Wilson loops in Supersymmetric Gauge Theories

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Abstract

This thesis is devoted to several exact computations in four-dimensional supersymmetric gauge field theories.

In the first part of the thesis we prove conjecture due to Erickson-Semenoff-Zarembo and Drukker-Gross which relates supersymmetric circular Wilson loop operators in the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with a Gaussian matrix model. We also compute the partition function and give a new matrix model formula for the expectation value of a supersymmetric circular Wilson loop operator for the pure $\mathcal{N} = 2$ and the $\mathcal{N} = 2^*$ supersymmetric Yang-Mills theory on a four-sphere. Circular supersymmetric Wilson loops in four-dimensional $\mathcal{N} = 2$ superconformal gauge theory are treated similarly.

In the second part we consider supersymmetric Wilson loops of arbitrary shape restricted to a two-dimensional sphere in the four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We show that expectation value for these Wilson loops can be exactly computed using a two-dimensional theory closely related to the topological two-dimensional Higgs-Yang-Mills theory, or two-dimensional Yang-Mills theory for the complexified gauge group.

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Contents

Chapter 1

Introduction

String theory [1–3], born from Veneziano amplitude [4] as an attempt to describe dynamics of strong interactions, presently is a main candidate for the unified quantum field theory. Nowadays string theory incorporates ideas of grand unification, quantum gravity, supersymmetry, compactification of extra dimensions, dualities and many others.

The Standard Model of fundamental interactions is based on non-abelian quantum gauge field theories. A coupling constant in such theories usually decreases at high energies and blows up at low energies. Hence, it is easy and valid to apply perturbation theory at high energies. However, as the energy decreases the perturbation theory works worse and completely fails to give any meaningful results at the energy scale called Λ_{QCD}.

Therefore, to understand the Λ_{QCD} scale physics, such as confinement, hadron mass spectrum and the dynamics of low-energy interactions, we need non-perturbative methods. The main such methods developed in string theory are based on supersymmetry and dualities.

Like any symmetry, supersymmetry imposes some constraints on dynamics of a physical system. The maximally supersymmetric four-dimensional gauge theory is $\mathcal{N} = 4$ supersymmetric Yang-Mills. In this theory the dynamics is severely restricted by the large amount of supersymmetry, but it is still very non-trivial theory and thus is interesting for theoretical study. Besides gravity dual conjecture [5–7], the theory is related to the geometrical Langlands program [8] and the theory of integrable systems [9, 10] and quantum groups [11].

Duality means an existence of two different descriptions of the same physical system. If the strong coupling limit at one side of the duality corresponds to the weak coupling limit at the other side, such duality is especially useful to study the theory. Indeed, in that case difficult computations in strongly coupled theory can be done perturbatively using the dual weakly coupled theory.

A key role in string theory dualities play objects called D-branes. The D-branes are solitonic-like non-perturbative objects in the closed sector of string theory. The open strings end on D-branes. If we integrate out massive string modes we get low-energy action for the massless fields [12–14]. In the leading order in α', where $(2\pi\alpha')^{-1}$ is the string tension, the low-energy dynamics of one D-brane is described by gauge theory coupled to scalar fields corresponding to the fluctuations of the D-brane in transversal directions. If we take N D-branes on top of each other, the gauge symmetry is enlarged to $U(N)$.

On N D3-branes we actually get $\mathcal{N}=4$ supersymmetric Yang-Mills theory with $U(N)$ gauge group. On the other hand, recalling that D-brane is a solitonic like object in the closed sector of string theory we can use gravity description, since gravity is contained in the closed string sector. Hence, the dynamics of D3-branes can be described in two different ways. The resulting duality, called nowadays AdS/CFT duality [6, 15], is similar to the old ideas to describe QCD by strings [16, 17].

From the viewpoint of closed strings, Dp-branes carry a charge with respect to Ramond-Ramond fields [3], which geometrically are $p+1$-forms. Due to supersymmetry, the mass and the charge of D-brane are related in such a way, that the metric near D3-brane is like a metric near extremal black hole extended in three-dimensions

$$ds^2 = (1 + R^4/r^4)^{-1/2} dx^2 + (1 + R^4/r^4)^{1/2} (dr^2 + r^2 d\Omega_5^2). \tag{1.0.1}$$

Here x^i for $i = 1, \ldots, 4$ are coordinates along the world-volume of the D3-branes, and Ω_5 and r are spherical coordinates for the six transversal directions. Near horizon the metric asymptotically is

$$ds^2 = R^2 \frac{dx^2 + dy^2}{y^2} + R^2 d\Omega_5^2, \tag{1.0.2}$$

where $y = \frac{R^2}{r}$.

Hence metric near D3-brane is the metric of the $AdS_5 \times S^5$ space with boundary located at $y = 0$. The AdS/CFT conjecture [6, 7, 15] claims exact equivalence between the theory defined on the boundary, which is $\mathcal{N} = 4$ supersymmetric Yang-Mills, and the theory in the bulk, which is IIB string theory. The 't Hooft coupling constant $\lambda = N g_{YM}^2$ relates to the string tension as $T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$, and string coupling constant $g_s = e^{\Phi} = 4\pi g_{YM}^2$. In the planar 't Hooft limit [16] $N \to \infty$, $\lambda =$ const the strings do not interact. In other words, only contributions of genus zero worldsheets do not vanish.

The 't Hooft idea [16] on how $U(N)$ gauge theory simplifies in the large N limit is the following. Let us denote propagators of gluons by double lines, such that each line corresponds to an index of the fundamental representation of $U(N)$. Then Feynman diagrams are equivalent to ribbon graphs. The color factor for each Feynman diagram is equal to N^f, where f is the number of faces. For each Feynman graph we can associate a Riemann surface on which this graph can be drown without intersections. Let coupling constant g_{YM} enters the Yang-Mills actions as $\frac{1}{4g_{YM}^2} \operatorname{tr} F^2$. Then a Feynman diagram which has v vertices, e edges and f faces has weight $N^f(\lambda/N)^{e-v}$. Using Euler character $\chi = 2-2g = f-e+v$ we obtain that Feynman graph of genus g contributes with the factor $N^{2-2g}\lambda^{E-V}$. So the genus expansion takes form $\sum_g N^{2-2g} F_g$ where F_g stands for the sum of all diagrams of genus g. This precisely corresponds to the genus expansion for the string theory, if we identify $1/N$ with the string coupling constant $g_s = e^{\varphi}$. (Here φ is the dilaton fields which enters the sigma model action on world-sheet Σ as $\frac{1}{2\pi}\int_\Sigma R\varphi$, where R is the scalar curvature of the world-sheet metric.)

Hence at the large N limit, higher genus contributions are suppressed. In other words, the large N limit corresponds to weakly coupled strings. The string model representing real QCD in the large N limit is still not found. However, the AdS/CFT conjecture claims that such dual description of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is given by IIB string theory in the $AdS_5 \times S^5$ background [6, 7, 15].

The $\mathcal{N} = 4$ supersymmetric Yang-Mills theory has global $SO(4,2)$ conformal symmetry and $SU(4)_R$ symmetry. The $SO(4,2) \times SU(4)$ symmetry corresponds to the isometry group of $AdS_5 \times S^5$.

If the coupling constant $\lambda = Ng^2_{YM}$ is small, the perturbation theory on the gauge theory side works well. On the other hand, if λ is large, then the curvature radius $R = \alpha\prime^{1/2}\lambda^{1/4}$ of the space-time in the dual description is large and then closed string, or, in the leading order, supergravity approximation works well.

This is an example of very non-trivial duality between gauge theory and gravity. Naively it seems that gauge theory in d dimensions and gravity in $d + 1$ dimension have very different configurational space of degrees of freedom and classically it seems they can not be dual theories. However, the key point here is that duality swaps the weak coupling limit with the strong coupling limit. The collective degrees of freedom at one side become fundamental degrees of freedom at the other side.

Such weak-strong coupling duality is very powerful, since it allows us to make non-perturbative predictions for each side of the story using the other side perturbatively. On the other hand, the same property makes it very difficult to prove or even test the duality conjecture. To test the duality conjecture on some physical observables we have to compute expectation value exactly, to all orders in the coupling constant, at least in one of the theory independently from the dual description.

Some other non-perturbative methods are required to make such computation. One of such methods of exact computation of the path integral for theories with suitable fermionic symmetry Q is localization on Q-invariant configurations [18, 19]. Mathematically, localization is integration of equivariantly closed forms using Duistermaat-Heckman [20] or Atiyah-Bott-Berline-Vergne [21, 22] theorem. (See [23, 24] for a review.) We shall use such approach for non-perturbative computations in the gauge theory.

The basic observable in gauge theories is Wilson loop operator. Mathematically it represents holonomy of a connection around a loop, physically it measures interaction between heavy quark and antiquark. Finding the expectation value of an arbitrary set of Wilson loop operators is a formidable problem; it is equivalent to the complete solution of the gauge theory. As mentioned above, a theory simplifies in presence of additional symmetries, a particular kind of which is a supersymmetry. The simplest case from the theoretical viewpoint is then maximally supersymmetric theory, i.e. $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

In supersymmetric theories the usual Wilson loop operator can be made supersymmetric by adding coupling to some scalar fields. The simplest such operator is supersymmetric circular loop

$$W_R(C) = \mathrm{tr}_R \,\mathrm{Pexp} \oint_C (A_\mu dx^\mu + i\Phi_0 ds). \tag{1.0.3}$$

Here R is a representation of the gauge group, Pexp is the path-ordered exponent, C is a circular loop, A_μ is the gauge field and Φ_0 is one of the scalar fields of the theory. All fields take values in the Lie algebra of the gauge group, i.e. in our conventions the covariant derivative is $\nabla_\mu = \partial_\mu + A_\mu$.

In [25] Erickson, Semenoff and Zarembo conjectured that the expectation value $\langle W_R(C)\rangle$ of the Wilson loop operator (1.0.3) in the four-dimensional $\mathcal{N} = 4$ $SU(N)$ gauge theory in the large N limit can be exactly computed by summing all rainbow diagrams in Feynman gauge. The combinatorics of rainbow diagrams can be represented by a Gaussian matrix model. In [25] the conjecture was tested at one-loop level in gauge theory. In [26] Drukker and Gross conjectured that the exact relation to the Gaussian matrix model holds for any N and argued that the expectation value of the Wilson loop operator (1.0.3) can be computed by a matrix model. However, Drukker-Gross argument does not prove that this matrix model is Gaussian.

In the context of the AdS/CFT correspondence [6, 7, 15] the conjecture was relevant for many works; see for example [27–51] and references there in. From the viewpoint of string dual description, the expectation value of the Wilson loop (1.0.3) is given by string partition function in $AdS_5 \times S^5$ which lands at the contour C at the $\mathbb{R}^4$ boundary of AdS_5.

In chapter 2 we prove the Erickson-Semenoff-Zarembo/Drukker-Gross conjecture for the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory formulated for an arbitrary gauge group

$$\langle W_R(C)\rangle_{\mathcal{N}\,=\,4 \text{ on } S^4} = \frac{\int_{\mathfrak{g}}[da]\, e^{-\frac{4\pi^2 r^2}{g^2_{YM}}(a,a)}\, \mathrm{tr}_R\, e^{2\pi r i a}}{\int_{\mathfrak{g}}[da]\, e^{-\frac{4\pi^2 r^2}{g^2_{YM}}(a,a)}}. \tag{1.0.4}$$

See introduction to the chapter 2 for details on our conventions and notations.

We also get a new formula for the $\langle W_R(C)\rangle$ in the $\mathcal{N} = 2$ and the $\mathcal{N} = 2^*$ supersymmetric Yang-Mills theory.

Our main result is

$$\boxed{Z^{\mathcal{N}}_{S^4}\langle W_R(C)\rangle_{\mathcal{N}} = \frac{1}{\mathrm{vol}(G)}\int_{\mathfrak{g}}[da]\, e^{-\frac{4\pi^2 r^2}{g^2_{YM}}(a,a)}\, Z^{\mathcal{N}}_{\text{1-loop}}(ia)|Z^{\mathcal{N}}_{\text{inst}}(r^{-1}, r^{-1}, ia)|^2\, \mathrm{tr}_R\, e^{2\pi r i a}}. \tag{1.0.5}$$

In chapter 3 we consider more interesting Wilson loops in $\mathcal{N} = 4$ Yang-Mills. Namely, we follow [41, 52] and consider supersymmetric Wilson loops of arbitrary shape located on a fixed two-sphere S^2 in the four-dimensional theory. Such supersymmetric Wilson loops preserve 4 out of 32 superconformal symmetries of the $\mathcal{N} = 4$ Yang-Mills. It turns out that the localization procedure works for such loops as well. The result is a certain two-dimensional theory resembling perturbative sector of the bosonic two-dimensional Yang-Mills. From another viewpoint this two-dimensional theory can interpreted as partially gauge-fixed two-dimensional

Yang-Mills for complexified gauge group, or as a certain sector of topological Higgs-Yang-Mills [53–55] theory related to the moduli space of the solutions to Hitching's equations [56].

The chapter 4 concludes the thesis.

Chapter 2

Circular Wilson loops

This chapter is devoted to the exact calculation of the expectation value of supersymmetric Wilson loop in $\mathcal{N} = 4$ and $\mathcal{N} = 2$ superconformal gauge theories. The main results, presented in this chapter were initially obtained in the work [57].

2.1 Introduction

Topological gauge theory can be obtained by a twist of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory [18]. The path integral of the twisted theory localizes to the moduli space of instantons and computes the Donaldson-Witten invariants of four-manifolds [18, 58, 59].

In a flat space the twisting does not change the Lagrangian. In [60] Nekrasov used a $U(1)^2$ subgroup of the $SO(4)$ Lorentz symmetry on $\mathbb{R}^4$ to define a $U(1)^2$-equivariant version of the topological partition function, or, equivalently, the partition function of the $\mathcal{N} = 2$ supersymmetric gauge theory in the Ω-deformed background [61]. The integral over moduli space of instantons $\mathcal{M}_{inst}$ localizes at the fixed point set of a group which acts on $\mathcal{M}_{inst}$ by Lorentz rotations of the space-time and gauge transformations at infinity. The partition function $Z_{\text{inst}}(\varepsilon_1, \varepsilon_2, a)$ depends on the parameters $(\varepsilon_1, \varepsilon_2)$, which generate $U(1)^2$ Lorentz rotations, and the parameter $a \in \mathfrak{g}$, which generates gauge transformations at infinity. By $\mathfrak{g}$ we denote the Lie algebra of the gauge group. This partition function is finite because the Ω-background effectively confines the dynamics to a finite volume $V_{\text{eff}} = \frac{1}{\varepsilon_1\varepsilon_2}$. In the limit of vanishing Ω-deformation ($\varepsilon_1, \varepsilon_2 \to 0$) the effective volume V_{eff} diverges as well as the free energy $F = -\log Z_{\text{inst}}$. But the specific free energy F/V_{eff} has a well-defined limit, which actually coincides with Seiberg-Witten low-energy effective prepotenial $\mathcal{F}(a)$ of the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory [62, 63]. In this way instanton counting gives a derivation of Seiberg-Witten prepotential from the first principles.

In this chapter we consider another interesting situation where an analytical computation of the partition function is possible. We consider the $\mathcal{N} = 2$, the $\mathcal{N} = 2^*$ and the $\mathcal{N} = 4$ Yang-Mills theory on a four-sphere S^4 equipped with the standard round metric.[1]

There are no zero modes for the gauge fields, because the first cohomology group of S^4 is trivial. There are no zero modes for the fermions. This follows from the fact that the Laplacian operator on a compact space is semipositve and the formula $\not{D}^2 = \Delta + \frac{R}{4}$, where by $\not{D}$ we denote the Dirac operator, by Δ the Laplacian, and by R the scalar curvature, which is positive on S^4. There are no zero modes for the scalar fields, because there is a mass term in the Lagrangian proportional to the scalar curvature.

Observing that there are no zero modes at all, we can try to integrate over all fields in the path integral and to compute the full partition function of the theory. In addition, we would like to compute expectation values of certain interesting observables.

In this chapter we are mostly interested in the observable defined by the supersymmetric circular Wilson loop operator (see Fig. 2.1)

$$W_R(C) = \operatorname{tr}_R \operatorname{Pexp} \oint_C (A_\mu dx^\mu + i\Phi_0^E ds). \tag{2.1.1}$$

Here R is a representation of the gauge group, Pexp is the path-ordered exponent,

[1] What we call $\mathcal{N} = 2$ supersymmetry on S^4 is explained in section 2.2. It would be interesting to extend the analysis to more general backgrounds [64].

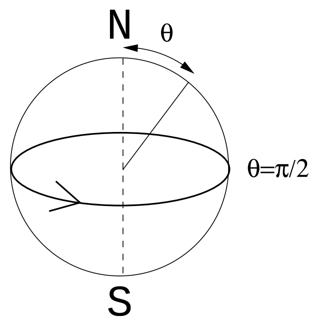

Figure 2.1: Wilson loop on the equator of S^4

C is a circular loop located at the equator of S^4, A_μ is the gauge field and $i\Phi_0^E$ is one of the scalar fields of the $\mathcal{N}=2$ vector multiplet. We reserve notation Φ_0^E for the scalar field in a theory obtained by dimensional reduction of a theory in Euclidean signature. Our conventions are that all fields take values in the real Lie algebra of the gauge group. For example, if the gauge group is $U(N)$, then all fields can be represented by antihermitian matrices. The covariant derivative is $D_\mu = \partial_\mu + A_\mu$ and the field strength is $F_{\mu\nu} = [D_\mu, D_\nu]$.

In this chapter, we prove the Erickson-Semenoff-Zarembo/Drukker-Gross conjecture [25, 26] for the $\mathcal{N}=4$ supersymmetric Yang-Mills theory formulated for an arbitrary gauge group. Let r be the radius of S^4. The conjecture states that

$$\langle W_R(C)\rangle_{\mathcal{N}=4 \text{ on } S^4} = \frac{\int_{\mathfrak{g}}[da]\, e^{-\frac{4\pi^2 r^2}{g_{YM}^2}(a,a)}\, \mathrm{tr}_R\, e^{2\pi r i a}}{\int_{\mathfrak{g}}[da]\, e^{-\frac{4\pi^2 r^2}{g_{YM}^2}(a,a)}}. \tag{2.1.2}$$

The finite dimensional integrals in this formula are taken over the Lie algebra $\mathfrak{g}$ of the gauge group, a denotes an element of $\mathfrak{g}$. By (a,a) for $a \in \mathfrak{g}$ we denote an invariant positive definite quadratic form on $\mathfrak{g}$. Our convention is that the kinetic term in the gauge theory is normalized as $\frac{1}{4g_{YM}^2}\int d^4x\sqrt{g}(F_{\mu\nu}, F^{\mu\nu})$. The formula (2.1.2) can be rewritten in terms of the integral over the Cartan subalgebra of $\mathfrak{g}$ with insertion of the usual Weyl measure $\Delta(a) = \prod_{\alpha\in\text{roots of }\mathfrak{g}} \alpha\cdot a$.

We also get a new formula for the $\langle W_R(C)\rangle$ in the $\mathcal{N}=2$ and the $\mathcal{N}=2^*$ supersymmetric Yang-Mills theory. As in the $\mathcal{N}=4$ case, the result can be written in terms of a matrix model. However, this matrix model is much more complicated than a Gaussian matrix model. We derive this matrix model action up to all orders in perturbation theory. Then we argue what is the non-perturbative contribution of all instanton/anti-instanton corrections.

Our main result is

$$\boxed{Z^{\mathcal{N}}_{S^4}\langle W_R(C)\rangle_{\mathcal{N}} = \frac{1}{\mathrm{vol}(G)}\int_{\mathfrak{g}}[da]\, e^{-\frac{4\pi^2 r^2}{g^2_{YM}}(a,a)} Z^{\mathcal{N}}_{\text{1-loop}}(ia)|Z^{\mathcal{N}}_{\text{inst}}(r^{-1},r^{-1},ia)|^2\, \mathrm{tr}_R\, e^{2\pi r i a}}. \tag{2.1.3}$$

Here $Z^{\mathcal{N}}_{S^4}$ is the partition function of the $\mathcal{N}=2$, the $\mathcal{N}=2^*$ or the $\mathcal{N}=4$ supersymmetric Yang-Mills theory on S^4, defined by the path integral over all fields in the theory, and $\langle W_R(C)\rangle_{\mathcal{N}}$ is the expectation value of $W_R(C)$ in the corresponding theory. In particular, if we take R to be the trivial one-dimensional representation, the formula says that the partition function $Z^{\mathcal{N}}_{S^4}$ is computed by the following finite-dimensional integral:

$$Z^{\mathcal{N}}_{S^4} = \frac{1}{\mathrm{vol}(G)}\int [da] e^{-\frac{4\pi^2 r^2}{g^2_{YM}}(a,a)} Z^{\mathcal{N}}_{\text{1-loop}}(ia)|Z^{\mathcal{N}}_{\text{inst}}(r^{-1},r^{-1},ia)|^2. \tag{2.1.4}$$

In other words, we show that the Wilson loop observable (2.1.1) is compatible with the localization of the path integral to the finite dimensional integral (2.1.3) and that

$$\boxed{\langle W_R(C)\rangle_{\text{4d theory}} = \langle \mathrm{tr}_R\, e^{2\pi r i a}\rangle_{\text{matrix model}}}, \tag{2.1.5}$$

where the matrix model measure $\langle \ldots \rangle_{\text{matrix model}}$ is given by the integrand in (2.1.4).

The factor $Z_{\text{1-loop}}(ia)$ is a certain infinite dimensional product, which appears as a determinant in the localization computation. It can be expressed in terms of a product of Barnes G-functions [65]. In the most general $\mathcal{N}=2^*$ case, the factor $Z_{\text{1-loop}}(ia)$ is given by the formula (2.4.48). The $\mathcal{N}=2$ and $\mathcal{N}=4$ cases can be obtained by taking respectively limits $m=\infty$ and $m=0$, where m is the hypermultiplet mass in the $\mathcal{N}=2^*$ theory. For the $\mathcal{N}=4$ theory we get $Z_{\text{1-loop}}=1$.

The factor $Z_{\text{inst}}(\varepsilon_1, \varepsilon_2, ia)$ is Nekrasov's partition function [61] of point instantons in the equivariant theory on $\mathbb{R}^4$. In the $\mathcal{N} = 2^*$ case it is given by the formula (2.5.12). In the limit $m = \infty$, one gets the $\mathcal{N} = 2$ case (2.5.1), in the limit $m = 0$ one gets the $\mathcal{N} = 4$ case. In the $\mathcal{N} = 4$ case, the instanton partition function (2.5.15) does not depend on a. Therefore in the $\mathcal{N} = 4$ case, instantons do not contribute to the expectation value $\langle W_R(C) \rangle$.

Our claim about vanishing of instanton corrections for the $\mathcal{N} = 4$ theory contradicts to the results of [35], where the first instanton correction for the $SU(2)$ gauge group was found to be non-zero. In [35] the authors introduced a certain cut-off on the instanton moduli space, which is not compatible with the relevant supersymmetry of the theory and the Wilson loop operator. Our instanton calculation is based on Nekrasov's partition function on $\mathbb{R}^4$. This partition function is regularized by a certain non-commutative deformation of $\mathbb{R}^4$ compatible with the relevant supersymmetry. Though we do not write down explicitly the non-commutative deformation of the theory on S^4, we assume that such deformation can be well defined. We also assume that in a small neighbourhood of the North or the South pole of S^4 this non-commutative deformation agrees with the non-commutative deformation used by Nekrasov [60] on $\mathbb{R}^4$.

Since both $Z_{\text{inst}}(\varepsilon_1, \varepsilon_2, ia)$ and its complex conjugate enter the formula, this means that we count both instantons and anti-instantons. The formula is similar to Ooguri-Strominger-Vafa relation between the black hole entropy and the topological string partition function [66, 67]

$$Z_{BH} \propto |Z_{top}|^2. \tag{2.1.6}$$

Actually the localization computation is compatible with more general observables than a single Wilson loop in representation R inserted on the equator (2.1.1). Let us fix two opposite points on the S^4 and call them the North and the South poles. Then we can consider a class of Wilson loops placed on circles of arbitrary radius such that they all have a common center at the North pole, and such that they all can be transformed to each other by a composition of a dilation in the North-South direction and by an anti-self-dual rotation in the $SU(2)_L$ left subgroup of the $SO(4)$ subgroup of the S^4 isometry group which fixes the North pole. However, for Wilson loops of not maximal size, we need to change the relative coefficient between the gauge and the scalar field terms in (2.1.1). Let C_θ be a circle located at an arbitrary polar angle θ measured from the North pole (at the equator $\sin\theta = 1$). Then we consider

$$W_R(C_\theta) = \mathrm{tr}_R \,\mathrm{Pexp} \oint_{C_\theta} (A_\mu dx^\mu + \frac{1}{\sin\theta}(i\Phi_0^E + \Phi_9 \cos\theta)ds), \tag{2.1.7}$$

where Φ_0^E and Φ_9 are the scalar fields of the $\mathcal{N} = 2$ vector multiplet.

Equivalently this can be rewritten as

$$W_R(C_\theta) = \mathrm{tr}_R \,\mathrm{Pexp} \oint_{C_\theta} (A_\mu dx^\mu + (i\Phi_0^E + \Phi_9 \cos\theta) r d\alpha). \tag{2.1.8}$$

where $\alpha \in [0, 2\pi)$ is an angular coordinate on the circle C. Formally, as the size of the circle vanishes ($\theta \to 0$) we get a "holomorphic" observable $W_R(C_{\theta\to 0}) = \mathrm{tr}_R \exp 2\pi r \Phi(\mathrm{N})$ where $\Phi(\mathrm{N})$ is the complex scalar field $i\Phi_0^E + \Phi_9$ evaluated at the North pole. In the opposite limit ($\theta \to \pi$) we get an "anti-holomoprhic" observable $W_R(C_{\theta\to\pi}) = \mathrm{tr}_R \exp 2\pi r \bar{\Phi}(\mathrm{S})$, where $\bar{\Phi}(S)$ is the conjugated scalar field $-i\Phi_0^E + \Phi_9$ evaluated at the South pole. However, in the actual computation of the path integral we will always assume a finite size of C, so that the operator $W_R(C)$ is well defined.

Then for an arbitrary set $\{W_{R_1}(C_{\theta_1}), \ldots, W_{R_n}(C_{\theta_n})\}$ of Wilson loops in the class we described above we obtain

$$\boxed{\langle W_{R_1}(C_{\theta_1}) \ldots W_{R_n}(C_{\theta_n})\rangle_{\text{4d theory}} = \langle \mathrm{tr}_{R_1} e^{2\pi r i a} \ldots \mathrm{tr}_{R_n} e^{2\pi r i a}\rangle_{\text{matrix model}}}. \tag{2.1.9}$$

The Drukker-Gross argument only applies to the case of a single circle which can be related to a straight line on $\mathbb{R}^4$ by a conformal transformation, but in the present approach we can consider several circles simultaneously.

So far we described the class of observables which we can compute in the massive $\mathcal{N} = 2^*$ theory. All these observables are invariant under the same operator Q generated by a conformal Killing spinor on S^4 of constant norm. This operator Q is a fermionic symmetry at quantum level.

Now we describe more general classes of circular Wilson loops which can be solved in $\mathcal{N} = 4$ theory. Thanks to the conformal symmetry of the $\mathcal{N} = 4$ theory there is a whole family of operators $\{Q(t)\}$ where t runs from 0 to ∞, which we can use for the localization computation. The case $t = 1$ corresponds to the conformal Killing spinor of constant norm and to the observables which we study in the $\mathcal{N} = 2^*$ theory. However, for a general t in the $\mathcal{N} = 4$ theory we can take

$$W_R(C_\theta, t) = \mathrm{tr}_R \, \mathrm{Pexp} \oint_{C_\theta} \left(A_\mu dx^\mu + \frac{1}{t \sin\theta} \left((\cos^2 \frac{\theta}{2} + t^2 \sin^2 \frac{\theta}{2}) i\Phi_0^E \right.\right.$$
$$\left.\left. + \Phi_9 (\cos^2 \frac{\theta}{2} - t^2 \sin^2 \frac{\theta}{2}) \right) ds \right). \quad (2.1.10)$$

At $t \sin\frac{\theta}{2} = \cos\frac{\theta}{2}$ we get the Wilson loop (2.1.1) with the same relative coefficient 1 between A_μ and $i\Phi_0^E$ but of arbitrary size. The $\mathcal{N} = 4$ theory with insertion of the operator $W_R(C_\theta, t)$ still localizes to the Gaussian matrix model.

The idea underlying localization is that in some situations the integral is exactly equal to its semiclassical approximation. For example, the Duistermaat-Heckman formula says [20]

$$\int_M \frac{\omega^n}{(2\pi)^n n!} e^{iH(\phi)} = i^n \sum_{p \in F} \frac{e^{iH(\phi)}}{\prod \alpha_i^p(\phi)},$$

where (M, ω) is a symplectic manifold, and $H : M \to g^*$ is a moment map[2] for a Hamiltonian action of $G = U(1)^k$ on M. The Duistermaat-Heckman formula is a particular case of a more general Atiyah-Bott-Berline-Vergne localization formula [21, 22]. Let an abelian group G act on a compact manifold M. We consider the complex of G-equivariant differential forms on M valued in functions on $\mathfrak{g}$ with

[2]In other words, $i_\phi \omega = dH(\phi)$ for any $\phi \in g$, where i_ϕ is a contraction with a vector field generated by ϕ.

the differential $Q = d - \phi^a i_a$. The differential squares to a symmetry transformation $Q^2 = -\phi^a \mathcal{L}_{v^a}$. Here $\mathcal{L}_{v^a}$ represents the action of G on M. Hence Q^2 annihilates G-invariant objects. Then for any Q-closed form α, Atiyah-Bott-Berline-Vergne localization formula is

$$\int_M \alpha = \int_F \frac{i_F^* \alpha}{e(N_F)},$$

where $F \overset{i}{\hookrightarrow} M$ is the G-fixed point set, and $e(N_F)$ is the equivariant Euler class of the normal bundle of F in M. When F is a discrete set of points, the equivariant Euler class $e(N_F)$ at each point $f \in F$ is simply the determinant of the representation in which $\mathfrak{g}$ acts on the tangent bundle of M at a point f.

Localization can be argued in the following way [18, 68]. Let Q be a fermionic symmetry of a theory. Let $Q^2 = \mathcal{L}_\phi$ be some bosonic symmetry. Let S be a Q-invariant action, so that $QS = 0$. Consider a functional V which is invariant under $\mathcal{L}_\phi$, so that $Q^2 V = 0$. Deformation of the action by a Q-exact term QV can be written as a total derivative and does not change the integral up to boundary contributions

$$\frac{d}{dt} \int e^{S+tQV} = \int \{Q, V\} e^{S+tQV} = \int \{Q, V e^{S+tQV}\} = 0.$$

As $t \to \infty$, the one-loop approximation at the critical set of QV becomes exact. Then for a sufficiently nice V, the integral is computed by evaluating S at critical points of QV and the corresponding one-loop determinant.

We apply this strategy to the $\mathcal{N} = 2$, the $\mathcal{N} = 2^*$ and the $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theories on S^4 and show that the path integral is localized to the constant modes of the scalar field Φ_0 with all other fields vanishing. In this way we also compute exactly the expectation value of the circular supersymmetric Wilson loop operator (2.1.1).

Remark. Most of the presented arguments in this work should apply to an $\mathcal{N} = 2$ theory with an arbitrary matter content. For a technical reasons related to the regularization issues, we limit our discussion to the $\mathcal{N} = 2$ theory with a single $\mathcal{N}=2$ massive hypermultiplet in the adjoint representation, also known as the $\mathcal{N}=2^*$. By taking the limit of vanishing or infinite mass we can respectively recover the $\mathcal{N}=4$ or the $\mathcal{N}=2$ theory.

Still we will give in (2.4.57) a formula for the factor $Z_{\text{1-loop}}$ for an $\mathcal{N}=2$ gauge theory with a massless hypermultiplet in such representation that the theory is conformal. Perhaps, one could check our result by the traditional Feynman diagram computations directly in the gauge theory. To simplify comparison, we will give an explicit expansion in g_{YM} up to the sixth order of the expectation value of the Wilson loop operator for the $\mathcal{N}=2$ theory with the gauge group $SU(2)$ and 4 hypermultiplets in the fundamental representation (see (2.4.58))

$$\langle e^{2\pi na}\rangle_{\text{matrix model}} = 1+\frac{3}{2\cdot 2^2}n^2g_{YM}^2+\frac{5}{8\cdot 2^4}n^4g_{YM}^4+\frac{7}{48\cdot 2^6}n^6g_{YM}^6 - \frac{35\cdot 12\cdot\zeta(3)}{2^4(4\pi)^2}n^2g_{YM}^6+O(g_{YM}^8), \quad (2.1.11)$$

In this formula $a\in\mathbb{R}$ is a coordinate on the Cartan algebra $\mathfrak{h}$ of $\mathfrak{g}$. By an integer $n\in\mathfrak{h}^*$ we denote a weight. For example, if the Wilson loop is taken in the spin-j representation, where j is a half-integer, the weights are $\{-2j,-2j+2,\ldots,2j\}$. Hence we get $\langle W_j(C)\rangle = \langle\sum_{m=-j}^{j} e^{4\pi ma}\rangle_{MM}$.

We shall note that the first difference between the $\mathcal{N}=2$ superconformal theory and the $\mathcal{N}=4$ theory appears at the order g_{YM}^6, up to which the Feynman diagrams in the $\mathcal{N}=4$ theory were computed in [69, 70]. Therefore a direct computation of Feynman diagrams in the $\mathcal{N}=2$ theory up to this order seems to be possible and would be a non-trivial test of our results.

Some unusual features in this work are: (i) the theory localizes not on a counting problem, but on a nontrivial matrix model, (ii) there is a one-loop factor involving an index theorem for transversally elliptic operators [71, 72].

In section 2 we give details about the $\mathcal{N}=2$, the $\mathcal{N}=2^*$ and $\mathcal{N}=4$ SYM theories on a four-sphere S^4. In section 3 we make a localization argument to compute the partition function for these theories. Section 4 explains the computation of the one-loop determinant [71, 72], or, mathematically speaking, of the equivariant Euler class of the infinite-dimensional normal bundle in the localization formula. In section 5 we consider instanton corrections.

There are some open questions and immediate directions in which one can proceed:

1. One can consider more general supersymmetric Wilson loops like studied in [41, 50, 52] and try to prove the conjectural relations of those with matrix models or two-dimensional super Yang-Mills theory. Perhaps it will be also possible to extend the analysis of those more general loops to (superconformal) $\mathcal{N} = 2$ theories like it is done in the present work.

2. Using localisation, one can try to solve exactly for an expectation value of a circular supersymmetric 't Hooft-Wilson operator (this is a generalization of Wilson loop in which the loop carries both electric and magnetic charges) [8, 73, 74]. The expectation values of such operators should transform in the right way under the S-duality transformation which replaces the coupling constant by its inverse and the gauge group G by its Langlands dual ${}^{L}G$. Perhaps this could tell us more on the four-dimensional gauge theory and geometric Langlands [8] where 't Hooft-Wilson loops play the key role.

3. It would be interesting to find more precise relation between our formulas, and Ooguri-Strominger-Vafa [66] conjecture (2.1.6). There could be a four-dimensional analogue of the tt^*-fusion [75].

2.2 Fields, action and symmetries

To write down the action of the $\mathcal{N} = 4$ SYM on S^4, we use dimensional reduction of the $\mathcal{N} = 1$ SYM [76] on $\mathbb{R}^{9,1}$. By G we denote the gauge group. By A_M with $M = 0, \ldots, 9$ we denote the components of the gauge field in ten dimensions, where we take the Minkowski metric $ds^2 = -dx_0^2 + dx_1^2 + \cdots + dx_9^2$. When we write formulas in Euclidean signature so that the metric is $ds^2 = dx_0^2 + dx_1^2 + \cdots + dx_9^2$, we use notation A_0^E for the zero component of the gauge field.

By Ψ we denote a sixteen real component ten-dimensional Majorana-Weyl fermion valued in the adjoint representation of G. (In Euclidean signature Ψ is not real, but its complex conjugate does not appear in the theory.) The ten-dimensional action $S = \int d^{10}x \mathcal{L}$ with the Lagrangian

$$\mathcal{L} = \frac{1}{2g_{YM}^2}\left(\frac{1}{2}F_{MN}F^{MN} - \Psi\Gamma^M D_M \Psi\right) \tag{2.2.1}$$

is invariant under the supersymmetry transformations

$$\begin{aligned}\delta_\varepsilon A_M &= \varepsilon\Gamma_M\Psi \\ \delta_\varepsilon \Psi &= \frac{1}{2}F_{MN}\Gamma^{MN}\varepsilon.\end{aligned}$$

Here ε is a constant Majorana-Weyl spinor parameterizing the supersymmetry transformations in ten dimensions. (See appendix A.1 for our conventions on the algebra of gamma-matrices.)

We do not write explicitly the color and spinor indices. We also assume that in all bilinear terms the color indices are contracted using some invariant positive definite bilinear form (Killing form) on the Lie algebra $\mathfrak{g}$ of the gauge group. Sometimes we denote this Killing form by $(\cdot,\cdot)$. In Euclidean signature we integrate over fields which all take value in the real Lie algebra of the gauge group. For example, for the $U(N)$ gauge group all fields are represented by the antihermitian matrices, and we can define the Killing from on $\mathfrak{g}$ as $(a,b) = -\operatorname{tr}_{\mathrm{F}} ab$, where $\operatorname{tr}_{\mathrm{F}}$ is the trace in the fundamental representation.

We take $(x_1, \ldots, x_4)$ to be the coordinates along the four-dimensional space-time, and we make dimensional reduction in the remaining directions: $0, 5, \ldots 8, 9$. Note that the four-dimensional space-time has Euclidean signature.

Now we describe the symmetries of the four-dimensional theory if we start from Minkowski signature in ten dimensions. Note that we make dimensional reduction along the time-like coordinate x_0. Therefore we get the wrong sign for the kinetic term for the scalar field Φ_0, where Φ_0 denotes the 0-th component of the gauge field A_M after dimensional reduction. To make sure that the path integral is well defined and convergent, in this case in the path integral for the four-dimensional theory we integrate over imaginary Φ_0. Actually this means that the path integral is the same as in the Euclidean signature with all bosonic fields taken real.

The ten-dimensional $Spin(9,1)$ Lorentz symmetry group is broken to $Spin(4) \times Spin(5,1)^R$, where the first factor is the four-dimensional Lorentz group acting on $(x_1, \ldots, x_4)$ and the second factor is the R-symmetry group acting on $(x_5, \ldots, x_9, x_0)$. It is convenient to split the four-dimensional Lorentz group as $Spin(4) = SU(2)_L \times SU(2)_R$, and brake the $Spin(5,1)^R$-symmetry group into $Spin(4)^R \times SO(1,1)^R = SU(2)^R_L \times SU(2)^R_R \times SO(1,1)^R$. The components of the ten-dimensional gauge field, which become scalars after the dimensional reduction are denoted by Φ_A with $A = 0, 5, \ldots, 9$. Let us write the bosonic fields and the symmetry groups under which they transform:

$$\overbrace{A_1, \ldots A_4}^{SU(2)_L \times SU(2)_R} \quad \overbrace{\Phi_5, \ldots, \Phi_8}^{SU(2)^R_L \times SU(2)^R_R} \quad \overbrace{\Phi_9, \Phi_0}^{SO(1,1)^R} .$$

Using a certain Majorana-Weyl representation of the Clifford algebra $\mathrm{C}l(9,1)$ (see appendix A.1 for our conventions), we write Ψ in terms of four four-dimensional chiral spinors as

$$\Psi = \begin{pmatrix} \psi^L \\ \chi^R \\ \psi^R \\ \chi^L \end{pmatrix} .$$

Each of these spinors $(\psi^L, \chi^R, \psi^R, \chi^L)$ has four real components. Their transformation properties are summarized in the table:

ε	Ψ	$SU(2)_L$	$SU(2)_R$	$SU(2)^R_L$	$SU(2)^R_R$	$SO(1,1)^R$
$*$	ψ^L	1/2	0	1/2	0	+
0	χ^R	0	1/2	0	1/2	+
$*$	ψ^R	0	1/2	1/2	0	−
0	χ^L	1/2	0	0	1/2	−

Let the spinor ε be the parameter of the supersymmetry transformations. We restrict the $\mathcal{N}=4$ supersymmetry algebra to the $\mathcal{N}=2$ subalgebra by taking ε in the $+1$-eigenspace of the operator Γ^{5678}. Such spinor ε has the structure

$$\varepsilon = \begin{pmatrix} * \\ 0 \\ * \\ 0 \end{pmatrix},$$

transforms in the spin-$\frac{1}{2}$ representation of the $SU(2)^R_L$ and in the trivial representation of the $SU(2)^R_R$.

With respect to the supersymmetry transformation generated by such ε, the $\mathcal{N}=4$ gauge multiplet splits in two parts

- $(A_1 \dots A_4, \Phi_9, \Phi_0,\ \psi^L, \psi^R)$ is the $\mathcal{N}=2$ vector multiplet
- $(\Phi_5 \dots \Phi_8,\ \chi^L, \chi^R)$ is the $\mathcal{N}=2$ hypermultiplet.

So far we considered dimensional reduction from $\mathbb{R}^{9,1}$ to the flat space $\mathbb{R}^4$. Now we would like to put the theory on a four-sphere S^4. We denote by A_μ with $\mu = 1,\dots,4$ the four-dimensional gauge field and by Φ_A with $A = 0,5,\dots,9$ the four-dimensional scalar fields. The only required modification of the action is a coupling of the scalar fields to the scalar curvature of space-time. Namely, the kinetic term must be changed as $(\partial\Phi)^2 \to (\partial\Phi)^2 + \frac{R}{6}\Phi^2$, where R is the scalar curvature. One way to see why this is the natural kinetic term for the scalar fields is to use the argument of the conformal invariance. Namely, one can check that $\int d^4x\,\sqrt{g}((\partial\Phi)^2 + \frac{R}{6}\Phi^2)$ is invariant under Weyl transformations of the metric $g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu}$ and scalar fields $\Phi \to e^{-\Omega}\Phi$. Then the action on S^4 of the $\mathcal{N}=4$ SYM is

$$S_{\mathcal{N}=4} = \frac{1}{2g_{YM}^2} \int_{S^4} \sqrt{g} d^4x \left(\frac{1}{2} F_{MN}F^{MN} - \Psi\gamma^M D_M \Psi + \frac{2}{r^2}\Phi^A\Phi_A \right), \tag{2.2.2}$$

where we used the fact that the scalar curvature of a d-sphere S^d of radius r is $\frac{d(d-1)}{r^2}$.

The action (2.2.2) is invariant under the $\mathcal{N}=4$ superconformal transformations

$$\delta_\varepsilon A_M = \varepsilon \Gamma_M \Psi \tag{2.2.3}$$

$$\delta_\varepsilon \Psi = \frac{1}{2} F_{MN}\Gamma^{MN}\varepsilon + \frac{1}{2}\Gamma_{\mu A}\Phi^A \nabla^\mu \varepsilon, \tag{2.2.4}$$

where ε is a conformal Killing spinor solving the equations

$$\nabla_\mu \varepsilon = \Gamma_\mu \tilde{\varepsilon} \tag{2.2.5}$$

$$\nabla_\mu \tilde{\varepsilon} = -\frac{1}{4r^2}\Gamma_\mu \varepsilon. \tag{2.2.6}$$

(See e.g. [77] for a review on conformal Killing spinors, and for the explicit solution of these equations on S^4 see appendix A.2.) To get intution about the meaning of ε and $\tilde{\varepsilon}$ we can take the flat space limit $r \to \infty$. In this limit $\tilde{\varepsilon}$ becomes covariantly constant spinor $\tilde{\varepsilon} = \hat{\varepsilon}_c$, while ε becomes a spinor with at most linear dependence on flat coordinates x^μ on $\mathbb{R}^4$: $\varepsilon = \hat{\varepsilon}_s + x^\mu \Gamma_\mu \hat{\varepsilon}_c$. By $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ we denote some constant spinors. Then $\hat{\varepsilon}_s$ generates supersymmetry transformations, while $\hat{\varepsilon}_c$ generates special superconformal symmetry transformations.

