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On Dualities in String Theory

Published papers format

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מוגש למועצה המדעית של
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Contents

Chapter 1

Summary

This is a thesis for the degree of Doctor of Philosophy submitted to the Scientific Council of the Weizmann Institute of Science. It is submitted in the Published Papers format and based on three publications [1–3]. These projects study some aspects of dualities between string theories and of the AdS/CFT correspondence.

The structure of this thesis is as follows: Chapter 3 is a general introduction to the role dualities play in modern high energy physics with respect to the publications in this thesis. The first section 3.1 is a brief historical introduction. The next three sections 3.2-3.4 expand on the main ideas behind the three papers [1–3] respectively, and put these ideas in the context of the general theme of this work. Finally, Part II includes reproductions of the the three papers [1–3].

Chapter 2

Acknowledgement

I am greatly in debt to my advisors, Professor Ofer Aharony and Professor Micha Berkooz, for their guidance and support throughout my Ph.D. studies. I consider myself very lucky to have had the opportunity to study from two scientists that are not only true experts in the field but also great teachers. Besides benefitting from their knowledge in physics and their brilliant ideas, I have also enjoyed their professional guidance and the enthusiasm they inspired in me.

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Quotes

Now, just to prove that I have been listening over all these years, here are some things heard at the weekly group meetings:

> *The strings don't want to stretch for no reason.*
>
> Prof. O. Aharony

> *Orientifolds are not something you can use every day.*
>
> Prof. B. Kol

> – *It would be very depressing if it turned out that what we should do is statistics over the vacua.*
>
> – *There is nothing in the anthropic principle that says that the laws of physics must be such that you will be happy.*
>
> Conversation between Prof. M. Berkooz and Prof. E. Rabinovici

> *... using the laws that Witten gave us...*
>
> Prof. O. Aharony

> *From my experience time is finite.*
>
> Prof. N. Itzhaki

Part I

Introduction

Chapter 3

Introduction

3.1 Brief History and General Introduction

Understanding strong coupling phenomena has been a central challenge of quantum field theory since its birth. In the field of particle physics, the problem of strong coupling arises most urgently in the study of the low energy regime of quantum chromodynamics. Many methods to attack this problem have been developed over the years with various degrees of success. Some qualitative methods such as non-standard perturbative expansion, simple truncations of an exact equation (applied to Schwinger-Dyson equations or to renormalization group equations), and some numerical methods (usually on a lattice) have proven to be potent in addressing some specific problems, yet to date even the most basic questions of spectrum and scattering amplitudes lie beyond the reach of analytical and numerical methods.

In string theory, the problem of strong coupling has an additional conceptual aspect: since string theory is only defined as a perturbative expansion, it is not clear at all whether the theory makes any sense beyond weak coupling.

The idea of duality between quantum field theories has origins that go as early as the 1960s, when the duality between the Sine-Gordon and the Thirring model was suggested [4, 5] and later proved [6]. Another milestone worth mentioning in this context is the duality conjectured by Montonen-Olive [7], which was the first of many examples known today of dualities between supersymmetric gauge theories in four dimensions (some reviews on supersymmetric field theory dualities are [8–16]). The idea in both of these examples and in general is that the strong coupling regime of a strongly coupled field theory can be described in terms of a weakly coupled field theory through some complicated nonlocal mapping (that more often than not is not known explicitly). Thus, dualities between field theories allow us to analyze strong coupling phenomena via translating between the strongly coupled model and the weakly coupled one.

Dualities play a more fundamental role in Superstring Theory (some reviews on string dualities are [17–21]). In the early days of Superstring Theory it was known that there are exactly five consistent theories in 10 dimensions: type IIA, type IIB, type I, Heterotic $E_8 \times E_8$ and Heterotic SO(32) [22]. Each of the theories can be compactified on a manifold and is also parameterized by its string coupling. This set of parameters (the coupling and the parameters of the compact manifold) are known as moduli. These moduli are the vacuum expectation values of various fields. Dualities in Superstring Theory is the idea that a point in the moduli space of one string theory can be described equivalently by a point in the moduli space of another string theory or by a different point of the moduli space of the same theory. These dualities typically mix up the perturbation expansions in the moduli and are thus useful for studying the non-perturbative structure of the theory. Some examples of dualities were known since the early days of the theory (see in [23]), e.g. it was known that there is a T-duality between types IIA and IIB and between the two heterotic theories. However, the discoveries of the early 1990s leading up to the so called “Second Superstring Revolution” [22] radically changed the way we think of the theory today.

A central property of Superstring Theories is that they all include non perturbative soliton-like objects known as branes. The growing understanding of the role these branes play in non-perturbative string theory has led to the discoveries that all five types of Superstring Theories are in fact dual to each other [24–26]. Furthermore, these works found another limit of string theory, where the low energy dynamics are that of eleven dimensional supergravity. This motivated a new perception of quantum gravity, which is now widely accepted. In this new perception there is one mysterious theory known as M-theory, which perhaps admits a matrix model description in some cases [27], and the various known string theories are special limits on the enormous moduli space of this theory.

Yet another advancement due to the new understanding of branes and non perturbative strings is the conjecture of the AdS/CFT correspondence [28–30]. This conjecture is the first example of a duality between a gravitating string theory and a non gravitating field theory. The idea of holography, i.e. that a theory of gravity can be described by a theory without gravity in one less dimension, was suggested earlier [31, 32] based on the proportionality of entropy to the surface area of black holes [33]. However, the explicit examples of AdS/CFT correspondences are the first that allow us to explore the mechanisms of the mapping between gravity and gauge theories. Furthermore, this correspondence provides another opportunity for studying strongly coupled gauge theories (such as QCD) via mapping them to weakly coupled string theories.

This thesis includes three published papers [1–3]. The subject of these papers are suggestions of new dualities between string theories, new examples of AdS/CFT correspondences and tests of new and old conjectured dualities. In what follows we give a more specific introduction to the subjects discussed in these papers. It is hoped that this work contributes to this very exciting developments in the study of dualities.

3.2 The non-AdS/non-CFT Correspondence Introduction to [1]

As mentioned above the AdS/CFT correspondence is the remarkable idea that a gravitational string theory is precisely equivalent to a non-gravitational field theory. In general, it relates a theory of quantum gravity in $d+1$ dimensional anti-de Sitter (AdS) space (times some compact manifold) to a d dimensional conformal field theory (CFT). The simplest example is the duality between type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and the $\mathcal{N}=4$ supersymmetric Yang-Mills (YM) theory in four dimensions with a gauge group $\mathrm{SU}(N)$.

The parameters of string theory in this background are the string coupling constant g_s and the radius of curvature R (of both the AdS and the sphere) measured in string length l_s. The parameters of the YM theory are the coupling constant g_{YM} and the number of colors N. The two sets of parameters are related by

$$4\pi g_s = g_{\mathrm{YM}}^2, \tag{3.1}$$

$$(R/l_s)^4 = g_{\mathrm{YM}}^2 N \equiv \lambda. \tag{3.2}$$

To leading order in $1/N$, we keep only planner diagrams in the field theory, which can be described by a perturbation expansion in λ. On the other hand, for small g_s we keep only the lowest genus in the string sigma model, and the theory is an expansion in the space-time curvature $l_s/R = \lambda^{-1/4}$. Thus, this is an example of a strong-weak duality.

Unfortunately, the $\mathcal{N} = 4$ theory (and any CFT in general) is quite different from QCD. The problem is not the different matter content, but the fact that most of the interesting phenomena of QCD (e.g. confinement, chiral symmetry breaking, the formation of a mass gap, bound states spectrum) are manifestly not scale invariant. However, there are reasons to believe that a string theory that is dual to QCD (or at least to a gauge theory with similar features) may be found, and that features of QCD-like theories may be studied using string theory.[1] The first step towards realizing this goal is to break conformal invariance, and this requires breaking at least part of the supersymmetry.

One popular method to achieve this is to deform the CFT in a way that breaks the symmetry, but is also controllable and corresponds to a known deformation on the AdS side. The AdS/CFT correspondence matches the SO(4, 2) group of isometries on AdS_5 to the conformal group in $d = 4$. Each gauge invariant local (single trace) operator in the CFT corresponds to a specific field in AdS. The conformal dimension of the operator Δ_i is related to the mass of the field m_i via

$$R^2 m_i^2 = \Delta_i(\Delta_i - 4).$$

Thus, matching deformations is easiest for chiral operators, whose conformal dimension is protected (basically, for non-chiral operator the correspondence is known explicitly only in very rare cases).

The $\mathcal{N} = 4$ theory includes a gauge vector A_μ, four fermions λ_a and six scalar fields ϕ^i. If one could "get rid" of the fermions and scalars, one would remain with pure YM theory, which shares many interesting characteristics with QCD. The naive way to achieve this goal would be to deform the original CFT by adding mass terms

$$\Delta\mathcal{L} = \text{Tr}(M_{ij}^2 \phi^i \phi^j + (m_{ab}\lambda_a\lambda_b + \text{c.c.})), \tag{3.3}$$

so that the low energy theory would be pure YM. The problem with the deformation (3.3) is that the scalar part is not a chiral operator (unless M_{ij}^2 is traceless, in which case the theory would not be stable and the gauge symmetry would be broken spontaneously). This means that for large $g_{\text{YM}}^2 N$ the

[1] A review of the advancement toward this goal is [34].

operator acquires large anomalous dimension and the deformation becomes irrelevant.

We are left with the possibility of adding masses for the fermions only. The first paper [1] studies a special case, where we add an equal mass to all the fermions. This deformation is of interest since it breaks conformal invariance and supersymmetry completely (hence, this theory is sometimes referred to as $\mathcal{N} = 0^*$) and dynamically generates a potential for the six real scalars, ϕ^i, in the theory.

In the paper, we have shown that the theory resulting is stable (perturbatively in the 't Hooft coupling), and that there are some indications that $\langle \phi \rangle = 0$ is the vacuum of the theory. On the field theory side we calculated the effective potential generated for the scalars by a direct quantum field theory calculation, and using the fact that the effective potential is known for $m_{ab} \to 0$. We also studied this deformation on the AdS side in the strong coupling limit (when the supergravity approximation is valid). The corresponding supergravity computation is done by probing the deformed $\mathrm{AdS}_5 \times S^5$ background with a brane probe. Despite being different limits of the theory, we found qualitative agreement between the two calculations.

3.3 Strings on AdS_2 and 0A Matrix Models Introduction to [2]

One of the earliest dualities conjectures in string theory is the duality between two dimensional bosonic string theory and matrix models (in the double scaling limit). This correspondence, which was suggested some twenty years ago (see [35, 36] for a thorough review), was recently given a new interpretation [37–39] that views the matrix model as the effective field theory on D0-branes that exist in the string theory, construing the correspondence as an open-closed duality (similarly to the construction of the AdS/CFT correspondence). Based on this understanding, a similar correspondence was conjectured for type 0 superstrings in two dimensions [40, 41].

Two dimensional superstring theory are described by $\mathcal{N} = 1$ worldsheet field theory coupled to worldsheet supergravity. The non chiral GSO projection gives the two type 0 theories. Since there are no NS-R or R-NS sectors in the theory, the type 0 theories have no fermions and are not supersymmetric. The two NS-NS sectors include a graviton, a dilaton and a tachyon. In type

0A the RR sectors include two 1-forms and in type 0B a 0-form and a pair of 2-forms. However, in two dimensions there are no transverse string oscillations and longitudinal oscillations are unphysical except at special values of the momentum. Thus, the tachyon is the only dynamical NS-NS sector field. The space time equations of motion are solved by a linear dilaton background

$$G_{\mu\nu} = \eta_{\mu\nu}, \qquad \Phi = -2x, \tag{3.4}$$

where x is the spatial coordinate.

Type 0 superstrings on this background are conjectured [40, 41] to be dual to a $c = 1$ matrix model. In analogy to the bosonic case, the conjecture is that the field theory on N D0-branes (with $N \to \infty$) in this background is a dual description of the full string theory. It is believed that the matrix model that gives the 0A theory is that of a $\mathrm{U}(N)_A \times \mathrm{U}(N)_B$ gauge theory with a complex matrix m in the bifundamental. The two $\mathrm{U}(N)$ groups originate from D0 and anti-D0 branes. After preforming the standard manipulation of diagonalizing the matrix (cf. [42]), the potential for the eigenvalues is

$$V(\lambda) = -\frac{1}{8}\lambda^2.$$

The motivation behind the second paper [2] is to study the AdS/CFT correspondence in the context of two dimensional string theory. A conjecture for this correspondence was suggested in [43], however we suggested a different picture and presented some evidence to support our assertions. The reasons to believe that the AdS/CFT correspondence can be studied in this context come from both the space-time and the matrix model points of view:

- On the space-time side, it has been suggested that type 0A string theory should have an AdS_2 solution with RR flux. This is based on the fact that the space-time effective action has an extremal black hole solution [44–46] whose near-horizon limit is AdS_2. There are two good reasons not to trust this solution: it is highly curved in part of the space-time and it requires turning on both of the 1-forms, which implies that one needs to add strings to the background and take their effect into account [47]. Still, one may hope that even after taking all corrections into account, because of the large symmetry, there would still be a limit where space-time is AdS_2.

- On the matrix model side, one can turn on RR flux by adjusting the potential for the matrix or the shape of the matrix. In either options, the end result in terms of the eigenvalues is changing the potential to

$$V(\lambda) = -\frac{1}{8}\lambda^2 + \frac{M}{2\lambda^2}.$$

  In the region of small λ the potential is dominated by the last term and the quantum mechanical system has conformal invariance (see [48] for a discussion of conformal invariance in quantum mechanics).

