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Harmonic Oscillator with a Step and/or a Ramp

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Abstract. We discuss the one-dimensional Schrödinger equation for a harmonic oscillator with a finite step at the origin and/or an external field described by a ramp function. The first half of this paper is a partial review of our recent work. The latter half is devoted to an extension of the problem, i.e., imposing an external linear field on the negative half line. The solvability of the problem via the Hermite polynomials is discussed. We demonstrate that a harmonic oscillator with a step and a ramp can have one eigenstate whose wavefunction is expressed in terms of Hermite polynomials of different orders. Explicit examples are also provided at appropriate places in the text.

1. Introduction

In quantum theories, a variety of exactly solvable models have extensively been considered to understand related physical phenomena. Among them, the harmonic oscillator is probably the most basic and important one. We discuss exact solutions of the time-independent, one-dimensional Schrödinger equation:

$$-\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \ , \tag{1}$$

particularly with deformations of the harmonic oscillator in this study.

The exactly solvable potentials for the Schrödinger equation (1) can be categorized into several classes. One major class would be the piecewise constant potentials, including the so-called point interaction models. The finite square-well potential and the Kronig–Penney model are two typical examples. The essence of their solution is the matching-of-wavefunctions technique. Another major class is analytical potentials whose eigenfunctions are written in closed analytic form. The shape-invariant potentials, such as the harmonic oscillator and the Coulomb problem, are classified in this devision. A notable feature of the shape-invariant potentials is that their solvability is guaranteed by the orthogonal polynomials. The problems lying in the intersection between these two classes, that is, potentials defined by piecewise analytic functions, have also attracted attention (For recent works, see, e.g., references [1, 2, 3, 4, 5, 6]).

One of the subclasses in this intersection is potentials made up of a harmonic oscillator plus singularity functions. In reference [7], a Dirac delta function at the origin $\delta(x)$ under a harmonic oscillator has been discussed. Moreover, our recent work [6] has dealt with a harmonic oscillator with a Heaviside step function $\theta(x)$. A natural extension of these problems is to consider a harmonic oscillator plus some other singularity functions $x\theta(x), x^2\theta(x), \ldots$ or $\delta^{(n)}(x) \equiv d^n\delta(x)/dx^n$. In this paper, we discuss the case with $x\theta(x)$, which is often referred to

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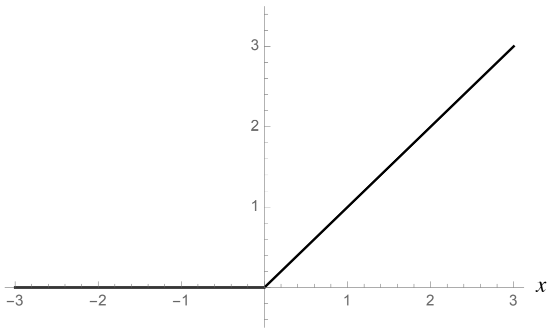

Figure 1. The ramp function, $x\,\theta(x)$. It is named after the shape of its graph.

as the ramp function (See figure 1). A potential described by the ramp function can be seen as an external linear field, such as an electric field, on a half line.

Since our problems are non-analytic at $x = 0$, the Schrödinger equation are to be solved under the following boundary condition (or matching condition) at $x = 0$:

$$\lim_{x\to 0^+} \psi(x) = \lim_{x\to 0^-} \psi(x) \ , \qquad \lim_{x\to 0^+} \frac{d\psi(x)}{dx} = \lim_{x\to 0^-} \frac{d\psi(x)}{dx} \ . \tag{2}$$

In addition, we summarize some of our notations here for later use. The general solution of the following second-order ordinary differential equation:

$$-\frac{d^2\psi(x)}{dx^2} + \left(x^2 - 1\right)\psi(x) = E\psi(x) \ , \tag{3}$$

is

$$\psi(x) = \mathrm{e}^{-\frac{x^2}{2}} \left[\alpha\, {}_1F_1\left(-\frac{E}{4}; \frac{1}{2}; x^2\right) + \beta x\, {}_1F_1\left(-\frac{E-2}{4}; \frac{3}{2}; x^2\right)\right] \ , \tag{4}$$

where α and β are constants, and ${}_1F_1(a; c; x)$ denotes the Kummer's confluent hypergeometric function. In the case of the ordinary harmonic oscillator, we require the square-integrability of $\psi(x)$ in $(-\infty, \infty)$, and get

$$E = E_n = 2n \ , \quad \psi(x) = \phi_n(x) = \mathrm{e}^{-\frac{x^2}{2}} H_n(x) \ , \quad n = 0, 1, 2, \ldots \ , \tag{5}$$

where $H_n(x)$ is the n-th order Hermite polynomial.