The superconformal algebra closes only on-shell. Let δ_ε^2 be the square of the fermionic transformation δ_ε generated by a spinor ε. After some algebra (see appendix A.3) we obtain

$$\begin{aligned}
\delta_\varepsilon^2 A_\mu &= -(\varepsilon\Gamma^\nu\varepsilon)F_{\nu\mu} - [(\varepsilon\Gamma^B\varepsilon)\Phi_B, D_\mu] \\
\delta_\varepsilon^2 \Phi_A &= -(\varepsilon\Gamma^\nu\varepsilon)D_\nu\Phi_A - [(\varepsilon\Gamma^B\varepsilon)\Phi_B, \Phi_A] + 2(\tilde{\varepsilon}\Gamma_{AB}\varepsilon)\Phi^B - 2(\varepsilon\tilde{\varepsilon})\Phi_A \\
\delta_\varepsilon^2 \Psi &= -(\varepsilon\Gamma^\nu\varepsilon)D_\nu\Psi - [(\varepsilon\Gamma^B\varepsilon)\Phi_B, \Psi] - \frac{1}{2}(\tilde{\varepsilon}\Gamma_{\mu\nu}\varepsilon)\Gamma^{\mu\nu}\Psi + \frac{1}{2}(\tilde{\varepsilon}\Gamma_{AB}\varepsilon)\Gamma^{AB}\Psi \\
&\quad -3(\tilde{\varepsilon}\varepsilon)\Psi + \mathrm{eom}[\Psi].
\end{aligned} \tag{2.2.7}$$

Here the term denoted by eom[Ψ] is proportional to the Dirac equation of motion for fermions Ψ

$$\text{eom}[\Psi] = \frac{1}{2}(\varepsilon\Gamma_N\varepsilon)\tilde{\Gamma}^N \not{D}\Psi - (\varepsilon\not{D}\Psi)\varepsilon. \tag{2.2.8}$$

The square of the supersymmetry transformation can be written as

$$\delta_\varepsilon^2 = -\mathcal{L}_v - R - \Omega. \tag{2.2.9}$$

The first term is the gauge covariant Lie derivative $\mathcal{L}_v$ in the direction of the vector field

$$v^M = \varepsilon\gamma^M\varepsilon. \tag{2.2.10}$$

For example, $\mathcal{L}_v$ acts on scalar fields as follows: $\mathcal{L}_v\Phi_A = v^M D_M \Phi = v^\mu D_\mu \Phi_A + v^B[\Phi_B, \Phi]$. Here D_μ is the usual covariant derivative $D_\mu = \partial_\mu + A_\mu$

To explain what the gauge covariant Lie derivative means geometrically, first we consider the situation when the gauge bundle, say E, is trivial. We fix some flat background connection $A_\mu^{(0)}$ and choose a gauge such that $A_\mu^{(0)} = 0$. For any connection A on E we define $\tilde{A} = A - A^{(0)}$. The field $\tilde{A}$ transforms as a one-form valued in the adjoint representation of E. The path integral over A is equivalent to the path integral over $\tilde{A}$. Then we can write the gauge covariant Lie derivative $\mathcal{L}_v$ as follows

$$\mathcal{L}_v = L_v + G_\Phi. \tag{2.2.11}$$

Here L_v is a usual Lie derivative in the direction of the vector field v^μ. The action of L_v on the gauge bundle is defined by the background connection $A^{(0)}$ which we set to zero. The second term G_Φ is the gauge transformation generated by the adjoint valued scalar field Φ where

$$\Phi = v^M \tilde{A}_M. \tag{2.2.12}$$

The gauge transformation G_Φ acts on the matter and the gauge fields in the usual way: $G_\Phi \Phi_A = [\Phi, \Phi_A]$, $G_\Phi \cdot A_\mu = [\Phi, D_\mu] = -D_\mu\Phi$.

The term denoted by R in (2.2.9) is a $Spin(5,1)^R$-symmetry transformation. It acts on scalar fields as $(R \cdot \Phi)_A = R_{AB}\Phi^B$, and on fermions as $R \cdot \Psi = \frac{1}{4}R_{AB}\Gamma^{AB}\Psi$, where $R_{AB} = 2\varepsilon\tilde{\Gamma}_{AB}\tilde{\varepsilon}$. When ε and $\tilde{\varepsilon}$ are restricted to the $\mathcal{N}=2$ subspace of $\mathcal{N}=4$ algebra, ($\Gamma^{5678}\varepsilon = \varepsilon$ and $\Gamma^{5678}\tilde{\varepsilon} = \tilde{\varepsilon}$), the matrix R_{AB} with $A, B = 5, \ldots, 8$ is an anti-self-dual (left) generator of $SO(4)^R$ rotations. In other words, when we restrict ε to the $\mathcal{N}=2$ subalgebra of the $\mathcal{N}=4$ algebra, the $SO(4)^R$ R-symmetry group restricts to its $SU(2)^R_L$ subgroup. The fermionic fields of the $\mathcal{N}=2$ vector multiplet (we call them ψ) transform in the trivial representation of R, while the fermionic fields of the $\mathcal{N}=2$ hypermultiplet (we call them χ) transform in the spin-$\frac{1}{2}$ representation of R.

Finally, the term denoted by Ω in (2.2.9) generates a local dilatation with the parameter $2(\varepsilon\tilde{\varepsilon})$, under which the gauge fields do not transform, the scalar fields transform with weight 1, and the fermions transform with weight $\frac{3}{2}$. (In other words, if we make Weyl transformation $g_{\mu\nu} \to e^{2\Omega}g_{\mu\nu}$, we should scale the fields as $A_\mu \to A_\mu, \Phi \to e^{-\Omega}\Phi, \Psi \to e^{-\frac{3}{2}\Omega}\Psi$ to keep the action invariant.)

Classically, it is easy to restrict the fields and the symmetries of the $\mathcal{N}=4$ SYM to the pure $\mathcal{N}=2$ SYM: one can discard all fields of the $\mathcal{N}=2$ hypermultiplet and restrict ε by the condition $\Gamma^{5678}\varepsilon = \varepsilon$. The resulting action is invariant under $\mathcal{N}=2$ superconformal symmetry. On quantum level the pure $\mathcal{N}=2$ SYM is not conformally invariant. We will be able to give a precise definition of the quantum $\mathcal{N}=2$ theory on S^4, considering it as the $\mathcal{N}=4$ theory softly broken by giving a mass term to the hypermultiplet, which we will send to the infinity in the end.

If we start from Minkowski signature in the ten dimensional theory, then classically the supersymmetry groups for the $\mathcal{N}=4$, the $\mathcal{N}=2$, and the $\mathcal{N}=2^*$ Yang-Mills theories on S^4 are the following.

In the $\mathcal{N}=2$ case, ε is a Dirac spinor on S^4. The equation (2.2.5) has 16 linearly independent solutions, which correspond to the fermionic generators of the

$\mathcal{N} = 2$ superconformal algebra. Intuitively, 8 generators out of these 16 correspond to 8 charges of $\mathcal{N} = 2$ supersymmetry algebra on $\mathbb{R}^4$, and the other 8 correspond to the remaining generators of $\mathcal{N} = 2$ superconformal algebra. The full $\mathcal{N} = 2$ superconformal group on S^4 is $SL(1|2,\mathbb{H})$.[3] Its bosonic subgroup is $SL(1,\mathbb{H}) \times SL(2,\mathbb{H}) \times SO(1,1)$. The first factor $SL(1,\mathbb{H}) \simeq SU(2)$ generates the R-symmetry $SU(2)^R_L$ transformations. The second factor $SL(2,\mathbb{H}) \simeq SU^*(4,\mathbb{C}) \simeq Spin(5,1)$ generates conformal transformations of S^4. The third factor $SO(1,1)^R$ generates the $SO(1,1)^R$ symmetry transformations. The fermionic generators of $SL(1,2|\mathbb{H})$ transform in the $\mathbf{2} + \mathbf{2}'$ of the $SL(2,\mathbb{H})$, where $\mathbf{2}$ denotes the fundamental representation of $SL(2,\mathbb{H})$ of quaternionic dimension two. This representation can be identified with the fundamental representation $\mathbf{4}$ of $SU^*(4)$ of complex dimension four, or with chiral (Weyl) spinor representation of the conformal group $Spin(5,1)$. The other representatfion $\mathbf{2}'$ corresponds to the other chiral spinor represenation of $Spin(5,1)$ of the opposite chirality.

In the $\mathcal{N} = 4$ case we do not impose the chirality condition on ε. Hence a sixteen component Majorana-Weyl spinor ε of $Spin(9,1)$ reduces to a pair of the four-dimensional Dirac spinors $(\varepsilon_\psi, \varepsilon_\chi)$, where ε_ψ and ε_χ are elements of the $+1$ and -1 eigenspaces of the chirality operator Γ^{5678} respectively. Each of the Dirac spinors ε_ψ and ε_χ independently satisfies the conformal Killing spinor equation (2.2.5) because the operators Γ_μ do not mix the $+1$ and -1 eigenspaces of Γ^{5678}. Then we get $16{+}16 = 32$ linearly independent conformal Killing spinors. Each of these spinors corresponds to a generator of the $\mathcal{N} = 4$ superconformal symmetry. One can check that the full $\mathcal{N} = 4$ superconformal group on S^4 is $PSL(2|2,\mathbb{H})$.

To describe the $\mathcal{N} = 2^*$ theory on S^4, which is obtained by giving mass to the hypermultiplet, we need some more details on Killing spinors on S^4. Because mass terms break conformal invariance, we should expect the $\mathcal{N} = 2^*$ theory to be

[3]By $SL(n,\mathbb{H})$ we mean group of general linear transformation $GL(n,\mathbb{H})$ over quaternions factored by $\mathbb{R}^*$, so that the real dimension of $SL(n,\mathbb{H})$ is $4n^2 - 1$.

invariant only under 8 out of 16 fermionic symmetries of the $\mathcal{N}=2$ superconformal group $SL(1,2|\mathbb{H})$. In other words, we should impose some additional restrictions on ε. Let us desribe this theory in more details.

First we explicitly give a general solution for the conformal spinor Killing equation on S^4. Let x^μ be the stereographic coordinates on S^4. The origin corresponds to the North pole, the infinity corresponds to the South pole. If r is the radius of S^4, then the metric has the form

$$g_{\mu\nu} = \delta_{\mu\nu} e^{2\Omega}, \quad \text{where} \quad e^{2\Omega} := \frac{1}{(1+\frac{x^2}{4r^2})^2}. \tag{2.2.13}$$

We use the vielbein $e^i_\mu = \delta^i_\mu e^\Omega$ where δ^i_μ is the Kronecker delta, the index $\mu = 1,\ldots,4$ is the space-time index, the index $i = 1,\ldots,4$ enumerates vielbein elements. The solution of the conformal Killing equation (2.2.5) is (see appendix A.2)

$$\varepsilon = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}(\hat{\varepsilon}_s + x^i\Gamma_i\hat{\varepsilon}_c) \tag{2.2.14}$$

$$\tilde{\varepsilon} = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}(\hat{\varepsilon}_c - \frac{x^i\Gamma_i}{4r^2}\hat{\varepsilon}_s), \tag{2.2.15}$$

where $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ are Dirac spinor valued constants.

Classically, the action of $\mathcal{N}=2$ SYM on $\mathbb{R}^4$ with a massless hypermultiplet is invariant under the $\mathcal{N}=2$ superconformal group, which has 16 fermionic generators. Turning on non-zero mass of the hypermultiplet breaks 8 superconformal fermionic symmetries, but preserves the other 8 fermionic symmetries which generate the $\mathcal{N}=2$ supersymmetry. These 8 charges are known to be preserved on quantum level [63]. The $\mathcal{N}=2$ supersymmetry algebra closes to the scale preserving transformations: the translations on $\mathbb{R}^4$. These scale preserving transformations are symmetries of the massive theory as well.

Following the same logic, we would like to find a subgroup, which will be called $\mathcal{S}$, of the $\mathcal{N}=2$ superconformal group on S^4 with the following properties. The supergroup $\mathcal{S} \subset SL(1|2,\mathbb{H})$ contains 8 fermionic generators, the bosonic transformations of $\mathcal{S}$ are the scale preserving transformations and are compatible with mass terms for the hypermultiplet. The group $\mathcal{S}$ is what we call the $\mathcal{N}=2$ supersymmetry group on S^4.

The conformal group of S^4 is $SO(5,1)$. The scale preserving subgroup of the $SO(5,1)$ is the $SO(5)$ isometry group of S^4. We require that the space-time bosonic part of $\mathcal{S}$ is a subgroup of this $SO(5)$. This means that for any conformal Killing spinor ε that generates a fermionic transformation of $\mathcal{S}$, the dilatation parameter $(\tilde{\varepsilon}\varepsilon)$ in the δ_ε^2 vanishes.

For a general ε in the $\mathcal{N}=2$ superconformal group, the transformation δ_ε^2 contains $SO(1,1)^R$ generator. Since the $SO(1,1)^R$ symmetry is broken explicitly by hypermultiplet mass terms, and since it is broken on quantum level in the usual $\mathcal{N}=2$ theory in the flat space[4], we require that $\mathcal{S}$ contains no $SO(1,1)^R$ transformations. In other words, the conformal Killing spinors ε which generate transformations of $\mathcal{S}$ are restricted by the condition that the $SO(1,1)^R$ generator in δ_ε^2 vanishes. By equation (2.2.7) this means $\tilde{\varepsilon}\Gamma^{09}\varepsilon=0$.

Using the explicit solution (2.2.14) we rewrite the equation $(\tilde{\varepsilon}\varepsilon)=(\tilde{\varepsilon}\Gamma^{09}\varepsilon)=0$ in terms of $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$

$$\begin{aligned} \hat{\varepsilon}_s\hat{\varepsilon}_c &= \hat{\varepsilon}_s\Gamma^{09}\hat{\varepsilon}_c = 0 \\ \hat{\varepsilon}_c\Gamma^\mu\hat{\varepsilon}_c &- \frac{1}{4r^2}\hat{\varepsilon}_s\Gamma^\mu\hat{\varepsilon}_s = 0. \end{aligned} \tag{2.2.16}$$

To solve the second equation, we take chiral $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ with respect to the four-dimensional chirality operator Γ^{1234}. Since the operators Γ^μ reverse the four-dimensional chirality, both terms in the second equation vanish automatically. There are two interesting cases: (i) the chirality of $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ is opposite, (ii) the chirality of $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ is the same. The main focus of this work is on the second case.

1. In the first case we can assume that

$$\varepsilon_s^L = 0, \quad \hat{\varepsilon}_c^R = 0.$$

[4] See e.g. [62, 63] keeping in mind that if we start from the Euclidean signature in ten dimensions, the $SO(1,1)^R$ group is replaced by the usual $U(1)^R$ symmetry of $\mathcal{N}=2$ theory.

Here by ε_s^L and ε_s^R we denote left/right four-dimensional chiral components. They are respectively defined as the $-1/+1$ eigenspaces of the chirality operator Γ^{1234}. In this case the first equation in (2.2.16) is also automatically satisfied. Moreover, the spinors ε and $\tilde{\varepsilon}$ also have opposite chirality over the whole S^4. Hence we have 8 generators, say $\hat{\varepsilon}_s^R$ and $\hat{\varepsilon}_c^L$, which anticommute to pure gauge transformations generated by the scalar field $\Phi := (\varepsilon\Gamma^A\varepsilon)\Phi_A$. The δ_ε-closed observables are the gauge invariant functions of Φ and their descendants. One could try to interpret such δ_ε as a cohomological BRST operator Q and to relate in this way the physical $\mathcal{N} = 2$ gauge theory on S^4 with the topological Donaldson-Witten theory. That does not work, because in the present case the conformal Killing spinor ε, generated by such $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ necessary vanishes somewhere on S^4. Of course, in the twisted theory [18, 78] the problem does not arise, since ε is a scalar and can be set to be a non-zero constant everywhere. However, our goal is to treat the non-twisted theory. Moreover, the circular Wilson loop operator $W_R(C)$ is not closed under such δ_ε. Thus we turn to the second case.

2. The spinors $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ have the same chirality, say left, and the first equation restricts them to be orthogonal

$$\hat{\varepsilon}_s^R = 0, \quad \hat{\varepsilon}_c^R = 0, \quad (\hat{\varepsilon}_s^L \hat{\varepsilon}_c^L) = 0.$$

The Killing vector field $v^\mu = \varepsilon\Gamma^\mu\varepsilon$, associated with the δ_ε^2, generates an anti-self-dual (left) rotation of S^4 around the North pole. In addition, δ_ε^2 generates a $SU(2)_L^R$-symmetry transformation and a gauge symmetry transformation. The spinor ε is chiral only at the North and the South poles of S^4, but not at any other point. At the North pole ε is left, at the South pole ε is right. We can find circular Wilson loop operators of the form (2.1.1) which are invariant under such δ_ε. Conversely, for any given circular Wilson loop $W_R(C)$ of the form (2.1.1) we can find a suitable conformal Killing spinor δ_ε which annihilates $W_R(C)$. (The North pole is picked up at the center of the $W_R(C)$.) If the spinors $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ are both non zero, then

ε is a nowhere vanishing spinor on S^4. We can use such δ_ε to relate the physical $\mathcal{N}=2$ gauge theory on S^4 to a somewhat unusual equivariant topological theory, and apply localization methods developed for topological theories [18, 58] to solve for $\langle W_R(C)\rangle$. The relation has the simplest form if the norm of ε is constant.

Before proceeding to this equivariant topological theory, we would like to finish our description of the supersymmetry group $\mathcal{S}$ of the $\mathcal{N}=2^*$ theory on S^4. First we find the maximal set of linearly independent conformal Killing spinors $\{\varepsilon^i\}$ that simultaneously satisfy the equations

$$\varepsilon^{(i}\tilde{\varepsilon}^{j)} = \varepsilon^{(i}\Gamma^{09}\tilde{\varepsilon}^{j)} = 0, \tag{2.2.17}$$

and then we find what superconformal group is generated by this set. One can show that the equivalent way to formulate the conformal Killing spinor equation for the spinors in the $+1$ space of the chirality operator Γ^{5678} is the following

$$D_\mu\varepsilon = \frac{1}{2r}\Gamma_\mu\Lambda\varepsilon, \tag{2.2.18}$$

where Λ is a generator of $SU(2)^R_L$-symmetry. For example, if we start from the ten-dimensional Minkowski signature we can take $\Lambda = \Gamma^0\Gamma_{ij}$ where $5 \le i < j \le 8$. If we start from the ten-dimensional Euclidean signature we can take $\Lambda = -i\Gamma^0\Gamma_{ij}$ where $5 \le i < j \le 8$. Equivalently, Λ is a real antisymmetric matrix, which acts in the $+1$ eigenspace of Γ^{5678}, satisfies $\Lambda^2 = -1$ and commutes with Γ^m for $m = 1, \dots, 4, 0, 9$. The equation (2.2.18) has 8 linearly independent solutions. Let V_Λ be the vector space that they span. Then the space of solutions of the conformal Killing spinor equations (2.2.5) is $V_\Lambda \oplus V_{-\Lambda}$, where we take $\tilde{\varepsilon} = \frac{1}{2r}\Lambda\varepsilon$.

The spinors in the space V_Λ satisfy our requirement (2.2.17), because Λ is antisymmetric and commutes with Γ^9. The generators $\{\delta_\varepsilon | \varepsilon \in V_\Lambda\}$ anticommute to generators of $Spin(5) \times SO(2)^R$, where $Spin(5)$ rotates S^4, and $SO(2)^R$ is a subgroup of the $SU(2)^R_L$-symmetry group. This $SO(2)^R$ subgroup is generated by Λ. The space V_Λ transforms in the fundamental representation of $Sp(4) \simeq Spin(5)$. We conclude that restricting the fermionic generators to the space V_Λ of (2.2.18) breaks the full $\mathcal{N}=2$ superconformal group $SL(1|2,\mathbb{H})$ to the supergroup $OSp(2|4)$, where the choice of the $SU(2)^R_L$ generator Λ defines the embedding of the $SO(2)_R$ in the $SU(2)^R_L$.

Besides the spaces V_Λ, obtained as solutions of (2.2.18), we can find other half-dimensional fermionic subspaces of the $\mathcal{N}=2$ superconformal group satisfying (2.2.17). These spaces can be obtained by $SO(1,1)_R$ twisting of V_Λ. Indeed, if the spinors ε and $\tilde{\varepsilon}$ satisfy (2.2.17), then so do the spinors $\varepsilon' = e^{\frac{1}{2}\beta\Gamma^{09}}\varepsilon$ and $\tilde{\varepsilon}' = e^{-\frac{1}{2}\beta\Gamma^{09}}\tilde{\varepsilon}$, where Γ^{09} generates $SO(1,1)_{R'}$, and β is a parameter of the twisting. The $SO(1,1)_R$ twisted space $V_{\Lambda,\beta}$ is equivalently a space of solutions to the twisted Killing equation

$$D_\mu \varepsilon = \frac{1}{2r}\Gamma_\mu e^{-\beta\Gamma^{09}}\Lambda\varepsilon. \tag{2.2.19}$$

We summarize, that restriction to the half-dimensional fermionic subspace by equation (2.2.17) breaks the $\mathcal{N}=2$ superconformal group $SL(1|2,\mathbb{H})$ down to $OSp(2|4)$. The choice of $OSp(2|4)$ is defined by the generator of $SU(2)_R$ symmetry Λ, and the generator of $SO(1,1)_{R'}$ symmetry β.

If we require that the Wilson loop operator is closed with respect to δ_ε with $\varepsilon \in V_{\Lambda,\beta}$, then the parameter β is related to the radius of the Wilson loop. In the ten-dimensional Minkowski conventions, the Wilson loop operator has the form

$$W_R(\rho) = \mathrm{tr}_R \,\mathrm{Pexp} \oint_C ((A_\mu \frac{dx^\mu}{ds} + \Phi_0)ds). \tag{2.2.20}$$

Let the circular contour C be $(x^1,x^2,x^3,x^4) = t(\cos\alpha, \sin\alpha, 0, 0)$ in the stereographic coordinates. Here $t = 2r\tan\frac{\theta_0}{2}$ for the Wilson loop located at the polar angle θ_0. The combination $v^M A_M = v^\mu A_\mu + v^A \phi_A$ is annihilated by δ_ε, since $(\varepsilon\Gamma^M\varepsilon)(\psi\Gamma_M\varepsilon)$ vanishes because of the triality identity (A.1.9). Then the Wilson loop (2.2.20) is δ_ε-closed if $(v^\mu, v^9, v^0) = (\frac{dx^\mu}{ds}, 0, 1)$. Using $\Gamma^0 = 1$ and the explicit form (2.2.16) for ε we get

$$\hat{\varepsilon}_c = \frac{1}{t}\Gamma_{12}\hat{\varepsilon}_s. \tag{2.2.21}$$

To satisfy (2.2.18) we must have

$$\hat{\varepsilon}_c = \frac{1}{2r} e^{-\beta\Gamma^{09}} \Lambda \hat{\varepsilon}_s. \tag{2.2.22}$$

Let chirality of $\hat{\varepsilon}_s, \hat{\varepsilon}_c$ be positive at $x = 0$. Then $\beta = \log \frac{t}{2r}$, and $(\Lambda - \Gamma_{12})\hat{\varepsilon}_s = 0$. This equation has a non-zero solution for $\hat{\varepsilon}_s$ only when $\det(\Lambda - \Gamma_{12}) = 0$. That determines Λ uniquely up to a sign. In other words, the choice of the position of the Wilson loop on S^4 determines the way the $SU(2)_R$ symmetry group breaks to $SO(2)$, and the size of the Wilson loop determines the $SO(1,1)$ twist parameter β. For the Wilson loop located at the equator $t = 2r$.

A very nice property of the conformal Killing spinor ε generating $OSp(2|4)$ is that it has a constant norm over S^4, similarly to a supersymmetry transformation on flat space. Since $OSp(2|4)$ has 8 fermionic generators, contains only scale preserving transformations, and it is generated by spinors of constant norm on S^4, we call it $\mathcal{N} = 2$ supersymmetry on S^4. So we have found that $\mathcal{S} = OSp(2|4)$.

Now we show that it is possible to add a mass term for the hypermultiplet fields and preserve the $OSp(2|4)$ symmetry. From now we will assume that the Wilson loop is located at the equator, so that ε has a constant norm. To generate such mass term in four dimensions we use Scherk-Schwarz reduction of ten-dimensional $\mathcal{N} = 1$ SYM. Namely, we turn on a Wilson line in the $SU(2)_R^R$ symmetry group along the coordinate x_0. The $\mathcal{N} = 2$ vector multiplet fields $A_\mu, \Phi_0, \Phi_9, \Psi$ are not charged under $SU(2)_R^R$, therefore their kinetic terms are not changed. The hypermultiplet fields χ and Φ_i with $i = 5, \ldots, 8$ transform in the spin-$\frac{1}{2}$ representation under $SU(2)_R^R$. Explicitly it means that we should replace $D_0\Phi_i$ by $D_0\Phi_i + M_{ij}\Phi_j$, and $D_0\chi$ by $D_0\chi + \frac{1}{4}M_{ij}\Gamma_{ij}\chi$, where an antisymmetric 4×4 matrix M_{ij} with $i, j = 5, \ldots, 8$ is a generator of the $SU(2)_R^R$ symmetry. Since F_{0i} is replaced by $[\Phi_0, \Phi_i] + M_{ij}\Phi_j$, the $F_{0i}F^{0i}$ term in the action generates mass for the scalars of the hypermultiplet.

On the flat space, the resulting action is still invariant under the usual $\mathcal{N} = 2$ supersymmetry. However, on S^4 we need to be more careful with the ε-derivative terms in the supersymmetry transformations. Let us explicitly compute variation of the Scherk-Schwarz deformed $\mathcal{N}=4$ theory on S^4. We use the conformal Killing spinor ε in the $\mathcal{N}=2$ superconformal subsector, i.e. $\Gamma^{5678}\varepsilon=\varepsilon$. Then ε is not charged under $SU(2)^R_R$, so $D_0\varepsilon=0$. Variation of (2.2.2) by (2.2.3) gives us (we write variation of the Lagrangian up to total derivative terms since they vanish after integration over the compact space S^4)

$$\begin{aligned}
&\delta_\varepsilon(\frac{1}{2}F_{MN}F^{MN}-\Psi\Gamma^M D_M\Psi+\frac{2}{r^2}\Phi_A\Phi^A)=\\
&=2D_M(\varepsilon\Gamma_N\Psi)F^{MN}+2\Psi\Gamma^M D_M(\frac{1}{2}F_{PQ}\Gamma^{PQ}\varepsilon-2\Phi_A\tilde{\Gamma}^A\tilde{\varepsilon})+\frac{4}{r^2}(\varepsilon\Gamma^A\psi)\Phi_A=\\
&=-2(\varepsilon\Gamma_N\Psi)D_M F^{MN}+\Psi D_M F_{PQ}\Gamma^M\Gamma^{PQ}\varepsilon+\Psi\Gamma^M\Gamma^{PQ}F_{PQ}D_M\varepsilon-4\Psi\Gamma^M\tilde{\Gamma}^A\tilde{\varepsilon}D_M\Phi_A+\\
&\qquad+\frac{1}{r^2}\Psi\Gamma^\mu\tilde{\Gamma}^A\Phi_A\Gamma_\mu\varepsilon+\frac{4}{r^2}(\varepsilon\Gamma^A\Psi)\Phi_A=\dots
\end{aligned}$$

Using

$$\Gamma^M\Gamma^{PQ}=\frac{1}{3}(\Gamma^M\Gamma^{PQ}+\Gamma^P\Gamma^{QM}+\Gamma^M\Gamma^{PQ})+2g^{M[P}\Gamma^{Q]} \tag{2.2.23}$$

and the Bianchi identity, we see that the first term cancels the second, and that the last two terms cancel each other. Then

$$\dots=\Psi\Gamma^\mu\Gamma^{PQ}\Gamma_\mu\tilde{\varepsilon}F_{PQ}-4\Psi\Gamma^M\tilde{\Gamma}^A\tilde{\varepsilon}D_M\Phi_A=4\Psi\tilde{\Gamma}^{MA}\tilde{\varepsilon}F_{MA}-4\Psi\Gamma^M\tilde{\Gamma}^A\tilde{\varepsilon}D_M\Phi_A$$

where we use the index conventions $M,N,P,Q=0,\dots,9$, $\mu=1,\dots,4$, $A=5,\dots,9,0$. In the absence of Scherk-Schwarz deformation we have $F_{MA}=D_M\Phi_A$ for all $M=0,\dots,9$ and $A=5,\dots,9,0$, hence the two terms cancel. After the deformation, we have $F_{0i}=D_0\Phi_i$, but $F_{i0}=-D_0\Phi_i=-[\Phi_0,\Phi_i]-M_{ij}\Phi_j=D_i\Phi_0-M_{ij}\Phi_j$. Therefore, the naively Scherk-Schwarz deformed $\mathcal{N}=4$ theory on S^4 is not invariant under arbitrary $\mathcal{N}=2$ superconformal transformation:

$$\delta_\varepsilon(\frac{1}{2}F_{MN}F^{MN}-\Psi\Gamma^M D_M\Psi+\frac{2}{r^2}\Phi_A\Phi^A)=-4\Psi\Gamma^i\tilde{\Gamma}^0\tilde{\varepsilon}M_{ij}\Phi_j. \tag{2.2.24}$$

This is the natural consequence of adding mass terms to the Lagrangian. Nevertheless, we can add some other terms to the action in such a way to make the action invariant under the $OSp(2|4)$ subgroup of $\mathcal{N}=2$ superconformal group on S^4. We use the fact that ε generating a transformation in the $OSp(2|4)$ subgroup satisfies the conformal Killing equation with $\tilde{\varepsilon} = \frac{1}{2r}\Lambda\varepsilon$, where Λ is a generator of $SU(2)^R_L$-group normalized as $\Lambda^2 = -1$. Let us take $\Lambda = \frac{1}{4}\Gamma_{kl}R_{kl}$ where R_{kl} is an anti-self-dual matrix normalized as $R_{kl}R^{kl} = 4$, where $k, l = 5, \ldots, 8$. Then we get

$$\begin{aligned}\delta_\varepsilon(\frac{1}{2}F_{MN}F^{MN} - \Psi\Gamma^M D_M\Psi + \frac{2}{r^2}\Phi_A\Phi^A) = \frac{1}{2r}\Psi\Gamma^0\Gamma^i\Gamma^{kl}\varepsilon R_{kl}M_{ij}\Phi_j = \\ = \frac{1}{2r}(\Psi\Gamma^i\varepsilon)R_{ki}M_{kj}\Phi_j = \frac{1}{2r}(\delta_\varepsilon\Phi^i)(R_{ki}M_{kj})\Phi_j\end{aligned} \tag{2.2.25}$$

Hence, the addition of $\frac{-1}{4r}(R_{ki}M_{kj})\Phi^i\Phi^j$ term to the Scherk-Schwarz deformed action on S^4 makes the action invariant under the $OSp(2|4)$.

Let us summarize. The action

$$S_{\mathcal{N}=2^*} = \frac{1}{2g^2_{YM}}\int d^4x\sqrt{g}\left(\frac{1}{2}F_{MN}F^{MN} - \Psi\Gamma^M D_M\Psi + \frac{2}{r^2}\Phi_A\Phi^A - \frac{1}{4r}(R_{ki}M_{kj})\Phi^i\Phi^j\right), \tag{2.2.26}$$

where $D_0\Phi^i = [\Phi_0, \cdot] + M_{ij}\Phi^j$ and $D_0\Psi = [\Phi_0, \Psi] + \frac{1}{4}\Gamma^{ij}M_{ij}\Psi$, is invariant under the $OSp(2|4)$ transformations, generated by conformal Killing spinors solving $D_\mu\varepsilon = \frac{1}{8r}\Gamma_\mu\Gamma^{0kl}R_{kl}\varepsilon$ with ε restricted to $\mathcal{N}=2$ subspace $\Gamma^{5678}\varepsilon = \varepsilon$.

Since δ^2_ε generates a covariant Lie derivative along the vector field $-v^M = -\varepsilon\Gamma^M\varepsilon$, in particular it is contributed by the gauge transformation along the 0-th direction. After we turned on mass for the hypermultiplet by Scherk-Schwarz mechanism, δ^2_ε gets new contributions on the hypermultiplet

$$\begin{aligned}\delta^2_\varepsilon\Phi_i = \delta^2_{\varepsilon,\mathrm{M}=0}\Phi_i - v^0 M_{ij}\Phi_j \\ \delta^2_\varepsilon\chi = \delta^2_{\varepsilon,\mathrm{M}=0}\chi - \frac{1}{4}v^0 M_{ij}\Gamma^{ij}\chi.\end{aligned} \tag{2.2.27}$$

So far we computed δ^2_ε on-shell. To use the localization method we need an off-shell closed formulation of the fermionic symmetry of the theory. The pure $\mathcal{N}=2$ SYM can be easily closed by means of three auxiliary scalar fields, but it is well known that the off-shell closure of $\mathcal{N}=2$ hypermultiplet is impossible with a finite number of auxiliary fields. For our purposes we do not need to close off-shell the whole $OSp(2|4)$ symmetry group. Since the localization computation uses only one fermionic generator Q_ε, it is enough to close off-shell only the symmetry generated by this ε.

To close off-shell the relevant supersymmetry of the $\mathcal{N}=4$ theory on S^4 we make the dimensional reduction of Berkovits method [79] used for the ten-dimensional $\mathcal{N}=1$ SYM, see also [80, 81]. The number of auxiliary fields compensates the difference between the number of fermionic and bosonic off-shell degrees of freedom modulo gauge transformations. In the $\mathcal{N}=4$ case we add $16-(10-1)=7$ auxiliary fields K_i with free quadratic action and modify the superconformal transformations to

$$\begin{aligned}
&\delta_\varepsilon A_M = \Psi\Gamma_M\varepsilon \\
&\delta_\varepsilon\Psi = \frac{1}{2}\gamma^{MN}F_{MN} + \frac{1}{2}\gamma^{\mu A}\phi_A D_\mu\varepsilon + K^i\nu_i \\
&\delta_\varepsilon K_i = -\nu_i\gamma^M D_M\Psi,
\end{aligned} \tag{2.2.28}$$

where spinors ν_i with $i=1,\ldots,7$ are required to satisfy

$$\varepsilon\Gamma^M\nu_i = 0 \tag{2.2.29}$$

$$\frac{1}{2}(\varepsilon\Gamma_N\varepsilon)\tilde{\Gamma}^N_{\alpha\beta} = \nu^i_\alpha\nu^i_\beta + \varepsilon_\alpha\varepsilon_\beta \tag{2.2.30}$$

$$\nu_i\Gamma^M\nu_j = \delta_{ij}\varepsilon\Gamma^M\varepsilon. \tag{2.2.31}$$

For any non-zero Majorana-Weyl spinor ε of $Spin(9,1)$ there exist seven linearly independent spinors ν_i, which satisfy these equations[5] [79]. They are determined up to an $SO(7)$ transformations. The equation (2.2.29) ensures closure on A_M, the equation (2.2.30) ensures closure on Ψ, and the equations (2.2.29) and (2.2.31) ensure closure on K

$$\delta^2_\varepsilon K_i = -(\varepsilon\gamma^M\varepsilon)D_M K^i - (\nu_{[i}\gamma^M D_M\nu_{j]})K^j - 4(\tilde{\varepsilon}\varepsilon)K_i. \tag{2.2.32}$$

[5] The author thanks N.Berkovits for communications.

If E_K is an $SO(7) \otimes \mathrm{ad}(G)$ vector bundle over S^4 whose sections correspond to the auxiliary fields K_i, then (2.2.32) can be interpreted as a covariant Lie derivative action along the vector field v^μ, or in other words as a lift of the L_v action on S^4 to the action on the vector bundle $E_K \to S^4$. A conformal Killing spinor ε generating a transformation of the $OSp(2|4)$ subgroup can be represented in the following form (see appendix A.2 for details)

$$\varepsilon(x) = \exp\left(\frac{\theta}{2} n_i(x)\Gamma^i\Gamma^9\right)\hat{\varepsilon}_s, \tag{2.2.33}$$

where x^i are the stereographic coordinates on S^4, n_i is the unit vector in the direction of the vector field $v_i = \frac{1}{r}x^i\omega_{ij}$. We use the conformal Killing spinor $\varepsilon(x)$ such that $(\varepsilon(x), \varepsilon(x)) = 1$ and $\Gamma^9\hat{\varepsilon}_s = \hat{\varepsilon}_s$. The matrix ω_{ij} is the anti-self-dual generator of $SU(2)_L \subset SO(4)$ rotation around the North pole in δ^2_ε. We see that the conformal Killing spinor $\varepsilon(x)$ at an arbitrary point x is obtained by $Spin(5)$ rotation $\exp(\frac{\theta}{2}n_i(x)\Gamma^i\Gamma^9)$ of its value at the origin $\varepsilon(0) = \hat{\varepsilon}_s$.

For the closure of $\mathcal{N} = 4$ symmetry we need seven spinors ν_i which satisfy (2.2.29)-(2.2.31). Following [79], at the origin we can take $\hat{\nu}_i = \Gamma^{i8}\hat{\varepsilon}_s$ for $i = 1 \ldots 7$, and then transform $\hat{\nu}_i$ to an arbitrary point on S^4 as

$$\nu_i(x) = \exp(\frac{\theta}{2}n_i(x)\Gamma^{i8})\hat{\varepsilon}_s. \tag{2.2.34}$$

Finally, we conclude that the action

$$\begin{aligned} S_{\mathcal{N}=2^*} = \frac{1}{2g_{YM}^2}\int d^4x\sqrt{g}\,\Bigg(&\frac{1}{2}F_{MN}F^{MN} - \Psi\Gamma^M D_M\Psi + \frac{2}{r^2}\Phi_A\Phi^A \\ &-\frac{1}{4r}(R_{ki}M_{kj})\Phi^i\Phi^j - K_iK_i\Bigg), \end{aligned} \tag{2.2.35}$$

is invariant under the off-shell supersymmetry Q_ε given by (2.2.28) with ν_i defined by (2.2.34). Though we will not need this fact, we remark that it is possible to simultaneously close four fermionic symmetries generating the $OSp(2|2)$ subgroup of $OSp(2|4)$. The space-time part of this $OSp(2|2)$ subgroup consists of anti-self-dual rotations around the North pole on S^4.

2.3 Localization

As explained in the introduction, to localize the theory we deform the action by a Q-exact term

$$S \to S + tQV. \tag{2.3.1}$$

Since we use Q which squares to a symmetry of the theory, and since the action and the Wilson loop observable are Q-closed, we can use the localization argument. For Q^2-invariant V, the deformation (2.3.1) does not change the expectation value of Q-closed observables. Hence, when we send t to infinity, the theory localizes to some set F of critical points of QV, over which we will integrate in the end. The measure in the integral over F comes from the restriction of the action S to F and the determinant of the kinetic term of QV which counts fluctuations in the normal directions to F.

To ensure convergence of the four-dimensional path integral, we compute it for a theory obtained by dimensional reduction from a theory in ten-dimensional Euclidean signature. To technically simplify the description of the symmetries in the previous section, we used ten-dimensional Minkowski signature. We can keep Minkowski metric g_{MN} and Minkowski gamma-matrices Γ_M and still get the same partition function as in Euclidean signature by making Wick rotation of the Φ_0 field. In other words, the path integral, computed with Minkowski metric g_{MN} but with Φ_0 substituted by $i\Phi_0^E$ where Φ_0^E is real, is convergent and is equal to the Euclidean path integral. We also integrate over imaginary contour for the auxilary fields K_i, so that $K_i = iK_i^E$, where K_i^E is real.

For localization computation we will take the following functional

$$V = (\Psi, \overline{Q\Psi}). \tag{2.3.2}$$

Then the bosonic part of the QV-term is a positive definite functional

$$S^Q|_{bos} = (Q\Psi, \overline{Q\Psi}). \tag{2.3.3}$$

Explicitly we have

$$\begin{aligned} Q\Psi &= \frac{1}{2}F_{MN}\Gamma^{MN}\varepsilon + \frac{1}{2}\Phi_A\Gamma^{\mu A}\nabla_\mu\varepsilon + K^i\nu_i \\ \overline{Q\Psi} &= \frac{1}{2}F_{MN}\tilde{\Gamma}^{MN}\varepsilon + \frac{1}{2}\Phi^A\tilde{\Gamma}^{\mu A}\nabla_\mu\varepsilon - K^i\nu_i, \end{aligned} \tag{2.3.4}$$

where $\tilde{\Gamma}^0 = -\Gamma^0, \tilde{\Gamma}^M = \Gamma^M$ for $M = 1, \ldots, 9$, and $\Gamma^{MN} = \tilde{\Gamma}^{[M}\Gamma^{N]}, \tilde{\Gamma}^{MN} = \Gamma^{[M}\tilde{\Gamma}^{N]}$.

Before proceeding to technical details of the computation, let us explicitly define the conformal Killing spinor ε which we will use, and find the vector field $v^M = \varepsilon\Gamma^M\varepsilon$ generated by the corresponding δ_ε^2. We take ε in the form (2.2.14), where $\hat{\varepsilon}_s$ is any spinor such that

1. The chirality operator Γ^{5678} acts on $\hat{\varepsilon}_s$ by 1
2. The chirality operator Γ^{1234} acts on $\hat{\varepsilon}_s$ it by -1
3. $\hat{\varepsilon}_s\hat{\varepsilon}_s = 1$

The first condition means that ε generates transformation inside the $\mathcal{N} = 2$ superconformal subgroup of $\mathcal{N} = 4$ superconformal group. The second condition ensures that ε is a four-dimensional left chiral spinor on the North pole of S^4. The third condition is a conventional normalization. In our conventions for the gamma-matrices (appendix A.1) we can take $\hat{\varepsilon}_s = (1, 0, \ldots, 0)^t$. Let the Wilson loop be located at the equator and invariant under anti-self-dual rotations in the $SO(4)$ group of rotations around the North pole. To be concrete, let the Wilson loop be placed in the (x_1, x_2) plane. Then we take $\hat{\varepsilon}_c = \frac{1}{2r}\Gamma^{12}\hat{\varepsilon}_s$. The conformal Killing spinor ε defined by such $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ has a constant unit norm over the whole four-sphere ($(\varepsilon\varepsilon) = 1$). At the North pole the spinor ε is purely left, at the South pole the spinor ε is purely right.

Now we compute the components of the vector field $v^M = \varepsilon\Gamma^M\varepsilon$. If we assume ten-dimensional Minkowski signature, then we get

$$\begin{aligned} v_t &= \sin\theta \\ v^0 &= 1 \\ v^9 &= \cos\theta \\ v^i &= 0 \quad \text{for} \quad i = 5, \ldots, 8, \end{aligned} \tag{2.3.5}$$

where θ is the polar angle on S^4 such that the Wilson loop is placed at $\theta = \frac{\pi}{2}$, the North pole is at $\theta = 0$, and the South pole is at $\theta = \pi$. The four-dimensional space-time component v_t of v^M has length $\sin\theta$ and is directed along the orbits of the $U(1) \subset SU(2)_L \subset SO(4)$ group which rotates the (x_1, x_2) plane. If we switch to the ten-dimensional Euclidean signature, then $v^0 = i$ while the other componens are the same as in Minkowski signature.

