

# Global 4-group symmetry and 't Hooft anomalies in topological axion electrodynamics

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Received September 3, 2021; Accepted November 17, 2021; Published November 22, 2021

We study higher-form global symmetries and a higher-group structure of a low-energy limit of  $(3 + 1)$ -dimensional axion electrodynamics in a gapped phase described by a topological action. We argue that the higher-form symmetries should have a semi-strict 4-group (3-crossed module) structure by consistency conditions of couplings of the topological action to background gauge fields for the higher-form symmetries. We find possible 't Hooft anomalies for the 4-group global symmetry, and discuss physical consequences.

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## 1. Introduction

Axions are hypothetical pseudo-scalar bosons and have been studied in various contexts in modern physics, such as particle physics, cosmology, string theory, hadron physics, and condensed matter physics. In the context of particle physics, the axion, called the QCD axion, was introduced as a candidate for the solution to the strong CP problem [1–7]. Later the axion was considered to be a candidate for dark matter as well [8–11]. The axion was also generalized to axion-like particles [12,13], which are not necessary to solve the strong CP problem, but have similar virtues to the QCD axion, as we will mention below. Such axion-like particles can naturally arise as moduli fields in 4D effective theories of string theory [14–19]. They have also been discussed as candidates for dark matter [20] or an inflaton that can cause inflation in the early universe [21,22]. In the context of condensed matter physics, axions have been regarded as quasi-particle excitations [23,24] or parameters that characterize topological insulators [25–27].<sup>1</sup>

One of the virtues of axions is topological coupling to a photon. Here, “topological coupling” means that it does not depend on the metric of the spacetime. Such a coupling exists as a result of a chiral anomaly of Dirac fermions coupled to the axion and photon. Thus, the coupling is stable against higher-order radiative corrections. Axion–photon coupling plays important roles in the above applications, such as the decay of axions to photons, magneto-electric responses, and so on [25–27,35–43]. The simplest model given by only an axion and a photon with topological coupling is called “axion electrodynamics” [25].

<sup>1</sup>See, e.g., Refs. [28–32] and Refs. [33,34] for reviews of axions in particle physics and condensed matter physics, respectively.

There can be several phases of axion electrodynamics according to the mass gaps of the axion and photon. In particular, the phase in which both the axion and photon have mass gaps has been investigated in the context of, e.g., topological superconductors in  $(3 + 1)$  dimensions [44–46]. One of the characteristic features of this gapped phase is that there can be topological solitons, i.e., topologically stable objects, in addition to magnetic monopoles and axionic strings that can always exist. When the photon is massive, there can be quantized magnetic fluxes or vortex strings, which are called Abrikosov–Nielsen–Olesen (ANO) vortex strings [47,48]. When the axion is massive, there can be axionic domain walls, which connect distinct vacua of the axion, and a single axionic string is attached by some axionic domain walls [49,50]. The electromagnetic properties of the topological objects have been investigated since they have been proposed. For systems with massive photons, an ANO vortex string exhibits the Aharonov–Bohm (AB) effect with a fractionally quantized phase due to its quantized magnetic flux. For systems with massive axions, an axionic domain wall has an induced electric charge when a magnetic flux is penetrated to the wall, which we will call the Sikivie effect in this paper [36] (see also Refs. [25,26,51]). Furthermore, the axionic domain wall exhibits an anomalous Hall effect: When an electric flux, instead of a magnetic flux, is applied along an axionic domain wall, the domain wall has an induced electric current whose direction is perpendicular to the electric field [25,26,36,51].

A natural question that arises is the following: What is the underlying structure of these electromagnetic effects for the topological solitons? The notion of extended symmetries may be one key ingredient to understand them. Recently, symmetries for extended objects and topological solitons have been investigated in the language of higher-form symmetries; higher  $p$ -form symmetries are symmetries under actions on  $p$ -dimensional extended objects [52–54] (see also Refs. [55–63]). The conventional symmetries can be understood as 0-form symmetries, since they act on local 0D operators. In contrast, the AB effect in the gapped phase can be understood as a 2-form symmetry, where the charged object is a worldsheet of a vortex line. Higher-form symmetries have been applied to various systems in quantum field theories [59,61,64–77].

As in the conventional symmetries, higher-form symmetries can be correlated to each other. Their correlations can be elegantly described by  $n$ -groups [78]. Roughly speaking, an  $n$ -group is a set of groups for  $0, \dots, (n - 1)$ -form symmetries with actions among them. Quantum field theories with global 2- and 3-group symmetries have been investigated in Refs. [64,79–100]. The higher-group symmetries can be efficiently found by coupling symmetry generators to background gauge fields for the higher-form symmetries. In general, background gauging of a  $p$ -form symmetry can be given by a  $(p + 1)$ -form gauge field [54]. The  $n$ -group symmetry can be found by nontrivial gauge transformation laws between 1-, 2-, ...,  $n$ -form gauge fields.

In particular, the  $(3 + 1)$ -dimensional axion electrodynamics with a massless axion and photon was found to be one of the simplest examples possessing a 3-group structure [101,102]. In this system, there are 0-, 1-, 2-form symmetries associated with equations of motion and Bianchi identities for the axion and photon. The 3-group structure has been found by correlation functions between symmetry generators as well as background gauging of the higher-form symmetries. If the axion and photon become massive, we expect that the higher-form symmetries are different from those of the massless axion and photon, since there can be symmetries associated with conservation of the ANO vortex strings and axionic domain wall in the gapped phase. Thus, it is nontrivial to ask what higher-form symmetries and the associated higher-group symmetry in axion electrodynamics in the gapped phase are.

In our previous paper [103], the low-energy effective action of the  $(3+1)$ -dimensional axion electrodynamics in the gapped phase was constructed. Since the action only contains topological terms that do not depend on the metric of the spacetime, the effective theory is referred to as “topological axion electrodynamics”. In this effective theory, it was found that there are 0-, 1-, 2-, and 3-form symmetries. Furthermore, symmetry generators of the 0- and 1-form symmetries have nontrivial correlations, similar to the current algebra of conventional symmetries. A 2-form symmetry generator is induced on an intersection of the 0- and 1-form symmetry generators. Similarly, a 3-form symmetry generator is obtained on intersections of two 1-form symmetry generators. Such nontrivial correlations are signals for higher-group symmetries and, in fact, a 4-group structure has been found.

In this paper, we investigate higher-form symmetries in topological axion electrodynamics in more detail. In particular, we discuss couplings of background gauge fields for the higher-form symmetries. We find that the gauging of each symmetry should be correlated. The 1-form symmetry cannot be solely gauged by preserving the large gauge invariance; we need to gauge and modify the 3-form symmetry simultaneously. Further, simultaneous gauging of the 0- and 1-form symmetries requires additional gauging and a modification of the 2-form symmetry. We determine the modifications by the large gauge invariance. By these modifications, we obtain a gauge theory of 1-, 2-, 3-, and 4-form gauge fields with correlations among them.

We then determine the higher-group structure of topological axion electrodynamics by the modified background gauge fields. The corresponding group is identified as a semi-strict 4-group or 3-crossed module [104]. We specify the ingredients of the semi-strict 4-group by using the structure of the modified gauge fields.

By background gauging, we show 't Hooft anomalies for the higher-form symmetries. In general, 't Hooft anomalies are obstructions to gauging global symmetries dynamically [105–107]. The presence of 't Hooft anomalies forbids a symmetry-preserving gapped vacuum, since such a vacuum does not have degrees of freedom that can match the anomalies. The 't Hooft anomalies can be expressed as the ambiguity of the choice of a 5D space on which an action of the background gauge fields is defined. It has been seen in many cases that 't Hooft anomalies in  $D$  spacetime dimensions can be canceled by adding a boundary of an appropriate  $(D+1)$ -dimensional action given by background gauge fields. In our case, there are mixed 't Hooft anomalies between the pair of the 0- and 3-form symmetries as well as the pair of the 1- and 2-form symmetries. Furthermore, there is a mixed 't Hooft anomaly between the 0- and 1-form symmetries. This type of 't Hooft anomaly is called a 2-group anomaly, since it depends on 0- and 1-form symmetries [84]. The 't Hooft anomalies can be expressed as a 5D action with the background gauge fields of the higher-form symmetries. We also discuss the physical consequences of the 't Hooft anomalies, such as topological order in the bulk and on the axionic domain wall. While the essence of the physical effects has been discussed in the previous paper [103], we give detailed derivations of correlation functions with the intersections of symmetry generators.

This paper is organized as follows. In Sect. 2, we review topological axion electrodynamics with a detailed derivation of the effective action. The higher-form symmetries in topological axion electrodynamics are then reviewed in Sect. 3. In Sect. 4, we consider the background gauging of the higher-form symmetries and 't Hooft anomalies. In Sect. 5, we discuss physical consequences that can be derived by the background gauging. Finally, we summarize this paper

in Sect. 6. In Appendix A, we give detailed derivations of correlation functions for the symmetry generators discussed in Sect. 3.

## 2. Topological axion electrodynamics

In this section, we give an action of the axion electrodynamics where both the axion and photon are massive. In the presence of non-zero mass terms, this theory is fully gapped. The low-energy effective theory can be described by a topological field theory where the axion and photon are topologically coupled with 3- and 2-form gauge fields, respectively. The topological field theory can be obtained by dual transformations.

### 2.1 Action

First, we introduce the action of the axion electrodynamics with mass terms of the axion and photon. We begin with the following effective action in a  $(3+1)$ -dimensional spacetime  $M_4$ :

$$S = - \int_{M_4} \left( \frac{v^2}{2} |d\phi|^2 + V(\phi) \star 1 + \frac{1}{2e^2} |da|^2 + \frac{v'^2}{2} |d\chi - qa|^2 - \frac{N}{8\pi^2} \phi da \wedge da \right). \quad (1)$$

Here, we use the notation of differential forms. We introduce  $|\omega_n|^2 := \omega_n \wedge \star \omega_n = \omega_{\mu_1 \dots \mu_n} \omega^{\mu_1 \dots \mu_n} d^4x$  for an  $n$ -form field  $\omega_n = (1/n!) \omega_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$ ;  $d$  is the exterior derivative and  $\star$  denotes the Hodge star operator. The quantities  $v$  and  $v'$  are mass dimension 1 parameters, the coupling constant  $e$  is a dimensionless parameter, and  $N$  is an integer. The axion  $\phi$  is assumed to be a  $2\pi$  periodic pseudo-scalar field,

$$\phi(\mathcal{P}) + 2\pi \sim \phi(\mathcal{P}), \quad (2)$$

for a point  $\mathcal{P}$  in the spacetime  $M_4$ . The periodicity can be regarded as gauge redundancy of the axion; i.e., the  $2\pi$  shift of the axion  $\phi \rightarrow \phi + 2\pi$  is a gauge transformation. This redundancy is called a  $(-1)$ -form gauge symmetry [53, 108, 109]. A gauge-invariant object given by  $\phi$  is a local point operator:

$$L(q_\phi, \mathcal{P}) := e^{iq_\phi \phi(\mathcal{P})}. \quad (3)$$

Here, the charge of  $q_\phi$  is quantized as

$$q_\phi \in \mathbb{Z} \quad (4)$$

due to the  $(-1)$ -form gauge invariance. Note that the axion operator  $\phi(\mathcal{P})$  itself is not a gauge-invariant operator. Since the axion is  $2\pi$  periodic, the axion can have a nontrivial winding number along a 1D closed subspace  $\mathcal{C}$ :

$$\int_{\mathcal{C}} d\phi \in 2\pi \mathbb{Z}. \quad (5)$$

In other words,  $\phi$  can be a multi-valued function. Let us look at this quantization from a different angle. We consider the two-point object of the axion  $e^{i\phi(\mathcal{P})} e^{-i\phi(\mathcal{P}')}$  with the lowest charge  $q_\phi = 1$ , and express it by using a line integral along a line  $\mathcal{C}_{\mathcal{P}, \mathcal{P}'}$ :

$$e^{i\phi(\mathcal{P})} e^{-i\phi(\mathcal{P}')} = e^{i \int_{\mathcal{C}_{\mathcal{P}, \mathcal{P}'}} d\phi}. \quad (6)$$

Here, the boundary of the line  $\mathcal{C}_{\mathcal{P}, \mathcal{P}'}$  is  $\mathcal{P}$  and  $\mathcal{P}'$ :  $\partial \mathcal{C}_{\mathcal{P}, \mathcal{P}'} = \mathcal{P} \cup \bar{\mathcal{P}'}$ , where  $\bar{\mathcal{P}'}$  is a point  $\mathcal{P}'$  with the opposite orientation. We have chosen the line  $\mathcal{C}_{\mathcal{P}, \mathcal{P}'}$ , but it is possible to choose another line  $\mathcal{C}'_{\mathcal{P}, \mathcal{P}'}$  with the same boundaries  $\partial \mathcal{C}'_{\mathcal{P}, \mathcal{P}'} = \mathcal{P} \cup \bar{\mathcal{P}'}$  as  $e^{i\phi(\mathcal{P})} e^{-i\phi(\mathcal{P}')} = e^{i \int_{\mathcal{C}'_{\mathcal{P}, \mathcal{P}'}} d\phi}$ . Since the two expressions should be identical, we have the condition

$$e^{i \int_{\mathcal{C}_{\mathcal{P}, \mathcal{P}'}} d\phi} e^{-i \int_{\mathcal{C}'_{\mathcal{P}, \mathcal{P}'}} d\phi} = e^{i \int_{\mathcal{C}} d\phi} = 1, \quad (7)$$

where  $\mathcal{C} = \mathcal{C}_{\mathcal{P}, \mathcal{P}'} \cup \tilde{\mathcal{C}}'_{\mathcal{P}, \mathcal{P}'}$  is a loop without boundaries. By this condition, we have the quantization in Eq. (5).

The mass of the axion is given by the potential term  $V_k(\phi)$ . Since we are interested in the axionic domain wall, we assume that the potential term has  $k$  of distinct minima at  $\phi = 2\pi n/k$  ( $n \in \mathbb{Z} \text{ mod } k$ ), i.e., the potential satisfies local stability conditions  $V'_k(2\pi n/k) = 0$  and  $V''_k(2\pi n/k) = M^4 > 0$ . In addition, we choose the minimum of the potential as  $V_k(2\pi n/k) = 0$ . We further assume that the potential has a symmetry under the shift  $\phi \rightarrow \phi + 2\pi/k$ :

$$V_k\left(\phi + \frac{2\pi}{k}\right) = V(\phi). \quad (8)$$

Since each of the  $k$  minima is physically different, we regard this discrete transformation in Eq. (8) as a global symmetry. A typical example is a cosine-type potential,  $V(\phi) \propto 1 - \cos k\phi$ , but we do not specify the details of the potential because we will consider the low-energy limit.