It should be emphasized that this background must include nonzero RR flux, thus, it is also an opportunity to study a simple example of strings on background with RR flux - a poorly understood subject in string theory.

In the paper [2] we studied the conformal limit of the 0A matrix model in order to find the properties of this conjectured AdS_2 background. We found that the spectrum of this theory is equal to that of a free fermion field on AdS_2, with a mass proportional to the RR flux q. This fermion originates from the eigenvalues of the matrix model which correspond to D0-anti-D0-brane pairs, so this spectrum suggests that the only excitations in this AdS_2 background are such uncharged D-brane pairs, and that there are no closed string excitations in this background. From the point of view of 0A backgrounds with flux which asymptote to a linear dilaton region, this implies that the closed string excitations cannot penetrate into the strongly coupled region which is dual to the conformal limit of the matrix model. This view is quite different than what was suggested before [43]. A more recent paper [49] that studied this system from a different perspective agreed with our results.

It would be interesting to try and verify the picture we proposed directly in string theory. An explicit computation would be to quantize the worldsheet theory and show that there are no physical excitations (except perhaps at special values of the momenta). The difficulty lies in the limited knowledge on quantizing strings on RR backgrounds, so this remains an open question at this point.

3.4 9d $\mathcal{N} = 1$ Backgrounds of M/String Theory Introduction to [3]

One of the most exciting discoveries of the second superstring revolution was that the strongly coupled limit of each 10 dimensional theory can be described equivalently by a weakly coupled string theory or by 11 dimensional supergravity. This means that for 10 dimensional strings (where the only parameter is the string coupling) there is a perturbative description in each corner of the moduli space.

For theories with 32 supercharges, it was well known that this remains true in 9 noncompact dimensions. For example, consider type IIA on $\mathbb{R}^9 \times S^1$. The moduli here are the string coupling g_s and the radius of the circle R_9. The IIA description is valid as long as $g_s \ll 1$ and $R \gg l_s$. For small radius, we can switch to the T-dual IIB picture, where the string coupling and $g_s l_s/R_9$ and the radius is l_s^2/R_9. The IIB description is valid as long as $g_s \ll R_9/l_s$ and $R_9 \ll l_s$. Another region of the moduli space is reached by starting from the IIA and increasing the string coupling. This leads to 11d supergravity (M theory) on a torus of radii are $(R_9, g_s l_s)$ with Planck length $l_p = g_s^{1/3} l_s$. Thus, this M theory on a torus description is valid for $g_s \ll (R_9/l_s)^3$ and $g_s \gg 1$. Finally, we can now reduce the theory on the other circle of the torus,[2] to find another IIA description with coupling $g_s^{-1/2}(R_9/l_s)^{3/2}$, string length $g_s^{1/2}(l_s/R_9)^{1/2} l_s$ and on a circle of radius $g_s l_s$. This is a valid description for $g_s \gg (R_9/l_s)^3$ and $g_s \gg (l_s/R_9)$. The missing part of the moduli space,$(R_9/l_s) \ll g_s \ll (l_s/R_9)$, is filled by another IIB description which is the T-dual of the last IIA and S-dual to the previous IIB. The parameters of this new IIB are the coupling $R_9/(l_s g_s)$ and the radius l_s^2/R_9 (the string length remains $g_s^{1/2}(l_s/R_9)^{1/2} l_s$). Thus, as demonstrated in figure 3.1, we have covered the entire moduli space, were all limits have a weakly coupled string or M theory description. It is also true that this exhausts all the theories with 32 supercharges in nine dimensions ($\mathcal{N} = 2$).

The next natural step is to study the moduli space of theories with 16 supercharges ($\mathcal{N} = 1$), which also give exact backgrounds of string and M

[2] This is actually superfluous, since the torus has a symmetry for exchanging the two radii, but we carry on this calculation anyhow in order to demonstrate how the various limits are covered.

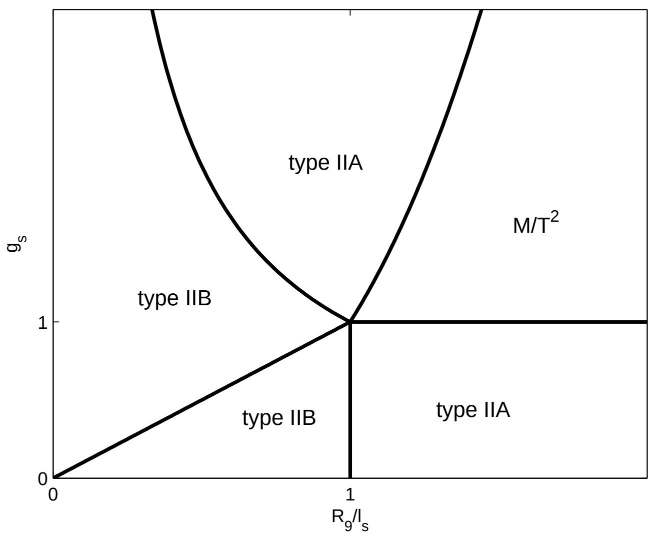

Figure 3.1: The moduli space of nine dimensional M and string theory with 32 supercharges. The parameters are the string coupling and the radius of the circle in the IIA description of the lower right corner. The upper-left half of the graph, which mirrors the lower-right half, is just a manifestation of the symmetry of exchanging the

theory. This was the first objective of the third paper [3] . The moduli space of such nine dimensional backgrounds with rank 18 (including type I and heterotic strings on a circle and M theory on a cylinder) has been extensively studied, and all its corners are understood (these is reviewed in subsection 2.1 of the paper); however it seems that no similar study has been done for backgrounds with lower rank (rank 10 or rank 2). These backgrounds include the compactification of M theory on a Klein bottle and on a Möbius strip.[3] In the paper we studied the moduli space of these backgrounds in detail. We encountered two surprises: we showed that the moduli space of backgrounds of rank 2 has two disconnected components, and that both for rank 2 and for rank 10 there is a region of the moduli space which had not previously been explored, and whose description requires interesting non-perturbative corrections to some orientifold planes in type IIA string theory.

[3]Some of these backgrounds were studied in [50].

Orientifold planes are important non-dynamical objects in string theory. In perturbative string theory they are arise by dividing by a parity reversal symmetry that also exchanges the orientation, i.e.

$$X^m(z,\bar{z}) \sim -X^m(\bar{z},z).$$

These objects are necessary ingredients of string theory, as can be deduced by studying how unoriented string theories transform under T-duality (see in [51]). One can also derive the charges of orientifolds under the various space-time fields. For examples, an eight-dimensional orientifold plane (denoted as an $O8$ planes) is a source for the dilaton and has exactly -8 times the charge of a D8-brane.

In the rank 18 case discussed above, some parts of the moduli space are described by a space time bounded between 2 orientifold planes and containing 16 D-branes, in a manner that cancels the overall tadpole for the dilaton. The branes can be moved around between the orientifolds to obtain different physics. An interesting thing happens when the branes are configured such that an $O8$-plane is at a point in space-time where the string coupling diverges: in this case a special mechanism enhances the gauge symmetry (for example, from SO(16) to E_8 [52–54]). In fact, orientifolds exhibit interesting phenomena at strong coupling in general (some examples of $O6$, $O7$ and $O8$ planes at strong coupling are discussed in [26, 52, 55, 56]). In [3] we argued that the missing part of the moduli space of theories with rank 2 and 10 can be understood by conjecturing the existence of a new type of orientifold plane which is neutral with respect to the dilaton. This new object can only exist in infinite coupling, and thus has no description in terms of perturbative string theory.

The second objective of [3] was to study the matrix model description of the rank 2 and 10 backgrounds. In the context of string theory, M(atrix) Theory is the idea that M theory can be described (at least in some cases) completely by a matrix model [27]. There is special motivation for finding the matrix model description of the $\mathcal{N}=1$ backgrounds. The theories we discussed (with rank n) have non-trivial duality groups of the form $\mathrm{SO}(n-1,1,\mathbb{Z})$. In their stringy descriptions these are simply the T-duality groups [57]. However, from the M theory point of view these dualities are quite non-trivial.

T-duality groups in space-time often map to interesting S-duality groups in the M(atrix) theory gauge theories. In the case of 32 supercharges, the modular invariance of the torus maps to the SL$(2, \mathbb{Z})$ S-duality of $\mathcal{N} = 4$ Super Yang-Mills theory in four dimensions [58]. It is thus interesting to study the manifestation of duality groups for the rank 2 and 10 cases in the M(atrix) theory description of these backgrounds.

In the paper we derived the M(atrix) theory for some of these backgrounds. By constructing the matrix description of these backgrounds and preforming a T-duality twice, we got to a 2+1 super Yang-Mills theory, where space-time is a cylinder (times time) and where the gauge coupling monotonically varies between the boundaries of the cylinder. In particular, at some finite critical length (which corresponds to the self duality of the relevant M theory) the gauge coupling diverges and the field theory ceases to make sense. The duality group of the M theory relates a cylinder with subcritical length to one with supercritical length, so the dual theory is not well defined. The paper concluded with this result and left further investigations into what happens 'beyond infinite coupling' to future work.

Chapter 4

Note on Papers

This note aims to delineate my part in each of the projects described above as required by the regulations of the Feinberg Graduate School. The research on all three papers was done under the supervision of my advisor Professor Ofer Aharony. The last two papers ([2] and [3]) were written together with Prof. Aharony and the part each of us played in the project is probably best described as the standard relationship between a student and his advisor. The first and last papers ([1] and [3]) were written together with Mr. Dori Reichmann and with Mr. Zohar Komargodski respectively, both students in the High Energy Theory Group at the Weizmann Institute. The research towards these projects in all its stages was done jointly with equal contribution from each of the students in each collaboration.

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Part II

Papers

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The Effective Potential of the $\mathcal{N} = 0^*$ Yang-Mills Theory

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Abstract: We study the $\mathcal{N} = 4$ SYM theory with SU(N) gauge group in the large N limit, deformed by giving equal mass to the four adjoint fermions. With this modification, a potential is dynamically generated for the six scalars in the theory, ϕ^i. We show that the resulting theory is stable (perturbatively in the 't Hooft coupling), and that there are some indications that $\langle\phi\rangle = 0$ is the vacuum of the theory. Using the AdS/CFT correspondence, we compare the results to the corresponding supergravity computation, i.e. brane probing a deformed $\mathrm{AdS}_5 \times S^5$ background, and we find qualitative agreement.

Keywords: ads.

1. Introduction

The AdS/CFT correspondence relates String Theory on $d+1$ dimensional anti-de Sitter (AdS) space (times some compact manifold) to a d-dimensional conformal field theory (CFT) (see [1, 2, 3] and the review [4]). In particular, it is conjectured that Type IIB String Theory on $\mathrm{AdS}_5 \times S^5$ is equivalent to $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory with gauge group SU(N). In the regime where $g^2_{\mathrm{YM}}N$ is large, supergravity is a good approximation to the full string theory. In this regime, the identification of partition functions in the two theories allows one to construct supergravity solutions that are dual to deformations of the $\mathcal{N}=4$ SYM theory.

Perhaps the main motivation for the study of such deformed theories is the prospect of finding a string theory dual to a field theory that is similar to QCD. This will allow the study of strong coupling phenomena of QCD-like theories via the understanding of weakly coupled supergravity (three possible paths to this goal were reviewed in [5]). Other motivations for studying such deformations are learning about strong coupling phenomena on both sides of the correspondence (by studying the weakly coupled dual perturbatively) and gaining insight into how the correspondence works by understanding how various phenomena are realized in the two dual descriptions. Eventually, one would hope to go beyond the supergravity approximation and find the string dual to QCD.

The $\mathcal{N}=4$ SYM theory includes four adjoint fermions ψ_a and six adjoint scalars ϕ^i, all massless. If somehow these fields would acquire masses, then at energies much lower than these masses one would remain with pure Yang-Mills theory, which is quite similar to QCD[1]. One way to do this is to add mass terms to all of these fields, i.e. adding to the SYM lagrangian the term $\mathcal{O} = M^2\,\mathrm{Tr}(\phi^i\phi^i) + m_{ab}\,\mathrm{Tr}(\bar{\psi}_a\psi_b) + \text{c.c.}$. The problem with this is that the scalar part is not a chiral operator, thus, this deformation cannot be studied in the supergravity approximation. Alternatively, one may try to add the chiral operator $(M^2)_{ij}\,\mathrm{Tr}(\phi^i\phi^j)$ with $(M^2)_{ii}=0$, but then some of the scalars would have negative mass-squares, leading to instabilities. A way around this is to introduce masses only for the fermions, breaking some (or all) of the supersymmetry. In this paper we focus on introducing an equal mass m for all four fermions. Such a deformation breaks the supersymmetry completely, hence we denote the theory one ends up with as $\mathcal{N}=0^*$ Yang-Mills theory. After the deformation, the scalar mass term isn't protected, so one expects that the scalars acquire mass quantum mechanically. From dimensional considerations, the acquired mass squared must be proportional to m^2. If it happens that the induced mass squared matrix is strictly positive, then $\langle\phi^i\rangle = 0$ will be a (perturbatively) stable vacuum of the theory, and at the IR (well below m^2) the theory becomes pure Yang-Mills with SU(N) gauge group. More generally, for this scenario to work, it is an essential requirement that there is a stable vacuum at a value of the scalars for which a non-Abelian gauge group remains unbroken. Thus, this paper studies the viability of the $\mathcal{N}=0^*$ theory as a way to learn about pure YM theory (or theories similar to it).