To simplify $S^Q|_{bos}$ we use the Bianchi identity for F_{MN}, the gamma-matrices algebra and integration by parts. The principal contribution to $S^Q|_{bos}$ is the curvature term

$$S_{FF} = \frac{1}{4}(\varepsilon\tilde{\Gamma}^N\Gamma^M\tilde{\Gamma}^P\Gamma^Q\varepsilon)F^{MN}F^{PQ} \tag{2.3.6}$$

The $F_{MN}K_i$ cross-terms vanish because $\nu_i\Gamma^0\Gamma^M\varepsilon = \nu_i\Gamma^M\varepsilon = 0$. Then we have a simple contribution from auxiliary KK-term

$$S_{KK} = -K_iK^i. \tag{2.3.7}$$

In the flat space limit, $r \to \infty$ the spinor ε is covariantly constant $\nabla_\mu\varepsilon = 0$. Therefore, in the flat space we simply have $S^Q|_{bos} = S_{FF} + S_{KK}$. Up to the total derivatives and $\nabla_\mu\varepsilon$-terms, using the Bianchi identity and the gamma-matrices algebra, we can see that S_{FF} is equivalent to the usual Yang-Mills action $\frac{1}{2}F^{MN}F_{MN}$. When the space is curved and $\nabla_\mu\varepsilon \neq 0$ we shall make more careful computation. Using (2.2.23) we get

$$S_{FF} = \frac{1}{2}F^{MN}F_{MN} + \frac{1}{4}\varepsilon\tilde{\Gamma}^N\Gamma^M\tilde{\Gamma}^P\Gamma^Q\varepsilon\frac{1}{3}\left(F_{MN}F_{PQ} + F_{PN}F_{QM} + F_{QN}F_{MP}\right). \tag{2.3.8}$$

To simplify the last term, first we break the indices into two groups: $M, N, P, Q = (1, \ldots, 4, 9, 0)$ and $M, N, P, Q = (5, \ldots, 8)$ describing respectively the fields of the vector and hyper multiplet. Using $\Gamma^{5678}\varepsilon = \varepsilon$ we can see that the nonvanishing terms have only zero, two or four of indices in the hypermultiplet range $(5, \ldots, 8)$. We call the resulting terms as vector-vector, vector-hyper and hyper-hyper respectively. First we consider vector-vector terms. For vector-vector terms we split indices to the gauge field part $(1, \ldots, 4)$ and to the scalar part $(0, 9)$. The nonvanishing gauge field terms all have different values of M, N, P, Q. Then their contribution is simplified to

$$\frac{1}{4}\cdot\frac{1}{3}\cdot 24\cdot\varepsilon\Gamma^{1234}\varepsilon(F^{21}F^{34}+F^{31}F^{42}+F^{41}F^{23}) = -\frac{1}{2}\varepsilon\Gamma^{1234}\varepsilon(F, *F) = -\frac{1}{2}\cos\theta(F, *F), \tag{2.3.9}$$

where $*F$ is the Hodge dual of F. All terms in which one of the indices is 0 vanish because Γ^{MPQ} is antisymmetric matrix, hence $\varepsilon\Gamma^0\Gamma^{MPQ}\varepsilon = 0$. Then the remaining vector-vector terms have $D_\mu\Phi_9 F$ structure. Integrating by parts and using Bianchi identity we get

$$-\frac{1}{3}D_\mu(\varepsilon\Gamma^9\Gamma^{\mu\nu\rho}\varepsilon)\Phi_9 F_{\nu\rho} + \text{cyclic}(\mu\nu\rho) = 4(\tilde{\varepsilon}\Gamma^9\Gamma^{\mu\nu}\varepsilon)\Phi_9 F_{\mu\nu}. \tag{2.3.10}$$

Doing similar algebra we get the contribution to the vector-hyper mixing terms in S_{FF}

$$-8\tilde{\varepsilon}\Gamma^9\Gamma^{ij}\varepsilon\Phi_i[\Phi_9, \Phi_j] - 6\tilde{\varepsilon}\Gamma^\mu\Gamma^{ij}\varepsilon\Phi_i D_\mu\Phi_j \tag{2.3.11}$$

We sum up all contributions to S_{FF} and obtain

$$\begin{aligned} S_{FF} = \frac{1}{2}F_{MN}F^{MN} - \frac{1}{2}\cos\theta F_{\mu\nu}(*F)^{\mu\nu} + 4(\tilde{\varepsilon}\Gamma^9\Gamma^{\mu\nu}\varepsilon)\Phi_9 F_{\mu\nu} \\ -8\tilde{\varepsilon}\Gamma^9\Gamma^{ij}\varepsilon\Phi_i[\Phi_9, \Phi_j] - 6\tilde{\varepsilon}\Gamma^\mu\Gamma^{ij}\Phi_i D_\mu\Phi_j. \end{aligned} \tag{2.3.12}$$

Next we consider the cross-terms between Φ^A and F_{MN} in $S^Q|_{bos}$

$$S_{F\Phi} = -\tilde{\varepsilon}\Gamma^A\tilde{\Gamma}^M\Gamma^N\varepsilon\Phi_A F_{MN} - \tilde{\varepsilon}\tilde{\Gamma}^A\Gamma^M\tilde{\Gamma}^N\varepsilon\Phi_A F_{MN}.$$

We consider separately the cases when the index A is in the set $\{0,9\}$ and in the set $\{5,\ldots,8\}$. The terms with index $A=0$ all vanish because $\tilde{\Gamma}^0=-\Gamma^0$ and because $\tilde{\varepsilon}\Gamma^M\varepsilon=0$ for our choice of ε in $OSp(2|4)$. Next we take index $A=9$. The only nonvanishing terms are

$$-2\tilde{\varepsilon}\Gamma^9\Gamma^{\mu\nu}\varepsilon\Phi_9F_{\mu\nu}-2\tilde{\varepsilon}\Gamma^9\Gamma^{ij}\varepsilon\Phi_9[\Phi_i,\Phi_j],$$

where $\mu,\nu=1,\ldots,4$ and $i,j=5,\ldots,8$. Finally, we consider the case when the index A is in the hypermultiplet range $5,\ldots 8$. The result is

$$4\tilde{\varepsilon}\Gamma^\mu\Gamma^{ij}\varepsilon\Phi_iD_\mu\Phi_j+4\tilde{\varepsilon}\Gamma^9\Gamma^{ij}\varepsilon\Phi_i[\Phi_9,\Phi_j].$$

Then

$$S_{F\Phi}=-2\tilde{\varepsilon}\Gamma^9\Gamma^{\mu\nu}\varepsilon\Phi_9F_{\mu\nu}+4\tilde{\varepsilon}\Gamma^\mu\Gamma^{ij}\varepsilon\Phi_iD_\mu\Phi_j+6\tilde{\varepsilon}\Gamma^9\Gamma^{ij}\varepsilon\Phi_i[\Phi_9,\Phi_j].$$

The $\Phi\Phi$ term is easy

$$S_{\Phi\Phi}=4\Phi^A\Phi^B\tilde{\varepsilon}\Gamma^A\tilde{\Gamma}^B\tilde{\varepsilon}=4\tilde{\varepsilon}\tilde{\varepsilon}\Phi^A\Phi_A.$$

Finally, we need the ΦK cross-term. Only Φ_0 contributes

$$S_{\Phi K}=2K_i\Phi_0\nu_i\tilde{\Gamma}^0\tilde{\varepsilon}-2K_i\Phi_0\nu_i\Gamma^0\tilde{\varepsilon}=-4K_i\Phi_0\nu_i\tilde{\varepsilon}.$$

The total result is

$$\begin{aligned}S^Q|_{bos}&=S_{FF}+S_{F\Phi}+S_{\Phi\Phi}+S_{\Phi K}+S_{KK}=\\&\frac{1}{2}F_{MN}F^{MN}-\frac{1}{2}\cos\theta F_{\mu\nu}(*F)^{\mu\nu}+2\tilde{\varepsilon}\Gamma^9\Gamma^{\mu\nu}\varepsilon\Phi_9F_{\mu\nu}-2\tilde{\varepsilon}\Gamma^9\Gamma^{ij}\varepsilon\Phi_i[\Phi_9,\Phi_j]-\\&-2\tilde{\varepsilon}\Gamma^\mu\Gamma^{ij}\varepsilon\Phi_iD_\mu\Phi_j+4(\tilde{\varepsilon}\tilde{\varepsilon})\Phi_A\Phi^A-4K_i\Phi_0\nu_i\tilde{\varepsilon}-K_iK^i\end{aligned}\tag{2.3.13}$$

The next step in the localization procedure is to find the critical points of the $S^Q|_{bos}$. Our strategy will be to represent $S^Q|_{bos}$ as a sum of semipositive terms (full squares) and find the field configurations which ensure vanishing all of them.

First we combine the four-dimensional curvature terms together with the Φ_9-mixing terms

$$\frac{1}{2}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\cos\theta F^{\mu\nu}(*F)_{\mu\nu} + 2\tilde{\varepsilon}\Gamma^9\Gamma^{\mu\nu}\varepsilon\Phi_9 F_{\mu\nu} + 4(\tilde{\varepsilon}\tilde{\varepsilon})\Phi_9\Phi^9 = \\ = \cos^2\frac{\theta}{2}(F^-_{\mu\nu} + w^-_{\mu\nu}\Phi_9)^2 + \sin^2\frac{\theta}{2}(F^+_{\mu\nu} + w^+_{\mu\nu}\Phi_9)^2. \quad (2.3.14)$$

where

$$w^-_{\mu\nu} = \frac{1}{\cos^2\frac{\theta}{2}}\tilde{\varepsilon}^L\Gamma^9\Gamma_{\mu\nu}\varepsilon^L \\ w^+_{\mu\nu} = \frac{1}{\sin^2\frac{\theta}{2}}\tilde{\varepsilon}^R\Gamma^9\Gamma_{\mu\nu}\varepsilon^R. \quad (2.3.15)$$

Next we make a full square with the terms

$$D_m\Phi_i D^m\Phi^i - 2\tilde{\varepsilon}\Gamma^9\Gamma^{ij}\varepsilon\Phi_i[\Phi_9, \Phi_j] - 2\tilde{\varepsilon}\Gamma^\mu\Gamma^{ij}\varepsilon\Phi_i D_\mu\Phi_j = \\ = (D_m\Phi_j - \tilde{\varepsilon}\Gamma_m\Gamma_{ij}\varepsilon\Phi^i)^2 - \Phi^i\Phi_i(\tilde{\varepsilon}\tilde{\varepsilon})(\varepsilon\varepsilon).$$

Finally we absorb the mixing term $K_i\Phi_0$ as follows

$$-4(\tilde{\varepsilon}\tilde{\varepsilon})\Phi_0\Phi_0 - 4\Phi_0 K_i(\nu^i\tilde{\varepsilon}) - K_i K^i = -(K_i + 2\Phi_0(\nu_i\tilde{\varepsilon}))^2.$$

We use the following relations through out the computation

$$(\varepsilon\varepsilon) = 1, \quad (\varepsilon^L\varepsilon^L) = \cos^2\frac{\theta}{2}, \quad (\varepsilon^R\varepsilon^R) = \sin^2\frac{\theta}{2}, \quad (\tilde{\varepsilon}\tilde{\varepsilon}) = \frac{1}{4r^2}, \\ w^-_{\mu\nu}w^{-\mu\nu} = w^+_{\mu\nu}w^{+\mu\nu} = \frac{1}{r^2}.$$

The final result is

$$S^Q|_{bos} = S^Q_{vect,bos} + S^Q_{hyper,bos}.$$

Here

$$S^Q_{vect,bos} = \cos^2\frac{\theta}{2}(F^-_{\mu\nu} + w^-_{\mu\nu}\Phi_9)^2 + \sin^2\frac{\theta}{2}(F^+_{\mu\nu} + w^+_{\mu\nu}\Phi_9)^2 + (D_\mu\Phi_a)^2 \\ + \frac{1}{2}[\Phi_a, \Phi_b][\Phi^a, \Phi^b] + (K^E_i + w_i\Phi^E_0)^2 \quad (2.3.16)$$

where the indices $a, b = 0, 9$ run over the scalars of the vector multiplet, the index $i = 5, 6, 7$ runs over the three auxiliary fields for the vector multiplet, and $w_i = 2(\nu_i\tilde{\varepsilon})$ has norm $w_i w^i = \frac{1}{r^2}$.

At this moment we also switched to the fields Φ_0^E, K_i^E which are related to the original fields in Minkowski signature as $\Phi_0 = i\Phi_0^E, K_i = iK_i^E$. Equivalently, we could make the computation in the Euclidean signature from the very beginning keeping all fields real. In this case some imaginary coefficients would appear in the supersymmetry transformations: we would write down i in front of the fields K_i and would replace the Γ^0 matrix by $i\Gamma^0$.

One could worry then that such supersymmetry transformations spoil the reality conditions on the fields. However, our localization computation is not affected. The Lagrangian and the theory is still invarint under such transformations if we understand the action as an analytically continued functional to the space of complexified fields. The path integral is understood as an integral of a holomorphic functional of fields over a certain real half-dimensional "contour of integration" in the complexified space of fields. Strictly speaking, the bar in the formula (2.3.4) for $\overline{Q\Psi}$ literally means complex conjugation only if we assume that we use that contour of integration which we described before: all fields are real except Φ_0 and K_i which are imaginary. For a general contour of integration in the path integral we just use the functional V (2.3.2) where $\overline{Q\Psi}$ is *defined* by the second line of (2.3.4). This means that the functional V holomorphically depends on all complexified fields. The bosonic part of QV is positive definite after restriction to the correct contour of integration.

From any point of view, we should stress that δ_ε squares to a complexified gauge transformation, whose scalar generator is $i\Phi_0^E + \cos\theta\Phi_9 + \sin\theta A_v$, where Φ_0^E, Φ_9^E and A_v take value in the real Lie algebra of the gauge group, and where A_v is the component of the gauge field in the direction of the vector field v^μ. The theory is similar to the Donaldson theory near the North pole where this generator becomes $i\Phi_0^E + \Phi_9$, and anti-Donaldson theory near the South pole where this generator becomes $i\Phi_0^E - \Phi_9$.

The hypermultiplet contribution is

$$S^Q_{hyper,bos} = (D_0\Phi_i)^2 + (D_m\Phi_j - f_{mij}\Phi^i)^2 + \frac{1}{2}[\Phi_i,\Phi_j][\Phi^i,\Phi^j] + \frac{3}{4r^2}\Phi^i\Phi_i + K_I^E K_I^E,$$

where $m = 1, \ldots, 5$, $i = 5, \ldots, 8$, $I = 1, \ldots 4$ and $f_{mij} = \tilde{\varepsilon}\Gamma_m\Gamma_{ij}\varepsilon$. We see that with our choice of the "integration contour" in the space of complexified fields (all fields are real except K_i, K_I and Φ_0 which are pure imaginary), all terms in the action $S^Q|_{bos}$ are semi-positive definite. Therefore, in the limit $t \to \infty$ we need to care in the path integral only about the locus at which all squares vanish and small fluctuations in the normal directions.

For the hypermultiplet action we get a simple "vanishing theorem": because of the quadratic term $\frac{3}{4r^2}\Phi^i\Phi_i$, the functional $S^Q_{hyper,bos}$ vanishes iff all fields Φ_i vanish.

Next consider zeroes of the term $S^Q_{vect,bos}$. The term $(D_\mu\Phi_9)^2$ ensures that the field Φ_9 must be covariantly constant. Away from the North and the South poles and requiring that the curvature terms vanish, we get the equations

$$F_{\mu\nu} = -w_{\mu\nu}\Phi_9$$

where $w_{\mu\nu} = w^-_{\mu\nu} + w^+_{\mu\nu}$. The curvature $F_{\mu\nu}$ satisfies Bianchi identity, hence we must have

$$d_{[\lambda}w_{\mu\nu]}\Phi_9 = 0. \tag{2.3.17}$$

It is easy to check that away from the North and the South poles, $d_{[\lambda}w_{\mu\nu]}$ does not vanish, hence Φ_9 and $F_{\mu\nu}$ must vanish. The kinetic term $(D_\mu\Phi_0^E)^2$ ensures that Φ_0^E is covariantly constant. Since $F_{\mu\nu} = 0$ we can assume that the gauge field vanish, then Φ_0^E is a constant field over S^4. We call this constant a_E and conclude, that up to a gauge transformation, at smooth configurations we must have

$$S^Q_{bos} = 0 \Rightarrow \begin{cases} A_\mu = 0 \quad \mu = 1, \ldots 4 \\ \Phi_i = 0 \quad i = 5, \ldots, 9 \\ \Phi_0^E = a_E \quad \text{constant over } S^4 \\ K_i^E = -w_i a_E \\ K_I = 0 \end{cases} . \tag{2.3.18}$$

This is the key step in the localization procedure and in the proof of the Erickson-Semenoff-Zarembo/Drukker-Gross conjecture about circular Wilson loop operators. The infinite-dimensional path integral localizes to the finite dimensional locus (2.3.18), and the integral over $a_E \in \mathfrak{g}$ is the resulting matrix model.

Let us evaluate the S_{YM} action (2.2.35) at (2.3.18). The nonvanishing terms are only

$$S_{YM}[a] = \frac{1}{2g_{YM}^2} \int d^4x \sqrt{g} \left(\frac{2}{r^2} (\Phi_0^E)^2 + (K_i^E)^2 \right) = \frac{1}{2g_{YM}^2} \operatorname{vol}(S^4) \frac{3}{r^2} a_E^2 = \frac{4\pi^2 r^2}{g_{YM}^2} a_E^2 \tag{2.3.19}$$

where we used $w_i w^i = \frac{1}{r^2}$ and the volume of the four-sphere $\frac{8}{3}\pi^2 r^4$. We obtained precisely the Drukker-Gross matrix model.

Let us check that the coefficient is correct. Recall, that the original action has the following propagators in Feynman gauge on $\mathbb{R}^4$

$$\langle A_\mu(x) A_\nu(x') \rangle = \frac{g_{YM}^2}{4\pi^2} \frac{g_{\mu\nu}}{(x-x')^2}$$
$$\langle \Phi_0^E(x) \Phi_0^E(x') \rangle = \frac{g_{YM}^2}{4\pi^2} \frac{1}{(x-x')^2}.$$

Hence, the correlator functions which appear in the perturbative expansion of the Wilson loop operator, have the structure

$$\langle A_\mu(\alpha)\dot{x}^\mu A_\nu(\alpha')\dot{x}^\nu + i\Phi_0^E(\alpha) i\Phi_0^E(\alpha') \rangle = -\frac{g_{YM}^2}{4\pi^2 r^2} \frac{\cos(\alpha-\alpha')-1}{4\sin^2\frac{\alpha-\alpha'}{2}} = -\frac{g_{YM}^2}{8\pi^2 r^2},$$

where α denotes an angular coordinate on the loop. That was the original motivation for Erickson-Semenoff-Zarembo conjecture [25]. We see that the first order perturbation theory agrees with the matrix model action derived by localization. The power of the localization computation is that it actually proves the relation between the field theory and the matrix model in all orders in perturbation theory. It is also capable of taking into account instanton effects, which we describe shortly after computing the fluctuation determinant near the locus (2.3.18) and confirming the exact solution.

We remark that for the $\mathcal{N} = 2^*$ theory, the same argument about zeroes of $S^Q|_{bos}$ holds. To ensure that all terms are positive definite, we take the mass parameter M_{ij} in the Scherk-Schwarz reduction to be pure imaginary antisymmetric self-dual matrix. Then the action of the mass deformed $\mathcal{N} = 2^*$ theory at configurations (2.3.18) reduces to the same matrix model action. However, as we will see shortly, when the mass parameter M_{ij} is non zero, the matrix model measure for the $\mathcal{N} = 2^*$ theory is corrected by a non-trivial determinant.

2.4 Determinant factor

2.4.1 Gauge-fixing complex

Because of the infinite-dimensional gauge symmetry of the action we need to work with the gauge-fixed theory. We use the Faddeev-Popov ghost fields and introduce the following BRST like complex with the differential δ:

$$\begin{aligned} &\delta X = -[c, X] \quad && \delta c = -a_0 - \frac{1}{2}[c, c] \quad && \delta \tilde{c} = b \quad && \delta \tilde{a}_0 = \tilde{c}_0 \quad && \delta b_0 = c_0 \\ & && \delta a_0 = 0 && \delta b = [a_0, \tilde{c}] && \delta \tilde{c}_0 = [a_0, \bar{a}_0] && \delta c_0 = [a_0, b_0]. \end{aligned} \tag{2.4.1}$$

Here X stands for all physical and auxiliary fields entering (2.2.35). All other fields are the gauge-fixing fields. By $[c, X]$ we denote a gauge transformation with a parameter c of any field X. (For the gauge fields A_μ we have $\delta A_\mu = -[c, \nabla_\mu]$. The gauge transformation of $\Phi = v^M A_M$ is $\delta\Phi = [v^\mu D_\mu + v^A \Phi_A, c] = \text{ad}(\Phi)c + L_v c$, where $\text{ad}(\Phi)c$ is the pointwise adjoint action of Φ on c involving no differential operators). The fields c and $\tilde{c}$ are the usual Faddeev-Popov ghost and anti-ghost. The bosonic field b is the standard Lagrange multiplier used in R_ξ-gauge, where the gauge fixing is done by adding terms like $(b, id^*A + \frac{\xi}{2}b)$ and $(\tilde{c}, d^*\nabla_A c)$ to the action. The fields c and $\tilde{c}$ actually have zero modes. To treat them systematically we add constant fields $c_0, \tilde{c}_0, a_0, \tilde{a}_0, b_0$ to the gauge-fixing complex. The field a_0 is interpreted as a ghost field for the ghost c. The fields $a_0, \tilde{a}_0, b_0$ are bosonic, and the fields $c_0, \tilde{c}_0$ are fermionic. The operator δ squares to the gauge transformation by the constant bosonic field a_0

$$\delta^2 \cdot = [a_0, \cdot].$$

The gauge invariant action and observable are δ-closed

$$\delta S_{YM}[X] = 0,$$

therefore their correlation functions are not changed when we add the δ-exact gauge-fixing term.

When we combine the gauge-fixing terms with the physical action, we will see that the convergence of the path integral requires the imaginary contour of integration for the constant field a_0. This field a_0 later will be identified with the zero mode of the physical field Φ_0 which is integrated over imaginary contour. To have consistent notations we set $a_0 = ia_0^E$ and assume that a_0^E is integrated over the real contour.

The δ-exact term

$$\begin{aligned} S^{\delta}_{g.f.} &= \delta((\tilde{c}, id^* A + \frac{\xi_1}{2} b + ib_0) - (c, \tilde{a}_0 - \frac{\xi_2}{2} a_0)) = \\ &= (b, id^*A + \frac{\xi_1}{2}b + ib_0) - (\tilde{c}, id^* \nabla_A c + ic_0 + \frac{\xi_1}{2}[a_0, \tilde{c}]) + (-ia_0^E + \frac{1}{2}[c,c], \tilde{a}_0 - \frac{\xi_2}{2} ia_0^E) + (c, i\tilde{c}_0) \end{aligned} \tag{2.4.2}$$

properly fixes the gauge.

Assuming that all bosonic fields are real, the bosonic part of gauge-fixed action has strictly positive definite quadratic term for all fields and ghosts at $\xi_1, \xi_2 > 0$.

By general arguments the partition function does not depend on the parameters ξ_1, ξ_2 in the δ-exact term. Let us fix $\xi_1 = 0$ and demonstrate explicitly independece on ξ_2 and equivalence with the standard gauge-fixing procedure. First we do Gaussian integral integral over a_0^E and get

$$(ia_0^E + \frac{1}{2}[c,c], i\tilde{a}_0 - \frac{\xi_2}{2} ia_0^E) \to +\frac{1}{2\xi_2}(\tilde{a}_0 - \frac{\xi_2}{4}[c,c])^2.$$

Then we do Gaussian integral over $\tilde{a}_0$ and the above term goes away completely. The determinant coming from the Gaussian integral over $\tilde{a}_0$ is inverse to the determinant coming from the Gaussian integral over a_0. Then we integrate the zero mode of b against b_0. Then integral over non-zero modes of b gives Dirac delta-functional inserted at the gauge-fixing hypersurface $d^*A = 0$. The remaining terms are

$$(\tilde{c}, id^*\nabla_A c) + i(\tilde{c}, c_0) + i(c, \tilde{c}_0).$$

We can integrate out c_0 with the zero mode of $\tilde{c}$, and $\tilde{c}_0$ with the zero mode of c. Then we are left with the integral over c and $\tilde{c}$ with the zero modes projected out and the gauge-fixing term

$$(\tilde{c}, id^*\nabla_A c).$$

This reproduces the usual Faddeev-Popov determinant $\det'(d^*\nabla_A)$ which we need to insert into the path integral for the partition function after restricting to the gauge-fixing hypersurface $d^*A = 0$. The symbol $'$ means that the determinant is computed on the space without the zero modes.

We summarize the gauge fixing procedure by the formula

$$\begin{aligned} Z = \frac{1}{\mathrm{vol}(\mathcal{G}, g_{YM})} \int [DX] e^{-S_{YM}[X]} = \\ = \frac{1}{\mathrm{vol}(\mathcal{G})} \int [DX] e^{-S_{YM}[X]} \int_{g\in\mathcal{G}'} [Dg]\, \delta_{Dirac}(d^*A^g) \det{}'(d^*\nabla_A) = \\ = \frac{\mathrm{vol}(\mathcal{G}', g_{YM})}{\mathrm{vol}(\mathcal{G}, g_{YM})} \int [DX\,Db' Dc' D\tilde{c}'] e^{-S_{YM}[X] - \int_{S^4}\sqrt{g}\, d^4x (i(b, d^*A) - (\tilde{c}, id^*\nabla_A c))} = \\ = \frac{1}{\mathrm{vol}\,(G, g_{YM})} \int [DX\; Db\, Db_0\; Dc\, Dc_0\; D\tilde{c}\, D\tilde{c}_0\; Da_0\; D\tilde{a}_0] e^{-S_{YM}[X] - S^{\delta}_{g.f.}[X, ghosts]}, \end{aligned} \tag{2.4.3}$$

where $\mathcal{G}' = \mathcal{G}/G$ is the coset of the group of gauge transformations by constant gauge transformations. We shall note that in our conventions for the gauge theory Lagrangian $\frac{1}{4g_{YM}^2}(F, F)$, where $F = dA + A \wedge A$, we need to take the volume of the group of gauge transformations with respect to the measure which is rescaled by a power of the coupling constant g_{YM}. In other words, we take $\mathrm{vol}(G, g_{YM}) = g_{YM}^{dimG}\,\mathrm{vol}(G)$, where $\mathrm{vol}(G)$ is the volume of the gauge group computed with respect to the Haar measure induced by the coupling constant independent Killing form $(,)$ on the Lie algebra.

2.4.2 Supersymmetry complex

To compute the path integral, it is convenient to bring the supersymmetry transformations to a cohomological form by a change of variables. (This change of variables involves no Jacobian, one can think about it as a change of notations.) We use the fact that conformal Killing spinor ε in (2.2.28) has constant unit norm at any point on S^4. Then the set of sixteen spinors consisting of $\{\Gamma^M \varepsilon\}$ for $M = 1, \ldots, 9$ and $\{\nu_i\}$ for $i = 1, \ldots, 7$ form an orthonormal basis for the space of $Spin(9,1)$ Majorana-Weyl spinors reduced on S^4. We expand Ψ over this basis

$$\Psi = \sum_{M=1}^{9} \Psi_M \Gamma^M \varepsilon + \sum_{i=1}^{7} \Upsilon_i \nu^i.$$

In new notations (Ψ_M, Υ_i), the supersymmetry transformations (2.2.28) take the following form:

$$\begin{cases} sA_M = \Psi_M \\ s\Psi_M = -(L_v + R + M + G_\Phi)A_M \end{cases}$$
$$\begin{cases} s\Upsilon_i = H^i \\ sH^i = -(L_v + R + M + G_\Phi)\Upsilon_i, \end{cases} \tag{2.4.4}$$

where

$$H^i \equiv K^i + w_i \Phi_0 + s_i(A_M). \tag{2.4.5}$$

Now s denotes δ_ε to distinguish it from the differential δ of the Faddeev-Popov complex. By L_v we denote the Lie derivative in the direction of the vector field v^μ, R denotes the R-symmetry transformation in SU_L^R, M denotes the mass-term induced transformation by M_{ij} in SU^R_R, and G_Φ denotes the gauge transformation by Φ. The functions $s_i(A_M)$ with $i = 1, \dots, 7$ are the "equations" of the equivariant theory

$$s_i(A_M) = \frac{1}{2} F_{MN} \nu_i \Gamma^{MN} \varepsilon + \frac{1}{2} \Phi_A \nu_i \Gamma^{\mu A} \nabla_\mu \varepsilon \quad \text{for} \quad M, N = 1, \dots, 9 \quad A = 5, \dots, 9. \tag{2.4.6}$$

Even shorter, we can write the supersymmetry complex like

$$\begin{aligned} sX &= X' \\ sX' &= [\phi + \varepsilon, X], \end{aligned} \tag{2.4.7}$$

and $s\phi = 0$, where we denoted $\phi = -\Phi$, $[\phi, X] = -G_\Phi X$ and $[\varepsilon, X] = -(L_v + R + M)X$.

All fields except Φ (2.2.12) are grouped in s-doublets (X, X'), where the fields X and X' have opposite statistics. We can think about fields X as coordinates on some infinite-dimensional supermanifold $\mathcal{M}$, on which group $\mathcal{G}$ acts. The fields X' can be interpreted as de Rham differentials $X' \equiv dX$, if we identify the operator s with the differential in the Cartan model of $\mathcal{G}$-equivariant cohomology on $\mathcal{M}$

$$s = d + \phi^a i_{v^a} \tag{2.4.8}$$

where ϕ^a are the coordinates on the Lie algebra $\mathfrak{g}$ of the group $\mathcal{G}$ with respect to some basis $\{e_a\}$, and i_{v^a} is the contraction with a vector field v^a representing action of e_a on $\mathcal{M}$. The differential s squares to the Lie derivative $\mathcal{L}_\phi$. In the present case, the group $\mathcal{G}$ is a semi-direct product

$$\mathcal{G} = \mathcal{G}_{gauge} \ltimes U(1) \tag{2.4.9}$$

of the infinite-dimensional group of gauge transformations $\mathcal{G}_{gauge}$ and the $U(1)$ subgroup of the $OSp(2|4)$ symmetry group generated by the conformal Killing spinor ε.

In the path integral (2.4.3) for the partition function Z_{phys}, we integrate s-equivariantly closed form e^S over $\mathcal{M}$ and then over ϕ. See [8, 78, 82] for twisted $\mathcal{N} =$

4 SYM related theories which have similar cohomological structure, and [83] where similar integration over the parameter of the equivariant cohomology is performed.

2.4.3 The combined Q-complex

So far we constructed separately the gauge-fixing complex with the differential δ and the supersymmetry complex with the differential s:

$$\begin{array}{llllll} \delta a_0 = 0 & \delta X = -[c,X] & \delta c = -a_0 - \frac{1}{2}[c,c] & \delta\tilde{c} = b & \delta\tilde{a}_0 = \tilde{c}_0 & \delta b_0 = c_0 \\ & \delta X' = -[c,X'] & \delta\phi = -[c+\varepsilon,\phi] & \delta b = [a_0,\tilde{c}] & \delta\tilde{c}_0 = [a_0,\tilde{a}_0] & \delta c_0 = [a_0,b_0] \\ sa_0 = 0 & sX = X' & sc = \phi & s\tilde{c} = 0 & s\tilde{a}_0 = 0 & sb_0 = 0 \\ & sX' = [\phi+\varepsilon,X] & s\phi = 0 & sb = [\varepsilon,\tilde{c}] & s\tilde{c}_0 = 0 & sc_0 = 0. \end{array} \tag{2.4.10}$$

Here we summarize the anticommutators for δ and s:

$$\begin{array}{ll} \{\delta,\delta\}X^{(\prime)} = [a_0, X^{(\prime)}] & \{\delta,\delta\}(ghost) = [a_0, ghost] \\ \{s,s\}X^{(\prime)} = [\phi+\varepsilon, X^{(\prime)}] & \{s,s\}(ghost) = 0 \\ \{s,\delta\}X^{(\prime)} = -[\phi, X^{(\prime)}] & \{s,\delta\}(ghost) = [\varepsilon, ghost]. \end{array} \tag{2.4.11}$$

In this formula $X^{(\prime)}$ stands for all physical and auxilary fields X and X', and *ghost* stands for any field of the BRST gauge fixing complex.

Now we combine the operators δ and s and define a fermionic operator Q:

$$Q = s + \delta.$$

Then we get

$$\begin{array}{lll} QX = X' - [c,X] & Qc = \phi - a_0 - \frac{1}{2}[c,c] & \\ QX' = [\phi+\varepsilon, X] - [c,X'] & Q\phi = -[c,\phi+\varepsilon] & \\ Q\tilde{c} = b & Q\tilde{a}_0 = \tilde{c}_0 & Qb_0 = c_0 \\ Qb = [a_0+\varepsilon,\tilde{c}] & Q\tilde{c}_0 = [a_0,\tilde{c}_0] & Qc_0 = [a_0,b_0] \\ Qa_0 = 0. & & \end{array} \tag{2.4.12}$$

This means that Q satisfies on all fields

$$Q^2 \cdot = [a_0 + \varepsilon, \cdot].$$

In other words, Q squares to a constant gauge transformation generated by a_0 and the $U(1)$ anti-self-dual Lorentz rotation around the North pole generated by ε.

Now, since $sS_{phys} = 0$ and $\delta S_{phys} = 0$ we have

$$QS_{phys} = 0.$$

We would like to make sure that the gauge-fixing term (2.4.2) is also Q-closed so that we could use the localization argument.

We will take the following Q-exact gauge-fixing term:

$$S^Q_{g.f.} = (\delta + s)((\tilde{c}, id^*A + \frac{\xi_1}{2} b + ib_0) - (c, \tilde{a}_0 - \frac{\xi_2}{2} a_0)) = S^\delta_{g.f} - (\tilde{c}, s(id^*A + \frac{\xi_1}{2} b + ib_0)) - (\phi, \tilde{a}_0) =$$
$$= S^\delta_{g.f} - (\tilde{c}, d^*\psi + \frac{\xi_1}{2}[\varepsilon, \tilde{c}]) - (\phi, \tilde{a}_0 - \frac{\xi_2}{2} a_0) \quad (2.4.13)$$

The replacement of $S^\delta_{g.f.}$ by $S^Q_{g.f.}$ does not change the partition function Z_{phys} (2.4.3). We can easily see this at $\xi_1 = 0$. Integrating over a_0 we get

$$(ia_0^E + \frac{1}{2}[c, c] - \phi, \tilde{a}_0 - \frac{\xi_2}{2} ia_0^E) \to \frac{1}{2\xi_2} \left(-\frac{\xi_2}{2} (\frac{1}{2}[c, c] - \phi) + i\tilde{a}_0 \right)^2 .$$

After we integrate over $\tilde{a}_0$ the above term goes away completely. The determinants for the Gaussian integrals over a_0 and $\tilde{a}_0$ cancel. Then we are left with the following gauge-fixing terms

$$i(b, d^*A + b_0) - i(\tilde{c}, d^*\nabla c + c_0) + i(c, \tilde{c}) - (\tilde{c}, d^*\psi),$$

where ψ is the fermionic one-form which is the superpartner of the gauge field A. Then we note that the term $(\tilde{c}, d^*\psi)$ does not change the fermionic determinant arising from the integral over $c, \tilde{c}, c_0$ and $\tilde{c}_0$. The reason is that all modes of c are coupled to $\tilde{c}$ by this quadratic action

$$i(\bar{c}, d^*\nabla c + c_0) + i(c, \tilde{c}),$$

and that there are no other terms in the gauge-fixed action which contain modes of c. In other words, if treat the term $(\tilde{c}, d^*\psi)$ as a perturbation to the usual gauge fixed action, all diagrams with it vanish because $\tilde{c}$ can be connected by a propagator only to c, but there are no other terms which generate vertices with c.

In other words we did the following. The action of the theory gauge-fixed in the standard way (2.4.3) is δ-closed, but not Q-closed. We make the action Q-closed by adding such terms to it which do not change the path integral. The fact that the partition function does not change can be also shown by making a change of variables which has trivial Jacobian.

We conclude that the total gauge-fixed action

$$\tilde{S}_{phys} = S_{phys} + S'_{g.f.} \tag{2.4.14}$$

is Q-closed

$$Q\tilde{S}_{phys} = 0, \tag{2.4.15}$$

and that the partition function defined by the path integral over all fields and ghosts with the action $\tilde{S}_{phys}$ is equal to the standard partition function with the usual gauge-fixing (2.4.3).

It is possible to write the operator Q in the canonical form; namely Q is the equivariant differential in the Cartan model for the $\tilde{G} = G \ltimes U(1)$ cohomology generated by a_0 and ε on the space of all other fields over which we integrate in the path integral (2.4.3). The multiplets $(\tilde{c}, b)$, $(\tilde{a}_0, \tilde{c}_0)$ and (b_0, c_0) are already in the canonical form. To bring the transformations of (X, X') and (c, ϕ) to the canonical form we make a change of variables

$$\begin{aligned} \tilde{X}' &= X' - [c, X] \\ \tilde{\phi} &= \phi - a_0 - \frac{1}{2}[c, c]. \end{aligned} \tag{2.4.16}$$

Such change of variables has trivial Jacobian and does not change the path integral. In terms of new fields, the Q-complex is canonical: all fields are grouped in doublets

$(Field, Field')$, while Q acts as

$$Q(Field) = (Field')$$
$$Q(Field') = [a_0 + \varepsilon, Field]. \tag{2.4.17}$$

Moreover, $Qa_0 = Q\varepsilon = 0$.

Now recall Atiyah-Bott-Berline-Vergne localization formula for the integrals of the equivariantly closed differential forms [21, 22]

$$\int_{\mathcal{M}} \alpha = \int_{F \subset \mathcal{M}} \frac{i_F^* \alpha}{e(\mathcal{N})}. \tag{2.4.18}$$

The numerator corresponds to the physical action evaluated at the critical locus of the tQV term. The equivariant Euler class of the normal bundle in the denominator is just a determinant, coming from the Gaussian integral using quadratic part of tQV in the normal directions $\mathcal{N}$. We will argue that this determinant can be expressed as a product of weights for the group action on $\mathcal{N}$ defined by (2.4.17). The basic difference with the usual localization formula (2.4.18) is that the manifold $\mathcal{M}$ in our problem is not a usual manifold, but an (infinite-dimensional) supermanifold. Hence, the equivariant Euler class must be understood in a super-formalism [84, 85]. In our case it is just a super-determinant. If we split the normal bundle to the bosonic and the fermionic subspaces, the resulting determinant is the product of weights on the bosonic subspace divided by the product of weights on the fermionic subspace.

Before making gauge-fixing procedure we argued previously that the theory localizes to the zero modes of the field Φ_0. The localization argument for the gauge-fixed theory remains the same, except that now we can identify the zero mode of the field Φ_0 with a_0. Indeed, if we first integrate over $\tilde{a}_0$ using gauge fixing terms at $\xi_2 = 0$

$$(ia_0^E + \frac{1}{2}[c, c] - i\phi^E, \tilde{a}_0),$$

we get the constraint that the zero mode of ϕ^E is equal to a_0^E.

2.4.4 Computation of the derminant by the index theory of transversally elliptic operators

We write the linearization of the Q-complex in the form

$$\begin{aligned} QX_0 &= X_0' \quad & QX_1 &= X_1' \\ QX_0' &= R_0 X_0 \quad & QX_1' &= R_1 X_1 \end{aligned} \tag{2.4.19}$$

where all bosonic and fermionic fields in the first line of (2.4.17) are denoted as X_0 and X_1 respectively, and their Q-differentials are denoted as X_0' and X_1'. So X_0, X_1' are bosonic, and X_0', X_1 are fermionic fields.

The quadratic part of the functional V is

$$V^{(2)} = \begin{pmatrix} X_0' \\ X_1 \end{pmatrix}^t \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X_0 \\ X_1' \end{pmatrix}, \tag{2.4.20}$$

where $D_{00}, D_{01}, D_{10}, D_{11}$ are some differential operators. Then we have

$$QV^{(2)} = (X_{bos}, K_{bos} X_{bos}) + (X_{ferm}, K_{ferm} X_{ferm}),$$

where the kinetic operators K_{bos}, K_{ferm} are expressed in terms of $D_{00}, D_{01}, D_{10}, D_{11}$ and R_0, R_1 in a certain way. The Gaussian integral gives

$$Z_{\text{1-loop}} = \left(\frac{\det K_{bos}}{\det K_{ferm}} \right)^{-\frac{1}{2}}. \tag{2.4.21}$$

Let E_0 and E_1 denote the vector bundles whose sections can be identified with fields X_0, X_1. Some linear algebra shows that this ratio of the determinants depends only on the representation structure R on the kernel and cokernel spaces of the operator $D_{10} : \Gamma(E_0) \to \Gamma(E_1)$. Namely we have

$$\frac{\det K_{bos}}{\det K_{ferm}} = \frac{\det_{\ker D_{10}} R}{\det_{\operatorname{coker} D_{10}} R}. \tag{2.4.22}$$

The operator D_{10} in our problem is not an ordinary elliptic operator, but a transversally elliptic operator with respect to the $U(1)$ rotation of S^4.

This means the following. Let E_0 and E_1 be vector bundles over a manifold X and $D : \Gamma(E_0) \to \Gamma(E_1)$ be a differential operator. (In our problem $X = S^4$.) Let a compact Lie group $\tilde{G}$ act on X such that its action preserves all structures. Let $\pi : T^*X \to X$ be the cotangent bundle of X. Then pullback $\pi^* E_i$ is a bundle over T^*X. By definition, a symbol of the differential operator $D : \Gamma(E_0) \to \Gamma(E_1)$ is a vector bundle homomorphism $\sigma(D) : \pi^* E_0 \to \pi^* E_1$, such that in local coordinates x_i, the symbol is defined by replacing all partial derivatives in the highest order component of D by momenta, so that $\frac{\partial}{\partial x^i} \to i p_i$, and then taking p_i to be coordinates on fibers of T^*X. The operator D is called elliptic if its symbol $\sigma(D)$ is invertible on $T^*X \setminus 0$, where 0 denotes the zero section. The kernel and cokernel of an elliptic operator are finite dimensional vector spaces. Using the Atiyah-Singer index theory [86–91] one can find a formal difference of representations in which $\tilde{G}$ acts on these spaces, as we will see in a moment. However, we will see as well that the operator D_{10} is not elliptic, so the ordinary Atiyah-Singer index theory does not apply. There is a generalization of Atiyah-Singer index theory for operators which are elliptic only in directions transverse to the $\tilde{G}$-orbits [71, 72]. Such operators are called transversally elliptic. In other words, for any point $x \in X$ we consider the subspace $T^*_{\tilde{G}} X_x$ of the T^*X_x, which consists of elements which are orthogonal to the $\tilde{G}$-orbit through x. We have

$$T^*_{\tilde{G}} X_x = \{ p \in T^* X_x \quad \text{such} \quad \text{that} \quad p \cdot v(\tilde{g}) = 0 \quad \forall \tilde{g} \in \mathrm{Lie}(\tilde{G}) \},$$

where $v(\tilde{G})$ denotes a vector field on X generated by an element $\tilde{g}$ of the Lie algebra of $\tilde{G}$. Then the family of the vector spaces $T^*_{\tilde{G}} X$ over X is defined as the union of $T^*_{\tilde{G}} X_x$ for all $x \in X$. The notion of a family of vector spaces over some base is similar to the notion of a vector bundle, except that dimension of fibers can jump. The operator D is called transversally elliptic if its symbol $\sigma(D)$ is invertible on $T^*_{\tilde{G}} X \setminus 0$. Computing the symbol of D_{10}, we will see explicitly in (2.4.30) that D_{10} is not an elliptic operator, but a transversally elliptic one. The kernel and the cokernel of such an operator are not generally finite dimensional vector spaces, but if we decompose them into irreducible representations, then each irreducible representation appears with a finite multiplicity [71, 72]. So we have

$$\begin{aligned} \ker D_{10} &= \oplus_\alpha m_\alpha^{(0)} R_\alpha \\ \operatorname{coker} D_{10} &= \oplus_\alpha m_\alpha^{(1)} R_\alpha, \end{aligned} \tag{2.4.23}$$

where α runs over irreducible representations of $\tilde{G}$, and m_α denotes the multiplicity of the irreducible representation R_α. Then

$$\frac{\det K_{bos}}{\det K_{ferm}} = \prod_\alpha (\det R_\alpha)^{m_\alpha^{(0)} - m_\alpha^{(1)}}. \tag{2.4.24}$$

Thus we need to know only the difference of multiplicities $m_\alpha^{(0)}$ and $m_\alpha^{(1)}$ of irreducible representations into which the kernel and cokernel of D_{10} can be decomposed. To find this difference we use Atiyah-Singer index theory [71, 72] for transversally elliptic operators, which generalizes the usual theory [86–91]. In our problem, R_α is an irreducible representation of the group $\tilde{G} = U(1) \times G$. We also denote this $U(1)$ group by H, so that $\tilde{G} = H \times G$. The relevant representations of G are those in which the physical fields transform (we will consider only the adjoint representation), but all representations of $H = U(1)$ arise. Let $q \in \mathbb{C}, |q| = 1$ denote an element of $U(1)$. Irreducible representations of $U(1)$ are labeled by integers n, so that the character of representation n is q^n. The $U(1)$-equivariant index of D_{10} is defined as

$$\operatorname{ind}(D_{10}) = \operatorname{tr}_{\ker D_{10}} R(q) - \operatorname{tr}_{\operatorname{coker} D_{10}} R(q) = \sum_n (m_n^{(0)} - m_n^{(1)}) q^n.$$

Hence, if we compute the equivariant index of D_{10} as a series in q, we will know $m_n^{(0)} - m_n^{(1)}$ and will be able to evaluate (2.4.24).

To compute the index of D_{10}, first we need to describe the bundles E_0, E_1 and the symbol of the operator $D_{10} : \Gamma(E_0) \to \Gamma(E_1)$. The collective notation X_0, X_0', X_1, X_1'

corresponds to the original fields in the following way

$$\begin{aligned} X_0 &= (A_M, \tilde{a}_0, b_0) \quad X_1 = (\Upsilon_i, c, \tilde{c}) \\ X_1' &= (\tilde{\Psi}_M, \tilde{c}_0, c_0) \quad X_1' = (\tilde{H}_i, \tilde{\phi}, b). \end{aligned} \tag{2.4.25}$$

The space of all fields decomposes in a way compatible with Q-action (2.4.19) into direct sum of two subspaces: the fields of vector multiplet and hypermultiplet. The vector subspace also includes fields of the gauge fixing complex. The vector subspace consists of

$$X_0^{vect} = (\Phi_9, A_M, \tilde{a}_0, b_0) \quad \text{for } M = 1, \dots, 4 \quad X_1^{vect} = (\Upsilon_i, c, \tilde{c}) \quad \text{for } i = 5, \dots, 7 \tag{2.4.26}$$

and their Q-superpartners. The hyper subspace consists of

$$X_0^{hyper} = (A_M) \quad \text{for } M = 5, \dots, 8 \quad X_1^{hyper} = (\Upsilon_i) \quad \text{for } i = 1, \dots, 4 \tag{2.4.27}$$

and their Q-superpartners. The operator D_{10} does not mix the vector and hyper subspaces. So the vector bundles split as $E_0 = E_0^{vect} \oplus E_0^{hyper}$, and $E_1 = E_1^{vect} \oplus E_1^{hyper}$, as well as the operator $D_{10} = D_{10}^{vect} + D_{10}^{hyper}$, where $D_{10}^{vect} : \Gamma(E_0^{vect}) \to \Gamma(E_1^{vect})$ and $D_{10}^{hyper} : \Gamma(E_0^{hyper}) \to \Gamma(E_1^{hyper})$.

First we consider the index of D_{10}^{vect}. The constant fields $(\tilde{a}_0, b_0)$ are in the kernel of D_{10}^{vect} and have zero $U(1)$ weights, hence their contribution to the index is 2. The remaining fields, denoted by $X_0^{vect'}$, are identified with sections of bundle $(T^* \oplus \mathcal{E}) \otimes \operatorname{ad} E$, where T^* is the cotangent bundle, and $\mathcal{E}$ is the rank one trivial bundles over S^4. The fields $X_1^{vect'}$ are identified with sections of $(\mathcal{E}^3 \oplus \mathcal{E}^2) \otimes \operatorname{ad} E$, where $\mathcal{E}^3$ is the rank three trivial bundle of auxiliary scalar fields, and $\mathcal{E}^2$ is the rank two trivial bundle of the gauge fixing fields c and $\tilde{c}$. Because of the difference due to $(\tilde{a}_0, b_0)$ contribution we have

$$\operatorname{ind}(D_{10}^{vect}) = \operatorname{ind}'(D_{10}^{vect}) + 2. \tag{2.4.28}$$

Now we compute the symbol of the operator D_{10}^{vect}. The relevant terms are

$$V^{(2)} = (\tilde{c}, d^* A) + (c, \nabla_\mu \mathcal{L}_v A_\mu) + (\Upsilon_i, (*F_{0i}) - F_{0i} \cos\theta + \nabla_i \Phi_9 \sin\theta), \tag{2.4.29}$$

where index i runs over vielbein elements on S^4.