2. Harmonic Oscillator with a Step

In this section, we summarize the first half of what are discussed in reference [6].

2.1. The potential

In reference [6], we have considered a potential, which is a combination of a harmonic oscillator and a finite step,

$$V(x) = \begin{cases} x^2 - 1 - a & (x < 0) \\ x^2 - 1 & (x > 0) \end{cases} \ , \tag{6}$$

where a is a positive constant. This is a confining potential, so it has infinitely many discrete eigenvalues, $\{E_n\}$ $(n = 0, 1, 2, \ldots)$, where the n-th excited state has the energy E_n. The corresponding wavefunctions $\{\psi_n(x)\}$ are square integrable, $\psi_n(x) \in L^2(\mathbb{R})$, which leads to the boundary conditions at $x \to \pm\infty$: $\psi(x) \to 0$.

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2.2. The solutions

For arbitrary $a > 0$, the n-th wavefunction is obtained as

$$\psi_n(x) = \begin{cases} \mathrm{e}^{-\frac{x^2}{2}} \left[\alpha\, {}_1F_1\left(-\frac{E_n+a}{4}; \frac{1}{2}; x^2\right) + \beta x\, {}_1F_1\left(-\frac{E_n+a-2}{4}; \frac{3}{2}; x^2\right)\right] & (x<0) \\ \mathrm{e}^{-\frac{x^2}{2}} \left[\alpha\, {}_1F_1\left(-\frac{E_n}{4}; \frac{1}{2}; x^2\right) + \beta x\, {}_1F_1\left(-\frac{E_n-2}{4}; \frac{3}{2}; x^2\right)\right] & (x>0) \end{cases} , \tag{7}$$

where α and β are constants, and E_n is a root of the following transcendental equation:

$$-\frac{\Gamma\left(-\frac{E}{4}\right)}{\Gamma\left(-\frac{E-2}{4}\right)} = \frac{\Gamma\left(-\frac{E+a}{4}\right)}{\Gamma\left(-\frac{E+a-2}{4}\right)} , \tag{8}$$

which comes from the boundary conditions at $x \to \pm\infty$. As is often the case with square-well potentials, we are to solve this equation graphically.

2.3. For specific choices of $a = 4\ell$ ($\ell = 1, 2, \ldots$)

When a takes 4ℓ ($\ell = 1, 2, \ldots$), the wavefunction (7) is reduced to a rather simple expression for all the non-negative energy states, that is, the eigenfunctions are expressed in terms of the Hermite polynomials. Also, these states are isospectral to the ordinary harmonic oscillator ($a \to 0$). The point, regarding the construction of such wavefunctions, is that the eigenfunctions of the ordinary harmonic oscillator have either zero or extremum at the origin regardless of the parameters, which is guaranteed by the parity of the potential.

The solutions are

$$E_n = 2(n-\ell) , \quad \psi_n(x) = \begin{cases} \mathcal{N}_n\, \mathrm{e}^{-\frac{x^2}{2}} H_{n+\ell}(x) & (x<0) \\ \mathrm{e}^{-\frac{x^2}{2}} H_{n-\ell}(x) & (x>0) \end{cases} , \quad n = \ell, \ell+1, \ell+2, \ldots , \tag{9}$$

where

$$\mathcal{N}_n = (-1)^\ell \frac{(n-\ell)!\left(\frac{n+\ell}{2}\right)!}{(n+\ell)!\left(\frac{n-\ell}{2}\right)!} \qquad \text{if } (n-\ell) \text{ is even} , \tag{10a}$$

$$\mathcal{N}_n = (-1)^\ell \frac{(n-\ell)!\left(\frac{n+\ell-1}{2}\right)!}{(n+\ell)!\left(\frac{n-\ell-1}{2}\right)!} \qquad \text{if } (n-\ell) \text{ is odd} . \tag{10b}$$