The photon is given by a  $U(1)$  1-form gauge field  $a = a_\mu dx^\mu$ , whose gauge transformation law is given by

$$a \rightarrow a + d\lambda. \quad (9)$$

Here,  $\lambda$  is a  $U(1)$  0-form gauge parameter: it is a  $2\pi$  periodic parameter  $\lambda(\mathcal{P}) + 2\pi \sim \lambda(\mathcal{P})$ , and it can have a winding number:

$$\int_{\mathcal{C}} d\lambda \in 2\pi\mathbb{Z}. \quad (10)$$

A gauge-invariant object made of  $a$  is a Wilson loop:

$$W(q_a, \mathcal{C}) = e^{iq_a \int_{\mathcal{C}} a}. \quad (11)$$

The invariance by a gauge parameter with a non-zero winding number in Eq. (10) requires quantization of the charge:

$$q_a \in \mathbb{Z}. \quad (12)$$

We can derive the flux quantization for the photon as in the case of the winding number of the axion. The Wilson loop can be expressed using 2D surfaces  $\mathcal{S}_{\mathcal{C}}$  and  $\tilde{\mathcal{S}}'_{\mathcal{C}}$  by the Stokes theorem as

$$W(q_a, \mathcal{C}) = e^{iq_a \int_{\mathcal{S}_{\mathcal{C}}} da} = e^{iq_a \int_{\tilde{\mathcal{S}}'_{\mathcal{C}}} da}. \quad (13)$$

By the expressions, we have

$$e^{iq_a \int_{\mathcal{S}_{\mathcal{C}}} da - iq_a \int_{\tilde{\mathcal{S}}'_{\mathcal{C}}} da} = e^{iq_a \int_{\mathcal{S}} da} = 1, \quad (14)$$

where  $\mathcal{S} = \mathcal{S}_{\mathcal{C}} \cup \tilde{\mathcal{S}}'_{\mathcal{C}}$  is a closed 2D space, and  $\tilde{\mathcal{S}}'_{\mathcal{C}}$  is  $\mathcal{S}'_{\mathcal{C}}$  with an opposite orientation. The condition implies

$$\int_{\mathcal{S}} da \in 2\pi\mathbb{Z}, \quad (15)$$

which means that there can be a magnetic monopole with a quantized charge. This is the flux quantization condition for  $a$ .

The mass of the photon is given by the Stückelberg mechanism, which is a low-energy description of the Higgs mechanism without a radial mode. This mechanism can be described by the scalar field  $\chi$  with the charge  $q \in \mathbb{Z}$ , which can be understood as a phase component of a charge  $q$  Higgs field. The gauge transformation law of  $\chi$  under Eq. (9) is given by

$$\chi \rightarrow \chi + q\lambda, \quad a \rightarrow a + d\lambda. \quad (16)$$

By this gauge transformation,  $\chi$  can be eaten by the gauge field  $a$ , and the gauge field becomes massive.

## 2.2 Dual 2-form gauge theory

Here, we dualize the action in Eq. (1) to the topological action in the low-energy limit. While the result has been shown in Ref. [103], we here discuss the dual transformation in detail. In the energy scale lower than the masses of the axion and photon, there is no local excitation, but there can be topological excitation such as an AB effect around the quantized magnetic vortices. In order to see the topological effects, it will be convenient to dualize the theory to a topological field theory. The topological field theory can be expressed by topological actions that consist of higher-form gauge fields. In the absence of the axion, the low-energy effective theory around the ground state can be described by a BF theory given by 1- and 2-form gauge fields [110,111]. In the absence of the photon, we can describe the topological theory for the axion by 0- and 3-form gauge fields [53,54,112].

First, we dualize the Stückelberg coupling to a BF coupling. We begin the following action that is written by the first-order derivative of  $\chi$ :

$$S' = - \int_{M_4} \left( \frac{v^2}{2} |d\phi|^2 + V(\phi) \star 1 + \frac{1}{2e^2} |da|^2 + \frac{v'^2}{2} |w|^2 - \frac{N}{8\pi^2} \phi da \wedge da \right. \\ \left. + \frac{1}{2\pi} h \wedge (w - d\chi + qa) \right). \quad (17)$$

Here, we have introduced the 3- and 1-form fields  $h$  and  $w$ , respectively. The action is classically equivalent to the original action in Eq. (1): By the equation of motion for  $h$ , i.e.,  $w - d\chi + qa = 0$ , the variables  $w$  and  $h$  can be eliminated, and we have the original action in Eq. (1). Instead, we can go to a dual theory by eliminating the scalar field  $\chi$  and the 1-form field  $w$  by their equations of motion. The equation of motion for  $\chi$  is  $dh = 0$ , which can be locally solved by using a 2-form gauge field  $b$  as

$$h = db. \quad (18)$$

The 2-form gauge field has a gauge redundancy,

$$b \rightarrow b + d\lambda_1, \quad (19)$$

where  $\lambda_1$  is a 1-form gauge parameter. The normalization of the 2-form gauge field  $b$  and  $\lambda_1$  is given by the quantization conditions:

$$\int_V db \in 2\pi\mathbb{Z}, \quad \int_S d\lambda_1 \in 2\pi\mathbb{Z}. \quad (20)$$

Next, we eliminate the variable  $w$ . The equation of motion for  $w$  is

$$v'^2 \star w - \frac{1}{2\pi} db = 0. \quad (21)$$

Therefore, we have

$$S_{\text{BF}} = - \int \left( \frac{v^2}{2} |d\phi|^2 + V(\phi) \star 1 + \frac{1}{2e^2} |da|^2 + \frac{1}{8\pi^2 v'^2} |db|^2 - \frac{N}{8\pi^2} \phi da \wedge da \right. \\ \left. - \frac{q}{2\pi} b \wedge da \right). \quad (22)$$

Thus, the scalar field  $\chi$  is dualized to the 2-form gauge field  $b$ .

### 2.3 Dual 3-form gauge theory

Next, we will dualize the potential term of the axion to a topological term given by the axion and a 3-form gauge field. When we dualize the action in Eq. (22), it will be convenient to include the configuration of the domain wall, since we can determine normalizations of dynamical fields according to the configuration of the domain wall. In the low-energy region where we can neglect the width of the domain wall, we can express the configuration of the domain walls as the delta function 1-form,

$$d\phi_W = \frac{2\pi}{k} \delta_1(\mathcal{V}), \quad (23)$$

where  $\mathcal{V}$  denotes the worldvolume of a domain wall. Here, the delta function  $(4-n)$ -form on an  $n$ -dimensional subspace  $\Sigma_n$  is defined by the relation

$$\int_{\Sigma_n} \omega_n = \int \omega_n \wedge \delta_{4-n}(\Sigma_n). \quad (24)$$

Before dualizing the action, we decompose the axion into the fluctuation part  $\phi_F$  and domain wall part  $\phi_W$ :

$$\phi = \phi_F + \phi_W. \quad (25)$$

We can expand the potential term around  $\phi_W$ . Except for the place of the domain walls, we can set  $V_k(2\pi n/k) = 0$ . By the local stability conditions,  $V'_k(2\pi n/k) = 0$  and  $V''_k(2\pi n/k) = M^4$ , the expansion of the potential up to the second order of  $\phi_F$  is as follows:

$$V_k(\phi) = \frac{1}{2} V''(\phi_W) \phi_F^2 + \mathcal{O}(\phi_F^3) = \frac{M^4}{2} (\phi - \phi_W)^2 + \mathcal{O}(\phi_F^3). \quad (26)$$

Therefore, the action in Eq. (22) can be effectively written as

$$\begin{aligned} S_{\text{BF,quad}} = - \int_{M_4} \left( & \frac{v^2}{2} |d\phi|^2 + \frac{M^4}{2} (\phi - \phi_W)^2 \star 1 \right. \\ & \left. + \frac{1}{2e^2} |da|^2 + \frac{1}{8\pi^2 v^2} |db|^2 - \frac{N}{8\pi^2} \phi da \wedge da - \frac{q}{2\pi} b \wedge da \right). \end{aligned} \quad (27)$$

Now, we dualize the action in Eq. (27). We replace  $\phi_W$  with a  $2\pi$  periodic pseudo-scalar field  $f$  by using a Lagrange multiplier 3-form field  $c$ :

$$\begin{aligned} S'_{\text{BF,quad}} = - \int_{M_4} \left( & \frac{v^2}{2} |d\phi|^2 + \frac{M^4}{2} (\phi - f)^2 \star 1 + \frac{1}{2e^2} |da|^2 - \frac{N}{8\pi^2} \phi da \wedge da \right. \\ & \left. + \frac{1}{8\pi^2 v^2} |db|^2 - \frac{q}{2\pi} b \wedge da - \frac{k}{2\pi} c \wedge d(f - \phi_W) \right). \end{aligned} \quad (28)$$

Here, we assume that  $f$  has the same boundary conditions as  $\phi_W$ ,  $f|_{\text{bd}} = \phi_W|_{\text{bd}}$ , where the symbol ‘ $|_{\text{bd}}$ ’ denotes the value at the boundary of the spacetime. The 3-form field  $c$  in the action in Eq. (28) can be regarded as a  $U(1)$  3-form gauge field, since the action has invariance under the gauge transformation:

$$c \rightarrow c + d\lambda_2. \quad (29)$$

Here,  $\lambda_2$  is a  $U(1)$  2-form gauge parameter with the normalization

$$\int_{\mathcal{V}} d\lambda_2 \in 2\pi\mathbb{Z}. \quad (30)$$

The 3-form gauge field is also normalized on a closed 4D space  $\Omega$  as

$$\int_{\Omega} dc \in 2\pi\mathbb{Z}. \quad (31)$$

The normalization of the Lagrange multiplier part  $\frac{k}{2\pi} \int_{M_4} c \wedge d(f - \phi_W)$  is determined so that it is invariant modulo  $2\pi$  under the large gauge transformation in Eq. (30).

We can go back to the original action in Eq. (27) by eliminating the 3-form gauge field  $c$  using the equation of motion of  $c$  with the boundary conditions of  $f$  and  $\phi_W$ . Instead, we can go to the dual theory by eliminating  $f$  by its equation of motion:

$$0 = \frac{k}{2\pi} dc - M^4(f - \phi) \star 1. \quad (32)$$

Substituting the equation into the action in Eq. (28), we obtain the dual action:

$$\begin{aligned} S_{\text{dual, wall}} = & - \int_{M_4} \left( \frac{v^2}{2} |d\phi|^2 + \frac{1}{2e^2} |da|^2 + \frac{1}{8\pi^2 v^2} |db|^2 + \frac{k^2}{8\pi^2 M^4} |dc|^2 \right) \\ & + \int_{M_4} \left( \frac{k}{2\pi} c \wedge d\phi + \frac{q}{2\pi} b \wedge da + \frac{N}{8\pi^2} \phi da \wedge da \right) - \int_{\mathcal{V}} c \\ & + \frac{k^2}{8\pi^2 M^4} \int_{M_4} d(c \wedge \star dc). \end{aligned} \quad (33)$$

In the dual action, the potential term of the axion has been dualized to the quadratic kinetic term for the 3-form gauge field  $|dc|^2$ .<sup>2</sup> Further, we obtain the topological term  $c \wedge d\phi$  between the axion and the 3-form gauge field. Moreover, the worldvolume of the domain walls  $d\phi_W = (2\pi/k)\delta_1(\mathcal{V})$  is now electrically coupled with the 3-form gauge field as  $\int_{\mathcal{V}} c$ . The normalization of the 3-form gauge field  $c$  is determined so that a single domain wall has a unit charge of the 3-form gauge field. The last term is the boundary term for the kinetic term, which is generally needed to have an energy-momentum tensor consistent with the equation of motion [116–119].

In a sufficiently lower-energy scale than the masses of the axion and photon, we can neglect the kinetic terms of  $\phi$ ,  $a$ ,  $b$ , and  $c$ . We thus arrive at the following topological action:

$$S_{\text{TAE}} = \int \left( \frac{k}{2\pi} c \wedge d\phi + \frac{q}{2\pi} b \wedge da + \frac{N}{8\pi^2} \phi da \wedge da \right). \quad (34)$$

Following our previous paper [103], we call this theory “topological axion electrodynamics,” since the action does not depend on the metric in the spacetime.

### 3. Higher-form symmetries in topological axion electrodynamics

In this section, we review higher-form global symmetries in topological axion electrodynamics [103]. The higher-form symmetries are found by the equations of motion and Bianchi identities of the dynamical fields.

#### 3.1 Electric symmetries

First, we show higher-form symmetries associated with the equations of motion. Following Ref. [54], we will call them electric symmetries, but we will often omit “electric” if there is no

<sup>2</sup>In our discussion, we have dualized the potential term after expanding the potential term around the vacua. It is possible to dualize the potential term without the expansion. In this case, the detail of the potential is dualized to higher-derivative corrections to the kinetic term of the 3-form gauge field [113–115].

confusion. The equations of motion for the dynamical fields, i.e.,  $\phi$ ,  $a$ ,  $b$ , and  $c$ , are

$$\frac{k}{2\pi}dc + \frac{N}{8\pi^2}da \wedge da = 0, \quad \frac{q}{2\pi}db + \frac{N}{4\pi^2}d\phi \wedge da = 0, \quad \frac{q}{2\pi}da = 0, \quad \frac{k}{2\pi}d\phi = 0, \quad (35)$$

respectively. The corresponding symmetry generators have the form

$$U_0(e^{i\alpha_0}, \mathcal{V}) = \exp\left(-i\alpha_0 \int_{\mathcal{V}} \left(\frac{k}{2\pi}c + \frac{N}{8\pi^2}a \wedge da\right)\right), \quad (36)$$

$$U_1(e^{i\alpha_1}, \mathcal{S}) = \exp\left(-i\alpha_1 \int_{\mathcal{S}} \left(\frac{q}{2\pi}b + \frac{N}{4\pi^2}\phi da\right)\right), \quad (37)$$

$$U_2(e^{i\alpha_2}, \mathcal{C}) = \exp\left(-i\alpha_2 \int_{\mathcal{C}} \frac{q}{2\pi}a\right), \quad (38)$$

$$U_3(e^{i\alpha_3}, (\mathcal{P}, \mathcal{P}')) = \exp\left(-i\alpha_3 \cdot \frac{k}{2\pi}(\phi(\mathcal{P}) - \phi(\mathcal{P}'))\right), \quad (39)$$

where  $e^{i\alpha_0}, \dots, e^{i\alpha_3}$  may be  $U(1)$  parameters, which will be determined below. Hereafter, we assume that the subspaces  $\mathcal{V}$ ,  $\mathcal{S}$ , and  $\mathcal{C}$  on which the symmetry generators are defined do not have self-intersections for simplicity.

