\omega_i)\right)}. \tag{61}$$

We expand (61), perform the integration over α and γ and change variables from β to ξ such that

$$\frac{1}{2}\int_0^{\pi} \sin\beta \mathrm{d}\beta = \frac{1}{2}\int_0^{\pi} 2\sin\frac{\beta}{2}\cos\frac{\beta}{2}\mathrm{d}\beta = 2\int_0^1 \xi \mathrm{d}\xi = \int_0^1 \mathrm{d}(\xi^2), \tag{62}$$

to rewrite it as

$$\delta_{\sum_i J_i x_i,0}\, \delta_{\sum_i J_i y_i,0} \left[\int_0^1 \mathrm{d}(\xi^2)\right] \left(\frac{1}{(4\pi)^3 \prod_i J_i \sqrt{\prod_i \Delta_i}}\right)^{\frac{1}{2}} \\ \times \sum_{s_i=\pm 1} \frac{1}{\sqrt{\prod_i s_i \imath^3}} e^{\imath \sum_i s_i\left(\frac{\phi_i}{2} + f_i\right)}, \tag{63}$$

where the index i runs from 1 to 3, $\delta_{\sum_i J_i x_i,0}$ is a Kronecker symbols and

$$f_i = J_i\,[\phi_i + x_i\psi_i - y_i\omega_i]. \tag{64}$$

We will derive the asymptotic behavior of Eq. (63) via an SPA with respect to ξ^2. Note that Eq. (59) involves two distinct $3j$ symbols. If one attempts to first set $M_i' = M_i$, and obtain a representation of the square of a single $3j$ symbol, one encounters a very serious technical problem. We will see in the sequel that there

Fig. 1 Angular momentum vectors

are two saddle points $\xi_\pm^2$ contributing to the asymptotic behavior of Eq. (63). If one starts by setting $M_i = M_i'$, one of the two saddle points $\xi_+^2 = 1$, and the second derivative in ξ_+^2 diverges. The contribution of this saddle point cannot be worked out by a simple Gaussian integration.

The SPA evaluation of the general case, Eq. (63), is a very lengthy computation. We will perform it using the classical angular momentum vectors. For large representation index J_i, there exists a classical angular momentum vector $\mathbf{J}_i$ in $\mathbb{R}^3$ of length $|\mathbf{J}_i| = J_i$ and projection on the Oz axis (of unit vector $\mathbf{n}$) $\mathbf{n} \cdot \mathbf{J}_i = M_i$. A $3j$ symbol is then associated to three vectors, $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$ with $|\mathbf{J}_i| = J_i$ and $\mathbf{n} \cdot \mathbf{J}_i = M_i = x_i J_i$. By the selection rules, the quantum numbers J_i respect the triangle inequalities, and $M_1 + M_2 + M_3 = 0$. This translates into the condition that the vectors $\mathbf{J}_i$ form a triangle $\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 = 0$ (and $\mathbf{n} \cdot [\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3] = 0$). The asymptotic behavior of the $3j$ symbol can be written in terms of the angular momentum vectors as given by the following statement:

Theorem 3 *For large representation indices J_i the $3j$ symbol has the asymptotic behavior*

$$\begin{pmatrix} J_1 & J_2 & J_3 \\ M_1 & M_2 & M_3 \end{pmatrix} = \frac{1}{\sqrt{\pi(\mathbf{n}\cdot\mathbf{S})}} \cos\left[\sum_i \left(J_i + \frac{1}{2}\right)\Phi_{\mathbf{n}}^i + (\mathbf{n}\cdot\mathbf{J}_1)\Psi_{\mathbf{n}}^{13} + (\mathbf{n}\cdot\mathbf{J}_2)\Psi_{\mathbf{n}}^{23} + \frac{\pi}{4}\right], \qquad (65)$$

with $\mathbf{S} = \mathbf{J}_1 \wedge \mathbf{J}_2 = \mathbf{J}_2 \wedge \mathbf{J}_3 = \mathbf{J}_3 \wedge \mathbf{J}_1$, twice the area of the triangle $\{\mathbf{J}_i\}$ and $\Phi_{\mathbf{n}}^i, \Psi_{\mathbf{n}}^{13}$ and $\Psi_{\mathbf{n}}^{23}$ five angles defined as

$$\begin{aligned} \imath\Phi_{\mathbf{n}}^i &= \ln \frac{\mathbf{n}\cdot(\mathbf{J}_i \wedge \mathbf{S}) + \imath J_i(\mathbf{n}\cdot\mathbf{S})}{S\sqrt{(\mathbf{n}\wedge\mathbf{J}_i)^2}}, \\ \imath\Psi_{\mathbf{n}}^{i3} &= \ln \frac{(\mathbf{n}\wedge\mathbf{J}_i)\cdot(\mathbf{n}\wedge\mathbf{J}_3) + \imath\mathbf{n}\cdot(\mathbf{J}_3\wedge\mathbf{J}_i)}{\sqrt{(\mathbf{n}\wedge\mathbf{J}_i)^2(\mathbf{n}\wedge\mathbf{J}_3)^2}}, \quad i = 1,2. \end{aligned} \qquad (66)$$

Before proceeding with the proof of Theorem 3, note that our starting Eq. (59) involves two distinct $3j$ symbols. They are each associated to a triple of vectors, $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$ ($|\mathbf{J}_i| = J_i$ and $\mathbf{n}\cdot\mathbf{J}_i = x_i J_i$) and $\mathbf{J}_1', \mathbf{J}_2', \mathbf{J}_3'$ ($|\mathbf{J}_i'| = J_i, \mathbf{n}\cdot\mathbf{J}_i' = y_i J_i$). Remarking that $|\mathbf{J}_i| = |\mathbf{J}_i'|$, the two triangles $\{\mathbf{J}_i\}$ and $\{\mathbf{J}_i'\}$ are congruent. Consequently there exists a rotation which overlaps them. Under this rotation the normal vector $\mathbf{n}$ turns into the unit vector $\mathbf{k}$. All the geometrical information can therefore be encoded into an *unique* triple of vectors, henceforth denoted $\mathbf{J}_i$, and the *two* unit vectors $\mathbf{n}$ and $\mathbf{k}$ such that $|\mathbf{J}_i| = J_i, \mathbf{n}\cdot\mathbf{J}_i = x_i J_i$ and $\mathbf{k}\cdot\mathbf{J}_i = y_i J_i$ (see Fig. 1).

Proof of Theorem 3 The proof follows the, by now familiar, routine of an SPA. We perform this evaluation at fixed angular momenta, i.e. at the fixed set of vectors $\mathbf{J}_i, \mathbf{n}, \mathbf{k}$.

The dominant saddle points. The saddle points governing the asymptotic behavior of Eq. (63) are solutions of the equation

$$0 = \partial_{(\xi^2)} \sum s_i(\imath f_i) = \imath \sum_i s_i J_i [\partial_{(\xi^2)} \phi_i + x_i \partial_{(\xi^2)} \psi_i - y_i \partial_{(\xi^2)} \omega_i)]. \tag{67}$$

A straightforward computation (see Appendix D.1) yields

$$\partial_{(\xi^2)} \sum s_i(\imath f_i) = -\frac{\imath}{\xi^2(1-\xi^2)} \sum_i s_i J_i \sqrt{\Delta_i}, \tag{68}$$

hence the saddle point equation is of the form

$$0 = s_1 J_1 \sqrt{\Delta_1} + s_2 J_2 \sqrt{\Delta_2} + s_3 J_3 \sqrt{\Delta_3}. \tag{69}$$

Introducing the angular momentum vectors, the saddle point equation becomes after a short calculation (see Appendix D.2)

$$\begin{aligned} &4\xi^4 S^2 - 4\xi^2 \left\{ S^2 + (\mathbf{n}\cdot\mathbf{k}) S^2 - (\mathbf{n}\cdot\mathbf{S})(\mathbf{k}\cdot\mathbf{S}) \right\} \\ &\quad + \Big\{ [1+(\mathbf{n}\cdot\mathbf{k})]^2 S^2 - 2(\mathbf{n}\cdot\mathbf{S})(\mathbf{k}\cdot\mathbf{S}) [1+(\mathbf{n}\cdot\mathbf{k})] \Big\} = 0, \end{aligned} \tag{70}$$

for all choices of signs s_1, s_2 and s_3. Dividing by $4S^2$, Eq. (70) can be factorized as

$$\left[\xi^2 - \frac{1+(\mathbf{n}\cdot\mathbf{k})}{2} \right] \left[\xi^2 - \left(\frac{1+(\mathbf{n}\cdot\mathbf{k})}{2} - \frac{(\mathbf{n}\cdot\mathbf{S})(\mathbf{k}\cdot\mathbf{S})}{S^2} \right) \right] = 0, \tag{71}$$

with roots,

$$\xi_+^2 = \frac{1+(\mathbf{n}\cdot\mathbf{k})}{2}, \quad \xi_-^2 = \frac{1+(\mathbf{n}\cdot\mathbf{k})}{2} - \frac{(\mathbf{n}\cdot\mathbf{S})(\mathbf{k}\cdot\mathbf{S})}{S^2}, \tag{72}$$

again independent of the signs s_1, s_2 and s_3. To identify the terms contributing to the asymptotic of Eq. (63) for fixed $\mathbf{J}_i, \mathbf{n}$ and $\mathbf{k}$ one needs to evaluate $J_i \sqrt{\Delta_i}$ for each of the two roots ξ_+^2 and ξ_-^2. Using Appendix D.3, we have

$$\begin{aligned} J_i^2 \Delta_i^+ &= \frac{1}{4} [\mathbf{J}_i \cdot (\mathbf{n}\wedge\mathbf{k})]^2 , \\ J_i^2 \Delta_i^- &= \frac{1}{4} \frac{\{\mathbf{J}_i \cdot [(\mathbf{S}\wedge\mathbf{n})(\mathbf{k}\cdot\mathbf{S}) + (\mathbf{S}\wedge\mathbf{k})(\mathbf{n}\cdot\mathbf{S})]\}^2}{S^4}. \end{aligned} \tag{73}$$

To any semiclassical state $\mathbf{J}_i, \mathbf{n}, \mathbf{k}$, we associate six signs, ε_i^+ and ε_i^- defined by

$$\begin{aligned} J_i \sqrt{\Delta_i^+} &= \varepsilon_i^+ \frac{1}{2} \mathbf{J}_i \cdot (\mathbf{n}\wedge\mathbf{k}), \\ J_i \sqrt{\Delta_i^-} &= \varepsilon_i^- \frac{1}{2} \frac{\mathbf{J}_i \cdot [(\mathbf{S}\wedge\mathbf{n})(\mathbf{k}\cdot\mathbf{S}) + (\mathbf{S}\wedge\mathbf{k})(\mathbf{n}\cdot\mathbf{S})]}{S^2}. \end{aligned} \tag{74}$$

Substituting $J_i\sqrt{\Delta_i^{\pm}}$ into the saddle point Eq. (69), the latter becomes

$$\frac{1}{2}\left(\sum_i s_i\varepsilon_i^{\pm}\mathbf{J}_i\right)\cdot\mathbf{A}^{\pm}, \tag{75}$$

with $\mathbf{A}^{+}=(\mathbf{n}\wedge\mathbf{k})$ and $\mathbf{A}^{-}=\frac{[(\mathbf{S}\wedge\mathbf{n})(\mathbf{k}\cdot\mathbf{S})+(\mathbf{S}\wedge\mathbf{k})(\mathbf{n}\cdot\mathbf{S})]}{S^2}$. As, on the other hand, $\sum_i\mathbf{J}_i=0$, we conclude that at fixed a semiclassical state we have two saddle points ξ_+^2 and two saddle points ξ_-^2 contributing

- The ξ_+^2 saddle point in the term $s_i=\varepsilon_i^{+}$ and that in the term $s_i=-\varepsilon_i^{+}$
- The ξ_-^2 saddle point in the term $s_i=\varepsilon_i^{-}$ and that in the term $s_i=-\varepsilon_i^{-}$

The SPA evaluation of Eq. (63) is the sum of these four contributions.

The second derivative. The derivative of Eq. (68) with respect to ξ^2 yields

$$\begin{aligned}\partial_{(\xi^2)}[\partial_{(\xi^2)}\sum_i s_i(\imath f_i)] = -\imath\partial_{(\xi^2)}\left(\frac{1}{\xi^2(1-\xi^2)}\right)\sum_i s_iJ_i\sqrt{\Delta_i}\\ -\frac{\imath}{\xi^2(1-\xi^2)}\sum_i s_iJ_i\frac{-(2\xi^2-1-x_iy_i)}{2\sqrt{\Delta_i}},\end{aligned} \tag{76}$$

and the term in the first line cancels (due to the saddle point equation) when evaluating the second derivative at the critical points. After Gaussian integration of the dominant saddle point contributions, the prefactor in the SPA approximation of Eq. (63) can be written

$$\frac{1}{\sqrt{K}},\quad K=32\,\pi^2 s_1s_2s_3\imath^3J_1J_2J_3\sqrt{\Delta_1\Delta_2\Delta_3}\left(-\partial^2_{(\xi^2)}\sum_i s_i(\imath f_i)\right). \tag{77}$$

The remainder of this paragraph is devoted to the evaluation of K for the two roots ξ_+^2 and ξ_-^2. Substituting the second derivative gives

$$K^{\pm}=-16\pi^2 s_1s_2s_3\imath^4\frac{J_1J_2J_3}{\xi_\pm^2(1-\xi_\pm^2)}\sum_i s_iJ_i\frac{\sqrt{\Delta_1\Delta_2\Delta_3}}{\sqrt{\Delta_i}}(2\xi_\pm^2-1-x_iy_i). \tag{78}$$