We chose a vielbein in such a way that $i = 1$ is the direction of the $U(1)$ vector field, and $i = 2, 3, 4$ are the remaining orthogonal directions. The term $(c, \nabla_\mu \mathcal{L}_v A_\mu)$ comes from the term $(\psi_\mu, \mathcal{L}_v A_\mu)$ and the relation $\psi_\mu = \tilde{\psi}_\mu - \nabla_\mu c$. Then the symbol $\sigma(D_{10}^{vect}) : \pi^* E_0^{vect} \to \pi^* E_1^{vect}$, where π denotes the projection of the cotangent bundle $\pi : T^*X \to X$, is represented by the following matrix

$$\begin{pmatrix} c \\ \tilde{c} \\ \Upsilon_1 \\ \Upsilon_2 \\ \Upsilon_3 \end{pmatrix} \leftarrow \begin{pmatrix} c_\theta p^2 & s_\theta \overrightarrow{p}^2 & -s_\theta p_2 p_1 & -s_\theta p_3 p_1 & -s_\theta p_4 p_1 \\ 0 & p_1 & p_2 & p_3 & p_4 \\ s_\theta p_2 & -c_\theta p_2 & c_\theta p_1 & -p_4 & p_3 \\ s_\theta p_3 & -c_\theta p_3 & p_4 & c_\theta p_1 & -p_2 \\ s_\theta p_4 & -c_\theta p_4 & -p_3 & p_2 & c_\theta p_1 \end{pmatrix} \begin{pmatrix} \Phi_9 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}. \tag{2.4.30}$$

Here p_i for $i = 1, \ldots, 4$ denotes coordinates on fibers of T^*X, $\overrightarrow{p} = (p_2, p_3, p_4)$ denotes coordinate on fibers of $T_H^* X$, and $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$. In other words, $\overrightarrow{p}$ is a momentum orthogonal to the direction of the $U(1)$ vector field on S^4. After a change of coordinates on fibers of bundles $E_0 \to T^*X$ and $E_1 \to T^*X$

$$\begin{aligned} c &\to c + s_\theta p_0 \tilde{c} \\ \Phi_9 &\to c_\theta \Phi_9 + s_\theta A_1 \\ A_1 &\to -s_\theta \Phi_9 + c_\theta A_1, \end{aligned} \tag{2.4.31}$$

the matrix of the symbol of D_{10}^{vect} takes the form

$$\begin{pmatrix} p^2 & 0 & 0 & 0 & 0 \\ s_\theta p_1 & c_\theta p_1 & p_2 & p_3 & p_4 \\ 0 & -p_2 & c_\theta p_2 & -p_4 & p_3 \\ 0 & -p_3 & p_3 & c_\theta p_2 & -p_2 \\ 0 & -p_4 & -p_3 & p_2 & c_\theta p_2. \end{pmatrix}. \tag{2.4.32}$$

The term $s_\theta p_1$ in the first column of the second line can be also removed by subtracting the first line multiplied by $s_\theta p_1 / p^2$. Then the notrivial part of the symbol is represented by the following 4×4 matrix

$$\sigma = \begin{pmatrix} c_\theta p_1 & p_2 & p_3 & p_4 \\ -p_2 & c_\theta p_2 & -p_4 & p_3 \\ -p_3 & p_3 & c_\theta p_2 & -p_2 \\ -p_4 & -p_3 & p_2 & c_\theta p_2. \end{pmatrix}. \tag{2.4.33}$$

The determinant of this matrix is $(\cos^2\theta p_1^2 + \overrightarrow{p}^2)^2$. First of all, we see that the symbol is not elliptic at the equator of S^4, since if $\cos\theta = 0$ we can take $(p_1 \neq 0, \overrightarrow{p} = 0)$ and the determinant will vanish. But the symbol is transversally elliptic with respect to the $H = U(1)$ group, since its determinant is always non-zero whenever $\overrightarrow{p} \neq 0$. Indeed, to check if the symbol is transversally elliptic, we need to consider only non-zero momenta orthogonal to the $U(1)$ orbits. In our notations that means $p_1 = 0, \overrightarrow{p} \neq 0$.

In a neighborhood of the North pole ($c_\theta = 1$) the symbol is equivalent to the elliptic symbol of the standard anti-self-dual complex (d, d^-)

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^-} \Omega^{2-}, \tag{2.4.34}$$

while in a neighborhood of the South pole ($c_\theta = -1$), the symbol is equivalent to the elliptic symbol of the standard self-dual complex (d, d^+)

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2+}. \tag{2.4.35}$$

Intuitively one can see that from the structure of the QV-action (2.3.14).

In the elliptic case, one could use Atiyah-Bott formula [90, 91] to compute the index as a sum of local contributions from H-fixed points on X. In the transversally elliptic case the situation is more complicated. By definition, the index is a sum of characters of irreducible representations. We have

$$\mathrm{ind}(D) = \sum_{n=-\infty}^{\infty} a_n q^n, \tag{2.4.36}$$

where $a_n = m_n^{(0)} - m_n^{(1)}$ is a difference of multiplicities in which irreducible representation n appears in the kernel and cokernel of D. In the elliptic case, only a finite number of a_n does not vanish, so that the index is a finite polynomial in q and q^{-1}. This also means that the index is a regular function on the group $H = U(1)$. In the transversally elliptic case, the series (2.4.36) can be infinite, so that index is generally not a regular function. However, Atiyah and Singer showed [71, 72] that in the tranversally elliptic case, all coefficients a_n are finite, and that the index is well defined as a distribution (a generalized function) on the group.

For example, consider the zero operator acting on functions on a circle $X = S^1$, so $D : C^\infty(S^1) \to 0$. This is a transversally elliptic operator with respect to the canonical $U(1)$ action on S^1. The kernel of the zero operator is the space of all functions on S^1, the cokernel is zero. Then $m_n^{(0)} = 1, m_n^{(1)} = 0$ for all n, so the index is $\sum_{n=-\infty}^{\infty} q^n$, which is the Dirac delta-function supported at $q = 1$.

The equivariant index theory can be generalized to the transversally elliptic case [71, 72, 92, 93]. The idea is that we can cut a H-manifold X into small neighborhoods of H-fixed points and the remaining subspace $Y \subset X$ on which H acts freely. By definition, at each H-fixed point the symbol of transversally elliptic operator is actually elliptic, so the ordinary equivariant index theory applies. Since H acts freely on Y, we can consider the quotient Y/H. A H-transversally elliptic operator on Y gives us an elliptic operator on Y/H. Then we can combine the representation theory of G and the usual index theory on the quotient Y/H to find the index of transversally elliptic operator on Y [71].

Let $R(H)$ be the space of regular functions on H (the space of finite polynomials in q and q^{-1}). Let $\mathcal{D}'(H)$ denote the space of distributions (generalized functions) on H (not necessarily finite series in q and q^{-1}). The space of distributions $\mathcal{D}'(H)$ is a module over the space of regular functions $R(H)$, since there is a well defined term by term multiplication of series in q and q^{-1} by finite polynomials in q and q^{-1}. Some singular generalized functions such as the Dirac delta-function $\sum_{n=-\infty}^{\infty} q^n$

can be annihilated by non-zero regular functions. For example, Dirac delta-function $\sum_{n=-\infty}^{\infty} q^n \in \mathcal{D}'(H)$ vanishes after multiplication to $(1-q)$. Such elements of $\mathcal{D}'(H)$ which can be annihilated by non-zero regular functions in $R(H)$ are called torsion elements.

To find the index of transversally elliptic operator up to a distribution supported at $q = 1$ (a torsion element of $\mathcal{D}'(H)$), we can use the usual Atiyah-Bott formula [89–91] (see appendix (A.4)). This formula gives a contribution to the index from each fixed point as a rational function of q. This function is generally singular at $q = 1$. For example, if $H = U(1)$ acts on $\mathbb{C}$ as $z \to qz$, then the Atiyah-Bott formula for the index of the $\bar{\partial}$-operator at the fixed point $z = 0$ gives

$$\mathrm{ind}(\bar{\partial})|_0 = \frac{1}{1 - q^{-1}}. \tag{2.4.37}$$

To get a distribution associated with this rational function, we need to expand it in series in q and q^{-1}. Of course, the result is not unique, but different expansions differ only by a distribtuion supported at $q = 1$. For $H = U(1)$, there are two basic ways, or regularizations, which fix the singular part [71]. The regularization $[f(q)]_+$ is defined by taking expansion at $q = 0$. This gives us a series infinite in positive powers of q. The regularization $[f(q)]_-$ is defined by taking expansion at $q = \infty$. This gives us a series infinite in negative powers of q. These two regularizations differ by a distribution supported at $q = 1$. For example, for the $\bar{\partial}$-operator we get as the difference the Dirac delta-function $[(1-q^{-1})^{-1}]_+ - [(1-q^{-1})^{-1}]_- = -\sum_{n=-\infty}^{n=\infty} q^n$.

Let $X = \mathbb{C}^n$ be a $H = U(1)$ module with positive weights $m_1, \ldots, m_n$, so that $U(1)$ acts as $z_i \to q^{m_i} z_i$, and let $Y = \{0\}$ be the H-fixed point set. Let v be the vector field generated by the $U(1)$ action on X. Let $\sigma(D)$ be an elliptic symbol defined on $T^*X|_Y$, i.e. defined on the fiber of the cotangent bundle to X at the origin. Atiyah showed [71] that we can use the vector field v in two different ways, called $[\cdot]_+$ and $[\cdot]_-$, to construct a transversally elliptic symbol $\tilde{\sigma} = [\sigma]_\pm$ on the whole space $T^*_H X$ such that $\tilde{\sigma}$ is an isomorphism outside of the origin Y. (If (x, p)

are coordinates on T^*X, then, loosely speaking, we take $\tilde{\sigma}(x,p) = \sigma(0, p \pm v)$. See appendix A.4 for more precise details). Then the index of the transversally elliptic symbol $\tilde{\sigma}$ is well defined as a distribution on H. Moreover, if $\operatorname{ind}(\sigma)$ is a rational function of q associated at the fixed point Y to the elliptic symbol σ by Atiyah-Bott formula, then

$$\operatorname{ind}([\sigma]_\pm) = [\operatorname{ind}(\sigma)]_\pm. \tag{2.4.38}$$

We apply this procedure to our problem. Namely, we use the vector field generated by the $H = U(1)$-action on $X = S^4$ to trivialize the symbol $\sigma(D_{10}^{vect})$ everywhere on T_H^*X except at the North and the South pole. Then the index is equal to the sum of contributions from the fixed points, where each contribution is expanded in positive or negative powers of q according to the (2.4.38). More concretely, we trivialize the transversally elliptic symbol $\sigma = \sigma(D_{10}^{vect})$ everywhere outside the North and the South poles on T_H^*X by replacing $c_\theta p_1$ by $c_\theta p_1 + v$ on the diagonal in (2.4.33), where $v = \sin\theta$. In other words, we deform the operator by adding the Lie derivative in the direction of the vector field v. The resulting symbol

$$\tilde{\sigma} = \begin{pmatrix} c_\theta p_1 + s_\theta & p_2 & p_3 & p_4 \\ -p_2 & c_\theta p_1 + s_\theta & -p_4 & p_3 \\ -p_3 & p_4 & c_\theta p_1 + s_\theta & -p_2 \\ -p_4 & -p_3 & p_2 & c_\theta p_1 + s_\theta \end{pmatrix}. \tag{2.4.39}$$

has determinant $(\overrightarrow{p}^2 + (c_\theta p_1 + s_\theta)^2)^2$ which is non-zero everywhere outside the North and the South poles at T_H^*X. (To check this, take $p_1 = 0$ and $s_\theta > 0$.) The index of $\tilde{\sigma}$ is equal to the index of σ, since $\tilde{\sigma}$ is a continous deformation of σ. On the other hand, since $\tilde{\sigma}$ is an isomorphism outside of the North and the South pole, to get the index of $\tilde{\sigma}$ we sum up contributions from the North and the South pole. At the North pole $\cos\theta = 1$. Therefore, in a small neighborhood of the North pole, the transversally elliptic symbol $\tilde{\sigma}$ coincides with the symbol associated to the elliptic symbol $\tilde{\sigma}_{\theta=0}$ by the $[\cdot]_+$ regularization. At the South pole $\cos\theta = -1$. Therefore, in a small neighborhood of the South pole, the transversally elliptic symbol $\tilde{\sigma}$ coincides with the symbol associated to the elliptic symbol $\tilde{\sigma}_{\theta=\pi}$ by the $[\cdot]_-$ regularization.

Finally we obtain

$$\mathrm{ind}'(D_{10}^{vect}) = \big[\mathrm{ind}(d, d^-)|_{\theta=0}\big]_+ + \big[\mathrm{ind}(d, d^+)|_{\theta=\pi}\big]_- . \tag{2.4.40}$$

One could probably also derive this result following the procedure in [94], where the index theorem for the Dirac operator was obtained using the deformation $\Gamma^\mu D_\mu \to \Gamma^\mu D_\mu + t\Gamma^\mu v_\mu$.

Let z_1, z_2 be complex coordinates in a small neighboorhod of the South pole, such that the $U(1)$ action is $z_1 \to qz_1, z_2 \to qz_2$. With respect to this action the complexified self-dual complex is isomorphic to the Dolbeault $\bar{\partial}$-complex twisted by the bundle $\mathcal{O} \oplus \Lambda^2 T^*_{1,0}$. Using the fact that the index of $\bar{\partial}$ operator is $(1-q^{-1})^{-2}$, we get

$$\mathrm{ind}'(D_{10}^{vect}) = \left[-\frac{1+q^2}{(1-q)^2}\right]_+ + \left[-\frac{1+q^2}{(1-q)^2}\right]_- , \tag{2.4.41}$$

where $[f(q)]_\pm$ respectively means to take expansion of $f(q)$ in positive or negative powers of q. In our conventions E_0 corresponds to the middle term of the standard (anti)-self dual complex (2.4.35), therefore we get an extra minus sign.

Finally,

$$\begin{aligned}\mathrm{ind}(D_{10}^{vect}) &= 2 + \mathrm{ind}'(D_{10}^{vect}) = \\ &= 2 - (1+q^2)(1+2q+3q^2+\dots) - (1+q^{-2})(1+2q^{-1}+3q^{-2}+\dots) = \\ &= -\sum_{n=-\infty}^{\infty} |2n| q^n.\end{aligned} \tag{2.4.42}$$

Note that in the computation of the index for the vector multiplet, the chirality of the complex coincides with the chirality of the $U(1)$ rotation near each of the fixed points.

Now we proceed to the hypermultiplet contribution to the index. The computation is similar to the vector multiplet. The transversally elliptic operator

$D_{10}^{hyper} : \Gamma(E_0^{hyper}) \to \Gamma(E_1^{hyper})$ can be trivialized everywhere over $T_G^* X$ except fixed points, where it is isomorphic to the self-dual complex at the North pole, or anti-self-dual complex at the South pole. For the hypermultiplet the chirality of the complex is opposite to the chirality of the $U(1)$ rotation near each of the fixed points. Then, using that the index of the twisted Dolbeault operator is $(1+qq^{-1})/((1-q)(1-q^{-1}))$, we get

$$\mathrm{ind}_q(D_{10}^{hyper}) = \left[-\frac{2}{(1-q)(1-q^{-1})} \right]_+ + \left[-\frac{2}{(1-q)(1-q^{-1})} \right]_- , \tag{2.4.43}$$

which results in

$$\mathrm{ind}_q(D_{10}^{hyper}) = + \sum_{n=-\infty}^{\infty} |2n| q^{-n}. \tag{2.4.44}$$

So far we considered the massless hypermultiplet. In this case its contribution to the index exactly cancels the vector multiplet. Hence, the determinant factor in the $\mathcal{N} = 4$ theory is trivial. This finishes the proof that the Erickson-Semenoff-Zarembo matrix model is exact in all orders of perturbation theory.

In the $\mathcal{N} = 2^*$ case the situation is more interesting. Now the hypermultiplet is massive. In the transformations (2.4.19) the action of R is contributed by the $SU(2)_R^R$ generator M_{ij}. We normalize it as $M_{ij}M^{ij} = 4m^2$. The hypermultiplet fields transform in the spin-$\frac{1}{2}$ representation of $SU(2)_R^R$. Therefore, in the massive case the index is multiplied by the spin-$\frac{1}{2}$ character relative to the massless case: $\frac{1}{2}(e^{im} + e^{-im})$. Hence all $U(1)$-eigenspaces split into half-dimensional subspaces with eigenvalues shifted by $\pm m$.

Finally, all fields transform in the adjoint representation of gauge group. Making a constant gauge transformation we can assume that the generator a_0 is in the Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of the gauge group. Then non-zero eigenvalues of a_0 in the adjoint representation are $\{\alpha \cdot a_0\}$, where α runs over all roots of $\mathfrak{g}$. Hence, combining all contributions to the index, we obtain for the $\mathcal{N} = 2^*$ theory

$$\left(\frac{\det K_{bos}}{\det K_{ferm}} \right)_{\mathcal{N}=2^*} = \prod_{\text{roots } \alpha} \prod_{n=-\infty}^{\infty} \left[\frac{(\alpha \cdot a_0 + n\varepsilon + m)(\alpha \cdot a_0 + n\varepsilon - m)}{(\alpha \cdot a_0 + n\varepsilon)^2} \right]^{|n|} .$$

Here we denote $\varepsilon = r^{-1}$. The term $n\varepsilon$ comes from a weight n representation of the $U(1)$, the term $\alpha \cdot a_0$ is an eigenvalue of a_0 acting on the eigensubspace of the adjoint representation corresponding to root α.

We argued before that to ensure convergence of the path integral the mass parameter and the scalar field Φ_0 should be taken imaginary if we work with ten-dimensional Minkowski signature. The parameter a_0 is also imaginary since it is identified with the zero mode of Φ_0. Let us denote $m = im_E, a_0 = ia_E \equiv ia_0^E$. Then, recalling (2.4.21) we get

$$Z^{\mathcal{N}=2^*}_{\text{1-loop}}(ia_E) = \prod_{\text{roots } \alpha} \prod_{n=1}^{\infty} \left[\frac{((\alpha \cdot a_E)^2 + \varepsilon^2 n^2)^2}{((\alpha \cdot a_E + m_E)^2 + \varepsilon^2 n^2)((\alpha \cdot a_E - m_E)^2 + \varepsilon^2 n^2)} \right]^{\frac{n}{2}}. \tag{2.4.45}$$

This product requires some regularization which we explain in a moment.

Recall the product formula for the Barnes G-function (see e.g. [65])

$$G(1+z) = (2\pi)^{z/2} e^{-((1+\gamma z^2)+z)/2} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^n e^{-z + \frac{z^2}{2n}}, \tag{2.4.46}$$

where γ is the Euler constant. Then we introduce a function $H(z) = G(1+z)G(1-z)$ and obtain

$$H(z) = e^{-(1+\gamma)z^2} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)^n \prod_{n=1}^{\infty} e^{\frac{z^2}{n}}. \tag{2.4.47}$$

Using this relation we obtain formally

$$Z^{\mathcal{N}=2^*}_{\text{1-loop}}(ia_E) = \exp\left(\frac{m_E^2}{\varepsilon^2} \left((1+\gamma) - \sum_{n=1}^{\infty} \frac{1}{n} \right) \right) \times \\ \times \prod_{\text{roots } \alpha} \frac{H\left(i\alpha \cdot a_E / \varepsilon\right)}{\left[H\left((i\alpha \cdot a_E + im_E)/\varepsilon\right) H\left((i\alpha \cdot a_E - im_E)/\varepsilon\right) \right]^{1/2}}. \tag{2.4.48}$$

The first factor $\exp(\dots)$ is divergent, but it does not depend on a_E. Therefore it cancels when we compute expectation value of the operators which localize to functions of a_E, such as the circular supersymmetric Wilson loop operator. Therefore we can remove this factor from the partition function. The resulting product of the G-functions is a well defined analytic function of a_E.

Our result is consistent with the renormalization properties of the gauge theory. To check that the β-function comes out right, we need asymptotic expansion of the G-function at large z

$$\log G(1+z) = \frac{1}{12} - \log A + \frac{z}{2}\log 2\pi + \left(\frac{z^2}{2} - \frac{1}{12}\right)\log z - \frac{3}{4}z^2 + \sum_{k=1}^{\infty}\frac{B_{2k+2}}{4k(k+1)z^{2k}}, \tag{2.4.49}$$

where A is a constant and B_n are Bernoulli numbers. Then

$$\frac{1}{2}\left(\log G(1+iz_E) + \log G(1-iz_E)\right) = \frac{1}{12} - \log A + \left(-\frac{z_E^2}{2} - \frac{1}{12}\right)\log z_E + \frac{3}{4}z_E^2 + \ldots \tag{2.4.50}$$

If we take a limit of very large mass of the hypermultiplet, we expect to get the minimal $\mathcal{N}=2$ theory at the energy scales much lower then the mass of the hypermultiplet. At large m, we expand the denominator in (2.4.48), corresponding to the hypermultiplet contribution to $Z_{\text{1-loop}}$, and get

$$Z_{\text{1-loop}}^{hyper} = const(m_E) + \left(const + \log\frac{m_E}{\varepsilon}\sum_{\alpha}\frac{(\alpha\cdot a_E)^2}{\varepsilon^2}\right) + \mathcal{O}(\frac{1}{m^2}). \tag{2.4.51}$$

The important dependence on a_E can be combined with the classical Gaussian action in the matrix model

$$\frac{4\pi^2 r^2}{g_{YM}^2}(a_E, a_E) \to \left(\frac{4\pi^2 r^2}{g_{YM}^2} - \frac{C_2}{\varepsilon^2}\log\frac{m_E}{\varepsilon}\right)(a_E, a_E), \tag{2.4.52}$$

where C_2 denotes the proportionality constant of the second Casimir $\mathrm{tr}_{\mathrm{Ad}}\, T_a T_b = C_2 \delta_a \delta_b$. We can write that as

$$\frac{1}{\tilde{g}_{YM}^2} = \frac{1}{g_{YM}^2} - \frac{C_2}{4\pi^2}\log\frac{m_E}{\varepsilon} \tag{2.4.53}$$

where $\tilde{g}_{YM}^2$ has a simple meaning of the renormalized coupling constant. In other words, the bare microscopical constant g_{YM}^2 is defined at the UV scale m_E and higher (in that region it does not run because of restored $\mathcal{N}=4$ supersymmetry). At scales less than m_E, the coupling constant runs by beta-function of pure $\mathcal{N}=2$

theory. Recall that the one-loop beta function for a gauge theory with N_f Dirac fermions and N_s complex scalars in adjoint representation is

$$\frac{\partial g(\mu)}{\partial \log \mu} = \beta(g) = -\frac{C_2 g^3}{(4\pi)^2}\left(\frac{11}{3} - \frac{4}{3}N_f - \frac{1}{3}N_c\right). \tag{2.4.54}$$

Taking $N_f = N_s = 1$ for a pure $\mathcal{N} = 2$ theory we get precisely the relation (2.4.53), which says that $\tilde{g}^2_{YM}$ is the running coupling constant at the IR scale $\varepsilon = r^{-1}$, which is the lowest scale for the theory on S^4 of radius r. This is also the scale of the Wilson loop operator, since it is placed on the equator.

We can check that the resulting integral over a_E is always convergent as long as the bare coupling constant g^2_{YM} is positive, in other words as long as the original action is positive definite. First of all, the Barnes function $G(1+z)$ does not have poles or zeroes on the imaginary contour $\mathrm{R}ez = 0$ over which we integrate. To see that the integral also behaves nicely at infinity we use the asymptotic expansion (2.4.50).

In the pure $\mathcal{N} = 2$ case the leading term in the exponent comes from the numerator of $Z_{\text{1-loop}}$ and is equal to $-\frac{1}{2}z_E^2 \log z_E$. This is a negative function which grows in absolute value faster than any other terms including the renormalized quadratic term (2.4.52) even if $\tilde{g}^2_{YM}$ formally becomes negative.

In the $\mathcal{N} = 2^*$ case we need to take asymptotic expansion at large z_E of both the numerator and denominator of (2.4.48) to check convergence at infinity. The leading terms $(\alpha \cdot a_E)^2 \log(\alpha \cdot a_E)$ cancel, and the next order term is proportional to $m_E^2 \log(\alpha \cdot a_E)$. This does not spoil the convergence insured by the Gaussian classical factor $\exp(-\frac{4\pi^2 r^2}{g^2_{YM}}(a_E, a_E))$.

To summarize, in the pure $\mathcal{N} = 2$ theory we need to insert the factor

$$Z^{\mathcal{N}=2}_{\text{1-loop}} = \prod_{\text{roots } \alpha} H\left(i\alpha \cdot a_E/\varepsilon\right), \tag{2.4.55}$$

under the integral in the matrix model and to substitute g_{YM} by the renormalized coupling constant $\tilde{g}_{YM}$ in the Gaussian classical action.

When we set $m = 0$ we get the $\mathcal{N} = 4$ theory. The numerator coming from the vector multiplet exactly cancels the denominator coming from the hypermultiplet in the formula (2.4.48) and we get

$$Z^{\mathcal{N}=4}_{\text{1-loop}} = 1. \tag{2.4.56}$$

We shall note that most of the above computations are generalized easily for the $\mathcal{N}=2$ theory with a massless hypermultiplet taken in an arbitrary representation. Let us denote this represenation by W. Analogously to the case of the adjoint representation, one can get a formula

$$Z^{\mathcal{N}=2,W}_{1-loop}(ia_E) = \frac{\prod_{\alpha\in\text{weights(Ad)}} H(i\alpha\cdot a_E/\varepsilon)}{\prod_{w\in\text{weights}(W)} H(iw\cdot a_E/\varepsilon)}. \tag{2.4.57}$$

Strictly speaking, this formula is valid in the situations when the infinite product of weights for the vector multiplet and hypermultiplet is proportional to the product of Barnes G-functions with the same divergent factor. That happens for such representations W when $\sum_\alpha(\alpha\cdot a)^2 = \sum_w (w\cdot a)^2$ for any $a\in\mathfrak{g}$. This is actually the condition of vanishing β-function for the $\mathcal{N}=2$ theory with a hypermultiplet in representation W. Therefore we claim that the formula (2.4.57) literally holds for all $\mathcal{N}=2$ superconformal theories. In a general $\mathcal{N}=2$ case, the one-loop determinant requires regularization similarly to what we did for the pure $\mathcal{N}=2$ theory.

It would be interesting to combine the factor Z_{1-loop} with the partition function of instanton corrections $|Z_{inst}|^2$ in an arbitrary $\mathcal{N}=2$ superconformal case, integrate over a_E and check predictions of the S-duality for these theories (see e.g.[73, 95]).

2.4.5 Example

Before turning to the instanton corrections, let us give a simplest example of a nontrival prediction of the formula (2.4.57), which perhaps can be checked using the traditional methods of the perturbation theory.

Take the $\mathcal{N}=2$ theory with with the $SU(2)$ gauge group and 4 hypermultiplets in the fundamental representation. We choose coordinate a on the Cartan subalgebra of the real Lie algebra of the gauge group $SU(2)$, such that an element a

is represented by an anti-hermitian matrix $diag(ia, -ia)$. Let the invariant bilinear form on the Lie algebra be minus the trace in the fundamental representation, and let the kinetic term of the Yang-Mills action be normalized as $\frac{1}{4g_{YM^2}}\int d^4x\sqrt{g}(F_{\mu\nu}, F^{\mu\nu})$. The weights w in the spin-j representation run from $-2j$ to $2j$. In the adjoint representaton ($j=1$) we have $\{\alpha\cdot a\} = \{-2a, 0, 2a\}$. In the fundamental representation ($j=\frac{1}{2}$) we have $\{w\cdot a\} = \{a, -a\}$. We also have $(a,a) = 2a^2$. The matrix model for the expectation value of the Wilson loop in the spin-j representation is

$$\langle \mathrm{tr}_j \,\mathrm{Pexp}(\int A dx + i\Phi_0 ds)\rangle = Z^{-1}\int_{-\infty}^{\infty} da e^{-\frac{8\pi^2}{g_{YM}^2}a^2}(2a)^2\frac{H(2ia)H(-2ia)}{(H(ia)H(-ia))^4}(\sum_{m=-j}^{j} e^{4\pi m a}),$$

where Z is a constant independent of the inserted Wilson loop operator. The extra factor $(2a)^2$ is the usual Vandermonde determinant appearing when we switch to the integral over the Cartan subalgebra from the integral over the whole Lie algebra. At the weak coupling $g_{YM}\to 0$ we can evaluate this integral as a series in g_{YM}. For the Barnes G-function we use Taylor series expansion at small z

$$\log G(1+z) = \frac{1}{2}(\log(2\pi)-1)z - (1+\gamma)\frac{z^2}{2} + \sum_{n=3}^{\infty}(-1)^{n-1}\zeta(n-1)\frac{z^n}{n}.$$

After some algebra one gets the following perturbative result for the expectation value of $e^{2\pi n a}$ in the matrix model (we write here $g = g_{YM}$)

$$\langle e^{2\pi n a}\rangle = 1 + \frac{3}{2\cdot 2^2}n^2g^2 + \frac{5}{8\cdot 2^4}n^4g^4 + \frac{7}{48\cdot 2^6}n^6g^6 + \frac{35}{2^4(4\pi)^2}t_2n^2g^6 + O(g^8), \quad (2.4.58)$$

where t_2 is the coefficient coming from the expansion of the Barnes G-function. It is expressed in terms of Riemann zeta-function

$$t_2 = -12\zeta(3).$$

To get this result we expanded the determinant factor in powers of a:

$$\log\left(\frac{H(2ia)H(-2ia)}{(H(ia)H(-ia))^4}\right) = -8\sum_{k=2}^{\infty}\frac{\zeta(2k-1)}{k}(2^{2k-2}-1)(-1)^k a^{2k} =: \sum_{k=2}^{\infty} t_k a^{2k}.$$

Then for a Gaussian measure $\int da\, e^{-\frac{1}{2\sigma^2}a^2}$ with $\sigma^2 = \frac{g_{YM}^2}{16\pi^2}$ we have

$$\left\langle a^2 \exp\left(\sum t_k a^{2k}\right) e^{qa} \right\rangle_{\text{gauss}} = \left(\frac{\partial}{\partial q}\right)^2 \exp\left(\sum t_k \left(\frac{\partial}{\partial q}\right)^k\right) e^{\frac{1}{2}q^2\sigma^2}.$$

The perturbative result for the $\mathcal{N}=4$ $SU(2)$ theory is given by the same formula but with $t_k = 0$:

$$\langle e^{qa}\rangle_{\mathcal{N}=4} = (1+\sigma^2 q^2)\exp(\frac{1}{2}\sigma^2 q^2) = 1 + \frac{3}{2}(\sigma q)^2 + \frac{5}{8}(\sigma q)^4 + \frac{7}{48}(\sigma q)^6 + O((\sigma q)^8).$$

Taking $q = 2\pi n$ and $\sigma = \frac{g_{YM}}{4\pi}$ we get the result (2.4.58) for the $\mathcal{N}=4$ theory with $t_2 = 0$. For a superconformal $\mathcal{N}=2$ theory the Gaussian matrix model action is corrected by the terms $t_k a^{2k}$. The first correction is quartic $t_2 a^4$, and at the lowest order it gives the result (2.4.58) for the $SU(2)$ theory with 4 hypermultiplets in the fundamental representation.

The first difference for $\langle W_R(C)\rangle$ between the $\mathcal{N}=2$ $SU(2)$ gauge theory with 4 fundamental hypermultiplets and the $\mathcal{N}=4$ $SU(2)$ gauge theory appears at the order g_{YM}^6. This is the order of the two-loop level Feynman diagram computations which have been done in the gauge theory for the $\mathcal{N}=4$ case [69, 70].

In the matrix model it is very easy to get the higher terms in the expansion over g_{YM}. On the other hand, the complexity of the Feynman diagram computations done directly in the gauge theory grows enormously with the number of loops.

Now we will argue that we can improve the matrix model by taking into account all instanton corrections of the theory, so that the result becomes non-perturbatively exact.

2.5 Instanton corrections

When we argued by (2.3.17) that the theory localizes to the trivial gauge field configurations, we used the fact that $d_{[\lambda}w_{\mu\nu]}$ does not vanish everywhere except at the North and the South poles and we assumed smooth gauge fields. Dropping the smoothness condition, we can only say that the gauge field strength must vanish everywhere away from the North and South poles. If we allow field configurations like Dirac-delta function, then the gauge field strength can be supported at the poles and still be consistent with vanishing tQV-term. From (2.3.16) we see that F^+ might be non zero at the North pole, where $\sin^2\frac{\theta}{2}$ vanish, while F^- might be non zero at the South pole, where $\cos^2\frac{\theta}{2}$ vanish. Thus, if we allow non-smooth gauge fields in the path integral, we should count configurations with point anti-instantons ($F^- = 0$) localized at the North pole, and point instantons ($F^+ = 0$) localized at the South pole. The Q-complex on S^4 in our problem in a neighborhood of the South/the North pole coincides with the Q-complex of the topological ($F^+ = 0$)/anti-topological ($F^- = 0$) gauge theory on $\mathbb{R}^4$ in the Ω-background studied by Nekrasov [60]. There the moduli space of solutions to $F^+ = 0$ modulo gauge transformations was taken equivariantly under the $U(1)^2$ action on $\mathbb{R}^4 \simeq \mathbb{C}^2$ by $z_1 \to e^{i\varepsilon_1} z_1, z_2 \to e^{i\varepsilon_2} z_2$, and gauge transformations at infinity with generator $a \in \mathfrak{g}$. Making the correspondence between the theory on S^4 in a local neighborhood of the North pole and the theory on $\mathbb{R}^4$ we should take $\varepsilon_1 = \varepsilon_2 = r^{-1}$, since for the problem on S^4, the chirality of the equations at the North pole coincides with the chirality of the generator of the Lorentz rotations $d_{[\mu} v_{\nu]}$. The same applies to the South pole: the chirality of the equations is reversed as well as the chirality of the generator of the Lorentz rotations.

In this section we consider only the case of the $U(N)$ gauge group. We use the following conventions. The solutions of the equations $F^+ = 0$ are called instantons. The solutions of the equations $F^- = 0$ are called anti-instantons.

We define the instanton charge as the second Chern class[6]

$$k = c_2 = -\frac{1}{8\pi^2}\int F\wedge F,$$

and modify the action by the θ-term

$$S_{YM} \to S_{YM} + \frac{i\theta}{8\pi^2}\int F\wedge F.$$

At $F^+ = 0$ we have $\sqrt{g}F_{\mu\nu}F^{\mu\nu}d^4x = 2F\wedge *F = -2F\wedge F$. Then the Yang-Mills action of instanon of charge k is

$$S_{YM}(k) = \frac{1}{4g_{YM}^2}\int \sqrt{g}d^4xF_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{8\pi^2}\int F\wedge F = \left(\frac{4\pi^2}{g_{YM}^2} - i\theta\right)k.$$

Its contribution to the partition function is proportional to

$$\exp(-S_{YM}(k)) = \exp\left(2\pi i\tau k\right) = q^k,$$

where we introduced the complexified coupling constant

$$\tau = \frac{2\pi i}{g_{YM}^2} + \frac{\theta}{2\pi},$$

and the expansion parameter

$$q = \exp(2\pi i\tau).$$

(The expansion parameter q in this section should not be confused with the formal generator of the $U(1)$ group used to compute the index of the transversally elliptic operator in the previous section).

Near the South pole the theory on S^4 looks like topological theory with the equations $F^+ = 0$, so that only point instantons contribute. Near the North pole the situation is opposite: the equations are replaced by $F^- = 0$, therefore we need

[6] For $U(N)$ bundles we have the total Chern class $c = \det(1+\frac{iF}{2\pi}) = \prod(1+x_i) = c_0 + c_1 + \dots$, where F is the curvature which takes value in the Lie algebra of the gauge group, x_i are the Chern roots, and c_k is polynomial of degree k in x_i. We have $c_2 = \sum_{i<j} x_i x_j = \frac{1}{2}(\sum x_i)^2 - \frac{1}{2}\sum x_i^2$. If $c_1 = \sum x_i$ vanishes, we get $c_2 = -\frac{1}{2}\int \operatorname{tr}\frac{iF}{2\pi}\wedge\frac{iF}{2\pi} = \frac{1}{8\pi^2}\int \operatorname{tr} F\wedge F = -\frac{1}{8\pi^2}\int (F,\wedge F)$, where the trace is taken in the fundamental representation. The parentheses $(a,b) = -\operatorname{tr} ab$ denote the positive definite bilinear form on the Lie algebra which is assumed in the most of the formulas.

to count anti-instantons. The generating function of anti-instantons is the same as the generating function of instantons with replacement of the expansion parameter q by its complex conjugate $\bar{q}$.

For the $U(N)$ gauge group the explicit formula for the equivariant instanton partition function on $\mathbb{R}^4$ is [60, 61, 96–99]

$$Z_{\text{inst}}^{\mathcal{N}=2}(\varepsilon_1, \varepsilon_2, a) = \sum_{\vec{Y}} \frac{q^{|Y|}}{\prod_{\alpha,\beta=1}^{N} n_{\alpha,\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2, \vec{a})}, \tag{2.5.1}$$

where we sum over an ordered set of N Young diagrams $\{Y_\alpha\}$ with $\alpha = 1 \ldots N$. By $|\vec{Y}|$ we denote the total size of all diagrams in a set $|\vec{Y}| = \sum |Y_\alpha|$. The total size is equal to the instanton number. The factor $n_{\alpha,\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2, \vec{a})$ denotes the equivariant Euler class of the tangent space to the instanton moduli space at the fixed point labeled by $\vec{Y}$. It is given by

$$\begin{aligned} n_{\alpha,\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2, \vec{a}) = \prod_{s \in Y_\alpha} (-h_{Y_\beta}(s)\varepsilon_1 + (v_{Y_\alpha}(s) + 1)\varepsilon_2 + a_\beta - a_\alpha) \times \\ \times \prod_{t \in Y_\beta} ((h_{Y_\alpha}(t) + 1)\varepsilon_1 - v_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha). \end{aligned} \tag{2.5.2}$$

(We assume that an element a in the Cartan subalgebra of $\mathfrak{u}(N)$ is represented by a diagonal matrix $(ia_1, \ldots, ia_N)$.) Here s and t run over squares of Young diagrams Y_α and Y_β. Let Y is a Young diagram $\nu_1 \geq \nu_2 \cdots \geq \nu_{\nu_1'}$, where ν_i is the length of the i-th column, ν_j' is the length of the j-th row. If a square $s = (i, j)$ is located at the i-th column and the j-th row then $v_Y(s) = \nu_i(Y) - j$ and $h_Y(s) = \nu_j'(Y) - i$. In other words, $v_Y(s)$ and $h_Y(s)$ is respectively the vertical and horizontal distance from the square s to the edge of the diagram Y. We can rewrite the product in the denominator of (2.5.1) as

$$\prod_{\alpha,\beta=1}^{N} n_{\alpha,\beta}^{\vec{Y}}(\varepsilon_1, \varepsilon_2, \vec{a}) = \prod_{\alpha,\beta=1}^{N} \prod_{s \in Y_\alpha} E_{\alpha\beta}(s)(\varepsilon_1 + \varepsilon_2 - E_{\alpha\beta}(s)), \tag{2.5.3}$$

where

$$E_{\alpha\beta}(s) = (-h_{Y_\beta}(s)\varepsilon_1 + (v_{Y_\alpha}(s) + 1)\varepsilon_2 + a_\beta - a_\alpha). \tag{2.5.4}$$

We will give a few simplest examples of evaluation of this formula. First consider $U(1)$ case. Then we sum over all Young diagrams of one color. At one instanton level $k = 1$, there is only one diagram $Y = (1)$. Then $E_{11} = \varepsilon_2$, so that

$$Z^{\mathcal{N}=2}_{k=1}(\varepsilon_1, \varepsilon_2, a) = \frac{q}{\varepsilon_2 \varepsilon_1}. \tag{2.5.5}$$

At two instanton level $k = 2$, there are two diagrams $Y = (2, 0)$ and $Y = (1, 1)$. Their contribution is

$$Z^{\mathcal{N}=2}_{k=2}(\varepsilon_1, \varepsilon_2, a_1) = \frac{1}{(2\varepsilon_2)(\varepsilon_1 - \varepsilon_2)(\varepsilon_2)(\varepsilon_1)} + \frac{1}{(-\varepsilon_1 + \varepsilon_2)(2\varepsilon_1)(\varepsilon_2)(\varepsilon_1)} = \frac{1}{2(\varepsilon_1 \varepsilon_2)^2} \tag{2.5.6}$$

At three instanton level $k = 3$, there are three diagrams $Y = (3, 0)$, $Y = (2, 1)$ and $Y = (1, 1, 1)$. Their contribution is

$$\begin{aligned} Z^{\mathcal{N}=2}_{k=3}(a, \varepsilon_1, \varepsilon_2) &= \frac{1}{(\varepsilon_2)(\varepsilon_1)(2\varepsilon_2)(\varepsilon_1 - \varepsilon_2)(3\varepsilon_2)(\varepsilon_1 - 2\varepsilon_2)} + \\ &+ \frac{1}{(\varepsilon_2)(\varepsilon_1)(2\varepsilon_2 - \varepsilon_1)(2\varepsilon_1 - \varepsilon_2)(\varepsilon_2)(\varepsilon_1)} + \frac{1}{(\varepsilon_2)(\varepsilon_1)(\varepsilon_2 - \varepsilon_1)(2\varepsilon_1)(\varepsilon_2 - 2\varepsilon_1)(3\varepsilon_2)} = \\ &= \frac{1}{6(\varepsilon_1 \varepsilon_2)^3} \end{aligned} \tag{2.5.7}$$

At an arbitrary instanton level k, the sum of all Young diagrams of order k simplifies to

$$Z^{\mathcal{N}=2}_{k}(\varepsilon_1, \varepsilon_2, a) = \frac{1}{k!(\varepsilon_1 \varepsilon_2)^k}, \tag{2.5.8}$$

hence

$$Z^{\mathcal{N}=2}_{U(1)}(\varepsilon_1, \varepsilon_2, a) = \sum_{k=1}^{\infty} \frac{q^k}{k!(\varepsilon_1 \varepsilon_2)^k} = \exp\left(\frac{q}{\varepsilon_1 \varepsilon_2}\right). \tag{2.5.9}$$

Now we consider a few instantons for the $U(2)$ gauge group. At one instanton there are two colored Young diagrams $((1), 0)$ and $(0, (1))$ contributing

$$\begin{aligned} Z^{\mathcal{N}=2}_{k=1}(\varepsilon_1, \varepsilon_2, a_1, a_2) &= \frac{1}{\varepsilon_1 \varepsilon_2 (a_2 - a_1 + \varepsilon_1 + \varepsilon_2)(a_1 - a_2)} + \\ &+ \frac{1}{(a_1 - a_2 + \varepsilon_1 + \varepsilon_2)(a_2 - a_1)\varepsilon_1 \varepsilon_2} = \frac{2}{\varepsilon_1 \varepsilon_2((\varepsilon_1 + \varepsilon_2)^2 - a^2)}, \end{aligned} \tag{2.5.10}$$

where we denoted $a = a_2 - a_1$. As the instanton number grows, its contribution becomes more and more complicated rational function of a_i. For example, at $k = 2$ we get (we set $a = ia_E$, where a_E is real)

$$Z_{k=2}^{\mathcal{N}=2}(\varepsilon_1, \varepsilon_2, ia_E) = \frac{(2a_E^2 + 8\varepsilon_1^2 + 8\varepsilon_2^2 + 17\varepsilon_1\varepsilon_2)}{((\varepsilon_1 + 2\varepsilon_2)^2 + a_E^2)((2\varepsilon_1 + \varepsilon_2)^2 + a_E^2)((\varepsilon_1 + \varepsilon_2)^2 + a_E^2)\varepsilon_1^2\varepsilon_2^2}. \tag{2.5.11}$$

Generally, instanton contributions are certain rational functions of a_i and ε_i. Contrary to the case $\varepsilon_1 = -\varepsilon_2 = \hbar$, which is often taken in the literature to simplify the instanton partition function [60, 61, 99], in our problem we get the same signs: $\varepsilon_1 = \varepsilon_2 = \frac{1}{r}$. Looking at the examples above, one can note an important property of the instanton contributions at $\varepsilon_1 = \varepsilon_2$; they do not have poles at the integration contour for a_i. Recall that in the matrix integral we integrate over imaginary $a = ia_E$, while ε_1 and ε_2 is real. Generally, the denominator contains factors $n_1\varepsilon_1 + n_2\varepsilon_2 + a$, where n_1 and n_2 are some numbers. There is a pole at the integration contour only if $n_1\varepsilon_1 + n_2\varepsilon_2 = 0$. Though it happens regularly at $\varepsilon_1 = -\varepsilon_2$, it never happens at $\varepsilon_1 = \varepsilon_2$. (This fact was checked explicitly up to $k = 5$ instantons for $U(2)$ gauge group and actually one can show it in general.[7]) Therefore the integrand in (2.1.3) is a smooth function everywhere at the integration domain and it also decreases rapidly at infinity. Thus the integral is convergent and well defined.

In the $\mathcal{N} = 2^*$ case, each instanton contribution is multiplied by a new factor. This factor is equal to the product of the same weights as in the denominator, but shifted by the hypermultiplet mass $m = im_E$:

$$Z_{\text{inst}}^{\mathcal{N}=2^*}(\varepsilon_1, \varepsilon_2, m, a) = \sum_{\vec{Y}} q^{|\vec{Y}|} \prod_{\alpha,\beta=1}^{N} \prod_{s\in Y_\alpha} \frac{(E_{\alpha\beta}(s) - m)(\varepsilon_1 + \varepsilon_2 - E_{\alpha\beta}(s) - m)}{E_{\alpha\beta}(s)(\varepsilon_1 + \varepsilon_2 - E_{\alpha\beta}(s))}. \tag{2.5.12}$$

[7] The author thanks H. Nakajima for a discussion.