As we explained in Eqs. (4) and (12), the parameters  $e^{i\alpha_2}$  and  $e^{i\alpha_3}$  are constrained as

$$e^{i\alpha_2} = e^{2\pi in_2/q} \in \mathbb{Z}_q, \quad e^{i\alpha_3} = e^{2\pi in_3/k} \in \mathbb{Z}_k, \quad (40)$$

respectively. Further, the parameters  $e^{i\alpha_0}$  and  $e^{i\alpha_1}$  are also subject to some constraints due to the large gauge invariance of the integrals. To make the integrand gauge invariant, we define  $U_0$  and  $U_1$  by using the Stokes theorem:

$$\begin{aligned} U_0(e^{i\alpha_0}, \mathcal{V}) &= \exp\left(-i\alpha_0 \int_{\Omega_{\mathcal{V}}} \left(\frac{k}{2\pi}dc + \frac{N}{8\pi^2}da \wedge da\right)\right), \\ U_1(e^{i\alpha_1}, \mathcal{S}) &= \exp\left(-i\alpha_1 \int_{\mathcal{V}_S} \left(\frac{q}{2\pi}db + \frac{N}{4\pi^2}d\phi \wedge da\right)\right). \end{aligned} \quad (41)$$

Here,  $\Omega_{\mathcal{V}}$  and  $\mathcal{V}_S$  are 4- and 3D manifolds whose boundaries are  $\mathcal{V}$  and  $\mathcal{S}$ , respectively. By the Stokes theorem, we have manifestly gauge-invariant integrands. However, we have chosen auxiliary spaces by hand. Therefore, we require that the symmetry generators should be independent of the choices of the auxiliary spaces. To see the conditions that satisfy the requirement, we choose other subspaces  $\Omega'_{\mathcal{V}}$  and  $\mathcal{V}'_S$  for the 0- and 1-form symmetry generators that satisfy  $\partial\Omega'_{\mathcal{V}} = \mathcal{V}$  and  $\partial\mathcal{V}'_S = \mathcal{S}$ , respectively. The independence of the choices can be expressed by the following integrals on closed 4- and 3D spaces  $\Omega = \Omega_{\mathcal{V}} \cup \bar{\Omega}'_{\mathcal{V}}$  and  $\mathcal{V} = \mathcal{V}_S \cup \bar{\mathcal{V}}'_S$  as

$$\begin{aligned} \exp\left(-i\alpha_0 \int_{\Omega} \left(\frac{k}{2\pi}dc + \frac{N}{8\pi^2}da \wedge da\right)\right) &= 1, \\ \exp\left(-i\alpha_1 \int_{\mathcal{V}} \left(\frac{q}{2\pi}db + \frac{N}{4\pi^2}d\phi \wedge da\right)\right) &= 1. \end{aligned} \quad (42)$$

Because of the flux quantization conditions in Eqs. (5), (15), (20), (31), and  $\int_{\Omega} da \wedge da \in 2 \cdot (2\pi)^2 \mathbb{Z}$  on a spin manifold, we find that the parameters  $e^{i\alpha_0}$  and  $e^{i\alpha_1}$  should belong to discrete groups,

$$e^{i\alpha_0} = e^{2\pi in_0/m} \in \mathbb{Z}_m, \quad e^{i\alpha_1} = e^{2\pi in_1/p} \in \mathbb{Z}_p, \quad (43)$$

where we have defined  $m = \gcd(N, k)$  and  $p = \gcd(N, q)$ . “gcd” stands for greatest common divisor.

To summarize, the gauge-invariant symmetry generators are given by

$$U_0(e^{2\pi i n_0/m}, \mathcal{V}) = \exp \left( -2\pi i \frac{n_0}{m} \int_{\mathcal{V}} \left( \frac{k}{2\pi} c + \frac{N}{8\pi^2} a \wedge da \right) \right), \quad (44)$$

$$U_1(e^{2\pi i n_1/p}, \mathcal{S}) = \exp \left( -2\pi i \frac{n_1}{p} \int_{\mathcal{S}} \left( \frac{q}{2\pi} b + \frac{N}{4\pi^2} \phi da \right) \right), \quad (45)$$

$$U_2(e^{2\pi i n_2/q}, \mathcal{C}) = \exp \left( -in_2 \int_{\mathcal{C}} a \right), \quad (46)$$

$$U_3(e^{2\pi i n_3/k}, (\mathcal{P}, \mathcal{P}')) = \exp \left( -in_3(\phi(\mathcal{P}) - \phi(\mathcal{P}')) \right). \quad (47)$$

They form  $\mathbb{Z}_m$  0-form,  $\mathbb{Z}_p$  1-form,  $\mathbb{Z}_q$  2-form, and  $\mathbb{Z}_k$  3-form global symmetries.

The charged objects on which the symmetry generators act are the Wilson loop and its analogues, given by

$$L(q_0, \mathcal{P}) = e^{iq_0\phi(\mathcal{P})}, \quad (48)$$

$$W(q_1, \mathcal{C}) = e^{iq_1 \int_{\mathcal{C}} a} = U_2(e^{-2\pi i q_1/q}, \mathcal{C}), \quad (49)$$

$$V(q_2, \mathcal{S}) = e^{iq_2 \int_{\mathcal{S}} b}, \quad (50)$$

$$D(q_3, \mathcal{V}) = e^{iq_3 \int_{\mathcal{V}} c}, \quad (51)$$

respectively. Here, the charges are integers  $q_0, \dots, q_3 \in \mathbb{Z}$  because of the large gauge invariance of charged objects. We remark that  $W(q_1, \mathcal{C})$  is identical to the symmetry generator  $U_2(e^{2\pi i q_1/q}, \mathcal{C})$ . We will use this property to show that topological axion electrodynamics is topologically ordered. The symmetry transformations are found by the correlation functions (see Appendix A for derivations):

$$\langle U_0(e^{2\pi i n_0/m}, \mathcal{V}) L(q_0, \mathcal{P}) \rangle = e^{2\pi i q_0 n_0 \text{Link}(\mathcal{V}, \mathcal{P})/m} \langle L(q_0, \mathcal{P}) \rangle, \quad (52)$$

$$\langle U_1(e^{2\pi i n_1/p}, \mathcal{S}) W(q_1, \mathcal{C}) \rangle = e^{2\pi i q_1 n_1 \text{Link}(\mathcal{S}, \mathcal{C})/p} \langle W(q_1, \mathcal{C}) \rangle, \quad (53)$$

$$\langle U_2(e^{2\pi i n_2/q}, \mathcal{C}) V(q_2, \mathcal{S}) \rangle = e^{2\pi i q_2 n_2 \text{Link}(\mathcal{S}, \mathcal{C})/q} \langle V(q_2, \mathcal{S}) \rangle, \quad (54)$$

$$\langle U_3(e^{2\pi i n_3/k}, (\mathcal{P}, \mathcal{P}')) D(q_3, \mathcal{V}) \rangle = e^{2\pi i q_3 n_3 \text{Link}((\mathcal{P}, \mathcal{P}'), \mathcal{V})/k} \langle D(q_3, \mathcal{V}) \rangle. \quad (55)$$

Here, we have defined a linking number between  $n$ - and  $(3-n)$ -dimensional subspaces  $\Sigma_n$  and  $\Sigma'_{3-n}$  as

$$\text{Link}(\Sigma_n, \Sigma'_{3-n}) = \int_{\Omega_{\Sigma_n}} \delta_{n+1}(\Sigma'_{3-n}) = \int_{M_4} \delta_{n+1}(\Sigma'_{3-n}) \wedge \delta_{3-n}(\Omega_{\Sigma_n}), \quad (56)$$

where  $\Omega_{\Sigma_n}$  is an  $(n+1)$ -dimensional subspace whose boundary is  $\Sigma_n$ .

### 3.2 Magnetic symmetries

In addition, we have the following symmetry generators associated with the Bianchi identities for the dynamical fields:

$$\begin{aligned} U_{2M}(e^{i\beta_2}, \mathcal{C}) &= e^{i\beta_2 \int_{\mathcal{C}} \frac{d\phi}{2\pi}}, & U_{1M}(e^{i\beta_1}, \mathcal{S}) &= e^{i\beta_1 \int_{\mathcal{S}} \frac{da}{2\pi}}, \\ U_{0M}(e^{i\beta_0}, \mathcal{V}) &= e^{i\beta_0 \int_{\mathcal{V}} \frac{db}{2\pi}}, & U_{-1M}(e^{i\beta_{-1}}, \Omega) &= e^{i\beta_{-1} \int_{\Omega} \frac{dc}{2\pi}}. \end{aligned} \quad (57)$$

Here,  $e^{i\beta_2}, \dots, e^{i\beta_{-1}}$  are  $U(1)$  parameters. We will call these symmetries magnetic symmetries, since they are associated with the Bianchi identities.

The charged objects for the 2-, 1-, and 0- form symmetries are a worldsheet of the axionic string with the winding number  $q_{2M}$  denoted as  $S(q_{2M}, \mathcal{S})$ , a charge  $q_{1M}$  't Hooft loop  $T(q_{1M}, \mathcal{C})$ , and a pair of charge  $\pm q_{0M}$  instantons  $I(q_{0M}, (\mathcal{P}, \mathcal{P}'))$ , respectively. We should remark that the charged objects for the magnetic symmetries should be boundaries of the electric symmetry generators, but we do not write the configurations of the electric symmetry generators since the configurations of the magnetic objects do not depend on them. Note that we do not consider a magnetic object for the 3-form gauge field, since the spacetime dimension of the object would be  $-1$ . The symmetry transformation laws are

$$\begin{aligned} \langle U_{0M}(e^{i\beta_0}, \mathcal{V})I(q_{0M}, (\mathcal{P}, \mathcal{P}')) \rangle &= e^{iq_{0M} \text{Link}(\mathcal{V}, (\mathcal{P}, \mathcal{P}'))} \langle I(q_{0M}, (\mathcal{P}, \mathcal{P}')) \rangle, \\ \langle U_{1M}(e^{i\beta_1}, \mathcal{S})T(q_{1M}, \mathcal{C}) \rangle &= e^{iq_{1M}\beta_1 \text{Link}(\mathcal{S}, \mathcal{C})} \langle T(q_{1M}, \mathcal{C}) \rangle, \\ \langle U_{2M}(e^{i\beta_2}, \mathcal{C})S(q_{2M}, \mathcal{S}) \rangle &= e^{iq_{2M}\beta_2 \text{Link}(\mathcal{S}, \mathcal{C})} \langle S(q_{2M}, \mathcal{S}) \rangle. \end{aligned} \quad (58)$$

In addition, there are  $U(1)$  0- and  $(-1)$ -form symmetries given by products of the currents for the magnetic symmetries:

$$U_{0CW}(e^{i\gamma_0}, \mathcal{V}) = \exp \left( i\gamma_0 \int_{\mathcal{V}} \frac{d\phi}{2\pi} \wedge \frac{da}{2\pi} \right), \quad (59)$$

$$U_{-1CW}^1(e^{i\gamma_{-1}}, \Omega) = \exp \left( i\gamma_{-1}^1 \int_{\Omega} \frac{1}{2} \frac{da}{2\pi} \wedge \frac{da}{2\pi} \right), \quad (60)$$

$$U_{-1CW}^2(e^{i\gamma_{-1}}, \Omega) = \exp \left( i\gamma_{-1}^2 \int_{\Omega} \frac{d\phi}{2\pi} \wedge \frac{db}{2\pi} \right). \quad (61)$$

They are called Chern–Weil global symmetries [95,98].

#### 4. Background gauging and 't Hooft anomalies

In this section, we discuss the background gauging of the higher-form global symmetries discussed in the previous section. The correlations between the symmetry generators can be efficiently discussed by background gauging. By background gauging, we show that the higher-form symmetries of topological axion electrodynamics possess a semi-strict 4-group structure. Furthermore, we can find possible 't Hooft anomalies for the higher-form global symmetries, which are obstructions to gauging the symmetries dynamically.

##### 4.1 Modification of background gauging

We consider the background gauging of the higher-form symmetries by introducing appropriate background gauge fields. Before performing the background gauging, it is useful to rewrite the action (34) by one defined on the boundary of an auxiliary 5D manifold  $X_5$ :

$$S_{TAE}[X_5] = \int_{X_5} \left( \frac{k}{2\pi} dc \wedge d\phi + \frac{q}{2\pi} db \wedge da + \frac{N}{8\pi^2} d\phi \wedge da \wedge da \right) \bmod 2\pi, \quad (62)$$

with  $\partial X_5 = M_4$ . Hereafter, we omit “mod  $2\pi$ ” of the actions given by 5D manifolds as it does not contribute to  $e^{iS_{TAE}[X_5]}$  in the path integral. This action is manifestly gauge invariant, reducing to the original one in Eq. (34) with the help of the Stokes theorem. Furthermore, the action does not depend on the choice of the 5D manifold  $X_5$ . To show the independence, we choose another 5D manifold  $X'_5$  satisfying  $\partial X'_5 = M_4$ . The difference between these two choices in the

path integral can be evaluated as

$$e^{iS_{\text{TAE}}[X_5]} e^{-iS_{\text{TAE}}[X'_5]} = e^{iS_{\text{TAE}}[Z_5]} = 1, \quad (63)$$

where  $Z_5 = X_5 \cup \bar{X}'_5$  is a 5D manifold without boundaries  $\partial Z_5 = \emptyset$ . Therefore, the action  $S_{\text{TAE}}[X_5]$  does not depend on the choice of  $X_5 \bmod 2\pi$ .

Now, we couple the action  $S_{\text{TAE}}[X_5]$  to background gauge fields. We first consider the electric symmetries discussed in Sect. 3.1. Since these higher-form symmetries correspond to shift symmetries of dynamical fields, the background gauge fields can be coupled with the dynamical fields by Stückelberg couplings.