Taking into consideration $s_1^2s_2s_3=\varepsilon_2^{\pm}\varepsilon_3^{\pm}, K^{\pm}$ can be expressed as

$$K^{\pm}=-(16\pi^2)\frac{\left[\varepsilon_2^{\pm}\varepsilon_3^{\pm}J_2\sqrt{\Delta_2^{\pm}}J_3\sqrt{\Delta_3^{\pm}}\left[(2\xi_\pm^2-1)J_1^2-J_1^{\mathbf{n}}J_1^{\mathbf{k}}\right]+\circlearrowleft_{123}\right]}{\xi_\pm^2(1-\xi_\pm^2)}, \tag{79}$$

where $\circlearrowleft_{123}$ denotes circular permutations on the indices $1,2$ and 3. Using Eq. (72), the denominator evaluates to, for the ξ_+^2 root,

$$\xi_+^2(1-\xi_+^2)=\frac{1-(\mathbf{n}\cdot\mathbf{k})^2}{4}, \tag{80}$$

while the numerator can be computed to (see Appendix D.4 for detailed computations and notations)

$$\varepsilon_2^+\varepsilon_3^+ J_2\sqrt{\Delta_2^+}J_3\sqrt{\Delta_3^+}\left[(2\xi_+^2-1)J_1^2-J_1^{\mathbf{n}}J_1^{\mathbf{k}}\right]+\circlearrowleft_{123}=-\frac{1}{4}S^{\mathbf{n}}S^{\mathbf{k}}(\mathbf{n}\wedge\mathbf{k})^2, \tag{81}$$

hence

$$K^+ = 16\pi^2 S^{\mathbf{n}}S^{\mathbf{k}}. \tag{82}$$

Evaluating the denominator in Eq. (79) for ξ_-^2, we obtain

$$\begin{aligned}\xi_-^2(1-\xi_-^2) &= \left(\frac{1+(\mathbf{n}\cdot\mathbf{k})}{2}-\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right)\left(\frac{1-(\mathbf{n}\cdot\mathbf{k})}{2}+\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right)\\ &= \frac{1}{4}\left\{(1-(\mathbf{n}\cdot\mathbf{k})^2+4(\mathbf{n}\cdot\mathbf{k})\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}-4\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^4}\right\},\end{aligned} \tag{83}$$

while a lengthy computation (see Appendix D.4) shows that the numerator is

$$\begin{aligned}&\varepsilon_2^-\varepsilon_3^- J_2\sqrt{\Delta_2^-}J_3\sqrt{\Delta_3^-}\left[(2\xi_-^2-1)J_1^2-J_1^{\mathbf{n}}J_1^{\mathbf{k}}\right]+\circlearrowleft_{123}\\ &=\frac{1}{4}S^{\mathbf{n}}S^{\mathbf{k}}\left\{1-(\mathbf{n}\cdot\mathbf{k})^2+4(\mathbf{n}\cdot\mathbf{k})\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}-4\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^4}\right\},\end{aligned} \tag{84}$$

proving that

$$K^- = -16\pi^2 S^{\mathbf{n}}S^{\mathbf{k}}. \tag{85}$$

Contribution of each saddle. To evaluate the contribution of each saddle point to the asymptote of Eq. (63), we first evaluate

$$\imath\sum_i s_i\left[\frac{\phi_i}{2}+f_i\right]=\sum_i s_i\left[\left(J_i+\frac{1}{2}\right)(\imath\phi_i^{\pm})+x_iJ_i(\imath\psi_i^{\pm})-y_iJ_i(\imath\omega_i^{\pm})\right]. \tag{86}$$

Recall that for a fixed semiclassical state only the terms with s_i equal to $\varepsilon_i^+, -\varepsilon_i^+, \varepsilon_i^-$ and $-\varepsilon_i^-$ contribute. We substitute $x_3J_3=-x_2J_2-x_1J_1$ and $y_3J_3=-y_1J_1-y_2J_2$ into Eq. (86) to bring it into the form

$$\pm\left\{\sum_i\left(J_i+\frac{1}{2}\right)(\imath\varepsilon_i^{\pm}\phi_i^{\pm})+x_1J_1(\imath\varepsilon_1^{\pm}\psi_1^{\pm}-\imath\varepsilon_3^{\pm}\psi_3^{\pm})+x_2J_2(\imath\varepsilon_2^{\pm}\psi_2^{\pm}-\imath\varepsilon_3^{\pm}\psi_3^{\pm})\right.$$

$$\left.-y_1J_1(\imath\varepsilon_1^{\pm}\omega_1^{\pm}-\imath\varepsilon_3^{\pm}\omega_3^{\pm})-y_2J_2(\imath\varepsilon_2^{\pm}\omega_2^{\pm}-\imath\omega_3^{\pm}\psi_3^{\pm})\right\}, \tag{87}$$

where $\phi_i^{\pm}$, $\psi_i^{\pm}$ and $\omega_i^{\pm}$ are the angles ϕ_i, ψ_i and ω_i evaluated at ξ_+^2 and ξ_-^2. For each choice $+$ or $-$ in the accolades, one must count *both* choices of the overall sign. The angles $\phi_i^{\pm}, \varepsilon_1^{\pm}\psi_1^{\pm}-\varepsilon_3^{\pm}\psi_3^{\pm}$, etc. are evaluated by a rather involved computation in Appendix D.5. The end results are synthesized below

$$\imath\varepsilon_i^{\pm}\phi_i^{\pm}=\imath\Phi_{\mathbf{n}}^i\mp\imath\Phi_{\mathbf{k}}^i,\quad \imath\Phi_{\mathbf{n}}^i=\ln\frac{\mathbf{n}\cdot(\mathbf{J}_i\wedge\mathbf{S})+\imath J_iS^{\mathbf{n}}}{S\sqrt{(\mathbf{n}\wedge\mathbf{J}_i)^2}}$$

$$\imath\varepsilon_j^{\pm}\psi_j^{\pm}-\imath\varepsilon_3^{\pm}\psi_3^{\pm}=\imath\Psi_{\mathbf{n}}^{j3},\quad \imath\Psi_{\mathbf{n}}^{j3}=\ln\frac{(\mathbf{n}\wedge\mathbf{J}_j)\cdot(\mathbf{n}\wedge\mathbf{J}_3)+\imath\mathbf{n}\cdot(\mathbf{J}_3\wedge\mathbf{J}_j)}{\sqrt{(\mathbf{n}\wedge\mathbf{J}_j)^2(\mathbf{n}\wedge\mathbf{J}_3)^2}},$$

$$j=1,2,$$

$$\imath\varepsilon_j^{\pm}\omega_j^{\pm}-\imath\varepsilon_3^{\pm}\omega_3^{\pm}=\pm\imath\Psi_{\mathbf{k}}^{j3}. \tag{88}$$

It is a now matter of substitution of Eqs. (88) into (87) to get

$$\pm\left\{\sum_i\left(J_i+\frac{1}{2}\right)\left(\imath\Phi_{\mathbf{n}}^i\mp\imath\Phi_{\mathbf{k}}^i\right)+(\mathbf{n}\cdot\mathbf{J}_1)\imath\Psi_{\mathbf{n}}^{13}+(\mathbf{n}\cdot\mathbf{J}_2)\imath\Psi_{\mathbf{n}}^{23}\right.$$

$$\left.\mp(\mathbf{k}\cdot\mathbf{J}_1)\imath\Psi_{\mathbf{k}}^{13}\mp(\mathbf{k}\cdot\mathbf{J}_2)\imath\Psi_{\mathbf{k}}^{23}\right\}=\pm(\Omega_{\mathbf{n}}\mp\Omega_{\mathbf{k}}), \tag{89}$$

where $\Omega_{\mathbf{n}}$ denotes

$$\imath\Omega_{\mathbf{n}}=\sum_i\left(J_i+\frac{1}{2}\right)\imath\Phi_{\mathbf{n}}^i+(\mathbf{n}\cdot\mathbf{J}_1)\imath\Psi_{\mathbf{n}}^{13}+(\mathbf{n}\cdot\mathbf{J}_2)\imath\Psi_{\mathbf{n}}^{23}. \tag{90}$$

Final evaluation. We put together Eqs. (82), (85) and (89) and, noting that the two contributions from the saddle ξ_-^2 are complex conjugate to one another, we obtain

$$\begin{pmatrix} J_1 & J_2 & J_3\\ M_1 & M_2 & M_3\end{pmatrix}\begin{pmatrix} J_1 & J_2 & J_3\\ M_1' & M_2' & M_3'\end{pmatrix}$$
$$\approx\frac{1}{\sqrt{\pi(\mathbf{n}\cdot\mathbf{S})}}\frac{1}{\sqrt{\pi(\mathbf{k}\cdot\mathbf{S})}}\frac{1}{4}\left(e^{\imath(\Omega_{\mathbf{n}}-\Omega_{\mathbf{k}})}+e^{-\imath(\Omega_{\mathbf{n}}-\Omega_{\mathbf{k}})}+\imath e^{\imath(\Omega_{\mathbf{n}}+\Omega_{\mathbf{k}})}-\imath e^{-\imath(\Omega_{\mathbf{n}}+\Omega_{\mathbf{k}})}\right). \tag{91}$$

Taking into consideration

$$\frac{1}{4}\left(e^{\imath(\Omega_{\mathbf{n}}-\Omega_{\mathbf{k}})}+e^{-\imath(\Omega_{\mathbf{n}}-\Omega_{\mathbf{k}})}+\imath e^{\imath(\Omega_{\mathbf{n}}+\Omega_{\mathbf{k}})}-\imath e^{-\imath(\Omega_{\mathbf{n}}+\Omega_{\mathbf{k}})}\right)$$
$$=\cos\left(\Omega_{\mathbf{n}}+\frac{\pi}{4}\right)\cos\left(\Omega_{\mathbf{k}}+\frac{\pi}{4}\right), \tag{92}$$

Theorem 3 follows.

6 Conclusion

Using the EM SPA method, we have determined the asymptotic behaviors at large spin J of Wigner matrix elements, Wigner $3j$ symbols and the character $\chi^J(g)$ of an $SU(2)$ group element g.

By far the most surprising fact about this computation is that our formula for the character $\chi^J(g)$ is exact. SPA reproducing the exact result for integrals are usually the consequence of a Duistermaat Heckman (28; 29; 30) localization property (one of the most famous example of this being the Harish Chandra Itzykson Zuber integral (31)). Recall that the Duistermaat–Heckman theorem states that a phase space integral

$$\int \Omega \; e^{-\imath H(p,q)}, \tag{93}$$

where Ω is the Liouville form, equals its leading order SPA estimation if the flow of the Hamiltonian vector field $\mathbf{X}$ ($i_{\mathbf{X}}\Omega = dH$) is $U(1)$. To our knowledge, all integrals exhibiting a localization property (i.e. equaling their leading order SPA approximation) fall in (some generalization of) this case. A standard example is the integration of the height function on the sphere (29; 30) which turns out to be the exact sum of the evaluation of the function on the north and south pole which are indeed the extrema the height function.

Note that the character of an $SU(2)$ group element can be expressed directly as a double integral by

$$\chi^J(g) = \sum_{M,t} e^{h(J,M,t)} \approx \frac{J}{2\pi} \int \mathrm{d}u \mathrm{d}x \, \sqrt{K(x,x,u)} e^{Jf(x,x,u)} + \text{E.M.} + \text{S.}, \tag{94}$$

where E.M. denotes corrections coming from the Euler–Maclaurin approximation, and S the corrections coming from sub leading terms in the Stirling approximation. The double integral in Eq. (94) is of the correct form, with symplectic form $\Omega = \sqrt{K(x,x,u)} dx \wedge du$ and Hamiltonian $f(x,x,u)$ generating the Hamiltonian flow

$$\frac{\mathrm{d}u}{\mathrm{d}\rho} = \sqrt{\frac{u^2(1+x-u)(1-x-u)}{1-x^2}} \ln\left\{ e^{-\imath(\alpha+\gamma)} \frac{(1+x)(1-x-u)}{(1-x)(1+x-u)} \right\} \tag{95}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\rho} = -\sqrt{\frac{u^2(1+x-u)(1-x-u)}{1-x^2}} \ln\left\{ e^{\imath\pi} \frac{(1-\xi^2)}{\xi^2} \frac{(1-x-u)(1+x-u)}{u^2} \right\}. \tag{96}$$

Our result can be explained if first, the above flow is $U(1)$ (thus the SPA of the double integral is exact) and second the EM and Stirling correction terms cancel, E.M. + S. = 0. The alternative, namely that the flow is not $U(1)$ would require an even more subtle cancellation of the sub leading correction terms. Either way, the exact result for the character we derive in this paper deserves further investigation.

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Appendix

In these appendices, we detail various technical points and computations.

A The Stirling Approximation

We provide here details on the passage from Eq. (17) to (12). Our starting point is

$$D^J_{MM'}(\alpha,\beta,\gamma) \approx \int \mathrm{d}t\, F(J,M,M',t), \tag{A.1}$$

with

$$\begin{aligned} F(J,M,M',t) &= e^{\imath\pi t} e^{-\imath\alpha M} e^{-\imath\gamma M'} \xi^{2J+M-M'-2t} \eta^{2t-M+M'} \\ &\times \frac{\sqrt{\Gamma(J+M+1)\Gamma(J-M+1)\Gamma(J+M'+1)\Gamma(J-M'+1)}}{\Gamma(J+M-t+1)\Gamma(J-M'-t+1)\Gamma(t+1)\Gamma(t-M+M'+1)}. \end{aligned} \tag{A.2}$$

We use the Stirling formula

$$\Gamma(n+1) = n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n = \sqrt{2\pi n}\, e^{n\ln n - n}, \tag{A.3}$$

for all Γ functions and re-scaled variables $M = xJ, M' = yJ, t = uJ$. Collecting all prefactors, we end up with

$$\left(\frac{\sqrt{(2\pi)^4 J^4 (1+x)(1-x)(1+y)(1-y)}}{(2\pi)^4 J^4 (1+x-u)(1-y-u)(u)(u-x+y)}\right)^{\frac{1}{2}} = \frac{1}{2\pi J}\sqrt{K(x,y,u)}, \tag{A.4}$$

and $K(x,y,u)$ takes the form as in Eq. (14). The "-n" terms in the Stirling approximation add to

$$\begin{aligned} &\frac{1}{2}\{-J(1+x) - J(1-x) - J(1+y) - J(1-y)\} \\ &\quad -\{-J(1+x-u) - J(1-y-u) - Ju - J(u-x+y)\} = 0, \end{aligned} \tag{A.5}$$

which also implies that the coefficient of $\ln J$ in the exponent cancels. The contribution of the Γ functions Eq. (A.2) is therefore

$$\begin{aligned} &\frac{J}{2}\{(1+x)\ln(1+x) + (1-x)\ln(1-x) + (1+y)\ln(1+y) + (1-y)\ln(1-y)\} \\ &\quad -J\{(1+x-u)\ln(1+x-u) + (1-y-u)\ln(1-y-u) \\ &\quad + u\ln(u) + (u-x+y)\ln(u-x+y)\}. \end{aligned} \tag{A.6}$$

The substitution of Eqs. (A.6) into (A.2) yields

$$F(J,xJ,yJ,uJ) \approx \frac{1}{2\pi J}\sqrt{K(x,y,u)}\, e^{Jf(x,y,u)}, \tag{A.7}$$

where $f(x,y,u)$ takes the form in Eq. (13), and

$$D^J_{MM'}(\alpha,\beta,\gamma) \approx \int \mathrm{d}t\, F(J,M,M',t) \approx \int \mathrm{d}t \frac{1}{2\pi J}\sqrt{K(x,y,u)}\, e^{Jf(x,y,u)}, \tag{A.8}$$

which reproduces Eq. (12) after changing the integration variable to $u = \frac{t}{J}$.