For example,

$$\begin{aligned} Z^{\mathcal{N}=2^*}_{k=1}(\varepsilon_1,\varepsilon_2,a_1,a_2) &= \frac{(\varepsilon_1-m)(\varepsilon_2-m)(a_2-a_1+\varepsilon_1+\varepsilon_2-m)(a_1-a_2-m)}{\varepsilon_1\varepsilon_2(a_2-a_1+\varepsilon_1+\varepsilon_2)(a_1-a_2)}+ \\ &+\frac{(a_1-a_2+\varepsilon_1+\varepsilon_2-m)(a_2-a_1-m)(\varepsilon_1-m)(\varepsilon_2-m)}{(a_1-a_2+\varepsilon_1+\varepsilon_2)(a_2-a_1)\varepsilon_1\varepsilon_2} = \\ &= \frac{2(m-\varepsilon_2)(m-\varepsilon_1)(m^2-a^2-m(\varepsilon_1+\varepsilon_2)+(\varepsilon_1+\varepsilon_2)^2)}{((\varepsilon_1+\varepsilon_2)^2-a^2)\varepsilon_1\varepsilon_2} \end{aligned} \tag{2.5.13}$$

The integrand is still a smooth function on the whole integration domain and decreases sufficiently fast at infinity.

Hence, we conclude that the matrix integral, with all instanton corrections included, is well defined in the $\mathcal{N}=2$, the $\mathcal{N}=2^*$ and the $\mathcal{N}=4$ cases, and that it gives the exact partition function of these theories on S^4. The expectation value of a supersymmetric circular Wilson operator on S^4 in an arbitrary representation is equal to the expectation value of the operator $\operatorname{tr}_R e^{2\pi i r a}$ in this matrix model.

In the general $\mathcal{N}=2^*$ case there is the non-trivial one-loop determinant factor and the non-trivial instanton corrections. However, in the $\mathcal{N}=4$ theory, the numerator and the denominator cancel each other both in $Z_{\text{1-loop}}$ and in each of the fixed point instanton contribution to Z_{inst}. More precisely, $Z^{\mathcal{N}=4}_{1-loop}=1$, and

$$Z^{\mathcal{N}=4}_{\text{inst}}(U(N)) = \sum_{\vec{Y}} q^{|\vec{Y}|} = \prod_{k=1}^{\infty}\frac{1}{(1-q^k)^N} \tag{2.5.14}$$

is the generating function for the number of N-colored partitions.

Since $Z^{\mathcal{N}=4}_{\text{inst}}$ does not depend on a, it gives the same contribution to the partition function and to the partition function with inserted Wilson loop operator $W_R(C)$. Therefore $|Z^{\mathcal{N}=4}_{\text{inst}}(q)|^2$ factors out of the Gaussian integral and cancels in the expectation value for $\langle W_R(C)\rangle$. In other words, we conclude that in the $\mathcal{N}=4$ theory there are no instanton corrections to the Gaussian integral conjecture (2.1.2).

Using the definition of the Dedekind eta-function $\eta(\tau)=q^{1/24}\prod_{k=1}^{\infty}(1-q^k)$ we can write

$$Z^{\mathcal{N}=4}_{\text{inst}} = \left(\frac{1}{q^{-1/24}\eta(\tau)}\right)^N. \tag{2.5.15}$$

Then the partition function of the $\mathcal{N}=4$ theory on S^4 with $U(N)$ gauge group is

$$Z_{S^4}^{\mathcal{N}=4} = \frac{1}{\mathrm{vol}(U(N))} \left| \left(\frac{1}{q^{-1/24}\eta(\tau)} \right)^N \right|^2 \int_{\mathfrak{g}} [da] e^{-\frac{4\pi^2 r^2}{g_{YM}^2}(a,a)} \tag{2.5.16}$$

The natural measure on the gauge group $U(N)$ includes g_{YM} coupling constant, so that $\mathrm{vol}(U(N)) \propto g_{YM}^{N^2}$. This factor is cancelled by the determinant coming from the Gaussian integral over a. Then the $\mathcal{N}=4$ partition function on S^4 as a function of the coupling constant is

$$Z_{S^4}^{\mathcal{N}=4} = \left| \left(\frac{1}{q^{-1/24}\eta(\tau)} \right)^N \right|^2 \tag{2.5.17}$$

This function does not transform well under S-duality $\tau \to -1/\tau$. However, we might recall that the theory can have c-number gravitational curvature terms which shift the action by a constant [78]. For example we can add the following R^2-term:

$$S_{YM} \to S_{YM} - 2\pi\tau_2 \frac{1}{24} \frac{N}{32\pi^2} \int_{S^4} R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda}. \tag{2.5.18}$$

Such R^2 terms generally appear as gravitational corrections to an effective action on a brane in string theory [100].

This R^2 term cancels the extra factor $q^{-1/24}$ in the partition function, so that we finally get

$$Z_{S^4}^{\mathcal{N}=4} = \frac{1}{|\eta(\tau)|^{2N}}. \tag{2.5.19}$$

This function transforms as a modular form of the weight $(N/2, N/2)$ under the S-duality $SL(2,\mathbb{Z})$ transformations generated by $\tau \to -\frac{1}{\tau}$ and $\tau \to \tau + 1$.

So far we discussed instanton corrections only to the partition function. Now we consider corrections to the Wilson loop operator. One can show that the Wilson loop $W(C)$ which we consider is in the same δ_ε cohomology class as the operator $\mathrm{tr}_R \exp(\frac{2\pi}{\varepsilon}\Phi)$ inserted at the North pole, where $\Phi = i\Phi_0^E + \Phi_9$. Instanton corrections to the operator $\exp(\beta\Phi)$ in the $\mathcal{N}=2$ equivariant theory on $\mathbb{R}^4$ for a given asymptotic of Φ at infinity were computed in [61, 99, 101, 102]. Using these results, one can actually see that if $\beta = \frac{2\pi n}{\varepsilon}$ where n is integer, there are no instanton corrections to the operator $\mathrm{tr}_R \exp(\beta\Phi)$. In other words, the operator $\mathrm{tr}_R \exp(\beta\Phi)$ in the field theory is replaced simply by the operator $\mathrm{tr}_R \exp(2\pi i r a)$ in the matrix model.

This is exactly the case of Wilson loop operator which we consider. In other words, even after taking into account the instanton corrections, we still conclude that the Wilson loop operator $W(C)$ corresponds to the operator $\mathrm{tr}_R \exp(2\pi i r a)$ in the matrix model. However, the expectation value of $W(C)$ in a generic $\mathcal{N} = 2$ theory receives corrections because the measure in the matrix integral (2.1.3) is corrected by the insertion of the instanton factor $|Z_{\mathrm{inst}}(ia, \varepsilon, \varepsilon)|^2$.

Chapter 3

Wilson loops on S^2

3.1 Wilson loops on S^2 subspace in four-dimensional $\mathcal{N} = 4$ super Yang-Mills

In this chapter we consider supersymmetric Wilson loops of arbitrary shape located on S^2 subspace in the four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Such Wilson loops were constructed in [41, 52] and there it was conjectured that expectation value of such Wilson loops can be computed by perturbative sector of the two-dimensional bosonic Yang-Mills theory.

In this chapter we prove that correlation functions of such Wilson loop operators are indeed computed by certain two-dimensional gauge theory closely related to the perturbative Yang-Mill theory and constrained topological Higgs-Yang-Mills theory[53–55] for the moduli space of solutions of Hitchin's equations [56].

3.1.1 The geometrical set up

Let X_i for $i = 1 \ldots 5$ be coordinates in $\mathbb{R}^5$ into which the S^4 is embedded as the hypersurface $\sum X_i^2 = r^2$. By x_i for $i = 1 \ldots 4$ we denote the standard coordinates on the stereographic image of S^4 on $\mathbb{R}^4$ such that the point $N := (0, 0, 0, 0, r)$ maps to the origin

$$\begin{aligned} X_i &= \frac{x_i}{1+\frac{x^2}{4r^2}} \\ X_5 &= r\frac{1-\frac{x^2}{4r^2}}{1+\frac{x^2}{4r^2}}. \end{aligned} \tag{3.1.1}$$

We define the S^3 subspace of the S^4 by the equation $X_5 = 0$. Equivalently, in the x_i coordinates on $\mathbb{R}^4$, this three-sphere is defined by the equation $x^2 = 4r^2$. Further we define the two-sphere $S^2 \subset S^3$ by the additional equation $X_1 = 0$. In the x_i coordinates, the S^2 is described by the equations $\{x_1 = 0, x_2^2 + x_3^2 + x_4^2 = 4r^2\}$.

We call the point $P = (0, r, 0, 0, 0)$ the North pole of the S^2. By $y_i, i = 1 \ldots 4$ we denote the standard coordinates on the stereographic image of S^4 such that the point P maps to the origin:

$$\begin{aligned} X_i &= \frac{y_i}{1+\frac{y^2}{4r^2}} \qquad i = 1, 3, 4 \\ X_5 &= \frac{-y_2}{1+\frac{y^2}{4r^2}} \\ X_2 &= r\frac{1-\frac{y^2}{4r^2}}{1+\frac{y^2}{4r^2}}. \end{aligned} \tag{3.1.2}$$

In x^i coordinates, the point P is $(0, 2r, 0, 0)$.

The $SO(5)$ isometry group of S^4 can be broken to $SO(2) \times SO(3)$ where $SO(2)$ acts on (X_1, X_5) and $SO(3)$ acts on (X_2, X_3, X_4). The two-sphere S^2 is the fixed point set of this $SO(2)$. Sometimes it is convenient to use the $SO(2) \times SO(3)$ spherical coordinates on S^4 in which metric has the form

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\tau^2 + \cos^2\theta\, d\Omega_2^2) \tag{3.1.3}$$

In other words, we represent the S^4 as a warped $S^2 \times S^1$ fibration over the interval $\theta \in [0, \pi/2]$, such that at $\theta = 0$ the S^1 shrinks to zero and the S^2 is of maximal size, while at $\theta = \pi/2$ the S^2 shrinks to zero and the S^1 is of maximal size. We will also use the reversed to θ coordinate $\xi = \pi/2 - \theta$. In the following, the $SO(2)$ acting on (X_1, X_5), will be denoted as $SO(2)_S$, and the $SO(3)$ acting on (X_2, X_3, X_4) will be denoted as $SO(3)_S$. (We shall use the subscript "S" to denote subgroups of the space-time symmetries, and the subscript "R" do denote subgroups of the R-symmetry. We also remark that the $SO(3)_S$ subgroup of the $SO(4)$ isometry group of $\mathbb{R}^4$ is not a chiral $SU(2)_L$ subgroup in the decomposition $SO(4) = SU(2)_L \times SU(2)_R$, but rather a diagonal embedding.)

3.1.2 Superconformal symmetries and conformal Killing spinors

Following [52] we shall study the following Wilson loops located on the three-sphere S^3 and, specifically, the more specialized case: Wilson loops restricted to the maximal two-sphere S^2 embedded into the S^3. Here we shall work in the $\mathbb{R}^4$ stereographic coordinates x_i (3.1.1). The definition of such Wilson loops and the condition for supersymmetry was found in [41, 50, 52]:

$$W_R(C) = \mathrm{tr}_R \, \mathrm{Pexp} \oint \left(A_\mu + i\sigma^A_{\mu\nu} \frac{x^\nu}{2r} \Phi_A \right) dx^\mu. \tag{3.1.4}$$

Here Φ_A runs over three of six scalar fields of the $\mathcal{N}=4$ super Yang Mills theory. In our conventions index A takes values $6,7,8$. The μ,ν are the space-times indices running over the range $1,\ldots 4$. The $\sigma^A_{\mu\nu}$ are the 't Hooft symbols: three 4×4 anti-self-dual matrices satisfying $\mathfrak{su}(2)$ commutation relations. Explicitly we choose

$$\sigma^{i+4}_{1i} = 1 \quad \sigma^{i+4}_{jk} = -\epsilon_{ijk} \quad \text{for} \quad i = 2,3,4, \tag{3.1.5}$$

where ϵ_{ijk} is the standard antisymmetric symbol with $\epsilon_{234} = 1$. The $SO(6)$ R-symmetry group is broken into $SO(3)_A \times SO(3)_B$. Our conventions are that the $SO(3)_A$ acts on scalars Φ_6, Φ_7, Φ_8 which couple to the Wilson loops (3.1.4). The $SO(3)_B$ acts on the remaining scalars Φ_5, Φ_9, Φ_0. The Wilson loop (3.1.4) is explicitly invariant under the $SO(3)_B$ symmetry, because the scalar fields Φ_5, Φ_9, Φ_0 do not appear in (3.1.4). In the case when the Wilson loop (3.1.4) is restricted to the two-sphere S^2 by the constraint $x_1 = 0$, it is also invariant under the diagonal

$SO(3)$ subgroup of the $SO(3)_S \times SO(3)_A$, i.e. under the simultaneous rotation of the coordinates x_i and the scalars Φ_{i+4} for $i=2,3,4$.

Let us find the supersymmetries which are preserved by the Wilson loops (3.1.4).

The conformal Killing spinor on $\mathbb{R}^4$ is parameterized by two constant spinors which we call $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$, where $\hat{\varepsilon}_s$ generates the usual Poincare supersymmetries, and $\hat{\varepsilon}_c$ generates the special superconformal symmetries

$$\varepsilon(x) = \hat{\varepsilon}_s + x^\rho \Gamma_\rho \hat{\varepsilon}_c. \tag{3.1.6}$$

The variation of the bosonic fields of the theory is

$$\delta A_M = \psi \Gamma_M \varepsilon. \tag{3.1.7}$$

The variation of generic Wilson loop (3.1.4) vanishes iff ε satisfies

$$(\Gamma_\mu + i\Gamma_A \sigma^A_{\mu\nu} \frac{x^\nu}{2r})(\hat{\varepsilon}_s + x^\rho \Gamma_\rho \hat{\varepsilon}_c)\dot{x}^\mu = 0 \tag{3.1.8}$$

for any point $x \in S^3$ and the tangent vector $\dot{x}$ constrained by $\dot{x}x = 0$. The terms linear in x give the equation

$$x^\mu \dot{x}^\rho (\Gamma_\mu \Gamma_\rho \hat{\varepsilon}_c + i\Gamma_A \sigma^A_{\mu\rho} \frac{\hat{\varepsilon}_s}{2r}) = 0. \tag{3.1.9}$$

Since the vectors x^μ and $\dot{x}_\mu$ are constrained only by $x^\mu \dot{x}_\mu = 0$, we get

$$\Gamma_{\mu\rho} \hat{\varepsilon}_c + i\Gamma^A \sigma^A_{\mu\rho} \frac{\hat{\varepsilon}_s}{2r} = 0. \tag{3.1.10}$$

The constant and quadratic in x terms give the equation

$$\dot{x}^\mu (\Gamma_\mu \hat{\varepsilon}_s + i\Gamma_{A\lambda} \frac{\sigma^A_{\mu\nu}}{2r} x^\nu x^\lambda \hat{\varepsilon}_c) = 0. \tag{3.1.11}$$

Multiplying by non-degenerate matrix $x^\rho \Gamma_\rho$ we get

$$\dot{x}^\mu x^\rho (\Gamma_{\rho\mu} \hat{\varepsilon}_s + i\Gamma_\rho \Gamma_{A\lambda} \frac{\sigma^A_{\mu\nu}}{2r} x^\nu x^\lambda \hat{\varepsilon}_c) = 0. \tag{3.1.12}$$

Using $x^\mu x_\mu = 4r^2$ and $\dot{x}^\mu x_\mu = 0$ we get

$$\Gamma_{\mu\rho} \hat{\varepsilon}_s + i\Gamma_A \sigma^A_{\mu\rho} (2r) \hat{\varepsilon}_c = 0. \tag{3.1.13}$$

The equation (3.1.13) is actually equivalent to (3.1.10) and to

$$2r\hat{\varepsilon}_c = i\sigma_{A\mu\rho}\Gamma_{A\mu\rho}\hat{\varepsilon}_s. \tag{3.1.14}$$

If Wilson loop is restricted to S^2, then (3.1.14) amounts to three maximally orthogonal projections in the spinor representation space $S^+ \oplus S^-$. Each projection operator reduces the dimension of the space of solutions by half. Starting from the dimension 32 of $S^+ \oplus S^-$ we get $32/2^3 = 4$-dimensional space of solutions for $(\hat{\varepsilon}_s, \hat{\varepsilon}_c)$. For generic Wilson loops on S^3 the dimension of the space of solutions is further reduced by two, so there are only 2 supersymmetries left.

For explicit computation we shall use the following 16×16 gamma-matrices representing Clifford algebra on S^+:

$$\begin{aligned}
\Gamma_M &= \begin{pmatrix} 0 & E_M^T \\ E_M & 0 \end{pmatrix}, \qquad M = 2\dots 9 \\
\Gamma_1 &= \begin{pmatrix} 1_{8\times 8} & 0 \\ 0 & -1_{8\times 8} \end{pmatrix}, \\
\Gamma_0 &= \begin{pmatrix} i1_{8\times 8} & 0 \\ 0 & i1_{8\times 8} \end{pmatrix},
\end{aligned} \tag{3.1.15}$$

Here E_M for $M = 2\dots 8$ are 8×8 matrices representing left multiplication of the octonions and $E_9 = 1_{8\times 8}$. (Let e_i for $i = 2, \dots, 9$ be the generators of the octonion algebra $\mathbb{O}$. We chose e_9 to be identity. Let c_{ij}^k be the structure constants of the left multiplication $e_i \cdot c_j = c_{ij}^k e_k$. Then $(E_i)_j^k = c_{ij}^k$. The multiplication table can be chosen by specifying cyclic triples (ijk) such that $e_i e_j = e_k$. We define the cyclic triples to be $(234), (256), (357), (458), (836), (647), (728)$.)

Explicitly, the four linearly independent solutions of (3.1.14), i.e. supersymmetries of Wilson loops on the S^2 are the following

$$\hat{\varepsilon}^s_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \otimes |1\rangle \quad \hat{\varepsilon}^s_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \otimes |1\rangle \quad \hat{\varepsilon}^s_{\bar{1}} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \otimes |1\rangle \quad \hat{\varepsilon}^s_{\bar{2}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \otimes |1\rangle .$$

$$\hat{\varepsilon}^c_1 = \frac{1}{2r} \begin{pmatrix} 0 \\ i \\ 0 \\ i \end{pmatrix} \otimes |1\rangle \quad \hat{\varepsilon}^c_2 = \frac{1}{2r} \begin{pmatrix} -i \\ 0 \\ i \\ 0 \end{pmatrix} \otimes |1\rangle \quad \hat{\varepsilon}^c_{\bar{1}} = \frac{1}{2r} \begin{pmatrix} 0 \\ -i \\ 0 \\ i \end{pmatrix} \otimes |1\rangle \quad \hat{\varepsilon}^c_{\bar{2}} = \frac{1}{2r} \begin{pmatrix} i \\ 0 \\ i \\ 0 \end{pmatrix} \otimes |1\rangle \tag{3.1.16}$$

In more generic case of Wilson loops on S^3, we get only the two-dimensional space of solutions, which is spanned by $\varepsilon_1, \varepsilon_2$. We use indices $1, 2$ and $\bar{1}, \bar{2}$ to enumerate the basis elements of the solutions to (3.1.14), but it is not assumed that $\varepsilon_{\bar{1}}$ or $\varepsilon_{\bar{2}}$ is complex conjugate to ε_1 or ε_2. Sixteen components of the spinors are written in the 4×4 block notations, where

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{3.1.17}$$

Let $Q_1, Q_2, Q_{\bar{1}}, Q_{\bar{2}}$ be the four conformal supersymmetries generated by conformal Killing spinors (3.1.6) with $\hat{\varepsilon}_s, \hat{\varepsilon}_c$ given by (3.1.16). Let R_{AB} be matrices in the fundamental representation of the $SO(6)$ R-symmetry generators. On scalar fields the generators R_{AB} act as

$$(\delta_{R_{AB}} \Phi)_A = R_{AB} \Phi_B. \tag{3.1.18}$$

The fermionic symmetries anti-commute according to (A.3.3),(A.3.6) as

$$\delta^2_\varepsilon \Phi_A = 2(\tilde{\varepsilon} \Gamma_{AB} \varepsilon) \Phi_B, \tag{3.1.19}$$

hence the R-symmetry part of the anti-commutators is

$$Q_{\{\alpha}Q_{\beta\}} = 2(\tilde{\varepsilon}_{\{\alpha}\Gamma_{AB}\varepsilon_{\beta\}})R_{AB} \tag{3.1.20}$$

For space-time rotations we have similar equation except for the sign. Let us consider a fixed point of the space-time rotation. Then, assuming that the $SO(4)_S$ generators $R_{\mu\nu}$ act on tangent space $\mathbb{R}^4$ in the same way as the $SO(6)_R$ generators R_{AB} act on the scalar target space $\mathbb{R}^6$, we get the space-time symmetry part of the anti-commutators

$$Q_{\{\alpha}Q_{\beta\}} = -2(\tilde{\varepsilon}_{\{\alpha}\Gamma_{\mu\nu}\varepsilon_{\beta\}})R_{\mu\nu}, \tag{3.1.21}$$

where ε and $\tilde{\varepsilon}$ are taken at the fixed point set of the space-time rotation. To summarize,

$$Q_{\{\alpha}Q_{\beta\}} = 2(\tilde{\varepsilon}_{\{\alpha}\Gamma_{AB}\varepsilon_{\beta\}})R_{AB} - 2(\tilde{\varepsilon}_{\{\alpha}\Gamma_{\mu\nu}\varepsilon_{\beta\}})R_{\mu\nu}. \tag{3.1.22}$$

At a fixed point of space-time rotation, the $SO(4)_S \times SO(6)_R$ generators act on spinors in the S^+ representation of $SO(10)$ as

$$\delta_{R_{MN}}\Psi = \frac{1}{4}R_{MN}\Gamma^{MN}\Psi. \tag{3.1.23}$$

Then there are the following anti-commutation relations

$$\begin{array}{ll}
\{Q_1, Q_1\} = \frac{2}{r}R_{05} - \frac{2}{r}iR_{59} & \{Q_{\bar{1}}, Q_{\bar{1}}\} = \frac{2}{r}R_{05} + \frac{2}{r}iR_{59} \\
\{Q_2, Q_2\} = -\frac{2}{r}R_{05} - \frac{2}{r}iR_{59} & \{Q_{\bar{2}}, Q_{\bar{2}}\} = -\frac{2}{r}R_{05} + \frac{2}{r}iR_{59} \\
\{Q_1, Q_2\} = \frac{2}{r}R_{09} & \{Q_{\bar{1}}, Q_{\bar{2}}\} = -\frac{2}{r}R_{09} \\
\{Q_1, Q_{\bar{1}}\} = -\frac{2}{r}R_{12} & \{Q_1, Q_{\bar{2}}\} = 0 \\
\{Q_2, Q_{\bar{1}}\} = 0 & \{Q_2, Q_{\bar{2}}\} = -\frac{2}{r}R_{12}
\end{array}. \tag{3.1.24}$$

These anticommutation relations can be packed into

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= \frac{2}{r}(C\sigma^I)_{\alpha\beta}R_I \\
\{Q_{\bar{\alpha}}, Q_{\bar{\beta}}\} &= \frac{2}{r}(\overline{C\sigma^I})_{\bar{\alpha}\bar{\beta}}R_I \\
\{Q_\alpha, Q_{\bar{\beta}}\} &= \frac{2}{r}\delta_{\alpha\bar{\beta}}R_0,
\end{aligned} \tag{3.1.25}$$

where σ^I for $I = 1, 2, 3$ are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1.26}$$

The C denotes "the charge conjugation" matrix $C = i\sigma_2$, the triplet of the $SO(3)_B$ generators is denoted by R_I such that $(R_1, R_2, R_3) := (R_{05}, -R_{59}, -R_{09})$, and the $SO(2)_S$ generator is called $R_0 := -R_{12}$.

The fermionic generators Q_α and $Q_{\bar{\alpha}}$ transform naturally in the representation $\mathbf{2}$ and $\bar{\mathbf{2}}$ of the $SO(3)_B$, while $SO(2)_S$ mixes them

$$\begin{aligned} [R_I Q_\alpha] &= -\frac{1}{2} i\sigma^I_{\alpha\beta} Q_\beta \quad & [R_0 Q_\alpha] &= -\frac{1}{2} i C_{\alpha\bar{\beta}} Q_{\bar{\beta}} \\ [R_I Q_{\bar{\alpha}}] &= \frac{1}{2} i\bar{\sigma}^I_{\bar{\alpha}\bar{\beta}} Q_{\bar{\beta}} \quad & [R_0 Q_{\bar{\alpha}}] &= \frac{1}{2} i C_{\bar{\alpha}\beta} Q_\beta. \end{aligned} \tag{3.1.27}$$

The relations (3.1.25) and (3.1.27) are the commutation relations of the Lie algebra $\mathfrak{su}(1|2)$ of the $SU(1|2)$ subgroup of the superconformal group [52]. The bosonic part of $\mathfrak{su}(1|2)$ is $\mathfrak{so}(2)_S \times \mathfrak{so}(3)_B$, spanned by R_0, R_I, the fermionic part is four-dimensional, spanned by $Q_\alpha, Q_{\bar{\alpha}}$.

If we take an arbitrary linear combination of the fermionic generators with complex coefficients $\varepsilon^\alpha, \varepsilon^{\bar{\alpha}} \in \mathbb{C}$,

$$Q = \varepsilon^\alpha Q_\alpha + \varepsilon^{\bar{\alpha}} Q_{\bar{\alpha}}, \tag{3.1.28}$$

we will find that Q squares to a real generator of the $SO(3)_B \times SO(2)_S$ if $\varepsilon^{\bar{\alpha}}$ is actually complex conjugate to ε^α. Such Q will be called hermitian. We will use this fact in the following in our choice of a nice generator Q for the localization computation. We shall also notice that if Q is hermitian, i.e. if $\varepsilon^{\bar{\alpha}}$ is complex conjugate to ε^α, then the norm of the $SO(2)_S$ generator and $SO(3)_B$ generator in Q^2 is proportional to the norm of ε. Hence, a non-zero hermitian Q always squares to a non-zero rotation generator in both $SO(2)_S$ and $SO(3)_B$.

For explicit computations we shall use the following generator Q

$$Q_\varepsilon = \frac{1}{2}(Q_1 + Q_{\bar{1}}). \tag{3.1.29}$$

It corresponds to the conformal Killing spinor associated with

$$\hat{\varepsilon}_s = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes |1\rangle \quad \hat{\varepsilon}_c = \frac{1}{2r} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} \otimes |1\rangle \,. \tag{3.1.30}$$

By (3.1.25) we have

$$Q^2 = \frac{1}{r}(R_{05} - R_{12}). \tag{3.1.31}$$

Clearly, since $[Q^2, Q] = 0$ we have

$$[R_{05} - R_{12}, Q] = 0 \quad \Rightarrow \quad (\Gamma_{05} - \Gamma_{12})\varepsilon(P) = 0. \tag{3.1.32}$$

The last equality is written for the conformal Killing spinor ε associated with Q at the point P in coordinate patch y^μ (3.1.2). The rotation of (y_1, y_2) plane corresponds in the global coordinates to the rotation of (X_5, X_1) plane, or the vector field $\frac{\partial}{\partial \tau}$ in the polar coordinates (3.1.3). Geometrically, the equation (3.1.32) means that the conformal Killing spinor ε is invariant under simultaneous rotation of the (X_5, X_1) plane and (Φ_5, Φ_0) plane.

From the condition (3.1.14) on ε and (3.1.5) it follows that ε is also invariant under diagonal rotations in the $SO(3)_S \times SO(3)_A$. Indeed, from (3.1.14) one has

$$\Gamma_{j+4}\Gamma_{ki}\hat{\varepsilon}_s = \Gamma_{i+4}\Gamma_{jk}\hat{\varepsilon}_s \tag{3.1.33}$$

for distinct i, j, k in the range $2, 3, 4$. Multiplying by $\Gamma_{j+4}\Gamma_{jk}$ both sides of this equation we get

$$\Gamma_{j,i}\hat{\varepsilon}_s = -\Gamma_{j+4,i+4}\hat{\varepsilon}_s, \tag{3.1.34}$$

which shows that ε is invariant under $SO(3)_S$ rotation of (X_2, X_3, X_4) and equal $SO(3)_A$ rotation of (X_6, X_7, X_8) using the isomorphism $\mathbb{R}^3 \to \mathbb{R}^3 : X_i \mapsto X_{i+4}$.

We shall remark that a generic supersymmetric Wilson loop on the S^3 is invariant only under the $OSp(1|2, \mathbb{C})$ subgroup of the complexified superconformal group

$PSU(2,2|4,\mathbb{C})$. The fermionic part of $OSp(1|2,\mathbb{C})$ is spanned by Q_α, i.e. by half of generators of $SU(1|2,\mathbb{C})$. The bosonic part of $\mathfrak{osp}(1|2,\mathbb{C})$ is $\mathfrak{sp}(2,\mathbb{C}) \simeq \mathfrak{su}(2,\mathbb{C})$ spanned by R_I. The commutation relations are represented by the first equation in (3.1.25) and in (3.1.27). However, there is no real structure on $OSp(1|2,\mathbb{C})$ such that it could be a subgroup of compact unitary $SU(1|2,\mathbb{R})$.[1]

So there exists no fermionic element Q in $OSp(1|2,\mathbb{C})$ such that Q^2 generates a unitary transformation in $SU(2)_B$. Since the localization method, which we are going to use in this work, requires that global transformation generated by Q^2 is unitary, we cannot treat the $OSp(1|2,\mathbb{C})$ case and generic Wilson loops on S^3 in the same way. So we restrict the detailed study to the case of Wilson loops on $S^2 \subset S^3$.

Let us summarize. We shall study supersymmetric Wilson loops on the $S^2 \subset S^3$ of the form (3.1.4). These Wilson loops are invariant under the subgroup $SU(1|2)$ of the superconformal group, where $U(1) = SO(2)_S$ rotates (X_1, X_5) plane, and $SU(2) = SU(2)_B$ rotates (Φ_5, Φ_9, Φ_0). The Wilson loops are also invariant under the diagonal of $SO(3)_S \times SO(3)_A$, where $SO(3)_S$ acts on (X_2, X_3, X_4) and $SO(3)_A$ acts on (Φ_6, Φ_7, Φ_8), i.e. on scalar fields entering the Wilson loop.

We choose hermitian generator Q, generated by the conformal Killing spinor ε as (3.1.29). The spinor ε is invariant under the diagonal subgroup of $SO(3)_S \times SO(3)_A$ by (3.1.34) and the diagonal subgroup of $SO(2)_S \times SO(2)_B$ by (3.1.32), where the $SO(2)_B \subset SO(3)_B$ acts on (Φ_5, Φ_0)-plane. .

Remark on 1/4 BPS circular Wilson loops

As discussed above, a Wilson loop (3.1.4) of an arbitrary shape on S^2 preserves 4 out of 32 superconformal symmetries, so it can be called $4/32 = 1/8$ BPS Wilson loop. In [40, 52] it was noted that a circular loop preserves more supersymmetries.

[1] If we use signature for $(5,9,0)$ directions $(+,+,-)$, then, since in this case gamma-matrices can be chosen real, we can get a real structure on $OSp(1|2,\mathbb{R})$ by taking all generators to be real. However, in this case, Q^2 is always light-like generator of the bosonic part of $SO(2,1) \simeq SL(2,\mathbb{R})$.

A Wilson loop on an arbitrary circle on S^2 preserves 8 supersymmetries. A Wilson loop on the equator preserves 16 supersymmetries. The Wilson loop on the equator actually is that circular supersymmetric loop which was studied in [25, 26]. There it was conjectured that expectation value of such operator can be computed in a Gaussian matrix model. In [26] an argument was given that the field theory localizes to matrix model, however that argument does not show that the matrix model is Gaussian. In [57] the Gaussian matrix model was obtain by localization.

In [49] it was conjectured that 1/4 BPS circular Wilson loops also can be computed using the Gaussian matrix model but with a rescaled coupling constant. Such 1/4 BPS circular Wilson loops can be considered as an intermediate case between maximally supersymmetric 1/2 BPS Wilson loops and 1/8 BPS Wilson loops of an arbitrary shape on S^2.

One may ask whether it is possible to directly localize field theory for 1/4 BPS circular Wilson loops to the Gaussian matrix model? We shall note that a new localization computation, different from localization computation for generic Wilson loops on S^2, might be possible only for a single 1/4 BPS Wilson on S^2. In other words, if we take two 1/4 BPS loops located at two distinct lattitudes β_1 and β_2 on S^2, then each Wilson loop preserves eight supesymmetries, but only four supersymmetries are preserved by both loops simultaneously. These four common supersymmetries are actually the same as for a generic 1/8 BPS Wilson loop on S^2. Hence, if we want to compute the connected correlator of two latitudes on S^2, we are back to the case of generic 1/8 BPS loops on S^2, where the four-dimensional theory localizes to a certain two-dimensional theory on S^2.

3.2 Localization

3.2.1 Introduction

We would like to show that expectation value of the Wilson loops (3.1.4) on the S^2 in four-dimensional $\mathcal{N} = 4$ Yang-Mills can be computed by a certain two-dimensional theory localized to S^2. The fermionic symmetry Q (3.1.29) is BRST-like generator of equivariantly cohomological field theory, thanks to the fact that Q squares to global unitary transformation and gauge transformation. This is valid off-shell after adding to the theory the necessary auxiliary fields. Then Q^2 is an off-shell symmetry of the theory and the Wilson loop observable. By well-known arguments, see e.g. [18, 23] for a general review and [57] for technical details on using localization to solve supersymmetric circular Wilson loops in $d = 4$ $\mathcal{N} = 4$ SYM, the theory localizes to the supersymmetric configurations $Q\Psi = 0$, where Ψ denotes fermionic fields of the theory. One can argue localization by deforming the action of the theory by Q-exact term $S_{YM} \to S(t) = S_{YM} + tQV$ with $V = (\Psi, \overline{Q\Psi})$ and sending t to infinity. Since the bosonic part of the deformed action is $S_{YM}^{bos} + t|Q\Psi|^2$, at the $t = +\infty$ limit the term $t|Q\Psi|^2$ dominates. So, at the $t = +\infty$ limit, in the path integral we shall integrate only over configurations solving $Q\Psi = 0$ with the measure coming from the one-loop determinant. On the other hand, the partition function and the expectation value of observables do not depend on the t-deformation. Indeed, let the partition function be $Z(t) = \int e^{S(t)}$. Then, if $S(t)$ is Q-closed and $\partial_t S(t)$ is Q-exact, we can integrate by parts in $\partial_t Z(t)$. If the space of fields is essentially compact (all fields decrease sufficiently fast at infinity) the boundary term vanishes and we obtain $\partial_t Z(t) = 0$.

In the present situation we use $V = (\Psi, \overline{Q\Psi})$. We recall, that Ψ is fermion of $\mathcal{N} = 4$ super Yang-Mills obtained by dimensional reduction of chiral sixteen-component $Spin(10)$ spinor transforming in the S^+ irreducible representation. The other irreducible spin representation S^- of $Spin(10)$ is dual to S^+. Therefore, there is a natural pairing $S^+ \otimes S^- \to \mathbb{C}$, so that if $\psi \in S^+$ and $\chi \in S^-$ are spinors of the opposite chirality, the bilinear (χ, ψ) is $Spin(10)$-invariant. (In components (χ, ψ) should be read as $\sum_{\alpha=1}^{16} \chi_\alpha \psi_\alpha$ with no complex conjugation operations).

In the Euclidean signature the representations S^+ and S^- of $Spin(10, \mathbb{R})$ are unitary, and therefore are complex conjugate to each other. Hence, if $\chi \in S^+$ and $\psi \in S^+$ are spinors of the same chirality, the bilinear $(\bar{\chi}, \psi) = \sum_{\alpha=1}^{16} \overline{\chi}_\alpha \psi_\alpha$ is invariant under $Spin(10, \mathbb{R})$. So, because of our choice of hermitian Q (3.1.29) and because Q squares to unitary global transformation in $SO(2)_S \times SO(2)_B$, the deformation term $V = (\overline{\Psi}, Q\Psi)$ is Q^2-invariant and can be used for the localization.

The localization from the four-dimensional $\mathcal{N} = 4$ SYM on S^4 to two-dimensional theory on $S^2 \subset S^4$ is done essentially in two steps. It is convenient to represent S^4 as $S^2 \times S^1$ warped fibration over an interval I as in (3.1.3).

Step 1. We argue that $Q\Psi = 0$ field configurations are invariant under the $SO(2)_S$ rotations which act by translations along the S^1 fibers: $\tau \to \tau + \text{const}$. Hence, the $\mathcal{N} = 4$ SYM on S^4 localizes to some three-dimensional theory on the manifold D^3 represented as a warped S^2 fibration over I. The metric on D^3 is

$$ds^2 = r^2(d\xi^2 + \sin\xi^2 d\Omega_2^2) \quad \text{where} \quad 0 \leq \xi \leq \pi/2. \tag{3.2.1}$$

One can see that $D^3 = (S^4 \setminus S^2)/SO(2)_S$ is a half of three-dimensional ball. Under the projection $\pi : S^4 \to D^3$ the $S^2 \subset S^4$ maps to the boundary of D^3, which is located at $\xi = \pi/2$.

The resulting three-dimensional theory on the manifold with boundary D^3 reminds a deformed version of certain cohomological field theory for extended Bogomolny equations which appeared in [8]. The interesting observables, i.e. the Wilson loops (3.1.4), are located at the boundary $\partial D^3 \simeq S^2$.

Step 2. We show that physical action S_{YM} for the theory on D^3 can be represented as a total derivative term modulo the equations $Q\Psi = 0$. Therefore, at the supersymmetric configurations, the value of the physical action S_{YM} is determined by the boundary conditions at S^2. The integral over the moduli space of solutions to $Q\Psi = 0$ reduces to an integral over boundary conditions on S^2. This is essentially the way how the two-dimensional theory appears. It turns out that the resulting two-dimensional theory is closely related to topological Higgs-Yang-Mills theory on S^2 studied in [53–55].

3.2.2 Equations

Metric on S^4 is represented as a warped product: $S^4 = D^3 \times_w S^1$. Here $w(\tilde{x})$ is the warp function $w(\tilde{x}) = r^2 \cos^2 \xi = r^2(1 - \tilde{x}^2/(4r^2))^2/(1 + \tilde{x}^2/(4r^2))^2$. On D^3 we introduce the $\mathbb{R}^3$ stereographic projection coordinates $\tilde{x}_i$. The metric in coordinates $\tilde{x}_i, \tau$ has the form

$$ds^2(S^4) = ds^2(D^3 \times_w S^1) = \frac{d\tilde{x}_i d\tilde{x}_i}{(1 + \frac{\tilde{x}^2}{4r^2})^2} + r^2 \frac{(1 - \frac{\tilde{x}^2}{4r^2})^2}{(1 + \frac{\tilde{x}^2}{4r^2})^2} d\tau^2 \quad i = 2, 3, 4 \qquad (3.2.2)$$

We shall remark that the $\mathbb{R}^4$ stereographic coordinates x_i for $i = 1 \ldots 4$ and the $D^3 \times_w S^1$ coordinates $(\tilde{x}_i, \tau)$ for $i = 2, 3, 4$ are related simply at the hypersurface $x_1 = \tau = 0$, there $x_i = \tilde{x}_i$ for $i = 2, 3, 4$. The generic relation between x_i and $(\tau, \tilde{x}_i)$ are the following. From (3.1.1) we have

$$x_i = \frac{2}{1 + X_5/r} X_i \quad \text{for} \quad i = 1 \ldots 4 \qquad (3.2.3)$$

The $SO(2)_S$ orbits are labeled by (X_2, X_3, X_4). The τ is the coordinate along $SO(2)_S$ orbits, and we have

$$\begin{aligned} X_1 &= R \sin\theta \sin\tau \\ X_5 &= R \sin\theta \cos\tau. \end{aligned} \qquad (3.2.4)$$

So, from (3.2.3) we get the $SO(2)_s$ orbits in the $\mathbb{R}^4$ coordinates x_i, and hence, the transformation from coordinates $(\tau, \tilde{x}_i)$ to coordinates (x_1, x_i)

$$\begin{aligned} x_i(\tau, \tilde{x}_i) &= \tilde{x}_i \frac{1 + \sin\theta}{1 + \sin\theta \cos\tau} \quad \text{for} \quad i = 2, \ldots, 4 \\ x_1(\tau, \tilde{x}_i) &= R \frac{2 \sin\theta \sin\tau}{1 + \sin\theta \cos\tau} \end{aligned} \qquad (3.2.5)$$

where

$$\sin\theta = \frac{1-\frac{\tilde{x}^2}{4r^2}}{1+\frac{\tilde{x}^2}{4r^2}}. \tag{3.2.6}$$

These $SO(2)_S$ orbits are actually round circles in the $\mathbb{R}^4$ coordinates x_i, which link with the two-sphere $S^2 = \{x_i | x_2^2 + x_3^2 + x_4^2 = 4r^2, x_1 = 0\}$. The orbits are labeled by points on $D^3 = \{\tilde{x}_i, \tilde{x}^2 < 4r^2\}$. For each $\tilde{x}_i$ the corresponding circle orbit in $\mathbb{R}^4$ is located along the two-plane spanned by vector $(1,0,0,0)$ and vector $(0,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4)$. The distance from the origin to the nearest point of the orbit is $|\tilde{x}|$, the distance to the furthest point is $\frac{4r^2}{\tilde{x}^2}$. The center is located at the point $x_1 = 0, x_i = \tilde{x}_i(\frac{1}{2} + \frac{r^2}{\tilde{x}^2})$. The diameter is $(4r^2 - \tilde{x}^2)/|\tilde{x}|$.

The supersymmetry equations $Q\Psi = 0$ are Weyl invariant. Indeed, given that under Weyl transformation of metric $g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu}$ the bosonic fields transform as $A_\mu \to A_\mu, \Phi_A \to \Phi_A e^{-2\Omega}, K_i \to K_i e^{-4\Omega}$ and the conformal Killing spinor transform as $\varepsilon \to e^{\frac{1}{2}\Omega}\varepsilon$, one gets that $Q_\varepsilon\Psi \to e^{-\frac{3}{2}\Omega} Q_\varepsilon \Psi$ which is a correct scaling dimension for fermions. Therefore, the localization procedure is essentially the same for two theories defined with respect to the metrics related by a smooth Weyl transformation. (We ask transformation to be smooth so that no conformal anomaly related to the infinity can appear).

In the coordinates $(\tilde{x}_i, \tau)$ the $SO(2)_S \times SO(3)_S$ symmetry is explicit, so we shall start from the metric in the form (3.2.2). Since $\tilde{x}$ is bounded $|\tilde{x}| < 2r$, the scale factor $(1+\tilde{x}^2/(4r^2))$ is non-zero and smooth everywhere over the D^3. It is convenient to get rid of this factor in the equations by making Weyl transformation of the metric $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = (1 + \tilde{x}^2/(4r^2))^2 g_{\mu\nu}$. So we shall study the equations $Q\Psi = 0$ on the space $D^3 \times_{\tilde{w}} S^1$ with the metric

$$ds^2(B^3 \times_{\tilde{w}} S^1) = d\tilde{x}_i d\tilde{x}_i + r^2 \left(1 - \frac{\tilde{x}^2}{4r^2}\right)^2 d\tau^2 \quad \text{where} \quad \tilde{x}_i^2 \leq 4r^2 \tag{3.2.7}$$

Here

$$\tilde{w}(x) = r(1 - \tilde{x}^2/(4r^2)) \tag{3.2.8}$$

is the warp factor for the warped product of the flat ball $B^3 \subset \mathbb{R}^3$ (with coordinates $\tilde{x}_i$) and the circle S^1 (with coordinate $\tau \in [0, 2\pi)$). For explicit computations we will use the following vielbein (an orthonormal basis in the cotangent space)

$$(e_i) = (\tilde{w}(x)d\tau, d\tilde{x}_i). \tag{3.2.9}$$

At $\tau = 0$ the coordinates $\tilde{x}_i$ and corresponding vielbein coincide with coordinates x_i. So we use the conformal Killing spinor ε on D^3

$$\varepsilon(\tilde{x}, \tau = 0) = \hat{\varepsilon}_s + \tilde{x}^i \Gamma_i \hat{\varepsilon}_c \tag{3.2.10}$$

to write equations at $\tau = 0$ and then $U(1) \subset SO(2)_S \times SO(2)_B$ symmetry to extend them to arbitrary τ. (The spinor ε on the whole space $D^3 \times_{\tilde{w}} S^1$ is invariant under the diagonal $U(1) \subset SO(2)_S \times SO(2)_B$, i.e. under simultaneous rotation of the (X_5, X_1) and the (Φ_5, Φ_0) planes.) A convenient change of variables with respect to this symmetry is

$$\begin{aligned} \Phi_T &= \cos\tau\Phi_0 - \sin\tau\Phi_5 \\ \Phi_R &= \sin\tau\Phi_0 + \cos\tau\Phi_5. \end{aligned} \tag{3.2.11}$$

Conformal Killing spinor ε satisfies equation

$$\nabla_\mu \varepsilon = \Gamma_\mu \tilde{\varepsilon}. \tag{3.2.12}$$

The off-shell transformation of fermions is given by

$$Q\Psi = \frac{1}{2} F_{MN} \Gamma^{MN} \varepsilon - 2\Phi_A \tilde{\Gamma}^A \tilde{\varepsilon} + i\nu_i K_i. \tag{3.2.13}$$

Explicitly, our choice of ε in components is

$$\varepsilon = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2r} \begin{pmatrix} 0 \\ i \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix} \tag{3.2.14}$$

and for $\tilde{\varepsilon}$ we have

$$\tilde{\varepsilon} = \frac{1}{2r}\begin{pmatrix}0\\0\\0\\i\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\\0\end{pmatrix}. \tag{3.2.15}$$

Also we need 7 auxiliary spinors ν_i which are used to write off-shell closure of the supersymmetry transformations (3.2.13) like in [79, 81]. It is easy to find such set of ν_i because only top 8 components of ε are non-zero. More invariantly, ε satisfies

$$(\Gamma^1 + i\Gamma^0)\varepsilon = 0, \tag{3.2.16}$$

i.e. it is chiral with respect to the $SO(8)$ acting on the vector indices $2, \ldots, 9$. Then, as a set of 7 spinors ν_i, one can choose

$$\nu_i = \Gamma_{9i}\varepsilon \quad \text{for} \quad i = 2, \ldots 8. \tag{3.2.17}$$

Such spinors ν_i also have non-zero only 8 top components.