For example, for the  $\mathbb{Z}_m$  0-form symmetry, we may replace  $d\phi$  by  $d\phi - A_1^m$  with a  $\mathbb{Z}_m$  1-form gauge field  $A_1^m$ . Here, the  $\mathbb{Z}_m$  1-form gauge field means that  $A_1^m$  is closed,  $dA_1^m = 0$ , and normalized as  $\int_{\mathcal{C}} A_1^m \in \frac{2\pi}{m} \mathbb{Z}$  on a 1D closed path  $\mathcal{C}$ . In other words,  $A_1^m$  can be locally expressed as

$$mA_1^m = dA_0^m, \quad (64)$$

where  $A_0^m$  is a 0-form gauge field with the normalization  $\int_{\mathcal{C}} dA_0^m \in 2\pi \mathbb{Z}$ . The combination  $d\phi - A_1^m$  is gauge invariant under

$$\phi \rightarrow \phi + \Lambda_0, \quad A_1^m \rightarrow A_1^m + d\Lambda_0, \quad A_0^m \rightarrow A_0^m + m\Lambda_0, \quad (65)$$

where  $\Lambda_0$  is a gauge parameter satisfying  $\int_{\mathcal{C}} d\Lambda_0 \in 2\pi \mathbb{Z}$ . Similar background gauging can be performed for the other higher-form symmetries. Therefore, a naive gauging would be given by

$$\begin{aligned} S_{\text{TAE},0}[X_5] = & \int_{X_5} \left( \frac{k}{2\pi} (dc - D_4^k) \wedge (d\phi - A_1^m) + \frac{q}{2\pi} (db - C_3^q) \wedge (da - B_2^p) \right. \\ & \left. + \frac{N}{8\pi^2} (d\phi - A_1^m) \wedge (da - B_2^p) \wedge (da - B_2^p) \right). \end{aligned} \quad (66)$$

Here, we have introduced the gauge fields  $B_2^p$ ,  $C_3^q$ , and  $D_4^k$ , which are  $\mathbb{Z}_p$  2-form,  $\mathbb{Z}_q$  3-form, and  $\mathbb{Z}_k$  4-form gauge fields satisfying

$$pB_2^p = dB_1^p, \quad qC_3^q = dC_2^q, \quad kD_4^k = dD_3^k, \quad (67)$$

with 1-, 2-, 3-form gauge fields,  $B_1^p$ ,  $C_2^q$ , and  $D_3^k$ , respectively. The gauge transformation laws are given by

$$a \rightarrow a + \Lambda_1, \quad B_2^p \rightarrow B_2^p + d\Lambda_1, \quad B_1^p \rightarrow B_1^p + p\Lambda_1, \quad (68)$$

$$b \rightarrow b + \Lambda_2, \quad C_3^q \rightarrow C_3^q + d\Lambda_2, \quad C_2^q \rightarrow C_2^q + q\Lambda_2, \quad (69)$$

$$c \rightarrow c + \Lambda_3, \quad D_4^k \rightarrow D_4^k + d\Lambda_3, \quad D_3^k \rightarrow D_3^k + k\Lambda_3. \quad (70)$$

Since  $A_0^m$ ,  $B_1^p$ ,  $C_2^q$ , and  $D_3^k$  are also gauge fields, they transform under their gauge transformations:

$$A_0^m \rightarrow A_0^m + 2\pi, \quad (71)$$

$$B_1^p \rightarrow B_1^p + d\Lambda_0^p, \quad (72)$$

$$C_2^q \rightarrow C_2^q + d\Lambda_1^q, \quad (73)$$

$$D_3^k \rightarrow D_3^k + d\Lambda_2^k. \quad (74)$$

These gauge fields are coupled to magnetic and Chern–Weil symmetries whose currents are

$$A_0^m : \frac{k}{m} \frac{dc}{2\pi} + \frac{N}{m} \frac{1}{2} \frac{da}{2\pi} \wedge \frac{da}{2\pi}, \quad (75)$$

$$B_1^p : \frac{q}{p} \frac{db}{2\pi} + \frac{N}{p} \frac{d\phi}{2\pi} \wedge \frac{da}{2\pi}, \quad (76)$$

$$C_2^q : \frac{da}{2\pi}, \quad (77)$$

$$D_3^k : \frac{d\phi}{2\pi}. \quad (78)$$

We can see that  $A_0^m$  and  $B_1^p$  are coupled to linear combinations of magnetic and Chern–Weil symmetries, while  $C_2^q$  and  $D_3^k$  are directly coupled to magnetic ones.

The gauge fields and gauge parameters are normalized as

$$\int_S dB_1^p, \int_V dC_2^q, \int_\Omega dD_3^k \in 2\pi\mathbb{Z}, \quad (79)$$

$$\int_S d\Lambda_1, \int_V d\Lambda_2, \int_\Omega d\Lambda_3, \int_C d\Lambda_0^p, \int_S d\Lambda_1^q, \int_V d\Lambda_2^k \in 2\pi\mathbb{Z}, \quad (80)$$

respectively. Like the  $\mathbb{Z}_m$  0-form symmetry, the background gauge fields are flat, but they can have fractional AB phases:

$$\int_S B_2^p \in \frac{2\pi}{p}\mathbb{Z}, \quad \int_V C_3^q \in \frac{2\pi}{q}\mathbb{Z}, \quad \int_\Omega D_4^k \in \frac{2\pi}{k}\mathbb{Z}. \quad (81)$$

However, the naive action in Eq. (66) has ambiguity in the choice of the auxiliary manifold  $X_5$  if  $N/(mp)$  or  $N/p^2$  are nontrivial fractional numbers. Such ambiguity can be found by evaluating the difference of the background actions between the two choices of 5D manifolds  $X_5$  and  $X'_5$ :

$$\begin{aligned} S_{\text{TAE},0}[X_5] - S_{\text{TAE},0}[X'_5] &= S_{\text{TAE},0}[Z_5] \\ &= \int_{Z_5} \left( \frac{k}{2\pi} (dc - D_4^k) \wedge (d\phi - A_1^m) + \frac{q}{2\pi} (db - C_3^q) \wedge (da - B_2^p) \right. \\ &\quad \left. + \frac{N}{8\pi^2} (d\phi - A_1^m) \wedge (da - B_2^p) \wedge (da - B_2^p) \right) \\ &= \int_{Z_5} \left( \frac{k}{2\pi} D_4^k \wedge A_1^m + \frac{q}{2\pi} C_3^q \wedge B_2^p - \frac{N}{8\pi^2} A_1^m \wedge B_2^p \wedge B_2^p \right. \\ &\quad \left. + \frac{N}{8\pi^2} d\phi \wedge B_2^p \wedge B_2^p + \frac{N}{4\pi^2} A_1^m \wedge da \wedge B_2^p \right). \end{aligned} \quad (82)$$

We have dropped terms proportional to  $2\pi$  in the last equality as they do not contribute to the weight of the path integral  $e^{iS_{\text{TAE},0}}$ . The first line in the last equation, which is independent of dynamical fields, may represent possible 't Hooft anomalies. However, the second line, depending on the dynamical fields  $\phi$  and  $a$ , leads to the inconsistency of the theory if  $N/p^2$  or  $N/(mp)$  are fractional. The inconsistency can also be understood as a violation of the large gauge invariance in the 4D spacetime [84,102]. To preserve the consistency

of the theory, we should modify the background fields such that there is no ambiguity in the choice of the 5D manifolds for terms containing dynamical fields in the action. Since the ambiguity is caused by the terms proportional to  $da$  and  $d\phi$ , we will modify the 3- and 4-form gauge fields.

We modify the 3-form gauge field  $C_3^q$  and 4-form gauge field  $D_4^k$  as follows:

$$C_3^q \rightarrow C_3^Q := C_3^q + \frac{N}{2\pi q} A_1^m \wedge B_2^p, \quad (83)$$

$$D_4^k \rightarrow D_4^K := D_4^k + \frac{N}{4\pi k} B_2^p \wedge B_2^p. \quad (84)$$

Equivalently, we can write

$$dC_2^q + \frac{N}{2\pi} A_1^m \wedge B_2^p = qC_3^Q, \quad (85)$$

$$dD_3^k + \frac{N}{4\pi} B_2^p \wedge B_2^p = kD_4^K. \quad (86)$$

As we will discuss in Sect. 4.3, these are the key equations in our higher-order group.

We should preserve the gauge invariance of the Stückelberg couplings  $db - C_3^Q$  and  $dc - D_4^K$ . We thus impose the modified gauge transformation laws

$$\begin{aligned} C_3^q &\rightarrow C_3^q + d\Lambda_2 - \frac{N}{2\pi q} (d\Lambda_0 \wedge B_2^p + A_1^m \wedge d\Lambda_1 + d\Lambda_0 \wedge d\Lambda_1), \\ C_2^q &\rightarrow C_2^q + q\Lambda_2 - \frac{N}{2\pi} (\Lambda_0 B_2^p + \Lambda_1 \wedge A_1^m + \Lambda_0 d\Lambda_1), \end{aligned} \quad (87)$$

$$\begin{aligned} D_4^k &\rightarrow D_4^k + d\Lambda_3 - \frac{N}{2\pi k} d\Lambda_1 \wedge B_2^p - \frac{N}{4\pi k} d\Lambda_1 \wedge d\Lambda_1, \\ D_3^k &\rightarrow D_3^k + k\Lambda_3 - \frac{N}{2\pi} \Lambda_1 \wedge B_2^p - \frac{N}{4\pi} \Lambda_1 \wedge d\Lambda_1. \end{aligned} \quad (88)$$

We remark that the modifications of the gauge transformation laws do not violate the  $2\pi$  periodicity of  $C_2^q$  and  $D_3^k$  in Eq. (80).

By the modifications of the background 3- and 4-form gauge fields, their fractional AB phases are modified as

$$\int_{\mathcal{V}} C_3^Q = \frac{2\pi}{q} \left( n_2 + \frac{N}{mp} n_{01} \right), \quad \int_{\Omega} D_4^K = \frac{2\pi}{k} \left( n_3 + \frac{N}{p^2} n_{11}^2 \right). \quad (89)$$

Here,  $n_2 = \frac{1}{2\pi} \int_{\mathcal{V}} dC_2^q$ ,  $n_{12} = \frac{1}{4\pi^2} \int_{\mathcal{V}} dA_1^m \wedge dB_2^p$ ,  $n_{11} = \frac{1}{8\pi^2} \int_{\Omega} dB_1^p \wedge dB_1^p$  are integers. Thus, the gauge fields  $C_3^Q$  and  $D_4^K$  are  $\mathbb{Z}_Q$  and  $\mathbb{Z}_K$  gauge fields, where the integers  $Q$  and  $K$  are defined by

$$Q = q \cdot \frac{mp}{\gcd(N, mp)}, \quad K = k \cdot \frac{p^2}{\gcd(N, p^2)}, \quad (90)$$

respectively. Here, the integers  $mp/\gcd(N, mp)$  and  $p^2/\gcd(N, p^2)$  are denominators of  $N/(mp)$  and  $N/p^2$ , which characterize the necessity of the modifications of the background gauge fields as in Eq. (82) if they are nontrivial.

After the above modifications, the gauged action becomes

$$\begin{aligned}
S_{\text{TAE, bg}}[X_5] &= \int_{X_5} \left( \frac{k}{2\pi} (dc - D_4^K) \wedge (d\phi - A_1^m) + \frac{q}{2\pi} (db - C_3^Q) \wedge (da - B_2^p) \right. \\
&\quad \left. + \frac{N}{8\pi^2} (d\phi - A_1^m) \wedge (da - B_2^p) \wedge (da - B_2^p) \right) \\
&= \int_{X_5} \left( \frac{k}{2\pi} (dc - D_4^k) \wedge (d\phi - A_1^m) + \frac{q}{2\pi} (db - C_3^q) \wedge (da - B_2^p) \right. \\
&\quad \left. + \frac{N}{8\pi^2} (d\phi - A_1^m) \wedge da \wedge da - \frac{N}{4\pi^2} d\phi \wedge da \wedge B_2^p \right. \\
&\quad \left. + \frac{N}{4\pi^2} A_1^m \wedge B_2^p \wedge B_2^p \right). \tag{91}
\end{aligned}$$

This action causes no ambiguity due to the dynamical fields.

We can further gauge the magnetic symmetries. However, further gauging might cause redundancy because  $A_0^m$ ,  $B_1^p$ ,  $C_2^q$ , and  $D_3^k$  couple to magnetic symmetries. In particular,  $C_2^q$  and  $D_3^k$  are directly coupled to magnetic 1- and 2-form symmetries shown in Eqs. (77) and (78). Let us look at this redundancy in detail by gauging the magnetic symmetries. In the absence of background gauge fields for the electric symmetries, the background gauging of the magnetic symmetries is given by adding the following action to  $S_{\text{TAE}}$ :

$$S_{\text{bg, M}}[X_5] = \frac{1}{2\pi} \int_{X_5} (dc \wedge d\Phi_0^M - db \wedge dA_1^M + da \wedge dB_2^M - d\phi \wedge dC_3^M). \tag{92}$$

Here,  $\Phi_0^M$ ,  $A_1^M$ ,  $B_2^M$ , and  $C_3^M$  are 0-, 1-, 2-, and 3-form  $U(1)$  gauge fields. The gauge transformations of the gauge fields are

$$\Phi_0^M \rightarrow \Phi_0^M + 2\pi, \quad A_1^M \rightarrow A_1^M + d\Lambda_0^M, \quad B_2^M \rightarrow B_2^M + d\Lambda_1^M, \quad C_3^M \rightarrow C_3^M + d\Lambda_2^M. \tag{93}$$

The gauge fields and gauge parameters are normalized as

$$\int_C d\Phi_0^M, \int_S dA_1^M, \int_V dB_2^M, \int_\Omega dC_3^M \in 2\pi\mathbb{Z}, \tag{94}$$

$$\int_C d\Lambda_0^M, \int_S d\Lambda_1^M, \int_V d\Lambda_2^M \in 2\pi\mathbb{Z}, \tag{95}$$

respectively. Under the normalization, one can show that the action  $S_{\text{bg, M}}[X_5]$  mod  $2\pi$  does not depend on a choice of the auxiliary space  $X_5$ .

Simultaneous gauging of the electric and magnetic symmetries can be done by the coupling of the electric background gauge fields to  $S_{\text{bg, M}}$ . The total background gauged action is

$$\begin{aligned}
S_{\text{EMbg}}[X_5] &= \int_{X_5} \left( \frac{k}{2\pi} (dc - D_4^K) \wedge (d\phi - A_1^m) + \frac{q}{2\pi} (db - C_3^Q) \wedge (da - B_2^p) \right. \\
&\quad \left. + \frac{N}{8\pi^2} (d\phi - A_1^m) \wedge (da - B_2^p) \wedge (da - B_2^p) \right) \\
&\quad + \frac{1}{2\pi} \int_{X_5} \left( (dc - D_4^K) \wedge d\Phi_0^M - (db - C_3^Q) \wedge dA_1^M \right. \\
&\quad \left. + (da - B_2^p) \wedge dB_2^M - (d\phi - A_1^m) \wedge dC_3^M \right). \tag{96}
\end{aligned}$$

Since  $dB_2^M$  and  $qC_3^Q = dC_2^q + \frac{N}{2\pi}A_1^m \wedge B_2^p$  are coupled to the same current  $\frac{1}{2\pi}(da - B_2^p)$ , we can absorb  $dB_2^M$  by shifting  $C_2^q \rightarrow C_2^q - B_2^M$  while preserving the flux quantization conditions in Eqs. (80) and (94). Similarly, we can also absorb  $dC_3^M$  by the shift  $D_3^k \rightarrow D_3^k - C_3^M$ . Therefore, the background gauging of magnetic  $U(1)$  1- and 2-form symmetries is redundant if we treat  $C_3^Q$ ,  $C_2^q$ ,  $D_4^K$ , and  $D_3^k$  as independent gauge fields. On the other hand,  $d\Phi_0^M$  and  $dA_1^M$  are coupled to currents different from those of  $A_1^m$  and  $B_2^p$ , so that  $d\Phi_0^M$  and  $dA_1^M$  cannot be absorbed by the shift of gauge fields. The resultant gauged action is

$$\begin{aligned} S_{\text{TAE,EMbg}}[X_5] = & \int_{X_5} \left( \frac{k}{2\pi} (dc - D_4^K) \wedge (d\phi - A_1^m) + \frac{q}{2\pi} (db - C_3^Q) \wedge (da - B_2^p) \right. \\ & + \frac{N}{8\pi^2} (d\phi - A_1^m) \wedge (da - B_2^p) \wedge (da - B_2^p) \Big) \\ & + \frac{1}{2\pi} \int_{X_5} \left( (dc - D_4^K) \wedge d\Phi_0^M - (db - C_3^Q) \wedge dA_1^M \right). \end{aligned} \quad (97)$$

#### 4.2 't Hooft anomalies

We have obtained the gauged action consistent with the gauge invariance of the dynamical fields. Meanwhile, we have ambiguity due to only the background gauge fields:

$$\begin{aligned} & \int_{Z_5} \left( \frac{k}{2\pi} (dc - D_4^K) \wedge (d\phi - A_1^m) + \frac{q}{2\pi} (db - C_3^Q) \wedge (da - B_2^p) \right. \\ & + \frac{N}{8\pi^2} (d\phi - A_1^m) \wedge (da - B_2^p) \wedge (da - B_2^p) \Big) \\ & + \frac{1}{2\pi} \int_{Z_5} \left( (dc - D_4^K) \wedge d\Phi_0^M - (db - C_3^Q) \wedge dA_1^M \right. \\ & + (da - B_2^p) \wedge dB_2^M - (d\phi - A_1^m) \wedge dC_3^M \Big) \\ = & \int_{Z_5} \left( \frac{k}{2\pi} D_4^k \wedge A_1^m + \frac{q}{2\pi} C_3^q \wedge B_2^p + \frac{N}{4\pi^2} A_1^m \wedge B_2^p \wedge B_2^p \right) \\ & + \frac{1}{2\pi} \int_{Z_5} \left( -D_4^K \wedge d\Phi_0^M + C_3^Q \wedge dA_1^M \right). \end{aligned} \quad (98)$$

This is an 't Hooft anomaly, which is an obstruction to gauging global symmetries dynamically. In our case, the term with the fractional number  $\frac{k}{2\pi} \int_{Z_5} D_4^k \wedge A_1^m \in \frac{2\pi}{m} \mathbb{Z}$  implies that we cannot gauge the pair of 0- and 3-form symmetries. Similarly, the presence of  $\frac{q}{2\pi} \int_{Z_5} C_3^q \wedge B_2^p \in \frac{2\pi}{p} \mathbb{Z}$  prevents us from gauging the pair of 1- and 2-form symmetries.