B Evaluations on the Critical Set

In this appendix, we present the various evaluations relevant for the proof of Theorem 1. We start by some preliminary computations. Let us recall that

$$\Delta = (1-\xi^2)(\xi^2 - xy) - \frac{(x-y)^2}{4} \geq 0. \tag{B.9}$$

As a preliminary, we calculate the absolute values of the four complex numbers

$$\begin{aligned} u_\pm &= 1-\xi^2+\frac{x-y}{2}\pm\imath\sqrt{\Delta}, \quad u_\pm - x + y = 1-\xi^2-\frac{x-y}{2}\pm\imath\sqrt{\Delta}, \\ 1+x-u_\pm &= \xi^2+\frac{x+y}{2}\mp\imath\sqrt{\Delta}, \quad 1-y-u_\pm = \xi^2-\frac{x+y}{2}\mp\imath\sqrt{\Delta}, \end{aligned} \tag{B.10}$$

which are

$$\begin{aligned} |u_\pm|^2 &= (1-\xi^2)(1+x)(1-y), \qquad |u_\pm - x + y|^2 = (1-\xi^2)(1-x)(1+y), \\ |1+x-u_\pm|^2 &= \xi^2(1+x)(1+y), \qquad |1-y-u_\pm|^2 = \xi^2(1-x)(1-y). \end{aligned} \tag{B.11}$$

B.1 Evaluation of f at the Critical Points

To establish Eqs. (25) and (26), we note that Eq. (24) at $u_\pm$ is

$$\begin{aligned} f(x,y,u_\pm) &= -\imath\alpha x - \imath\gamma y + (2+x-y)\ln\xi + (-x+y)\ln\eta \\ &+\frac{1}{2}(1-x)\ln(1-x)+\frac{1}{2}(1+x)\ln(1+x)+\frac{1}{2}(1-y)\ln(1-y) \\ &+\frac{1}{2}(1+y)\ln(1+y)-(1+x)\ln(1+x-u_\pm)-(1-y)\ln(1-y-u_\pm) \\ &-(-x+y)\ln(u_\pm - x + y). \end{aligned} \tag{B.12}$$

The real part of $f(x,y,u_\pm)$ is

$$\begin{aligned} \Re f(x,y,u_\pm) &= (2+x-y)\ln\xi + (-x+y)\ln\eta \\ &+\frac{1}{2}(1-x)\ln(1-x)+\frac{1}{2}(1+x)\ln(1+x)+\frac{1}{2}(1-y)\ln(1-y) \\ &+\frac{1}{2}(1+y)\ln(1+y)-(1+x)\ln|1+x-u_\pm|-(1-y)\ln|1-y-u_\pm| \\ &-(-x+y)\ln|u_\pm - x + y|, \end{aligned} \tag{B.13}$$

and substituting the absolute values computed in Eq. (B.11) leads to

$$\begin{aligned} \Re f(x,y,u_\pm) &= (2+x-y)\ln\xi + (-x+y)\ln\eta \\ &+\frac{1}{2}(1-x)\ln(1-x)+\frac{1}{2}(1+x)\ln(1+x)+\frac{1}{2}(1-y)\ln(1-y) \\ &+\frac{1}{2}(1+y)\ln(1+y)-\frac{(1+x)}{2}\ln\left[\xi^2(1+x)(1+y)\right] \\ &-\frac{(1-y)}{2}\ln\left[\xi^2(1-x)(1-y)\right]-\frac{(-x+y)}{2}\ln\left[(1-\xi^2)(1-x)(1+y)\right]. \end{aligned} \tag{B.14}$$

Keeping in mind that $1-\xi^2=\eta^2$, one notes that the coefficients of both $\ln\xi$ and $\ln(1-\xi^2)$ cancel. Furthermore, a direct inspection shows that the coefficients of all $\ln(1-x), \ln(1+x), \ln(1-y)$ and $\ln(1+y)$ cancel. Hence

$$\Re f(x,y,u_\pm)=0. \tag{B.15}$$

Therefore $f(x,y,u_\pm)$ is a purely imaginary number

$$\begin{aligned} f(x,y,u_\pm) \;=\; & -\imath\alpha x-\imath\gamma y-(1+x)\ln\frac{1+x-u_\pm}{|1+x-u_\pm|}-(1-y)\ln\frac{1-y-u_\pm}{|1-y-u_\pm|} \\ & -(-x+y)\ln\frac{u_\pm-x+y}{|u_\pm-x+y|}. \end{aligned} \tag{B.16}$$

which assumes the form

$$f(x,y,u_\pm)=-\imath\alpha x-\imath\gamma y\pm\imath(\phi+x\psi-y\omega), \tag{B.17}$$

where the three angles ϕ,ψ and ω read off

$$\begin{aligned} \imath\phi &= -\ln\frac{(1+x-u_+)}{|1+x-u_+|}-\ln\frac{(1-y-u_+)}{|1-y-u_+|}, \\ \imath\psi &= -\ln\frac{1+x-u_+}{|1+x-u_+|}+\ln\frac{(u_+-x+y)}{|u_+-x+y|}, \\ \imath\omega &= -\ln\frac{(1-y-u_+)}{|1-y-u_+|}+\ln\frac{u_+-x+y}{|u_+-x+y|}. \end{aligned} \tag{B.18}$$

As the two roots u_+ and u_- are complex conjugate, one can absorb the various signs in Eq. (B.18) and then writes

$$\begin{aligned} \imath\phi &= \ln\frac{(1+x-u_-)(1-y-u_-)}{|1+x-u_-||1-y-u_-|}, \quad \imath\psi=\ln\frac{(1+x-u_-)(u_+-x+y)}{|1+x-u_-||u_+-x+y|}, \\ \imath\omega &= \ln\frac{(1-y-u_-)(u_+-x+y)}{|1-y-u_-||u_+-x+y|}. \end{aligned} \tag{B.19}$$

One by one ϕ,ψ and ω can be computed by substituting Eqs. (B.10) and (B.11) to

$$\begin{aligned} \imath\phi \;&=\; \ln\frac{\left(\xi^2+\frac{x+y}{2}+\imath\sqrt{\Delta}\right)\left(\xi^2-\frac{x+y}{2}+\imath\sqrt{\Delta}\right)}{\sqrt{\xi^4(1-x^2)(1-y^2)}} \\ &= \ln\frac{\xi^4-\frac{(x+y)^2}{4}+2\xi^2\imath\sqrt{\Delta}-(1-\xi^2)(\xi^2-xy)+\frac{(x-y)^2}{4}}{\sqrt{\xi^4(1-x^2)(1-y^2)}} \\ &= \ln\frac{2\xi^2-1-xy+2\imath\sqrt{\Delta}}{\sqrt{(1-x^2)(1-y^2)}}, \end{aligned} \tag{B.20}$$

and

$$
\begin{aligned}
\imath\psi &= \ln \frac{\left(\xi^2 + \frac{x+y}{2} + \imath\sqrt{\Delta}\right)\left(1-\xi^2 - \frac{x-y}{2} + \imath\sqrt{\Delta}\right)}{\sqrt{\xi^2(1-\xi^2)(1-x^2)(1+y)^2}} \\
&= \ln\left\{ \frac{\xi^2(1-\xi^2) - \frac{x+y}{2} + y\xi^2 + \frac{x^2-y^2}{4} - (1-\xi^2)(\xi^2 - xy)}{\sqrt{\xi^2(1-\xi^2)(1-x^2)(1+y)^2}} \right. \\
&\qquad \left. + \frac{\frac{(x-y)^2}{4} + \imath(1+y)\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-x^2)(1+y)^2}} \right\} \\
&= \ln \frac{-x(1+y)\xi^2 + \frac{x+y}{2} + xy + \frac{y^2-xy}{2} + \imath(1+y)\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-x^2)(1+y)^2}} \\
&= \ln \frac{\frac{x+y}{2} - x\xi^2 + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-x^2)}},
\end{aligned}
\tag{B.21}
$$

and finally

$$
\begin{aligned}
\imath\omega &= \ln \frac{\left(\xi^2 - \frac{x+y}{2} + \imath\sqrt{\Delta}\right)\left(1-\xi^2 - \frac{x-y}{2} + \imath\sqrt{\Delta}\right)}{\sqrt{\xi^2(1-\xi^2)(1-y^2)(1-x)^2}} \\
&= \ln\left\{ \frac{\xi^2(1-\xi^2) - \frac{x+y}{2} + y\xi^2 + \frac{x^2-y^2}{4} - (1-\xi^2)(\xi^2 - xy)}{\sqrt{\xi^2(1-\xi^2)(1-y^2)(1-x)^2}} \right. \\
&\qquad \left. + \frac{\frac{(x-y)^2}{4} + \imath(1-x)\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-y^2)(1-x)^2}} \right\} \\
&= \ln \frac{y(1-x)\xi^2 - \frac{x+y}{2} + xy + \frac{x^2-xy}{2} + \imath(1-x)\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-y^2)(1-x)^2}} \\
&= \ln \frac{-\frac{x+y}{2} + y\xi + \imath\sqrt{\Delta}}{\sqrt{\xi^2(1-\xi^2)(1-y^2)}}.
\end{aligned}
\tag{B.22}
$$

B.2 Evaluation of the Second Derivative

From Eq. (27), we have

$$
-\partial_u^2 f(x,y,u) = \frac{1}{1+x-u} + \frac{1}{1-y-u} + \frac{1}{u} + \frac{1}{u-x+y}. \tag{B.23}
$$

Each term can be evaluated at the critical points as

$$
\begin{aligned}
\frac{1}{1+x-u_\pm} &= \frac{1+x-u_\mp}{|1+x-u_\pm|^2} = \frac{\xi^2 + \frac{x+y}{2} \pm \imath\sqrt{\Delta}}{\xi^2(1+x)(1+y)} \\
\frac{1}{1-y+u_\pm} &= \frac{1-y+u_\mp}{|1-y+u_\mp|^2} = \frac{\xi^2 - \frac{x+y}{2} \pm \imath\sqrt{\Delta}}{\xi^2(1-x)(1-y)} \\
\frac{1}{u_\pm - x + y} &= \frac{u_\mp - x + y}{|u_\mp - x + y|^2} = \frac{(1-\xi^2) - \frac{x-y}{2} \mp \imath\sqrt{\Delta}}{(1-\xi^2)(1-x)(1+y)} \\
\frac{1}{u_\pm} &= \frac{u_\mp}{|u_\mp|^2} = \frac{(1-\xi^2) + \frac{x-y}{2} \mp \imath\sqrt{\Delta}}{(1-\xi^2)(1+x)(1-y)}.
\end{aligned}
\tag{B.24}
$$

The real part of (B.23) is, therefore,

$$\frac{\xi^2+\frac{x+y}{2}}{\xi^2(1+x)(1+y)}+\frac{\xi^2-\frac{x+y}{2}}{\xi^2(1-x)(1-y)} \\ +\frac{(1-\xi^2)-\frac{x-y}{2}}{(1-\xi^2)(1-x)(1+y)}+\frac{(1-\xi^2)+\frac{x-y}{2}}{(1-\xi^2)(1+x)(1-y)}, \tag{B.25}$$

and can be rewritten as

$$\Re(-\partial_u^2 f)|_{x,y,u_\pm}=\frac{4\Delta}{(1-x^2)(1-y^2)\xi^2(1-\xi^2)}. \tag{B.26}$$

The imaginary part of Eq. (B.23) is

$$\pm\imath\sqrt{\Delta}\left(\frac{1}{\xi^2(1+x)(1+y)}+\frac{1}{\xi^2(1-x)(1-y)} \right. \\ \left. -\frac{1}{(1-\xi^2)(1-x)(1+y)}-\frac{1}{(1-\xi^2)(1+x)(1-y)}\right), \tag{B.27}$$

which, after some algebra, can be put in the form

$$\Im(-\partial_u^2 f)|_{x,y,u_\pm}=\pm\imath 2\sqrt{\Delta}\ \frac{1-2\xi^2-xy}{(1-x^2)(1-y^2)\xi^2(1-\xi^2)}. \tag{B.28}$$

B.3 Evaluation of K

The prefactor $K|_{x,y,u_\pm}$ is

$$K=\frac{\sqrt{(1-x^2)(1-y^2)}}{(1+x-u_\pm)(1-y-u_\pm)(u_\pm)(u_\pm-x+y)}, \tag{B.29}$$

which is, using Eq. (B.24),

$$K = \frac{\sqrt{(1-x^2)(1-y^2)}}{\xi^4(1-\xi^2)^2(1-x^2)^2(1-y^2)^2} \\ \times\left(\xi^2+\frac{x+y}{2}\pm\imath\sqrt{\Delta}\right)\left(\xi^2-\frac{x+y}{2}\pm\imath\sqrt{\Delta}\right) \\ \times\left((1-\xi^2)-\frac{x-y}{2}\mp\imath\sqrt{\Delta}\right)\left((1-\xi^2)+\frac{x-y}{2}\mp\imath\sqrt{\Delta}\right), \tag{B.30}$$

and a straightforward computation proves Eq. (30).

