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Ternary Associativity and Ternary Lie Algebras at Cube Roots of Unity

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Abstract: We propose a new approach to extend the notion of commutator and Lie algebra to algebras with ternary multiplication laws. Our approach is based on the ternary associativity of the first and second kinds. We propose a ternary commutator, which is a linear combination of six triple products (all permutations of three elements). The coefficients of this linear combination are the cube roots of unity. We find an identity for the ternary commutator that holds due to the ternary associativity of either the first or second kind. The form of this identity is determined by the permutations of the general affine group $GA(1,5) \subset S_5$. We consider this identity as a ternary analog of the Jacobi identity. Based on the results obtained, we introduce the concept of a ternary Lie algebra at cube roots of unity and provide examples of such algebras constructed using ternary multiplications of rectangular and three-dimensional matrices. We also highlight the connection between the structure constants of a ternary Lie algebra with three generators and an irreducible representation of the rotation group. The classification of two-dimensional ternary Lie algebras at cube roots of unity is proposed.

Keywords: Lie algebra; ternary associativity of the first and second kinds; ternary algebra; ternary commutator; ternary Lie algebra at cube roots of unity; general affine group

MSC: 17A40; 20N10

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1. Introduction

The concept of a group endowed with the structure of a smooth manifold, along with its tangent space at the identity of a group structured as a Lie algebra, plays a crucial role in differential geometry, classical mechanics, and theoretical physics. The development of the theory of Lie groups and algebras began with Sophus Lie's work on the symmetries of differential equations and can be seen as an analog to Galois theory for differential equations. The development of the theory of Lie groups and algebras is closely intertwined with the development of theoretical physics. The development of supersymmetric field theories that emerged in the 1970s is based on the concept of Lie superalgebra, which can be considered a generalization of the concept of Lie algebra.

The development of the theory of Lie algebras has led to numerous generalizations of the concept. One such generalization arose from extending the concept of Lie algebra to algebraic structures with n-ary multiplication laws. This generalization is referred to as n-Lie algebra and it was proposed and developed by Filippov [1]. Independently, Nambu proposed a generalization of Hamiltonian mechanics based on the notion of an n-ary Poisson bracket [2]. It was later demonstrated that an n-ary Poisson bracket in Nambu's framework satisfies an n-Lie algebra identity (now referred to as the Filippov–Jacobi or Fundamental Identity) and thus induces an n-Lie algebra structure on a vector space of smooth functions. It should be noted that the concept of n-Lie algebra turned out to be fruitful, and in the early 2000s, this structure was used in the theory of M2-branes [3,4]. It is interesting to note that the quark model served as a motivation for Nambu to construct a generalization of Hamiltonian mechanics.

In this paper, we propose a new approach to extend the concept of Lie algebra to algebraic structures with ternary multiplication laws. Our approach is distinct from the Filippov–Nambu approach. To explain this difference, we briefly recall the main properties of n-Lie algebras. First, an n-ary Lie bracket of n-Lie algebra is completely skew-symmetric, and, second, the Filippov–Jacobi identity is an extension of the Leibniz rule to a double n-ary Lie bracket. The main examples of n-ary Lie brackets in n-Lie algebras are constructed using determinants. By this, we mean that the theory of n-Lie algebras lacks an important construction that makes it possible to construct a Lie algebra using a commutator. It is well known that if $\mathcal{A}$ is an associative algebra over a field of real or complex numbers with multiplication $(u,v) \in \mathcal{A} \to u \cdot v \in \mathcal{A}$ then one can construct a Lie algebra by equipping $\mathcal{A}$ with a Lie bracket defined via the commutator $[u,v] = u \cdot v - v \cdot u$. The commutator satisfies the Jacobi identity, as follows:

$$\big[[u,v],w\big] + \big[[v,w],u\big] + \big[[w,u],v\big] = 0, \tag{1}$$

and $\mathcal{A}$ becomes the Lie algebra. This construction is very important since it opens up the possibility of constructing a wide and important class of matrix Lie algebras.

Let us consider in more detail the above construction of a Lie algebra via the commutator. First of all, we are interested in why the commutator satisfies the Jacobi identity. It is easy to verify that when we expand all the double commutators in the left-hand side of the Jacobi identity (1), each permutation of the three elements u, v, w appears twice in the resulting expression—once in the form of the product $(u \cdot v) \cdot w$, and again as $u \cdot (v \cdot w)$, with these two products having opposite signs. Thus, due to associativity, where $(u \cdot v) \cdot w = u \cdot (v \cdot w)$, the result is zero. Our goal in this paper is to extend this construction to ternary algebras, that is, to construct a ternary commutator and find an identity for this ternary commutator (analogous to the Jacobi identity), based on ternary associativity.

Let $\mathcal{T}$ be a vector space over the field of complex numbers $\mathbb{C}$ endowed with a ternary multiplication, as follows:

$$(a,b,c) \in \mathcal{T} \times \mathcal{T} \times \mathcal{T} \mapsto a \cdot b \cdot c \in \mathcal{T}.$$

By ternary multiplication, we refer to a trilinear mapping that assigns to each triple of elements in a complex vector space $\mathcal{T}$ a uniquely defined element within the same space $\mathcal{T}$. In the case of ternary multiplication, there are two kinds of associativity. Ternary multiplication is said to be associative of the first kind if it satisfies the following conditions:

$$(a \cdot b \cdot c) \cdot d \cdot f = a \cdot (b \cdot c \cdot d) \cdot f = a \cdot b \cdot (c \cdot d \cdot f), \tag{2}$$

and associative of the second kind if we have the following:

$$(a \cdot b \cdot c) \cdot d \cdot f = a \cdot (d \cdot c \cdot b) \cdot f = a \cdot b \cdot (c \cdot d \cdot f). \tag{3}$$

In the present paper, we use the terminology proposed in [5]. As pointed out in [6], it should be noted that the associativity types of the second kind are also called weak associativity [7], quasi-associativity [8], para-associativity [9], type B associativity [10], pseudo-associativity [11], and generalized associativity [6]. Note that in the case of the associativity of the first kind, shifting the round brackets from left to right does not change the order of the factors in a product, and in the case of the associativity of the second kind, shifting the round brackets from left to right swaps elements b and d. If we do not assume that $\mathcal{T}$ has a vector space structure, i.e., $\mathcal{T}$ is a set, then $\mathcal{T}$ together with a ternary multiplication that satisfies the associativity of the second kind is referred to as a semi-heap [12]. In what follows, a complex vector space $\mathcal{T}$ equipped with a ternary multiplication that satisfies either the associativity of the first (2) or second kind (3) will be referred to as a ternary algebra. Hence, all ternary algebras considered in this paper are assumed to be over the field of complex numbers $\mathbb{C}$.

In the case of an algebra with a binary multiplication law, there are two ways to place brackets in a product $u \cdot v \cdot w$, indicating the order of multiplication: u, v, w, that is, $(u \cdot v) \cdot w$ and $u \cdot (v \cdot w)$. If the multiplication is associative, these two products are equal. Therefore, by multiplying one of these products by -1 and adding them together, we obtain zero due to associativity. This principle forms the basis of the Jacobi identity.

With ternary multiplication, there are three ways to place round brackets in a product of five elements. In the case of the first kind associativity, we can do as follows:

$$(a \cdot b \cdot c) \cdot d \cdot f, \quad a \cdot (b \cdot c \cdot d) \cdot f, \quad a \cdot b \cdot (c \cdot d \cdot f), \tag{4}$$

and similar considerations apply in the case of the second kind of associativity:

$$(a \cdot b \cdot c) \cdot d \cdot f, \quad a \cdot (d \cdot c \cdot b) \cdot f, \quad a \cdot b \cdot (c \cdot d \cdot f), \tag{5}$$

The main idea of the present paper is that we can identify an analog of the Jacobi identity for ternary algebras if we follow the previously described scheme, that is, we multiply each of the three products in (4) or in (5) by some number, add three resulting products and, by virtue of the ternary associativity of the first or second kind, obtain zero. We believe that using -1 in the ternary case looks unnatural, as it makes the whole construction asymmetrical and awkward. But the cube roots of unity fit perfectly into this scheme. Indeed, there are three cube roots of unity $1, \omega, \overline{\omega}$, where ω is a primitive cube root of unity and $\overline{\omega}$ is its complex conjugate. Now, if we multiply each product in (4) and (5) by a distinct cube root of unity and then add the resulting products, we obtain zero due to ternary associativity and the property of cube roots of unity, where $1 + \omega + \overline{\omega} = 0$. For example, we have the following:

$$1\,(a \cdot b \cdot c) \cdot d \cdot f + \omega\, a \cdot (b \cdot c \cdot d) \cdot f + \overline{\omega}\, a \cdot b \cdot (c \cdot d \cdot f) = 0, \tag{6}$$

and this equality can be considered a ternary analog of the binary one $(u \cdot v) \cdot w - u \cdot (v \cdot w) = 0$.

The above reasoning leads to the conclusion that, in the case of ternary algebra $\mathcal{T}$, an analog of the binary commutator can be constructed via the cube roots of unity. We propose a ternary commutator, which is a linear combination of all six permutations of its arguments, and the coefficients of this linear combination are the cube roots of unity. Thus, we endow a ternary algebra $\mathcal{T}$ with the ternary commutator defined by the following formula:

$$[a, b, c] = a \cdot b \cdot c + \omega\, b \cdot c \cdot a + \overline{\omega}\, c \cdot a \cdot b + c \cdot b \cdot a + \overline{\omega}\, b \cdot a \cdot c + \omega\, a \cdot c \cdot b. \tag{7}$$

The structure of this ternary commutator is in line with the ideas, methods, and structures developed in References [13–15]. The ternary commutator (7) proposed in the present paper differs in its properties from a 3-Lie bracket of 3-Lie algebra. Indeed, the ternary commutator (7) is not skew-symmetric; therefore, the presence of two equal arguments does not make it identically zero. However, in the case where all three arguments are equal, it is identically zero. Here, we see an analogy with the ternary generalization of the Pauli exclusion principle proposed by Kerner [16]. According to this principle, a wave function of a quantum system of three particles does not vanish in the case of two particles with identical quantum characteristics, but it vanishes identically when the system contains three of such particles.