For n's lower than ℓ, the wavefunctions are no longer expressed by the Hermite polynomials, and we need to go back to equation (7) itself. On the other hand, as for the energy eigenvalues, the transcendental equation (8) is reduced to an algebraic equation of degree ℓ:

$$-\prod_{k=1}^{\ell}(E+4k-2) = \prod_{k=1}^{\ell}(E+4k) . \tag{11}$$

Note that the ℓ roots of this equation has the following property: *if $E = -1-2\ell+\alpha$ is a root, $E = -1-2\ell-\alpha$ is also a root.* For odd ℓ, $E = -1-2\ell$ is also a root, which corresponds to $\alpha = 0$.

Remarks. For explicit examples, see reference [6]. In reference [6], the authors have further investigated the spectral properties of the potential (6). They have discussed several isospectral transformations of the potential, and showed that it is possible to construct infinitely many potentials whose energy spectra coincide completely with that of the ordinary harmonic oscillator.

3. Harmonic Oscillator with a Step and a Ramp

In this paper, we consider a harmonic oscillator with a step function and a ramp function. This potential can describe the system (6) under an external field, such as an electric field.

3.1. The potential

We add a linear potential $-gx$ to the potential (6) for $x < 0$,

$$V(x) = \begin{cases} x^2 - 1 - a - gx = \left(x - \frac{g}{2}\right)^2 - 1 - a - \frac{g^2}{4} & (x < 0) \\ x^2 - 1 & (x > 0) \end{cases} , \tag{12}$$

where g is a real constant. This is also a confining potential and has infinitely many discrete eigenvalues $\{E_n\}$. Taking $g = 0$ coincides with the potential (6).

Remark. In this paper, we restrict ourselves to $a > 0$ in the potential (12). Note that unlike the case of a harmonic oscillator with a step (section 2), the constraint on a breaks the generality. For $a < 0$, discussions are almost parallel to those for $a > 0$ (See the following), but a kind of double-well potentials appear and they are more likely to be physically applicable.

3.2. The solutions

One can construct the eigenfunctions for arbitrary a and g:

$$\psi_n(x) = \begin{cases} \mathrm{e}^{-\frac{(x-\frac{g}{2})^2}{2}} \left[\alpha_- \, {}_1F_1\left(-\frac{E_n + a + \frac{g^2}{4}}{4}; \frac{1}{2}; \left(x - \frac{g}{2}\right)^2\right) \right. \\ \qquad \left. + \beta_- \left(x - \frac{g}{2}\right) {}_1F_1\left(-\frac{E_n + a + \frac{g^2}{4} - 2}{4}; \frac{3}{2}; \left(x - \frac{g}{2}\right)^2\right) \right] & (x < 0) \\ \mathrm{e}^{-\frac{x^2}{2}} \left[\alpha_+ \, {}_1F_1\left(-\frac{E_n}{4}; \frac{1}{2}; x^2\right) + \beta_+ \, x \, {}_1F_1\left(-\frac{E_n - 2}{4}; \frac{3}{2}; x^2\right) \right] & (x > 0) \end{cases} , \tag{13}$$

in which $\alpha_\pm, \beta_\pm$ are constants. From the boundary conditions at $x = 0$, α_+ and β_+ are

$$\alpha_+ = \psi_n(0^-) \ , \quad \beta_+ = \frac{d\psi_n(0^-)}{dx} \ .$$

On the other hand, those at $x \to \pm\infty$ yield the following simultaneous transcendental equations:

$$\frac{1}{\psi_n(0^-)} \frac{d\psi_n(0^-)}{dx} = -\frac{2\Gamma\left(-\frac{E_n - 2}{4}\right)}{\Gamma\left(-\frac{E_n}{4}\right)} \ , \quad \frac{\beta_-}{\alpha_-} = \frac{2\Gamma\left(-\frac{E_n + a + \frac{g^2}{4} - 2}{4}\right)}{\Gamma\left(-\frac{E_n + a + \frac{g^2}{4}}{4}\right)} \ , \tag{14}$$

which are to be solved graphically, and determine the energy eigenvalues $\{E_n\}$ as is shown in the following example.