To compute $Q\Psi$ in components it is convenient to split sixteen component spinors into two eight-component spinors on which Γ^1 acts by $+1$ or -1 respectively. (We will use interchangeably space-time index 1 or τ to denote direction along the coordinate τ in (3.2.7).) According to our choice of gamma-matrices (3.1.15) the eight-component spinors will be called Ψ^t and Ψ^b, so that

$$\Psi = \begin{pmatrix}\Psi^t\\ \Psi^b\end{pmatrix}. \tag{3.2.18}$$

Then we also have

$$\varepsilon = \begin{pmatrix}\varepsilon^t\\ 0\end{pmatrix} \quad \tilde{\varepsilon} = \begin{pmatrix}0\\ \tilde{\varepsilon}^b\end{pmatrix}. \tag{3.2.19}$$

Next, we will represent eight-component spinors Ψ^t and Ψ^b by octonions $\mathbb{O}$. The spinor

$$\Psi^t = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \dots \\ \Psi_8 \end{pmatrix} \tag{3.2.20}$$

we shall write as

$$\Psi^t = \Psi_1^t e_9 + \Psi_2^t e_2 + \cdots + \Psi_8^t e_8, \tag{3.2.21}$$

where $e_9, e_2, \dots, e_8$ are the basis elements of $\mathbb{O}$, see explanation after (3.1.15). Similarly,

$$\Psi^b = \Psi_1^b \tilde{e}_9 + \Psi_2^b \tilde{e}_2 + \cdots + \Psi_8^b \tilde{e}_8, \tag{3.2.22}$$

where $\tilde{e}_9, \tilde{e}_2, \dots, \tilde{e}_8$ are the basis elements in the second copy of $\mathbb{O}$ representing bottom components of Ψ. In these notations

$$\varepsilon = e_9 - \frac{i}{2r}\tilde{x}_i e_{i+4} \tag{3.2.23}$$

and

$$\tilde{\varepsilon} = \frac{i}{2r}\tilde{e}_5. \tag{3.2.24}$$

3.2.3 Bottom equations

First we consider the bottom components of the equations (3.2.13).

Taking into account the chiral structure of gamma-matrices (3.1.15) and spinors $\varepsilon, \tilde{\varepsilon}$ as in (3.2.19), we get

$$\begin{aligned} Q\Psi^b = \sum_{m=\hat{2}\dots\hat{9}} (F_{\hat{0}\hat{m}}\Gamma^{\hat{0}\hat{m}} + F_{\hat{1}\hat{m}}\Gamma^{\hat{1}\hat{m}})\varepsilon - 2\Phi_0\tilde{\Gamma}^0\tilde{\varepsilon} = \\ -(iF_{\hat{0}\hat{m}} + F_{\hat{1}\hat{m}})E_{\hat{m}}\varepsilon + 2i\Phi_0\tilde{\varepsilon} = -(iF_{\hat{0}\hat{m}} + F_{\hat{1}\hat{m}})e_{\hat{m}}(e_9 - \frac{i}{2r}\tilde{x}_i e_{i+4}) + 2i\Phi_0\frac{i}{2r}\tilde{e}_5 \end{aligned} \tag{3.2.25}$$

We use indices with hat to denote vector components with respect to the orthonormal vielbein (3.2.9), e.g. $F_{\hat{1}\hat{m}} = \tilde{w}(x)^{-1}F_{\tau\hat{m}}$. For simplicity we consider equations along the radial line $(\tau, \tilde{x}) = (0, \tilde{x}_2, 0, 0)$. Using $SO(2)_S$ and $SO(3)_S$ symmetry we can extend them to the whole space $B^3 \times_{\tilde{w}} S^1$. At $\tilde{x}_2 < 2r$ six equations corresponding to the components $\hat{m} = 3, 4, 6, 7, 8, 9$ are linearly independent and imply

$$iF_{\hat{0}\hat{m}} + F_{\hat{1}\hat{m}} = 0 \quad \text{for} \quad \hat{m} = 3, 4, 6, 7, 8, 9. \tag{3.2.26}$$

We can make diagonal transformation in $SO(2)_S \times SO(2)_B$ like in (3.2.11) to transform (3.2.26) to an arbitrary τ

$$iF_{\hat{m}T} + \frac{1}{r(1 - \frac{\tilde{x}^2}{4r^2})} F_{\hat{m}\tau} = 0 \quad \hat{m} = 3, 4, 6, 7, 8, 9 \tag{3.2.27}$$

where we replaced index $\hat{1}$ by τ using the scaling function $\tilde{w}(\tilde{x})$, and where $F_{T\hat{m}} = [\Phi_T, \nabla_{\hat{m}}] = -\nabla_{\hat{m}}\Phi_T$. Next we consider the remaining two components in (3.2.25) for basis elements e_2 and e_5. At $\tau = 0$ we have

$$\begin{aligned} iF_{\hat{0}\hat{2}} + F_{\hat{1}\hat{2}} - \frac{i}{2r}\tilde{x}_2(iF_{\hat{0}\hat{5}} + F_{\hat{1}\hat{5}}) = 0 \quad (\text{on } e_2) \\ iF_{\hat{0}\hat{5}} + F_{\hat{1}\hat{5}} + \frac{i}{2r}\tilde{x}_2(iF_{\hat{0}\hat{2}} + F_{\hat{1}\hat{2}}) - \frac{1}{r}\Phi_0 = 0 \quad (\text{on } e_5) \end{aligned} \tag{3.2.28}$$

Again we shift to an arbitrary τ by making diagonal $U(1) \in SO(2)_S \times SO(2)_B$

$$\begin{aligned} (iF_{T\hat{2}} + \tilde{w}^{-1}F_{\tau\hat{2}}) - \frac{i}{2r}\tilde{x}_2(iF_{TR} + \tilde{w}^{-1}(F_{\tau R} - \Phi_T)) = 0 \\ (iF_{TR} + \tilde{w}^{-1}(F_{\tau R} - \Phi_T)) + \frac{i}{2r}\tilde{x}_2(iF_{T2} + \tilde{w}^{-1}F_{\tau 2}) + \frac{1}{r}\Phi_T = 0 \end{aligned} \tag{3.2.29}$$

The first line plus the second multiplied by $i\tilde{x}_2/2r$ is

$$i(1 - \frac{\tilde{x}^2}{4r^2})F_{T2} + \frac{1}{r}F_{\tau 2} + i\frac{\tilde{x}_2}{2r^2}\Phi_T = 0 \tag{3.2.30}$$

Introducing a rescaled field

$$\tilde{\Phi}_T = r(1 - \frac{\tilde{x}^2}{4r^2})\Phi_T, \tag{3.2.31}$$

the equation (3.2.31) is rewritten as

$$i\nabla_2\tilde{\Phi}_T + F_{2\tau} = 0 \tag{3.2.32}$$

The remaining equation from (3.2.29) is then

$$i(1-\frac{\tilde{x}^2}{4r^2})F_{TR}+\frac{1}{r}F_{\tau R}=0 \tag{3.2.33}$$

We can summarize the 8 equations (3.2.27),(3.2.31),(3.2.33) resulting from $Q\Psi^b=0$:

$$[\nabla_{\hat{m}},\nabla_\tau+i\tilde{\Phi}_T]=0 \quad \text{for} \quad \hat{m}=2,3,4,R,6,7,8,9. \tag{3.2.34}$$

One can introduce complexified connection $\nabla^{\mathbb{C}}_\tau=\nabla_\tau+i\tilde{\Phi}_T$ and interpret the equations (3.2.34), as vanishing of the electric field (three equations $F^{\mathbb{C}}_{\tau\hat{i}}=0$ for $i=2,3,4$) and covariant time independence of the remaining five scalars ($\nabla^{\mathbb{C}}_\tau\Phi_{R,6,7,8,9}=0$), assuming τ is the time coordinate.

Since Q^2 generates translations along τ, we can interpret Q^2 as the Hamiltonian. The equations (3.2.34) say that momenta of all fields vanish and that the theory localizes to some three-dimensional theory. This three-dimensional theory is defined on a three-dimensional ball B^3 whose boundary is the two-sphere S^2 where interesting Wilson loop operators are located.

The supersymmetric configurations in this three-dimensional theory are determined by the top eight components of the equations $Q\Psi=0$.

3.2.4 Top equations

For the top eight components of $Q\Psi$ we get

$$\begin{aligned} Q\Psi^t &= F_{\hat{0}\hat{1}}\Gamma^{\hat{0}\hat{1}}\varepsilon^t+\sum_{2\le m<n\le 9}F_{mn}\Gamma^{mn}\varepsilon^t-2\tilde{E}_A\Phi_A\tilde{\varepsilon}^b+\sum_{1\le I\le 8}iK_I\Gamma^{9I}\varepsilon^t= \\ &= -iF_{\hat{0}\hat{1}}\varepsilon^t+(F_{9I}+iK_I)E_I\varepsilon^t-\sum_{2\le I<J\le 8}F_{IJ}E_IE_J\varepsilon^t-2\tilde{E}_A\Phi_A\tilde{\varepsilon}^b \end{aligned} \tag{3.2.35}$$

In the following we shall use indices $I,J=2,\dots,8$ and $i,j,k,p,q=2,\dots,4$. In this section we put $r=1/2$ to avoid extra factors. We also do not write tilde over x understanding that x^i for $i=2,3,4$ are the coordinates on the flat unit ball

$B^3 \subset \mathbb{R}^3$. The antisymmetric symbol ϵ_{ijk} is defined as $\epsilon_{234} = 1$. The following multiplication table of octonions is helpful

$$\begin{aligned}
&e_i e_j = \epsilon_{ijk} e_k - \delta_{ij} e_9 \\
&e_{i+4} e_i = e_5 \qquad e_i e_5 = e_{i+4} \qquad e_5 e_{i+4} = e_i \\
&e_k e_{i+4} = -\epsilon_{kij} e_{j+4} - \delta_{ik} e_5 \quad e_{i+4} e_{j+4} = -\epsilon_{ijk} e_k - \delta_{ij} e_9 \quad e_{j+4} e_k = \delta_{jk} e_5 - \epsilon_{jki} e_{i+4}
\end{aligned} \tag{3.2.36}$$

After some algebra we get the first term

$$Q\Psi^{t(1)} = -iF_{\hat{0}\hat{1}}\varepsilon = -iF_{\hat{0}\hat{1}}(e_9 - ix_j e_{j+4}), \tag{3.2.37}$$

the second term

$$\begin{aligned}
Q\Psi^{t(2)} = (F_{9I} + iK_I)E_I\varepsilon = (F_{9I} + iK_I)e_I(e_9 - ix_j e_{j+4}) = \\
(F_{9i} + iK_i)(e_i + ix^j \epsilon_{ijk} e_{k+4} + ix^j \delta_{ij} e_5) + \\
(F_{95} + iK_5)(e_5 - ix_j e_j) + \\
(F_{9\,i+4} + iK_{i+4})(e_{i+4} + ix^j \epsilon_{ijk} e_k + ix^j \delta_{ij} e_9),
\end{aligned} \tag{3.2.38}$$

the third term

$$\begin{aligned}
Q\Psi^{t(3)} &= -F_{I<J} E_I E_J \varepsilon = \\
&= \left[-\frac{1}{2}(F_{ij} - F_{i+4\,j+4})\epsilon_{ijk} e_k + F_{i\,j+4}\epsilon_{ijk} e_{k+4} + F_{i\,i+4} e_5 - F_{5\,k+4} e_k - F_{k5} e_{k+4} \right] \\
&\quad + i \Big[F_{ij} x_i e_{j+4} + \frac{1}{2} F_{ij} x_k \epsilon_{ijk} e_5 \\
&\quad + F_{i5} x_k \epsilon_{ikj} e_j - F_{i5} x_i e_9 - F_{i\,j+4} x_i e_j - F_{i\,j+4} x_j e_i + F_{i\,i+4} x_k e_k + F_{i\,j+4} x_k \epsilon_{ijk} e_9 \\
&\quad + F_{5,i+4} x_k \epsilon_{ikj} e_{j+4} - F_{5j+4} x_j e_5 + F_{i+4\,j+4} x_i e_{j+4} - \frac{1}{2} F_{i+4j+4} \epsilon_{ijk} x_k e_5 \Big]
\end{aligned} \tag{3.2.39}$$

and the fourth term

$$Q\Psi^{t(4)} = -2\tilde{E}_A \Phi_A \tilde{\varepsilon}^b = -2i(\Phi_9 e_5 + \Phi_5 e_9 + \Phi_{i+4} e_i). \tag{3.2.40}$$

Now we analyze the equations. We have eight complex (sixteen real) equations on eight physical fields $A_{2,3,4}, \Phi_{R,6,7,8,9}$ and seven auxiliary fields K_i. Here we shall impose the reality condition on all fields. That is also clear from the localization argument. Indeed, the action is deformed by $t|Q\Psi|^2$ which vanishes on the real integration contour iff both imaginary and complex parts of $Q\Psi$ vanish. Hence, the top equations $Q\Psi^t = 0$ naively imply 16 real equations. We shall see shortly that only 15 equations are independent. Seven auxiliary fields can be easily integrated out. Then we are left with eight equations. One of these eight equations gives real constraint on the complexified time connection:

$$[\nabla_\tau, \tilde{\Phi}_T] = 0. \tag{3.2.41}$$

(This equation together with (3.2.34) completes our claim that the field configurations are all τ-invariant up to a gauge transformation).

What remains is the system of seven first order differential equations on three components of the gauge field and five scalars. The equations are gauge invariant. Modulo gauge transformations, the system is elliptic in the interior of the three-dimensional ball B^3. The system is closely related to the extended three-dimensional Bogomolny equations studied in [74].

Now we shall give technical details on the equations. First we eliminate $\operatorname{Im} Q\Psi^t|e_9$ by adding to it $-x_i \operatorname{Re} Q\Psi|_{e_{i+4}}$

$$\begin{aligned} \operatorname{Im} Q\Psi|e_9 - x_i \operatorname{Re} Q\Psi|_{e_{i+4}} = -F_{\hat{0}\hat{1}} + F_{9\,i+4}x_i - F_{i5}x_i + F_{ij+4}x_k\epsilon_{ijk} - 2\Phi_5 \\ - (-F_{\hat{0}\hat{1}}x^2 + F_{9\,i+4}x_i - F_{i5}x_i + F_{i\,j+4}x_k\epsilon_{ijk}) = \\ = -F_{\hat{0}\hat{1}}(1-x^2) - 2\Phi_5 = 2[\nabla_\tau \Phi_T] \end{aligned} \tag{3.2.42}$$

This is the real equation which completes the system of time-invariance equations (3.2.34).

Next we consider $\operatorname{Re} Q\Psi^t|_{e_9}$:

$$\operatorname{Re} Q\Psi^t|e_9 = -K_{i+4}x_i \tag{3.2.43}$$

This equation is one constraint on the auxiliary fields K_i. We are left with 14 more equations $\operatorname{Im} Q\Psi^t|_{e_I} = 0$ and $\operatorname{Re} Q\Psi^t|_{e_I} = 0$ for $I = 2, \ldots, 8$. Using $\operatorname{Im} Q\Psi^t|_{e_I} = 0$

we shall solve for K_I in terms of the physical fields A and Φ, and we will see actually that the constraint (3.2.43) is automatically implied.

The seven equations $\mathrm{Im}\, Q\Psi^t|_{e_I} = 0$ imply

$$\begin{aligned} K_k &= F_{95}x_k - F_{9\,i+4}\epsilon_{ijk}x_j - F_{i5}x_j\epsilon_{ijk} + F_{i\,k+4}x_i + F_{k\,i+4}x_i - F_{i\,i+4}x_k + 2\Phi_{k+4} \\ K_5 &= -F_{9i}x_i - \frac{1}{2}F_{ij}x_k\epsilon_{ijk} + F_{5\,j+4}x_j + \frac{1}{2}F_{i+4\,j+4}x_k\epsilon_{ijk} + 2\Phi_9 \\ K_{k+4} &= -F_{9i}\epsilon_{ijk}x_j - F_{ik}x_i - F_{5\,i+4}x_j\epsilon_{ijk} - F_{i+4\,k+4}x_i. \end{aligned} \tag{3.2.44}$$

The seven components $\mathrm{Re}\, Q\Psi^t|_{e_I} = 0$ are

$$\begin{aligned} \mathrm{Re}\, Q\Psi^t|_{e_k} &= F_{9k} - \frac{1}{2}(F_{ij} - F_{i+4\,j+4})\epsilon_{ijk} - F_{5\,k+4} + K_5x_k - K_{i+4}x_j\epsilon_{ijk} \\ \mathrm{Re}\, Q\Psi^t|_{e_5} &= F_{95} + F_{i\,i+4} - K_ix_i \\ \mathrm{Re}\, Q\Psi^t|_{e_{k+4}} &= F_{9\,k+4} + F_{i\,j+4}\epsilon_{ijk} - F_{k5} + 2\Phi_5(1-x^2)^{-1}x_k - K_ix_j\epsilon_{ijk} \end{aligned} \tag{3.2.45}$$

After plugging in (3.2.45) the expressions for K_I (3.2.44) we get

$$\begin{aligned} \mathrm{Re}\, Q\Psi^t|_{e_k} &= F_{9k}(1-x^2) - \frac{1}{2}F_{ij}\epsilon_{ijk}(1+x^2) + \frac{1}{2}F_{i+4\,j+4}\epsilon_{ijp}(\delta_{pk} - x^2\delta_{pk} + 2x_px_k) - \\ &\qquad F_{5\,j+4}(\delta_{jk} + x^2\delta_{jk} - 2x_jx_k) + 2\Phi_9x_k \\ \mathrm{Re}\, Q\Psi^t|_{e_5} &= F_{95}(1-x^2) + F_{i\,j+4}(\delta_{ij} + \delta_{ij}x^2 - 2x_ix_j) - 2\Phi_{j+4}x_j \\ \mathrm{Re}\, Q\Psi^t|_{e_{k+4}} &= F_{9\,i+4}(\delta_{ik} + x_ix_k - x^2\delta_{ik}) - F_{i5}(\delta_{ik} - x_ix_k + x^2\delta_{ik}) + 2\Phi_5(1-x^2)^{-1}x_k \\ &\qquad + F_{i\,j+4}(\epsilon_{ijk} - x_ix_p\epsilon_{jpk} - x_jx_p\epsilon_{ipk}) - 2\Phi_{i+4}\epsilon_{ijk}x_je_{k+4} \end{aligned} \tag{3.2.46}$$

The above calculations are done at the slice $\tau = 0$. For an arbitrary τ the field Φ_5 should be replaced by Φ_R as in (3.2.11).

Let us analyze the equations $\mathrm{Re}\, Q\Psi^t|_{e_I} = 0$ using (3.2.46). At the origin, i.e. at $x = 0$, the equations are

$$- * (F - \Phi \wedge \Phi) - d_A\Phi_9 + [\Phi, \Phi_5] = 0 \tag{3.2.47}$$

$$*d_A\Phi - d_A\Phi_5 - [\Phi, \Phi_9] = 0 \tag{3.2.48}$$

$$d_A * \Phi + [\Phi_9, \Phi_5] = 0 \tag{3.2.49}$$

where we identified the three scalar fields Φ_{i+4} with components of one-form on $\mathbb{R}^3$, i.e. $\Phi = \Phi_{i+4} dx^i$, and $*$ is the Hodge operator on $\mathbb{R}^3$ with the standard metric.

Let us combine the gauge field A and the one-form Φ into a complexified connection $A_c = A + i\Phi$, and similarly combine the scalars Φ_5 and Φ_9 into complexified scalar $\Phi_c = \Phi_5 + i\Phi_9$. Then the equations (3.2.47)(3.2.48) can be written as

$$- * \operatorname{Re} F_c - \operatorname{Re} d_{A_c} \Phi_c = 0 \tag{3.2.50}$$

$$* \operatorname{Im} F_c - \operatorname{Im} d_{A_c} \Phi_c = 0. \tag{3.2.51}$$

This pair of real equations can be combined into the complex equation

$$* \overline{F_c} + d_{A_c} \Phi_c = 0. \tag{3.2.52}$$

The equation (3.2.52) was called extended Bogomolny equation in [8].

Hence, we see that at the origin of $\mathbb{R}^3$, the equations (3.2.46) resemble some known system of elliptic equations. Away from $x = 0$ the equations are deformed into more complicated system. We shall try to make some simple rescaling of variables to convert the equations to more standard form on the whole domain.

For this purpose we make a change of variables for the scalar fields Φ_{i+4} for $i = 2, 3, 4$

$$\Phi_{i+4} = \tilde{\Phi}_j \left(\delta_{ij} + \frac{2x_i x_j}{1 - x^2} \right). \tag{3.2.53}$$

This change of variables is smooth in the interior of the ball B^3. In terms of $\tilde{\Phi}_{i+4}$ the first equation in (3.2.46) becomes

$$-\frac{1}{2}(1 + x^2)\epsilon_{ijk}(F_{ij} - [\tilde{\Phi}_i, \tilde{\Phi}_j]) - \nabla_k((1 - x^2)\Phi_9) + (1 + x^2)[\tilde{\Phi}_k, \Phi_5] = 0. \tag{3.2.54}$$

The second equation in (3.2.46) becomes

$$(1 - x^2)\epsilon_{ijk}\nabla_i \tilde{\Phi}_j - \nabla_k((1 - x^2)\Phi_5) - \frac{1 - x^2}{1 + x^2}\left((1 - x^2)\delta_{ik} + \frac{4x_i x_k}{1 - x^2} \right)[\tilde{\Phi}_i, \tilde{\Phi}_9] = 0. \tag{3.2.55}$$

Finally, the third equation in (3.2.46) is

$$(1 + x^2)\nabla_i \tilde{\Phi}_i + 2\frac{x^2 + 3}{1 - x^2} x_i \tilde{\Phi}_i + (1 - x^2)[\Phi_9, \Phi_5] = 0 \tag{3.2.56}$$

In the localization computation we need to integrate over the moduli space $\mathcal{M}$ of smooth solutions to (3.2.54)(3.2.55)(3.2.56) with finite Yang-Mills action. Clearly, the zero configuration $A = \tilde{\Phi} = 0, \Phi_5 = \Phi_9 = 0$ is a solution. Let us analyze the linearized problem near the zero configuration, in other words, let us find the fiber of the tangent space $T\mathcal{M}_0$. The linearized equations (3.2.54),(3.2.55),(3.2.56) are

$$(1+x^2) *_{\mathbb{R}^3} dA + d((1-x^2)\Phi_9) = 0 \qquad (3.2.57)$$

$$(1-x^2) *_{\mathbb{R}^3} d\tilde{\Phi} - d((1-x^2)\Phi_5) = 0 \qquad (3.2.58)$$

$$(1+x^2) d^*_{\mathbb{R}^3}\tilde{\Phi} + 2\frac{x^2+3}{1-x^2}(x,\tilde{\Phi}) = 0 \qquad (3.2.59)$$

Here we by $*_{\mathbb{R}^3}$ we denoted the Hodge star operation with respect to the standard metric on R^3. It is possible to absorb extra $(1 \pm x^2)$ factors in the Hodge star operation using a rescaled metric. We will use the metric

$$ds^2(S^3) = \frac{dx_i dx_i}{(1+x^2)^2}, \quad |x| < 1 \qquad (3.2.60)$$

which is a metric on a half of round three-sphere S^3, and

$$ds^2(H_3) = \frac{dx_i dx_i}{(1-x^2)^2}, \quad |x| < 1 \qquad (3.2.61)$$

which is a metric on hyperbolic space H^3 in Poincare coordinates. Then the first two equations in (3.2.57) turn into

$$*_{S^3} dA + d\tilde{\Phi}_9 = 0 \qquad (3.2.62)$$

$$*_{H^3} d\tilde{\Phi} - d\tilde{\Phi}_5 = 0, \qquad (3.2.63)$$

where

$$\tilde{\Phi}_5 = (1-x^2)\Phi_5 \qquad (3.2.64)$$

$$\tilde{\Phi}_9 = (1-x^2)\Phi_9. \qquad (3.2.65)$$

The equation (3.2.62) implies that $\tilde{\Phi}_9$ is harmonic for the S^3 metric

$$\Delta_{S^3}\tilde{\Phi}_9 = 0, \qquad (3.2.66)$$

and the equation (3.2.63) implies that $\tilde{\Phi}_5$ is harmonic for the H_3 metric

$$\Delta_{H^3}\tilde{\Phi}_5 = 0. \tag{3.2.67}$$

We need to consider only such solutions that the fields Φ_5, Φ_9 are not singular at the boundary. (Singular solutions can be considered too, but they correspond to the disorder surface operator [103] inserted on the two-sphere $S^2 = \partial B^3$. In this work we aim to compute the expectation value of Wilson loop operators on S^2 in the absence of any surface operators. Hence we require Φ_5 and Φ_9 fields to be finite at the S^2.) If Φ_5 and Φ_9 fields are finite at $|x| = 1$, then $\tilde{\Phi}_5$ and $\tilde{\Phi}_9$ vanish there by (3.2.66),(3.2.67). Hence we have the Laplacian problem (3.2.66)(3.2.67) with Dirichlet boundary conditions

$$\tilde{\Phi}_5|_{\partial B^3} = \tilde{\Phi}_9|_{\partial B^3} = 0. \tag{3.2.68}$$

Since a harmonic function $Y(x)$ vanishing on the boundary must vanish (it can be shown integrating by parts $\int_B dY \wedge *dY = \int_{\partial B} Y \wedge *dY$), we conclude that there is no nontrivial finite solution for the fields Φ_5, Φ_9, so

$$\Phi_5 = \Phi_9 = 0. \tag{3.2.69}$$

One might worry that this argument might fail for the H^3 space where we have to deal with the infinite boundary. However, the explicit solution of the Laplace equation in spherical coordinates on the H^3 space shows that all radial wave-functions, which are smooth in the interior of H^3, do not vanish at the boundary. In spherical coordinates, the H^3 metric is

$$ds^2 = \frac{d\xi^2 + \sin^2\xi d\Omega_2^2}{\cos^2\xi} \tag{3.2.70}$$

where ξ is the radial coordinate $0 \le \xi < \pi/2$ and $d\Omega_2^2$ is the standard metric on the unit two-sphere. Then

$$\Delta_{H^3} f = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j)f = \frac{\cos^3\xi}{\sin^2\xi}\partial_\xi\left(\frac{\sin^2\xi}{\cos\xi}\partial_\xi f\right) + \frac{\cos^2\xi}{\sin^2\xi}\Delta_{S^2} f \tag{3.2.71}$$

If $f_s(\xi)$ is the radial wave-function for the angular momentum s on the S^2 then $\Delta_{S^2} f_s = -s(s+1)f_s$. So the equation (3.2.71) is a special case of the Laplace equation in the (p,q) polyspherical coordinates (see e.g. [104] p.499)

$$\frac{1}{\cos^p\xi\sin^q\xi}\frac{\partial}{\partial\xi}\left(\cos^p\xi\sin^q\xi\frac{\partial u}{\partial\xi}\right)-\left(\frac{r(r+p-1)}{\cos^2\xi}+\frac{s(s+q-1)}{\sin^2\xi}-l(l+p+q)\right)u=0 \tag{3.2.72}$$

for $q=2, p=-1, r=0, l=0$. The solutions of (3.2.72) non-singular at $\xi=0$ are

$$u=\tan^s\xi F\left(\frac{s-l+r}{2},\frac{s-l-r-p+1}{2},s+\frac{q+1}{2};-\tan^2\xi\right), \tag{3.2.73}$$

where $F(\alpha,\beta,\gamma;z)$ is the ${}_2F_1$ hypergeometric function. In our case we have

$$f_s(\xi)=\tan^s\xi F(s/2,s/2+1,s+3/2,-\tan^2\xi). \tag{3.2.74}$$

Using identity

$$F(\alpha,\beta,\gamma,z)=(1-z)^{-\alpha}F(\alpha,\gamma-\beta,\gamma;\frac{z}{z-1}) \tag{3.2.75}$$

we can rewrite (3.2.74) as

$$f_s(\xi)=\sin^s\xi F(s/2,s/2+1/2,s+3/2,\sin^2\xi) \tag{3.2.76}$$

The function $f_s(\xi)$ has asymptotic ξ^s at $\xi\longrightarrow 0$ and a finite non-zero value at $\xi=\pi/2$:

$$\lim_{\xi\to\pi/2}f_s(\xi)=\frac{\Gamma(s+3/2)\Gamma(1)}{\Gamma(s/2+3/2)\Gamma(s/2+1)}. \tag{3.2.77}$$

This confirms our argument that there are no non-trivial solutions to the Laplace equation on H^3 with zero asymptotic at the boundary.

Now, given that Φ_5 and Φ_9 vanish, the linearized equations (3.2.62)(3.2.63) turn into

$$dA=0 \tag{3.2.78}$$

$$d\Phi=0. \tag{3.2.79}$$

That means that the complexified gauge connection $A_c = A + i\Phi$ is flat. The third equation in (3.2.57) is effectively a partial gauge fixing condition on the imaginary part of A_c. It is actually possible to rewrite this partial gauge fixing condition in terms of the d^* operator with respect to a rescaled metric. Namely, for this metric on $\mathbb{R}^3$

$$g_{ij} = f(|x|)\delta_{ij} \tag{3.2.80}$$

the d^*_f operator acts on one-form $\tilde{\Phi}$ as

$$d^*_f\tilde{\Phi} = f^{-1}(\partial_i\tilde{\Phi}_i + \frac{1}{2}f^{-1}f'\tilde{\Phi}_i x_i/|x|), \tag{3.2.81}$$

where $f' = df(|x|)/dx$. Comparing (3.2.81) with (3.2.59) we get the scale factor. The result is

$$g_{ij} = f(|x|)\delta_{ij} \quad \text{where} \quad f(|x|) = \frac{(1+x^2)^2}{(1-x^2)^4}. \tag{3.2.82}$$

Hence, the partial gauge fixing equation (3.2.59) is rewritten as

$$d^*_f\tilde{\Phi} = 0 \tag{3.2.83}$$

Now we can find all solutions to the linearized problem as follows. From (3.2.79) we solve $\tilde{\Phi}$ in terms of some scalar potential p

$$\tilde{\Phi} = dp. \tag{3.2.84}$$

The gauge fixing equation (3.2.83) implies then

$$d^*_f dp = 0, \tag{3.2.85}$$

i.e. that p is a harmonic function with respect to the metric (3.2.82). We can find explicitly the harmonic modes in spherical coordinates. The metric (3.2.80) is

$$ds^2 = \frac{d\xi^2 + \sin^2\xi d\Omega_2^2}{\cos^4\xi}, \tag{3.2.86}$$

so the Laplacian equation (3.2.85) on spherical mode $p_s(\xi)$ with angular momentum s is

$$\cot^2\xi\frac{\partial}{\partial\xi}\left(\tan^2\xi\frac{\partial p_s(\xi)}{\partial\xi}\right) - \frac{s(s+1)}{\sin^2\xi}p_s(\xi) = 0. \tag{3.2.87}$$

Again, this is the Laplacian equation in the (p,q) polyspherical coordinates (3.2.72) with $p=-2, q=2, r=0, l=0$. The solution regular at $\xi=0$ is

$$\begin{aligned} p_s(\xi) = \tan^s \xi F(s/2, s/2+3/2, s+3/2, -\tan^2\xi) = \\ = \sin^s \xi F(s/2, s/2, s+3/2, \sin^2\xi). \end{aligned} \tag{3.2.88}$$

The solution is finite at $\xi=0$ for any s, hence the components of $\tilde{\Phi}$ tangent to the boundary ∂B^3 are also finite. To find asymptotic of the normal component of $\tilde{\Phi}$ we need to know expansion of (3.2.88) at $\theta=\pi/2-\xi$ at $\theta=0$. For this purpose we rewrite (3.2.88) using identity on hypergeometric functions (see e.g. [105] p.160)

$$\begin{aligned} F(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} F(\alpha,\beta,\alpha+\beta-\gamma+1,1-z) \\ + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1,1-z). \end{aligned} \tag{3.2.89}$$

We get

$$\begin{aligned} p_s(\xi) = \sin^s(\xi)\Bigg(\frac{\Gamma(s+3/2)\Gamma(3/2)}{\Gamma(s/2+3/2)^2} F(s/2, s/2, -1/2, \cos^2\xi) \\ + \frac{\Gamma(s+3/2)\Gamma(-3/2)}{\Gamma(s/2)^2}(\cos^2\xi)^{3/2} F(s/2+3/2, s/2+3/2, 5/2, \cos^2\xi)\Bigg) \end{aligned} \tag{3.2.90}$$

Near $\theta=0$ we obtain

$$p_s(\theta) = \cos^s\theta(A + B\sin^2\theta + C\sin^3\theta + \dots), \tag{3.2.91}$$

where A, B, C some constants. Therefore

$$\tilde{\Phi}_\theta = \frac{\partial p_s(\theta)}{\partial\theta} = (-As + B)\theta + O(\theta^2). \tag{3.2.92}$$

This means that the normal component of $\tilde{\Phi}$ at the boundary vanishes as the first power of θ or $(1-x^2)$. Hence, the original scalars, related to $\tilde{\Phi}$ by (3.2.53), are all finite at the boundary S^2.

So all solutions of the linearized equations (3.2.57)(3.2.58)(3.2.59) modulo gauge transformations are parametrized by the scalar potential p (modulo zero modes of

p), which is a harmonic function in the three-dimensional ball with respect to the metric (3.2.86). A harmonic functions p is uniquely defined by its boundary value on the S^2. Hence we see that that tangent space to the moduli space of solutions at the origin $T\mathcal{M}_0$ is equal to the space of adjoint-valued scalar functions on the S^2 modulo zero modes.

Now we consider the full non-abelian equations (3.2.54)(3.2.55)(3.2.56). Looking back at our solution of the linearized problem (3.2.69), we shall suggest an ansatz $\Phi_5 = \Phi_9 = 0$ for the exact solution. Then the remaining equations on the complexified connection $A_c = A + i\tilde{\Phi}$ are

$$F_A - \tilde{\Phi} \wedge \tilde{\Phi} = 0 \tag{3.2.93}$$

$$d_A \tilde{\Phi} = 0 \tag{3.2.94}$$

$$d_A^{*_f} \tilde{\Phi} = 0, \tag{3.2.95}$$

which can be combined into the complex equation of flat curvature

$$F(A_c) = 0 \tag{3.2.96}$$

and a partial gauge-fixing equation using the metric (3.2.86)

$$d_A *_f \tilde{\Phi} = 0. \tag{3.2.97}$$

The first equation can be solved in terms of a scalar function valued in the complexified gauge group $g_c : B^3 \to G^{\mathbb{C}}$

$$A_c = g_c^{-1} dg_c. \tag{3.2.98}$$

The partial gauge-fixing condition can be complemented by a real gauge fixing $d^* A = 0$. That gives a non-linear analogue of the harmonic equation (3.2.85)

$$d_A *_f (g_c^{-1} dg_c) = 0. \tag{3.2.99}$$

The solutions of this second order differential equation are parameterized by the boundary value of g_c. Hence, the tangent space of solutions to the full non-abelian equations constrained by $\Phi_5 = \Phi_9 = 0$ coincides with the moduli space of the linearized problem.

We conclude, that the solutions of (3.2.99) represent complete moduli space $\mathcal{M}$ of finite solutions of the supersymmetry equations (3.2.46). Hence, the space of gauge orbits $\mathcal{M}/G_{gauge}$ can be parameterized by the boundary value of the $G^{\mathbb{C}}/G$-valued potential function g_c.

Equivalently, we can parameterize $\mathcal{M}/G$ by the space of complex flat connections on the boundary S^2 modulo the gauge transformations restricted on S^2

$$A_c|_{S^2} = g_c^{-1} dg_c|_{S^2}. \tag{3.2.100}$$

Hence, the localization of the path integral of the four-dimensional $\mathcal{N} = 4$ SYM theory to the moduli space $\mathcal{M}/G$ can be represented by a path integral over the space of complex flat connections on the B^3 boundary S^2. The action of this two-dimensional theory is determined by values of the four-dimensional Yang-Mills functional on the field configurations representing points on $\mathcal{M}$.

We will show, that the $\mathcal{N} = 4$ Yang-Mill action restricted to the fields configurations in $\mathcal{M}$ is actually a total derivative. Hence the Yang-Mills action S_{YM} restricted to $\mathcal{M}$ can be represented in terms of a two-dimensional functional on the boundary S_{2d}.

We conclude that the result of the localization procedure for the partition function of the four-dimensional theory is a two-dimensional path integral over the space of complex flat connections on S^2.

Now we will find the two-dimensional action S_{2d}. The measure of integration in the two-dimensional theory is then $\exp(-S_{2d})$ times the induced volume form from the four-dimensional theory on the moduli space $\mathcal{M}$.

3.3 Two-dimensional theory

The bosonic part of the $\mathcal{N}=4$ Yang-Mills action on S^4 in coordinates (3.2.2) is

$$S_{YM} = \frac{1}{2g_{YM}^2}\int_0^{2\pi} d\tau \int_{|x|<1} d^3x\sqrt{g}(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\Phi_A D^\mu\Phi_A + \frac{1}{2}[\Phi_A,\Phi_B]^2 + \frac{R}{6}\Phi_A^2 + K^2). \tag{3.3.1}$$

Here R is the scalar curvature. For S^4 of radius 1/2 we have $R = 12/(1/2)^2 = 48$. First we make Weyl transformation to get the action on the space with metric (3.2.7)

$$g[S^4] = e^{2\Omega}g[\mathbb{R}^3 \times_{\tilde{w}} S^1] \tag{3.3.2}$$

$$\Phi_A[S^4] = e^{-\Omega}\Phi_A[\mathbb{R}^3 \times_{\tilde{w}} S^1] \tag{3.3.3}$$

$$K_I[S^4] = e^{-2\Omega}K_I[\mathbb{R}^3 \times_{\tilde{w}} S^1] \tag{3.3.4}$$

where

$$e^{2\Omega} = (1+x^2)^{-2}. \tag{3.3.5}$$

In terms of the fields on $\mathbb{R}^3 \times_{\tilde{w}} S^1$ the bosonic action is

$$S_{YM} = \frac{1}{2g_{YM}^2}\int_0^{2\pi} d\tau \int_{|x|<1} d^3x \left(\frac{1}{2}(1-x^2)\times (\frac{1}{2}F_{ij}^2 + g^{\tau\tau}F_{\tau i}^2 + g^{\tau\tau}(D_\tau\Phi_A)^2 + (D_i\Phi_A)^2 + \frac{2}{(1-x^2)}\Phi_A^2 + \frac{1}{2}[\Phi_A\Phi_B]^2 + K^2) + D_i\left(\frac{1-x^2}{1+x^2}x_i\Phi_A^2\right)\right) \tag{3.3.6}$$

The last term is the total derivative which vanishes because the factor $(1-x^2)$ vanishes at the integration boundary $|x|=1$. The action on $\mathbb{R}^3 \times_{\tilde{w}} S^1$ can be also written starting from (3.3.1) and substituting the metric (3.2.7). The scalar curvature on $\mathbb{R}^3 \times_{\tilde{w}} S^1$ can be computed easily using a general formula for the scalar curvature on a warped product of two manifold $M \times_f N$, see e.g. [106]. If g_M and g_N are the metrics on M and N, and if $g_M \oplus f^2 g_N$ is the metric on $M \times_f N$, then

$$R_{M\times_f N}u = -\frac{4n}{n+1}\Delta_M u + R_M u + R_N u^{\frac{n-3}{n+1}} \quad \text{where} \quad n = \dim N, \quad u = f^{\frac{n+1}{2}}, \quad \Delta_M \text{ is Laplacian on M} \tag{3.3.7}$$

In the case $\mathbb{R}^3 \times_{\tilde{w}} S^1$ we get $n = \dim N = 1$, so $u = f = \frac{1}{2}(1 - x^2)$. Then, for the radius 1/2, we get

$$R[\mathbb{R}^3 \times_{\tilde{w}} S^1] = -u^{-1}\Delta u = \frac{12}{1 - x^2}, \tag{3.3.8}$$

which agrees with (3.3.1) and (3.3.6).