[1] This statement is only true for small $g^2_{\mathrm{YM}}N$. At large $g^2_{\mathrm{YM}}N$, one finds that Λ_{YM} is of the same order of magnitude as the masses of the fields we are trying to get rid of, thus, there is no separation of scales in this case.

In this paper we study the $\mathcal{N}=0^*$ theory and compare the results to those obtained on the supergravity side of the correspondence. In section 2 we study the scalar effective potential and find that it has a perturbatively stable vacuum. We also analyze the flow of the scalar mass-squared and find some indications that it becomes positive at low energies, implying that $\langle \phi^i \rangle = 0$ is indeed a stable vacuum. In section 3 we review the analogue computation preformed on the deformed AdS by brane probing the supergravity background (the results described were obtained in [6]), and compare it with the weak coupling computation.

2. The Deformed $\mathcal{N}=4$ SYM

The $\mathcal{N}=4$ SYM lagrangian (with $\theta=0$) is given by

$$\mathcal{L} = \frac{1}{g_{\mathrm{YM}}^2}\,\mathrm{Tr}\Big\{ -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - i\bar{\psi}^a \not{D}\psi_a - D_\mu\phi^i D^\mu\phi^i + \\ + \sum_{k=1}^{3}\mathrm{Tr}\left(C^k_{ab}\bar{\psi}_a[\phi^{2k-1},\psi_b] + iB^k_{ab}\bar{\psi}_a\gamma_5[\phi^{2k},\psi_b]\right) + \frac{1}{2}[\phi^i,\phi^j][\phi^i,\phi^j]\Big\} \quad (2.1)$$

where ψ_a are the four Majorana fermions of the theory in Dirac notation. The 4×4 Yukawa coupling matrices B^k_{ab} and C^k_{ab} satisfy the algebra

$$[C^k,C^l] = 2\varepsilon^{klm}C^m, \qquad [B^k,B^l] = 2\varepsilon^{klm}B^m, \qquad [C^k,B^l]=0,$$
$$\{C^k,C^l\} = -2\delta^{kl}, \qquad \{B^k,B^l\} = -2\delta^{kl}.$$

The exact form of the B and C matrices we use may be read from (2.8) below. The full $\mathrm{SO}(6)_R$ is hidden in this notation, since the fermions transform non trivially.

We deform the $\mathcal{N}=4$ SYM theory by adding the operator

$$\mathcal{O}_\beta = \frac{m}{2g_{\mathrm{YM}}^2}\sum_{a=1}^{4}\bar{\psi}_a\left(\cos\beta + i\gamma_5\sin\beta\right)\psi_a \quad (2.2)$$

This deformation breaks the global symmetry $\mathrm{SU}(4)_R \to \mathrm{SO}(4) \simeq \mathrm{SO}(3)\times\mathrm{SO}(3)\times\mathbb{Z}_2$. The six scalars break into two groups of three that transform under the $\mathrm{SO}(3)\times\mathrm{SO}(3)$ as $(3_v,1)$ and $(1,3_v)$. An $\mathrm{SO}(6)_R$ transformation that rotates between the two triplets with an angle $\delta\beta$ leaves all terms in the original lagrangian invariant, but changes (2.2) by $\beta\to\beta+\delta\beta$.

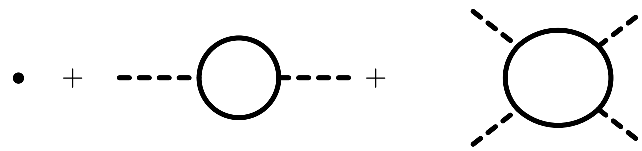
In the following we calculate the one-loop corrections to the scalar effective potential in a specific direction, $\phi^i = \chi\delta^{i,1}T_1$, where T_1 is the $\mathrm{SU}(N)$ generator

$$T_1 = \frac{1}{\sqrt{2N(N-1)}}\,\mathrm{diag}(1,1,\ldots,1,1-N)$$

Notice that studying the scalar potential in a different direction in the ϕ^1-ϕ^2 plane is equivalent to studying the potential in the ϕ^1 direction with a different angle β in the deformation. On the supergravity side of the correspondence the above choice describes putting $N-1$ branes at the origin and probing with one brane away from the others. The supergravity side is discussed in the next section.

2.1 The Effective Potential

In the supersymmetric theory ($m = 0$) the effective potential is flat in this specific direction to all orders in perturbation theory. Thus, in order to find the 1-loop effective potential, we calculate the contribution from diagrams with fermion loops:

$V_\psi =$  $+ \cdots,$

and then subtract the same result with $m = 0$, in order to take the bosonic loops contribution into account. Using a procedure similar to that of Coleman and Weinberg [7], the contribution from the massive fermions at one-loop is

$$V_\psi(\chi, \beta) = -i \sum_{n=1}^{\infty} \frac{1}{2n} \operatorname{Tr}(T_{1G}^{2n}) \chi^{2n} (-1)^n \operatorname{Tr}(C_1^{2n}) \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr} \left\{ \left[\frac{\gamma_\mu k^\mu + m(c_\beta - i s_\beta \gamma^5)}{k^2 - m^2} \right]^{2n} \right\}$$

With $c_\beta = \cos\beta$, $s_\beta = \sin\beta$, T_{1G} is the adjoint representation matrix of the generator T_1 and C_1 is the Yukawa coupling matrix defined in (2.1). They satisfy

$$\operatorname{Tr}(C_1^{2n}) = 4(-1)^n \qquad \operatorname{Tr}(T_{1G})^{2n} = \frac{N^n}{(2(N-1))^{n-1}} \approx 2^{1-n} N$$

where we have taken the large N limit[2].

In order to calculate $\operatorname{Tr}\left[\gamma_\mu k^\mu + m(c_\beta - i s_\beta \gamma^5)\right]^{2n}$, consider the general term in the trinomial expansion: it has s_1 factors of $m c_\beta$, s_2 factors of $\not{k}$ and s_3 factors of $-im\gamma^5 s_\beta$. Let us denote the number of $\not{k}$s to the left most γ^5 as l_1, the number of $\not{k}$s from the left most γ^5 to the next γ^5 as l_2, and so on till l_{s_3+1}, e.g.

$$\operatorname{Tr}[\underbrace{\not{k} \cdots \not{k}}_{l_1} \gamma^5 \underbrace{\not{k} \cdots \not{k}}_{l_2} \gamma^5 \not{k} \cdots \underbrace{\cdots \not{k}}_{l_{s_3}} \gamma^5 \underbrace{\not{k} \cdots \not{k}}_{l_{s_3+1}}] = \begin{cases} (-1)^{l_2 + l_4 + \cdots + l_{s_3}} 4(k^2)^{s_2/2} & s_2, s_3 \in 2\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Thus, the combinatorial problem may be defined as follows: put $s_2 = 2q_2$ balls in $s_3 + 1 = 2q_3 + 1$ cells. Find the difference between the number of configuration where the number of balls in the first q_3 cells is even and the configuration where the same quantity is odd. This number is

$$\sum_{p=0}^{2q_2} (-1)^p \binom{2q_2 - p + q_3}{2q_2 - p} \binom{p + q_3 - 1}{p} = \sum_{p=0}^{2q_2} (-1)^p \binom{-q_3 - 1}{2q_2 - p} \binom{-q_3}{p} =$$
$$= \sum_{p=0}^{q_2} (-1)^p \binom{-1}{2q_2 - 2p} \binom{-q_3}{p} = \sum_{p=0}^{q_2} (-1)^p \binom{-q_3}{p} = \binom{q_3 + q_2}{q_2}$$

[2]We are taking the large N limit since we are interested in comparing our results to the string theory results which are valid at large N. However, our one-loop computations in this section can be easily performed also for finite N.

(we use generalized binomial coefficients). The second equality is a formula taken from [8]. We find,

$$\mathrm{Tr}\left[\gamma_\mu k^\mu + m(c_\beta - i s_\beta \gamma^5)\right]^{2n} = \\ = 4\sum_{q_2=0}^{n}\sum_{q_3=0}^{n-q_2}\binom{2n}{2q_2+2q_3}\binom{q_2+q_3}{q_2}\left(c_\beta^2 m^2\right)^{n-q_2}(k^2)^{q_2}(-)^{q_3}\tan^{2q_3}\beta$$

Thus,

$$V_\psi(\chi,\beta) = -16Ni\int\frac{d^4k}{(2\pi)^4}\times \\ \times\sum_{n=1}^{\infty}\sum_{q_2=0}^{n}\sum_{q_3=0}^{n-q_2}\frac{1}{n}\binom{2n}{2q_2+2q_3}\binom{q_2+q_3}{q_2}\left(\frac{\chi^2}{2}\right)^n(-)^{q_3}\tan^{2q_3}\beta\frac{\left(k^2\right)^{q_2}\left(c_\beta^2 m^2\right)^{n-q_2}}{(k^2-m^2)^{2n}}$$

This sum may be reorganized

$$\sum_{n=1}^{\infty}\sum_{q_2=0}^{n}\sum_{q_3=0}^{n-q_2} = \sum_{q_2,q_3=0}^{\infty}\sum_{n=\max(1,q_2+q_3)}^{\infty} = \sum_{n=1}^{\infty}\delta_{q_2,0}\delta_{q_3,0} + \sum_{q_3=1}^{\infty}\sum_{n=q_3}^{\infty}\delta_{q_2,0} + \sum_{q_2=1}^{\infty}\sum_{q_3=0}^{\infty}\sum_{n=q_2+q_3}^{\infty}$$

Respectively, these three terms yield

$$V_\psi(X,\beta) = -16iNm^4\int\frac{d^4q}{(2\pi)^4}\Big\{-\log\Big[\frac{(q^2-1)^2 - X^2\cos^2\beta}{(q^2-1)^2}\Big] - \\ -\frac{1}{2}\log\Big[\frac{(q^2-1)^4 - 2(q^2-1)^2X^2\cos 2\beta + X^4}{((q^2-1)^2 - X^2\cos^2\beta)^2}\Big] - \\ -\frac{1}{2}\log\Big[\frac{(q^2-1)^2(q^4+1+X^4-2q^2(1+X^2)-2X^2\cos 2\beta)}{(q^2-1)^4 - 2(q^2-1)^2X^2\cos 2\beta + X^4}\Big]\Big\} = \\ = 8iNm^4\int\frac{d^4q}{(2\pi)^4}\log\Big[\frac{q^4+1+X^4-2q^2(1+X^2)-2X^2\cos 2\beta}{(q^2-1)^2}\Big]$$

where we have introduced the dimensionless quantities

$$X = \frac{1}{\sqrt{2}}\frac{\chi}{m}, \qquad q^\mu = \frac{k^\mu}{m}.$$

As noted above, adding the bosonic loops contribution to the effective potential is the same as subtracting the fermionic contribution with $m = 0$, so the effective potential (up to counter-terms) is

$$V_{\text{eff}} = 8iNm^4\int\frac{d^4q}{(2\pi)^4}\log\Big[\frac{q^4+1+X^4-2q^2(1+X^2)-2X^2\cos 2\beta}{(q^2-1)^2}\Big] - \\ -8iNm^4\int\frac{d^4q}{(2\pi)^4}\log\Big[\frac{(q^2-X^2)^2}{q^4}\Big] \tag{2.3}$$

Rotating to Euclidean space ($q_0 = -iq_4$) and introducing the 't Hooft coupling $g_t^2 = g_{\text{YM}}^2 N$,

$$g_{\text{YM}}^2 V_{\text{eff}} = 8g_t^2 m^4\int\frac{d^4q}{(2\pi)^4}\log\Big[\frac{q^4\big(q^4+1+X^4+2q^2(1+X^2)-2X^2\cos 2\beta\big)}{(q^2+X^2)^2(q^2+1)^2}\Big] \tag{2.4}$$

In general, one should introduce counter-terms into an effective potential for all relevant and marginal operators consistent with the symmetry, this is how renormalization is reflected in the Coleman-Weinberg formalism. The result we got in (2.4) has only a Λ^2 divergence (as a function of a cutoff Λ), the $\log \Lambda^2$ vanishes since the potential must vanish in the original theory ($m = 0$). Thus, we only need to add a mass counter-term, which we choose such that

$$M_1^2 \cos^2 \beta + M_2^2 \sin^2 \beta = \frac{1}{g_{\rm YM}^2} \frac{d^2 V}{d\chi^2}\Big|_{\chi=0} \tag{2.5}$$

We have

$$\frac{g_{\rm YM}^2 V_{\rm eff}}{m^4} = \frac{M_1^2 \cos^2 \beta + M_2^2 \sin^2 \beta}{m^2} X^2 +$$
$$+\frac{g_t^2}{\pi^2}\Big\{ \int dq\, q^3 \log\Big[\frac{q^4(q^4+1+X^4+2q^2(1+X^2)-2X^2\cos 2\beta)}{(q^2+1)^2(q^2+X^2)^2}\Big] + 2(2+\cos 2\beta) X^2 \log\Lambda - X^2 \Big\}$$

Performing the integration we find

$$\frac{g_{\rm YM}^2 V_{\rm eff}}{m^4} = \frac{M_1^2 \cos^2 \beta + M_2^2 \sin^2 \beta}{m^2} X^2 + \frac{g_t^2}{4\pi^2}\Big\{ -2X^2 - 12X^2\cos^2\beta - X^4 \log X^4 +$$
$$+ 4X(1+X^2)\cos\beta \log \frac{1+2X\cos\beta + X^2}{1-2X\cos\beta+X^2}$$
$$+ (1+2X^2+4X^2\cos^2\beta + X^4)\log\left(1-2X^2\cos 2\beta + X^4\right)\Big\} \tag{2.6}$$

So far we have been treating the angle β as a parameter of the deformation (2.2). However, by a SO(6) rotation, the calculation preformed above with a specific β is equivalent to deforming the $\mathcal{N} = 4$ lagrangian by a standard mass term (putting $\beta = 0$ in (2.2)) and studying the potential in a direction $\phi^1 + i\phi^2 = \sqrt{2} m X e^{i\beta}$. Thus (2.6) is the 1-loop effective potential in the (ϕ_1, ϕ_2) plane (which is related by the remaining R-symmetry to the other scalar directions).