C Real Saddle Points

In this section, we present the SPA evaluation of a matrix element with

$$\Delta=(1-\xi^2)(\xi^2-xy)-\frac{(x-y)^2}{4}<0. \tag{C.31}$$

For convenience we denote $\Delta'=-\Delta>0$. In this range of parameters the two saddle points

$$u_\pm=h_\pm(x;y)=(1-\xi^2)+\frac{x-y}{2}\pm\sqrt{\Delta'}, \tag{C.32}$$

Fig. 2 The function $\Xi = \Phi + x\Psi - y\Omega$ (*red*) is negative, and vanishes (plane $z = 0$ *light blue*) when Δ' (*dark blue*) vanishes, for $\xi = 0.1, 0.5$ and 0.9, from *left* to the *right*

are real. For simplicity suppose that $0 < x \leq y < 1$. A straightforward computation shows that $0 < u_- < u_+ < 1 - y$, hence both roots are in the integration interval. Using the results of Appendix B.1, the function finds, at the two saddle points, the evaluation such that

$$f|_{u_\pm} = -\imath\alpha x - \imath\beta y \pm (\Phi + x\Psi - y\Omega), \tag{C.33}$$

with

$$\Phi = \ln \frac{(2\xi^2 - 1 - xy + 2\sqrt{\Delta'})}{\sqrt{(1-x^2)(1-y^2)}}, \tag{C.34}$$

$$\Psi = \ln \frac{(-x\xi^2 + \frac{x+y}{2} + \sqrt{\Delta'})}{\sqrt{\xi^2(1-\xi^2)(1-x^2)}}, \tag{C.35}$$

$$\Omega = \ln \frac{(\xi^2 y - \frac{x+y}{2} + \sqrt{\Delta'})}{\sqrt{\xi^2(1-\xi^2)(1-y^2)}}. \tag{C.36}$$

From Appendix B.2, we obtain

$$-\partial_u^2 f \,|_{u_\pm} = \frac{-4\Delta' \mp 2\sqrt{\Delta'}(2\xi^2 - 1 - xy)}{\xi^2(1-\xi^2)(1-x^2)(1-y^2)}, \tag{C.37}$$

which shows in particular that the maximum of f is u_- (as $-\partial_u^2 f|_{u_-} < 0$), and the SPA is dominated by the latter. In Fig. 2, we represent the function $\Xi = \Phi + x\Psi - y\Omega$ as a function of x and y,

The prefactor can be evaluated as, using Appendix B.3,

$$K|_{u-} = \frac{-\sqrt{(1-x^2)(1-y^2)}\left(2\xi^2 - 1 - xy - 2\sqrt{\Delta}\right)^2}{\xi^2(1-\xi^2)(1-x^2)^2(1-y^2)^2}, \tag{C.38}$$

hence we get the asymptotic estimate

$$D^J_{xJ,yJ}(\alpha,\beta,\gamma) \approx -\frac{1}{\sqrt{2\pi J}}\left(\frac{1}{2\sqrt{\Delta'}}\right)^{1/2} e^{-\imath\alpha J x - \imath\gamma J y} e^{-\frac{\Phi}{2}} e^{-J(\Phi + x\Psi - y\Omega)}, \tag{C.39}$$

which is indeed suppressed for large J.

The case $\Delta' = 0$ is special. A straightforward calculation shows that under these circumstances

$$\Phi = \Psi = \Omega = 0. \tag{C.40}$$

In addition, Eq. (C.37) implies $\partial_u^2 f|_{u_\pm} = 0$. One needs to push the Taylor development around the root

$$u_0 = 1 - \xi^2 + \frac{x-y}{2}, \tag{C.41}$$

to the third order

$$f(u,x,y) = f\,|_{u_0} + \frac{1}{6}(u - u_0)^3[\partial_u^3 f]\,|_{u_0} + O(u^3), \tag{C.42}$$

and the Wigner matrix elements has an asymptotic behavior (see (1))

$$\int \mathrm{d}u \sqrt{K(u,x,y)} e^{Jf} \approx e^{Jf|_{u_0}} \left\{ \mathrm{Ai}(a(x,y)[\imath J]^{\frac{2}{3}})[\imath J]^{-\frac{1}{3}} + \mathrm{Ai}'(a(x,y)[\imath J]^{\frac{2}{3}})[\imath J]^{-\frac{2}{3}} \right\}, \tag{C.43}$$

where $a(x,y)$ is some non vanishing smooth real function (determined by K and f evaluated at u_0, see (1)), Ai is the Airy function of the first kind and Ai$'$ its derivative. At large argument, the Airy functions behave like

$$\mathrm{Ai}(\zeta) \approx \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\,\zeta^{\frac{1}{4}}} \approx -\mathrm{Ai}'(\zeta). \tag{C.44}$$

The term Ai$'$ is therefore sub-leading and we have

$$\int \mathrm{d}u \sqrt{K(u,x,y)} e^{Jf} \approx \frac{e^{\imath J\left(\alpha x+\gamma y-\frac{2}{3}(a(x,y))^{\frac{3}{2}}\right)}}{\sqrt{\imath J}(a(x,y))^{1/4}}. \tag{C.45}$$

D Computations for the $3j$ Symbol

In this appendix, we detail at length the various computations required for the proof of Theorem 3.

D.1 The First Derivative

To compute the derivative $\partial_{\xi^2}\sum_i s_i(\imath f_i)$, note that $\partial_{(\xi^2)}\Delta_i = -(2\xi^2-1-x_iy_i)$. The partial derivative of $\imath\phi_i$ is then

$$\begin{aligned}
\imath\partial_{(\xi^2)}\phi_i &= \frac{2+2\imath\frac{\partial_{(\xi^2)}\Delta_i}{2\sqrt{\Delta_i}}}{(2\xi^2-1-x_iy_i+2\imath\sqrt{\Delta_i})} \\
&= \frac{\left(2-\imath\frac{2\xi^2-1-x_iy_i}{\sqrt{\Delta_i}}\right)(2\xi^2-1-x_iy_i-2\imath\sqrt{\Delta_i})}{(2\xi^2-1-x_iy_i)^2+4\Delta_i} \\
&= \frac{-\imath}{\sqrt{\Delta_i}},
\end{aligned} \tag{D.46}$$

while the derivative of $\imath\psi_i$ is

$$\begin{aligned}
\imath\partial_{(\xi^2)}\psi_i &= \frac{-x_i+\imath\frac{\partial_{(\xi^2)}\Delta_i}{2\sqrt{\Delta_i}}}{\frac{x_i+y_i}{2}-x_i\xi^2+\imath\sqrt{\Delta_i}} - \frac{1-2\xi^2}{2\xi^2(1-\xi^2)} \\
&= \imath\frac{\left[-(2\xi^2-1-x_iy_i)+\imath 2x_i\sqrt{\Delta_i}\right]\left[\frac{x_i+y_i}{2}-x_i\xi^2-\imath\sqrt{\Delta_i}\right]}{2\sqrt{\Delta_i}\xi^2(1-\xi^2)(1-x_i^2)} \\
&\quad -\frac{1-2\xi^2}{2\xi^2(1-\xi^2)}.
\end{aligned} \tag{D.47}$$

We first evaluate $2x_i\Delta_i - \left(2\xi^2-1-x_iy_i\right)\left(\frac{x_i+y_i}{2}-x_i\xi^2\right)$ as

$$\begin{aligned}
&= 2x_i\left[-\xi^4+\xi^2(1+x_iy_i)-\frac{(x_i+y_i)^2}{4}\right]-\left[2\xi^2-1-x_iy_i\right]\left(\frac{x_i+y_i}{2}-x_i\xi^2\right) \\
&= \xi^2\left[x_i(1+x_iy_i)-x_i-y_i\right]-\frac{x_i+y_i}{2}\left[x_i(x_i+y_i)-1-x_iy_i\right] \\
&= (1-x_i^2)\left(\frac{x_i+y_i}{2}-\xi^2y_i\right),
\end{aligned} \tag{D.48}$$

hence Eq. (D.47) can be translated as

$$
\begin{aligned}
&\imath\frac{(1-x_i^2)(\frac{x_i+y_i}{2}-\xi^2 y_i)}{2\sqrt{\Delta_i}\xi^2(1-\xi^2)(1-x_i^2)}+\imath\frac{\imath\sqrt{\Delta_i}\left[x_i^2-2x_i^2\xi^2+2\xi^2-1\right]}{2\sqrt{\Delta_i}\xi^2(1-\xi^2)(1-x_i^2)}-\frac{(1-2\xi^2)}{2\xi^2(1-\xi^2)}\\
&=\imath\frac{(\frac{x_i+y_i}{2}-\xi^2 y_i)}{2\sqrt{\Delta_i}\xi^2}. \qquad \text{(D.49)}
\end{aligned}
$$

Noting that $\omega(x_i,y_i)=\psi(-y_i,-x_i)$ the derivative of $\imath\omega$ is simply

$$
\imath\partial_{(\xi^2)}\omega_i=\frac{\imath(-\frac{x_i+y_i}{2}+\xi^2 x_i)}{2\sqrt{\Delta_i}\xi^2(1-\xi^2)}. \qquad \text{(D.50)}
$$

The derivative of $\sum_i s_i(\imath f_i)$ is then

$$
\begin{aligned}
\partial_{(\xi^2)}\sum_i s_i(\imath f_i) &= \imath\sum_i s_i J_i\left(\frac{-1}{\sqrt{\Delta_i}}+x_i\frac{(\frac{x_i+y_i}{2}-\xi^2 y_i)}{2\sqrt{\Delta_i}\xi^2(1-\xi^2)}\right.\\
&\qquad\left.-y_i\frac{(-\frac{x_i+y_i}{2}+\xi^2 x_i)}{2\sqrt{\Delta_i}\xi^2(1-\xi^2)}\right)\\
&= \imath\sum_i s_i J_i\left(\frac{-2\xi^2(1-\xi^2)+\frac{(x_i+y_i)^2}{2}-2\xi^2 x_i y_i}{2\sqrt{\Delta_i}\xi^2(1-\xi^2)}\right)\\
&= \imath\sum_i s_i J_i\left(-2\frac{\Delta_i}{2\sqrt{\Delta_i}\xi^2(1-\xi^2)}\right)\\
&= -\frac{\imath}{\xi^2(1-\xi^2)}\sum_i s_i J_i\sqrt{\Delta_i}. \qquad \text{(D.51)}
\end{aligned}
$$

D.2 The Saddle Point Equation

We will use in the sequel the short hand notation $A^{\mathbf{B}} := \mathbf{A}\cdot\mathbf{B}$ for all vectors $\mathbf{A}$ and $\mathbf{B}$. Squaring twice the saddle point Eq. (69) we obtain, for all signs s_i,

$$
\left[J_3^2\Delta_3-J_1^2\Delta_1-J_2^2\Delta_2\right]^2=4J_1^2J_2^2\Delta_1\Delta_2. \qquad \text{(D.52)}
$$

We first translate Eq. (D.52) in terms of angular momentum vectors

$$
J_i^2\Delta_i=(1-\xi^2)\xi^2J_i^2+\xi^2J_i^{\mathbf{n}}J_i^{\mathbf{k}}-\frac{1}{4}(J_i^{\mathbf{n}+\mathbf{k}})^2, \qquad \text{(D.53)}
$$

and this allows us to write the sum $J_3^2\Delta_3-J_1^2\Delta_1-J_2^2\Delta_2$ as

$$
\begin{aligned}
&(1-\xi^2)\xi^2\left[J_3^2-J_1^2-J_2^2\right]+\xi^2\left[J_3^{\mathbf{n}}J_3^{\mathbf{k}}-J_1^{\mathbf{n}}J_1^{\mathbf{k}}-J_2^{\mathbf{n}}J_2^{\mathbf{k}}\right]\\
&-\frac{1}{4}\left[(J_3^{\mathbf{n}+\mathbf{k}})^2-(J_1^{\mathbf{n}+\mathbf{k}})^2-(J_2^{\mathbf{n}+\mathbf{k}})^2\right], \qquad \text{(D.54)}
\end{aligned}
$$

and using $\mathbf{J}_3=-\mathbf{J}_1-\mathbf{J}_2$, Eq. (D.52) becomes

$$
\begin{aligned}
&\left\{2(1-\xi^2)\xi^2\mathbf{J}_1\cdot\mathbf{J}_2+\xi^2\left(J_1^{\mathbf{n}}J_2^{\mathbf{k}}+J_2^{\mathbf{n}}J_1^{\mathbf{k}}\right)-\frac{1}{2}J_1^{\mathbf{n}+\mathbf{k}}J_2^{\mathbf{n}+\mathbf{k}}\right\}^2\\
&=\left[2(1-\xi^2)\xi^2J_1^2+2\xi^2J_1^{\mathbf{n}}J_1^{\mathbf{k}}-\frac{1}{2}(J_1^{\mathbf{n}+\mathbf{k}})^2\right]\\
&\quad\times\left[2(1-\xi^2)\xi^2J_2^2+2\xi^2J_2^{\mathbf{n}}J_2^{\mathbf{k}}-\frac{1}{2}(J_2^{\mathbf{n}+\mathbf{k}})^2\right]. \qquad \text{(D.55)}
\end{aligned}
$$

Collecting all terms on the LHS, we get

$$\begin{aligned}
&4(1-\xi^2)^2\xi^4\left[J_1^2J_2^2-(\mathbf{J}_1\cdot\mathbf{J}_2)^2\right]\\
&\quad+4(1-\xi^2)\xi^4\left[J_1^2J_2^{\mathbf{n}}J_2^{\mathbf{k}}+J_2^2J_1^{\mathbf{n}}J_1^{\mathbf{k}}-\mathbf{J}_1\cdot\mathbf{J}_2\left(J_1^{\mathbf{n}}J_2^{\mathbf{k}}+J_2^{\mathbf{n}}J_1^{\mathbf{k}}\right)\right]\\
&\quad-(1-\xi^2)\xi^2\left[J_1^2(J_2^{\mathbf{n}+\mathbf{k}})^2+J_2^2(J_1^{\mathbf{n}+\mathbf{k}})^2-2\mathbf{J}_1\cdot\mathbf{J}_2J_1^{\mathbf{n}+\mathbf{k}}J_2^{\mathbf{n}+\mathbf{k}}\right]\\
&\quad+\xi^4\left[4J_1^{\mathbf{n}}J_1^{\mathbf{k}}J_2^{\mathbf{n}}J_2^{\mathbf{k}}-\left(J_1^{\mathbf{n}}J_2^{\mathbf{k}}+J_2^{\mathbf{n}}J_1^{\mathbf{k}}\right)^2\right]\\
&\quad-\xi^2\left[J_1^{\mathbf{n}}J_1^{\mathbf{k}}(J_2^{\mathbf{n}+\mathbf{k}})^2+J_2^{\mathbf{n}}J_2^{\mathbf{k}}(J_1^{\mathbf{n}+\mathbf{k}})^2-\left(J_1^{\mathbf{n}}J_2^{\mathbf{k}}+J_2^{\mathbf{n}}J_1^{\mathbf{k}}\right)J_1^{\mathbf{n}+\mathbf{k}}J_2^{\mathbf{n}+\mathbf{k}}\right]=0.
\end{aligned} \tag{D.56}$$

which is again

$$\begin{aligned}
&4(1-\xi^2)^2\xi^4\left[\mathbf{J}_1\wedge\mathbf{J}_2\right]^2+4(1-\xi^2)\xi^4\left[\mathbf{n}\wedge(\mathbf{J}_1\wedge\mathbf{J}_2)\right]\cdot\left[\mathbf{k}\wedge(\mathbf{J}_1\wedge\mathbf{J}_2)\right]\\
&\quad-(1-\xi^2)\xi^2\left[(\mathbf{n}+\mathbf{k})\wedge(\mathbf{J}_1\wedge\mathbf{J}_2)\right]^2-\xi^4\left[(\mathbf{n}\wedge\mathbf{k})\cdot(\mathbf{J}_1\wedge\mathbf{J}_2)\right]^2\\
&\quad-\xi^2\left[\mathbf{n}\cdot\left[(\mathbf{n}+\mathbf{k})\wedge(\mathbf{J}_1\wedge\mathbf{J}_2)\right]\right]\left[\mathbf{k}\cdot\left[(\mathbf{n}+\mathbf{k})\wedge(\mathbf{J}_1\wedge\mathbf{J}_2)\right]\right]=0.
\end{aligned} \tag{D.57}$$