We find an identity for the ternary commutator (7). The left-hand side of the Jacobi identity is the sum of three double commutators obtained by cyclic permutations of arguments. Hence, the structure of the left-hand side of the Jacobi identity is determined by the subgroup of cyclic permutations of three elements, $\mathbb{Z}_3 \subset S_3$. It is natural to assume that an identity for the ternary commutator (7) should be also based on a subgroup of the symmetric group S_5, and it is. The identity we found is based on the general affine group, $GA(1,5) \subset S_5$. Thus, the left-hand side of identity has 20 terms, but in some cases of ternary multiplication with commutativity with respect to the first two arguments, it decreases to 10 terms. The identity has the following form:

$$\circlearrowleft \Big(\big[[a,b,c],d,f\big] + \big[[a,d,b],f,c\big] + \big[[a,f,d],c,b\big] + \big[[a,c,f],b,d\big] \Big) = 0, \quad (8)$$

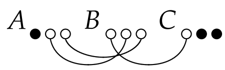
where the symbol $\circlearrowleft$ stands for cyclic permutations of five elements and $a,b,c,d,f \in \mathcal{T}$.

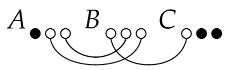
Motivated by this result, we propose a notion of ternary Lie algebra at cubic roots of unity or, more briefly, ternary ω-Lie algebra, where ω is a primitive cube root of unity. We present the definition of a ternary ω-Lie algebra, construct several examples via associative ternary multiplications of rectangular and three-dimensional (cubic) matrices, and identify the complete classification of two-dimensional ternary ω-Lie algebras, containing four non-isomorphic algebras. In order to construct the examples, we use the ternary multiplication of rectangular matrices. This ternary multiplication is defined on a vector space of complex rectangular $m \times n$-matrices $A, B, C \in M_{m,n}(\mathbb{C})$, as follows:

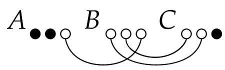
$$A \cdot B \cdot C = A\,B^T\,C,$$

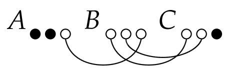
where the right-hand side of the above formula is the usual matrix product of $m \times n$-matrix A, $n \times m$-matrix B^T (transposed of B), and $m \times n$-matrix C. It is easy to verify that this ternary multiplication is associative of the second kind. The rest of the ternary multiplications that we will use are defined on a vector space of complex three-dimensional (cubic) matrices, and they are given in the following theorem: [17].

Theorem 1. *Let A, B, C be nth-order complex three-dimensional matrices. Then, there are only four different triple products of nth-order complex three-dimensional matrices that obey the associativity of the second kind. These are as follows:*

(1) $(A \odot B \odot C)_{ijk} = A_{ilm}B_{nlm}C_{njk}, \quad A \odot B \odot C \rightarrow$ 

(2) $(A \odot B \odot C)_{ijk} = A_{ilm}B_{nml}C_{njk}, \quad A \odot B \odot C \rightarrow$ 

(3) $(A \odot B \odot C)_{ijk} = A_{ijl}B_{nml}C_{mnk}, \quad A \odot B \odot C \rightarrow$ 

(4) $(A \odot B \odot C)_{ijk} = A_{ijl}B_{mnl}C_{mnk}, \quad A \odot B \odot C \rightarrow$ 

In the diagrammatic representation of ternary multiplication, one should sum over a pair of indices depicted by empty circles connected by arcs, while black-filled circles represent free indices.

Finally, we propose the complete classification of two-dimensional ternary ω-Lie algebras at cube roots of unity. This classification consists of four non-isomorphic two-dimensional ternary ω-Lie algebras. In this classification, we use the structure constants C^m_{ijk} of a ternary ω-Lie algebra. The structure constant is a (1,3)-tensor and we derive the system of equations for this tensor from Identity (8).

2. Ternary Commutator and Its Symmetries

In this section, we explain why we call the expression on the right-hand side of (7) a ternary commutator. In addition, we describe the symmetries of the ternary commutator, define its conjugate ternary commutator, and derive a formula for the ternary commutator using sixth-order roots of unity.

The concept of a commutator is closely related to the notion of commutativity. In the case of binary multiplication, $u \cdot v$, two elements, u, v, are referred to as commuting if the equality $u \cdot v = v \cdot u$ holds. Now, one introduces a commutator as an expression that vanishes on commuting elements, that is, $[u,v] = u \cdot v - v \cdot u$. In the case of n-ary multiplication, where $n > 2$, commutativity can be defined in different ways, depending on how we interpret commutativity in the binary case. For our purposes, it is convenient to interpret the binary commutativity as follows: We split a product into two parts; rearranging these parts does not change the value of a product. In this form, commutativity can be extended to n-ary multiplication laws. Assume that we have an n-ary product $a_1 \cdot a_2 \cdot \ldots \cdot a_n$ and we split it into two parts as follows:

$$\underbrace{a_1 \cdot a_2 \cdot \ldots \cdot a_i}\cdot\underbrace{a_{i+1} \cdot a_{i+2} \cdot \ldots \cdot a_n}, \tag{9}$$

where $i = 1, 2, \ldots, n-1$. Then, an analog to the previously mentioned interpretation of binary commutativity could be an n-ary multiplication with the following property: for any partition of an n-ary product into two parts (9), rearranging these two parts does not alter the value of the product, i.e.,

$$a_1 \cdot a_2 \cdot \ldots \cdot a_i \cdot a_{i+1} \cdot a_{i+2} \cdot \ldots \cdot a_n = a_{i+1} \cdot a_{i+2} \cdot \ldots \cdot a_n \cdot a_1 \cdot a_2 \cdot \ldots \cdot a_i.$$

Applying this approach to the case of ternary multiplication results in only one condition, that is, for any three elements, a, b, c, of a ternary algebra, the following holds:

$$a \cdot b \cdot c = b \cdot c \cdot a. \tag{10}$$

Thus, this relation shows that any cyclic permutation of arguments in a triple product $a \cdot b \cdot c$ does not change the value of this product. Property (10) of a ternary multiplication will be referred to as the cyclic commutativity of a triple product. Generally, if a ternary multiplication is cyclic-commutative, then for any three elements, a, b, c, of a ternary algebra, there are two sets of equal triple products, $a \cdot b \cdot c = b \cdot c \cdot a = c \cdot a \cdot b$, and $c \cdot b \cdot a = b \cdot a \cdot c = a \cdot c \cdot b$. If we additionally assume that a ternary multiplication is abelian [18], that is, $a \cdot b \cdot c = c \cdot b \cdot a$, then all six permutations of three arguments in a triple product will be equal. A simple example of a ternary algebra that is both cyclic-commutative and abelian can be constructed with the help of complex diagonal three-dimensional matrices of the second order, along with ternary multiplication 3, as given in Theorem 1. A three-dimensional second-order matrix A will be written as follows [19]:

$$A = \left(\begin{array}{cc|cc} \alpha_{111} & \alpha_{112} & \alpha_{211} & \alpha_{212} \\ \alpha_{121} & \alpha_{122} & \alpha_{221} & \alpha_{222} \end{array} \right).$$

A three-dimensional second-order matrix $A = (\alpha_{ijk})$ is referred to as diagonal if the entries $\alpha_{111}, \alpha_{222}$ are different from zero, but all other entries are equal to zero. Let $A = (\alpha_{ijk}), B = (\beta_{ijk}), C = (\gamma_{ijk})$ be diagonal three-dimensional second-order matrices. We denote the following:

$$\alpha_{111} = a,\ \alpha_{222} = u, \quad \beta_{111} = b,\ \beta_{222} = v, \quad \gamma_{111} = c,\ \gamma_{222} = w.$$

Then, by applying ternary multiplication 3 defined in Theorem 1 (denoted by $\cdot$), we obtain the triple product, as follows:

$$A \cdot B \cdot C = \left(\begin{array}{cc|cc} a\,b\,c & 0 & 0 & 0 \\ 0 & 0 & 0 & u\,v\,w \end{array} \right), \tag{11}$$

which is also a diagonal three-dimensional second-order matrix. From this, it follows that the vector space of diagonal three-dimensional second-order matrices endowed with ternary multiplication 3 (Theorem 1) is an abelian cyclic-commutative ternary algebra. Although ternary multiplication 3 has the associativity of the second kind, when restricted to diagonal three-dimensional matrices, it also has the associativity of the first kind.

Now, in order to construct a ternary commutator, we combine all six products into a linear combination in such a way that, first, we use the cube roots of unity, and, second, when the cyclic-commutative condition (10) is satisfied, this combination vanishes. Hence, we consider the following expression:

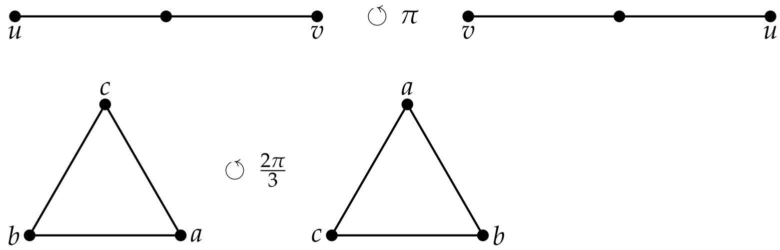
$$a \cdot b \cdot c + \omega\, b \cdot c \cdot a + \overline{\omega}\, c \cdot a \cdot b + c \cdot b \cdot a + \overline{\omega}\, b \cdot a \cdot c + \omega\, a \cdot c \cdot b, \tag{12}$$

where a, b, c are elements of a ternary algebra and ω is a primitive third-order root of unity. We will refer to this expression as ternary commutator and denote it using square brackets, as follows:

$$[a, b, c] = a \cdot b \cdot c + \omega\, b \cdot c \cdot a + \overline{\omega}\, c \cdot a \cdot b + c \cdot b \cdot a + \overline{\omega}\, b \cdot a \cdot c + \omega\, a \cdot c \cdot b. \tag{13}$$

We would like to draw another analogy between the binary commutator and the ternary commutator (13); this analogy is based on geometric reasoning. In the case of binary multiplication $u \cdot v$, it is natural to place the factors u, v at the ends of a segment and compose their products by taking the element that stands at the left end of the segment as the first factor. The permutation of factors in a product $u \cdot v$ corresponds to the rotation of a segment around the center by the angle of π. Thus, the binary commutator can be interpreted in such a way that we take the product $u \cdot v$ determined by the initial position of the segment and add to it the product determined by the segment rotated by the angle of π, multiplied by the coefficient $e^{i\pi}$, that is, $u \cdot v + e^{i\pi}\, v \cdot u = u \cdot v - v \cdot u = [u, v]$.