Let us analyze the effective potential we obtained without considering its range of validity. For large X, $g_{\rm YM}^2 V_{\rm eff} \approx g_t^2 m^4 X^2 \log X^4$, i.e. this flat direction is lifted to a stable potential at one-loop. For small X,

$$\frac{g_{\rm YM}^2 V_{\rm eff}}{m^4} = \frac{M_1^2 \cos^2 \beta + M_2^2 \sin^2 \beta}{m^2} X^2 - \frac{g_t^2}{4\pi^2} X^4 \log X^4 + O(X^4) \tag{2.7}$$

For $M_1^2, M_2^2 \geq 0$ the effective potential has a minimum at $X = 0$, this is a stable vacuum. If $M_i^2 < 0$ for $i = 1$ or 2 (or both), $X = 0$ becomes a saddle point (maximum) and we find a minimum for some $X > 0$. In the latter case both the gauge and the R-symmetry groups are broken (notice that even if $M_1^2 = M_2^2$ the potential isn't radially symmetric although the expansion (2.7) is, i.e. R-symmetry is completely broken).

There are two types of logarithms in this effective potential. The first, $\log X$, diverges at $X = 0$, and the other logarithms diverge at the four points $X = \pm e^{\pm i\beta}$ (notice that X is defined to be real, so these points are not physical unless $\beta = 0$). The $X = 0$ divergence is well expected since the $\mathcal{N} = 0^*$ theory is IR divergent. The other divergences are easy to see by using Weyl notation for the fermions. The terms in the lagrangian that are quadratic in the fermions may be written $im\,\mathrm{Tr}(A_{ab}\lambda_a^T\sigma^2\lambda_b)$+c.c. (the Tr is over the gauge group), where

$$A = g_{\rm YM}^{-2}\begin{pmatrix} e^{i\beta} & X^1+iX^2 & X^3+iX^4 & -X^5+iX^6 \\ X^1+iX^2 & e^{i\beta} & X^5+iX^6 & X^3-iX^4 \\ X^3+iX^4 & X^5+iX^6 & e^{i\beta} & X^1-iX^2 \\ -X^5+iX^6 & X^3-iX^4 & X^1-iX^2 & e^{i\beta} \end{pmatrix} \tag{2.8}$$

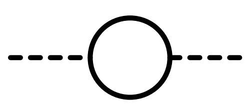
and $X^i = \phi^i/m$. By turning off all the scalars except X^1 and X^2, and treating X^1+iX^2 as a complex scalar, we see that the points $(X^1+iX^2) = \pm\, e^{\pm i\beta}$ are exactly the values where (2.8) has a zero eigenvalue (exactly one eigenvalue vanishes at each root). Therefore, the divergences of the effective potential at these values correspond to values of the VEV where one fermion becomes massless.

Higher loop corrections to this potential would be polynomials in $g_t^2\log X$ and $g_t^2\log(X-r_i)$ for $r_i = \pm e^{\pm i\beta}$. At constant $X\neq 0$ equation (2.6) would be a good approximation to the full potential as long as g_t is small enough. Thus, although the potential near $X=0$ is not trustworthy, the existence and location (or lack of existence) of stable vacua for $X>0$ are reliable for small enough g_t.

2.2 Renormalization Group Flow

In the above we saw that the location of the stable vacuum of the theory depends on the sign of the arbitrary mass renormalization parameters M_1^2 and M_2^2, introduced in (2.5). At the UV the $\mathcal{N}=0^*$ theory is expected to flow to the $\mathcal{N}=4$ theory, hence, it makes sense to define the scalar masses counter terms such that the scalar masses vanish at the UV and then follow its sign as we flow to the IR. To do this we follow the standard procedure of renormalization (in momentum space) in order to analyze the flow of M_i^2 to lowest order.

We calculate the scalar two-point function in the same direction ($\phi^i = \chi\delta^{i,1}T_1$) as before. The diagram with massive fermions is

$\Pi(m) =$  $=$

$$= \mathrm{Tr}[(C_1)^2] f_{1ml}f_{1ml}\,\mathrm{Tr}\int \frac{d^dk}{(2\pi)^d}\frac{k_\mu\gamma^\mu + m(c_\beta - is_\beta\gamma_5)}{k^2-m^2+i\varepsilon}\frac{(p+k)_\nu\gamma^\nu + m(c_\beta - is_\beta\gamma_5)}{(p+k)^2-m^2+i\varepsilon} =$$

$$= -\frac{ig_t^2}{\pi^2 g_{\rm YM}^2}\cdot\frac{\hat p^2 + m^2(4+2\cos 2\beta)}{\varepsilon} + O(\varepsilon^0)$$

where $\hat p$ is the Wick rotated momentum and we have employed dimensional regularization ($\varepsilon = 4-d$). The singular part of the scalar two-point function vanishes in the original $\mathcal{N}=4$ theory. Thus, to get the full scalar 2-point function (including the bosonic contributions), we subtract the same with $m=0$,

$$\Pi_{XX} = \Pi(m) - \Pi(0) = -\frac{ig_t^2}{\pi^2 g_{\rm YM}^2}\cdot\frac{m^2(4+2\cos 2\beta)}{\varepsilon} + O(\varepsilon^0)$$

From this counter term, one finds the RG equations for the scalar masses,

$$\mu\frac{\partial M_1^2}{\partial\mu} = -\frac{6g_t^2m^2}{\pi^2} + o(g_t^4) \qquad \mu\frac{\partial M_2^2}{\partial\mu} = -\frac{2g_t^2m^2}{\pi^2} + o(g_t^4)$$

$$\Rightarrow \quad M_1^2 = -\frac{6g_t^2m^2}{\pi^2}\log(\mu/\mu_0) + o(g_t^4) \qquad M_2^2 = -\frac{2g_t^2m^2}{\pi^2}\log(\mu/\mu_0) + o(g_t^4)$$

Thus, if one sets the parameters at the cutoff μ_0 to be such that the theory at μ_0 will be $\mathcal{N} = 4$ SYM deformed by the mass term (2.2), then the theory will flow such that the scalars get a positive mass-squared. This remains valid as long as $g_t^2\log(\mu/\mu_0)$ remains small. It is then reasonable to assume that the free parameters, M_i^2, introduced in (2.6) should be positive, implying that $X = 0$ is a stable vacuum.

3. The Scalar Potential from AdS/CFT Correspondence

We briefly review the formalism used in [6] to calculate the scalar potential from the AdS side of the correspondence.

3.1 Deformed AdS

The AdS/CFT correspondence maps between $\mathcal{N} = 4$ Super Yang-Mills deformed by a chiral operator and type IIB Superstrings on a modified $AdS_5\times S^5$ string theory background, which can be approximated by supergravity at large 't Hooft coupling. Deforming the CFT by a scalar (and chiral) operator $\mathcal{O}$ with scaling dimension Δ is dual to turning on a scalar field in the $AdS_5\times S^5$ with mass:

$$m^2 = \Delta(\Delta - 4) \tag{3.1}$$

A consistent and relatively easy process to find the deformed supergravity background is to find a corresponding AdS_5 $\mathcal{N} = 8$ gauged supergravity background and rely on the consistent truncation conjecture to lift the solutions to the full 10D Type IIB Supergravity. The scalar field corresponding to the $\mathcal{N} = 0^*$ deformation (2.2) was identified in [9] as the scalar generated by the lowest spherical harmonic of the $\mathbf{10_c}$ representation of the global SO(6). The relevant equation of motion can be derived from the gauged supergravity action:

$$S = \int d^5x\sqrt{g}\left(R + \frac{1}{2}\partial_\mu\lambda\partial^\mu\lambda + V(\lambda)\right), \tag{3.2}$$

$$V(\lambda) = -\frac{3}{2}\left(1 + \cosh^2\lambda\right). \tag{3.3}$$

We assume the following ansatz for the 5-d metric:

$$ds^2 = e^{2A(r)}dx^\mu dx_\mu + dr^2 \qquad \mu = 0, 1\ldots 4 \tag{3.4}$$

Noticing that λ must depend solely on r, the equation of motion reduces to:

$$\lambda'' + 4A'\lambda' = \frac{\partial V}{\partial\lambda} = -2\cosh\lambda\sinh\lambda \tag{3.5}$$

$$6A'^2 = \lambda'^2 - 2V(\lambda) = \lambda'^2 + 3\left(1 + \cosh^2\lambda\right) \tag{3.6}$$

At large r the solution must be asymptotically AdS$_5$, e.g. $A(r) \xrightarrow{r\to\infty} r$ and $\lambda(r) \xrightarrow{r\to\infty} 0$. The equation of motion can be solved to first order at this limit to find the asymptotic behavior:

$$V(\lambda) \approx -3 - \frac{3}{2}\lambda^2 + 0(\lambda^4) \qquad \Rightarrow \qquad \lambda(r \to \infty) = \mathcal{M}e^{-r} + \mathcal{K}e^{-3r} \tag{3.7}$$

The exact equation of motion (3.5),(3.6) can be solved numerically (a full discussion of the solutions for different boundary conditions $(\mathcal{M}, \mathcal{K})$ is given in [6]). The parameters $\mathcal{M}$ and $\mathcal{K}$ correspond respectively to the coefficient of the deformation (2.2) in the CFT and the VEV of this operator. It was shown in [6] that the numerical solutions of the equations of motion are singular at finite r for all values of $(\mathcal{M}, \mathcal{K})$. Thus, supergravity breaks down near this point and one should really find a full string background which should be non-singular (as was suggested in [10] for the $\mathcal{N} = 1^*$ theory). Still, one expects that far from the singularity the full background will be similar to the supergravity background, so supergravity should be a good approximation for the computations performed there. The parameter $\mathcal{K}$ is determined in principle by the behavior of the solution for small r, but since the solution is singular, it remains undetermined in the supergravity approximation.

3.2 Brane Probe Potential

The scalar potential in the strongly coupled $\mathcal{N} = 0^*$ theory can be calculated in the deformed AdS$_5\times$S^5 background by the method of brane probing. Using the Born-Infeld action for a D3-brane probe separated by a distance r from the center of the AdS$_5 \times S^5$, it is easy to find the induced potential on the radial coordinate of the probe location. The radial coordinate of the probe is mapped to the location of the VEV X, discussed in section 2. The exact lifting of the solution to 10D and the probe potential calculation was done in [6]. The result is

$$V_{\text{probe}}(\chi) = \tau_3 e^{4A(r)} \left[\xi(r) - \frac{dA}{dr}\right], \tag{3.8}$$

where $A(r)$, $\lambda(r)$ are the solutions to (3.5),(3.6) and τ_3 is the D3-brane tension. The 10D deformed AdS metric is divided to a part tangent to the D3-brane probe ($ds^2_{1,4}$) and a part to it

$$ds^2_{10} = \xi^{1/2} ds^2_{1,4} + \xi^{-\frac{3}{2}} \left[\cos^2\alpha\, \xi_- d\Omega^2_+ + \sin^2\alpha\, \xi_+ d\Omega^2_- + \xi^2 d\alpha^2\right]$$

$$\begin{aligned} \xi_\pm &= \cosh^2\lambda \pm \cos^2(2\alpha)\sinh^2\lambda \\ \xi^2 &= \xi_+\xi_- = \cosh^4\lambda - \cos^2(2\alpha)\sinh^4\lambda \end{aligned}$$

In the above, the 6-dimensional space perpendicular to the D3-brane probe is parameterized in terms of two S^2 spheres, a radial coordinate r and an angle α. The two S^2 spheres are realized by the two constraints

$$\sum_{i=1}^{3} \left(U^i_+\right)^2 = r^2 \cos\alpha \qquad , \qquad \sum_{i=1}^{3} \left(U^i_-\right)^2 = r^2 \sin\alpha$$