Next we rewrite the action in terms of the twisted scalars Φ_T, Φ_R and Φ_m for $m = 6, 7, 8, 9$ (3.2.11)

$$\begin{aligned} S_{YM} = \frac{1}{2g_{YM}^2} \int_0^{2\pi} d\tau \int_{|x|<1} d^3x \frac{1}{2}(1 - x^2)(g^{\tau\tau} F_{i\tau}^2 + (D_i\Phi_T)^2 \\ + g^{\tau\tau}(D_\tau\Phi_R - \Phi_T)^2 + [\Phi_T, \Phi_R]^2 + g^{\tau\tau}(D_\tau\Phi_m)^2 + [\Phi_T, \Phi_m]^2 \\ g^{\tau\tau}(D_\tau\Phi_T + \Phi_R)^2 + \frac{1}{2}F_{ij}^2 + (D_i\Phi_m)^2 + (D_i\Phi_R)^2 + \frac{1}{2}[\Phi_m, \Phi_n]^2 + [\Phi_R, \Phi_m]^2 \\ + \frac{2}{(1 - x^2)}(\Phi_m^2 + \Phi_T^2 + \Phi_R^2) + K_I^2). \end{aligned} \tag{3.3.9}$$

Then we restrict the action onto the $U(1) \subset SO(2)_S SO(2)_B$ invariant configurations using (3.2.34) and (3.2.41). We also assume that $\Phi_T = 0$ in the supersymmetric background, otherwise Φ_T has first order singularity near the S^2 which would mean insertion of surface operator. Removing all commutators with ∇_τ and Φ_T from the action (3.3.9), we arrive to this three-dimensional action for the gauge field A_i and five scalars Φ_R, Φ_m for $m = 6, 7, 8, 9$

$$\begin{aligned} S_{YM}^{\text{inv}}(B^3) = \frac{1}{2g_{YM}^2} 2\pi \int_{|x|<1} d^3x \frac{1}{2}(1 - x^2)(\frac{4}{(1 - x^2)^2}\Phi_R^2 + \frac{1}{2}F_{ij}^2 + (D_i\Phi_m)^2 \\ + (D_i\Phi_R)^2 + \frac{1}{2}[\Phi_m, \Phi_n]^2 + [\Phi_R, \Phi_m]^2 + \frac{2}{(1 - x^2)}(\Phi_m^2 + \Phi_R^2) + K_I^2). \end{aligned} \tag{3.3.10}$$

Now let us show that modulo supersymmetry equations the action (3.3.10) is a total derivative. We try the following ansatz

$$S^{\text{inv}}_{susy}(B^3) = \frac{1}{4g^2_{YM}} 2\pi \int_{|x|<1} d^3x$$
$$((-\frac{1}{2}(F_{ij} - [\Phi_{i+4}\Phi_{j+4}])\epsilon_{ijk} + K_5 x_k - K_{i+4}x_j\epsilon_{ijk})\cdot$$
$$\cdot(-\frac{1}{2}(F_{ij} - [\Phi_{i+4}\Phi_{j+4}])\epsilon_{ijk} - K_5 x_k + K_{i+4}x_j\epsilon_{ijk})$$
$$+ (\nabla_i\Phi_{i+4} - K_i x_i)(\nabla_j\Phi_{j+4} + K_j x_j)$$
$$+ ((\nabla_i\Phi_{j+4} - K_i x_j)\epsilon_{ijk})(\delta_{k\tilde{k}} - x_k x_{\tilde{k}})((\nabla_{\tilde{i}}\Phi_{\tilde{j}+4} + K_{\tilde{i}}x_{\tilde{j}})\epsilon_{\tilde{i}\tilde{j}\tilde{k}})$$
$$+ (K_k - (x_i\nabla_i\Phi_{k+4} + x_i\nabla_k\Phi_{i+4} - x_k\nabla_i\Phi_{i+4} + 2\Phi_{k+4}))\cdot$$
$$\cdot(K_k + (x_i\nabla_i\Phi_{k+4} + x_i\nabla_k\Phi_{i+4} - x_k\nabla_i\Phi_{i+4} + 2\Phi_{k+4}))$$
$$+ (K_5 + \frac{1}{2}x_k\epsilon_{ijk}(F_{ij} - [\Phi_{i+4}\Phi_{j+4}]))(K_5 - \frac{1}{2}x_k\epsilon_{ijk}(F_{ij} - [\Phi_{i+4}\Phi_{j+4}]))$$
$$+ (K_{k+4} + x_i(F_{ik} + [\Phi_{i+4}\Phi_{k+4}]))(K_{k+4} - x_i(F_{ik} + [\Phi_{i+4}\Phi_{k+4}]))$$
$$- (x_i K_{i+4})^2) \quad (3.3.11)$$

Each term above corresponds to one of the top supersymmetry equations (3.2.43),(3.2.44) and (3.2.45) multiplied by a suitable factor to match the kinetic term of the reduced Yang-Mills action (3.3.10). Therefore at the supersymmetric configurations $S^{\text{inv}}_{susy}(B^3)$ vanishes. On the other hand, after some algebra , one can show that the actions (3.3.10) and (3.3.11) differ on a total derivative

$$S^{\text{inv}}_{susy}(B^3) = S^{\text{inv}}_{YM}(B^3) + \frac{2\pi}{4g^2_{YM}} \int d^3x_{|x|<1}(\nabla_i((1-x^2)\Phi_{i+4}\nabla_j\Phi_{j+4} - \Phi_{j+4}\nabla_j\Phi_{i+4})$$
$$- 4\nabla_j(x_i x_k\Phi_{k+4}\nabla_i\Phi_{j+4} - x_i x_j\Phi_i\nabla_{k+4}\Phi_{k+4})$$
$$- 6\nabla_j(x_i\Phi_{i+4}\Phi_{j+4})) \quad (3.3.12)$$

Integrating the total derivative term we get a boundary action

$$S^{\text{inv}}_{YM}(B^3) = S^{\text{inv}}_{susy}(B^3) + \frac{2\pi}{4g^2_{YM}} \int_{S^2:\,|x|=1} d\Omega\,(4\Phi_n(\nabla_n\Phi_n - \nabla_i\Phi_{i+4}) + 6\Phi^2_n), \quad (3.3.13)$$

where Φ_n is the normal component to the S^2 of the one-form Φ, i.e. $\Phi_n = n_i\Phi_{i+4}$, and ∇_n is the derivative in the normal direction, $n_i = x_i/|x|$. Using the equation

(3.2.45) for $\mathrm{Re}\, Q\Psi^t|_{e_5}$ with K_i substituted from (3.2.44) we get a constraint on Φ_n on the boundary

$$\nabla_n \Phi_n - \nabla_i \Phi_{i+4} = -\Phi_n. \tag{3.3.14}$$

Hence, the boundary action (3.3.13) simplifies to

$$S^{\rm inv}_{YM}(B^3) = S^{\rm inv}_{susy}(B^3) + \frac{\pi}{g^2_{YM}} \int_{S^2:|x|=1} d\Omega\, \Phi_n^2, \tag{3.3.15}$$

where $d\Omega$ is the standard volume form on S^2. On supersymmetric configuration $S^{\rm inv}_{susy}(B^3)$ vanishes, thus the $\mathcal{N} = 4$ Yang-Mills localizes to the two-dimensional theory on S^2 with the action

$$S_{2d} = \frac{\pi}{g^2_{YM}} \int_{S^2:|x|=1} d\Omega\, \Phi_n^2. \tag{3.3.16}$$

Equivalently we can express the action in terms of the tangent to S^2 components of Φ using the constraint (3.3.14)

$$S_{2d} = \frac{\pi}{g^2_{YM}} \int_{S^2:|x|=1} d\Omega\, (d^{*2d}_A \Phi_t)^2. \tag{3.3.17}$$

We recall that the scalar fields in (3.3.2) - (3.3.17) are the fields for the four-dimensional theory on $\mathbb{R}^3 \times_{\tilde{w}} S^1$. In terms of the original fields of the $\mathcal{N} = 4$ Yang-Mills on S^4 we have $\Phi[\mathbb{R}^3 \times_{\tilde{w}} S^1] = (1 + x^2)^{-1} \Phi[S^4]$, so

$$S_{2d} = \frac{\pi}{4g^2_{YM}} \int_{S^2:|x|=1} d\Omega (d^{*2d}_A \Phi_t^{S^4})^2. \tag{3.3.18}$$

Above was assumed that the radius $r = \frac{1}{2}$. To restore r we need to insert a power of factor $(2r)$ to get the correct dimension

$$S_{2d} = (2r)^2 \frac{\pi}{4g^2_{YM}} \int_{S^2:|x|=2r} \sqrt{g_{S^2}} d^2\sigma (d^{*2d}_A \Phi_t^{S^4})^2. \tag{3.3.19}$$

The Wilson loop operator (3.1.4) descends to the Wilson loop operator in the two-dimensional theory

$$W_R(C) = tr_R\, \mathrm{Pexp} \oint (A - i * \Phi) \tag{3.3.20}$$

We introduce complexified connection

$$\tilde{A}_c = A - i * \Phi, \tag{3.3.21}$$

so the Wilson loop operator (3.3.22) is holonomy of $\tilde{A}_c$

$$W_R(C) = tr_R \, \mathrm{Pexp} \oint \tilde{A}_c \tag{3.3.22}$$

Let $F_{\tilde{A}_c}$ be the curvature of $\tilde{A}_c$, then

$$F_(\tilde{A}_c) = d\tilde{A}_c + \tilde{A}_c \wedge \tilde{A}_c = F_A - \Phi \wedge \Phi - i d_A * \Phi \tag{3.3.23}$$

By (3.2.93) at the localized configurations we have $F_A - \Phi \wedge \Phi = 0$, then

$$d_A * \Phi = i F_{\tilde{A}_c} \quad \text{for localized configurations} \tag{3.3.24}$$

Then the action of the two-dimensional theory (3.3.19) is equivalent to the action of the bosonic Yang-Mills for complexified connection $\tilde{A}_c$

$$S_{2d} = -\frac{1}{2g_{2d}^2} \int_{S^2} d\Omega \, F_{\tilde{A}_c}^2, \tag{3.3.25}$$

where the two-dimensional coupling constant is denoted g_{2d}

$$g_{YM}^2 = 2\pi r^2 g_{2d}^2. \tag{3.3.26}$$

So the original four-dimensional problem has been reduced to complexified two-dimensional bosonic Yang-Mills theory (3.3.25) with the standard Wilson loop observables (3.3.22). However, the complexified connection $\tilde{A}_c = A - i*\Phi$ is constrained to the localization locus by (3.2.93)

$$\mathrm{Re}\, F_{\tilde{A}_c} = 0 \tag{3.3.27}$$

$$d_{\mathrm{Re}\, \tilde{A}_c} * \mathrm{Im}\, \tilde{A}_c = 0. \tag{3.3.28}$$

The two real constraints remove two real degrees of freedom from the four real degrees of freedom of complex one-form $\tilde{A}_c$ (we do not subtract gauge symmetry in this counting). Therefore, the path integral is taken over a certain half-dimensional subspace of complexified connections $\tilde{A}_c$.

We can interpret the path integral for the usual two-dimensional Yang-Mills for real connections as a contour integral in the space of complexified connections, where the contour is given by the constraint that the imaginary part of the connection vanishes: $\operatorname{Im} \tilde{A}_c = 0$.

Our assertion is that the complexified theory (3.3.25) with constraints (3.3.27) is equivalent to the real theory by a change of the integration contour in the space of complexified connections.

Since perturbative correlation functions of holomorphic observables do not depend on deformation of the contour of integration, we conclude that the expectation value of Wilson loop observables (3.3.22) perturbatively coincides with the expectation values of Wilson loops in the ordinary two-dimensional Yang-Mills.

We shall look at the complexified two-dimensional Yang-Mills theory with constraints from the slightly broader viewpoint of so called topological Higgs-Yang-Mills theory [53–55] which deals with then moduli space of solutions to Hitchin equations.

3.3.1 Higgs-Yang-Mills theory

Here we will review Higgs-Yang-Mills theory following [8, 53–55]. Let Σ be a Riemann surface, A be a gauge field for the gauge group G (G is a compact Lie group) and Φ be a one-form taking value in the Lie algebra $\mathfrak{g}$ of G.

Let φ be a scalar field taking value in $\mathfrak{g}$. The field φ can be thought as an element of the Lie algebra $\mathfrak{g}_{gauge}$ of the infinite-dimensional group of gauge transformations G_{gauge}. Let M be the space of fields (A, Φ). Using the invariant Killing form on $\mathfrak{g}$ we identify g with g^*. Then locally M is $T^*\Omega^1(\Sigma, \operatorname{ad}\mathfrak{g})$.

We notice (see [8, 53–55, 83]) that the space M can be equipped with a triplet of symplectic structures ω_i and a triplet of corresponding Hamiltonian moment maps μ_i for G_{gauge} acting on M.

Explicitly we define the symplectic structure ω_i as follows. Let δ be the differential on M. Then

$$\omega_1(\delta A_1, \delta\Phi_1; \delta A_2, \delta\Phi_2) = \mathrm{tr}\int_\Sigma \delta A_1 \wedge \delta A_2 - \delta\Phi_1 \wedge \delta\Phi_2 \tag{3.3.29}$$

$$\omega_2(\delta A_1, \delta\Phi_1; \delta A_2, \delta\Phi_2) = \mathrm{tr}\int_\Sigma \delta A_1 \wedge \delta\Phi_2 - \delta A_2 \wedge \delta\Phi_1 \tag{3.3.30}$$

$$\omega_3(\delta A_1, \delta\Phi_1; \delta A_2, \delta\Phi_2) = \mathrm{tr}\int_\Sigma \delta A_1 \wedge *\delta\Phi_2 - \delta A_2 \wedge *\delta\Phi_1, \tag{3.3.31}$$

where $*$ is the Hodge star on Σ. (Here subscripts $1, 2$ denote an argument of the functional two-form ω and they should not be confused with world-sheet indices, e.g. δA_1 should be read as one-form on Σ.)

A functional $\mu : M \to \mathfrak{g}^*_{gauge}$ is called a moment map if

$$i_\phi\omega = \mu(\phi) \quad \text{for all} \quad \phi \in \mathfrak{g}_{gauge}, \tag{3.3.32}$$

where i_ϕ denotes a contraction with a vector field generated on M by an element $\phi \in \mathfrak{g}_{gauge}$.

The group G_{gauge} acts on M by the usual gauge transformations

$$\begin{aligned} \delta A &= -d_A\phi \\ \delta\Phi &= [\phi, \Phi]. \end{aligned} \tag{3.3.33}$$

One can check that the functionals

$$\mu_1(\phi) = \mathrm{tr}\int (\phi, F - \Phi\wedge\Phi) \tag{3.3.34}$$

$$\mu_2(\phi) = \mathrm{tr}\int (\phi, d_A\Phi) \tag{3.3.35}$$

$$\mu_3(\phi) = \mathrm{tr}\int (\phi, d_A * \Phi) \tag{3.3.36}$$

are the moment maps for the symplectic structure $\omega_1, \omega_2, \omega_3$ correspondingly.

The space M has natural linear flat structure and the corresponding flat metric is

$$g(\delta A_1, \delta\Phi_1; \delta A_2, \delta\Phi_2) = \mathrm{tr}\int \delta A_1 \wedge *\delta A_2 + \delta\Phi_1 \wedge *\delta\Phi_2. \tag{3.3.37}$$

Using the metric g on M, to each symplectic structure ω_i we can associate a complex structure I_i in the usual way $\omega(\cdot,\cdot) = g(I\cdot,\cdot)$.

Comparing

$$\mathrm{tr}\int_\Sigma I(\delta A_1)\wedge *\delta A_2 + I(\delta\Phi_1)\wedge *\delta\Phi_2 \tag{3.3.38}$$

with (3.3.29)- (3.3.31) we get

$$I_1(\delta A) = *\delta A \qquad I_1(\delta\Phi) = -*\delta\Phi \tag{3.3.39}$$

$$I_2(\delta A) = *\delta\Phi \qquad I_2(\delta\Phi) = *\delta A \tag{3.3.40}$$

$$I_3(\delta A) = -\delta\Phi \qquad I_3(\delta\Phi) = \delta A \tag{3.3.41}$$

Notice that the following linear combinations span the holomorphic subspaces ($+i$-eigenspaces) of the corresponding complex structures:

$$\begin{aligned} I_1(A - i*A) &= i(A - i*A) \\ I_2(A - i*\Phi) &= i(A - i*\Phi) \\ I_3(A + i\Phi) &= i(A + i\Phi). \end{aligned} \tag{3.3.42}$$

One can also check that the complex structures satisfy $I_3 = I_2 I_1, I_1 = I_3 I_2, I_2 = I_1 I_3$. Hence the space M is the hyperKahler space.

We can use four-dimensional notations. Let us denote

$$\Phi_1 \equiv A_4 \quad \Phi_2 \equiv A_3, \tag{3.3.43}$$

then the three moment maps (3.3.34) correspond to the components of the self-dual part F_A^+ of the four-dimensional curvature F_A:

$$\begin{aligned} F - \Phi\wedge\Phi &= (F_{12} + F_{34})dx^1\wedge dx^2 \\ d_A\Phi &= (F_{13} - F_{24})dx^1\wedge dx^2 \\ d_A*\Phi &= (F_{14} + F_{23})dx^1\wedge dx^2 \end{aligned} \tag{3.3.44}$$

Clearly, the space $\mathbb{R}^4$ (or more generally $T^*\Sigma$) is hyperKahler, so it is equipped with $\mathbb{CP}^1$ family of complex structures. Let $z_1, \bar{z}_1, z_2, \bar{z}_2$ be complex coordinates with respect to some complex structure, e.g. $z_1 = x_1 + ix_2, z_2 = x_3 + ix_4$. Then, in terms of $A_{\bar{z}_1} = \frac{1}{2}(A_1 + iA_2)$, etc, we can write

$$\begin{aligned} F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} &= \frac{i}{2}(F_{12} + F_{34}) = \frac{i}{2}\mu_1 \\ F_{\bar{z}_1\bar{z}_2} &= \frac{1}{4}(F_{13} - F_{24}) + \frac{i}{4}(F_{23} + F_{14}) = \frac{1}{4}(\mu_2 + i\mu_3) \end{aligned} \tag{3.3.45}$$

Constrained Higgs-Yang-Mill theory

For the related story see [53, 54].

Consider the following path integral over ϕ and the space M of fields (A, Φ)

$$Z_{cHYM} = \int_{M|\mu_1=\mu_2=0} D\phi e^{i(\omega_3 - \mu_3(\phi)) - \frac{t_2}{2}\int \mathrm{tr}\,\phi^2}. \tag{3.3.46}$$

Later we will insert Wilson loop observables for the holomorphic part of the complexified connection with respect to the complex structure I_2. Explicitly such observables have form

$$W_R(C) = \mathrm{tr}_R\, \mathrm{Pexp} \oint_C (A - i * \Phi), \tag{3.3.47}$$

were C is a contour on Σ and R is representation of G.

We would like to look at this theory as a hyperKahler rotation of another theory

$$Z_{YM} = \int_{M|\mu_2=\mu_3=0} D\phi e^{i(\omega_1 - \mu_1(\phi)) - \frac{t_2}{2}\int \mathrm{tr}\,\phi^2}, \tag{3.3.48}$$

which is almost equivalent to bosonic two-dimensional Yang-Mills. Let Σ be a Riemann sphere. The constraint $\mu_2 = \mu_3 = 0$ means $d_A^*\Phi = d_A\Phi = 0$. For a generic connection A, the only solution to these constraints is $\Phi = 0$. Then the path integral (3.3.48) reduces to the 2d bosonic Yang-Mills integral over A and ϕ written in the first order formalism as in [83].

We can insert Wilson loop observables (3.3.47) into the path integral. Since Φ vanishes because of the constraint, the Wilson loop (3.3.47) reduces to the ordinary Wilson loop of the connection A. Therefore, the expectation value of Wilson loops

(3.3.47) naively is computed by the standard formulas of the two-dimensional Yang-Mills theory [83, 107, 108] modulo subtleties which are related to non-generic connections for which there are non-trivial solutions of the constraint $d_A^*\Phi = d_A\Phi = 0$. Such connections precisely correspond to unstable instantons, i.e. configurations with covariantly constant curvature F_A. It is well known that the partition function of bosonic two-dimensional Yang-Mills can be written as a sum of contributions from such unstable instantons [83, 109, 110]. A contribution of a classical solution with nontrivial curvature F enters with a weight $\exp(-\frac{1}{2}g^2\rho(\Sigma)F^2)$ where $\rho(\Sigma)$ is the area of Σ. In the weak coupling limit such instanton contributions are exponentially suppressed and do not contribute to the perturbation theory.

Hence, we conclude that perturbatively the constrained Higgs-Yang-Mills theory (3.3.48) is equivalent to the ordinary two-dimensional Yang-Mills.

In [83] Witten has related the physical two-dimensional Yang-Mills theory (3.3.48) with the topological two-dimensional Yang-Mills. The key point is that the path integral for the physical Yang-Mills theory can be represented as an integral of the equivariantly closed form with respect to the following operator Q

$$\begin{aligned} QA &= \psi \\ Q\psi &= -d_A\phi \\ Q\phi &= 0. \end{aligned} \tag{3.3.49}$$

In other words, the $\omega_1 - \mu_1(\phi)$ is the equivariantly closed form constructed from the symplectic structure ω_1 and the Hamiltonian moment map μ_1 for the gauge group acting on the space of connections. Then localization method can be used to compute the integral of such equivariantly closed form [20–22, 83].

Though the Wilson loop observable is not Q-closed, its expectation value can be still solved exactly. That gives a hope that we can also find exact expectation value of Wilson loops (3.3.47) in constrained Higgs-Yang-Mills theory (3.3.48) and its rotated version (3.3.46). See [53–55] for computation of correlation functions for the Q-closed observables $\operatorname{tr}\phi^n$.

First let us focus on the partition function (3.3.46). We can try to proceed in two directions. The first one is to try to use the localization method and relate the theory to some topological theory and computations with Q-equivariant cohomology. Though the Wilson loop operators are not Q-closed, we can try to solve for at least non-intersecting Wilson loops $\{C_1, \ldots, C_k\}$ by: (i) finding topological wave-function $\Psi(U_1, \ldots, U_k)$ on the boundary of the Riemann surface with Wilson loops deleted $\Sigma \backslash \{C_1 \cup \ldots C_k\}$, and (ii) then integrating over the space of holonomies $\{U_1, \ldots, U_k\}$. For the study of wave-functions in Higgs-Yang-Mills theory see [53, 54].

The second approach is to explicitly solve the constraint $\mu_1 = \mu_2 = 0$, which means that the complexified connection $A^c = A + i\Phi$ is flat, in the form

$$A + i\Phi = g_c^{-1} dg_c, \tag{3.3.50}$$

where g_c takes value in the complexified gauge group $G^{\mathbb{C}}$. The gauge transformations

$$A + i\Phi \to g^{-1}(A + i\Phi)g + g^{-1}dg \tag{3.3.51}$$

can be represented by the right multiplications $g_c \to g_c g$, where g takes value in the compact gauge group G. Hence the configurational space of the theory is the same as of gauged WZW model on the coset $G^{\mathbb{C}}/G$.

We shall not proceed these ideas further in this work. Instead we will give one more argument why the perturbative expecation value of Wilson loop (3.3.47) in the theory (3.3.48) and its hyperKahler rotated version (3.3.46) is the same.

First we rewrite the path integral of a constrained theory by means of Lagrangian multipliers. Consider the theory (3.3.48). We introduce scalar auxiliary fields H_2, H_3 and their superpartners χ_2, χ_3. The superpartners of A and Ψ are fermionic adjoined valued one-forms on Σ. Then we consider the usual complex for equivariant cohomology

$$\begin{aligned} QA &= \psi_A & Q\chi_{2,3} &= H_{2,3} \\ Q\psi_A &= -d_A\phi & QH_{2,3} &= [\phi, \chi_{2,3}] \end{aligned} \tag{3.3.52}$$

with

$$Q\phi = 0. \tag{3.3.53}$$

The theory (3.3.48) can be rewritten as

$$Z = \int D\phi\, DAD\psi_A\, D\Phi D\psi_\Phi DHD\chi$$
$$\exp(\int i(\psi_A \wedge \psi_A - \psi_\Phi \wedge \psi_\Phi - (F - \Phi \wedge \Phi))\phi - \frac{t_2}{2}\phi \wedge *\phi$$
$$+ S_c,) \tag{3.3.54}$$

where

$$S_c = iQ(\int d_A\Phi \wedge \chi_2 + d_A * \Phi \wedge \chi_3) =$$
$$i\int (d_A\psi_\Phi + [\psi_A, \Phi]) \wedge \chi_2 + (d_A * \psi_\Phi + [\psi_A, *\Phi]) \wedge \chi_3 + d_A\Phi \wedge H_2 + d_A * \Phi \wedge H_3$$
$$\tag{3.3.55}$$

If we integrate out the Lagrange multipliers H_2, H_3 and Φ, and their fermionic partners χ_2, χ_3 and ψ_A, the resulting determinants cancel, while Φ becomes restricted to the slice $d_A\Phi = d_A^*\Phi = 0$, and similarly ψ_Φ is restricted to $d_A\psi_\Phi + [\psi_A, \Phi] = 0$ and $d_A * \psi_\Phi + [\psi_A, *\Phi] = 0$. Since $\Phi = 0$ we get $\psi_\Phi = 0$. Then what remains is

$$Z = \int DAD\psi_A D\phi \exp(\int i(\psi_A \wedge \psi_A - F\phi) - \frac{t_2}{2}\phi \wedge *\phi), \tag{3.3.56}$$

which is the usual action of bosonic Yang-Mills in the first order formalism [83].

Now consider the constrained Higgs-Yang-Mills theory (3.3.46) . Actually we shall consider slightly different version:

$$Z_{cHYM} = \int_{M|\mu_1=\mu_2=0} D\phi e^{i(\omega_3 + i\omega_1 - (\mu_3(\phi) + \mu_1(\phi)) - \frac{t_2}{2}\int \operatorname{tr}\phi^2}. \tag{3.3.57}$$

Here we added to the action the term $\mu_1(\phi)$ and its supersymmetric extension ω_1. Since $\mu_1(\phi) = 0$ by constraint, classically this is the same theory as (3.3.46). The symplectic structure $\omega_1 - i\omega_3$ is holomorphic $(2,0)$ two-form with respect to the second complex structure in (3.3.42).

Let us make a change of variables from (A, Φ) to the variables $(\tilde{A}_c, \Phi)$ where

$$\tilde{A}_c = A - i * \Phi \tag{3.3.58}$$

Perturbatively we can rotate the integration contour for Φ to the imaginary axis, then $\tilde{A}_c$ is real valued. The jacobian for this change of variable is trivial.

The symplectic structure $\omega_1 - i\omega_3$ can be written as

$$\omega_1 - i\omega_3 = \mathrm{tr} \int_\Sigma \delta\tilde{A}_c \wedge \delta\tilde{A}_c, \tag{3.3.59}$$

and the moment map $\mu_1 - i\mu_3$ is actually the curvature of $\tilde{A}_c$

$$\mu_1 - i\mu_3 = F(\tilde{A}_c) \tag{3.3.60}$$

One can see that if Σ is a sphere, than constraints $\mu_1 = 0, \mu_2 = 0$ determine Φ uniquely for each $\tilde{A}_c$. Hence, the path integral (3.3.57) reduces to the integral over the fields $\tilde{A}_c$ with the measure induced by the symplectic structure (3.3.59). That is the standard bosonic Yang-Mills theory in the first order formalism for the connection $\tilde{A}_c$. The correlation function of Wilson loop operators (3.3.47) perturbatively are computed as in the usual bosonic two-dimensional Yang-Mills.

Chapter 4

Conclusion

In this thesis we considered the basic non-local observables in supersymmetric gauge theories – Wilson loop operators. We have shown that correlation function for certain supersymmetric Wilson loops in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal four-dimensional gauge theories can be computed exactly using the localization method. In particular we prove Erickson-Semenoff-Zarembo/Drukker-Gross conjecture which relates circular supersymmetric Wilson loops in the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory to correlation function in Gaussian matrix models. We consider the four-dimensional field theory on compact Euclidean space-time S^4, and we show that the matrix in the matrix model can be simply interpreted as the constant mode of one of the scalar fields of the theory.

We generalize Erickson-Semenoff-Zarembo/Drukker-Gross conjecture about circular Wilson loops to an arbitrary superconformal $\mathcal{N} = 2$ theory. In that case the computations are again localized to matrix model but with much more complicated, but still explicit potential. The potential combines Nekrasov's ε-deformed partition function of instantons and certain one-loop factor expressed in terms of Barnes G-functions.

We also generalized ESZ/DG conjecture to the more complicated case of supersymmetric Wilson loops of arbitrary planar shape in four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills. By planar shape we mean an arbitrary contour restricted to some two-sphere in the four-dimensional space-time. In this case the theory localizes to a certain two-dimensional theory which is closely related to the partially gauge-fixed version of two-dimensional Yang-Mills for the complexified gauge group, or, topological Higgs-Yang-Mills theory related to the Hitchin's equations on Riemann surface.

All-order exact results in gauge theory are interesting on its own, as well as they shed some more light on the central topic in string theory – gauge/string duality conjecture.

Appendix A

Appendix

A.1 Clifford algebra

We use the following conventions to denote symmetrized and antisymmetrized tensors:

$$\begin{aligned} a_{[i}b_{j]} &= \frac{1}{2}(a_i b_j - a_j b_i) \\ a_{\{i}b_{j\}} &= \frac{1}{2}(a_i b_j + a_j b_i), \end{aligned} \tag{A.1.1}$$

where a and b are any indexed variables.

Let us summarize here our conventions on gamma-matrices in ten dimensions. We start with Minkowski metric $ds^2 = -dx_0^2 + dx_1^2 + \ldots dx_9^2$. Capital letters from the middle of the Latin alphabet normally are used to denote ten-dimensional space-time indices $M, N, P, Q = 0, \ldots, 9$. Let γ^M for $M = 0, \ldots, 9$ be 32×32 matrices representing the Clifford algebra $\mathrm{C}l(9,1)$. They satisfy the standard anticommutation relations

$$\gamma^{\{M}\gamma^{N\}} = g^{MN}, \tag{A.1.2}$$

where g^{MN} is the metric. The corresponding representation of $Spin(9,1)$ has rank 32 and can be decomposed into irreducible spin representations $\mathcal{S}^+$ and $\mathcal{S}^-$ of rank 16. The chirality operator

$$\gamma^{11} = \gamma^1\gamma^2 \ldots \gamma^9\gamma^0$$

acts on $\mathcal{S}^+$ and $\mathcal{S}^-$ as multiplication by 1 and -1, respectively. The gamma-matrices Γ^M reverse chirality, so $\Gamma^M : \mathcal{S}^\pm \to \mathcal{S}^\mp$. We can write γ^M in the block form

$$\gamma^M = \begin{pmatrix} 0 & \tilde{\Gamma}^M \\ \Gamma^M & 0 \end{pmatrix}, \tag{A.1.3}$$

assuming that we write the rank 32 spin representation of $Spin(9,1)$ as

$$\begin{pmatrix} \mathcal{S}^+ \\ \mathcal{S}^- \end{pmatrix}. \tag{A.1.4}$$

Let Γ^M and $\tilde{\Gamma}^M$ be the chiral "half" gamma-matrices appearing in (A.1.3). Then

$$\tilde{\Gamma}^{\{M}\Gamma^{N\}} = g^{MN}, \quad \Gamma^{\{M}\tilde{\Gamma}^{N\}} = g^{MN}. \tag{A.1.5}$$

We define γ^{MN}, Γ^{MN} and $\tilde{\Gamma}^{MN}$ as follows

$$\gamma^{MN} = \gamma^{[M}\gamma^{N]} = \begin{pmatrix} \tilde{\Gamma}^{[M}\Gamma^{N]} & 0 \\ 0 & \Gamma^{[M}\tilde{\Gamma}^{N]} \end{pmatrix} =: \begin{pmatrix} \Gamma^{MN} & 0 \\ 0 & \tilde{\Gamma}^{MN} \end{pmatrix}. \tag{A.1.6}$$

Using anticommutation relations we get

$$\Gamma^M\Gamma^{PQ} = 4g^{M[P}\Gamma^{Q]} + \tilde{\Gamma}^{PQ}\Gamma^M. \tag{A.1.7}$$

For computations in the four-dimensional theory, we will often need to split the ten-dimensional space-time indices into two groups. The first group contains four-dimensional space-time indices in the range $1, \ldots, 4$, which we denote by Greek latter in the middle of the alphabet μ, ν, λ, ρ. The second group contains the indices for the normal directions, running over $5, \ldots, 9, 0$, which we denote by capital letters from the beginning of the Latin alphabet A, B, C, D. As usual, the repeated index means summation over it. Then we have the following identities

$$\begin{aligned} &\Gamma_{\mu A}\tilde{\Gamma}^\mu = -4\tilde{\Gamma}_A \\ &\Gamma^\mu\Gamma_{\nu\rho}\tilde{\Gamma}_\mu = 0 \\ &\Gamma^\mu\Gamma_{\nu A}\tilde{\Gamma}_\mu = 2\tilde{\Gamma}_{\nu A} \\ &\Gamma^\mu\Gamma_{AB}\tilde{\Gamma}_\mu = 4\tilde{\Gamma}_{AB} \end{aligned} \tag{A.1.8}$$

We choose matrices Γ_M and $\tilde{\Gamma}^M$ to be symmetric:

$$(\Gamma^M)^T = \Gamma_M \quad (\tilde{\Gamma}^M)^T = \tilde{\Gamma}^M.$$

Then we get $(\Gamma^{MN})^T = -\tilde{\Gamma}^{MN}$, so the representations $\mathcal{S}^+$ and $\mathcal{S}^-$ are dual to each other.

There is a very important "triality identity" which appears in the computations involving ten-dimensional supersymmetry:

$$(\Gamma_M)_{\alpha_1\{\alpha_2}(\Gamma^M)_{\alpha_3\alpha_4\}} = 0, \tag{A.1.9}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 1, \ldots, 16$ are the matrix indices of Γ^M.

All gamma-matrices relations above are valid both for Minkowski and Euclidean signature.

The difference between gamma-matrices for Minkowski and Euclidean signature is the following. In Minkowski signature we choose Γ^M to be real. In Euclidean signature we use the following matrices $\{i\Gamma^0, \Gamma^1, \ldots, \Gamma^9\}$. Therefore all Euclidean gamma-matrices are real except Γ^0, which is imaginary. In Euclidean signagure the representation $\mathcal{S}^+$ and $\mathcal{S}^-$ are unitary. Since in Euclidean signature they are also dual to each other, we conclude that in Euclidean signature $\mathcal{S}^+$ and $\mathcal{S}^-$ are complex conjugate representations.

It is convenient to use octonions to explicitly write down Γ^M. In Minkowski signature we choose

$$\begin{aligned}
\Gamma^i &= \begin{pmatrix} 0 & E_i^T \\ E_i & 0 \end{pmatrix}, \quad i = 1 \ldots 7 \\
\Gamma^9 &= \begin{pmatrix} 1_{8\times 8} & 0 \\ 0 & -1_{8\times 8} \end{pmatrix}, \\
\Gamma^0 &= \begin{pmatrix} 1_{8\times 8} & 0 \\ 0 & 1_{8\times 8} \end{pmatrix},
\end{aligned} \tag{A.1.10}$$

where E_i for $i = 1 \ldots 8$ are 8×8 matrices representing left multiplication of the octonions.

Let e_i with $i = 1 \dots 8$ be the generators of the octonion algebra with the octonionic structure constants c_{ij}^k defined by the multiplication table $e_i \cdot e_j = c_{ij}^k e_k$. Then $(E_i)_j^k = c_{ij}^k$. The element e_1 is the identity. To be concrete, we define the multiplication table by specifying the triples which have cyclic multiplication table: $(234), (256), (357), (458), (836), (647), (728)$ (e.g. $e_2 e_3 = e_4$, etc.). Then one can check that E_i have the following form

$$\begin{aligned} E_\mu &= \begin{pmatrix} J_\mu & 0 \\ 0 & \bar{J}_\mu \end{pmatrix}, \quad \mu = 1 \dots 4 \\ E_A &= \begin{pmatrix} 0 & -J_A^T \\ J_A & 0 \end{pmatrix}, \quad A = 5 \dots 8, \end{aligned} \tag{A.1.11}$$

where J_μ for $\mu = 1 \dots 4$ are the 4×4 matrices representing generators of quaternion algebra by the left action, while $\bar{J}_\mu$ are the 4×4 matrices representing generators of quaternion algebra by the right action. Concretely we obtain

$$(J_1, J_2, J_3, J_4) = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right), \tag{A.1.12}$$

with the relations

$$J_i J_j = \varepsilon_{ijk} J_k, \quad i, j, k = 2 \dots 4,$$

and

$$(\bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{J}_4) = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right), \tag{A.1.13}$$

with the relations

$$\bar{J}_i \bar{J}_j = -\varepsilon_{ijk} \bar{J}_k, \quad i, j, k = 2 \dots 4.$$

Similarly,

$$(J_5, J_6, J_7, J_8) = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right). \tag{A.1.14}$$

We choose orientation in the $(1\dots4)$-plane and the $(5\dots8)$-plane by saying that 1234 and 5678 are the the positive cycles.

Then the matrices $\Gamma^{\mu\nu}$ for $\mu,\nu=1\dots4$ and Γ^{ij} for $i,j=5\dots8$ have the following block decomposition:

$$\Gamma_{\mu\nu}=\begin{pmatrix} E^T_{[\mu}E_{\nu]} & 0 \\ 0 & E_{[\mu}E^T_{\nu]}\end{pmatrix}=\begin{pmatrix} J^-_{\mu\nu} & 0 & 0 & 0 \\ 0 & \bar{J}^+_{\mu\nu} & 0 & 0 \\ 0 & 0 & -J^+_{\mu\nu} & 0 \\ 0 & 0 & 0 & -\bar{J}^-_{\mu\nu}\end{pmatrix},$$
$$\Gamma_{ij}=\begin{pmatrix} E^T_{[i}E_{i]} & 0 \\ 0 & E_{[i}E^T_{i]}\end{pmatrix}=\begin{pmatrix} -\bar{J}^-_{ij} & 0 & 0 & 0 \\ 0 & -J^+_{ij} & 0 & 0 \\ 0 & 0 & -\bar{J}^-_{ij} & 0 \\ 0 & 0 & 0 & -J^+_{ij}\end{pmatrix}, \tag{A.1.15}$$

where the $\pm$-superscript denotes the self-dual and anti-self-dual tensors; $J_{12}=J_1^TJ_2=J_1$, etc.

Then we define the four-dimensional chirality operator acting in tangent directions to the four-dimensional space-time :

$$\Gamma^{(\overline{14})}=\Gamma_1\Gamma_2\Gamma_3\Gamma_4.$$

It is represented by the matrix

$$\Gamma^{(\overline{14})}=\begin{pmatrix} 1_{4\times4} & 0 & 0 & 0 \\ 0 & -1_{4\times4} & 0 & 0 \\ 0 & 0 & -1_{4\times4} & 0 \\ 0 & 0 & 0 & 1_{4\times4}\end{pmatrix}. \tag{A.1.16}$$

Similarly, we define the four-dimensional chirality operator

$$\Gamma^{(\overline{58})}=\Gamma_5\Gamma_6\Gamma_7\Gamma_8,$$

acting in four normal directions $M = 5\dots 8$. It is represented by the matrix

$$\Gamma^{(\overline{58})} = \begin{pmatrix} 1_{4\times 4} & 0 & 0 & 0 \\ 0 & -1_{4\times 4} & 0 & 0 \\ 0 & 0 & 1_{4\times 4} & 0 \\ 0 & 0 & 0 & -1_{4\times 4} \end{pmatrix}. \qquad \text{(A.1.17)}$$

Finally, we define the eight-dimensional chirality operator

$$\Gamma^9 = \Gamma^{(\overline{14})}\Gamma^{(\overline{58})}.$$

It is represented by the matrix

$$\Gamma^9 = \begin{pmatrix} 1_{4\times 4} & 0 & 0 & 0 \\ 0 & 1_{4\times 4} & 0 & 0 \\ 0 & 0 & -1_{4\times 4} & 0 \\ 0 & 0 & 0 & -1_{4\times 4} \end{pmatrix}. \qquad \text{(A.1.18)}$$

The representation $\mathbf{16} = \mathcal{S}^+$ (a sixteen component Majorana-Weyl fermion of $Spin(9,1)$) then splits as $\mathbf{16} = \mathbf{8} + \mathbf{8}'$ with respect to the $Spin(8) \subset Spin(9,1)$ acting in the directions $M = 1,\dots,8$. Then we brake $Spin(8)$ as $Spin(8) \hookleftarrow Spin(4) \times Spin(4)^R$, where the group $Spin(4)$ acts in the directions $M = 1,\dots,4$, while the group $Spin(4)^R$ acts in the directions $M = 5,\dots,8$. We write the $Spin(4)$ as $Spin(4) = SU(2)_L \times SU(2)_R$ and the $Spin(4)^R$ as $Spin(4)^R = SU(2)^R_L \times SU(2)^R_R$. With respect to these $SU(2)$-subgroups, the representation $\mathbf{16} = \mathcal{S}^+$ of $Spin(9,1)$ transforms as

$$\mathbf{16} = (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}).$$

As we mentioned before, the only difference between the gamma-matrices in the Euclidean and Minkowski case is that we multiply the matrix Γ^0 by $i \equiv \sqrt{-1}$, so the Euclidean gamma-matrices are:

$$\Gamma^M = \begin{pmatrix} 0 & E_M^T \\ E_M & 0 \end{pmatrix}, \quad M = 1\dots 7$$
$$\Gamma^9 = \begin{pmatrix} 1_{8\times 8} & 0 \\ 0 & -1_{8\times 8} \end{pmatrix}, \tag{A.1.19}$$
$$\Gamma^0 = \begin{pmatrix} i1_{8\times 8} & 0 \\ 0 & i1_{8\times 8} \end{pmatrix}.$$

A.2 Conformal Killing spinors on S^4

The explicit form of the Killing spinor on S^4 depends on the vielbein. For solution in spherical coordinates see [111]. In stereographic coordinates the solution has simpler form and is easily related to the flat limit.

Pick up a point on S^4, call it the North pole, and call the opposite point the South pole. Let x^μ be the stereographic coordinates on S^4 in the neighborhood of the North pole. The metric has the following form

$$g_{\mu\nu} = \delta_{\mu\nu} e^{2\Omega}, \quad \text{where} \quad e^{2\Omega} := \frac{1}{(1+\frac{x^2}{4r^2})^2}. \tag{A.2.1}$$

By θ we denote the polar angle in spherical coordinates measure from the North pole. In other words, $\theta = 0$ is the North pole, $\theta = \frac{\pi}{2}$ is the equator, and $\theta = \pi$ is the South pole. We have $|x| = 2r\tan\frac{\theta}{2}$ and $e^{\Omega} = \cos^2\frac{\theta}{2}$. Fix the vielbein[1] $e^{\hat{\mu}}_{\lambda} = \delta^{\hat{\mu}}_{\lambda} e^{\Omega}$. The spin connection $\omega^{\hat{\mu}}_{\hat{\nu}\lambda}$ induced by the Levi-Civita connection can be computed using the Weyl transformation of the flat metric $\delta_{\mu\nu} \mapsto e^{2\Omega}\delta_{\mu\nu}$. Under such transformation $\omega^{\hat{\mu}}_{\hat{\nu}\mu} \mapsto \omega^{\hat{\mu}}_{\hat{\nu}\lambda} + (e^{\hat{\mu}}_{\lambda} e^{\nu}_{\hat{\nu}}\Omega_\nu - e_{\hat{\nu}\lambda} e^{\hat{\mu}\nu}\Omega_\nu)$. Since in the flat case $\omega^{\hat{\mu}}_{\hat{\nu}\lambda} = 0$, we get

$$\omega^{\hat{\mu}}_{\hat{\nu}\lambda} = (e^{\hat{\mu}}_{\lambda} e^{\nu}_{\hat{\nu}}\Omega_\nu - e_{\hat{\nu}\lambda} e^{\hat{\mu}\nu}\Omega_\nu), \tag{A.2.2}$$

[1] In this section we use the indices $\hat{\mu}, \hat{\nu} = 1, \dots, 4$ to enumerate the vielbein basis elements, that is $e^{\hat{\mu}}_{\lambda} e^{\hat{\nu}}_{\nu} = \delta^{\hat{\mu}\hat{\nu}}$ where $\delta^{\hat{\mu}\hat{\nu}}$ is the four-dimensional Kronecker symbol. Then $\Gamma^{\hat{\mu}}$ are the four-dimensional gamma-matrices normalized as $\Gamma^{(\hat{\mu}}\Gamma^{\hat{\nu})} = \delta^{\hat{\mu}\hat{\nu}}$,

where $\Omega_\nu := \partial_\nu \Omega$.

The conformal Killing spinor equation takes the explicit form

$$\begin{aligned}(\partial_\lambda + \frac{1}{4}\omega_{\hat\mu\hat\nu\lambda}\Gamma^{\hat\mu\hat\nu})\varepsilon &= \Gamma_\lambda \tilde\varepsilon \\ (\partial_\lambda + \frac{1}{4}\omega_{\hat\mu\hat\nu\lambda}\Gamma^{\hat\mu\hat\nu})\tilde\varepsilon &= -\frac{1}{4r^2}\Gamma_\lambda \varepsilon;\end{aligned} \tag{A.2.3}$$

At the flat limit $r = \infty$ the equations simplify as $\partial_\lambda \varepsilon = \Gamma_\lambda \tilde\varepsilon$ and $\partial_\lambda \tilde\varepsilon = 0$; hence the flat space solution is

$$\begin{aligned}\varepsilon &= \hat\varepsilon_s + x^{\hat\mu}\Gamma_{\hat\mu}\hat\varepsilon_c \\ \tilde\varepsilon &= \hat\varepsilon_c,\end{aligned} \tag{A.2.4}$$

where $\hat\varepsilon_s, \hat\varepsilon_c$ are constant spinors on $\mathbb{R}^4$. The spinor $\hat\varepsilon_s$ generates usual supersymmetry transformations, the spinor $\hat\varepsilon_c$ generates special superconformal transformations.

For an arbitrary r the solution is

$$\varepsilon = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}(\hat\varepsilon_s + x^{\hat\mu}\Gamma_{\hat\mu}\hat\varepsilon_c) \tag{A.2.5}$$

$$\tilde\varepsilon = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}(\hat\varepsilon_c - \frac{x^{\hat\mu}\Gamma_{\hat\mu}}{4r^2}\hat\varepsilon_s), \tag{A.2.6}$$

where $\hat\varepsilon_s$ and $\hat\varepsilon_c$ are arbitrary spinor parameters.

Consider the case when ε is the conformal Killing spinors generating a transformation of an $OSp(2|4)$ subgroup. We take chiral $\hat\varepsilon_s$ and $\hat\varepsilon_c$, such that $\Gamma^9\hat\varepsilon_s = \hat\varepsilon_s$ and $\Gamma^9\hat\varepsilon_c = \hat\varepsilon_c$, so

$$\varepsilon = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}(\hat\varepsilon_s + x^{\hat\mu}\Gamma_{\hat\mu}\Gamma^9\hat\varepsilon_c). \tag{A.2.7}$$

Moreover, for such spinor ε we have $\hat\varepsilon_c = \frac{1}{2r}\frac{1}{4}\omega_{\hat\mu\hat\nu}\Gamma^{\hat\mu\hat\nu}\hat\varepsilon_s$, where $\omega_{\hat\mu\hat\nu}$ is an anti self-dual generator of $SO(4)$ normalized $\omega_{\hat\mu\hat\nu}\omega^{\hat\mu\hat\nu} = 4$.

This means that δ_ε squares to a rotation around the North pole generated by ω. Then $(\hat\varepsilon_c, \hat\varepsilon_c) = \frac{1}{4r^2}(\hat\varepsilon_s, \hat\varepsilon_s)$, and thus $(\varepsilon, \varepsilon)$ is constant over S^4.

Take $(\hat\varepsilon_s, \hat\varepsilon_s) = 1$. Then we get the vector field $v_{\hat\nu} = \varepsilon\Gamma_{\hat\nu}\varepsilon = 2\hat\varepsilon_s\Gamma_{\hat\nu}\Gamma_{\hat\mu}x^{\hat\mu}\hat\varepsilon^c = 2\hat\varepsilon_s\Gamma_{\hat\nu}\Gamma_{\hat\mu}x^{\hat\mu}\frac{1}{2r}\frac{1}{4}\omega_{\hat\rho\hat\lambda}\Gamma^{\hat\rho\hat\lambda}\hat\varepsilon_s = \frac{1}{r}x^{\hat\mu}\omega_{\hat\mu\hat\nu}(\hat\varepsilon_s\hat\varepsilon_s) = \frac{1}{r}x^{\hat\mu}\omega_{\hat\mu\hat\nu}$. Using this identity we can rewrite conformal Killing spinor $\varepsilon \equiv \varepsilon(x)$ as

$$\varepsilon(x) = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}(\hat{\varepsilon}_s + \frac{1}{2r}\frac{1}{4}x^{\hat{\mu}}\Gamma_{\hat{\mu}}\omega_{\hat{\rho}\hat{\lambda}}\Gamma^{\hat{\rho}\hat{\lambda}}\Gamma^9\hat{\varepsilon}_s) = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}(\hat{\varepsilon}_s + \frac{1}{2r}x^{\hat{\rho}}\Gamma^{\hat{\lambda}}\omega_{\hat{\rho}\hat{\lambda}}\Gamma^9\hat{\varepsilon}_s) = \tag{A.2.8}$$

$$= \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}(\hat{\varepsilon}_s + \frac{1}{2}v_{\hat{\lambda}}\Gamma^{\hat{\lambda}}\Gamma^9\hat{\varepsilon}_s) = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}(\hat{\varepsilon}_s + \frac{|x|}{2r}n_{\hat{\lambda}}(x)\Gamma^{\hat{\lambda}}\Gamma^9\hat{\varepsilon}_s) = \tag{A.2.9}$$

$$= \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}(n_{\hat{\lambda}}(x)\Gamma^{\hat{\lambda}}\Gamma^9)\right)\hat{\varepsilon}_s = \exp\left(\frac{\theta}{2}n_{\hat{\lambda}}(x)\Gamma^{\hat{\lambda}}\Gamma^9\right)\hat{\varepsilon}_s, \tag{A.2.10}$$

where $n_{\hat{\lambda}}$ is the unit vector in the direction of the vector field $v_{\hat{\lambda}}$. The aim of these manipulations was to represent the spinor $\varepsilon(x)$ at an arbitrary point x by an explicit $Spin(5)$ rotation $R(x) = \exp\frac{\theta}{2}(n_{\hat{\lambda}}(x)\Gamma^{\hat{\lambda}}\Gamma^9)$ of its value at the origin $\varepsilon(0) = \hat{\varepsilon}_s$.