Example 1: $a = 2, \ g = 1$. We first solve equations (14) with $a = 2$ and $g = 1$ to obtain the energy spectrum (See figure 2). The first several energy eigenvalues are displayed in the caption of figure 2 with six digits. With the knowledge of the energy spectrum, one can determine the coefficients $\alpha_\pm, \beta_\pm$ for each n, and therefore the eigenfunction $\psi_n(x)$. The solution of the Schrödinger equation (1) is summarized in figure 3.

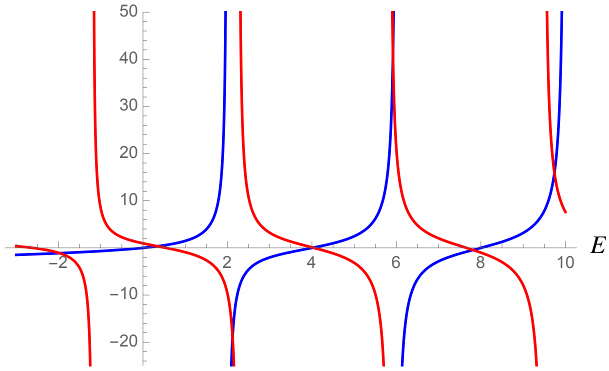


Figure 2. Graphical solution of (14) for *Example 1*. The blue curves correspond to the left hand side of the equation, while the red ones are the right hand side. The first several energy eigenvalues are determined as follows: $E_0 \approx -1.97196$, $E_1 \approx 0.343665$, $E_2 \approx 2.12101$, $E_3 \approx 4.02740$, $E_4 \approx 5.91817$, $E_5 \approx 7.81348$, $E_6 \approx 9.74097$.

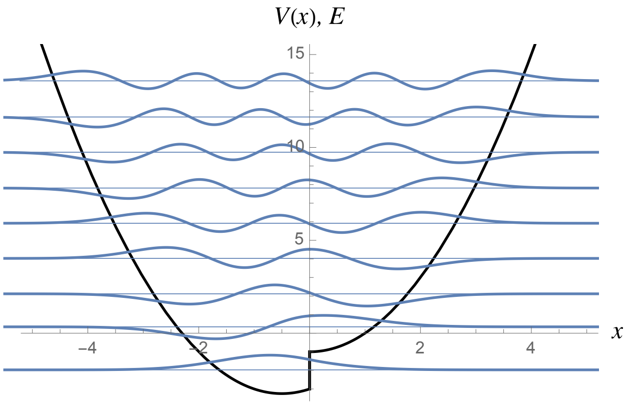


Figure 3. The solution of the Schrödinger equation (1) for *Example 1*. Thin blue lines show the energy spectrum, and the blue curve on each line is the corresponding eigenfunction. The potential (12) with $a = 2$, $g = 1$ is also plotted in this figure by a black curve.

3.3. Case $a \to 0$: Harmonic oscillator with a ramp

Here, let us concentrate on the case $a \to 0$, where the potential consists of a harmonic oscillator plus a ramp function only. In this subsection, we show how the energy spectrum changes as the external field is imposed. Remember that for the case of a harmonic oscillator plus homogeneous external field, what happens is a constant shift of energies. However, for our present case, figure 4 shows that that is not the case and the spectrum is never equidistant except for $g = 0$. For each n, the energy eigenvalue E_n increases monotonically in g.

3.4. Quasi Hermite-polynomial solvability

A potential is said to be *quasi*-exactly solvable, when several eigenstates are explicitly obtained whereas the others are not [8, 9]. In our model (12), we can make only one state solvable via the Hermite polynomials, while for other states the wavefunctions are expressed only by the confluent hypergeometric functions and they are not reduced to any orthogonal polynomials. This situation is similar to those in references [4, 5].