Furthermore, gauging of the pair of 0- and 1-form symmetries is forbidden in the presence of  $\frac{N}{4\pi^2} \int_{Z_5} A_1^m \wedge B_2^p \wedge B_2^p \in \frac{4\pi N}{mp} \mathbb{Z}$ . This type of anomaly is called the 2-group anomaly [84]. We also have 't Hooft anomalies due to the simultaneous gauging of the electric and magnetic symmetries. Both terms represent the mixed 't Hooft anomalies that forbid the dynamical gauging of the electric and magnetic symmetries associated with the equations of motion and Bianchi identities for the dynamical fields. In the presence of the 't Hooft anomalies, a symmetry-preserving

gapped vacuum is forbidden. This is consistent with the fact that the axion has  $m$  of degenerated vacua connected by domain walls  $U_0$ , and the photon is in a topologically ordered phase, as we discuss in Sect. 5.

The 't Hooft anomalies can also be seen from the viewpoint of 5D theory as follows. We consider the following topological action:

$$S_{5D}[Z_5] = \int_{Z_5} \left( \frac{k}{2\pi} D_4^K \wedge A_1^m + \frac{q}{2\pi} C_3^Q \wedge B_2^p - \frac{N}{8\pi^2} A_1^m \wedge B_2^p \wedge B_2^p \right) + \frac{1}{2\pi} \int_{Z_5} \left( - D_4^K \wedge d\Phi_0^M + C_3^Q \wedge dA_1^M \right), \quad (99)$$

which is gauge invariant if  $Z_5$  does not have boundaries. If  $Z_5$  has boundaries, the gauge invariance is violated at the boundaries. The violation of the gauge invariance matches the 't Hooft anomalies in topological axion electrodynamics. This means that the 't Hooft anomalies in topological axion electrodynamics can be canceled by  $S_{5D}$  via the anomaly inflow mechanism [120].

#### 4.3 Modified gauge fields as higher-group gauge fields

We here discuss the underlying mathematical structure for the modifications of the background gauge fields. Following Ref. [85], we refer to a set of  $0, \dots, (n-1)$ -form symmetry groups,  $\{G_0, \dots, G_{n-1}\}$ , with nontrivial correlations as an  $n$ -group. In the context of physics, it will be clearer to express the  $n$ -group structure in terms of gauge theory: we refer to a set of  $1, \dots, n$ -form gauge fields with mixed gauge transformation laws as an  $n$ -group gauge theory. Therefore, the  $n$ -group can be characterized as a set of groups whose gauge theory organizes an  $n$ -group gauge theory.

In our case, we can argue that the higher-form symmetries of topological axion electrodynamics organize a 4-group, since we have the  $1, \dots, 4$ -form background gauge fields for the global symmetries, which have mixed gauge transformation laws. Furthermore, we can specify a detailed mathematical structure of the 4-group by the field strengths as follows.

The key equations (85) and (86) in addition to  $dB_1^p = pB_2^p$  can be expressed as<sup>3</sup>

$$dB_1^p = \partial_1 B_2^p, \quad dC_2^q + A_1^m \triangleright C_2^q = \partial_2 C_3^Q, \quad dD_3^k + \{B_2^p, B_2^p\} = \partial_3 D_4^K, \quad (100)$$

where we have defined

$$\partial_1 B_2^p = pB_2^p, \quad \partial_2 C_3^Q = qC_3^Q, \quad \partial_3 D_4^K = kD_4^K, \quad (101)$$

$$A_1^m \triangleright C_2^q = \frac{N}{2\pi} A_1^m \wedge B_2^p, \quad \{B_2^p, B_2^p\} = \frac{N}{4\pi} B_2^p \wedge B_2^p. \quad (102)$$

From these data, we find that our 4-group can be classified into a so-called semi-strict 4-group or 3-crossed module denoted as  $(G_3 \xrightarrow{\partial_3} G_2 \xrightarrow{\partial_2} G_1 \xrightarrow{\partial_1} G_0, \triangleright, \{-, -\})$  in the mathematical literature [104]. We explain the ingredients of the semi-strict 4-group as follows:<sup>4</sup>

- (1)  $G_n$  are groups, which are  $G_0 = \mathbb{Z}_m \times U(1)$ ,  $G_1 = \mathbb{Z}_p \times U(1)$ ,  $G_2 = \mathbb{Z}_Q \times U(1)$ , and  $G_3 = \mathbb{Z}_K$ . The corresponding gauge fields are  $(A_1^m, B_1^p)$ ,  $(B_2^p, C_2^q)$ ,  $(C_3^Q, D_3^k)$ , and  $D_4^K$ , with mixed gauge transformation laws given by Eqs. (65), (68), (69), (70), (87), and (88).

<sup>3</sup>The conditions in Eq. (100) are called vanishing fake curvature conditions [121–123].

<sup>4</sup>We may include  $G_{-1} = U(1)$  whose gauge field is  $A_0^m$ . In this case, the boundary map  $\partial_0: G_0 \rightarrow G_{-1}$  is given by  $\partial_0(e^{2\pi i n_0/m}, e^{i\theta_0}) = e^{m \cdot 2\pi i n_0/m} = 1$ .

(2) Boundary maps  $\partial_n: G_n \rightarrow G_{n-1}$  are maps from electric symmetries to magnetic symmetries, which satisfy  $\partial_{n-1} \circ \partial_n(g_n) = 1 \in G_{n-2}$  for  $g_n \in G_n$ . Concretely, for  $(e^{2\pi i n_2/Q}, e^{i\theta_2}) \in \mathbb{Z}_Q \times U(1) = G_2$  and  $e^{2\pi i n_3/K} \in \mathbb{Z}_K = G_3$ , the maps are

$$\partial_2(e^{2\pi i n_2/Q}, e^{i\theta_2}) = (1, e^{2\pi i n_2 q/Q}) \in \mathbb{Z}_p \times U(1) = G_1, \quad (103)$$

$$\partial_3 e^{2\pi i n_3/K} = (1, e^{2\pi i n_3 k/K}) \in \mathbb{Z}_Q \times U(1) = G_2. \quad (104)$$

In contrast,  $\partial_1$  has no nontrivial structure because it sends the element of  $\mathbb{Z}_p$  to the identity element:  $\mathbb{Z}_p \times U(1) \ni (e^{2\pi i n_1/p}, e^{i\theta_1}) \mapsto (1, e^{2\pi i n_1}) = (1, 1) \in \mathbb{Z}_m \times U(1)$ . Kernels of  $\partial_n$  represent groups of higher-form global symmetries. In particular, the kernels of  $\partial_2$  and  $\partial_3$ ,  $\ker \partial_2 = \mathbb{Z}_q \times U(1) \subset G_2$  and  $\ker \partial_3 = \mathbb{Z}_k \subset G_3$ , are groups of 2- and 3-form symmetries, respectively.

(3) There is a group action of  $G_0$  on  $G_n$  denoted by  $\mathbf{x}$ . In our model, the only action on  $G_1$  is nontrivial. In practice, for  $(e^{2\pi i n_0/m}, e^{i\theta_0}) \in \mathbb{Z}_m \times U(1) = G_0$  and  $(e^{2\pi i n_1/p}, e^{i\theta_1}) \in \mathbb{Z}_p \times U(1) = G_1$ , the action of  $G_0$  is

$$(e^{2\pi i n_0/m}, e^{i\theta_0}) \triangleright (e^{2\pi i n_1/p}, e^{i\theta_1}) = (e^{2\pi i n_1/p}, e^{2\pi i n_0 n_1 N/(mp)} e^{i\theta_1}). \quad (105)$$

(4) A map  $\{ -, - \}: G_1 \times G_1 \rightarrow G_2$  is called Peiffer lifting [124]. For elements of  $G_1$ ,  $(e^{2\pi i n_1/p}, e^{i\theta_1})$ ,  $(e^{2\pi i n'_1/p}, e^{i\theta'_1}) \in \mathbb{Z}_p \times U(1) = G_1$ , the Peiffer lifting reads

$$\{(e^{2\pi i n_1/p}, e^{i\theta_1}), (e^{2\pi i n'_1/p}, e^{i\theta'_1})\} = (1, e^{2\pi i n_1 n'_1 N/(2p^2)}) \in \mathbb{Z}_Q \times U(1) = G_2. \quad (106)$$

(5) Operators  $\{ -, - \}$ ,  $\mathbf{x}$ , and  $\partial_n$  satisfy several consistency conditions (axioms). For example, the group action  $\mathbf{x}$  is consistent with the Peiffer lifting:

$$g \triangleright \{h_1, h_2\} = \{g \triangleright h_1, g \triangleright h_2\}, \quad (107)$$

where  $g \in G_0$  and  $h_1, h_2 \in G_1$ . The consistency conditions mean that the symmetry generators do not depend on the order of deformation.

Note that we can check that our 4-group satisfies the axioms given in Ref. [104] by using the above definitions. This class of higher groups has been found in the context of quantum chromodynamics [125], where a field strength of a 3-form gauge field is modified by a wedge product of a flat 2-form gauge field and a 4-form gauge field similar to Eq. (86).<sup>5</sup> Meanwhile, the 4-group structure discussed in our paper may be the first example where all of the 0-, ..., 3-form symmetry groups are nontrivially correlated.

We comment on a physical interpretation of semi-strict higher groups. In the language of gauge theories for the semi-strict higher groups, the field strengths of higher-form gauge fields are modified by quadratic forms of lower-form gauge fields [121,122,126–129]. Physically, there are boundaries of symmetry generators on the intersections of two symmetry generators, since the field strengths of the background gauge fields specify the configurations of boundaries of the symmetry generators. These modifications can be understood as natural extensions of non-Abelian gauge theories of ordinary non-Abelian groups, where the field strengths should have the quadratic terms of gauge fields if the structure constants are non-zero.

<sup>5</sup>Note that the structure of the 4-group in Ref. [125] can be identified as a semi-strict 4-group  $(G_3 \xrightarrow{\partial_3} G_2 \rightarrow G_1 \rightarrow G_0, \{ -, - \})$ , where  $G_3 = \mathbb{Z}_N p$ ,  $G_2 = U(1)$ ,  $G_1 = \mathbb{Z}_N$ ,  $G_0 = 1$ ,  $\partial_3 e^{2\pi i n_3/(Np)} = e^{2\pi i n_3/N}$ , and  $\{e^{2\pi i n_1/N}, e^{2\pi i n'_1/N}\} = e^{2\pi i n_1 n'_1 / 2N} \in G_2$ . This group structure can be derived by the modified field strength of a 3-form gauge field,  $dD_3^p + \frac{N}{4\pi} B_2^N \wedge B_2^N = pD_4^p$  in our notation.

More generally, higher groups can be weak: field strengths of higher-form gauge fields are modified by field strengths or cubic (or higher) forms of lower-form gauge fields [85]. In particular, structures of weak 2-groups have been investigated in detail, where the modifications of field strengths of 2-form gauge fields are given by Chern–Simons forms or Postnikov classes [64, 78, 84, 130]. The modifications can be understood as generalizations of 2-form gauge fields in heterotic string theories, whose field strengths are modified by the Chern–Simons terms of the Yang–Mills and local Lorentz gauge fields via the Green–Schwarz mechanism [131].

## 5. Physical effects in topological axion electrodynamics

In this section, we discuss some physical effects in topological axion electrodynamics by using both the background gauged actions and correlation functions of symmetry generators.

### 5.1 Topological order in bulk

Here, we argue that topological axion electrodynamics in the bulk exhibits an Abelian type of topological order for  $p = \gcd(q, N) \neq 1$ . In particular, we show that the fractional statistics is given by  $\mathbb{Z}_p$ , which is in contrast to the fractional phase in the ordinary Abelian Higgs model whose fractional statistics is given by the charge of the Higgs field,  $\mathbb{Z}_q$ .

In terms of the background gauging, the existence of the topological order can be directly seen by the topological term  $\int_{Z_5} \frac{q}{2\pi} B_2^p \wedge C_3^q \in \frac{2\pi}{p} \mathbb{Z}$  in Eq. (98), which expresses the mixed 't Hooft anomaly between 1- and 2-form symmetries. Since the 't Hooft anomaly is  $\mathbb{Z}_p$ -valued, we conclude that the ground state has degeneracy classified by the configurations of  $B_2^p$  and  $C_3^q$  as well as the topology of a spatial manifold. For example, if the spatial manifold is  $S^2 \times S^1$ , we have  $p$ -fold degeneracy (see, e.g., Ref. [72] for details).

The discussion based on the 't Hooft anomaly is direct and straightforward, but it may not be physically intuitive. In the following, we explain the topological order in terms of symmetry generators, which will be more intuitive than the above argument.

**5.1.1 Non-local order parameters and fractional linking statistics.** In order to find the topological order in  $(3+1)$  dimensions, we should find non-local order parameters, which are topological and have fractional linking statistics. We will call them topological order parameters. Since symmetry generators are non-local and topological, they are candidates for the order parameters. In the following, we show that the symmetry generators  $U_1$  and  $U_2$  can be regarded as topological order parameters.