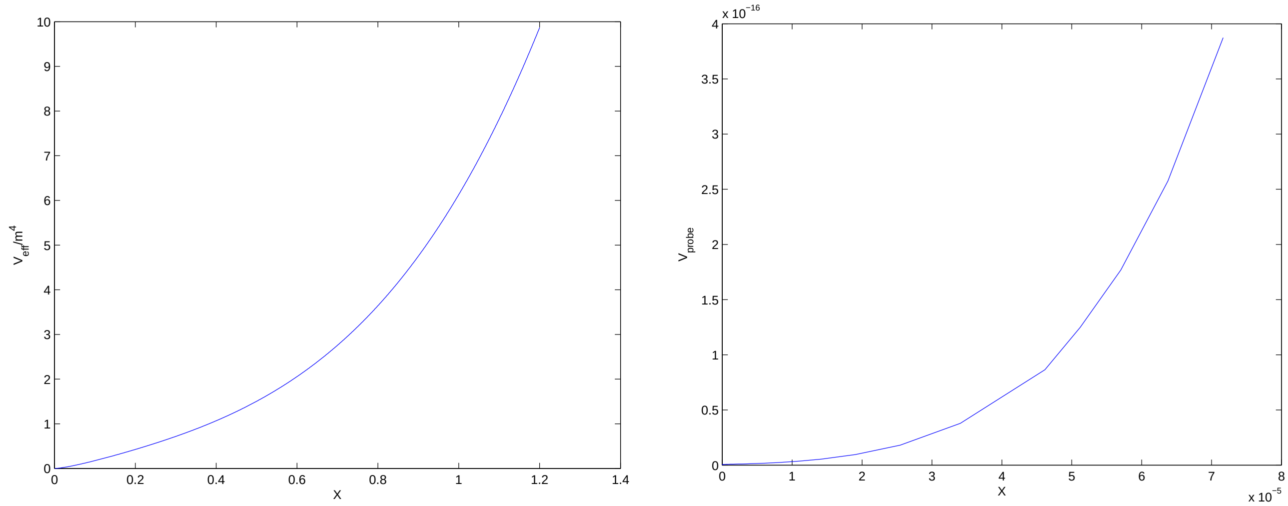


Figure 1: The scalar effective potential in the deformed CFT with $M = 0$ and $\beta = -\pi/4$ (left) and the probe potential in the deformed AdS with $\alpha = 0$ (right)

Where the coordinates U_+^i, U_-^i maps to ϕ^1, ϕ^3, ϕ^5 and ϕ^2, ϕ^4, ϕ^6 of section 2, up to re-parametrization of the radial coordinate r. This re-parametrization between the radial coordinate r (the distance of the D3-brane from the center of the AdS_5) and the VEV X (which is used in section 2) is found by redefining the r field in the Born-Infeld action such that it is normalized exactly as X in section 2. This re-parametrization is given by

$$X(r) = \frac{1}{2\pi\sqrt{2m^2\alpha'}} \int_0^r e^{A(r')}\xi^{1/2}(r')dr' \tag{3.9}$$

The probe potential is maximal at $\alpha = \pi/4$ and has a period of π. Comparing to the behavior in β of the $\mathcal{N} = 0^*$ effective potential we find the identification $\alpha = \beta + \pi/4$.

A numerical computation produces the probe potential shown in Figure 1 (right side). The numerical computation fails at a small value of χ_0, and the graph is produced by cutting the potential for $\chi < \chi_0$. Due to the numerical difficulty the translation between χ and r is defined up to addition of a small constant value (which cannot be computed using this approach). The solution of the equations of motion depends on boundary conditions related to the asymptotic behavior of the field λ. The numerical analysis was done for $(\mathcal{M}, \mathcal{K}) = (1, 0)$. As discussed at the end of 3.1, the value of $\mathcal{K}$ remains undetermined in the supergravity analysis (one expects $\mathcal{K}$, which corresponds to the gluino condensate in the CFT, to be of order unity in units of m^3). Fortunately, the qualitative behavior of the potential does not seem to depend much on this number. Allowing other values for the parameters ($\mathcal{K} > 0$) does not change the qualitative features shown in figure 1.

4. Conclusions

We have calculated the effective potential of the $\mathcal{N} = 0^*$ theory and shown that 1-loop corrections make the potential stable in specific directions that are flat at tree level (i.e. flat in the unmodified $\mathcal{N} = 4$ SYM theory). Note that although the gauge symmetry and the global R-symmetry restrict the general form of the potential, they do not fix it completely and there remain unexplored directions which were flat in $\mathcal{N} = 4$ SYM.

It is interesting that the scalar potential we found in the $\mathcal{N} = 0^*$ theory is qualitatively similar to the probe potential in its supergravity dual, although their ranges of validity do not coincide.

We have seen some indications that the vacuum of the $\mathcal{N} = 0^*$ theory is at $\langle \phi^i \rangle = 0$, implying that both the gauge symmetry and the R-symmetry remain unbroken. The location of the vacuum depends on the sign of the parameters M_1^2 and M_2^2 introduced in (2.6). In subsection 2.2 we gave arguments why these parameters should be positive, but they are not a proof, since they fail at low energies (where the theory becomes strongly coupled and the perturbative description fails). The brane probe potential in the supergravity approximation also indicates the same conclusion (a stable symmetric vacuum). However, it too fails in the interior of the AdS, implying the approximation breaks down and should not be trusted for small X.

The failure of the supergravity approximation in the interior of the AdS is a hint for stringy physics in this area. The true string vacuum dual to $\mathcal{N} = 0^*$ is likely to be described by some extended brane configuration, analogous to the configurations found by Polchinski and Strassler [10] for the string dual of the $\mathcal{N} = 1^*$ theory. Among its other advantages, knowing the full string background should allow one to calculate the value of the parameter $\mathcal{K}$ (introduced in (3.7)), thus picking the right solution asymptotically for a given $\mathcal{M}$. It is also interesting to see if the qualitative similarity between the effective potential in the $\mathcal{N} = 0^*$ theory and the probe potential in its supergravity dual, as depicted in Figure 1, will abide outside the supergravity approximation.

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The Conformal Limit of the 0A Matrix Model and String Theory on AdS_2

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Abstract: We analyze the conformal limit of the matrix model describing flux backgrounds of two dimensional type 0A string theory. This limit is believed to be dual to an AdS_2 background of type 0A string theory. We show that the spectrum of this limit is identical to that of a free fermion on AdS_2, suggesting that there are no closed string excitations in this background.

Contents

1. Introduction and Summary

Two dimensional non-critical string theories are useful toy models for studying various aspects of string theory (for reviews see [1, 2, 3]). In particular, the two dimensional type 0 string theories are useful for this, since they are non-perturbatively stable and they have a known matrix model dual [4, 5]. Since the type 0 theories have Ramond-Ramond (RR) fields, they can be used to study RR backgrounds, whose worldsheet description in string theory is still poorly understood.

The type 0A theory has RR 2-form field strengths which are sourced by D0-branes. It has been suggested that this theory should have an AdS_2 solution with RR flux. This is based both on the fact that the space-time effective action has an extremal black hole solution whose near-horizon limit is AdS_2, and on the fact that the matrix model for the 0A theory with flux has a limit in which it is a conformally invariant quantum mechanical system (which one expects to be dual via the AdS/CFT correspondence to an AdS_2 background of string theory) [6, 7]. If such a background exists, it is interesting to study it, both as a simple example of a RR background (the large symmetry may help in finding a useful worldsheet description of this background) and as an example of the correspondence between AdS_2 backgrounds and conformal quantum mechanics, which is still not as well understood as other examples of the AdS/CFT correspondence.

In this note we study the conformal limit of the 0A matrix model in order to find the properties of this conjectured AdS_2 background. We find that the spectrum of this theory is equal to that of a free fermion field on AdS_2, with a mass proportional to the RR flux q. This fermion originates from the eigenvalues of the matrix model which correspond to D0-anti-D0-brane pairs, so this spectrum suggests that the only excitations in this AdS_2 background are such uncharged D-brane-pairs, and that there are no closed string excitations in this background. From the point of view of 0A backgrounds with flux which asymptote to a linear dilaton region, this implies that the closed string excitations cannot penetrate into the strongly coupled region which is dual to the conformal limit of the matrix model.

We begin in section 2 by reviewing two dimensional type 0A string theory, its matrix model description, and its extremal black hole background. In section 3 we analyze the conformal limit of the matrix model and compute its spectrum. In section 4 we discuss the implications of this spectrum for the dual string theory on AdS_2, and in section 5 we discuss bosonizations of the fermionic system we find, which may be useful for studying the conformal quantum mechanics with a non-zero Fermi level. In the appendix we provide a detailed computation of the spectrum of a spinor field on AdS_2.

2. A Brief Review of Type 0A Superstrings

2.1 Type 0A Spacetime Effective Action

Two dimensional fermionic strings are described by $\mathcal{N} = 1$ supersymmetric world-sheet field theories coupled to worldsheet supergravity. The non chiral GSO projection gives the two type 0 theories. Because there are no NS-R or R-NS sectors in the theory, the type 0 theories have no fermions and are not supersymmetric in spacetime. The NS-NS sector includes a graviton, a dilaton and a tachyon. In two dimensions, there are no transverse string oscillations, and longitudinal oscillations are unphysical except at special values of the momentum. Thus, the tachyon is the only physical NS-NS sector field. In type 0A, the R-R sector contributes two 1-forms, and the action is [4] (with $\alpha' = 2$)

$$S = \int d^2x\sqrt{-g}\Bigg[\frac{e^{-2\Phi}}{2\kappa^2}\left(4 + R + 4(\nabla\Phi)^2 - \frac{1}{2}(\nabla T)^2 + \frac{1}{2}T^2 + \cdots\right) - \\ - \pi e^{-2T}(F^{(+)})^2 - \pi e^{2T}(F^{(-)})^2 + \cdots\Bigg]. \quad (2.1)$$

The equations of motion are solved by a linear dilaton solution

$$G_{\mu\nu} = \eta_{\mu\nu}, \qquad\qquad \Phi = x, \tag{2.2}$$

where x is the spatial coordinate. A possible deformation is to turn on the tachyon (whose mass is lifted to zero by the linear dilaton) $T = \mu e^x$.

The general equations of motion are

$$\begin{aligned} 0 &= 2R_{\mu\nu} + 4\nabla_\mu\nabla_\nu\Phi - \nabla_\mu T\nabla_\nu T + 2\pi\kappa^2 e^{2\Phi-2T}[g_{\mu\nu}(F^{(+)})^2 - 4(F^{(+)})_\mu^\beta(F^{(+)})_{\nu\beta}]+ \\ &+ 2\pi\kappa^2 e^{2\Phi+2T}[g_{\mu\nu}(F^{(-)})^2 - 4(F^{(-)})_\mu^\beta(F^{(-)})_{\nu\beta}], \end{aligned} \tag{2.3a}$$

$$0 = -2 - \nabla^2\Phi + 2(\nabla\Phi)^2 - \frac{1}{4}T^2 - \pi\kappa^2 e^{2\Phi}[e^{-2T}(F^{(+)})^2 + e^{2T}(F^{(-)})^2], \tag{2.3b}$$

$$0 = \nabla^2 T - 2\nabla\Phi\cdot\nabla T + T + 4\pi\kappa^2 e^{2\Phi}[e^{-2T}(F^{(+)})^2 - e^{2T}(F^{(-)})^2], \tag{2.3c}$$

$$0 = \nabla^\nu(e^{\mp 2T}F^{(\pm)}_{\mu\nu}). \tag{2.3d}$$

Equation (2.3d) implies

$$0 = \nabla_\nu(e^{\mp 2T}F^{(\pm)\mu\nu}) = \partial_\nu(\sqrt{-g}e^{\mp 2T}F^{(\pm)\mu\nu}) = \begin{cases} -\partial_0(\sqrt{-g}e^{\mp 2T}F^{(\pm)01}) \\ \partial_1(\sqrt{-g}e^{\mp 2T}F^{(\pm)01}) \end{cases}, \tag{2.4}$$

i.e. $\sqrt{-g}e^{\mp 2T}F^{(\pm)01}$ is constant. Thus, we see that the zero modes are the only degrees of freedom of the vector field. Furthermore, due to the non trivial coupling in front of $(F^{\pm})^2$, a time independent field strength can only be turned on for the vector field whose coupling $e^{\pm 2T}$ does not vanish at infinity. For example, in the linear dilaton background, the solutions of (2.4) are

$$F^{(+)}_{01} = q^+ e^{+2T}, \qquad\qquad F^{(-)}_{01} = q^- e^{-2T}. \tag{2.5}$$

Suppose that $\mu < 0$. As $x \to \infty$, $T \to -\infty$ and $F^{(-)}$ becomes singular, while $F^{(+)}$ is regular. Thus, turning on $F^{(-)}$ requires having D-branes as a source for this field, while no such branes are needed for $F^{(+)}$. These D-branes carry RR 1-form charge, so these are D0-branes (known as ZZ-branes). For positive μ the situation is reversed. Thus, both q^+ and q^- are quantized. For $\mu = 0$ it seems that we can turn on both fields, however, as shown in [8], this requires the insertion of q^+q^- strings that stretch from $x = -\infty$ to the strongly coupled region. This leads to additional terms in the effective action (2.1) (see also [9]).

2.2 Matrix Model Description

In analogy to the bosonic case [10], it was conjectured in [4, 5] that the field theory on N such D0-branes (with $N \to \infty$) is a dual description of the full string theory. It was argued there that the matrix model that gives the linear dilaton background of the 0A theory is a $\mathrm{U}(N)_A\times\mathrm{U}(N)_B$ gauge theory with a complex matrix m in the bifundamental representation. The two $U(N)$ groups originate from D0 and anti-D0 branes.

There are two ways of introducing RR flux [4]. First, we can modify the gauge group to $\mathrm{U}(N)_A\times\mathrm{U}(N+q^-)_B$. This leads to $q^-\neq 0$, $q^+=0$ and corresponds to placing $N+q^-$ ZZ-branes and N anti-ZZ-branes at $x=+\infty$. As long as the Fermi level is below the barrier ($\mu<0$), we will have q^- charged ZZ-branes left over after the open string tachyon condenses, so this is expected to correspond to the background with q^- units of $F^{(-)}$ flux and no $F^{(+)}$ flux. When reducing this rectangular matrix model to fermionic eigenvalue dynamics one finds that the potential for the eigenvalues is [11]

$$V(\lambda)=-\frac{1}{8}\lambda^2+\frac{M}{2\lambda^2}, \tag{2.6}$$

where $M=(q^-)^2-1/4$ and λ stands for the positive square roots of the eigenvalues of $m^\dagger m$. λ should be thought of as a radial coordinate, i.e. $\lambda\in[0,\infty)$.