In the case of ternary multiplication, we should use a regular triangle to graphically represent a triple product $a \cdot b \cdot c$. We will arrange the three factors, a, b, c, of this product at the vertices of a triangle, placing the first factor, a, at the lower right vertex and going around the triangle clockwise. Then, rotating the triangle around its center by the angle of $2\pi/3$ counterclockwise will give us the first cyclic permutation, $b \cdot c \cdot a$, and rotating it by the angle of $4\pi/3$ will give us the second, $c \cdot a \cdot b$. So, they must enter into the expression for the ternary commutator with the factors $e^{2\pi i/3} = \omega, e^{4\pi i/3} = \overline{\omega}$. The second part of the expression for the ternary commutator is obtained by mirroring the described construction:



It is useful to introduce the conjugate ternary commutator $[-,-,-]^*$ which is calculated on elements (a, b, c) of a ternary algebra as follows:

$$[a, b, c]^* = a \cdot b \cdot c + \overline{\omega}\, b \cdot c \cdot a + \omega\, c \cdot a \cdot b + c \cdot b \cdot a + \omega\, b \cdot a \cdot c + \overline{\omega}\, a \cdot c \cdot b. \tag{14}$$

Thus, the conjugate ternary commutator $[-,-,-]^*$ is obtained from the ternary commutator (13) by replacing each cube root of unity with its conjugate. Then, we have the following: $[a, b, c]^* = [c, b, a]$. It is easy to verify that the ternary commutator (13) and its conjugate (14) transform under cyclic permutations of their arguments, as follows:

$$[a, b, c] = \omega\, [b, c, a], \;\; [a, b, c] = \overline{\omega}\, [c, a, b], \tag{15}$$

$$[a, b, c]^* = \overline{\omega}\, [b, c, a]^*, \;\; [a, b, c]^* = \omega\, [c, a, b]^*. \tag{16}$$

From (15), it follows that the sum of three ternary commutators obtained by cyclic permutations of arguments is equal to zero, as follows:

$$[a, b, c] + [b, c, a] + [c, a, b] = 0. \tag{17}$$

Concerning this important property of the ternary commutator (13), we have to make three remarks. The first remark concerns Lie triple systems that arose in differential geometry in connection with the study of totally geodesic submanifolds [20]. Although property (17) of the ternary commutator (13) has the same form as one of the requirements in the definition of a Lie triple system, the ternary commutator (13) is not a Lie triple system because it is not skew-symmetric in the first two arguments and does not satisfy the Filippov–Jacobi identity.

The second remark concerns 3-Lie algebras. The ternary Lie bracket in 3-Lie algebra is skew-symmetric in its arguments and, thus, in general, it does not satisfy Equation (17). In the next section, we will identify an identity for the ternary commutator (13) and demonstrate how this identity differs from the Filippov–Jacobi identity. Thus, the ternary commutator (13) imparts a structure on a ternary algebra $\mathcal{T}$ that is different from that of a 3-Lie algebra.

The third remark concerns a relation with theoretical physics. It is easy to see that two equal arguments in our ternary commutator (13) do not make it vanish identically. But in the case, where all three arguments (a,b,c) are equal, our ternary commutator vanishes identically, i.e., $[a,a,a]=0$. Here, we see a possible connection with the ternary generalization of the Pauli exclusion principle proposed by Kerner [16]. We will discuss this connection in Section 6.

It is worth mentioning that Nambu, in Reference [2], which is devoted to the generalization of Hamiltonian mechanics, discussed the problem of quantizing generalized Hamiltonian mechanics. In this context, he considered the skew-symmetric ternary commutator as follows:

$$[A,B,C] = A\,B\,C + B\,C\,A + C\,A\,B - C\,B\,A - B\,A\,C - A\,C\,B, \tag{18}$$

where A,B,C are linear operators. This version of a ternary commutator can be considered a direct extension of the skew-symmetry of the binary commutator to the case of ternary multiplication. However, to our knowledge, no analog of the Jacobi identity based on ternary associativity has been found for such a ternary commutator. It is interesting that our ternary commutator (13) can also be written in a form where the three even permutations have a plus sign, and the three odd permutations have a minus sign. For this purpose, we will need a primitive sixth root of unity, which will be denoted by ε. We take $\omega=\varepsilon^2, \overline{\omega}=\varepsilon^4$. Among other relations, we have the following:

$$\varepsilon+\overline{\varepsilon}=1,\;\; \omega=-\overline{\varepsilon},\;\; \overline{\omega}=-\varepsilon. \tag{19}$$

Now, we can write the ternary commutator (13) in the following form::

$$[a,b,c] = a\cdot b\cdot c - \varepsilon\, b\cdot a\cdot c + \varepsilon^2\, b\cdot c\cdot a - \varepsilon^3\, c\cdot b\cdot a + \varepsilon^4\, c\cdot a\cdot b - \varepsilon^5\, a\cdot c\cdot b. \tag{20}$$

In this formula, even permutations have the plus sign and are multiplied by even powers of the sixth root of unity ε, and odd permutations have the minus sign and are multiplied by odd powers of the sixth root of unity. Now, the symmetries of the ternary commutator can be written in the following form:

$$\begin{aligned}&[a,b,c]=\varepsilon^2\,[b,c,a]=\varepsilon^4\,[c,a,b], \quad [a,b,c]^*=\varepsilon^4\,[b,c,a]^*=\varepsilon^2\,[c,a,b]^*,\\&[a,b,c]=-\varepsilon\,[b,a,c]^*,\;\; [a,b,c]=-\varepsilon^3\,[c,b,a]^*,\;\; [a,b,c]=-\varepsilon^5\,[a,c,b]^*.\end{aligned}$$

Formula (20) can be written in the following form:

$$[a,b,c] = (a\cdot b\cdot c - \varepsilon\, b\cdot a\cdot c) + \omega\,(b\cdot c\cdot a - \varepsilon\, c\cdot b\cdot a) + \overline{\omega}\,(c\cdot a\cdot b - \varepsilon\, a\cdot c\cdot b). \tag{21}$$

We can use this formula to justify the term "ternary commutator", which we use in relation to the expression on the right-hand side of (13). In the above formula, each of the three terms enclosed in round brackets can be interpreted as measuring the non-commutativity of the ternary multiplication with respect to the first two arguments in relation to the last, which does not change its position. Geometrically, it would be convenient to depict the three elements (a, b, c) of a ternary algebra $\mathcal{T}$ as the vertices of a regular triangle. Then, the above formula "measures" the non-commutativity of a ternary multiplication on each side of the triangle with respect to the opposite vertex. Thus, geometrically, the transition from binary multiplication—where two factors can be represented as points on a line—to ternary multiplication can be described as leaving a line and going to a plane, figuratively speaking. This explains why the above formula contains sixth-order roots of unity and conjugation. To measure ternary non-commutativity correctly, we need to use plane rotations and reflections. It should be noted here that in [18] the authors developed an interesting graphical and diagrammatic approach for representing ternary associative multiplication by using triangles in the plane.

In particular, if a ternary multiplication is commutative with respect to some pair of arguments, for example, the first pair, that is, $a \cdot b \cdot c = b \cdot a \cdot c$, then Formula (21) reduces to a shorter form containing only three terms. Indeed, we have the following:

$$\begin{aligned}[a, b, c] &= a \cdot b \cdot c + \omega\, b \cdot c \cdot a + \overline{\omega}\, c \cdot a \cdot b + c \cdot b \cdot a + \overline{\omega}\, b \cdot a \cdot c + \omega\, a \cdot c \cdot b \\ &= (1 + \overline{\omega})\, a \cdot b \cdot c + (1 + \omega)\, b \cdot c \cdot a + (\omega + \overline{\omega})\, c \cdot a \cdot b \\ &= -\omega\, (a \cdot b \cdot c + \omega\, b \cdot c \cdot a + \overline{\omega}\, c \cdot a \cdot b). \end{aligned} \tag{22}$$

3. General Affine Group, Basic Identity, and Ternary Lie Algebra at the Cube Root of Unity

The concept of a Lie algebra consists of two important components: firstly, a Lie bracket (or, specifically, the binary commutator) and its properties regarding permutations of arguments along with the Jacobi identity. Since we have the ternary commutator defined and considered in the previous section, our goal is to now identify an identity for the ternary commutator (13) based on ternary associativity. Following the analogy with the binary commutator, we could estimate how many terms a possible identity could contain. If we consider the binary case then each double commutator, when expanded, yields four products. But if we expand all the double commutators at the left-hand side of identity, then in the resulting expression, each product of three elements (in total, we have six permutations) will occur twice (the brackets are either on the left or on the right). Thus, we will have twelve products on the left-hand side of the identity. Dividing twelve by four, we conclude that an identity consists of three double commutators, as is the case with the Jacobi identity.

A similar calculation can be made in the case of the ternary commutator (13). If we expand the double ternary commutator $[[a, b, c], d, f]$, we obtain thirty-six terms. On the other hand, we have one hundred and twenty permutations of five elements. Due to ternary associativity, each permutation must occur at least three times (brackets on the left, in the center, and on the right) with coefficients $1, \omega, \overline{\omega}$. Thus, dividing three hundred and sixty by thirty-six gives ten. Note that this is the minimum number of terms in a possible identity. In this calculation, we do not take into account such an important structure of the ternary commutator as conjugation. Obviously, if we take this structure into account, we will have to double the number of terms in the identity, i.e., we can expect that a possible identity will contain twenty terms.