The topological order can be characterized by the following correlation function, which can be evaluated by the same procedure summarized in Appendix A:

$$\begin{aligned} \langle U_1(e^{2\pi i n_1/p}, \mathcal{S}) U_2(e^{2\pi i n_2/q}, \mathcal{C}) \rangle &= e^{-2\pi i \frac{n_1 n_2}{p} \text{Link}(\mathcal{S}, \mathcal{C})} \langle U_2(e^{2\pi i n_2/q}, \mathcal{C}) \rangle \\ &= e^{-2\pi i \frac{n_1 n_2}{q} \cdot \frac{q}{p} \text{Link}(\mathcal{S}, \mathcal{C})} \langle U_1(e^{2\pi i n_1/p}, \mathcal{S}) \rangle \\ &= e^{-2\pi i \frac{n_1 n_2}{p} \text{Link}(\mathcal{S}, \mathcal{C})}. \end{aligned} \quad (108)$$

We explain the physical meanings of Eq. (108). The right-hand side of the first line shows that  $U_2$  is charged under the action of  $U_1$  with the charge  $-n_2$ . The second line implies that  $U_1$  is also charged under  $U_2$  with the charge  $-n_1 q/p$ . Note that  $U_1$  belongs to the representation of  $\mathbb{Z}_q$  parameterized by  $\mathbb{Z}_p = \mathbb{Z}_{\gcd(N, q)}$  while the symmetry generator  $U_2$  is parameterized by the group  $\mathbb{Z}_q$ . The third line means that the symmetry generators  $U_1$  and  $U_2$  have a fractional

linking phase. It is the AB effect with a fractional phase: an electrically charged test particle receives a fractional phase when it encircles a string-like quantized magnetic field.

Since the topological order parameters develop non-zero vacuum expectation values (VEVs)  $\langle U_2(e^{2\pi i n_2/q}, \mathcal{C}) \rangle = \langle U_1(e^{2\pi i n_1/p}, \mathcal{S}) \rangle = 1$  and they have fractional linking phases, topological axion electrodynamics is topologically ordered. The symmetry generators consist of groups, so there is no nontrivial fusion rule, which implies that this is an Abelian type of topological order. This topologically ordered phase can be understood as a symmetry-broken phase of both the  $\mathbb{Z}_p$  1-form and  $\mathbb{Z}_q$  2-form symmetries, since the charged objects develop non-zero VEVs. Furthermore, the symmetry-breaking pattern can be classified as type-B spontaneous symmetry breaking, since the charged objects are symmetry generators [132–136].

**5.1.2 Comparison to topological order in Abelian Higgs model.** Here, we discuss the difference of the topologically ordered phases between the topological axion electrodynamics and Abelian Higgs models. The Abelian Higgs model with a charge  $q$  Higgs field can be topologically ordered in the low-energy limit [137]. On the one hand, the fractional linking phase is determined by the charge of the Higgs field as  $\mathbb{Z}_q$  for the Abelian Higgs model. On the other hand, the linking phase is deformed by the axion–photon coupling as  $\mathbb{Z}_p = \mathbb{Z}_{\gcd(N, q)}$  for topological axion electrodynamics. Therefore the global 1-form symmetries are different between the topological axion electrodynamics and Abelian Higgs models. Physically, the axion and Higgs fields screen  $N$  and  $q$  of quantized magnetic fields, respectively.

## 5.2 Topological order on axionic domain wall

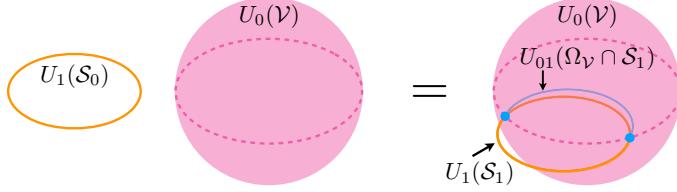
Next, we consider the topological order on the axionic domain wall from the viewpoint of the background gauge field. The nontrivial ordered phase corresponds to the 't Hooft anomaly  $\frac{N}{8\pi^2} \int_{Z_5} A_1^m \wedge B_2^p \wedge B_2^p$  in Eq. (99). The topological term in five dimensions means that the 1-form symmetry generators have a nontrivial linking phase ( $\sim B_2^p \wedge B_2^p$ ) on a worldvolume of the axionic domain wall represented by  $A_1^m$ . The 't Hooft anomaly implies that the ground state in the existence of the domain wall is not uniquely gapped. By the fractional phases of the flat gauge fields, the ground state exhibits topological order characterized by the  $\mathbb{Z}_P$  group, where  $P := mp^2/\gcd(N, mp^2)$  is the nontrivial denominator of  $N/(mp^2)$ .

**5.2.1 Intersection of 0- and 1-form symmetry generators.** In the following, we give a detailed review of the intersection of symmetry generators to discuss the topological order on the axionic domain wall [103]. In order to show the topological order, we need to intersect the symmetry generators  $U_0$  and  $U_1$ . As we will see below, we should treat the intersection of the symmetry generators carefully.

First, we naively consider a correlation function of 0- and 1-form symmetry generators with intersections  $\langle U_0(e^{2\pi i n_0/m}, \mathcal{V}) U_1(e^{2\pi i n_1/p}, \mathcal{S}_1) \rangle$ , where  $\mathcal{V}$  and  $\mathcal{S}_1$  are 3- and 2D closed subspaces without self-intersections. We assume that  $\mathcal{V} \cap \mathcal{S}_1$  is a closed 1D subspace. We can evaluate a correlation function absorbing  $U_1$  and  $U_0$  into the action by the redefinition of  $a$  and  $\phi$  as

$$\langle U_0(e^{2\pi i n_0/m}, \mathcal{V}) U_1(e^{2\pi i n_1/p}, \mathcal{S}_1) \rangle = \langle e^{-2\pi i \frac{N n_0 n_1}{mp} \int_{\Omega_V} \frac{da}{2\pi} \wedge \delta_2(\mathcal{S}_1)} \rangle. \quad (109)$$

However, the object on the right-hand side may violate the large gauge invariance of the photon, if the coefficient  $N n_0 n_1 / mp$  is fractional. The violation of the large gauge invariance can



**Fig. 1.** The intersection of 0- and 1-form symmetry generators. This figure shows a time slice of the symmetry generators. The 0- and 1-form symmetry generators are introduced on 3- and 2D subspaces, which are temporally and spatially extended. The pink sphere and orange line correspond to the 0- and 1-form symmetry generators on the time slice, respectively. The blue line in the right panel is a time slice of an induced static surface  $U_{01}$ . The blue dots denote a time slice of a temporally extended loop, which is the boundary of  $\Omega_V \cap S_1$  on the time slice, corresponding to induced anyons on the domain wall. We have abbreviated the parameters of the symmetry generators for simplicity.

be shown by the ambiguity of the choice of  $\Omega_V$ . In the above correlation function, we can choose another 4D subspace  $\Omega'_V$  whose boundary is  $\mathcal{V}$ . Since the left-hand side of the correlation function does not depend on the choice, the right-hand side should also be independent of the choice. However, when we replace the 4D manifold, we have an additional phase  $e^{-2\pi i \frac{N n_0 n_1}{mp} \int_{\Omega} \frac{da}{2\pi} \wedge \delta_2(S_1)}$ ,

$$e^{-2\pi i \frac{N n_0 n_1}{mp} \int_{\Omega_V} \frac{da}{2\pi} \wedge \delta_2(S_1)} = e^{-2\pi i \frac{N n_0 n_1}{mp} \int_{\Omega} \frac{da}{2\pi} \wedge \delta_2(S_1)} e^{-2\pi i \frac{N n_0 n_1}{mp} \int_{\Omega'_V} \frac{da}{2\pi} \wedge \delta_2(S_1)}, \quad (110)$$

and the phase can be nontrivial if we include an 't Hooft line in the correlation function. Therefore, we treat the intersections of symmetry generators with respect to the large gauge invariance carefully.

In order to discuss the intersection carefully, we take two symmetry generators  $U_0(e^{2\pi i n_0/m}, \mathcal{V})$  and  $U_1(e^{2\pi i n_1/p}, \mathcal{S}_0)$ , assuming that they are not intersected with each other,  $\mathcal{V} \cap \mathcal{S}_0 = \emptyset$  (see Fig. 1). We also assume that  $\mathcal{S}_0$  does not have any self-intersections. Since each of the symmetry generators is contractible, the correlation function given by the two symmetry generators becomes trivial:

$$\langle U_0(e^{2\pi i n_0/m}, \mathcal{V}) U_1(e^{2\pi i n_1/p}, \mathcal{S}_0) \rangle = 1. \quad (111)$$

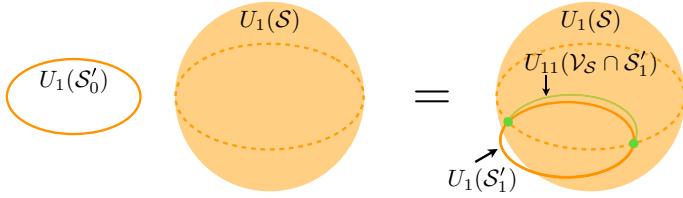
We now intersect them by deforming the worldsheet  $\mathcal{S}_0$  to  $\mathcal{S}_1$  with the condition  $\mathcal{S}_0 \cap \mathcal{S}_1 = \emptyset$ . Here, we assume that the intersection  $\mathcal{V} \cap \mathcal{S}_1$  is a 1D closed subspace. The deformation can be done by interpolating them with a 3D subspace  $\mathcal{V}_{01}$  satisfying  $\partial \mathcal{V}_{01} = \mathcal{S}_0 \cup \bar{\mathcal{S}}_1$ . We also assume that  $\mathcal{V}_{01}$  does not intersect with any singularity such as an 't Hooft line. Under the deformation, we can rewrite the 1-form symmetry generator as  $U_1(e^{2\pi i n_1/p}, \mathcal{S}_0) = U_1(e^{2\pi i n_1/p}, \partial \mathcal{V}_{01}) U_1(e^{2\pi i n_1/p}, \mathcal{S}_1)$ . The correlation function can be rewritten as

$$\begin{aligned} & \langle U_0(e^{2\pi i n_0/m}, \mathcal{V}) U_1(e^{2\pi i n_1/p}, \mathcal{S}_0) \rangle \\ &= \langle U_0(e^{2\pi i n_0/m}, \mathcal{V}) U_1(e^{2\pi i n_1/p}, \partial \mathcal{V}_{01}) U_1(e^{2\pi i n_1/p}, \mathcal{S}_1) \rangle. \end{aligned} \quad (112)$$

The symmetry generator  $U_1(e^{2\pi i n_1/p}, \partial \mathcal{V}_{01})$  can be absorbed into the action by the redefinition  $a + \frac{2\pi n_1}{p} \delta_1(\mathcal{V}_{01}) \rightarrow a$  as

$$\begin{aligned} & \langle U_0(e^{2\pi i n_0/m}, \mathcal{V}) U_1(e^{2\pi i n_1/p}, \mathcal{S}_0) \rangle \\ &= \langle e^{2\pi i \frac{N}{mp} n_0 n_1 \int_{\Omega_V} \frac{da}{2\pi} \wedge \delta_2(S_1)} U_0(e^{2\pi i n_0/m}, \mathcal{V}) U_1(e^{2\pi i n_1/p}, \mathcal{S}_1) \rangle, \end{aligned} \quad (113)$$

where  $\Omega_V$  is a 4D subspace whose boundary is  $\mathcal{V}$ , and we have used  $d\delta_1(\mathcal{V}_{01}) = -\delta_2(\mathcal{S}_0) + \delta_2(\mathcal{S}_1)$ . Since  $\Omega_V \cap \mathcal{S}_1$  is a 2D subspace whose boundary is  $\mathcal{V} \cap \mathcal{S}_1$ , we have a 2D object with the



**Fig. 2.** The intersection of two 1-form symmetry generators at a time slice. The 1-form symmetry generators are introduced on 2D subspaces. One of them is given as an instantaneous sphere  $\mathcal{S}$ , which is shown by an orange sphere in the figure. The other is given as a temporally and spatially extended 2D closed subspace  $\mathcal{S}'_{0,1}$ , which can be seen as a circle on the time slice as shown by an orange circle in the figure. The green line in the right panel is a time slice of an induced surface  $U_{11}$ , which is extended to spatial and temporal directions. The green dots on the boundaries of the green line are instantaneous objects representing an induced 3-form symmetry generator. The dots can be physically interpreted as induced axions. We have again abbreviated the parameters of the symmetry generators.

boundary:

$$U_{01}(e^{2\pi i \frac{N}{mp} n_0 n_1}, \Omega_V \cap \mathcal{S}_1) = e^{2\pi i \frac{N}{mp} n_0 n_1 \int_{\Omega_V} \frac{da}{2\pi} \wedge \delta_2(\mathcal{S}_1)}. \quad (114)$$

Therefore, we should have an additional object on a 2D subspace if we try to intersect them. Since the correlation function in Eq. (113) is trivial, there should be an electrically charged object with a fractional charge  $Nn_0 n_1 / (mp)$ .

Physically, a fractional charge on the intersection means the Sikivie effect and anomalous Hall effect. If we take  $\mathcal{S}_1$  as a spatially and temporally extended object, the symmetry generator represents a worldsheet of a quantized magnetic flux. The Sikivie effect implies that there is an induced electric charge on the intersection of the axionic domain wall and the magnetic flux [36]. If we instead take  $\mathcal{S}_1$  as an instantaneous surface, the 1-form symmetry generator can be understood as an external electric field. The anomalous Hall effect implies that there is an induced electric current on the axionic domain wall [25–27, 36]. Since the axionic domain wall can be understood as a fractional quantum Hall system because of the Chern–Simons term in  $U_0$ , the induced electric charge or current can be identified as an anyon.

The necessity of the additional object  $U_{01}$  can be naturally understood as a natural consequence of the deformation of the 3-form gauge field  $C_3^Q$  in Eq. (83). Since  $A_1^m \wedge B_2^p$  implies the intersection of the 0- and 1-form symmetry generators, the modification in Eq. (83) means that there should be a 2-form symmetry generator on the intersection. Since Eq. (113) is trivial, we have an object canceling  $U_{01}$ , which can be identified as a 2-form symmetry generator by the Stokes theorem.

**5.2.2 Intersection of two 1-form symmetry generators.** In order to discuss the topological order on the domain walls, we need to consider a link of anyons. This configuration can be constructed by using 1-form symmetry generators, which are intersected with each other in the bulk. As in the above discussion, we take two 1-form symmetry generators  $U_1(e^{2\pi i n_1/p}, \mathcal{S})$  and  $U_1(e^{2\pi i n'_1/p}, \mathcal{S}'_0)$ , which are not intersected,  $\mathcal{S} \cap \mathcal{S}'_0 = \emptyset$ , and do not have any self-intersection (see Fig. 2).