Using $\mathbf{S}=\mathbf{J}_1\wedge\mathbf{J}_2$, twice the oriented area of the triangle $\{\mathbf{J}_i\}$, the saddle point equation can be written

$$\begin{aligned}
0\,=\,&4(1-\xi^2)^2\xi^4S^2+4(1-\xi^2)\xi^4\left[\mathbf{n}\wedge\mathbf{S}\right]\cdot\left[\mathbf{k}\wedge\mathbf{S}\right]\\
&-(1-\xi^2)\xi^2\left[(\mathbf{n}+\mathbf{k})\wedge\mathbf{S}\right]^2-\xi^4\left[(\mathbf{n}\wedge\mathbf{k})\cdot\mathbf{S}\right]^2\\
&-\xi^2\left[\mathbf{S}\cdot(\mathbf{n}\wedge\mathbf{k})\right]\left[\mathbf{S}\cdot(\mathbf{k}\wedge\mathbf{n})\right],
\end{aligned} \tag{D.58}$$

and dividing by $(1-\xi^2)\xi^2$, we obtain

$$0=4(1-\xi^2)\xi^2S^2+4\xi^2\left[\mathbf{n}\wedge\mathbf{S}\right]\cdot\left[\mathbf{k}\wedge\mathbf{S}\right]+\left[\mathbf{S}\cdot(\mathbf{n}\wedge\mathbf{k})\right]^2-\left[(\mathbf{n}+\mathbf{k})\wedge\mathbf{S}\right]^2, \tag{D.59}$$

that is

$$\begin{aligned}
0\,=\,&4\xi^4S^2-4\xi^2\left[S^2+(\mathbf{n}\cdot\mathbf{k})S^2-S^{\mathbf{n}}S^{\mathbf{k}}\right]\\
&-S^2(\mathbf{n}\wedge\mathbf{k})^2+\left[\mathbf{S}\wedge(\mathbf{n}\wedge\mathbf{k})\right]^2+S^2(\mathbf{n}+\mathbf{k})^2-(S^{\mathbf{n}}+S^{\mathbf{k}})^2.
\end{aligned} \tag{D.60}$$

The last line in Eq. (D.60) can be simplified as

$$\begin{aligned}
&-S^2+S^2(\mathbf{n}\cdot\mathbf{k})^2+(S^{\mathbf{n}})^2+(S^{\mathbf{k}})^2-2(\mathbf{n}\cdot\mathbf{k})S^{\mathbf{n}}S^{\mathbf{k}}\\
&\quad+2S^2+2S^2(\mathbf{n}\cdot\mathbf{k})-(S^{\mathbf{n}})^2-(S^{\mathbf{k}})^2-2S^{\mathbf{n}}S^{\mathbf{k}}\\
&=\left[1+(\mathbf{n}\cdot\mathbf{k})\right]^2S^2-2\left[1+(\mathbf{n}\cdot\mathbf{k})\right]S^{\mathbf{n}}S^{\mathbf{k}},
\end{aligned} \tag{D.61}$$

from which Eq. (70) follows.

D.3 Evaluation of $J_i^2\Delta_i^{\pm}$

Recall that $J_i^2\Delta_i$ is

$$J_i^2\Delta_i=(1-\xi^2)\xi^2J_i^2+\xi^2J_i^{\mathbf{n}}J_i^{\mathbf{k}}-\frac{1}{4}(J_i^{\mathbf{n}+\mathbf{k}})^2. \tag{D.62}$$

Evaluated for $\xi_+^2=\frac{1+(\mathbf{n}\cdot\mathbf{k})}{2}$, Eq. (D.62) gives

$$J_i^2\Delta_i^+=\frac{1-(\mathbf{n}\cdot\mathbf{k})^2}{4}J_i^2+\frac{1+(\mathbf{n}\cdot\mathbf{k})}{2}J_i^{\mathbf{n}}J_i^{\mathbf{k}}-\frac{1}{4}(J_i^{\mathbf{n}}+J_i^{\mathbf{k}})^2, \tag{D.63}$$

which can be simplified further to

$$
\begin{aligned}
J_i^2\Delta_i^+ &= \frac{1}{4}\left\{(\mathbf{n}\wedge\mathbf{k})^2 J_i^2 + 2(\mathbf{n}\cdot\mathbf{k})J_i^{\mathbf{n}}J_i^{\mathbf{k}} - (J_i^{\mathbf{n}})^2 - (J_i^{\mathbf{k}})^2\right\} \\
&= \frac{1}{4}\left[(\mathbf{n}\wedge\mathbf{k})^2 J_i^2 + J_i^{\mathbf{n}}\left[(\mathbf{n}\wedge\mathbf{k})\cdot(\mathbf{k}\wedge\mathbf{J}_i)\right] - J_i^{\mathbf{k}}\left[(\mathbf{n}\wedge\mathbf{k})\cdot(\mathbf{n}\wedge\mathbf{J}_i)\right]\right].
\end{aligned}
\tag{D.64}
$$

Combining the last two terms, this is

$$
\begin{aligned}
&\frac{1}{4}\left\{(\mathbf{n}\wedge\mathbf{k})^2 J_i^2 + (\mathbf{n}\wedge\mathbf{k})\cdot\left[(\mathbf{J}_i\wedge(\mathbf{k}\wedge\mathbf{n}))\wedge\mathbf{J}_i\right]\right\} \\
&\quad = \frac{1}{4}\left\{(\mathbf{n}\wedge\mathbf{k})^2 J_i^2 + (\mathbf{n}\wedge\mathbf{k})\cdot\left[\mathbf{J}_i\left(\mathbf{J}_i\cdot(\mathbf{n}\wedge\mathbf{k})\right) - (\mathbf{n}\wedge\mathbf{k})\mathbf{J}_i^2\right]\right\},
\end{aligned}
\tag{D.65}
$$

hence for ξ_+^2, we get

$$
J_i\Delta_i^+ = \frac{1}{4}\left[\mathbf{J}_i\cdot(\mathbf{n}\wedge\mathbf{k})\right]^2. \tag{D.66}
$$

Evaluated in $\xi_-^2 = \frac{1+(\mathbf{n}\cdot\mathbf{k})}{2} - \frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}$, $J_i^2\Delta_i$ is of the form

$$
\begin{aligned}
J_i\Delta_i^- &= \left(\frac{1-(\mathbf{n}\cdot\mathbf{k})}{2} + \frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right)\left(\frac{1+(\mathbf{n}\cdot\mathbf{k})}{2} - \frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right)J_i^2 \\
&\quad + \left(\frac{1+(\mathbf{n}\cdot\mathbf{k})}{2} - \frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right)J_i^{\mathbf{n}}J_i^{\mathbf{k}} - \frac{1}{4}(J_i^{\mathbf{n}+\mathbf{k}})^2.
\end{aligned}
\tag{D.67}
$$

Combining all the terms common to the RHS in Eqs. (D.63) and (D.67), one obtains

$$
\begin{aligned}
J_i\Delta_i^- &= \frac{1}{4}\left[\mathbf{J}_i\cdot(\mathbf{n}\wedge\mathbf{k})\right]^2 + \frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\left[(\mathbf{n}\cdot\mathbf{k})J_i^2 - J_i^{\mathbf{n}}J_i^{\mathbf{k}}\right] - J_i^2\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^4} \\
&= \frac{1}{4S^2}\left\{\left[\mathbf{S}\left(\mathbf{J}_i\cdot(\mathbf{n}\wedge\mathbf{k})\right)\right]^2 + 4S^{\mathbf{n}}S^{\mathbf{k}}\left[(\mathbf{n}\wedge\mathbf{J}_i)\cdot(\mathbf{k}\wedge\mathbf{J}_i)\right] - 4J_i^2\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^2}\right\}.
\end{aligned}
\tag{D.68}
$$

But remarking that $\mathbf{S}\cdot\mathbf{J}_i = 0$, the first term on the RHS above can be written as a double vector product, i.e.

$$
\begin{aligned}
&J_i\Delta_i^- \\
&\quad = \frac{1}{4S^2}\left\{\left[\mathbf{J}_i\wedge(\mathbf{S}\wedge(\mathbf{n}\wedge\mathbf{k}))\right]^2 + 4S^{\mathbf{n}}S^{\mathbf{k}}\left[(\mathbf{n}\wedge\mathbf{J}_i)\cdot(\mathbf{k}\wedge\mathbf{J}_i)\right] - 4J_i^2\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^2}\right\} \\
&\quad = \frac{1}{4S^2}\left\{\left[\mathbf{J}_i\wedge\left(\mathbf{n}S^{\mathbf{k}} + \mathbf{k}S^{\mathbf{n}}\right)\right]^2 - 4J_i^2\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^2}\right\} \\
&\quad = \frac{1}{4S^4}\left\{S^2\left[\mathbf{J}_i\wedge\left(\mathbf{n}S^{\mathbf{k}} + \mathbf{k}S^{\mathbf{n}}\right)\right]^2 - 4J_i^2(S^{\mathbf{n}}S^{\mathbf{k}})^2\right\}.
\end{aligned}
\tag{D.69}
$$

Then, since $A^2B^2 = (\mathbf{A}\cdot\mathbf{B})^2 + (\mathbf{A}\wedge\mathbf{B})^2$, we have

$$
\begin{aligned}
J_i\Delta_i^- &= \frac{1}{4S^4}\left\{\left[\mathbf{S}\cdot\left[\mathbf{J}_i\wedge\left(\mathbf{n}S^{\mathbf{k}} + \mathbf{k}S^{\mathbf{n}}\right)\right]\right]^2 + \left[\mathbf{J}_i(S^{\mathbf{n}}S^{\mathbf{k}} + S^{\mathbf{k}}S^{\mathbf{n}})\right]^2 - 4J_i^2(S^{\mathbf{n}}S^{\mathbf{k}})^2\right\} \\
&= \frac{\left\{\mathbf{J}_i\cdot\left[(\mathbf{n}\wedge\mathbf{S})S^{\mathbf{k}} + (\mathbf{k}\wedge\mathbf{S})S^{\mathbf{n}}\right]\right\}^2}{4S^4} = \frac{\left\{\mathbf{J}_i\cdot\left[(\mathbf{S}\wedge\mathbf{n})S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})S^{\mathbf{n}}\right]\right\}^2}{4S^4}.
\end{aligned}
\tag{D.70}
$$

D.4 Second Derivative

Using $J_i\sqrt{\Delta_i^+}$ from Eq. (74) and ξ_+^2, the following is valid

$$\begin{aligned}
&\varepsilon_2^+\varepsilon_3^+ J_2\sqrt{\Delta_2^+}J_3\sqrt{\Delta_3^+}\left[(2\xi_+^2-1)J_1^2 - J_1^{\mathbf{n}}J_1^{\mathbf{k}}\right] + \circlearrowleft_{123} \\
&= \frac{1}{4}\left\{J_2^{\mathbf{n}\wedge\mathbf{k}}J_3^{\mathbf{n}\wedge\mathbf{k}}[(\mathbf{n}\wedge\mathbf{J}_1)\cdot(\mathbf{k}\wedge\mathbf{J}_1)] + J_3^{\mathbf{n}\wedge\mathbf{k}}J_1^{\mathbf{n}\wedge\mathbf{k}}[(\mathbf{n}\wedge\mathbf{J}_2)\cdot(\mathbf{k}\wedge\mathbf{J}_2)]\right. \\
&\quad \left. + J_1^{\mathbf{n}\wedge\mathbf{k}}J_2^{\mathbf{n}\wedge\mathbf{k}}[(\mathbf{n}\wedge\mathbf{J}_3)\cdot(\mathbf{k}\wedge\mathbf{J}_3)]\right\}. \qquad \text{(D.71)}
\end{aligned}$$

Substituting in the equation above $\mathbf{J}_3 = -\mathbf{J}_1 - \mathbf{J}_2$, the RHS can be written

$$\begin{aligned}
&\frac{1}{4}\left\{-J_2^{\mathbf{n}\wedge\mathbf{k}}J_2^{\mathbf{n}\wedge\mathbf{k}}[(\mathbf{n}\wedge\mathbf{J}_1)\cdot(\mathbf{k}\wedge\mathbf{J}_1)] - J_2^{\mathbf{n}\wedge\mathbf{k}}J_1^{\mathbf{n}\wedge\mathbf{k}}[(\mathbf{n}\wedge\mathbf{J}_1)\cdot(\mathbf{k}\wedge\mathbf{J}_1)]\right. \\
&\quad - J_1^{\mathbf{n}\wedge\mathbf{k}}J_1^{\mathbf{n}\wedge\mathbf{k}}[(\mathbf{n}\wedge\mathbf{J}_2)\cdot(\mathbf{k}\wedge\mathbf{J}_2)] - J_2^{\mathbf{n}\wedge\mathbf{k}}J_1^{\mathbf{n}\wedge\mathbf{k}}[(\mathbf{n}\wedge\mathbf{J}_2)\cdot(\mathbf{k}\wedge\mathbf{J}_2)] \\
&\quad + J_1^{\mathbf{n}\wedge\mathbf{k}}J_2^{\mathbf{n}\wedge\mathbf{k}}[(\mathbf{n}\wedge\mathbf{J}_1)\cdot(\mathbf{k}\wedge\mathbf{J}_1) + (\mathbf{n}\wedge\mathbf{J}_1)\cdot(\mathbf{k}\wedge\mathbf{J}_2) \\
&\quad \left. + (\mathbf{n}\wedge\mathbf{J}_2)\cdot(\mathbf{k}\wedge\mathbf{J}_1) + (\mathbf{n}\wedge\mathbf{J}_2)\cdot(\mathbf{k}\wedge\mathbf{J}_2)]\right\}, \qquad \text{(D.72)}
\end{aligned}$$

canceling the appropriate cross terms, the remaining expression admits the factorization

$$\begin{aligned}
&-\frac{1}{4}\left\{\left[J_2^{\mathbf{n}\wedge\mathbf{k}}(\mathbf{n}\wedge\mathbf{J}_1) - J_1^{\mathbf{n}\wedge\mathbf{k}}(\mathbf{n}\wedge\mathbf{J}_2)\right]\cdot\left[J_2^{\mathbf{n}\wedge\mathbf{k}}(\mathbf{k}\wedge\mathbf{J}_1) - J_1^{\mathbf{n}\wedge\mathbf{k}}(\mathbf{k}\wedge\mathbf{J}_2)\right]\right\} \\
&= -\frac{1}{4}\left\{\mathbf{n}\wedge((\mathbf{n}\wedge\mathbf{k})\wedge(\mathbf{J}_1\wedge\mathbf{J}_2))\right\}\cdot\left\{\mathbf{k}\wedge((\mathbf{n}\wedge\mathbf{k})\wedge(\mathbf{J}_1\wedge\mathbf{J}_2))\right\}, \qquad \text{(D.73)}
\end{aligned}$$

developing the double vector products and taking into account that $\mathbf{n}\cdot(\mathbf{n}\wedge\mathbf{k}) = \mathbf{k}\cdot(\mathbf{n}\wedge\mathbf{k}) = 0$, we conclude

$$\varepsilon_2^+\varepsilon_3^+ J_2\sqrt{\Delta_2^+}J_3\sqrt{\Delta_3^+}\left[(2\xi_+^2-1)J_1^2 - J_1^{\mathbf{n}}J_1^{\mathbf{k}}\right] + \circlearrowleft_{123} = -\frac{1}{4}S^{\mathbf{n}}S^{\mathbf{k}}(\mathbf{n}\wedge\mathbf{k})^2. \qquad \text{(D.74)}$$