Since the identity we are looking for is a sum of double ternary commutators of the form $[[a, b, c], d, f]$, the second assumption, which seems very natural, is that an identity must be based on a subgroup of a symmetric group, S_5. Taking into account the above, we conclude that there are two potential candidates for the subgroups of the symmetric group S_5; these are the dihedral group D_{10} (10 elements) or the general affine group $GA(1, 5)$ (20 elements). Moreover, the dihedral group is a subgroup of the general affine group, that is, $D_{10} \subset GA(1, 5)$.

The general affine group $GA(1, 5)$ has several different representations. In this article, we will use the representation of this group by permutations of five elements. The minimal set of permutations that generates the entire group consists of two cycles, which we denote as follows:

$$\sigma = (1\,2\,3\,4\,5), \quad \tau = (2\,4\,5\,3).$$

Hence, we have the following:

$$\sigma(1) = 2,\ \sigma(2) = 3,\ \sigma(3) = 4,\ \sigma(5) = 1, \tag{23}$$

and

$$\tau(1) = 1,\ \tau(2) = 4,\ \tau(3) = 2,\ \tau(4) = 5,\ \tau(5) = 3. \tag{24}$$

Then, we have the following:

$$GA(1,5) =< \sigma, \tau \mid \sigma^5 = e, \tau^4 = e, \tau\,\sigma\,\tau^{-1} = \sigma^2 >,$$

where e is the identity element of the group $GA(1,5)$. All elements of the group can be written in the following form:

$$e,\ \sigma,\ \sigma^2,\ \sigma^3,\ \sigma^4, \tag{25}$$
$$\tau,\ \tau\sigma^3,\ \tau\sigma,\ \tau\sigma^4,\ \tau\sigma^2, \tag{26}$$
$$\tau^2,\ \tau^2\sigma^4,\ \tau^2\sigma^3,\ \tau^2\sigma^2,\ \tau^2\sigma, \tag{27}$$
$$\tau^3,\ \tau^3\sigma^2,\ \tau^3\sigma^4,\ \tau^3\sigma,\ \tau^3\sigma^3. \tag{28}$$

We will use this representation to write the identity. In this representation, all elements of the general affine group are divided into four sets (25)–(28), and in each of these sets, the second element is obtained by a cyclic permutation of five elements in the first, the third by a cyclic permutation in the second, and so on. For a more compact representation of an identity, we will use the symbol $\circlearrowleft$. This symbol means that an expression that follows contains five elements and must be subjected to the following procedure. One should form the sum of five expressions, starting with the initial one, where each subsequent one is a cyclic permutation of five elements of the previous one. Thus, we have the following:

$$\begin{aligned} \circlearrowleft \big[[a,b,c],d,f\big] \;=\; & \big[[a,b,c],d,f\big] + \big[[b,c,d],f,a\big] + \big[[c,d,f],a,b\big] + \big[[d,f,a],b,c\big] \\ & + \big[[f,a,b],c,d\big], \end{aligned}$$

where a,b,c,d,f are elements of a ternary algebra $\mathcal{T}$. We have the following:

Theorem 2. *Let $\mathcal{T}$ be a ternary algebra. Then, for any $a,b,c,d,f \in \mathcal{T}$, the ternary commutator (13) and its conjugate (14) have the following property:*

$$[a,b,c] = \omega\,[b,c,a] = \overline{\omega}\,[c,a,b], \quad [a,b,c]^* = \overline{\omega}\,[b,c,a]^* = \omega\,[c,a,b]^*$$

and the ternary commutator satisfies the identity, as follows:

$$\circlearrowleft \Big(\big[[a,b,c],d,f\big] + \big[[a,d,b],f,c\big] + \big[[a,f,d],c,b\big] + \big[[a,c,f],b,d\big] \Big) = 0.$$

In what follows, we refer to the statement of Theorem 2 as the basic identity:

Proof. We prove Theorem 2 by direct computation, that is, by applying Formula (13) twice to each term of the basic identity and using a ternary associativity of multiplication. We carried out this computation using a computer program containing a non-commutative symbolic calculus. The computer program makes it possible to study the structure of the basic identity. A study of the structure of the basic identity shows that it holds due to the reasoning explained previously with the help of Formulas (4)–(6). Let us denote $a = a_1, b = a_2, c = a_3, d = a_4, f = a_5$. The computer program enables us to determine in which terms of the basic identity a particular product of elements a_1, a_2, a_3, a_4, a_5 appears, along with the associated coefficients and the placement of the round brackets. For example, the product $a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5$ appears six times as follows:

$$\begin{array}{ccc} [[a_1,a_2,a_3],a_4,a_5], & [[a_2,a_3,a_4],a_5,a_1], & [[a_3,a_4,a_5],a_1,a_2], \\ (a_1 \cdot a_2 \cdot a_3) \cdot a_4 \cdot a_5, & \overline{\omega}\, a_1 \cdot (a_2 \cdot a_3 \cdot a_4) \cdot a_5, & \omega\, a_1 \cdot a_2 \cdot (a_3 \cdot a_4 \cdot a_5), \end{array}$$

and

$$\begin{array}{ccc} [[a_5,a_4,a_3],a_2,a_1], & [[a_4,a_3,a_2],a_1,a_5], & [[a_3,a_2,a_1],a_5,a_4], \\ a_1 \cdot a_2 \cdot (a_3 \cdot a_4 \cdot a_5), & \overline{\omega}\, a_1 \cdot (a_2 \cdot a_3 \cdot a_4) \cdot a_5, & \omega\, (a_1 \cdot a_2 \cdot a_3) \cdot a_4 \cdot a_5. \end{array}$$

Here, in the first line, we show the double ternary commutators of the basic identity, and below them, we show in which forms, that is, the coefficient and position of the round brackets, the product $a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5$ appears in the corresponding double commutator. By adding up the six terms obtained in this case and assuming the associativity of the first kind, the result is zero.

In the case of the ternary associativity of the second kind, in addition to the above table, we should also consider the set of double ternary commutators on the left-hand side of the basic identity that includes the product $a_1 \cdot a_4 \cdot a_3 \cdot a_2 \cdot a_5$, summarized as follows:

$$\begin{array}{ccc} [[a_3,a_1,a_4],a_2,a_5], & [[a_2,a_3,a_4],a_5,a_1], & [[a_2,a_5,a_3],a_1,a_4], \\ \omega\, (a_1 \cdot a_4 \cdot a_3) \cdot a_2 \cdot a_5, & \overline{\omega}\, a_1 \cdot (a_4 \cdot a_3 \cdot a_2) \cdot a_5, & a_1 \cdot a_4 \cdot (a_3 \cdot a_2 \cdot a_5), \end{array}$$

and

$$\begin{array}{ccc} [[a_4,a_1,a_3],a_5,a_2], & [[a_4,a_3,a_2],a_1,a_5], & [[a_3,a_5,a_2],a_4,a_1], \\ (a_1 \cdot a_4 \cdot a_3) \cdot a_2 \cdot a_5, & \overline{\omega}\, a_1 \cdot (a_4 \cdot a_3 \cdot a_2) \cdot a_5, & \omega\, a_1 \cdot a_4 \cdot (a_3 \cdot a_2 \cdot a_5). \end{array}$$

A comparison of the columns in the center of these tables immediately shows that—in the case of the ternary associativity of the second kind—we obtain the same type of sum, which is equal to zero. □

Thus, the basic identity consists of 20 double ternary commutators. The general affine group, considered as a subgroup of the permutations of the symmetric group S_5, is generated by two cycles σ, τ. The double ternary commutators are as follows:

$$\big[[a_1,a_2,a_3],a_4,a_5\big],\ \big[[a_1,a_4,a_2],a_5,a_3\big],\ \big[[a_1,a_5,a_4],a_3,a_2\big],\ \big[[a_1,a_3,a_5],a_2,a_4\big] \quad (29)$$

are determined by the permutations e, τ, τ^2, τ^3, that is, by the first elements of the general affine group $GA(1,5)$ in Formulas (25)–(28). The cyclic permutations of the double ternary commutators (29) are determined by the elements in (25), (26), (27), and (28), respectively, starting from the second element. Note that the elements in (25) and (27) form the dihedral subgroup D_{10} of the symmetric group S_5.

Theorem 2 motivates the introduction of the following notion:

Definition 1. *Let $\mathcal{L}$ be a vector space over the field of complex numbers. Then, $\mathcal{L}$ is said to be a ternary Lie algebra at cube roots of unity if $\mathcal{L}$ is endowed with a ternary bracket $(x,y,z) \in \mathcal{L} \times \mathcal{L} \times \mathcal{L} \mapsto [x,y,z] \in \mathcal{L}$, which transforms under the cyclic permutations of its arguments as follows:*

$$[x,y,z] = \omega\, [y,z,x] = \overline{\omega}\, [z,x,y],\ \ [x,y,z]^* = \overline{\omega}\, [y,z,x]^* = \omega\, [z,x,y]^*, \quad (30)$$

where ω is a primitive cube root of unity, $\overline{\omega}$ is its conjugate, $[x,y,z]^* = [z,y,x]$, and a ternary bracket satisfies the identity, as follows:

$$\circlearrowleft \Big([[x,y,z],u,v] + [[x,u,y],v,z] + [[x,v,u],z,y] + [[x,z,v],y,u] \Big) = 0. \quad (31)$$

In order to simplify the terminology, a ternary Lie algebra at cube roots of unity will also be referred to as a ternary ω-Lie algebra. Property (30) will be referred to as ω-symmetry of the ternary bracket. Identity (31) will be referred to the basic identity.