The construction is as follows. First we choose g such that $\phi_m(x - g/2)$ has one of either zeros or extrema at $x = 0$ (See table 1). Suppose that $\phi_m(-g/2)$ is the j-th zero [extremum] from the left. Then, when the remaining model parameter a is set to

$$a = 2k - \frac{g^2}{4} , \tag{15}$$

where $k \in \mathbb{Z}_{>0}$ is smaller than or equal to, and of the opposite parity to [same parity as] m, the $(j + \frac{m-k-1}{2})$-th [$(j + \frac{m-k}{2} - 1)$-th] excited state is Hermite-polynomially solvable. Such state is of the energy eigenvalue

$$E_{j+\frac{m-k-1}{2}} = 2j + m - k - 1 \quad \left[E_{j+\frac{m+k}{2}-1} = 2j + m - k - 2\right] , \tag{16}$$

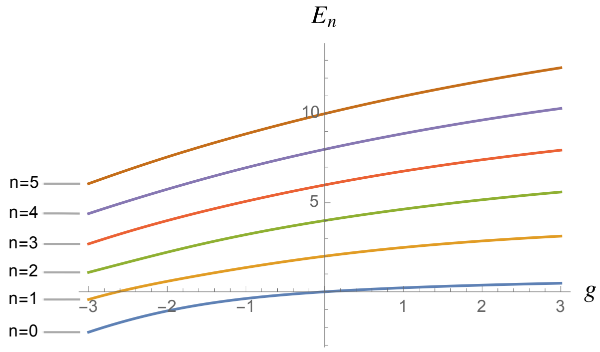


Figure 4. The first six eigenvalues E_n as functions of g for $a = 0$. They are all monotonically increasing in g, but never be equidistant for $g \neq 0$.

and the corresponding wavefunction is

$$\psi_{j+\frac{m-k-1}{2}}(x)\left[\psi_{j+\frac{m+k}{2}-1}(x)\right] = \begin{cases} \mathcal{N}^{(-)}\mathrm{e}^{-\frac{(x-\frac{g}{2})^2}{2}} H_m\left(x-\frac{g}{2}\right) & (x<0) \\ \mathcal{N}^{(+)}\mathrm{e}^{-\frac{x^2}{2}} H_{m-k}(x) & (x>0) \end{cases} \tag{17}$$

with $\mathcal{N}^{(\pm)}$ are constants to be determined from the boundary condition at $x = 0$.

Example 2: $g = \mp\sqrt{2},\ k = 1.$ Let us pick such g's that $\phi_2(x - g/2)$ has an extremum at $x = 0$. There are two extrema, $j = \{1, 2\}$, and $-g/2 = \pm 1/\sqrt{2}$. Then, only $k = 1$ is allowed, and a is specified as $a = 3/2$.

For $g = -\sqrt{2}$, the first excited-state wavefunction consists of Hermite polynomials,

$$\psi_1(x) = \begin{cases} 2\mathrm{e}^{\frac{1}{4}}\mathrm{e}^{-\frac{\left(x+\frac{1}{\sqrt{2}}\right)^2}{2}} H_2\left(x+\frac{1}{\sqrt{2}}\right) & (x<0) \\ \mathrm{e}^{-\frac{x^2}{2}} H_1(x) & (x>0) \end{cases}, \tag{18}$$

and the energy is $E_1 = 2$. On the other hand, for $g = \sqrt{2}$, the Hermite polynomials constitute the second excited-state wavefunction with the energy $E_2 = 4$:

$$\psi_2(x) = \begin{cases} -2\mathrm{e}^{\frac{1}{4}}\mathrm{e}^{-\frac{\left(x-\frac{1}{\sqrt{2}}\right)^2}{2}} H_2\left(x-\frac{1}{\sqrt{2}}\right) & (x<0) \\ -\mathrm{e}^{-\frac{x^2}{2}} H_1(x) & (x>0) \end{cases}. \tag{19}$$

In order to obtain the other eigenstates, one needs to follow the solution explained in section 3.2. The solutions of the Schrödinger equation (1) are plotted in figure 5.