For the ξ_-^2 root, we have

$$\begin{aligned}
&\varepsilon_2^-\varepsilon_3^- J_2\sqrt{\Delta_2}J_3\sqrt{\Delta_3}\left[(2\xi_-^2-1)J_1^2 - (\mathbf{n}\cdot\mathbf{J}_1)(\mathbf{k}\cdot\mathbf{J}_1)\right] + \circlearrowleft_{123} \\
&= \frac{1}{4}\frac{\left[J_2^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_2^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right]}{S^2}\frac{\left[J_3^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_3^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right]}{S^2} \\
&\quad \times\left[(\mathbf{n}\wedge\mathbf{J}_1)\cdot(\mathbf{k}\wedge\mathbf{J}_1) - 2J_1^2\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right] \\
&\quad + \frac{1}{4}\frac{\left[J_1^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_1^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right]}{S^2}\frac{\left[J_3^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_3^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right]}{S^2} \\
&\quad \times\left[(\mathbf{n}\wedge\mathbf{J}_2)\cdot(\mathbf{k}\wedge\mathbf{J}_2) - 2J_2^2\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right] \\
&\quad + \frac{1}{4}\frac{\left[J_1^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_1^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right]}{S^2}\frac{\left[J_2^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_2^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right]}{S^2} \\
&\quad \times\left[(\mathbf{n}\wedge\mathbf{J}_3)\cdot(\mathbf{k}\wedge\mathbf{J}_3) - 2J_3^2\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right]. \qquad \text{(D.75)}
\end{aligned}$$

We substitute again in the equation above $\mathbf{J}_3 = -\mathbf{J}_1 - \mathbf{J}_2$. The coefficient of $\frac{1}{4S^2}$ can be calculated, canceling the appropriate cross terms,

$$
\begin{aligned}
&-\left[J_2^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_2^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right]^2 (\mathbf{n}\wedge\mathbf{J}_1)\cdot(\mathbf{k}\wedge\mathbf{J}_1)\\
&-\left[J_1^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_1^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right]^2 (\mathbf{n}\wedge\mathbf{J}_2)\cdot(\mathbf{k}\wedge\mathbf{J}_2)\\
&+\left[J_1^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_1^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right]\left[J_2^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_2^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right]\\
&\times\left[(\mathbf{n}\wedge\mathbf{J}_1)\cdot(\mathbf{k}\wedge\mathbf{J}_2) + (\mathbf{n}\wedge\mathbf{J}_2)\cdot(\mathbf{k}\wedge\mathbf{J}_1)\right],
\end{aligned}
\tag{D.76}
$$

while the coefficient of $-\frac{S^{\mathbf{k}}S^{\mathbf{n}}}{2S^6}$ is

$$
\begin{aligned}
&-J_1^2\left[J_2^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_2^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right]^2 - J_2^2\left[J_1^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_1^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right]^2\\
&+2\mathbf{J}_1\cdot\mathbf{J}_2\left[J_1^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_1^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right]\left[J_2^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_2^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right].
\end{aligned}
\tag{D.77}
$$

The RHS of Eq. (D.75) becomes

$$
\begin{aligned}
&\frac{-1}{4S^4}\left[\left(J_2^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_2^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right)(\mathbf{n}\wedge\mathbf{J}_1) - \left(J_1^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_1^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right)(\mathbf{n}\wedge\mathbf{J}_2)\right]\\
&\cdot\left[\left(J_2^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_2^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right)(\mathbf{k}\wedge\mathbf{J}_1) - \left(J_1^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_1^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right)(\mathbf{k}\wedge\mathbf{J}_2)\right]\\
&+\frac{S^{\mathbf{k}}S^{\mathbf{n}}}{2S^6}\left[\mathbf{J}_1\left(J_2^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_2^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right) - \mathbf{J}_2\left(J_1^{\mathbf{S}\wedge\mathbf{n}}S^{\mathbf{k}} + J_1^{\mathbf{S}\wedge\mathbf{k}}S^{\mathbf{n}}\right)\right]^2,
\end{aligned}
\tag{D.78}
$$

which can be again rewritten, combining the appropriate terms into double vector products as

$$
\begin{aligned}
&\frac{-1}{4S^4}\left\{\mathbf{n}\wedge\left[(\mathbf{S}\wedge\mathbf{n})\wedge(\mathbf{J}_1\wedge\mathbf{J}_2)S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})\wedge(\mathbf{J}_1\wedge\mathbf{J}_2)S^{\mathbf{n}}\right]\right\}\\
&\cdot\left\{\mathbf{k}\wedge\left[(\mathbf{S}\wedge\mathbf{n})\wedge(\mathbf{J}_1\wedge\mathbf{J}_2)S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})\wedge(\mathbf{J}_1\wedge\mathbf{J}_2)S^{\mathbf{n}}\right]\right\}\\
&+\frac{S^{\mathbf{k}}S^{\mathbf{n}}}{2S^6}\left[(\mathbf{S}\wedge\mathbf{n})\wedge(\mathbf{J}_1\wedge\mathbf{J}_2)S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})\wedge(\mathbf{J}_1\wedge\mathbf{J}_2)S^{\mathbf{n}}\right]^2.
\end{aligned}
\tag{D.79}
$$

Recalling that $\mathbf{J}_1\wedge\mathbf{J}_2 = \mathbf{S}$, the above equation is again

$$
\begin{aligned}
&\frac{-1}{4S^4}\left\{\mathbf{n}\wedge\left[(\mathbf{S}\wedge\mathbf{n})\wedge\mathbf{S}S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})\wedge\mathbf{S}S^{\mathbf{n}}\right]\right\}\\
&\cdot\left\{\mathbf{k}\wedge\left[(\mathbf{S}\wedge\mathbf{n})\wedge\mathbf{S}S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})\wedge\mathbf{S})S^{\mathbf{n}}\right]\right\}\\
&+\frac{S^{\mathbf{k}}S^{\mathbf{n}}}{2S^6}\left[(\mathbf{S}\wedge\mathbf{n})\wedge\mathbf{S}S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})\wedge\mathbf{S}S^{\mathbf{n}}\right]^2.
\end{aligned}
\tag{D.80}
$$

We develop the double vector products in the first line and take into account that $\mathbf{n}\cdot(\mathbf{S}\wedge\mathbf{n}) = \mathbf{k}\cdot(\mathbf{S}\wedge\mathbf{k}) = 0$. For the second line we use $(\mathbf{S}\wedge\mathbf{A})^2 = S^2A^2 - (\mathbf{S}\cdot\mathbf{A})^2$ and $\mathbf{S}\cdot(\mathbf{S}\wedge\mathbf{n}) = \mathbf{S}\cdot(\mathbf{S}\wedge\mathbf{k}) = 0$ to rewrite the equation as

$$
\begin{aligned}
&\frac{-S^{\mathbf{n}}S^{\mathbf{k}}}{4S^4}\left\{-\left[\mathbf{n}\cdot(\mathbf{S}\wedge\mathbf{k})\right]\mathbf{S} + (\mathbf{S}\wedge\mathbf{n})S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})S^{\mathbf{n}}\right]\right\}\\
&\cdot\left\{-\left[\mathbf{k}\cdot(\mathbf{S}\wedge\mathbf{n})\right]\mathbf{S} + (\mathbf{S}\wedge\mathbf{n})S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})S^{\mathbf{n}}\right]\right\}\\
&+\frac{S^{\mathbf{k}}S^{\mathbf{n}}}{2S^4}\left[(\mathbf{S}\wedge\mathbf{n})S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})S^{\mathbf{n}}\right]^2.
\end{aligned}
\tag{D.81}
$$

Noting that the cross term in the first scalar product cancel (again as $\mathbf{S}\cdot(\mathbf{S}\wedge\mathbf{n}) = \mathbf{S}\cdot(\mathbf{S}\wedge\mathbf{k}) = 0$), and combining the remaining three terms, we get

$$
\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{4S^4}S^2\left[\mathbf{S}\cdot(\mathbf{n}\wedge\mathbf{k})\right]^2 + \frac{S^{\mathbf{k}}S^{\mathbf{n}}}{4S^4}\left[(\mathbf{S}\wedge\mathbf{n})S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})S^{\mathbf{n}}\right]^2.
\tag{D.82}
$$

Factoring $\mathbf{S}$ in the second term and using $A^2B^2 = (\mathbf{A}\cdot\mathbf{B})^2 + (\mathbf{A}\wedge\mathbf{B})^2$, this expression can be rewritten as

$$\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{4}\left[(\mathbf{n}\wedge\mathbf{k})^2 - \frac{[\mathbf{n}\mathbf{S}^{\mathbf{k}} - \mathbf{k}\mathbf{S}^{\mathbf{n}}]^2}{S^2} + \frac{[\mathbf{n}S^{\mathbf{k}} + \mathbf{k}S^{\mathbf{n}}]^2}{S^2} - \frac{4(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^4}\right], \tag{D.83}$$

thus we conclude

$$\begin{aligned}&\varepsilon_2^-\varepsilon_3^- J_2\sqrt{\Delta_2}J_3\sqrt{\Delta_3}\left[(2\xi_-^2-1)J_1^2 - (\mathbf{n}\cdot\mathbf{J}_1)(\mathbf{k}\cdot\mathbf{J}_1)\right] + \circlearrowleft_{123}\\ &= \frac{S^{\mathbf{n}}S^{\mathbf{k}}}{4}\left[(\mathbf{n}\wedge\mathbf{k})^2 + 4(\mathbf{n}\cdot\mathbf{k})\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2} - \frac{4(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^4}\right].\end{aligned} \tag{D.84}$$

D.5 Function at the Saddle Points

We evaluate the relevant angles at the points $\xi_\pm^2$ by substituting Eqs. (72) and (74) into (26).

D.5.1 The Angles $\phi_i^\pm$

For the angles $\phi_i^\pm$, the direct substitution yields

$$\begin{aligned}\iota\varepsilon_i^+\phi_i^+ &= \ln\frac{(\mathbf{n}\wedge\mathbf{J}_i)\cdot(\mathbf{k}\wedge\mathbf{J}_i) + \iota J_i\left[\mathbf{J}_i\cdot(\mathbf{n}\wedge\mathbf{k})\right]}{\sqrt{(\mathbf{n}\wedge\mathbf{J}_i)^2}\sqrt{(\mathbf{k}\wedge\mathbf{J}_i)^2}}\\ \iota\varepsilon_i^-\phi_i^- &= \ln\frac{(\mathbf{n}\wedge\mathbf{J}_i)\cdot(\mathbf{k}\wedge\mathbf{J}_i) - 2J_i^2\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2} + \iota J_i\frac{\mathbf{J}_i\cdot\left[(\mathbf{S}\wedge\mathbf{n})S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})S^{\mathbf{n}}\right]}{S^2}}{\sqrt{(\mathbf{n}\wedge\mathbf{J}_i)^2}\sqrt{(\mathbf{k}\wedge\mathbf{J}_i)^2}}.\end{aligned} \tag{D.85}$$

Consider first the denominator of $\iota\phi_i^-$ multiplied by S^2, namely

$$\begin{aligned}&S^2(\mathbf{n}\wedge\mathbf{J}_i)\cdot(\mathbf{k}\wedge\mathbf{J}_i) - 2J_i^2S^{\mathbf{n}}S^{\mathbf{k}} + \iota J_i\mathbf{J}_i\cdot\left[(\mathbf{S}\wedge\mathbf{n})S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})S^{\mathbf{n}}\right]\\ &= [\mathbf{S}\wedge(\mathbf{n}\wedge\mathbf{J}_i)]\cdot[\mathbf{S}\wedge(\mathbf{k}\wedge\mathbf{J}_i)] + [\mathbf{S}\cdot(\mathbf{n}\wedge\mathbf{J}_i)][\mathbf{S}\cdot(\mathbf{k}\wedge\mathbf{J}_i)]\\ &\quad -2J_i^2S^{\mathbf{n}}S^{\mathbf{k}} + \iota J_i\mathbf{J}_i\cdot\left[(\mathbf{S}\wedge\mathbf{n})S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})S^{\mathbf{n}}\right]\\ &= \left[\mathbf{n}\cdot(\mathbf{J}_i\wedge\mathbf{S}) + \iota J_iS^{\mathbf{n}}\right]\left[\mathbf{k}\cdot(\mathbf{J}_i\wedge\mathbf{S}) + \iota J_iS^{\mathbf{k}}\right],\end{aligned} \tag{D.86}$$

hence

$$\iota\varepsilon_i^-\phi_i^- = \iota\Phi_{\mathbf{n}}^i + \iota\Phi_{\mathbf{k}}^i \quad \iota\Phi_{\mathbf{n}}^i = \ln\frac{\mathbf{n}\cdot(\mathbf{J}_i\wedge\mathbf{S}) + \iota J_iS^{\mathbf{n}}}{S\sqrt{(\mathbf{n}\wedge\mathbf{J}_i)^2}}. \tag{D.87}$$