The correlation function of the two symmetry generators is trivial, since both of them can be continuously contracted:

$$\left\langle U_1\left(e^{2\pi i n_1/p}, \mathcal{S}\right) U_1\left(e^{2\pi i n'_1/p}, \mathcal{S}'_0\right) \right\rangle = 1. \quad (115)$$

Now, we deform  $\mathcal{S}'_0$  to  $\mathcal{S}'_1$ , which is intersected with  $\mathcal{S}$ . We can deform it by interpolating with a 3D subspace  $\mathcal{V}'_{01}$  satisfying  $\partial\mathcal{V}'_{01} = \mathcal{S}'_0 \cup \bar{\mathcal{S}}'_1$ . By using  $U_1(e^{2\pi in'_1/p}, \mathcal{S}'_0) = U_1(e^{2\pi in'_1/p}, \partial\mathcal{V}'_{01})U_1(e^{2\pi in'_1/p}, \mathcal{S}'_1)$  and by absorbing  $U_1(e^{2\pi in'_1/p}, \partial\mathcal{V}'_{01})$  into the action, we obtain

$$\begin{aligned} & \left\langle U_1(e^{2\pi in_1/p}, \mathcal{S}) U_1(e^{2\pi in'_1/p}, \mathcal{S}'_0) \right\rangle \\ &= \left\langle e^{2\pi i \frac{N}{p^2} n_1 n'_1 \int_{\mathcal{V}_S} \frac{d\phi}{2\pi} \wedge \delta_2(\mathcal{S}'_1)} U_1(e^{2\pi in_1/p}, \mathcal{S}) U_1(e^{2\pi in'_1/p}, \mathcal{S}'_1) \right\rangle. \end{aligned} \quad (116)$$

Here,  $\mathcal{V}_S$  is a 3D subspace whose boundary is  $\mathcal{S}$ , and we have used  $\mathcal{V}_S \cap \partial\mathcal{V}'_{01} = \mathcal{V}_S \cap \bar{\mathcal{S}}'_1$ . We thus obtain an object,

$$U_{11} \left( e^{2\pi i \frac{N}{p^2} n_1 n'_1}, \mathcal{V}_S \cap \mathcal{S}'_1 \right) := e^{2\pi i \frac{N}{p^2} n_1 n'_1 \int_{\mathcal{V}_S} \frac{d\phi}{2\pi} \wedge \delta_2(\mathcal{S}'_1)}, \quad (117)$$

on the 1D subspace  $\mathcal{V}_S \cap \mathcal{S}'_1$  whose boundary is  $\mathcal{S} \cap \mathcal{S}'_1$ . Thus, we should add this object when we try to intersect the 1-form symmetry generators. Physically, the presence of the induced object means the production of the axion, since  $\mathbf{E} \cdot \mathbf{B}$  becomes non-zero on the transversal intersections of the 1-form symmetry generators, and  $\mathbf{E} \cdot \mathbf{B}$  can be understood as a source of the axion.

**5.2.3 Fractional linking phase on the domain wall.** Finally, we consider the following cubic but trivial correlation function to show the topological order on the axionic domain wall:

$$1 = \left\langle U_1(e^{2\pi in_1/p}, \mathcal{S}_0) U_1(e^{2\pi in'_1/p}, \mathcal{S}'_0) U_0(e^{2\pi in_0/m}, \mathcal{V}) \right\rangle, \quad (118)$$

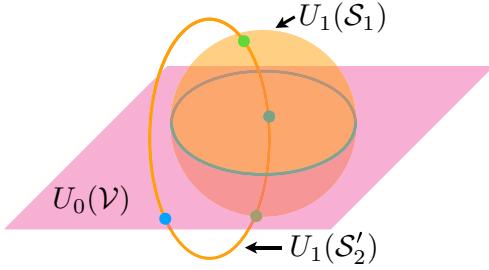
where the three symmetry generators are not intersected,  $\mathcal{S}_0 \cap \mathcal{S}'_0 = \mathcal{S}_0 \cap \mathcal{V} = \mathcal{S}'_0 \cap \mathcal{V} = \emptyset$ . As discussed in Sect. 5.2.1, we deform the subspaces by using  $\mathcal{V}_{01}$  and  $\mathcal{V}'_{01}$  satisfying  $\partial\mathcal{V}_{01} = \mathcal{S}_0 \cup \bar{\mathcal{S}}_1$  and  $\partial\mathcal{V}'_{01} = \mathcal{S}'_0 \cup \bar{\mathcal{S}}'_1$ , where  $\mathcal{S}_1$  and  $\mathcal{S}'_1$  intersect with  $\mathcal{V}$  but  $\mathcal{S}_1 \cap \mathcal{S}'_1 = \emptyset$ :

$$\begin{aligned} & \left\langle U_1(e^{2\pi in_1/p}, \mathcal{S}_0) U_1(e^{2\pi in'_1/p}, \mathcal{S}'_0) U_0(e^{2\pi in_0/m}, \mathcal{V}) \right\rangle \\ &= \left\langle U_{01} \left( e^{2\pi i \frac{N}{mp} n_0 n_1}, \Omega_{\mathcal{V}} \cap \mathcal{S}_1 \right) U_{01} \left( e^{2\pi i \frac{N}{mp} n_0 n'_1}, \Omega_{\mathcal{V}} \cap \mathcal{S}'_1 \right) \right. \\ & \quad \times \left. U_1(e^{2\pi in_1/p}, \mathcal{S}_1) U_1(e^{2\pi in'_1/p}, \mathcal{S}'_1) U_0(e^{2\pi in_0/m}, \mathcal{V}) \right\rangle. \end{aligned} \quad (119)$$

We then deform  $\mathcal{S}'_1$  to  $\mathcal{S}'_2$  by using  $\mathcal{V}'_{12}$  whose boundaries are given as  $\partial\mathcal{V}'_{12} = \mathcal{S}'_1 \cup \bar{\mathcal{S}}'_2$ , and  $\mathcal{S}'_2$  intersects with  $\mathcal{S}_1$  transversally. The final configuration is illustrated in Fig. 3.

By the deformation, we have

$$\begin{aligned} & \left\langle U_1(e^{2\pi in_1/p}, \mathcal{S}_0) U_1(e^{2\pi in'_1/p}, \mathcal{S}'_0) U_0(e^{2\pi in_0/m}, \mathcal{V}) \right\rangle \\ &= \left\langle U_{01} \left( e^{2\pi i \frac{N}{mp} n_0 n_1}, \Omega_{\mathcal{V}} \cap \mathcal{S}_1 \right) U_{01} \left( e^{2\pi i \frac{N}{mp} n_0 n'_1}, \Omega_{\mathcal{V}} \cap \mathcal{S}'_1 \right) \right. \\ & \quad \times \left. U_1(e^{2\pi in_1/p}, \mathcal{S}_1) U_1(e^{2\pi in'_1/p}, \partial\mathcal{V}'_{12}) U_1(e^{2\pi in'_1/p}, \mathcal{S}'_2) U_0(e^{2\pi in_0/m}, \mathcal{V}) \right\rangle. \end{aligned} \quad (120)$$



**Fig. 3.** The intersection of a 0-form symmetry generator and two 1-form symmetry generators at a time slice. The configuration of the 0- and 1-form symmetry generators are the same as the right panel of Figs. 1 and 2, respectively. The blue dots and circle represent the intersections of 0- and 1-form symmetry generators on the time slice, respectively, and they are linked with each other on the worldvolume  $\mathcal{V}$ . The green dots mean the intersections of the 1-form symmetry generators. We have omitted the induced objects  $U_{01}$  and  $U_{11}$  to avoid complicating the figure. They exist such that their boundaries are the blue dots and blue circle for  $U_{01}$  and the green dots for  $U_{11}$ . We have again abbreviated the parameters of the symmetry generators.

By the redefinition  $a + \frac{2\pi n'_1}{p} \delta_1(\mathcal{V}'_{12}) \rightarrow a$ , we obtain induced objects  $U_{01}$  and  $U_{11}$  as well as constant phases:

$$\begin{aligned}
 & \left\langle U_1 \left( e^{2\pi i n_1/p}, \mathcal{S}_0 \right) U_1 \left( e^{2\pi i n'_1/p}, \mathcal{S}'_0 \right) U_0 \left( e^{2\pi i n_0/m}, \mathcal{V} \right) \right\rangle \\
 &= e^{-2\pi i \frac{N}{mp^2} n_0 n_1 n'_1 \int_{\Omega_{\mathcal{V}}} d\delta_1(\mathcal{V}'_{12}) \wedge d\delta_1(\mathcal{V}_{01})} e^{-2\pi i \frac{N}{mp^2} n_0 n'_1 \int_{\Omega_{\mathcal{V}}} d\delta_1(\mathcal{V}'_{12}) \wedge d\delta_1(\mathcal{V}_{01})} \\
 & \times \left\langle U_{01} \left( e^{2\pi i \frac{N}{mp} n_0 n_1}, \Omega_{\mathcal{V}} \cap \mathcal{S}_1 \right) U_{01} \left( e^{2\pi i \frac{N}{mp} n_0 n'_1}, \Omega_{\mathcal{V}} \cap \mathcal{S}'_2 \right) \right. \\
 & \times U_{11} \left( e^{2\pi i \frac{N}{p^2} n_1 n'_1}, \mathcal{V}_{\mathcal{S}_1} \cap \mathcal{S}'_2 \right) \\
 & \left. \times U_1 \left( e^{2\pi i n_1/p}, \mathcal{S}_1 \right) U_1 \left( e^{2\pi i n'_1/p}, \mathcal{S}'_2 \right) U_0 \left( e^{2\pi i n_0/m}, \mathcal{V} \right) \right\rangle. \tag{121}
 \end{aligned}$$

Due to the relations

$$\begin{aligned}
 \int_{\Omega_{\mathcal{V}}} d\delta_1(\mathcal{V}'_{12}) \wedge d\delta_1(\mathcal{V}_{01}) &= \int_{\Omega_{\mathcal{V}}} \delta_2(\mathcal{S}'_2) \wedge \delta_2(\mathcal{S}_1) = - \int_{\mathcal{V}} \delta_2(\mathcal{S}_1) \wedge \delta_1(\mathcal{V}_{\mathcal{S}'_2}) \\
 &=: - \text{Link}(\mathcal{S}_1, \mathcal{S}'_2) |_{\mathcal{V}}, \tag{122}
 \end{aligned}$$

$$\int_{\Omega_{\mathcal{V}}} d\delta_1(\mathcal{V}'_{12}) \wedge d\delta_1(\mathcal{V}'_{01}) = \int_{\Omega_{\mathcal{V}}} (\delta_2(\mathcal{S}'_1) - \delta_2(\mathcal{S}'_2)) \wedge (\delta_2(\mathcal{S}'_0) - \delta_2(\mathcal{S}'_1)) = 0, \tag{123}$$

we find

$$\begin{aligned}
 & \left\langle U_1 \left( e^{2\pi i n_1/p}, \mathcal{S}_0 \right) U_1 \left( e^{2\pi i n'_1/p}, \mathcal{S}'_0 \right) U_0 \left( e^{2\pi i n_0/m}, \mathcal{V} \right) \right\rangle \\
 &= e^{2\pi i \frac{N}{mp^2} n_0 n_1 n'_1 \text{Link}(\mathcal{S}_1, \mathcal{S}'_2) |_{\mathcal{V}}} \\
 & \times \left\langle U_{01} \left( e^{2\pi i \frac{N}{mp} n_0 n_1}, \Omega_{\mathcal{V}} \cap \mathcal{S}_1 \right) U_{01} \left( e^{2\pi i \frac{N}{mp} n_0 n'_1}, \Omega_{\mathcal{V}} \cap \mathcal{S}'_2 \right) U_{11} \left( e^{2\pi i \frac{N}{p^2} n_1 n'_1}, \mathcal{V}_{\mathcal{S}_1} \cap \mathcal{S}'_2 \right) \right. \\
 & \left. \times U_1 \left( e^{2\pi i n_1/p}, \mathcal{S}_1 \right) U_1 \left( e^{2\pi i n'_1/p}, \mathcal{S}'_2 \right) U_0 \left( e^{2\pi i n_0/m}, \mathcal{V} \right) \right\rangle. \tag{124}
 \end{aligned}$$

Here,  $\mathcal{V}_{\mathcal{S}'_2}$  is a 3D subspace whose boundary is  $\mathcal{S}'_2$ , and the symbol “ $\text{Link}(\mathcal{S}_1, \mathcal{S}'_2)|_{\mathcal{V}}$ ” is a linking number of  $\mathcal{S}_1$  and  $\mathcal{S}'_2$  on the closed 3D subspace  $\mathcal{V}$ . In other words, using Eq. (118), we have

$$\begin{aligned} & \left\langle U_{01} \left( e^{2\pi i \frac{N}{mp} n_0 n_1}, \Omega_{\mathcal{V}} \cap \mathcal{S}_1 \right) U_{01} \left( e^{2\pi i \frac{N}{mp} n_0 n'_1}, \Omega_{\mathcal{V}} \cap \mathcal{S}'_2 \right) U_{11} \left( e^{2\pi i \frac{N}{p^2} n_1 n'_1}, \mathcal{V}_{\mathcal{S}_1} \cap \mathcal{S}'_2 \right) \right. \\ & \quad \times U_1 \left( e^{2\pi i n_1/p}, \mathcal{S}_1 \right) U_1 \left( e^{2\pi i n'_1/p}, \mathcal{S}'_2 \right) U_0 \left( e^{2\pi i n_0/m}, \mathcal{V} \right) \Big\rangle \\ & = e^{-2\pi i \frac{N}{mp^2} n_0 n_1 n'_1 \text{Link}(\mathcal{S}_1, \mathcal{S}'_2)|_{\mathcal{V}}}. \end{aligned} \quad (125)$$

The final form implies that the anyons on the domain wall induced by magnetic fluxes have a fractional linking phase, which implies the topological order.

## 6. Summary and discussion

In this paper, we have investigated higher-form symmetries in topological axion electrodynamics in  $(3+1)$  dimensions. We have coupled the background gauge fields for 0-, 1-, 2-, and 3-form symmetries to the action. By the gauge invariance for the axion and photon, we have found that gauging of the 1-form symmetry requires simultaneous gauging of the 3-form symmetry, and simultaneous gauging of the 0- and 1-form symmetries requires gauging of the 2-form symmetry. These requirements modify the fractional AB phases of the background gauge fields for the 2- and 3-form symmetries. By these modifications, we have found that the groups of the higher-form symmetries organize a semi-strict 4-group or 3-crossed module.

We further have derived 't Hooft anomalies of the 4-group symmetry. There are mixed 't Hooft anomalies between the 0- and 3-form symmetries as well as the 1- and 2-form symmetries. Furthermore, we have found a mixed 't Hooft anomaly between the 0- and 1-form symmetries, which is a 2-group anomaly. We have then discussed physical consequences derived by the 't Hooft anomalies. In particular, we have shown the topological order on the axionic domain walls by using the 2-group anomaly. We have also given a detailed derivation of the topological order in terms of the symmetry generators with careful treatment of the intersections of symmetry generators.

There are several avenues for future work. We can develop mathematical foundations of 4-group gauge theories based on the semi-strict 4-group. It would be nontrivial to ask how we can treat several types of the Peiffer lifting proposed in Ref. [104] to construct gauge theories. In particular, the 3-crossed module may have other types of Peiffer lifting such as  $G_1 \times G_2 \rightarrow G_3$  [104]. We expect that they will also express the presence of boundaries of symmetry generators on the intersections of symmetry generators.