Definition 2. *Let $\mathcal{L}$ be a ternary ω-Lie algebra and $\mathcal{I} \subset \mathcal{L}$ be its subspace. Then, $\mathcal{I}$ is said to be an ideal of a ternary ω-Lie algebra if, for any $a \in \mathcal{I}$ and $x, y \in \mathcal{L}$, it holds $[a, x, y] \in \mathcal{I}$. A ternary ω-Lie algebra is said to be simple if it has no non-trivial ideals, that is, it has no ideals other than $\{0\}$ and $\mathcal{L}$.*

Let $\mathcal{L}$ be a ternary ω-Lie algebra, where $\mathcal{L}$ is an n-dimensional vector space, and $e_1, e_2, \ldots, e_n$ be the basis for a vector space $\mathcal{L}$. In analogy with the binary case, we introduce the structure constants of a ternary ω-Lie algebra as follows:

$$[e_i, e_k, e_l] = C^m_{ikl}\, e_m, \quad [e_i, e_k, e_l]^* = \tilde{C}^m_{ikl}\, e_m \tag{32}$$

where $C^m_{ikl}, \tilde{C}^m_{ijk}$ will be referred to as structure constants of a ternary ω-Lie algebra $\mathcal{L}$. It is easy to see that $\tilde{C}^m_{ijk} = C^m_{kji}$. In (32), we used the Einstein convention of summation over repeated indices. Obviously, the structure constants of a ternary ω-Lie algebra can be considered as complex-valued tensors of type $(1, 3)$. This tensor has the ω-symmetry with respect to the cyclic permutations of its three subscripts, as follows:

$$C^m_{ikl} = \omega\, C^m_{kli} = \overline{\omega}\, C^m_{lik}, \quad \tilde{C}^m_{ikl} = \overline{\omega}\, \tilde{C}^m_{kli} = \omega\, \tilde{C}^m_{lik}. \tag{33}$$

For every value of the superscript $m = 1, 2, \ldots, n$, the structure constants of a ternary ω-Lie algebra $\mathcal{L}$, that is, both C^m_{ijk} and $\tilde{C}^m_{ijk}$, satisfy the following equation:

$$T_{ijk} + T_{jki} + T_{kij} = 0, \tag{34}$$

where T_{ijk} is a covariant tensor of order 3. It is evident that the third-order covariant tensors defined on the vector space $\mathcal{L}$, which satisfy Equation (34), form the subspace in the vector space of covariant tensors of order 3. This subspace will be denoted by $\mathfrak{T}^3(\mathcal{L})$.

Formula (33) clearly shows that for any superscript m, the structure constants C^m_{ijk}, $\tilde{C}^m_{ijk}$ are the eigenvectors of the linear operator in $\mathfrak{T}^3(\mathcal{L})$ induced by the cyclic permutation $(1\,2\,3)$ with eigenvalues $\omega, \overline{\omega}$, respectively. Thus, we have the following:

$$\mathfrak{T}^3(\mathcal{L}) = \mathfrak{T}^3_{\omega}(\mathcal{L}) \oplus \mathfrak{T}^3_{\overline{\omega}}(\mathcal{L}), \tag{35}$$

where

$$\mathfrak{T}^3_{\omega}(\mathcal{L}) = \{T_{ijk} \in \mathfrak{T}^3(\mathcal{L}) : T_{ijk} = \omega T_{jki}\}, \quad \mathfrak{T}^3_{\overline{\omega}}(\mathcal{L}) = \{T_{ijk} \in \mathfrak{T}^3(\mathcal{L}) : T_{ijk} = \overline{\omega}\, T_{jki}\}.$$

Thus, for each value of the superscript m, the structure constants C^m_{ijk} of a ternary ω-Lie algebra $\mathcal{L}$ belong to subspace $\mathfrak{T}^3_{\omega}(\mathcal{L})$, and the structure constants $\tilde{C}^m_{ijk}$ belong to subspace $\mathfrak{T}^3_{\overline{\omega}}(\mathcal{L})$.

Here, we would like to note an important connection between the structure constants of a three-dimensional ternary ω-Lie algebra and irreducible representations of the rotation group. Let $n = 3$, i.e., we are considering a three-dimensional ternary ω-Lie algebra. Let $A = (A^i_j) \in \mathrm{SO}(3)$ be a real orthogonal matrix with determinant 1. Then, we have the following formula:

$$T_{ijk} \;\rightarrow\; T'_{prs} = A^i_p A^j_r A^k_s\, T_{ijk}, \tag{36}$$

where $T_{ijk}, T'_{prs} \in \mathfrak{T}^3(\mathcal{L})$, defines a linear representation of the rotation group $\mathrm{SO}(3)$ in the space $\mathfrak{T}^3(\mathcal{L})$. If we add to Equation (34) the condition of tracelessness of a tensor T_{ijk} for any pair of subscripts, then Formula (36) defines a twice repeated irreducible representation of the rotation group in the corresponding subspace of third-order covariant tensors [21]. Now, the decomposition (35) splits this two-fold irreducible representation into two irreducible ones, respectively, in subspaces $\mathfrak{T}^3_{\omega}(\mathcal{L})$ and $\mathfrak{T}^3_{\overline{\omega}}(\mathcal{L})$ (with the additional condition that a tensor T_{ijk} is traceless). Note that the subspace of traceless tensors in $\mathfrak{T}^3_{\omega}(\mathcal{L})$ is a five-dimensional Hermitian space and the explicit description of this space can be found in [14]. In the next paper, we plan to use this connection with irreducible representations of the rotation group to classify three-dimensional ternary ω-Lie algebras.

It follows from the basic identity (31) that the structure constants of a ternary ω-Lie algebra $\mathcal{L}$ satisfy the system of equations, as follows:

$$\circlearrowleft (C^m_{\underline{i}\,\underline{k}\,\underline{l}}\,C^p_{m\,\underline{r}\,\underline{s}} + C^m_{\underline{i}\,\underline{r}\,\underline{k}}\,C^p_{m\,\underline{s}\,\underline{l}} + C^m_{\underline{i}\,\underline{s}\,\underline{r}}\,C^p_{m\,\underline{l}\,\underline{k}} + C^m_{\underline{i}\,\underline{l}\,\underline{s}}\,C^p_{m\,\underline{k}\,\underline{r}}) = 0. \tag{37}$$

In this formula, the symbol $\circlearrowleft$ means that in an expression that follows it, one should perform the five cyclic permutations of the underlined subscripts and then take the sum of obtained expressions. For instance, if we apply $\circlearrowleft$ to the first term in (37), we obtain the following:

$$\circlearrowleft C^m_{\underline{i}\,\underline{k}\,\underline{l}}\,C^p_{m\,\underline{r}\,\underline{s}} = C^m_{ikl}\,C^p_{mrs} + C^m_{klr}\,C^p_{msi} + C^m_{lrs}\,C^p_{mik} + C^m_{rsi}\,C^p_{mkl} + C^m_{sik}\,C^p_{mlr}.$$

4. Examples of Ternary Lie Algebra at Cube Roots of Unity

In this section, we give some important examples of ternary associative algebras and consider ternary ω-Lie algebras that are induced by the ternary commutator (13). A wide class of ternary associative algebras can be constructed using square matrices. Indeed, if A, B, C are square matrices of order n, we can consider the ternary product $A\,B\,C$. This definition is correct since matrix multiplication is associative. Obviously, in this case, we obtain ternary multiplication with the associativity of the first kind. However, from our point of view, this example is of little interest from the ternary point of view, because, firstly, ternary multiplication is constructed using binary, that is, binary is more fundamental than ternary, and, secondly, for square matrices, there is a deeply developed theory of (binary) Lie algebras. Therefore, in this section, we will consider examples of ternary algebras constructed using either rectangular (two-dimensional) matrices or cubic (three-dimensional) matrices. Thus, the notion of a ternary ω-Lie algebra proposed in this paper can be considered as an extension of the concept of Lie algebra to rectangular and cubic matrices. Note that, firstly, the ternary multiplications considered in this section cannot be reduced to binary ones, and, secondly, they are associative of the second kind.

One of the simplest examples of ternary algebra with the associativity of the second kind is an n-dimensional complex vector space $\mathbb{C}^n$ with a $\mathbb{C}$-valued bilinear symmetric form B defined on it. Then, the ternary multiplication in $\mathbb{C}^n$ will be defined as follows:

$$x \cdot y \cdot z = B(x, y)\, z. \tag{38}$$

It is easy to verify that this ternary product is associative of the second kind. Indeed, we have the following:

$$\begin{aligned}
(x \cdot y \cdot z) \cdot u \cdot v &= (B(x,y)\, z) \cdot u \cdot v = B(x,y)\, B(z,u)\, v,\\
x \cdot (u \cdot z \cdot y) \cdot v &= x \cdot (B(u,z)\, y) \cdot v = B(u,z)\, B(x,y)\, v,\\
x \cdot y \cdot (z \cdot u \cdot v) &= x \cdot y \cdot (B(z,u)\, v) = B(z,u)\, B(x,y)\, v,
\end{aligned}$$

and, due to the symmetry $B(z,u) = B(u,z)$, we see that all three products are equal. Hence, if we endow a vector space $\mathbb{C}^n$ with the ternary commutator (13), then according to

Theorem 2, it becomes a ternary ω-Lie algebra. In this case, the ternary commutator can be written as follows:

$$\begin{aligned}[x,y,z] &= x\cdot y\cdot z+\omega\, y\cdot z\cdot x+\overline{\omega}\, z\cdot x\cdot y+z\cdot y\cdot x+\overline{\omega}\, y\cdot x\cdot z+\omega\, x\cdot z\cdot y\\ &= B(x,y)\,z+\omega\,B(y,z)\,x+\overline{\omega}\,B(z,x)\,y+B(z,y)x+\overline{\omega}\,B(y,x)\,z+\omega\,B(x,z)\,y\\ &= (1+\overline{\omega})\,B(x,y)\,z+(1+\omega)\,B(y,z)\,x+(\omega+\overline{\omega})\,B(z,x)\,y\\ &= -\big(B(z,x)\,y+\omega\,B(x,y)\,z+\overline{\omega}\,B(y,z)\,x\big).\end{aligned}$$

Omitting the irrelevant factor -1, we can consider the ternary commutator (13) and its conjugate in a reduced form, as follows:

$$[x,y,z] \;= B(z,x)\,y+\omega\,B(x,y)\,z+\overline{\omega}\,B(y,z)\,x, \tag{39}$$
$$[x,y,z]^* = B(z,x)\,y+\overline{\omega}\,B(x,y)\,z+\omega\,B(y,z)\,x. \tag{40}$$

It is easy to verify that the reduced ternary commutator (39) and its conjugate (40) have the same transformation properties under cyclic permutations of arguments (15) and (16) as the full-length commutator (13) and its conjugate. It is interesting to note that in this particular case the reduced ternary commutator (39) satisfies a reduced version of the basic identity, which contains only ten terms, as follows:

$$\circlearrowleft\,\big[[x,y,z],u,v\big]+\circlearrowleft\,\big[[x,u,y],v,z\big]=0. \tag{41}$$

The basic identity contains two copies of the dihedral group D_{10}. The dihedral group D_{10} contains a subgroup of cyclic permutations $\mathbb{Z}_5$. Thus, the reduced identity (41) is obtained by reducing each copy of the dihedral group D_{10} to its cyclic subgroup $\mathbb{Z}_5$.