A second way to introduce flux, for $q^+\neq 0$, $q^-=0$, is to add a term of the form

$$S=S_0+iq^+\int(\mathrm{Tr}\,A-\mathrm{Tr}\,B)dt, \tag{2.7}$$

where A and B are the gauge fields of the $\mathrm{U}(N)_A$ and $\mathrm{U}(N)_B$ gauge groups respectively. This has the effect of constraining the eigenvalues of m to move in a plane, all with angular momentum q^+. Surprisingly, the reduction to eigenvalues of $m^\dagger m$ gives exactly the potential (2.6) with $M=(q^+)^2-1/4$.

We may try to turn on both q^+ and q^- at the same time. It was shown in [8] that this leads again to the same potential with $M=(|q^+|+|q^-|)^2-1/4$. Thus, we see that from the point of view of the matrix model, the theory depends on $|q|\equiv|q^+|+|q^-|$ alone. Due to this and supported by arguments from the target space theory[1], it was argued in [8] that physics depends only on $|q|$. This point will be important in the next subsection where we discuss the two dimensional black hole solution, which requires turning on both fluxes.

2.3 The Extremal Black Hole Solution

The equations of motion (2.3a)-(2.3d) also admit a solution that is often referred to as the 2d extremal black hole solution [6, 12, 13]

$$ds^2=\left[1+\frac{q^2}{8}\left(\Phi-\Phi_0-\frac{1}{2}\right)e^{2\Phi}\right](-dt^2+dx^2), \tag{2.8a}$$

$$F^{(+)}=F^{(-)}=\frac{q}{2}dt\wedge dx, \tag{2.8b}$$

$$T=0, \tag{2.8c}$$

[1] This is also important for consistency with T-duality to type 0B.

where $\Phi_0 = -\log\frac{q}{4}$ and Φ is given implicitly by the ODE

$$\frac{1}{2}\frac{d\Phi}{dx} = 1 + \frac{q^2}{8}\left(\Phi - \Phi_0 - \frac{1}{2}\right)e^{2\Phi}. \tag{2.9}$$

The boundary conditions are set such that at the asymptotic region $x \to -\infty$ the solution approaches the linear dilaton solution. In the $x \to +\infty$ region the solution becomes AdS_2 with string coupling $g_s = 4/q$:

$$ds^2 \to \frac{1}{8x^2}(-dt^2 + dx^2), \qquad F^{(+)} = F^{(-)} = \frac{q}{2}dt \wedge dx, \qquad \Phi \to \Phi_0. \tag{2.10}$$

There are two major problems with this solution. The first is that the curvature becomes large as $x \to +\infty$, specifically, at the AdS_2 region of this solution $R = -8$ (in string units), so higher order corrections to (2.1) are important and the solution is invalid there. Note that unlike the linear dilaton background, this solution is not an exact CFT. The second problem is that this solution requires turning on both of the 0A vector fields, which, as we mentioned before, implies that one needs to add strings to the background and take their effects into account. Furthermore, the study of the background $(q^+, q^-) = (q^+, 0)$ shows that there is no entropy and no classical absorption. Thus, if the physics indeed depends only on $|q|$, then a black hole is not expected to exist.

It was suggested in [7] that perhaps, despite these problems, a solution that interpolates between a linear dilaton region and an AdS_2 region exists for the full theory. If this is the case, then by the AdS/CFT correspondence [14], the AdS_2 region of the solution would be dual to a one dimensional CFT (or conformal quantum mechanics (CQM)), and it was conjectured in [7] that this CQM is the conformal limit of the 0A matrix model. In the next sections we will analyze this CQM to see what the properties of such a solution must be.

3. The Conformal Limit of the Matrix Model

The 0A matrix model eigenvalues move in the potential (see Figure 1)

$$V(\lambda) = -\frac{1}{8}\lambda^2 + \frac{M}{2\lambda^2}\ . \tag{3.1}$$

At large λ the second term is negligible and the dynamics are as in the original $q = 0$ linear dilaton background. At small λ one can ignore the λ^2 term and remain with the action

$$S = \frac{1}{2}\int dt \left(\dot{\lambda}^2 - \frac{M}{\lambda^2}\right). \tag{3.2}$$

This action was studied in detail in [15] as the simplest example of a (nontrivial) conformal field theory in one dimensional space (only time). It is invariant under

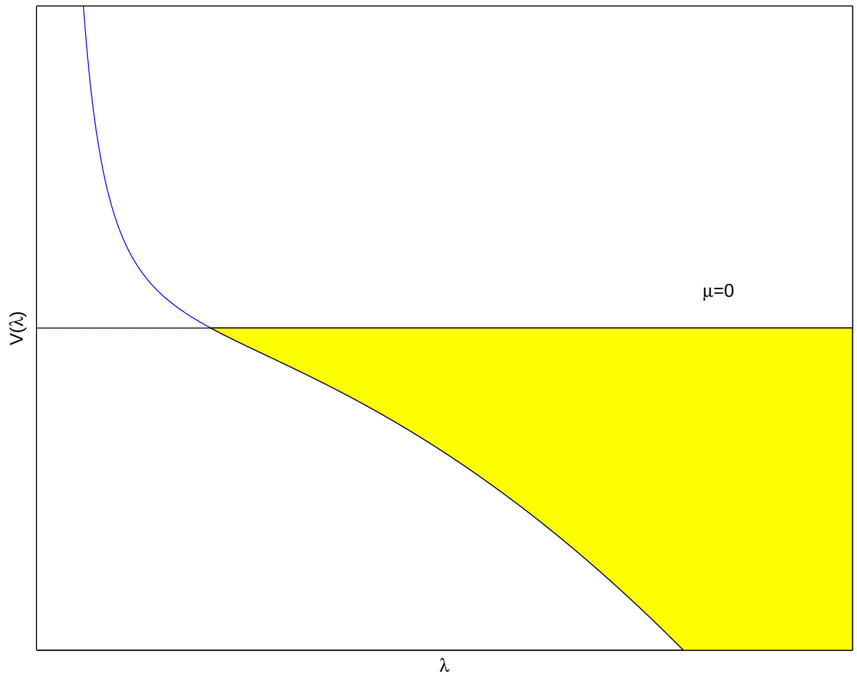

Figure 1: The potential for the eigenvalues of the matrix model describing 0A string theory in a background with non-zero flux, with Fermi level $\mu = 0$.

the SL(2,$\mathbb{R}$) group of transformations given by

$$t \to t' = \frac{at+b}{ct+d}, \qquad \lambda(t) \to \lambda'(t') = (ct+d)^{-1}\lambda(t), \tag{3.3}$$

with $ad - bc = 1$. The generators of these transformations are

$$H = \frac{1}{2}\dot{\lambda}^2 + \frac{M}{2\lambda^2}, \qquad K = \frac{1}{2}\lambda^2, \qquad D = -\frac{1}{4}(\lambda\dot{\lambda} + \dot{\lambda}\lambda). \tag{3.4}$$

Note that the AdS/CFT correspondence maps H to time evolution in the Poincaré time coordinate of AdS_2. It is important to emphasize that the AdS_2 vacuum is supposed to be mapped to the solution with $\mu = 0$, since a finite Fermi sea would break conformal invariance. The relation to SO(2,1) is made evident by taking the linear combinations

$$L_{01} = S = \frac{1}{2}\left(\frac{1}{r}K - rH\right), \qquad L_{02} = D, \qquad L_{12} = R = \frac{1}{2}\left(\frac{1}{r}K + rH\right). \tag{3.5}$$

(for any constant r). Here $L_{\mu\nu}$ are rotations in the $\mu\nu$ plane, thus, the operator R is compact. It should also be noticed that the three operators are related by a constraint in this system

$$\frac{1}{2}(HK + KH) - D^2 = \frac{M}{4} - \frac{3}{16} = \frac{q^2 - 1}{4} \tag{3.6}$$

(this is why one seems to have three constants of motion in a two dimensional phase space).

The essential observation of [15] is that by a reparametrization of time and field,

$$d\tau = \frac{dt}{u + vt + wt^2}, \qquad \tilde{\lambda}(\tau) = \frac{\lambda(t)}{\sqrt{u + vt + wt^2}}, \tag{3.7}$$

one may transform the action to a different action where the new corresponding Hamiltonian (τ-translation operator) is $G = uH + vD + wK$. Specifically, by choosing $(u, v, w) = (r, 0, r^{-1})$ one gets the transformation

$$\tilde{\lambda}(\tau) = \frac{\sqrt{r}}{\sqrt{r^2 + t^2}}\lambda(t), \qquad \tau = \arctan(t/r). \tag{3.8}$$

The transformed action is

$$S = \frac{1}{2}\int d\tau \left[(\partial_\tau \tilde{\lambda})^2 - \tilde{\lambda}^2 - \frac{M}{\tilde{\lambda}^2}\right], \tag{3.9}$$

whose Hamiltonian is the compact operator R. The spectrum of R is found using standard algebraic methods to be[2] [15]

$$r_n = r_0 + n, \qquad r_0 = \frac{1}{2}(1 + \sqrt{M + 1/4}) = \frac{1 + |q|}{2}. \tag{3.10}$$

The relations of the R eigenstates to the H eigenstates is also given in [15], as well as general methods of computing transition matrix elements exactly. Obviously, the spectrum of H is continuous and D-transformations rescale the eigenvalues of H.

As usual in the AdS/CFT correspondence, time evolution by R is supposed to be dual to time evolution of 0A string theory on AdS_2 in global coordinates [7]. The SL(2,$\mathbb{R}$) generators in the language of the new time parameter generate the transformations

$$D \qquad \delta\tau = 2\sin\tau \qquad \delta\lambda = \lambda \tag{3.11a}$$

$$H \qquad \delta\tau = \frac{1}{r}(1 + \cos\tau) \qquad \delta\lambda = 0 \tag{3.11b}$$

$$K \qquad \delta\tau = r(1 - \cos\tau) \qquad \delta\lambda = -r\tan(\tau/2)\lambda \tag{3.11c}$$

Indeed, this corresponds to the generators of the AdS_2 isometries near the boundary (A.9). A more complete analysis of the relation between the isometries and different parametrizations of AdS_2 is given in [16].

The original matrix model gives N non-interacting fermions moving in the full potential (2.6). After taking the CQM limit, we expect some number of these fermions

[2] More accurately, r_0 may also take the value $(1 - \sqrt{M + 1/4})/2$ if $M = 0$. However, $M = 0$ is not relevant for our study since it means that $q = \pm 1/2$, and q is quantized in integer units.

to “live” in the small λ region, leading in the large N limit to a second quantized version of (3.2). Note that the gauge-invariant operators in the matrix model are given by

$$\Lambda_n = \mathrm{Tr}\left[(m^\dagger m)^n\right] = \sum_{i=1}^{N} \lambda_i^{2n} \tag{3.12}$$

and in the CQM their mass dimension is $(-n)$ (recall that λ_i are the positive roots of the eigenvalues of $m^\dagger m$).

4. The AdS_2 Dual of the CQM

We have seen that the spectrum of the operator R in the CQM is $r_n = r_0 + n$ (see (3.10)). Thus, since the eigenvalues are fermionic, the spectrum of R in the second quantized CQM is given by stating for each level, r_n, whether it is occupied or vacant. We would like to identify this with the spectrum of some global AdS_2 solution of type 0A string theory, which would be a corrected version of (2.10). In fact, this is precisely the same as as the spectrum of a single free fermion field on AdS_2. Recall (as we review in the appendix) that the spectrum of a spinor field of mass m in global AdS_2 (with radius of curvature r) is[3]

$$E_n = \frac{1}{2} + |mr| + n. \tag{4.1}$$

The spectra match if we take a spinor on AdS_2 with mass $mr = |q|/2$. We suggest (based on the origin of the eigenvalues) that the excitations of this spinor field are brane-anti-brane pairs. Since these states account for the full spectrum of the CQM, we suggest that all other fields (in particular the tachyon) have no physical excitations in AdS_2. We have reached this conclusion by analyzing the spectrum in global coordinates, but of course it should apply to the Poincaré coordinates of AdS_2 as well.

We expect that the matrix model with the original potential (2.6) and with Fermi level $\mu = 0$ should correspond to a flux background of 0A which interpolates between a weakly coupled linear dilaton region and a Poincaré patch of AdS_2 (as in the extremal black hole solutions of section 2.3). We can check if the absence of the tachyon field in the AdS_2 region is consistent with this expectation. In the linear dilaton region, tachyon excitations are mapped to excitations of the surface of the Fermi sea in the matrix model. Before taking the “near horizon” limit the classical trajectory of a fermion moving in the potential (2.6) with energy $E = (\alpha')^{-1/2}\varepsilon$ has a turning point at

$$\lambda_{\max} = (\alpha')^{1/4}\sqrt{\sqrt{2M+\varepsilon^2} - \varepsilon} \tag{4.2}$$

[3]The computation in the appendix is for a field in a fixed AdS_2 background. Of course, in our case we expect to have a theory of gravity on AdS_2, but, as in other two dimensional backgrounds, the graviton-dilaton sector has no physical excitations and including it leads to the same results.