Note that

$$\begin{aligned}&\left[\mathbf{n}\cdot(\mathbf{J}_i\wedge\mathbf{S}) + \iota J_iS^{\mathbf{n}}\right]\left[\mathbf{k}\cdot(\mathbf{J}_i\wedge\mathbf{S}) - \iota J_iS^{\mathbf{k}}\right]\\ &= [\mathbf{S}\cdot(\mathbf{n}\wedge\mathbf{J}_i)][\mathbf{S}\cdot(\mathbf{k}\wedge\mathbf{J}_i)] + J_i^2S^{\mathbf{n}}S^{\mathbf{k}}\\ &\quad +\iota J_i\mathbf{J}_i\cdot\left[\mathbf{S}\wedge\left(\mathbf{k}S^{\mathbf{n}} - \mathbf{n}S^{\mathbf{k}}\right)\right]\\ &= S^2(\mathbf{n}\wedge\mathbf{J}_i)\cdot(\mathbf{k}\wedge\mathbf{J}_i) - [\mathbf{S}\wedge(\mathbf{n}\wedge\mathbf{J}_i)]\cdot[\mathbf{S}\wedge(\mathbf{k}\wedge\mathbf{J}_i)]\\ &\quad +J_i^2S^{\mathbf{n}}S^{\mathbf{k}} + \iota J_i\mathbf{J}_i\cdot[\mathbf{S}\wedge(\mathbf{S}\wedge(\mathbf{k}\wedge\mathbf{n}))],\end{aligned} \tag{D.88}$$

and developing the double vector products, taking into account $\mathbf{S}\cdot\mathbf{J}_i = 0$, we deduce

$$\iota\varepsilon_i^+\phi_i^+ = \iota\Phi_{\mathbf{n}}^i - \iota\Phi_{\mathbf{k}}^i. \tag{D.89}$$

D.5.2 The Angles $\varepsilon_1^\pm \psi_1^\pm - \varepsilon_3^\pm \psi_3^\pm$

We will denote in this section $\mathbf{A} \wedge \mathbf{B} = \mathbf{A}^{\wedge \mathbf{B}}$. A direct substitution of ξ_+^2 and ξ_-^2 yields

$$\begin{aligned} \iota\varepsilon_i^+ \psi_i^+ &= \ln \frac{J_i^{\mathbf{k}} - J_i^{\mathbf{n}}(\mathbf{n}\cdot\mathbf{k}) + \iota \mathbf{J}_i \cdot (\mathbf{n}\wedge\mathbf{k})}{\sqrt{[1-(\mathbf{n}\cdot\mathbf{k})^2]\,(\mathbf{n}\wedge\mathbf{J}_i)^2}} \\ \iota\varepsilon_i^- \psi_i^- &= \ln \frac{J_i^{\mathbf{k}} - J_i^{\mathbf{n}}(\mathbf{n}\cdot\mathbf{k}) + 2J_i^{\mathbf{n}} \frac{S^{\mathbf{n}} S^{\mathbf{k}}}{S^2} + \iota \frac{\mathbf{J}_i \cdot \left[(\mathbf{S}\wedge\mathbf{n})S^{\mathbf{k}} + (\mathbf{S}\wedge\mathbf{k})S^{\mathbf{n}}\right]}{S^2}}{\sqrt{\left[1-(\mathbf{n}\cdot\mathbf{k})^2 + 4(\mathbf{n}\cdot\mathbf{k})\frac{S^{\mathbf{n}} S^{\mathbf{k}}}{S^2} - 4\frac{(S^{\mathbf{n}} S^{\mathbf{k}})^2}{S^4}\right](\mathbf{n}\wedge\mathbf{J}_i)^2}}. \end{aligned} \tag{D.90}$$

To evaluate $\iota\varepsilon_1^+ \psi_1^+ - \iota\varepsilon_3^+ \psi_3^+$, we take apart the numerator

$$\begin{aligned} &\left[J_1^{\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})} + \iota J_1^{\mathbf{n}\wedge\mathbf{k}}\right]\left[J_3^{\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})} - \iota J_3^{\mathbf{n}\wedge\mathbf{k}}\right] \\ &\quad = -\mathbf{J}_1^{\wedge[\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})]}\cdot \mathbf{J}_3^{\wedge[\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})]} + \mathbf{J}_1\cdot\mathbf{J}_3\,(\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n}))^2 + J_1^{\mathbf{n}\wedge\mathbf{k}} J_3^{\mathbf{n}\wedge\mathbf{k}} \\ &\qquad + \iota\left(J_1^{\mathbf{n}\wedge\mathbf{k}} J_3^{\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})} - J_1^{\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})} J_3^{\mathbf{n}\wedge\mathbf{k}}\right). \end{aligned} \tag{D.91}$$

Taking into account $\mathbf{n}\cdot(\mathbf{k}\wedge\mathbf{n}) = 0$, this can be expressed as

$$\begin{aligned} &-(\mathbf{k}\wedge\mathbf{n})^2 J_1^{\mathbf{n}} J_3^{\mathbf{n}} + \mathbf{J}_1\cdot\mathbf{J}_3(\mathbf{k}\wedge\mathbf{n})^2 + \iota\mathbf{J}_1\cdot[\mathbf{J}_3\wedge[(\mathbf{n}\wedge\mathbf{k})\wedge(n\wedge(\mathbf{k}\wedge\mathbf{n}))]] \\ &\quad = (\mathbf{k}\wedge\mathbf{n})^2(\mathbf{n}\wedge\mathbf{J}_1)\cdot(\mathbf{n}\wedge\mathbf{J}_3) - \iota\mathbf{J}_1\cdot\left[\mathbf{J}_3\wedge\left[\mathbf{n}(\mathbf{n}\wedge\mathbf{k})^2\right]\right], \end{aligned} \tag{D.92}$$

hence

$$\iota\varepsilon_1^+ \psi_1^+ - \iota\varepsilon_3^+ \psi_3^+ = \ln \frac{(\mathbf{n}\wedge\mathbf{J}_1)\cdot(\mathbf{n}\wedge\mathbf{J}_3) - \iota\mathbf{n}\cdot(\mathbf{J}_1\wedge\mathbf{J}_3)}{\sqrt{(\mathbf{n}\wedge\mathbf{J}_1)^2(\mathbf{n}\wedge\mathbf{J}_3)^2}} = \iota\Psi_{\mathbf{n}}^{13}. \tag{D.93}$$

To evaluate $\iota\varepsilon_1^- \psi_1^- - \iota\varepsilon_3^- \psi_3^-$, we again consider apart the numerator

$$\left(J_1^{\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})+2\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}} + \iota J_1^{\mathbf{S}\wedge\mathbf{n}\frac{S^{\mathbf{k}}}{S^2}+\mathbf{S}\wedge\mathbf{k}\frac{S^{\mathbf{n}}}{S^2}}\right)\left(J_3^{\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})+2\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}} - \iota J_3^{\mathbf{S}\wedge\mathbf{n}\frac{S^{\mathbf{k}}}{S^2}+\mathbf{S}\wedge\mathbf{k}\frac{S^{\mathbf{n}}}{S^2}}\right). \tag{D.94}$$

The real part is

$$\begin{aligned} &J_1^{\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})+2\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}} J_3^{\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})+2\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}} + J_1^{\mathbf{S}\wedge\mathbf{n}\frac{S^{\mathbf{k}}}{S^2}+\mathbf{S}\wedge\mathbf{k}\frac{S^{\mathbf{n}}}{S^2}} J_3^{\mathbf{S}\wedge\mathbf{n}\frac{S^{\mathbf{k}}}{S^2}+\mathbf{S}\wedge\mathbf{k}\frac{S^{\mathbf{n}}}{S^2}} \\ &\quad = -\mathbf{J}_1^{\wedge\left[\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})+2\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right]}\cdot\mathbf{J}_3^{\wedge\left[\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})+2\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right]} \\ &\qquad + \mathbf{J}_1\cdot\mathbf{J}_3\left(\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n}) + 2\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right)^2 - \mathbf{J}_1^{\wedge\left[\mathbf{S}\wedge\mathbf{n}\frac{S^{\mathbf{k}}}{S^2}+\mathbf{S}\wedge\mathbf{k}\frac{S^{\mathbf{n}}}{S^2}\right]}\cdot\mathbf{J}_3^{\wedge\left[\mathbf{S}\wedge\mathbf{n}\frac{S^{\mathbf{k}}}{S^2}+\mathbf{S}\wedge\mathbf{k}\frac{S^{\mathbf{n}}}{S^2}\right]} \\ &\qquad + \mathbf{J}_1\cdot\mathbf{J}_3\left(\mathbf{S}\wedge\mathbf{n}\frac{S^{\mathbf{k}}}{S^2} + \mathbf{S}\wedge\mathbf{k}\frac{S^{\mathbf{n}}}{S^2}\right)^2, \end{aligned} \tag{D.95}$$

which is, holding in mind that $\mathbf{S}\cdot\mathbf{J}_i=0$,

$$
\begin{aligned}
&-\left(\mathbf{n}J_1^{\mathbf{k}\wedge\mathbf{n}}-(\mathbf{k}\wedge\mathbf{n})J_1^{\mathbf{n}}+2\mathbf{J}_1\wedge\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right)\\
&\quad\cdot\left(\mathbf{n}J_3^{\mathbf{k}\wedge\mathbf{n}}-(\mathbf{k}\wedge\mathbf{n})J_3^{\mathbf{n}}+2\mathbf{J}_3\wedge\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right)\\
&\quad-\mathbf{S}\left(J_1^{\mathbf{n}}\frac{S^{\mathbf{k}}}{S^2}+J_1^{\mathbf{k}}\frac{S^{\mathbf{n}}}{S^2}\right)\cdot\mathbf{S}\left(J_3^{\mathbf{n}}\frac{S^{\mathbf{k}}}{S^2}+J_3^{\mathbf{k}}\frac{S^{\mathbf{n}}}{S^2}\right)\\
&\quad+\mathbf{J}_1\cdot\mathbf{J}_3\left[[\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})]^2+4\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^4}+S^2\frac{(\mathbf{n}S^{\mathbf{k}}+\mathbf{k}S^{\mathbf{n}})^2}{S^4}-4\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^4}\right].
\end{aligned}
\tag{D.96}
$$

Developing the products in the first line, we get

$$
\begin{aligned}
&-J_1^{\mathbf{k}\wedge\mathbf{n}}J_3^{\mathbf{k}\wedge\mathbf{n}}-(\mathbf{n}\wedge\mathbf{k})^2J_1^{\mathbf{n}}J_3^{\mathbf{n}}-4\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^2}(\mathbf{J}_1\wedge\mathbf{n})\cdot(\mathbf{J}_3\wedge\mathbf{n})\\
&+2\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\left[(\mathbf{J}_1\wedge\mathbf{n})\cdot(\mathbf{k}\wedge\mathbf{n})J_3^{\mathbf{n}}+(\mathbf{J}_3\wedge\mathbf{n})\cdot(\mathbf{k}\wedge\mathbf{n})J_1^{\mathbf{n}}\right]\\
&-\frac{1}{S^2}\left(J_1^{\mathbf{n}}S^{\mathbf{k}}+J_1^{\mathbf{k}}S^{\mathbf{n}}\right)\left(J_3^{\mathbf{n}}S^{\mathbf{k}}+J_3^{\mathbf{k}}S^{\mathbf{n}}\right)+\mathbf{J}_1\cdot\mathbf{J}_3\left((\mathbf{n}\wedge\mathbf{k})^2+\frac{(\mathbf{n}S^{\mathbf{k}}+\mathbf{k}S^{\mathbf{n}})^2}{S^2}\right),
\end{aligned}
\tag{D.97}
$$

which is, expanding the second line,

$$
\begin{aligned}
&(\mathbf{J}_1\wedge\mathbf{n})\cdot(\mathbf{J}_3\wedge\mathbf{n})\left((\mathbf{n}\wedge\mathbf{k})^2-4\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^2}\right)-J_1^{\mathbf{k}\wedge\mathbf{n}}J_3^{\mathbf{k}\wedge\mathbf{n}}\\
&+2\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\left[J_1^{\mathbf{k}}J_3^{\mathbf{n}}+J_1^{\mathbf{n}}J_3^{\mathbf{k}}-2(\mathbf{n}\cdot\mathbf{k})J_1^{\mathbf{n}}J_3^{\mathbf{n}}\right]\\
&-\frac{1}{S^2}\left(J_1^{\mathbf{n}}S^{\mathbf{k}}+J_1^{\mathbf{k}}S^{\mathbf{n}}\right)\left(J_3^{\mathbf{n}}S^{\mathbf{k}}+J_3^{\mathbf{k}}S^{\mathbf{n}}\right)+\mathbf{J}_1\cdot\mathbf{J}_3\frac{(\mathbf{n}S^{\mathbf{k}}+\mathbf{k}S^{\mathbf{n}})^2}{S^2}.
\end{aligned}
\tag{D.98}
$$

Combining the cross terms in the second line and using $J_1^{\mathbf{k}\wedge\mathbf{n}}J_3^{\mathbf{k}\wedge\mathbf{n}}=J_1^{\mathbf{n}\wedge\mathbf{k}}J_3^{\mathbf{n}\wedge\mathbf{k}}$, we obtain

$$
\begin{aligned}
&(\mathbf{J}_1\wedge\mathbf{n})\cdot(\mathbf{J}_3\wedge\mathbf{n})\left((\mathbf{n}\wedge\mathbf{k})^2-4\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^2}\right)-4(\mathbf{n}\cdot\mathbf{k})J_1^{\mathbf{n}}J_3^{\mathbf{n}}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\\
&-J_1^{\mathbf{n}\wedge\mathbf{k}}J_3^{\mathbf{n}\wedge\mathbf{k}}-\frac{1}{S^2}J_1^{\mathbf{S}\wedge(\mathbf{n}\wedge\mathbf{k})}J_3^{\mathbf{S}\wedge(\mathbf{n}\wedge\mathbf{k})}+\mathbf{J}_1\cdot\mathbf{J}_3\frac{(\mathbf{n}S^{\mathbf{k}}+\mathbf{k}S^{\mathbf{n}})^2}{S^2}.
\end{aligned}
\tag{D.99}
$$

Computing the middle term on the second line using $\mathbf{S}\cdot\mathbf{J}_i=0$, the same expression is

$$
\begin{aligned}
&(\mathbf{J}_1\wedge\mathbf{n})\cdot(\mathbf{J}_3\wedge\mathbf{n})\left((\mathbf{n}\wedge\mathbf{k})^2-4\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^2}\right)-4(\mathbf{n}\cdot\mathbf{k})J_1^{\mathbf{n}}J_3^{\mathbf{n}}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\\
&+\mathbf{J}_1\cdot\mathbf{J}_3\frac{(\mathbf{n}S^{\mathbf{k}}+\mathbf{k}S^{\mathbf{n}})^2-(\mathbf{n}S^{\mathbf{k}}-\mathbf{k}S^{\mathbf{n}})^2}{S^2},
\end{aligned}
\tag{D.100}
$$

hence the real part is

$$
(\mathbf{J}_1\wedge\mathbf{n})\cdot(\mathbf{J}_3\wedge\mathbf{n})\left[(\mathbf{n}\wedge\mathbf{k})^2-4\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^2}+4\frac{S^{\mathbf{n}}S^{\mathbf{k}}(\mathbf{n}\cdot\mathbf{k})}{S^2}\right].
\tag{D.101}
$$