Let us consider a special case of a ternary ω-Lie algebra constructed using ternary multiplication (38). Let us consider n-dimensional vectors of $\mathbb{C}^n$ as row matrices. Then, we can put $B(x,y)=x\,y^T$, where y^T is the column matrix. Thus, we have the ternary ω-Lie algebra, where the underlying vector space is the n-dimensional complex vector space $\mathbb{C}^n$ and the ternary commutator is defined by the following formula:

$$[x,y,z]=z\,x^T\,y+\omega\;x\,y^T\,z+\overline{\omega}\;y\,z^T\,x. \tag{42}$$

In this particular case, we can easily compute the structure constants of the ternary ω-Lie algebra. Indeed let $e_1,e_2,\dots,e_n$ be the canonical basis for $\mathbb{C}^n$, that is, the ith coordinate of a vector e_i is 1, all other coordinates are equal to zero. Then, the structure constants of this ternary ω-Lie algebra are as follows:

$$C^m_{ijk}=\delta_{ki}\,\delta^m_j+\omega\;\delta_{ij}\,\delta^m_k+\overline{\omega}\;\delta_{jk}\,\delta^m_i. \tag{43}$$

If we calculate the structure constants of the ternary ω-Lie algebra (42) for the simplest case of $n=2$, then we obtain the following:

$$[e_1,e_2,e_1]=e_2,\quad [e_2,e_1,e_2]=e_1. \tag{44}$$

We denote the two-dimensional ternary ω-Lie algebra with commutation relations (44) by $\mathcal{L}_2$.

Let $M_n(\mathbb{C})$ be a vector space of complex nth-order square matrices. Then, the ternary product (38) can be applied to $M_n(\mathbb{C})$ if we take $B(\Phi,\Psi)=\mathrm{Tr}\,(\Phi\,\Psi)$, where $\Phi,\Psi\in M_n(\mathbb{C})$. Then, the ternary commutator (39) takes on the following form:

$$[\Phi,\Psi,\Omega]=\mathrm{Tr}\,(\Omega\,\Phi)\;\Psi+\omega\;\mathrm{Tr}\,(\Phi\,\Psi)\;\Omega+\overline{\omega}\;\mathrm{Tr}\,(\Psi\,\Omega)\;\Phi. \tag{45}$$

Hence, the ternary commutator (45) induces a structure of ternary ω-Lie algebra on a complex vector space $M_n(\mathbb{C})$. It is interesting to note that the ternary commutator, which is also constructed via the trace and cyclic permutations of arguments, is as follows:

$$[\![\Phi, \Psi, \Omega]\!] = \mathrm{Tr}\,(\Phi)\ [\Psi, \Omega] + \mathrm{Tr}\,(\Psi)\ [\Omega, \Phi] + \mathrm{Tr}\,(\Omega)\ [\Phi, \Psi], \tag{46}$$

where square brackets on the right-hand side of this formula denote the commutators of two matrices, that is, $[\Phi, \Psi] = \Phi\,\Psi - \Psi\,\Phi$, inducing a structure of 3-Lie algebra on a vector space $M_n(\mathbb{C})$. The ternary commutator (46) was introduced in [22] to construct a quantization for generalized Hamiltonian mechanics proposed by Nambu. It should be mentioned that the ternary commutators (45) and (46) have different properties with respect to permutations of arguments. That is, our ternary commutator (45) has ω-symmetry, while the ternary commutator (46) is totally skew-symmetric.

The example of a ternary ω-Lie algebra with ternary commutator (42) is a special case of a more general construction. In other words, we can extend the ternary commutator (42) to rectangular matrices of arbitrary dimensions. Let $M_{m,n}(\mathbb{C})$ be a vector space of complex $m \times n$-matrices. One can define the ternary product of three $m \times n$-matrices $A, B, C \in M_{m,n}(\mathbb{C})$ as follows:

$$A \cdot B \cdot C = A\,B^T\,C,$$

where, in this formula, the right side denotes the usual matrix multiplication, and B^T stands for transposed matrix. It is easy to verify that this ternary product of $m \times n$-matrices has the associativity of the second kind. Hence, we can endow the complex vector space $M_{m,n}(\mathbb{C})$ with the following ternary commutator:

$$[A, B, C] = A\,B^T\,C + \omega\ B\,C^T\,A + \overline{\omega}\ C\,A^T\,B + C\,B^T\,A + \overline{\omega}\ B\,A^T\,C + \omega\ A\,C^T\,B, \tag{47}$$

and the complex vector space $M_{m,n}(\mathbb{C})$ of rectangular $m \times n$-matrices becomes a ternary ω-Lie algebra.

The ternary multiplications of three-dimensional matrices given in Theorem 1 make it possible to construct numerous examples of ternary ω-Lie algebras. The simplest example can be constructed using diagonal three-dimensional matrices of the second order. Taking ternary multiplication 3 (Theorem 1), we obtain, as shown above, an abelian cyclic-commutative ternary algebra. It follows that for any three diagonal three-dimensional matrices, their ternary commutator is zero. If the ternary commutator of any three elements in a ternary ω-Lie algebra is zero, then this algebra is termed Abelian. Hence, the ternary ω-algebra consisting of diagonal three-dimensional second-order matrices provides an example of a two-dimensional Abelian ternary ω-Lie algebra.

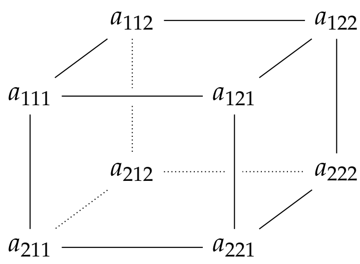
The next example of a two-dimensional ternary ω-Lie algebra will be constructed using three-dimensional traceless matrices of the second order. As a ternary product of three-dimensional matrices, we will use ternary multiplication 3 (Theorem 1), although it is worth noting that we could equally well use ternary multiplication 4. Let $A = (a_{ijk})$ be a three-dimensional matrix of the second order, that is, $i, j, k = 1, 2$. We will call a three-dimensional matrix A traceless if the trace of this matrix with respect to any pair of subscripts is zero. Hence, for any $k = 1, 2$ we have the following:

$$a_{iik} = a_{iki} = a_{kii} = 0, \tag{48}$$

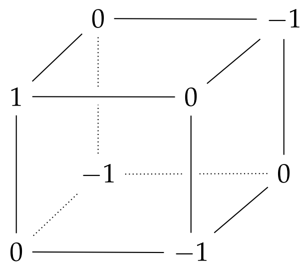
where $a_{iik} = a_{11k} + a_{22k}, a_{iki} = a_{1k1} + a_{2k2}, a_{kii} = a_{k11} + a_{k22}$. The ternary ω-Lie algebra of three-dimensional matrices of the second order is an 8-dimensional algebra. Traceless matrices form a two-dimensional subspace in this algebra, and it is easy to show that this two-dimensional subspace is closed under the ternary commutator (13); that is, traceless three-dimensional matrices of the second order form a subalgebra of the ternary ω-Lie algebra of three-dimensional matrices of the second order. From condition (48), it follows that—in the case of a three-dimensional matrix of the second-order A—we have two independent parameters, a_{111}, a_{222}, and all other entries of the matrix are expressed through them, that is,

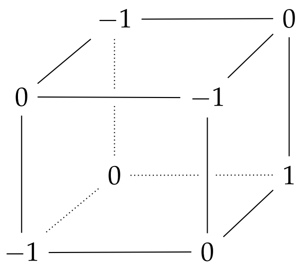
$$a_{221} = a_{212} = a_{122} = -a_{111}, \quad a_{112} = a_{121} = a_{211} = -a_{222}.$$

We arrange the entries of a three-dimensional matrix of the second-order A in space, that is, in the vertices of the cube, as follows:



Thus, as generators of the ternary ω-Lie algebra of three-dimensional traceless matrices of the second order, we can take two three-dimensional matrices $F_1 = -\frac{i}{2\sqrt{2}}\, E_1, F_2 = -\frac{i}{2\sqrt{2}}\, E_2$, where we have the following:

$E_1 =$ 

$E_2 =$ 

We calculate the ternary commutator as follows:

$$\begin{aligned}[A, B, C]_{ijk} \;=\; & A_{ijl} B_{nml} C_{mnk} + \omega\; B_{ijl} C_{nml} A_{mnk} + \overline{\omega}\; C_{ijl} A_{nml} B_{mnk} \\ & + C_{ijl} B_{nml} A_{mnk} + \overline{\omega}\; B_{ijl} A_{nml} C_{mnk} + \omega\; A_{ijl} C_{nml} B_{mnk},\end{aligned}$$

we identify the commutation relations of the ternary ω-Lie algebra of three-dimensional traceless matrices of the second order, as follows:

$$[F_1, F_2, F_1] = F_2, \quad [F_2, F_1, F_2] = F_1. \tag{49}$$

Thus, we construct one more realization of the two-dimensional ternary ω-Lie algebra $\mathcal{L}_2$ using three-dimensional traceless matrices of the second order.