A.3 Off-shell supersymmetry

Let δ_ε be the supersymmetry transformation generated by a Killing spinor ε. Then the square of δ_ε is computed as follows

$$\delta_\varepsilon^2 A_M = \delta_\varepsilon(\varepsilon\Gamma_M\Psi) = \varepsilon\Gamma_M(\frac{1}{2}\Gamma^{PQ}\varepsilon F_{PQ} + \frac{1}{2}\Gamma^{\mu A}\Phi_A D_\mu\varepsilon). \tag{A.3.1}$$

Since

$$\varepsilon\Gamma_M\Gamma_{PQ}\varepsilon = \varepsilon\Gamma^T_{PQ}\Gamma_M\varepsilon = -\varepsilon\tilde{\Gamma}_{PQ}\Gamma_M\varepsilon = \frac{1}{2}\varepsilon(\Gamma_M\Gamma_{PQ} - \tilde{\Gamma}_{PQ}\Gamma_M) = 2g_{M[P}\varepsilon\Gamma_{Q]}\varepsilon,$$

the first term for $\delta_\varepsilon^2 A_M$ gives $-\varepsilon\Gamma^N\varepsilon F_{NM}$. The second term is

$$\frac{1}{2}\varepsilon\Gamma_M\Gamma^{\mu A}\Phi_A D_\mu\varepsilon = -2\varepsilon\Gamma_M\tilde{\Gamma}_A\varepsilon\Phi^A.$$

Then

$$\delta_\varepsilon^2 A_M = -(\varepsilon\Gamma^N\varepsilon)F_{NM} - 2\varepsilon\Gamma_M\tilde{\Gamma}_A\varepsilon\Phi^A. \tag{A.3.2}$$

Restricting the index m to the range of μ or A we get respectively

$$\begin{aligned} \delta_\varepsilon^2 A_\mu &= -v^\nu F_{\nu\mu} - [v^B \Phi_B, D_\mu] \\ \delta_\varepsilon^2 \Phi_A &= -v^\nu D_\nu \Phi_A - [v^B \Phi_B, \Phi_A] - 2\varepsilon\tilde{\Gamma}_{AB}\tilde{\varepsilon}\Phi^B - 2\varepsilon\tilde{\varepsilon}\Phi_A, \end{aligned} \tag{A.3.3}$$

where we introduced the vector field v

$$v^\mu \equiv \varepsilon\Gamma^\mu\varepsilon, \quad v^A \equiv \varepsilon\Gamma^A\varepsilon. \tag{A.3.4}$$

Therefore

$$\delta_\varepsilon^2 = -L_v - G_{v^M A_M} - R - \Omega. \tag{A.3.5}$$

Here L_v is the Lie derivative in the direction of the vector field v^μ. The transformation $G_{v^M A_M}$ is the gauge transformation generated by the parameter $v^M A_M$. On matter fields G acts as $G_u \cdot \Phi \equiv [u, \Phi]$, on gauge fields G acts as $G_u \cdot A_\mu = -D_\mu u$. The transformation R is the rotation of the scalar fields $(R\cdot\Phi)_A = R_{AB}\Phi^B$ with the generator $R_{AB} = 2\varepsilon\tilde{\Gamma}_{AB}\tilde{\varepsilon}$. Finally, the transformation Ω is the dilation transformation with the parameter $2(\varepsilon\tilde{\varepsilon})$.

The δ_ε^2 acts on the fermions as follows

$$\begin{aligned} \delta_\varepsilon^2 \Psi = D_M(\varepsilon\Gamma_N\Psi)\Gamma^{MN}\varepsilon + \frac{1}{2}\Gamma^{\mu A}(\varepsilon\Gamma_A\Psi)D_\mu\varepsilon = \\ = (\varepsilon\Gamma_N D_M\Psi)\Gamma^{MN}\varepsilon + ((D_\mu\varepsilon)\Gamma_N\Psi)\Gamma^{\mu N}\varepsilon + \frac{1}{2}\Gamma^{\mu A}(\varepsilon\Gamma_A\Psi)D_\mu\varepsilon. \end{aligned} \tag{A.3.6}$$

From the "triality identity" we have $\Gamma_{N\alpha_2(\alpha_1}\Gamma^N_{\alpha_3)\xi} = -\frac{1}{2}\Gamma^N_{\alpha_2\xi}\Gamma_{N\alpha_1\alpha_3}$. Then the first term gives

$$\begin{aligned} (\varepsilon\Gamma_N D_M\Psi)(\Gamma^{MN}\varepsilon)_{\alpha_4} &= (\varepsilon\Gamma_N D_M\Psi)((\tilde{\Gamma}^M\Gamma^N\varepsilon)_{\alpha_4} - g^{MN}\varepsilon_{\alpha_4}) = \\ &= \varepsilon^{\alpha_1}\Gamma_{N\alpha_1\alpha_2}D_M\Psi^{\alpha_2}\tilde{\Gamma}^M_{\alpha_4\xi}\Gamma^N_{\xi\alpha_3}\varepsilon^{\alpha_3} - (\varepsilon\Gamma^N D_N\Psi)\varepsilon_{\alpha_4} = \\ &= -\frac{1}{2}(\varepsilon^{\alpha_1}\Gamma_{N\alpha_1\alpha_3}\varepsilon^{\alpha_3})(\tilde{\Gamma}^M_{\alpha_4\xi}\Gamma^N_{\alpha_2\xi}D_M\Psi^{\alpha_2}) - (\varepsilon\Gamma^N D_N\Psi)\varepsilon_{\alpha_4} = \\ &= -\frac{1}{2}(\varepsilon\Gamma_N\varepsilon)(\tilde{\Gamma}^M\Gamma^N D_M\Psi)_{\alpha_4} - (\varepsilon\Gamma^N D_N\Psi)\varepsilon_{\alpha_4} = \\ &= -\frac{1}{2}(\varepsilon\Gamma_N\varepsilon)(-\tilde{\Gamma}^N\Gamma^M D_M\Psi + 2D_N\Psi)_{\alpha_4} - (\varepsilon\Gamma^N D_N\Psi)\varepsilon_{\alpha_4} = \\ &= \frac{1}{2}(\varepsilon\Gamma_N\varepsilon)\tilde{\Gamma}^N(\not{D}\Psi)_{\alpha_4} - (\varepsilon\Gamma^N\varepsilon)(D_N\Psi)_{\alpha_4} - (\varepsilon\not{D}\Psi)\varepsilon_{\alpha_4}. \end{aligned} \tag{A.3.7}$$

The first and the third term in the last line vanish on-shell. When we add auxiliary fields, they will cancel the first and the third term explicitly. Then we get

$$\delta_\varepsilon^2\Psi = -(\varepsilon\Gamma^N\varepsilon)D_N\Psi + (\Psi\Gamma_N D_\mu\varepsilon)\Gamma^{\mu N}\varepsilon + \frac{1}{2}\Gamma^{\mu A}(\varepsilon\Gamma_A\Psi)D_\mu\varepsilon + \mathrm{eom}[\Psi], \quad \text{(A.3.8)}$$

where eom[Ψ] stands for the terms proportional to the Dirac equation of motion for Ψ. Then we rewrite the last two terms as follows

$$\begin{aligned}
&(\Psi\Gamma_N\Gamma_\mu\tilde{\varepsilon})\Gamma^{\mu N}\varepsilon + \frac{1}{2}\Gamma^{\mu A}(\varepsilon\Gamma_A\Psi)\Gamma_\mu\tilde{\varepsilon} = \\
&= (\Psi\Gamma_N\Gamma_\mu\tilde{\varepsilon})(\tilde{\Gamma}^\mu\Gamma^N - g^{\mu N})\varepsilon - 2(\varepsilon\Gamma_A\Psi)\tilde{\Gamma}^A\tilde{\varepsilon} = (\Psi\Gamma_N\Gamma_\mu\tilde{\varepsilon})\tilde{\Gamma}^\mu\Gamma^N\varepsilon - 4(\Psi\tilde{\varepsilon})\varepsilon - 2(\varepsilon\Gamma_A\Psi)\tilde{\Gamma}^A\tilde{\varepsilon} \\
&\overset{triality}{=} -(\tilde{\varepsilon}\tilde{\Gamma}_\mu\Gamma_N\varepsilon)\tilde{\Gamma}^\mu\Gamma^N\Psi - (\varepsilon\Gamma_N\Psi)\tilde{\Gamma}^\mu\Gamma^N\Gamma_\mu\tilde{\varepsilon} - 4(\Psi\tilde{\varepsilon})\varepsilon - 2(\varepsilon\Gamma_A\Psi)\tilde{\Gamma}^A\tilde{\varepsilon} = \\
&= -(\tilde{\varepsilon}\tilde{\Gamma}_\mu\Gamma_\nu\varepsilon)\tilde{\Gamma}^\mu\Gamma^\nu\Psi - (\tilde{\varepsilon}\tilde{\Gamma}_\mu\Gamma_A\varepsilon)\tilde{\Gamma}^\mu\Gamma^A\Psi + 2(\varepsilon\Gamma_\nu\Psi)\Gamma^\nu\tilde{\varepsilon} + 4(\varepsilon\Gamma_A\Psi)\tilde{\Gamma}^A\tilde{\varepsilon} - 4(\Psi\tilde{\varepsilon})\varepsilon - 2(\varepsilon\Gamma_A\Psi)\tilde{\Gamma}^A\tilde{\varepsilon} = \\
&- (\tilde{\varepsilon}\Gamma_{\mu\nu}\varepsilon)\Gamma^{\mu\nu}\varepsilon - 4(\varepsilon\tilde{\varepsilon})\Psi \; - (\tilde{\varepsilon}\Gamma_{\mu A}\varepsilon)\Gamma^{\mu A}\varepsilon + 2(\varepsilon\Gamma_\nu\Psi)\tilde{\Gamma}^\nu\tilde{\varepsilon} + 2(\varepsilon\Gamma_A\Psi)\tilde{\Gamma}^A\tilde{\varepsilon} - 4(\Psi\tilde{\varepsilon})\varepsilon = \\
&= -\frac{1}{2}(\tilde{\varepsilon}\Gamma_{\mu\nu}\varepsilon)\Gamma^{\mu\nu}\Psi - \frac{1}{2}(\tilde{\varepsilon}\Gamma_{\mu\nu}\varepsilon)\Gamma^{\mu\nu}\Psi - 4(\varepsilon\tilde{\varepsilon})\Psi - (\tilde{\varepsilon}\Gamma_{\mu A}\varepsilon)\Gamma^{\mu A}\varepsilon - \frac{1}{2}(\tilde{\varepsilon}\Gamma_{AB}\varepsilon)\Gamma^{AB}\Psi + \\
&\qquad + \frac{1}{2}(\tilde{\varepsilon}\Gamma_{AB}\varepsilon)\Gamma^{AB}\Psi + 2(\varepsilon\Gamma_N\Psi)\tilde{\Gamma}^N\tilde{\varepsilon} - 4(\Psi\tilde{\varepsilon})\varepsilon = \\
&\qquad = \left(-\frac{1}{2}(\tilde{\varepsilon}\Gamma_{\mu\nu}\varepsilon)\Gamma^{\mu\nu}\Psi + \frac{1}{2}(\tilde{\varepsilon}\Gamma_{AB}\varepsilon)\Gamma^{AB}\Psi\right) + \\
&\qquad + \left(-\frac{1}{2}(\tilde{\varepsilon}\Gamma_{MN}\varepsilon)\Gamma^{MN}\Psi - 4(\varepsilon\tilde{\varepsilon})\Psi - 4(\Psi\tilde{\varepsilon})\varepsilon + 2(\varepsilon\Gamma_N\Psi)\tilde{\Gamma}^N\tilde{\varepsilon}\right) \quad \text{(A.3.9)}
\end{aligned}$$

The first term is a part of the Lie derivative along the vector field $v^\mu = (\varepsilon\Gamma^\mu\varepsilon)$ acting on Ψ. The second term correspond to the rotations of the scalar fields Φ^A by the generator R_{AB} and the properly induced action on the fermions.

In the $\mathcal{N} = 4$ case we use Fierz identity for $\Gamma^{MN}_{\alpha_1\alpha_2}\Gamma_{MN\,\alpha_3\alpha_4}$ in the last line of (A.3.9) to see that all term in the second pair of parentheses are canceled except for $-3(\varepsilon\tilde{\varepsilon})\Psi$, so that

$$\delta_\varepsilon^2\Psi = -(\varepsilon\Gamma^N\varepsilon)D_N\Psi - \frac{1}{2}(\tilde{\varepsilon}\Gamma_{\mu\nu}\varepsilon)\Gamma^{\mu\nu}\Psi - \frac{1}{2}(\varepsilon\tilde{\Gamma}_{AB}\tilde{\varepsilon})\Gamma^{AB}\Psi - 3(\tilde{\varepsilon}\varepsilon)\Psi + \mathrm{eom}[\Psi]. \quad \text{(A.3.10)}$$

To achieve off-shell closure in the $\mathcal{N} = 4$ case we add seven auxiliary fields K_i

with $i = 1, \ldots, 7$ and modify the transformations as

$$\begin{aligned}\delta_\varepsilon \Psi &= \frac{1}{2}\Gamma^{MN} F_{MN} + \frac{1}{2}\Gamma^{\mu A}\Phi_A D_\mu \varepsilon + K^i \nu_i \\ \delta_\varepsilon K_i &= -\nu_i \Gamma^M D_M \Psi.\end{aligned} \tag{A.3.11}$$

Here we introduced seven spinors ν_i. They depend on choice of the conformal Killing spinor ε and are required to satisfy the following relations:

$$\varepsilon \Gamma^M \nu_i = 0 \tag{A.3.12}$$

$$\frac{1}{2}(\varepsilon \Gamma_N \varepsilon)\tilde{\Gamma}^N_{\alpha\beta} = \nu^i_\alpha \nu^i_\beta + \varepsilon_\alpha \varepsilon_\beta \tag{A.3.13}$$

$$\nu_i \Gamma^M \nu_j = \delta_{ij} \varepsilon \Gamma_M \varepsilon. \tag{A.3.14}$$

The equation (A.3.12) ensures closure on A_M, the equation (A.3.13) ensures closure on Ψ.

The new term in the transformations for Ψ modifies the last line of (A.3.7) as

$$\delta_\varepsilon(K^i \nu_i) = -(\nu_i \not{D} \Psi)\nu_i.$$

Then the terms in $\delta^2_\varepsilon \Psi$ which were not taken into an account in (A.3.18) are

$$-(\nu_i \not{D}\Psi)\nu_i + \frac{1}{2}(\varepsilon \Gamma_N \varepsilon)\tilde{\Gamma}^N \not{D}\Psi - (\varepsilon \not{D}\Psi)\varepsilon. \tag{A.3.15}$$

This expression is identically zero because of (A.3.13). Hence, after inclusion of the auxiliary fields K_i, the formula (A.3.10) for $\delta^2_\varepsilon \Psi$ is valid off-shell.

For the transformation $\delta^2_\varepsilon K_i$ we get

$$\delta^2_\varepsilon K_i = -\nu_i \Gamma^M [(\varepsilon \Gamma_M \Psi), \Psi] - \nu_i \Gamma^M D_M (\frac{1}{2}\Gamma^{PQ} F_{PQ}\varepsilon + \frac{1}{2}\Gamma^{\mu A}\Phi_A D_\mu \varepsilon + K^i \nu_i). \tag{A.3.16}$$

Using the gamma matrix "triality identity" the first term is transformed to $\frac{1}{2}(\nu_i \Gamma^M \varepsilon)[(\Psi, \Gamma^M \Psi)]$, which vanishes because of (A.3.12). The second term with derivative acting on F is equal by Bianchi identity to $(\nu_i \Gamma_N \varepsilon) D_M F^{MN}$ and vanishes because of (A.3.12). Then we use (A.1.8) to simplify the remaining terms

$$\delta_\varepsilon^2 K_i = -\frac{1}{2}\nu_i\Gamma^\mu\Gamma^{PQ}\Gamma_\mu\tilde{\varepsilon}F_{PQ} - \frac{1}{2}(\nu_i\Gamma^M\Gamma_{\mu A}\Gamma^\mu\tilde{\varepsilon})D_M\Phi_A - \frac{1}{2}(-\frac{1}{4r^2})\Phi_A\nu_i\Gamma^\nu\Gamma^{\mu A}\Gamma_\mu\Gamma_\nu\varepsilon -$$
$$- \nu_i\Gamma^M(D_M K^j)\nu_j - (\nu_i\Gamma^\mu D_\mu\nu_j)K^j =$$
$$= -\frac{1}{2}(4)\nu_i\tilde{\Gamma}^{MB}\tilde{\varepsilon}D_M\Phi_B - \frac{1}{2}(-4)\nu_i\tilde{\Gamma}^{MB}\tilde{\varepsilon}D_M\Phi_B + (\frac{2}{r^2})\nu_i\Gamma^A\varepsilon\Phi_A +$$
$$- (\nu_i\Gamma^M\nu_j)D_M K^j - (\nu_i\Gamma^\mu D_\mu\nu_j)K^j = -(\varepsilon\Gamma^M\varepsilon)D_M K^j - (\nu_i\Gamma^M D_M\nu_j)K^j =$$
$$= -(\varepsilon\Gamma^M\varepsilon)D_M K^i - (\nu_{[i}\Gamma^\mu D_\mu\nu_{j]})K^j - 4(\tilde{\varepsilon}\varepsilon)K_i. \quad \text{(A.3.17)}$$

To get the last line we use the differential of (A.3.14), i.e. $\nu_{(i}\not{D}\nu_{j)} = 4(\varepsilon\tilde{\varepsilon})\delta_{ij}$.

Now we consider separately the case of pure $\mathcal{N} = 2$ Yang-Mills. First we rewrite the last terms in (A.3.9) as follows (here d is the dimension of uncompactified theory)

$$(\tilde{\varepsilon}\Gamma_{MN}\varepsilon)\Gamma^{MN}\Psi = (\tilde{\varepsilon}\tilde{\Gamma}_M\Gamma_N\varepsilon)\Gamma^{MN}\Psi = (\tilde{\varepsilon}\tilde{\Gamma}_M\Gamma_N\varepsilon)\tilde{\Gamma}^M\Gamma^N\Psi - d(\tilde{\varepsilon}\varepsilon)\Psi \overset{triality}{=}$$
$$- (\varepsilon\Gamma_N\Psi)\tilde{\Gamma}^M\Gamma^N\tilde{\Gamma}_M\tilde{\varepsilon} - (\Psi\Gamma_N\tilde{\Gamma}_M\tilde{\varepsilon})\tilde{\Gamma}^M\Gamma^N\varepsilon - d(\tilde{\varepsilon}\varepsilon)\Psi =$$
$$= (d-2)(\varepsilon\Gamma_N\Psi)\tilde{\Gamma}^N\tilde{\varepsilon} - (\Psi\Gamma_N\tilde{\Gamma}_M\tilde{\varepsilon})\tilde{\Gamma}^M\Gamma^N\varepsilon - d(\tilde{\varepsilon}\varepsilon)\Psi. \quad \text{(A.3.18)}$$

For the pure $\mathcal{N} = 2$ theory in four-dimensions we take $d = 6$ and get

$$\left(-\frac{1}{2}(\tilde{\varepsilon}\Gamma_{MN}\varepsilon)\Gamma^{MN}\Psi - 4(\varepsilon\tilde{\varepsilon})\Psi - 4(\Psi\tilde{\varepsilon})\varepsilon + 2(\varepsilon\Gamma_N\Psi)\tilde{\Gamma}^N\tilde{\varepsilon}\right) =$$
$$- \frac{1}{2}\left(4(\varepsilon\Gamma_N\Psi)\tilde{\Gamma}^N\tilde{\varepsilon} - (\Psi\Gamma_N\tilde{\Gamma}_M\tilde{\varepsilon})\tilde{\Gamma}^M\Gamma^N\varepsilon - 6(\tilde{\varepsilon}\varepsilon)\Psi\right) -$$
$$- 4(\varepsilon\tilde{\varepsilon})\Psi - 4(\Psi\tilde{\varepsilon})\varepsilon + 2(\varepsilon\Gamma_N\Psi)\tilde{\Gamma}^N\tilde{\varepsilon} = \frac{1}{2}(\Psi\Gamma_N\tilde{\Gamma}_M\tilde{\varepsilon})\tilde{\Gamma}^M\Gamma^N\varepsilon - (\varepsilon\tilde{\varepsilon})\Psi - 4(\Psi\tilde{\varepsilon})\varepsilon =$$
$$= \frac{1}{2}(\Psi(-\Gamma_M\tilde{\Gamma}_N + 2g_{MN})\tilde{\varepsilon})\tilde{\Gamma}^M\Gamma^N\varepsilon - (\varepsilon\tilde{\varepsilon})\Psi - 4(\Psi\tilde{\varepsilon})\varepsilon =$$
$$= -\frac{1}{2}(\Psi\Gamma_M\tilde{\Gamma}_N\tilde{\varepsilon})\tilde{\Gamma}^M\Gamma^N + 6(\Psi\tilde{\varepsilon})\varepsilon - (\varepsilon\tilde{\varepsilon})\Psi - 4(\Psi\tilde{\varepsilon})\varepsilon =$$
$$= -\frac{1}{2}(\Psi\Gamma_M\tilde{\Gamma}_N\tilde{\varepsilon})\tilde{\Gamma}^M\Gamma^N\varepsilon + 2(\Psi\tilde{\varepsilon})\varepsilon - (\varepsilon\tilde{\varepsilon})\Psi. \quad \text{(A.3.19)}$$

We express the first term in terms of the triplet of matrices Λ^i, which are defined as a set of three antisymmetric matrices such that

$$\Lambda^i_{\alpha_1\alpha_3}\Lambda^j_{\alpha_2\alpha_3} = \epsilon^{ijk}\Lambda^k_{\alpha_1\alpha_2} + \delta^{ij}1_{\alpha_1\alpha_2}, \quad i,j,k = 1,\ldots,3. \tag{A.3.20}$$

$$[\Lambda_i, \Gamma^M] = 0 \tag{A.3.21}$$

$$\frac{1}{2}\Gamma^M_{\alpha_1\alpha_2}\tilde{\Gamma}_{M\,\alpha_3\alpha_4} = \delta_{\alpha_2(\alpha_1}\delta_{\alpha_3)\alpha_4} - \Lambda^i_{\alpha_2(\alpha_1}\Lambda^i_{\alpha_3)\alpha_4}. \tag{A.3.22}$$

Then we get

$$(\Psi\Gamma_M\tilde{\Gamma}_N\tilde{\varepsilon})\tilde{\Gamma}^M\Gamma^N\varepsilon = 4(\Psi\tilde{\varepsilon})\varepsilon + 4(\varepsilon\tilde{\varepsilon})\Psi + 4(\varepsilon\Lambda^i\tilde{\varepsilon})\Lambda^i\Psi, \tag{A.3.23}$$

and finally the equation (A.3.19) turns into

$$-2(\Psi\tilde{\varepsilon})\varepsilon - 2(\varepsilon\tilde{\varepsilon})\Psi - 2(\varepsilon\Lambda^i\tilde{\varepsilon})\Lambda^i\Psi + 2(\Psi\tilde{\varepsilon})\varepsilon - (\varepsilon\tilde{\varepsilon})\Psi = -2(\varepsilon\Lambda^i\tilde{\varepsilon})\Lambda^i\Psi - 3(\tilde{\varepsilon}\varepsilon)\Psi. \tag{A.3.24}$$

That completes simplification of δ^2_ε acting on fermions

$$\delta^2_\varepsilon\Psi = -(\varepsilon\Gamma^N\varepsilon)D_N\Psi - \frac{1}{2}(\tilde{\varepsilon}\Gamma_{\mu\nu}\varepsilon)\Gamma^{\mu\nu}\Psi - \frac{1}{2}(\varepsilon\tilde{\Gamma}_{AB}\tilde{\varepsilon})\Gamma^{AB}\Psi - 2(\varepsilon\Lambda^i\tilde{\varepsilon})\Lambda^i\Psi - 3(\tilde{\varepsilon}\varepsilon)\Psi. \tag{A.3.25}$$

It has the structure

$$\delta^2_\varepsilon\Psi = -L_v\Psi - G_{v^NA_N}\Psi - R\Psi - R'\Psi - \Omega\Psi, \tag{A.3.26}$$

where the notations for the generators are the same as in the bosonic case. The only new generator here is R', corresponding to the term $\delta^2_\varepsilon\Psi = -2(\varepsilon\Lambda^i\tilde{\varepsilon})\Lambda^i\Psi$. It generates an $SU(2)_L$ R-symmetry transformation of $\mathcal{N}=2$ which acts trivially on the bosonic fields of the theory, and as $\Psi \mapsto e^{r_i\Lambda_i}\Psi$ on fermionic fields.

To achieve off-shell closure in $\mathcal{N}=2$ case we add a triplet of auxiliary fields K_i and modify the transformations as

$$\begin{aligned} \delta_\varepsilon\Psi &= \frac{1}{2}\Gamma^{MN}F_{MN} + \frac{1}{2}\Gamma^{\mu A}\Phi_A D_\mu\varepsilon + K^i\Lambda_i\varepsilon \\ \delta_\varepsilon K_i &= \varepsilon\Lambda_i\Gamma^M D_M\Psi, \end{aligned} \tag{A.3.27}$$

The new term in the transformations for Ψ modifies the last line of (A.3.7) as

$$\delta_\varepsilon(K^i\Lambda_i\varepsilon) = (\varepsilon\Lambda_i \not{D}\Psi)\Lambda_i\varepsilon.$$

Then the terms in $\delta_\varepsilon^2\Psi$ which were not taken into an account in (A.3.18) are

$$(\varepsilon\Lambda_i \not{D}\Psi)\Lambda_i\varepsilon + \frac{1}{2}(\varepsilon\Gamma_N\varepsilon)\tilde{\Gamma}^N \not{D}\Psi - (\varepsilon\not{D}\Psi)\varepsilon. \tag{A.3.28}$$

This expression is identically zero because of the relation (A.3.20). Hence, after inclusion of the auxiliary fields K_i, the formula (A.3.10) for $\delta_\varepsilon^2\Psi$ is valid off-shell.

Remark. The second equation (A.3.13) follows from the first equation (A.3.12) and the third equation (A.3.12) as follows. Let

$$M_{\alpha\beta} = \nu_\alpha^i \nu_\beta^i + \varepsilon_\alpha\varepsilon_\beta.$$

We want to show that $M_{\alpha\beta} = \frac{1}{2}v_N\tilde{\Gamma}^N_{\alpha\beta}$, that is the matrix $M_{\alpha\beta}$ can be expanded over the matrices $\tilde{\Gamma}^N_{\alpha\beta}$ with the coefficients $\frac{1}{2}v_N$. Fix the positive definite metric on the space $\mathbb{R}^{16\times 16}$ of 16×16 matrices as $(M,M) := M_{\alpha\beta}M_{\alpha\beta}$. Since $\tilde{\Gamma}^N = \Gamma_N$ and $\Gamma_M^{\alpha\beta}\tilde{\Gamma}^N_{\alpha\beta} = 16\delta_M^N$, the set of 10 matrices $\frac{1}{4}\Gamma_N$ is orthonormal in $\mathbb{R}^{16\times 16}$. Complete this set to the basis of $\mathbb{R}^{16\times 16}$. Then the coefficient m_N of $\frac{1}{4}\Gamma_N$ in the expansion of M over this basis is given by the scalar product

$$m_N = (M, \frac{1}{4}\Gamma_N) = \frac{1}{4}(\nu^i\Gamma_N\nu^i + \varepsilon\Gamma_N\varepsilon) = 2v_N.$$

Therefore we have $M = 2v_N(\frac{1}{4}\Gamma_N) + (\dots)$, where $(\dots)$ stands for possible other terms in the expansion over the completion of the set $\{\frac{1}{4}\Gamma_N\}$ to the basis of $\mathbb{R}^{16\times 16}$. To prove that all other terms vanish, compare the norm of M

$$(M,M) = (\varepsilon\varepsilon)(\varepsilon\varepsilon) + (v_iv_j)(v_iv_j) = (\varepsilon\varepsilon) + \delta_{ij}(\varepsilon\varepsilon)\delta_{ij}(\varepsilon\varepsilon) = 8(\varepsilon\varepsilon)(\varepsilon\varepsilon)$$

with the $\sum_N m_N^2$

$$\sum_N m_N^2 = 4v_Nv_N = 4(\varepsilon\Gamma_N\varepsilon)(\varepsilon\tilde{\Gamma}^N\varepsilon) = 4((\varepsilon\Gamma_N\varepsilon)(\varepsilon\Gamma^N\varepsilon) + 2(\varepsilon\varepsilon)(\varepsilon\varepsilon)) = 8(\varepsilon\varepsilon)(\varepsilon\varepsilon).$$

Since the norms are the same, $(M,M) = \sum_N m_N^2$, and the metric is positive definite, we conclude that all other coefficients vanish.

A.4 Index of transversally elliptic operators

Here we collect some facts about indices of transversally elliptic operators mostly following Atiyah's book [71]. See also [72].

Let $\cdots \to E^i \xrightarrow{D_i} E^{i+1} \to \dots$ be an elliptic complex of vector bundles over a manifold X. Let a Lie group G act on X and bundles E^i. This means that for any transformation $g : X \to X$, which sends a point $x \in X$ to $g(x)$, we are given a vector bundle homomorphisms $\gamma^i : g^* E^i \to E^i$. Then we have natural linear maps $\hat{\gamma}^i : \Gamma(E^i) \to \Gamma(E^i)$ defined by $\hat{\gamma}^i = \gamma^i \circ g^*$. On any section $s(x) \in \Gamma(E^i)$ the map $\hat{\gamma}^i$ acts by the formula $(\hat{\gamma} s)(x) = \gamma_x s(g(x))$. We assume that $\hat{\gamma}$ commutes with the differential operators D_i of the complex E. Then $\hat{\gamma}$ descends to a well-defined action on the cohomology groups $H^i(E)$.

The G-equivariant index is defined as

$$\mathrm{ind}_g(E) = \sum_i (-1)^i \operatorname{tr}_{H^i} \hat{\gamma}^i. \tag{A.4.1}$$

In the case when the set of G-fixed points is discrete and the action of G is nice in a neighborhood of each of the fixed point, the Atiyah-Bott fixed point formula says [89–91]

$$\mathrm{ind}_g(E) = \sum_{x \in \text{fixed point set}} \frac{\sum (-1)^i \operatorname{tr} \gamma_x^i}{|\det(1 - dg(x))|}. \tag{A.4.2}$$

This formula can be easily argued in the following way (see [112] for a derivation using supersymmetric quantum mechanics). For an illustration we consider the case when the complex E consists of two vector bundles $E^0 \xrightarrow{D} E^1$, and we assume that the bundles are equipped with a hermitian G-invariant metric. Let $D : \Gamma(E^0) \to \Gamma(E^1)$ be the differential. Then we consider the Laplacian $\Delta = DD^* + D^*D$. The zero modes of the Laplacian are identified with the cohomology groups of E, which are in this case: $H^0(E) = \ker D$ and $H^1(E) = \operatorname{coker} D$. Hence, the index can be computed as

$$\mathrm{ind}_g(E) = \lim_{t \to \infty} \operatorname{str}_{\Gamma(E)} \hat{\gamma} e^{-t\Delta}.$$

Here the supertrace for operators acting on $\Gamma(E)$ is defined assuming even parity on $\Gamma(E^0)$ and odd parity on $\Gamma(E^1)$. However, the expression under the limit sign actually does not depend on t because $[\Delta, \hat{\gamma}] = 0$. Taking the limit $t \to 0$ we get supertrace of $\hat{\gamma}$. The trace can be easily taken in the coordinate representation. By definition, the operator $\hat{\gamma}$ has kernel $\hat{\gamma}(x, y) = \gamma_x \delta(g(x) - y)$ if we write $(\hat{\gamma} s)(x) = \int_X \hat{\gamma}(x, y) s(y)$. Here $\delta(x)$ is the Dirac delta-function. Taking the trace we get Atiyah-Bott result

$$\begin{aligned}\operatorname{ind}_g(E) = \lim_{t \to 0} \operatorname{str}_{\Gamma(E)} \hat{\gamma} e^{-t\Delta} = \int dx \operatorname{str}_{E_x} \hat{\gamma}(x, x) = \int dx \operatorname{str}_{E_x} \gamma_x \delta(g(x) - x) = \\ = \sum_{g(x)=x} \frac{\operatorname{str}_{E_x} \gamma_x}{|\det(1 - dg(x))|}. \quad \text{(A.4.3)}\end{aligned}$$

Let X be a complex manifold of dimension n. Consider the complex of $(0, p)$-forms with the differential $\bar{\partial}$. Let $G = U(1)$ acts on X holomorphically. In a neighborhood of a fixed point we can choose such coordinates $z^1, \dots, z^n$ that an element $g \in G$ acts by $z^i \to q_i z^i$. If z^i transforms in a $U(1)$ representation $m_i \in \mathbb{Z}$, and we parameterize $U(1)$ by a unit circle $\{|q| = 1, q \in \mathbb{C}\}$, then $q_i = q^{m_i}$. One-forms $f_{\bar{i}}$ transform as $f_{\bar{i}} \to \bar{q}_i^{-1} f_{\bar{i}}$. Since $|q| = 1$ we have $f_{\bar{i}} \to q_i f_{\bar{i}}$. Computing the supertrace for the numerator on external powers of the anti-holomorphic subspace of the fiber of the cotangent bundle at the origin, we get $\operatorname{str}_{\Omega^{0,\bullet}} q = \prod_{i=1}^n (1 - q_i)$. The denominator is $\prod_{i=1}^n (1 - q_i)(1 - q_i^{-1})$. Then contribution of a fixed point with weights $\{q_1, \dots, q_n\}$ to the index of $\bar{\partial}$ is

$$\operatorname{ind}_q(\bar{\partial})|_0 = \frac{1}{\prod_{i=1}^n (1 - q_i^{-1})}.$$

Let $\pi : T^*X \to X$ be the cotangent bundle. Then $\pi^* E_i$ are the bundles over T^*X. The symbol of the differential operator $D : \Gamma(E_0) \to \Gamma(E_1)$ is a vector bundle homomorphism $\sigma(D) : \pi^* E_0 \to \pi^* E_1$. In local coordinates x_i it is defined by replacing all partial derivatives in the highest order component of D by momenta, so $\frac{\partial}{\partial x^i} \to i p_i$, and then taking p_i to be coordinates on fibers of T^*X. Let the family of the vector spaces $T^*_G X$ be a union of vector spaces $T^*_G X_x$ over all points $x \in X$, where $T^*_G X_x$ denotes a subspace of T^*X transversal to the G-orbit through x. The operator D is transversally elliptic if its symbol $\sigma(D)$ is invertible on $T^*_G X \setminus 0$, where 0 denotes the zero section.

We need a few notions of K-theory [113]. Let $\mathrm{Vect}(X)$ be the set of isomorphism classes of vector bundles on X. It is an abelian semigroup where addition is defined as the direct sum of vector bundles. For any abelian semigroup A we can associate an abelian group $K(A)$ by taking all equivalence classes of pairs $(a, b) \sim (a+c, b+c)$, where $a, b, c \in A$. Taking $\mathrm{Vect}(X)$ as A we define the K-theory group $K(X)$. Its elements are pairs of isomorphism classes of vector bundles (E_0, E_1) over X up to the equivalence relation $(E_1, E_1) \sim (E_0 \oplus H, E_1 \oplus H)$ for all vector bundles H over X. If X is a space with a basepoint x_0, then we define $\tilde{K}(X)$ as a kernel of the map $i^* : K(X) \to K(x_0)$ where $i : x_0 \to X$ is the inclusion map. Next we define relative K-theory group $K(X, Y)$ for a compact pair of spaces (X, Y). Let X/Y be the space obtained by considering all points in Y to be equivalent and taking this equivalence class as a basepoint. Then $K(X, Y)$ is defined as $\tilde{K}(X/Y)$. Equivalently, $K(X, Y)$ consists of pairs of vector bundles (E_0, E_1) over X such that E_0 is isomorphic to E_1 over Y, and considered up to the equivalence relation $(E_0, E_1) \sim (E_0 \oplus H, E_1 \oplus H)$ for all vector bundles H over X. For a non-compact space, such as a total space of vector bundle $V \to X$, we define $K(V)$ as $\tilde{K}(X^V)$, where X^V is a one-point compactification of V, or equivalently $B(V)/S(V)$, where $B(V)$ and $S(V)$ is respectively a unit ball and unit sphere on V.

If a group G acts on X we can consider the set of isomorphism classes of G-vector bundles over X. It is an abelian semi-group, to which we associate an abelian group $K_G(X)$. All constructions above can be done in G-equivariant fashion.

Since the symbol of a transversally elliptic operator is an isomorphism $\sigma(D) : \pi^*E \to \pi^*F$ of vector bundles over $T^*_G X$ outside of zero section, by definition it represents an element of $K_G(T^*_G X)$. One can show that the index of transversally elliptic operator does not depend on continuous deformations of it symbol, hence it depends only on the homotopy type of the symbol. The index vanishes for a symbol which is induced by an isomorphism of vector bundles E and F. Therefore the index of D depends only on an element of $K_G(T^*_G X)$ which represents symbol $\sigma(D)$.

The equivariant index was defined for any group element g as an alternating sum of traces of g in representations R^i in which G acts on the cohomology groups H^i of the complex E. One can show that for transversally elliptic operators the representations R^i can be decomposed into a direct sum of irreducible representations where each irreducible representation enters with a finite multiplicity. In the elliptic case the number of irreducible representations which appear is finite since cohomology groups H^i have finite dimensions. Let χ_α be a character for each irreducible representation α. Then the index of transversally elliptic operator is $\sum_\alpha m_\alpha \chi_\alpha$ where m_α are finite integer multiplicities. Thus the index can be regarded as a distribution on G, so that the multiplicities m_α are coefficients in its Fourier series expansion. Let $\mathcal{D}'(G)$ be the space of distributions on G.

Consider an example. Let X be a circle S^1 on which group $G = U(1)$ acts in a natural way. Let E_0 be the trivial rank one bundle $\mathcal{E}$ over S^1, and E_1 be the zero bundle. Let $D : \Gamma(\mathcal{E}) \to 0$ be the zero operator. Then the cohomology group H^0 is the space of all functions on a circle, and H^1 vanishes. Functions on a circle can be decomposed into Fourier modes labeled by integers, so that each mode corresponds to an irreducible representation of $U(1)$. If $q = e^{i\alpha}$ for $\alpha \in [0, 2\pi)$ denotes an element of $U(1)$, then we obtain the index

$$\operatorname{ind} 0 = \sum_{-\infty}^{\infty} q^n = \sum_{-\infty}^{\infty} e^{in\alpha} = 2\pi\delta(\alpha).$$

We see that the index is not a smooth function on $U(1)$, but a distribution – the Dirac delta-function.

We learned that the index is a map from K-theory group of $T^*_G X$ to distributions on G

$$\operatorname{ind} : K_G(T^*_G X) \to \mathcal{D}'(G).$$

Moreover, the index is a group homomorphism with respect to the abelian group structure on $K_G(T^*_G X)$ and the addition operation on $\mathcal{D}'(G)$. The abelian groups $\mathcal{D}'(G)$ and $K_G(T^*_G X)$ are modules over the character ring $R(G)$. Indeed, $K_G(pt) = R(G)$ since elements of $R(G)$ are formal linear combinations of irreducible representations of G, and $K_G(X)$ has a module structure over $K_G(pt)$, since we can take tensor products of vector bundles representing $K_G(X)$ with trivial vector bundles representing $K_G(pt)$. The module $\mathcal{D}'(G)$ has a torsion submodule. For example, the Dirac delta-function on a circle supported at $q = 1$ is a torsion element of $\mathcal{D}'(U(1))$, because it is annihilated by $q - 1$. One can show that the support of the index is a subset of points $g \in G$ for which $X^g \neq \varnothing$, where $X^g \subset X$ is the g-fixed set. If G acts freely on X then the index is supported at the identity of G, hence it is a pure torsion element.

From now we consider the case $G = U(1)$. We can find torsion free part of the index if we know it as a function on a generic group element $g \neq \mathrm{Id}$. If X^g consists of non-degenerate points, then we can repeat the argument used in the elliptic case and obtain the formula (A.4.3). In the elliptic case, separate contributions from fixed points are not well defined at $q = 1$, but the total sum is well defined, since the index is a finite polynomial in q and q^{-1}. In the transversally elliptic case, if we add contributions of fixed points formally defined by the formula (A.4.3), we will obtain correctly only the torsion free part of the index. In other words, we will obtain the index up to a singular distribution supported at $q = 1$.

To fix the torsion part, we should find a way in which we associate distributions to rational functions given by the formula (A.4.3). This procedure is explained in details in [71]. For example, the contribution to the index of $\bar{\partial}$ operator from the origin of $\mathbb{C}$ as a rational function is

$$\mathrm{ind}_q(\bar{\partial})|_0 = \frac{1}{1 - q^{-1}}. \tag{A.4.4}$$

There are two basic ways to associate a distribution to it, which we call expansions in positive or negative powers of q:

$$\left[\frac{1}{1-q^{-1}}\right]_+ = -\frac{q}{1-q} = -\sum_{n=1}^{\infty} q^n \tag{A.4.5}$$

$$\left[\frac{1}{1-q^{-1}}\right]_- = -\sum_{n=0}^{\infty} q^{-n}. \tag{A.4.6}$$

These two regularizations differ by a torsion element (a distribution supported at $q = 1$):

$$\left[\frac{1}{1-q^{-1}}\right]_+ - \left[\frac{1}{1-q^{-1}}\right]_- = -\sum_{n=-\infty}^{\infty} q^n = -2\pi i \delta(q-1).$$

The decomposition of $K_G(T^*_G X)$ to the torsion part and the torsion free part can be described by the exact sequence

$$0 \to K_G(T^*_G(X \setminus Y)) \to K_G(T^*_G X) \to K_G(T^*X|_Y) \to 0, \tag{A.4.7}$$

where Y is the fixed point set in X. Since G acts freely on $X \setminus Y$, the image of $K_G(T^*_G(X \setminus Y))$ under the index homomorphism is a torsion submodule of $\mathcal{D}'(G)$. The last term of the sequence is the torsion free quotient determined completely by the fixed point set Y. Using a vector field v on X generated by action of G, it is possible to construct two homomorphisms

$$\theta^{\pm} : K_G(T^*X|_Y) \to K_G(T^*_G X),$$

where $\pm$ signs correspond to a choice of the direction of the vector field. First, given a symbol $\sigma : \pi^* E_0 \to \pi^* E_1$, representing an element of $K_G(T^*X|_Y)$, we extend it to an open neighborhood U of Y. It is an isomorphism outside of the zero section. Second, we define a symbol $\tilde{\sigma}$, restricting symbol σ to fibers of $T^*_G X$ shifted in the direction of the vector field v

$$\tilde{\sigma}(x,p) = \sigma(x, p + v e^{-p^2}),$$

where (x,p) are local coordinates on T^*X in a neighborhood of Y. Outside of Y the symbol $\tilde{\sigma}$ is an isomorphism for all points on fibers of $T^*_G X$ (not only outside of zero section). In other words, $\tilde{\sigma}$ is an isomorphism everywhere in the neighborhood U outside of the fixed point set Y. Hence $\tilde{\sigma}$ represents an element of $K_G(T^*_G U)$. Since U is open in X, using the natural homomorphism $K_G(T^*_G U) \to K_G(T^*_G X)$ we get an element of $K_G(T^*_G X)$.

Applying this construction to the space $X = \mathbb{C}^n$ on which $U(1)$ acts with positive weights $m_1, \ldots, m_n$, and taking generator of $K(T^*\mathbb{C}^n|_0)$ associated with $\bar{\partial}$ operator, we get its images under $\theta^{\pm}$ in $K_G(T^*_G\mathbb{C}^n)$. A direct computation shows that

$$\operatorname{ind}\theta^{\pm}[\bar{\partial}] = \left[\frac{1}{\prod(1-q^{-m_i})}\right]_{\pm}.$$

Now assume that using the vector field v it is possible to trivialize a transversally elliptic operator everywhere on $T^*_G X$ outside of the fixed point set Y, and that in a neighborhood of the fixed point set the trivialization is isomorphic to just described with some choice of $\pm$ signs for each fixed point. Then the index is computed by summing contributions from the set of fixed points, where each contribution is regularized by an expansion in positive or negative powers of q, according to the choice of sign for the θ homomorphism.

For example in this way we get the $U(1)$ index of the following operator on $\mathbb{CP}^1$:

$$\operatorname{ind}(f(\theta)\bar{\partial} + (1-f(\theta))\partial) = \left[\frac{1}{1-q^{-1}}\right]_{+} + \left[\frac{1}{1-q^{-1}}\right]_{-}. \tag{A.4.8}$$

Here θ denotes the polar angle on $\mathbb{CP}^1$ measured from the North pole, and $f(\theta) = \cos^2(\theta/2)$, so that the operator is approximately $\bar{\partial}$ at the North pole and ∂ at the South pole. It fails to be elliptic at the equator, but it is transversally elliptic with respect to the canonical $U(1)$ action on $\mathbb{CP}^1$ whose fixed points are the North and South poles.

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