(where we have reinstalled α'). Since the quadratic term in (2.6) has a coefficient $1/(4\alpha')$, the conformal limit is achieved by taking $\alpha' \to \infty$. We wish to consider what happens to a state in the matrix model (2.6) corresponding to a tachyon excitation (with a finite energy in string units) when we take this limit, so we keep ε fixed. Clearly, the limit drives the turning point to infinity. Namely, a finite excitation of the surface of the Fermi sea in the asymptotic region of the original matrix model does not penetrate into the region that we are interested in. On the other hand, fermions with very high excitation energies can penetrate into the CQM region, and we identify them with the fermionic excitations on AdS_2 discussed above.

This result suggests that the string dual of this AdS_2 background, expected to be a strongly coupled theory on the worldsheet, has the property that all closed string states in the theory are non-physical (except for, perhaps, discrete states at special momenta). We also predict that the brane-anti-brane excitations of this theory have masses proportional to the RR flux. This suggests that perhaps the string coupling of the dual is $g_s \sim 1/q$ as in (2.10) [6, 7], but it is not clear how to define the string coupling in the absence of string states.

Naively one may have expected that the theory on AdS_2 should contain bosonic fields which are dual to the gauge-invariant operators (3.12) of the matrix model, as usual in the AdS/CFT correspondence. At first sight this seems to be inconsistent with the fact that we find no bosonic fields in the bulk. Presumably, the operators (3.12) are mapped to complicated combinations of the fermion field we found.

One of the general mysteries associated with AdS_2 backgrounds in string theory is the fact that they can fragment into multiple copies of AdS_2 (related to the possibility for extremal black holes to split) [17]. Here we find no sign of this phenomenon. Presumably this is related to the fact that there are no transverse directions for the D-branes to be separated in.

5. Remarks on Bosonization

Even though we found that the spectrum of the CQM can be identified with that of a free fermion field on AdS_2, it is interesting to ask if there could also be an alternative bosonic description of the same theory, which could perhaps be interpreted as a closed string dual description. The context in which it seems most likely that such a description would exist is when we look at the CQM (3.2) with time evolution by H, and turn on a finite positive Fermi level μ. This will clearly break the $SL(2,\mathbb{R})$ conformal symmetry (and hence the spacetime isometry on the string side), so it should no longer be dual to an AdS_2 background (note that such a state has infinite energy, so perhaps the corresponding background is not even asymptotically AdS_2). In such a configuration there are excitations of the surface of the Fermi sea with arbitrarily low energies, and these could perhaps be mapped to the tachyon field, as was the case in (2.6) before taking the "near horizon" limit. Thus, it is interesting to search for a possible alternative description of this state (except for filling the Fermi sea of the brane-anti-brane excitations).

A method that has proven to be very fruitful for studying the matrix model in the past is that of the "collective field formalism". This method of bosonization, achieved by studying the dynamics of the Fermi liquid in phase space, has led to the computation of scattering amplitudes and other important quantities in the matrix model (e.g. [3, 18, 19]). However, there is an important difference between the CQM and the standard linear dilaton matrix model: in the CQM for momentum p and $t \to \pm\infty$ we have $\lambda \sim p\,t$, while in the linear dilaton case we have $p \sim \mp\lambda$. Suppose that at some finite time we construct a small pulse perturbing the shape of the Fermi sea. The relation $\lambda \sim p\,t$ means that after (and before) a finite amount of time the pulse will "break up", or more explicitly, the pulse will no longer admit a description in terms of the upper and lower surfaces of the Fermi liquid[4]. We can follow through the steps of the collective field formalism in order to analyze propagation of pulses along short time intervals or calculate some other quantities[5], but the elementary excitations of the bosonized version will not be asymptotic states.

This asymptotic behavior of the classical solutions is, of course, a consequence of the fact that the potential is constant as $\lambda \to +\infty$. One may thus seek other methods of bosonization that are suitable for systems with this property. However, the resulting bosonic systems always seem to have non-local interactions (for instance, this happens in the method presented in [20]). We have not been able to find a bosonic theory with local interactions, which could be interpreted as a bosonic field on some spacetime dual to the CQM with finite μ. It would be interesting to investigate this further.

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[4]The collective field formalism breaks down when the Fermi liquid in phase space can no longer be described in terms of its upper and lower surface $p_+(\lambda)$ and $p_-(\lambda)$.

[5]For examples, one can calculate the free energy, which turns out to be $E_0 = (4/3)\mu^2 L$, where L is an IR-cutoff.

A. Spinor Fields on AdS_2

A.1 Definitions of AdS_2

The metric on AdS_2 may be written in so called global coordinates as

$$ds^2 = \frac{r^2}{\cos^2\theta}(-d\tau^2 + d\theta^2), \tag{A.1}$$

where $\tau \in \mathbb{R}$, $\theta \in [-\pi/2, \pi/2]$, and r is the AdS radius of curvature. This is (conformally) an infinite strip, the boundaries at $\theta = \pm\pi/2$ being each a line. Another set of coordinates is the Poincaré coordinates ($t \in \mathbb{R}$ and $x \in \mathbb{R}^+$), where the line element is given by

$$ds^2 = \frac{r^2}{x^2}(-dt^2 + dx^2). \tag{A.2}$$

The relation between the coordinate sets is

$$x = \frac{r\cos\theta}{\cos\tau - \sin\theta}, \qquad t = \frac{r\sin\tau}{\cos\tau - \sin\theta}, \tag{A.3}$$

or

$$\tan\tau = \frac{2rt}{x^2 - t^2 + r^2}, \qquad \tan\theta = \frac{t^2 - x^2 + r^2}{2rx}. \tag{A.4}$$

The generators of isometries in the Poincaré coordinates are

$$H = i\partial_t, \qquad D = i(t\partial_t + x\partial_x), \qquad K = i((t^2 + x^2)\partial_t + 2tx\partial_x), \tag{A.5}$$

and in global coordinates

$$rH = i[(1 - \cos\tau\sin\theta)\partial_\tau + \sin\tau\cos\theta\partial_\theta], \tag{A.6}$$

$$K/r = i[(1 + \cos\tau\sin\theta)\partial_\tau - \sin\tau\cos\theta\partial_\theta], \tag{A.7}$$

$$D = i[-\sin\tau\sin\theta\partial_\tau + \cos\tau\cos\theta\partial_\theta]. \tag{A.8}$$

Near the boundaries these become

$$rH = i(1 \mp \cos\tau)\partial_\tau, \qquad K/r = i(1 \pm \cos\tau)\partial_\tau, \qquad D = \mp i\sin\tau\partial_\tau. \tag{A.9}$$

A.2 Quantization of Spinor Fields on AdS_2 in Global Coordinates

Consider a spinor field on AdS_2 (in global coordinates). In this section we work with signature $(+,-)$. The vierbein is

$$V_\mu^a = \frac{r}{\cos\theta}\delta_\mu^a \tag{A.10}$$

(μ is a tensor index and a is an index in the local inertial frame). The spin connection is

$$\Gamma_\mu = \frac{1}{2}\Sigma^{ab}V_a^\nu\nabla_\mu V_{b\nu} = \delta_\mu^0\tan\theta\Sigma^{01}, \tag{A.11}$$

where

$$\Sigma^{01} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \tag{A.12}$$

The γ matrices are chosen to be

$$\gamma^0 = (r^{-1}\cos\theta)\sigma^1, \qquad \gamma^1 = (r^{-1}\cos\theta)i\sigma^2, \tag{A.13}$$

so that the Dirac equation, $(i\gamma^\mu\nabla_\mu - m)\psi = 0$, may be written as

$$\begin{pmatrix} imr\sec\theta & \partial_0 - \partial_1 - \frac{1}{2}\tan\theta \\ \partial_0 + \partial_1 + \frac{1}{2}\tan\theta & imr\sec\theta \end{pmatrix}\psi = 0. \tag{A.14}$$

Let $\psi(\theta,\tau) = e^{-i\omega\tau}\cos^{1/2}\theta(e^{i\omega\theta}u(\theta), e^{-i\omega\theta}v(\theta))$. Equation (A.14) becomes

$$\frac{d}{d\theta}\begin{pmatrix} u \\ v \end{pmatrix} = imr\sec\theta\begin{pmatrix} 0 & -e^{-2i\omega\theta} \\ e^{2i\omega\theta} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{A.15}$$

This yields the second order equation

$$\cos^2\theta u'' + \cos\theta(2i\omega\cos\theta - \sin\theta)u' - (mr)^2 u = 0. \tag{A.16}$$

Substituting $z = (1 + i\tan\theta)/2$, we have the hypergeometric equation

$$z(1-z)u'' + (\frac{1}{2} + \omega - z)u' + (mr)^2 u = 0. \tag{A.17}$$

The Dirac norm is

$$(\psi_\omega, \psi_{\omega'}) = \int_{-\pi/2}^{\pi/2} d\theta\sqrt{-g}\,\psi_\omega^\dagger\psi_{\omega'} = r^2\int_{-\pi/2}^{\pi/2}\frac{d\theta}{\cos\theta}(u^*u + v^*v), \tag{A.18}$$

so we shall require the solutions of (A.17) to vanish at the boundaries ($z \to \infty$). We will show that this requirement implies that

$$|\omega| = |mr| + \frac{1}{2} + n, \qquad n = 0, 1, 2, \cdots \tag{A.19}$$

Equation (A.17) has three (regular) singular points $\{0, 1, \infty\}$. The pairs of exponents at each point are, respectively,

$$\{(0, 1/2 - \omega), (0, 1/2 + \omega), (mr, -mr)\}. \tag{A.20}$$

When the difference between two exponents at a given point is not an integer the equation has two power-law solutions at that point (when this happens we shall call this point "normal"). When the difference is an integer there is one power-law and one log solution (when this happens we shall call the relevant point "special"). Thus, we see that the proposed result (A.19) implies that we have three distinct cases: (i) $2mr \notin \mathbb{Z}$, in which case all three points are normal; (ii) $mr \in \mathbb{Z}$, in which case all points are special; and (iii) $mr + 1/2 \in \mathbb{Z}$, in which case $z = \infty$ is special and $z = 0, 1$ are normal.

(i) $2mr \notin \mathbb{Z}$

In this case there are two power-law solutions at $z = \infty$. According to the sign of m, one of the solutions diverges at the boundary and the other one is

$$u_1(z) = z^{-|mr|} F(|mr|, |mr| + \frac{1}{2} - \omega, 2|mr| + 1; z^{-1}). \tag{A.21}$$

As θ goes from $-\pi/2$ to $\pi/2$, the path drawn by $z^{-1}(\theta)$ is a unit circle around $z^{-1} = 1$. Along this circle the hypergeometric function has a pole at $z^{-1} = 0$ (which we have taken care of), but may also has a branch cut along $z^{-1} \in [1, \infty)$. We must make sure that the solution (A.21) is continuous (single valued) along the circle. One option is to take $|mr| + 1/2 - \omega = -n$ $(n \in \mathbb{N})$, so that the power expansion of the hypergeometric function in (A.21) terminates after a finite amount of terms and the function degenerates into a polynomial. In this case we can choose the branch-cut to be $(-\infty, 0]$, which makes $z^{-|mr|}$ single valued, and so (A.21) is single valued.

When $|mr| + 1/2 - \omega$ is not a non-positive integer the hypergeometric function will have a branch cut at $[1, \infty)$ and we must try to cancel this discontinuity with the discontinuity in $z^{-|mr|}$. For $\omega + 1/2 \notin \mathbb{Z}$, the solution at $z = \infty$ is related to the solutions at $z = 1$ through the identity

$$\begin{aligned} &F(|mr|, |mr| + \frac{1}{2} - \omega, 2|mr| + 1; z^{-1}) = \\ &\quad = \frac{\Gamma(1 + 2|mr|)\Gamma(1/2 + \omega)}{\Gamma(1 + |mr|)\Gamma(1/2 + |mr| + \omega)} F(|mr|, |mr| + \frac{1}{2} - \omega, \frac{1}{2} - \omega; 1 - z^{-1}) + \\ &+ \frac{\Gamma(1 + 2|mr|)\Gamma(-\frac{1}{2} - \omega)}{\Gamma(|mr|)\Gamma(|mr| + \frac{1}{2} - \omega)} (1 - z^{-1})^{1/2+\omega} F(1 + |mr|, |mr| + \frac{1}{2} + \omega, \frac{3}{2} + \omega; 1 - z^{-1}). \end{aligned} \tag{A.22}$$

The first of the two terms above has no branch singularity as z^{-1} circles the point $z^{-1} = 1$. The second term has $\Delta \arg = 2\pi(1/2 + \omega)$. Since $\Delta \arg(z^{-|mr|}) = 2\pi |mr|$ (notice we are moving z^{-1} and not z), we must take $\omega = -1/2 - |mr| - n$ $(n \in \mathbb{N})$, in which case, the first term above vanishes and the second term cancels out with the $z^{-|mr|}$. In conclusion, we need

$$|\omega| = |mr| + 1/2 + n, \qquad n = 0, 1, 2, \cdots \tag{A.23}$$

When $\omega + 1/2 \in \mathbb{Z}$ the identity (A.22) is invalid and should be replaced by either

$$\begin{aligned} F(|mr|, 1 + |mr| - k, 2|mr| + 1; z^{-1}) = -\frac{\Gamma(1 + 2|mr|)}{\Gamma(|mr|)\Gamma(1 + |mr| - k)} \times \\ \times (z^{-1} - 1)^k \log(1 - z^{-1}) \sum_{n=0}^{\infty} \frac{(|mr| + k)_n (1 + |mr|)_n}{n!(n + k)!} (1 - z^{-1})^n + G(1 - z^{-1}) \end{aligned} \tag{A.24a}$$

or

$$F(|mr|, 1+|mr|+k, 2|mr|+1; z^{-1}) = -\frac{(-1)^k\Gamma(1+2|mr|)}{\Gamma(|mr|+1)\Gamma(|mr|-k)}\times$$
$$\times (z^{-1}-1)^k \log(1-z^{-1}) \sum_{n=0}^{\infty} \frac{(|mr|+k+1)_n(|mr|)_n}{n!(n+k)!}(1-z^{-1})^n + G(1-z^{-1}), \tag{A.24b}$$

for $k = 0, 1, 2, \ldots$, where the function $G(1-z^{-1})$ is some known function that is single valued as z^{-1} circles $z^{-1} = 1$. Due to the $\log(1-z^{-1})$ these expressions change by a number that isn't just a phase, so there is no way to cancel it out with the phase from $z^{-|mr|}$. Thus, for $\omega + 1/2 = k \in \mathbb{Z}$ there is no way to construct a single-valued solution, and we conclude that (A.23) is the only possibility for $2mr \in \mathbb{Z}$.