3.5. Construction of a sequence of quasi Hermite-polynomial solvable potentials

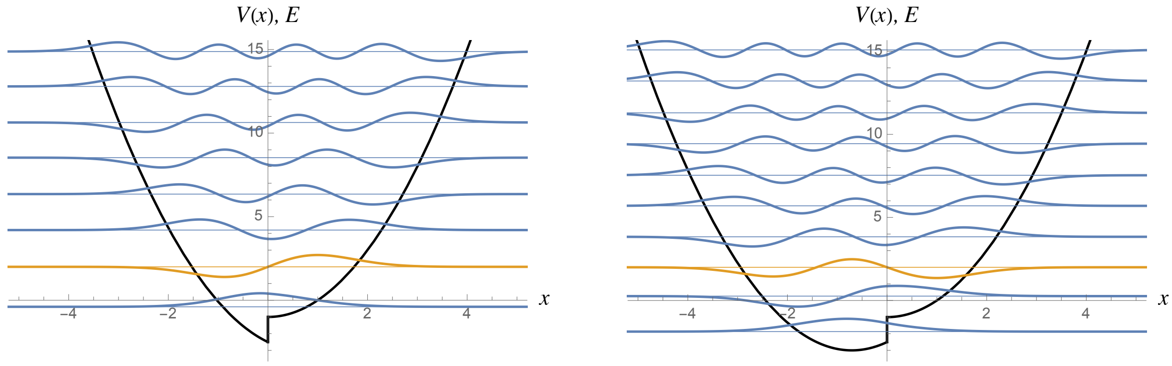
One application of our present work is to construct a sequence of the solvable potentials where only the ground-state wavefunction can be expressed by Hermite polynomials of different orders.

Such sequence is constructed as follows. First we choose g such that $\phi_m(-g/2)$ is the first extremum from the left of $\phi_m(x - g/2)$. Here, m can be any non-negative integer, and we choose $k = m$. Then we identify the parameter a using equation (15). In this manner, one can construct infinitely many potentials whose ground-state wavefunctions are expressed by the Hermite polynomials but other eigenfunctions are not. We show first several potentials $V_m(x)$, $m = 1, 2, 3, 4$, in figure 6a and the ground-state eigenfunctions $\psi_0^{(m)}(x)$:

$$\psi_0^{(m)}(x) = \begin{cases} \mathcal{N}_m\mathrm{e}^{-\frac{(x-\frac{g}{2})^2}{2}} H_m\left(x-\frac{g}{2}\right) & (x<0) \\ \mathrm{e}^{-\frac{x^2}{2}} & (x>0) \end{cases}, \tag{20}$$

Table 1. The zeros and extrema of $\phi_n(x) = \mathrm{e}^{-x^2/2} H_n(x)$ for $n = 0, 1, 2, 3, 4$.

Order n	0	1	2	3	4	$\cdots$
Zeros	—	$x = 0$	$x = \pm\frac{1}{\sqrt{2}}$	$x = 0, \pm\sqrt{\frac{3}{2}}$	$x = \pm\sqrt{\frac{3}{2} \pm \sqrt{\frac{3}{2}}}$	
Extrema	$x = 0$	$x = \pm 1$	$x = 0, \pm\sqrt{\frac{5}{2}}$	$x = \pm\frac{\sqrt{9 \pm \sqrt{57}}}{2}$	$x = 0, \pm\sqrt{\frac{7}{2} \pm \sqrt{\frac{11}{2}}}$	



(a) $g = -\sqrt{2}$. (b) $g = \sqrt{2}$.

Figure 5. The solutions for *Example 2* with (a) $g = -\sqrt{2}$ and (b) $g = \sqrt{2}$. The potential (12) is displayed in this figure by a black curve. Thin lines show the energy spectrum, and the colored curve on each line is the corresponding eigenfunction. The states plotted in yellow possess the Hermite-polynomial solvability, while that colored in blue does not.

with $\mathcal{N}_m$ being a constant (See table 2) in figure 6b. They all have the energy $E_0 = 0$. Taking $m = 0$ means the ordinary harmonic oscillator.

Note that a similar procedure can be applied to construct a sequence of potentials such that only the N-th excited states can be expressed by Hermite polynomials of different orders.