To apply the semi-strict 4-group to physics, it would be useful to understand the 4-group diagrammatically. We may extend a diagrammatic expression of the semi-strict 3-group proposed in Ref. [102] to the 4-group. It is also possible to apply our framework to a low-energy effective theory of topological superconductors in  $(3+1)$  dimensions, since they can be described by massive photons and axions with topological couplings between them [44–46].

## Acknowledgments

R.Y. thanks Ryohei Kobayashi, Tatsuki Nakajima, Tadakatsu Sakai, and Yuya Tanizaki for helpful discussions. This work is supported in part by a Japan Society of Promotion of Science (JSPS) Grant-in-Aid for Scientific Research (KAKENHI Grants No. JP17H06462, JP18H01211 (Y.H.), JP18H01217 (M.N.), JP21J00480, JP21K13928 (R.Y.)).

## Funding

Open Access funding: SCOAP<sup>3</sup>.

## Appendix A. Derivations of symmetry transformations

We have shown the symmetry transformations of higher-form symmetries in terms of correlation functions in Eqs. (52)–(55). Here, we summarize the derivations of the symmetry transformations. The derivations are based on reparameterizations of dynamical fields, which are finite versions of Schwinger–Dyson equations.

Before showing the derivations, it will be convenient to denote 0-, ..., 3-form symmetry generators by using conserved currents  $j_3, \dots, j_0$  as follows:

$$U_0(e^{2\pi i n_0/m}, \mathcal{V}) = e^{\frac{2\pi i n_0}{m} \int_{\mathcal{V}} j_3}, \quad (\text{A1})$$

$$U_1(e^{2\pi i n_1/p}, \mathcal{S}) = e^{\frac{2\pi i n_1}{p} \int_{\mathcal{S}} j_2}, \quad (\text{A2})$$

$$U_2(e^{2\pi i n_2/q}, \mathcal{C}) = e^{\frac{2\pi i n_2}{q} \int_{\mathcal{C}} j_1}, \quad (\text{A3})$$

$$U_3(e^{2\pi i n_3/k}, (\mathcal{P}, \mathcal{P}')) = e^{\frac{2\pi i n_3}{k} (j_0(\mathcal{P}) - j_0(\mathcal{P}'))}, \quad (\text{A4})$$

where the integrands of the 0-, ..., 3-form symmetry generators are

$$j_3 = -\frac{k}{2\pi} c - \frac{N}{8\pi^2} a \wedge da, \quad (\text{A5})$$

$$j_2 = -\frac{q}{2\pi} b - \frac{N}{4\pi^2} \phi da, \quad (\text{A6})$$

$$j_1 = -\frac{q}{2\pi} a, \quad (\text{A7})$$

$$j_0 = -\frac{k}{2\pi} \phi, \quad (\text{A8})$$

respectively. These currents are closed  $dj_i = 0$  by the equations of motion. Note that the subscripts of the currents indicate the degrees of the differential forms.

### A.1 $\mathbb{Z}_m$ 0-form symmetry

Now, we derive the symmetry transformations. First, we consider the symmetry transformation of the 0-form symmetry. In the path-integral formalism, the left-hand side of Eq. (52) can be expressed as

$$\langle U_0(e^{2\pi i n_0/m}, \mathcal{V}) L(q_0, \mathcal{P}) \rangle = \mathcal{N} \int \mathcal{D}[\phi, a, b, c] e^{iS_{\text{TAE}}[\phi, a, b, c] + \frac{2\pi i n_0}{m} \int_{\mathcal{V}} j_3 + i q_0 \phi(\mathcal{P})}. \quad (\text{A9})$$

Here,  $\mathcal{N}$  is the normalization factor so that  $\langle 1 \rangle = 1$ , and the symbol “ $\mathcal{D}[\phi, a, b, c]$ ” stands for the integral measure  $\mathcal{D}\phi \mathcal{D}a \mathcal{D}b \mathcal{D}c$ .

The correlation function can be evaluated by absorbing the symmetry generator into the action. To see this, we express  $\int_{\mathcal{V}} j_3$  by using the Stokes theorem

$$\int_{\mathcal{V}} j_3 = \int_{\Omega_{\mathcal{V}}} dj_3 = \int_{M_4} dj_3 \wedge \delta_0(\Omega_{\mathcal{V}}) \quad (\text{A10})$$

for a 4D space  $\Omega_{\mathcal{V}}$  satisfying  $\partial \Omega_{\mathcal{V}} = \mathcal{V}$ . By using the relation

$$S_{\text{TAE}} \left[ \phi - \frac{2\pi n_0}{m} \delta_0(\Omega_{\mathcal{V}}), a, b, c \right] = S_{\text{TAE}}[\phi, a, b, c] + \frac{2\pi n_0}{m} \int_{M_4} dj_3 \wedge \delta_0(\Omega_{\mathcal{V}}), \quad (\text{A11})$$

we can absorb the symmetry generator into the action as

$$\langle U_0(e^{2\pi i n_0/m}, \mathcal{V}) L(q_0, \mathcal{P}) \rangle = \mathcal{N} \int \mathcal{D}[\phi, a, b, c] e^{iS_{\text{TAE}}[\phi - \frac{2\pi n_0}{m} \delta_0(\Omega_{\mathcal{V}}), a, b, c] + iq_0 \phi(\mathcal{P})}. \quad (\text{A12})$$

By the reparameterization  $\phi - \frac{2\pi n_0}{m} \delta_0(\Omega_{\mathcal{V}}) \rightarrow \phi$ , we obtain the relation in Eq. (52):

$$\langle U_0(e^{2\pi i n_0/m}, \mathcal{V}) L(q_0, \mathcal{P}) \rangle = e^{i \frac{2\pi q_0 n_0}{m} \text{Link}(\mathcal{V}, \mathcal{P})} \langle L(q_0, \mathcal{P}) \rangle. \quad (\text{A13})$$

Here, we have used  $\phi(\mathcal{P}) = \int_{M_4} \phi(x) \delta_4(\mathcal{P})$  with  $\delta_4(\mathcal{P}) = \delta^4(x - \mathcal{P}) dx^0 \wedge \dots \wedge dx^3$ , and the definition of the linking number:

$$\text{Link}(\mathcal{V}, \mathcal{P}) = \int_{\Omega_{\mathcal{V}}} \delta_4(\mathcal{P}). \quad (\text{A14})$$

### A.2 $\mathbb{Z}_p$ 1-form symmetry

Secondly, we study the 1-form symmetry transformation in Eq. (53). The correlation function in Eq. (53) can be expressed as

$$\langle U_1(e^{2\pi i n_1/p}, \mathcal{S}) W(q_1, \mathcal{C}) \rangle = \mathcal{N} \int \mathcal{D}[\phi, a, b, c] e^{iS_{\text{TAE}}[\phi, a, b, c] + \frac{2\pi i n_1}{p} \int_{\mathcal{S}} j_2 + iq_1 \int_{\mathcal{C}} a}. \quad (\text{A15})$$

As in the case of the 0-form symmetry transformation, we absorb the symmetry generator into the action. By using the Stokes theorem

$$\int_{\mathcal{S}} j_2 = \int_{\mathcal{V}_{\mathcal{S}}} dj_2 = \int_{M_4} dj_2 \wedge \delta_1(\mathcal{V}_{\mathcal{S}}) \quad (\text{A16})$$

and the relation

$$\begin{aligned} S_{\text{TAE}} \left[ \phi, a + \frac{2\pi n_1}{p} \delta_1(\mathcal{V}_{\mathcal{S}}), b, c \right] \\ = S_{\text{TAE}}[\phi, a, b, c] + \frac{2\pi n_1}{p} \int_{M_4} dj_2 \wedge \delta_1(\mathcal{V}_{\mathcal{S}}) + \frac{N}{8\pi^2} \cdot \left( \frac{2\pi n_1}{p} \right)^2 \int_{M_4} \phi \delta_2(\mathcal{S}) \wedge \delta_2(\mathcal{S}) \end{aligned} \quad (\text{A17})$$

the correlation function can be written as

$$\langle U_1(e^{2\pi i n_1/p}, \mathcal{S}) W(q_1, \mathcal{C}) \rangle = \mathcal{N} \int \mathcal{D}[\phi, a, b, c] e^{iS_{\text{TAE}}[\phi, a + \frac{2\pi n_1}{p} \delta_1(\mathcal{V}_{\mathcal{S}}), b, c] + iq_1 \int_{\mathcal{C}} a}. \quad (\text{A18})$$

Here, we have used the assumption that  $\mathcal{S}$  does not have self-intersections,  $\delta_2(\mathcal{S}) \wedge \delta_2(\mathcal{S}) = 0$ .

By the reparameterization  $a + \frac{2\pi n_1}{p} \delta_1(\mathcal{V}_{\mathcal{S}}) \rightarrow a$ , we arrive at

$$\langle U_1(e^{2\pi i n_1/p}, \mathcal{S}) W(q_1, \mathcal{C}) \rangle = e^{\frac{2\pi i q_1 n_1}{p} \text{Link}(\mathcal{S}, \mathcal{C})} \langle W(q_1, \mathcal{C}) \rangle, \quad (\text{A19})$$

where we have used

$$\text{Link}(\mathcal{S}, \mathcal{C}) = \int_{\mathcal{V}_{\mathcal{S}}} \delta_3(\mathcal{C}) = - \int_{M_4} \delta_1(\mathcal{V}_{\mathcal{S}}) \wedge \delta_3(\mathcal{C}). \quad (\text{A20})$$

### A.3 $\mathbb{Z}_q$ 2-form symmetry

Thirdly, we discuss the 2-form symmetry transformation in Eq. (54). The derivation is similar to those of the 0- and 1-form symmetry transformations, as we discussed above. The correlation function in Eq. (54) can be written as

$$\langle U_2(e^{2\pi i n_2/q}, \mathcal{C}) V(q_2, \mathcal{S}) \rangle = \mathcal{N} \int \mathcal{D}[\phi, a, b, c] e^{iS_{\text{TAE}}[\phi, a, b, c] + \frac{2\pi i n_2}{q} \int_{\mathcal{C}} j_1 + iq_2 \int_{\mathcal{S}} b}. \quad (\text{A21})$$

By using the Stokes theorem

$$\int_{\mathcal{C}} j_1 = \int_{\mathcal{S}_{\mathcal{C}}} dj_1 = \int_{M_4} dj_1 \wedge \delta_2(\mathcal{S}_{\mathcal{C}}) \quad (\text{A22})$$

and the relation

$$S_{\text{TAE}} \left[ \phi, a, b - \frac{2\pi n_2}{q} \delta_2(\mathcal{S}_C), c \right] = S_{\text{TAE}} [\phi, a, b, c] + \frac{2\pi n_2}{q} \int dj_1 \wedge \delta_2(\mathcal{S}_C), \quad (\text{A23})$$

we obtain

$$\langle U_2(e^{2\pi i n_2/q}, \mathcal{C}) V(q_2, \mathcal{S}) \rangle = \mathcal{N} \int \mathcal{D}[\phi, a, b, c] e^{i S_{\text{TAE}} [\phi, a, b - \frac{2\pi n_2}{q} \delta_2(\mathcal{S}_C), c] + iq_2 \int_{\mathcal{S}} b}. \quad (\text{A24})$$

By the reparameterization  $b - \frac{2\pi n_2}{q} \delta_2(\mathcal{S}_C) \rightarrow b$ , we obtain

$$\langle U_2(e^{2\pi i n_2/q}, \mathcal{C}) V(q_2, \mathcal{S}) \rangle = e^{\frac{2\pi i q_2 n_2}{q} \text{Link}(\mathcal{C}, \mathcal{S})} \langle V(q_2, \mathcal{S}) \rangle, \quad (\text{A25})$$

where we have used

$$\text{Link}(\mathcal{C}, \mathcal{S}) = \int_{\mathcal{S}_C} \delta_2(\mathcal{S}) = \int \delta_2(\mathcal{S}_C) \wedge \delta_2(\mathcal{S}). \quad (\text{A26})$$

#### A.4 $\mathbb{Z}_k$ 3-form symmetry

Finally, we consider the 3-form symmetry transformation in Eq. (55). The correlation function in Eq. (55) can be written as

$$\langle U_3(e^{2\pi i n_3/q}, (\mathcal{P}, \mathcal{P}')) D(q_3, \mathcal{V}) \rangle = \mathcal{N} \int \mathcal{D}[\phi, a, b, c] e^{i S_{\text{TAE}} [\phi, a, b, c] + \frac{2\pi i n_3}{k} (j_0(\mathcal{P}) - j_0(\mathcal{P}')) + iq_3 \int_{\mathcal{V}} c}. \quad (\text{A27})$$

By using the line integral on a 1D subspace  $\mathcal{C}_{\mathcal{P}, \mathcal{P}'}$  satisfying  $\partial \mathcal{C}_{\mathcal{P}, \mathcal{P}'} = \mathcal{P} \cup \bar{\mathcal{P}'}$

$$j_0(\mathcal{P}) - j_0(\mathcal{P}') = \int_{\mathcal{C}_{\mathcal{P}, \mathcal{P}'}} dj_0 = \int_{M_4} dj_0 \wedge \delta_3(\mathcal{C}_{\mathcal{P}, \mathcal{P}'}) \quad (\text{A28})$$

and the relation

$$S_{\text{TAE}} \left[ \phi, a, b, c + \frac{2\pi n_3}{k} \delta_3(\mathcal{C}_{\mathcal{P}, \mathcal{P}'}) \right] = S_{\text{TAE}} [\phi, a, b, c] + \frac{2\pi n_3}{k} \int_{M_4} dj_0 \wedge \delta_3(\mathcal{C}_{\mathcal{P}, \mathcal{P}'}), \quad (\text{A29})$$

we obtain

$$\langle U_3(e^{2\pi i n_3/q}, (\mathcal{P}, \mathcal{P}')) D(q_3, \mathcal{V}) \rangle = \mathcal{N} \int \mathcal{D}[\phi, a, b, c] e^{i S_{\text{TAE}} [\phi, a, b, c + \frac{2\pi n_3}{k} \delta_3(\mathcal{C}_{\mathcal{P}, \mathcal{P}'})] + iq_3 \int_{\mathcal{V}} c}. \quad (\text{A30})$$

By the reparameterization  $c + \frac{2\pi n_3}{k} \delta_3(\mathcal{C}_{\mathcal{P}, \mathcal{P}'}) \rightarrow c$ , we obtain

$$\langle U_3(e^{2\pi i n_3/q}, (\mathcal{P}, \mathcal{P}')) D(q_3, \mathcal{V}) \rangle = e^{\frac{2\pi i q_3 n_3}{k} \text{Link}((\mathcal{P}, \mathcal{P}'), \mathcal{V})} \langle D(q_3, \mathcal{V}) \rangle, \quad (\text{A31})$$

where we have used

$$\text{Link}((\mathcal{P}, \mathcal{P}'), \mathcal{V}) = \int_{\mathcal{C}_{\mathcal{P}, \mathcal{P}'}} \delta_1(\mathcal{V}) = - \int \delta_3(\mathcal{C}_{\mathcal{P}, \mathcal{P}'}) \wedge \delta_1(\mathcal{V}). \quad (\text{A32})$$

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