The imaginary part of the numerator (D.94) assumes the form

$$J_1^{\mathbf{S}\wedge\mathbf{n}\frac{S^{\mathbf{k}}}{S^2}+\mathbf{S}\wedge\mathbf{k}\frac{S^{\mathbf{n}}}{S^2}} J_3^{\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})+2\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}} - J_1^{\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})+2\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}} J_3^{\mathbf{S}\wedge\mathbf{n}\frac{S^{\mathbf{k}}}{S^2}+\mathbf{S}\wedge\mathbf{k}\frac{S^{\mathbf{n}}}{S^2}}. \tag{D.102}$$

We start by expressing it as

$$\begin{aligned} &-\mathbf{J}_1^{\wedge\left[\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})+2\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right]}\cdot\mathbf{J}_3^{\wedge\left[\mathbf{S}\wedge\mathbf{n}\frac{S^{\mathbf{k}}}{S^2}+\mathbf{S}\wedge\mathbf{k}\frac{S^{\mathbf{n}}}{S^2}\right]} \\ &+\mathbf{J}_3^{\wedge\left[\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})+2\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right]}\cdot\mathbf{J}_1^{\wedge\left[\mathbf{S}\wedge\mathbf{n}\frac{S^{\mathbf{k}}}{S^2}+\mathbf{S}\wedge\mathbf{k}\frac{S^{\mathbf{n}}}{S^2}\right]}, \end{aligned} \tag{D.103}$$

as the cross terms in the development of the two scalar products cancel. This computes further to

$$\begin{aligned} &-\left(\mathbf{n}J_1^{\mathbf{k}\wedge\mathbf{n}} - (\mathbf{k}\wedge\mathbf{n})J_1^{\mathbf{n}} + 2\mathbf{J}_1\wedge\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right)\cdot\mathbf{S}\left(J_3^{\mathbf{n}}\frac{S^{\mathbf{k}}}{S^2}+J_3^{\mathbf{k}}\frac{S^{\mathbf{n}}}{S^2}\right) \\ &+\left(\mathbf{n}J_3^{\mathbf{k}\wedge\mathbf{n}} - (\mathbf{k}\wedge\mathbf{n})J_3^{\mathbf{n}} + 2\mathbf{J}_3\wedge\mathbf{n}\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right)\cdot\mathbf{S}\left(J_1^{\mathbf{n}}\frac{S^{\mathbf{k}}}{S^2}+J_1^{\mathbf{k}}\frac{S^{\mathbf{n}}}{S^2}\right). \end{aligned} \tag{D.104}$$

Grouping together similar terms, one should end up with

$$\begin{aligned} &\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}(\mathbf{J}_3\wedge\mathbf{J}_1)\cdot((\mathbf{k}\wedge\mathbf{n})\wedge\mathbf{n}) + \frac{(S^{\mathbf{n}})^2}{S^2}(\mathbf{J}_3\wedge\mathbf{J}_1)\cdot((\mathbf{k}\wedge\mathbf{n})\wedge\mathbf{k}) \\ &-[(\mathbf{k}\wedge\mathbf{n})\cdot\mathbf{S}]\frac{S^{\mathbf{n}}}{S^2}(\mathbf{J}_3\wedge\mathbf{J}_1)\cdot(\mathbf{n}\wedge\mathbf{k}) \\ &+2\frac{S^{\mathbf{n}}(S^{\mathbf{k}})^2}{S^4}[(\mathbf{n}\wedge(\mathbf{J}_3\wedge\mathbf{J}_1))\wedge\mathbf{n}]\cdot\mathbf{S} + 2\frac{(S^{\mathbf{n}})^2S^{\mathbf{k}}}{S^4}[(\mathbf{k}\wedge(\mathbf{J}_3\wedge\mathbf{J}_1))\wedge\mathbf{n}]\cdot\mathbf{S}. \end{aligned} \tag{D.105}$$

Recognizing $\mathbf{S} = \mathbf{J}_3\wedge\mathbf{J}_1$, the above can be written

$$\begin{aligned} &\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}[S^{\mathbf{n}}(\mathbf{n}\cdot\mathbf{k}) - S^{\mathbf{k}}] + \frac{(S^{\mathbf{n}})^2}{S^2}[S^{\mathbf{n}} - S^{\mathbf{k}}(\mathbf{n}\cdot\mathbf{k})] + \frac{S^{\mathbf{n}}}{S^2}[\mathbf{S}\cdot(\mathbf{n}\wedge\mathbf{k})]^2 \\ &\quad +2\frac{S^{\mathbf{n}}(S^{\mathbf{k}})^2}{S^4}\left(-(S^{\mathbf{n}})^2+S^2\right) + 2\frac{(S^{\mathbf{n}})^2S^{\mathbf{k}}}{S^4}\left(-S^{\mathbf{n}}S^{\mathbf{k}} + (\mathbf{n}\cdot\mathbf{k})S^2\right) \\ &= S^{\mathbf{n}}\left[(\mathbf{n}\wedge\mathbf{k})^2 - \frac{(\mathbf{n}S^{\mathbf{k}} - \mathbf{k}S^{\mathbf{n}})^2}{S^2} - \frac{(S^{\mathbf{k}})^2}{S^2} + \frac{(S^{\mathbf{n}})^2}{S^2} - 4\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^4}\right. \\ &\quad \left. +2\frac{(S^{\mathbf{k}})^2}{S^2} + 2(\mathbf{n}\cdot\mathbf{k})\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2}\right] \\ &= S^{\mathbf{n}}\left[(\mathbf{n}\wedge\mathbf{k})^2 + 4\frac{S^{\mathbf{n}}S^{\mathbf{k}}}{S^2} - 4\frac{(S^{\mathbf{n}}S^{\mathbf{k}})^2}{S^4}\right]. \end{aligned} \tag{D.106}$$

In conclusion, $\imath\varepsilon_1^-\psi_1^- - \imath\varepsilon_3^-\psi_3^-$ is

$$\imath\varepsilon_1^-\psi_1^- - \imath\varepsilon_3^-\psi_3^- = \ln\frac{(\mathbf{n}\wedge\mathbf{J}_1)\cdot(\mathbf{n}\wedge\mathbf{J}_3) + \imath\mathbf{n}\cdot(\mathbf{J}_3\wedge\mathbf{J}_1)}{\sqrt{(\mathbf{n}\wedge\mathbf{J}_1)^2(\mathbf{n}\wedge\mathbf{J}_3)^2}} = \imath\Psi_{\mathbf{n}}^{13}. \tag{D.107}$$

Following similar manipulations, we get

$$\imath\varepsilon_2^{\pm}\psi_2^{\pm} - \imath\varepsilon_3^{\pm}\psi_3^{\pm} = \ln\frac{(\mathbf{n}\wedge\mathbf{J}_2)\cdot(\mathbf{n}\wedge\mathbf{J}_3) + \imath\mathbf{n}\cdot(\mathbf{J}_3\wedge\mathbf{J}_2)}{\sqrt{(\mathbf{n}\wedge\mathbf{J}_2)^2(\mathbf{n}\wedge\mathbf{J}_3)^2}} = \imath\Psi_{\mathbf{n}}^{23}. \tag{D.108}$$

For the angles ω_i, recall that $\omega_{\mathbf{n},\mathbf{k}}$ can be written in terms of $\psi_{-\mathbf{k},-\mathbf{n}}$. Note that due to the choice of the determination of the $\sqrt{\Delta_i^+}$ the correct relation is $\omega_{\mathbf{n},\mathbf{k}}^+ = -\psi_{-\mathbf{k},-\mathbf{n}}^+$ and $\omega_{\mathbf{n},\mathbf{k}}^- = \psi_{-\mathbf{k},-\mathbf{n}}^-$. Moreover, as $\Psi_{-\mathbf{k}} = -\Psi_{\mathbf{k}}$, we conclude

$$\imath\varepsilon_1^{\pm}\omega_1^{\pm} - \imath\varepsilon_3^{\pm}\omega_3^{\pm} = \pm\imath\Psi_{\mathbf{k}}^{13}, \quad \imath\varepsilon_2^{\pm}\omega_2^{\pm} - \imath\varepsilon_3^{\pm}\omega_3^{\pm} = \pm\imath\Psi_{\mathbf{k}}^{23}. \tag{D.109}$$

E Boundary Terms in the Euler Maclaurin Formula

Using the short hand notation $F(t)$ for $F(J,M,M',t)$, the remainder terms in the EM formula are expressed as follows:

$$-B_1\left[F(t_{\max})+F(t_{\min})\right]+\sum_k \frac{B_{2k}}{(2k)!}\left[F^{(2k-1)}(t_{\max})-F^{(2k-1)}(t_{\min})\right]. \tag{E.110}$$

In this section, we deal with generic Wigner matrices, namely we consistently assume that $0<\xi^2<1$. Note that

$$t_{\min}=\max\{0,M-M'\},\quad t_{\max}=\min\{J+M,J-M'\}. \tag{E.111}$$

For simplicity, we will detail the diagonal matrix elements $M=M'$. By continuity, the region in which our results apply extends to some strip $|M-M'|<P$. For such elements $t_{\min}=0$ and, for $M>0, t_{\max}=J-M$. The Stirling approximations become easily upper and lower bounds, at the price of some constants, thus by Appendix A we obtain

$$\frac{C^{\min}}{J}\sqrt{K(x,x,u)}e^{J\Re f(x,x,u)}<|F(t)|<\frac{C^{\max}}{J}\sqrt{K(x,x,u)}e^{J\Re f(x,x,u)}, \tag{E.112}$$

with

$$\begin{aligned} f(x,x,u) \;=\; & -\imath(\alpha+\gamma)x+\imath\pi u+(1-u)\ln\xi^2+u\ln(1-\xi^2)\\ & +(1-x)\ln(1-x)+(1+x)\ln(1+x)-2u\ln u\\ & -(1+x-u)\ln(1+x-u)-(1-x-u)\ln(1-x-u). \end{aligned} \tag{E.113}$$

and

$$K(x,u)=\frac{(1-x^2)}{(1+x-u)(1-x-u)u^2}. \tag{E.114}$$

The behavior of the higher derivative terms in the EM formula is governed by $F^{(k)}(t_{\min})$ and $F^{(k)}(t_{\max})$. To see this, collect all factors depending on t in $F(t)$ and write

$$\begin{aligned} F(t) &= q(J,M)\; p_{J,M}(t),\\ p_{J,M}(t) &= \frac{e^{At}}{\Gamma(J+M-t+1)\Gamma(J-M-t+1)[\Gamma(t+1)]^2},\\ A &:= \imath[\pi-2\ln\xi-2\ln\eta]. \end{aligned} \tag{E.115}$$

Hence $F^{(k)}=q(J,M)p^{(k)}_{J,M}$, and the first derivative can be expressed in terms of

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}p_{J,M}(t) &= p_{J,M}(t)\left\{A+\frac{\Gamma'(J+M-t+1)}{\Gamma(J+M-t+1)}\right.\\ &\qquad \left.+\frac{\Gamma'(J-M-t+1)}{\Gamma(J-M-t+1)}-2\frac{\Gamma'(t+1)}{\Gamma(t+1)}\right\}\\ &= p_{J,M}(t)\left\{A+\psi^{(0)}(J+M-t+1)\right.\\ &\qquad \left.+\psi^{(0)}(J-M-t+1)-2\psi^{(0)}(t+1)\right\}, \end{aligned} \tag{E.116}$$

with $\psi^{(0)}(t)$ denoting the digamma function. For integer arguments

$$\psi^{(0)}(m+1) = -\gamma_0 + \sum_{k=1}^{m} \frac{1}{k}, \tag{E.117}$$

hence $|F'(t)| < C \ln J \, |F(t)|$ for some constant C. Higher order derivatives of Eq. (E.116) can be written in terms of higher order polygamma functions $\psi^{(n)} = d^n \psi^{(0)}/dt^n$. For all k, $\psi^{(2k)}(X) \leq \psi^{(0)}(X)$ at large X, therefore the k'th derivative is dominated by

$$F^{(2k-1)}(t) = F(t) \left\{ \left[A + \sum_{i=1}^{4} \pm \psi^{(0)}(X_i) \right]^{2k-1} + \cdots \right\}. \tag{E.118}$$

Then $|F^{(k)}| < C(\ln J)^k |F(t)|$ for some constant C.

From Eq. (E.112), we conclude that both $|F(t_{\min})|$ and $|F(t_{\max})|$, as well as all their derivatives are a priori exponentially suppressed in the region where $\Re f(x,y,u_{\min}) < 0$ and $\Re f(x,y,u_{\max}) < 0$. As

$$\begin{aligned} \Re f(x,x,0) &= \ln \xi^2 \\ \Re f(x,x,1-x) &= x \ln \xi^2 + (1-x)\ln(1-\xi^2) \\ &\quad + (1+x)\ln(1+x) - (1-x)\ln(1-x) - 2x\ln(2x), \end{aligned} \tag{E.119}$$

we infer that the derivative corrections coming from $t_{\min} = 0$ are always suppressed term by term. However the situation is markedly different for the corrections coming from $t_{\max} = J - M$. At fixed ξ^2, the corrections are exponentially suppressed for x close enough to either 0 or 1, but the maximum of $\Re f(x,x,1-x)$, achieved for $x = \frac{\xi}{\sqrt{4-3\xi^2}}$ is $\ln \frac{(\xi+\sqrt{4-3\xi^2})^2}{4} > 0$, hence there exists some interval in which, term by term, the derivative terms are bounded from below by an exponential blow up. In this region our EM SPA approximation should a priori fail (see also Fig. 3).

A second set of EM derivative terms come when passing from Eqs. (36) to (37), involving derivatives $\frac{\partial^n}{(\partial_x)^n} D^J_{xJ,xJ}|_{x=\pm 1}$. Using Appendix C, Eq. (C.39) we note that all these derivatives yield some function times $D^J_{xJ,xJ}$. As

Fig. 3 *Shaded region* where the EM corrections are exponentially suppressed

$$D^{J}_{-J,-J}(g) = \xi^{2J} e^{+\imath(\alpha+\gamma)J}, \quad D^{J}_{JJ}(g) = \xi^{2J} e^{-\imath(\alpha+\gamma)J}, \tag{E.120}$$

all such derivative terms are exponentially suppressed for large J.

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