5. Classification of Two-Dimensional Ternary ω-Lie Algebras

The main purpose of this section is to classify two-dimensional ternary ω-Lie algebras. We prove a theorem stating that, in dimension 2, up to isomorphism, there are precisely four ternary ω-Lie algebras. One of these ternary ω-Lie algebras is $\mathcal{L}_2$ constructed either via two-dimensional complex vectors (44) or traceless three-dimensional second-order matrices (49).

Theorem 3. *If $\mathcal{F}$ is a two-dimensional ternary ω-Lie algebra, then it is isomorphic to one of the four two-dimensional ternary ω-Lie algebras given by their structure constants in the following table:*

Number	C^1_{121}	C^2_{121}	C^1_{212}	C^2_{212}
I	0	0	0	0
II	0	1	1	0
III	0	1	0	0
IV	1	0	0	−1

Proof. Let $\mathcal{F}$ be a two-dimensional ternary ω-Lie algebra and e_1, e_2 be generators for this algebra. Due to the properties of a ternary ω-Lie algebra, a ternary bracket of this algebra containing three equal arguments is equal to zero. Hence, we have the following:

$$[e_1, e_1, e_1] = [e_2, e_2, e_2] = 0. \tag{50}$$

Moreover, due to the ω-symmetries of a ternary bracket of a ternary ω-Lie algebra, we have the following:

$$[e_1, e_2, e_1] = \omega\,[e_2, e_1, e_1] = \overline{\omega}[e_1, e_1, e_2], \qquad [e_2, e_1, e_2] = \omega\,[e_1, e_2, e_2] = \overline{\omega}[e_2, e_2, e_1].$$

Thus, we have two independent and possibly non-trivial ternary brackets, that is, $[e_1, e_2, e_1]$, $[e_2, e_1, e_2]$, which completely determine the structure of a two-dimensional ternary ω-Lie algebra. We can expand these two ternary brackets via the structure constants as follows:

$$[e_1, e_2, e_1] = C^1_{121}\, e_1 + C^2_{121}\, e_2, \qquad [e_2, e_1, e_2] = C^1_{212}\, e_1 + C^2_{212}\, e_2. \tag{51}$$

Thus, the structure of a two-dimensional ternary ω-Lie algebra $\mathcal{F}$ is completely determined by four independent structure constants $C^1_{121}, C^2_{121}, C^1_{212}, C^2_{212}$. First of all, the structure constants of a two-dimensional ω-Lie algebra have the following symmetries:

$$C^1_{121} = \omega\, C^1_{211} = \overline{\omega}\, C^1_{112}, \quad C^2_{121} = \omega\, C^2_{211} = \overline{\omega}\, C^2_{112}, \tag{52}$$

$$C^1_{212} = \omega\, C^1_{122} = \overline{\omega}\, C^1_{221}, \quad C^2_{212} = \omega\, C^2_{122} = \overline{\omega}\, C^2_{221}. \tag{53}$$

From (52) and (53), we have the following:

$$C^m_{121} + C^m_{211} + C^m_{112} = 0, \quad C^m_{212} + C^m_{221} + C^m_{122} = 0, \qquad m = 1, 2. \tag{54}$$

It follows from the basic identity that the structure constants of $\mathcal{F}$ must satisfy the following equations:

$$\circlearrowleft (C^m_{\underline{i}\,\underline{k}\,\underline{l}}\, C^p_{m\,\underline{r}\,\underline{s}} + C^m_{\underline{i}\,\underline{r}\,\underline{k}}\, C^p_{m\,\underline{s}\,\underline{l}} + C^m_{\underline{i}\,\underline{s}\,\underline{r}}\, C^p_{m\,\underline{l}\,\underline{k}} + C^m_{\underline{i}\,\underline{l}\,\underline{s}}\, C^p_{m\,\underline{k}\,\underline{r}}) = 0. \tag{55}$$

We claim that, in the case of a two-dimensional ternary ω-Lie algebra $\mathcal{F}$, the basic identity does not impose additional conditions on the structure constants of $\mathcal{F}$, that is, Equation (55) is satisfied for any integers (i, k, l, r, s), independently running the values $1, 2$, by virtue of the properties (54) of the structure constants.

Indeed, the number of ordered sequences of integers (i, k, l, r, s), where each integer can be either 1 or 2, is $2^5 = 32$. We can discard two of these sequences $(1, 1, 1, 1, 1), (2, 2, 2, 2, 2)$, since due to property (50), the left-hand side of Equation (37) in this case is equal to zero. Since each sequence of integers (i, k, l, r, s) determines an Equation (37), formally, we have 30 equations, but not all of these equations are distinct. We will go through all possible sequences of integers (i, k, l, r, s) using the following scheme. First, we will consider all sequences containing four 1s and one 2 (there will be 5 of these), then three 1s and two 2s (there will be 10 of these), then two 1s and three 2s (there will be 10 of these), and finally one 1 and four 2s (there will be 5 of these). We start with the sequence $(1, 2, 1, 1, 1)$. Through straightforward calculations, we identify that this sequence leads to the following:

$$C^m_{121}\, C^p_{m11} + C^m_{211}\, C^p_{m11} + C^m_{111}\, C^p_{m12} + C^m_{111}\, C^p_{m21} + C^m_{112}\, C^p_{m11} = 0. \tag{56}$$

First, note that this equation contains all ordered sequences of integers (i, k, l, r, s) containing four 1s and one 2. This means that any such sequence yields Equation (56). Second, the left-hand side of this equation is identically zero, which is easy to see if we take into account that $C^m_{111} = 0$ and write the left-hand side of (56) in the following form:

$$(C^m_{121} + C^m_{211} + C^m_{112})\, C^p_{m11}.$$

Because of property (54), this expression vanishes and we conclude that (56) is an identity and does not impose additional constraints on the structure constants.

Next, we consider the sequence $(1, 2, 1, 1, 2)$. This sequence leads to the following equation:

$$C^m_{121}\, C^p_{m12} + C^m_{211}\, C^p_{m21} + C^m_{112}\, C^p_{m12} + C^m_{121}\, C^p_{m21} + C^m_{212}\, C^p_{m11} \\ + C^m_{112}\, C^p_{m21} + C^m_{122}\, C^p_{m11} + C^m_{221}\, C^p_{m11} + C^m_{211}\, C^p_{m12} + C^m_{111}\, C^p_{m22} = 0. \quad (57)$$

As in the previous case, the left-hand side of the above equation contains all ordered sequences of integers containing three 1s and two 2s (there are ten of such sequences). Since we perform cyclic permutations on the five subscripts, this means that by taking any sequence and substituting it into (37), we obtain Equation (57). Since $C^m_{111} = 0$, the last term on the left-hand side of Equation (57) is zero, and by collecting the remaining nine terms, i.e.,

$$(C^m_{121} + C^m_{211} + C^m_{112})\, (C^p_{m12} + C^p_{m21}) + (C^m_{221} + C^m_{212} + C^m_{122})\, C^p_{m11}$$

we see that by property (54), the left-hand side of Equation (57) is identically zero. We still have two sets of sequences left, that is, the set of sequences containing two 1s and three 2s, and the set of sequences containing one 1 and four 2s. By choosing sequences $(1, 2, 1, 2, 2)$ and $(2, 1, 2, 2, 2)$ as representatives of these two sets and substituting them into (37), we obtain the following equations:

$$C^m_{121}\, C^p_{m22} + C^m_{212}\, C^p_{m21} + C^m_{122}\, C^p_{m12} + C^m_{221}\, C^p_{m21} + C^m_{212}\, C^p_{m12} \\ + C^m_{122}\, C^p_{m21} + C^m_{222}\, C^p_{m11} + C^m_{221}\, C^p_{m12} + C^m_{211}\, C^p_{m22} + C^m_{112}\, C^p_{m22} = 0,$$
$$C^m_{212}\, C^p_{m22} + C^m_{122}\, C^p_{m22} + C^m_{222}\, C^p_{m21} + C^m_{222}\, C^p_{m12} + C^m_{221}\, C^p_{m22} = 0,$$

respectively. The analysis of these two equations is similar to that given above and shows that we do not obtain any additional equation on the structure constants.

Now, we will study how the structure constants C^m_{ikl} behave when we pass to another basis of two-dimensional space. Obviously, the structure constants change, transforming as a (1,3)-tensor, but the structure of a ternary ω-Lie algebra $\mathcal{F}$ remains the same. Let e'_1, e'_2 be another basis of generators for the two-dimensional ternary ω-Lie algebra $\mathcal{F}$, where we have the following:

$$e_1 = \alpha^1_1\, e'_1 + \alpha^2_1\, e'_2, \qquad e_2 = \alpha^1_2\, e'_1 + \alpha^2_2\, e'_2. \quad (58)$$

Let us denote a transition matrix by A as follows:

$$A = \begin{pmatrix} \alpha^1_1 & \alpha^1_2 \\ \alpha^2_1 & \alpha^2_2 \end{pmatrix}. \quad (59)$$

Obviously, A is a regular matrix, that is, $\mathrm{Det}\, A \neq 0$. Thus, A belongs to the group of regular second-order complex matrices, that is, $A \in \mathrm{GL}_2(\mathbb{C})$. Let us denote the structure constants of $\mathcal{F}$ in the basis e'_1, e'_2 by $C'^{\,m}_{ikl}$, that is,

$$[e'_1, e'_2, e'_1] = C'^{\,1}_{121}\, e'_1 + C'^{\,2}_{121}\, e'_2, \quad [e'_2, e'_1, e'_2] = C'^{\,1}_{212}\, e'_1 + C'^{\,2}_{212}\, e'_2.$$

Through straightforward calculations, we identify the following:

$$\begin{pmatrix} C'^{1}_{121} \\ C'^{2}_{121} \\ C'^{1}_{212} \\ C'^{2}_{212} \end{pmatrix} = \frac{1}{(\mathrm{Det}\,A)^2} \begin{pmatrix} \alpha^1_1\,\alpha^2_2 & \alpha^1_2\,\alpha^2_2 & \alpha^1_1\,\alpha^2_1 & \alpha^1_2\,\alpha^2_1 \\ \alpha^2_1\,\alpha^2_2 & (\alpha^2_2)^2 & (\alpha^2_1)^2 & \alpha^2_1\,\alpha^2_2 \\ \alpha^1_1\,\alpha^1_2 & (\alpha^1_2)^2 & (\alpha^1_1)^2 & \alpha^1_1\,\alpha^1_2 \\ \alpha^1_2\,\alpha^2_1 & \alpha^1_2\,\alpha^2_2 & \alpha^1_1\,\alpha^2_1 & \alpha^1_1\,\alpha^2_2 \end{pmatrix} \begin{pmatrix} C^1_{121} \\ C^2_{121} \\ C^1_{212} \\ C^2_{212} \end{pmatrix}. \tag{60}$$

Obviously, Formula (60) defines the tensor representation of the Lie group $\mathrm{GL}_2(\mathbb{C})$ in the space of (1,3)-tensors with symmetries (52) and (53).