4. Conclusion

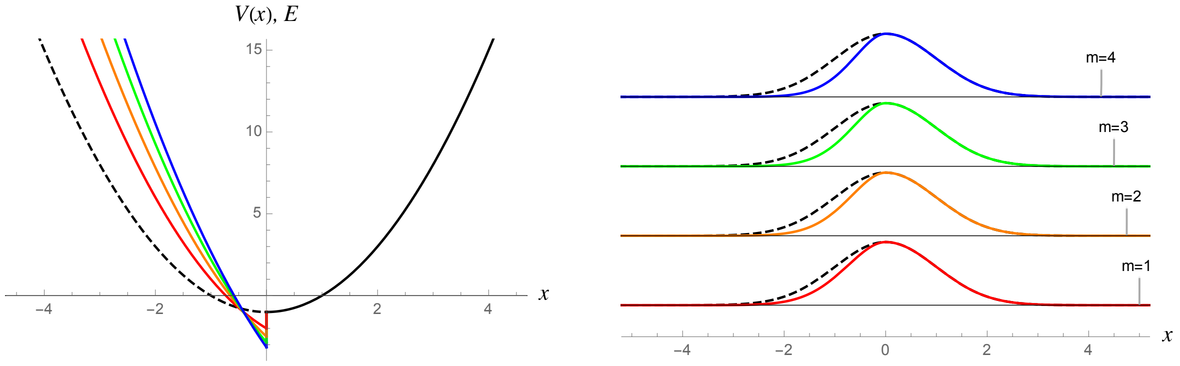
In this paper, we have investigated the Schrödinger equation for a harmonic oscillator with a finite step at the origin and/or an external linear field on the negative half line. The analytic solutions are obtained by applying the matching-of-wavefunctions technique to the general solutions (4) of equation (3) under the matching condition (2).

The eigenfunctions are not of closed form in general, but when $g = 0$ and $a = 4\ell$, the excited states higher than ℓ-th are expressed by the Hermite polynomials [6]. Moreover, when g is selected such that $\phi_m(x - g/2)$ has one of the zeros or the extrema at $x = 0$ and $a = 2k - g^2/4$, only one eigenstate is solvable in terms of the Hermite polynomials. This situation is similar to that in references [4, 5]. It would be quite a challenge to understand this type of quasi-exact solvability, including the one discussed in reference [10], in an integrated manner.

At the end, we make a comment on a problem of a harmonic oscillators with an $x^2\theta(x)$-type singularity function. Such potential is of different angular frequencies on $x < 0$ and $x > 0$ respectively. This problem has already been considered in, e.g., references [11, 12].

Table 2. Parameters of the wavefunction $\psi_0^{(m)}(x)$ for $m = 1, 2, 3, 4$.

m	1	2	3	4
g	2	$2\sqrt{\frac{5}{2}}$	$\sqrt{9+\sqrt{57}}$	$2\sqrt{\frac{7}{2}+\sqrt{\frac{11}{2}}}$
$\mathcal{N}_m$	$-\frac{\sqrt{\mathrm{e}}}{2}$	$\frac{\mathrm{e}^{5/4}}{8}$	$-\frac{\mathrm{e}^{\frac{9+\sqrt{57}}{8}}}{2\sqrt{6(39+5\sqrt{57})}}$	$\frac{\mathrm{e}^{\frac{7+\sqrt{22}}{4}}}{32(4+\sqrt{22})}$



(a) Potentials $V_m(x)$. (b) The ground-state wavefunctions $\psi_0^{(m)}(x)$.

Figure 6. The sequence $\{V_m(x)\}$. (a) The potentials for $m = 1, 2, 3, 4$ are plotted in red, orange, green and blue respectively, and $m = 0$ (harmonic oscillator) by black dashed curve. For $x > 0$, they all share the same function, so we plotted them in the same color: black. (b) The ground-state wavefunctions of those potentials, whose energies are zero, are expressed in terms of Hermite polynomials with different orders. They are plotted in the same colors as the potentials. The black dashed curves are the ground-state wavefunctions of the harmonic oscillator.

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