We will consider the structure constants, ordered as follows $(C^1_{121}, C^2_{121}, C^1_{212}, C^2_{212})$, as vectors of the four-dimensional complex vector space $\mathbb{C}^4$. We will exclude the trivial case where all structure constants are zero. In this case, we will refer to the corresponding ternary ω-Lie algebra as Abelian. In the four-dimensional complex vector space of structure constants, vectors satisfying the condition $C^1_{121} = C^2_{212}$ form the three-dimensional subspace, which will be denoted by $\mathcal{W}$. Thus, we have the following:

$$\mathcal{W} = \{(C^1_{121}, C^2_{121}, C^1_{212}, C^2_{212}) \in \mathbb{C}^4 : C^1_{121} = C^2_{212}\}.$$

As follows, from (60), the subspace $\mathcal{W}$ is invariant with respect to transformations (60). Let us denote by $\mathcal{V}$ the one-dimensional subspace of $\mathbb{C}^4$ spanned by the vector $(1, 0, 0, -1)$. Then, $\mathbb{C}^4 = \mathcal{W} \oplus \mathcal{V}$. It follows from (60) that the vector $(1, 0, 0, -1)$ is an eigenvector of all transformations (60), i.e., $\mathcal{V}$ is an invariant one-dimensional subspace. Thus, the tensor representation (60) is reducible and the one-dimensional subspace $\mathcal{V}$ defines the two-dimensional ternary ω-Lie algebra of our classification. The non-trivial commutation relations of this algebra will be written as follows:

$$[e_1, e_2, e_2] = e_1, \quad [e_2, e_1, e_2] = -e_2. \tag{61}$$

Now, we study the structure of the three-dimensional subspace $\mathcal{W}$. To simplify the presentation, we introduce the following notations:

$$a = C'^{1}_{121}, \quad b = C'^{2}_{121}, \quad c = C'^{1}_{212}, \quad x = \alpha^1_1, \quad y = \alpha^1_2, \quad z = \alpha^2_1, \quad u = \alpha^2_2.$$

Since $A \in \mathrm{GL}_2(\mathbb{C})$, it holds $x\,u - y\,z \neq 0$. First, we note that the vector $(0, 1, 1, 0)$ of structure constants of the two-dimensional ternary ω-Lie algebra $\mathcal{L}_2$ belongs to the subspace $\mathcal{W}$. Thus, by applying to this vector all possible transformations (60), we obtain the set of vectors lying in the subspace $\mathcal{W}$ that define the same algebra $\mathcal{L}_2$. Therefore, a vector $(a, b, c, a) \in \mathcal{W}$ defines the algebra $\mathcal{L}_2$ if the system of equations, i.e.,

$$\begin{aligned} \frac{x\,z + y\,u}{(x\,u - y\,z)^2} &= a, \\ \frac{z^2 + u^2}{(x\,u - y\,z)^2} &= b, \\ \frac{x^2 + y^2}{(x\,u - y\,z)^2} &= c, \end{aligned} \tag{62}$$

obtained from (60), has at least one solution. We mean that x, y, z, u are unknown (elements of a basis transformation matrix A), and a, b, c are given numbers (structure constants $C'^{1}_{121}, C'^{2}_{121}, C'^{1}_{212}$). We will prove that if the condition $a^2 = b\,c$ is satisfied, then system (62) has no solutions, and, consequently, the vector $(0, 1, 1, 0)$ cannot be transformed by transformation (60) into the vector (a, b, c, a). Thus, the vectors (a, b, c, a) satisfying the condition $a^2 = b\,c$ determine two-dimensional ternary ω-Lie algebras (in fact, one algebra) that are not isomorphic to $\mathcal{L}_2$.

The system of Equation (62) can be written in matrix form, as follows:

$$\frac{1}{(\mathrm{Det}\,A)^2}\,A\,A^T=\begin{pmatrix} c & a\\ a & b\end{pmatrix},\quad A=\begin{pmatrix} x & y\\ z & u\end{pmatrix}. \tag{63}$$

By calculating the determinants of both sides of this matrix equation, we obtain the following:

$$\frac{1}{(\mathrm{Det}\,A)^2}=\mathrm{Det}\begin{pmatrix} c & a\\ a & b\end{pmatrix}.$$

Thus, assuming that $a^2=b\,c$ and system (62) has a solution, we arrive at a contradiction since the left-hand side of the above equality cannot be zero ($\mathrm{Det}\,A\neq 0$), while due to the condition $a^2=b\,c$ the right-hand side is zero. A vector of the subspace $\mathcal{W}$ satisfying $a^2=b\,c$ can be written as $(\pm\sqrt{bc},b,c,\pm\sqrt{bc})$. We can show that the vector $(0,1,0,0)\in\mathcal{W}$ can be transformed by (60) to any vector of the form $(\pm\sqrt{bc},b,c,\pm\sqrt{bc})$. Indeed, if $b\neq 0$, then the transformation matrix, i.e.,

$$A=\begin{pmatrix} \frac{1}{\sqrt{b}} & \frac{\sqrt{c}}{\sqrt{b}}\\ 0 & 1\end{pmatrix}$$

induces the transformation (60) that sends the vector $(0,1,0,0)$ to the vector $(\sqrt{bc},b,c,\sqrt{bc})$. If $b=0$, but $c\neq 0$, then the transformation matrix

$$A=\begin{pmatrix} 0 & 1\\ \frac{1}{\sqrt{c}} & 0\end{pmatrix},$$

induces the transformation (60) that sends the vector $(0,1,0,0)$ to $(0,0,c,0)$. From this, we conclude that the set of vectors of type $(\pm\sqrt{bc},b,c,\pm\sqrt{bc})$ gives one more two-dimensional ternary ω-Lie algebra that is not isomorphic to either algebra $\mathcal{L}_2$ or algebra (61). The commutation relations of this algebra can be written as follows:

$$[e_1,e_2,e_1]=e_2,\quad [e_2,e_1,e_2]=0. \tag{64}$$

The last possibility we must consider involves the vectors (a,b,c,a) of the subspace $\mathcal{W}$ that satisfy condition $a^2\neq b\,c$. In a similar way to what we did above, we can prove that for any vector $(a,b,c,a)\in\mathcal{W}$ that satisfies the condition $a^2\neq b\,c$, the system of Equation (62) has solutions. This means that each vector determines the two-dimensional ternary ω-Lie algebra $\mathcal{L}_2$. □

In the classification table of Theorem 3, the two-dimensional ternary ω-Lie algebra labeled I is Abelian, and the two-dimensional ternary ω-Lie algebra labeled II is $\mathcal{L}_2$. Recall that the two-dimensional ternary ω-Lie algebra $\mathcal{L}_2$ was constructed in Section 4 in two ways. The first way uses vectors of the complex plane, and the second way uses traceless three-dimensional matrices of the second order. It is easy to verify that the two-dimensional ternary ω-Lie algebras labeled II ($\mathcal{L}_2$) and IV are simple ternary ω-Lie algebras and algebra III possesses a non-trivial ideal spanned by the second generator e_2.

6. Discussion

In this paper, we propose an extension of the concept of Lie algebra to algebras with ternary multiplication laws. Our approach is based on the concept of a ternary commutator, which we construct by analogy with a binary commutator, that is, we form six triple products using all permutations of three arguments of the ternary commutator and then form their linear combinations using the third-order roots of unity as coefficients. Due to the properties of third-order roots of unity, the proposed ternary commutator identically vanishes when all three of its arguments are equal. However, in the case when two of its three arguments are equal, it is generally not equal to zero. In addition, the ternary commutator we propose in the present paper has the so-called ω-symmetry property, that is, a cyclic permutation of the three arguments of the ternary commutator entails multiplying the ternary commutator by ω or its conjugate, where ω is a primitive cube root of unity. Here, we would like to point out an analogy with the ternary generalization of the Pauli exclusion principle proposed by Kerner in [16]. From this generalization of the Pauli exclusion principle, it follows that a wave function of a quantum system of three particles is transformed via a primitive cube root of unity under cyclic permutations of the arguments of a wave function. Therefore, in the case where all three particles of a quantum system have identically equal quantum characteristics, the wave function of such a system vanishes. However, in the case where two particles have equal quantum characteristics, the wave function does not necessarily vanish. We believe that the concept of ternary ω-Lie algebra that we propose correlates with the ternary generalization of the Pauli exclusion principle and its consequences.

We would also like to highlight the potential application of the ternary commutator proposed in this paper to the quantum theory of Nambu mechanics. In [23], the author constructed a representation of the Nambu–Heisenberg relation via the cubic roots of unity. The left-hand side of this relation is the ternary commutator (18), where A, B, C are linear operators. Linear operators A, B, C act in a vector space of states constructed via the ring $\mathbb{Z}(\omega)$. It is possible that using cubic roots of unity as coefficients in the ternary commutator could enhance the construction of a quantum theory of Nambu mechanics.

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