(ii) $2mr \in \mathbb{Z}$ **and** $m \neq 0$

Let $k/2 = |mr| \neq 0$ (i.e. $k = 1, 2...$). In this case the second solution near $z = \infty$ is

$$u_2(z) = -(-z)^{-k/2}F(k-\frac{1}{2}, k-\omega, 2k; z^{-1})\log(-z) + (-z)^{-k/2}\sum_{s=0}^{\infty} c_s z^{-s} +$$
$$+(-z)^{k/2}\frac{(-1)^{k-1}k!}{\Gamma(k/2)\Gamma(k/2+1/2-\omega)}\sum_{n=0}^{k-1}(-1)^n\frac{(k-n-1)!}{n!}\Gamma(n-k/2)\Gamma(n-k/2+1/2-\omega)z^{-n}. \tag{A.25}$$

This solution diverges and should not be considered, so we are left with the solution (A.21). For half-integer mr the argument of case (i) may be repeated, yielding the same result (A.23). For integer mr, z^{-mr} is single valued, so we need the hypergeometric function to be single valued as well. We can do this by taking $\omega = |mr| + 1/2 + n$ ($n \in \mathbb{N}$) as before, in which case the hypergeometric function degenerates into a polynomial. Equation (A.22) shows that the function is never single valued if $\omega + 1/2 \notin \mathbb{Z}$. By (A.24a) and (A.24b), we can achieve a single valued function for $\omega + 1/2 = k = 1 + |mr| + n$ or $-\omega - 1/2 = k = |mr| + n$ ($n \in \mathbb{N}$), in which case the multi-valued term with the log vanishes. Together this reduces to the previous result (A.23).

(iii) $mr = 0$

In this case it is most easy to observe that the original equation (A.14) is solved by

$$\psi = e^{-i\omega\tau}\cos^{1/2}\theta\begin{pmatrix} Ae^{i\omega\theta} \\ Be^{-i\omega\theta}\end{pmatrix}$$

and the two chiral components decouple as expected. These solutions are only δ-function normalizable, as expected of massless fields. Again, requiring single-valuedness leads to the condition (A.23).

A.3 The Dirac Equation in Poincaré Coordinates

We next consider fermions quantized on the Poincaré patch, in order to relate the mass of the fermion with the weight of the corresponding operator in the CQM. The vierbein is

$$V^a_\mu = \frac{r}{x}\delta^a_\mu.$$

The spin connection is

$$\Gamma_\mu = \frac{1}{2}\Sigma^{ab}V^\nu_a \nabla_\mu V_{b\nu} = \frac{1}{x}\delta^0_\mu \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \tag{A.26}$$

The γ matrices are chosen to be

$$\gamma^0 = (x/r)\sigma^1, \qquad \gamma^1 = (x/r)i\sigma^2, \tag{A.27}$$

so that the Dirac equation, $(i\gamma^\mu\nabla_\mu - m)\psi = 0$, may be written as

$$\begin{pmatrix} imr & x(\partial_0 + \partial_1) - \frac{1}{2} \\ x(\partial_0 - \partial_1) + \frac{1}{2} & imr \end{pmatrix} \psi = 0. \tag{A.28}$$

Let $\psi(x,t) = e^{-i\omega t}x^{1/2}(e^{-i\omega x}u(x), e^{i\omega x}v(x))$. We have

$$x\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & imre^{2i\omega x} \\ -imre^{-2i\omega x} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{A.29}$$

This yields the second order equation

$$x^2u'' + x(1 - 2i\omega x)u' - (mr)^2u = 0. \tag{A.30}$$

In general, this is solved by

$$u = x^{-1/2}e^{i\omega x}(A_+M_{1/2,mr}(2i\omega x) + A_-M_{1/2,-mr}(2i\omega x))$$

where $M_{k,m}$ is the Whittaker function. The $A_\pm$ terms behave near $x = 0$ as $x^{\pm mr}$.

The boundary condition at infinity should thus be defined by

$$\lim_{x\to 0}\psi(t,x) = x^{1/2+|mr|}\psi_0(t)$$

with a finite $\psi_0(t)$, where $\psi_0(t)$ is identified with a source for an operator $\mathcal{O}$ in the dual CFT, $\int dt\psi_0(t)\mathcal{O}(t)$. Therefore, the corresponding operator $\mathcal{O}$ has conformal (mass) dimension $1/2 - |mr|$. By relating the spectrum of the CFT (see (3.10)) to the spectrum of the spinor field in global coordinates, we find in section 4 that the mass of the fermion should be $|mr| = |q|/2$, so the operator $\mathcal{O}$ has mass dimension $(1 - |q|)/2$.

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The Moduli Space and M(atrix) Theory of 9d $\mathcal{N} = 1$ Backgrounds of M/String Theory

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Abstract: We discuss the moduli space of nine dimensional $\mathcal{N} = 1$ supersymmetric compactifications of M theory / string theory with reduced rank (rank 10 or rank 2), exhibiting how all the different theories (including M theory compactified on a Klein bottle and on a Möbius strip, the Dabholkar-Park background, CHL strings and asymmetric orbifolds of type II strings on a circle) fit together, and what are the weakly coupled descriptions in different regions of the moduli space. We argue that there are two disconnected components in the moduli space of theories with rank 2. We analyze in detail the limits of the M theory compactifications on a Klein bottle and on a Möbius strip which naively give type IIA string theory with an uncharged orientifold 8-plane carrying discrete RR flux. In order to consistently describe these limits we conjecture that this orientifold non-perturbatively splits into a D8-brane and an orientifold plane of charge (-1) which sits at infinite coupling. We construct the M(atrix) theory for M theory on a Klein bottle (and the theories related to it), which is given by a $2 + 1$ dimensional gauge theory with a varying gauge coupling compactified on a cylinder with specific boundary conditions. We also clarify the construction of the M(atrix) theory for backgrounds of rank 18, including the heterotic string on a circle.

Keywords: Superstring Vacua; String Duality; M-Theory; M(atrix) Theories.

Contents

1. Introduction

In this paper we discuss in detail the structure of the moduli space of nine dimensional $\mathcal{N} = 1$ supersymmetric backgrounds of M theory and string theory, and their M(atrix) theory construction. There are two main motivations for this study :

- The global structure of the moduli space of (maximally supersymmetric) toroidal compactifications of M/string theory has been studied extensively, and all of its corners have been mapped out. Less is known about compactifications which preserve only half of the supersymmetry (16 supercharges).[1] The moduli space

[1]Some of these backgrounds were studied in [1].

of such nine dimensional backgrounds with rank 18 (including heterotic strings on a circle) has been extensively studied, and all its corners are understood; however it seems that no similar study has been done for backgrounds with lower rank (rank 10 or rank 2). These backgrounds include the compactification of M theory on a Klein bottle and on a Möbius strip. In this paper we study the moduli space of these backgrounds in detail. We will encounter two surprises : we will see that the moduli space of backgrounds of rank 2 has two disconnected components,[2] and we will see that both for rank 2 and for rank 10 there is a region of the moduli space which has not previously been explored, and whose description requires interesting non-perturbative corrections to some orientifold planes in type IIA string theory.

- The theories we discuss (with rank n) have non-trivial duality groups of the form $\mathrm{SO}(n-1,1,\mathbb{Z})$. In their heterotic descriptions these are simply the T-duality groups. However, from the M theory point of view these dualities are quite non-trivial. In particular, it is interesting to study the manifestation of these duality groups in the M(atrix) theory description of these backgrounds; T-duality groups in space-time often map to interesting S-duality groups in the M(atrix) theory gauge theories. In this paper we will derive the M(atrix) theory for some of these backgrounds; the detailed discussion of the realization of the duality in M(atrix) theory is postponed to future work.

We begin in section 2 with a detailed discussion of the structure of the moduli space of 9d $\mathcal{N}=1$ backgrounds. We review the structure of the moduli space of rank 18 backgrounds, since this will have many similarities to the moduli spaces of reduced rank, and we then discuss in detail all the corners of the moduli spaces of reduced rank. In section 3 we construct the M(atrix) theory for the backgrounds corresponding to M theory on a cylinder (with a specific light-like Wilson line) and on a Klein bottle. The case of the cylinder has been constructed before [3], but we clarify its derivation and the mapping of parameters from space-time to the M(atrix) theory. Our construction for the Klein bottle is new. We end in section 4 with our conclusions and some open questions. Four appendices contain various technical details.

2. The moduli space of nine dimensional $\mathcal{N}=1$ backgrounds

In this section we review the moduli space of compactifications of string/M theory to nine dimensions which preserve $\mathcal{N}=1$ supersymmetry. On general grounds, if such a compactification has a rank n gauge group in its nine dimensional low-energy effective action, its moduli space takes the form $\mathrm{SO}(n-1,1,\mathbb{Z})\backslash \mathrm{SO}(n-1,1)/\mathrm{SO}(n-1)\times\mathbb{R}$,

[2]This was independently discovered by A. Keurentjes [2].

with the first component involving $n-1$ real scalars sitting in $n-1$ vector multiplets and the second component involving the scalar sitting (together with an additional U(1) vector field) in the graviton multiplet. In different regions of this moduli space there are different weakly coupled descriptions of the physics. We begin by reviewing the $n = 18$ case which is well-known. We then discuss the cases of $n = 2$ and $n = 10$, where we will encounter some surprises and some regions which have not previously been analyzed.

2.1 A review of nine dimensional $\mathcal{N} = 1$ theories with rank 18

In this subsection we review the different regions of the moduli space of nine dimensional compactifications of M theory and string theory with 16 supercharges and a rank 18 group, including M theory on $\mathbb{R}^9 \times S^1 \times (S^1/\mathbb{Z}_2)$, the two heterotic strings compactified on a circle, the type I string on a circle and the type I' string. The results are mainly from [4] and [5]. The full moduli space for these theories is $\mathrm{SO}(17,1,\mathbb{Z})\backslash \mathrm{SO}(17,1)/\mathrm{SO}(17) \times \mathbb{R}$. Naturally, different descriptions are valid in different regions of this space. For simplicity, we shall restrict our discussion to the subspaces of moduli space that have enhanced $E_8 \times E_8$ and enhanced SO(32) gauge symmetries at low energies. These subspaces take the form $\mathrm{SO}(1,1,\mathbb{Z})\backslash \mathrm{SO}(1,1) \times \mathbb{R}$, so they are analogous to the rank 2 case, which is the main subject of this paper, and we will see many similarities between these two cases. However, a reader that is familiar with rank 18 compactifications is welcome to skip to the next subsection.

We define M theory on $\mathbb{R}^9 \times S^1 \times (S^1/\mathbb{Z}_2)$ by periodically identifying the coordinates x^9 and x^{10} with periodicities $2\pi R_9$ and $2\pi R_{10}$, and also identifying $x^{10} \simeq -x^{10}$ and requiring that the M theory 3-form $C_{\mu\nu\rho}$ change sign under this reflection. This space has two boundaries/orientifold planes, at $x^{10} = 0$ and at $x^{10} = \pi R_{10}$. The orientifold breaks half of the eleven dimensional supersymmetry, leaving 16 supercharges. This eleven dimensional orientifold is not anomaly-free: there is a gravitational anomaly in the 10d theory obtained by reducing on x^{10} that comes from the boundaries, and must be cancelled by additional massless modes that are restricted to the fixed planes. As in the 10 dimensional case, part of the anomaly can be cancelled by a generalized Green-Schwarz mechanism (using the $C_{\mu\nu(10)}$ form), and the remaining anomaly is cancelled by the addition of 496 vector multiplets. In ten dimensions this restricts the gauge group to be either SO(32) or $E_8 \times E_8$. However, in the 11d case the anomaly must be divided equally between the two fixed planes, so there is [4] an E_8 gauge group living at each orientifold plane. When the two radii are large (compared to the 11d Planck length), the low energy limit is described by 11d supergravity on the cylinder, coupled to two $\mathcal{N} = 1$ E_8 SYM theories on the two boundaries.

In the limit where R_9 is large and R_{10} is small, one obtains [4] the heterotic $E_8 \times E_8$ string, with string coupling $g_h = R_{10}^{3/2}$ and string length $R_{10}^{-1/2}$ (here and henceforth we suppress numerical constants of order one, and measure all lengths in

11d Planck units). This is a valid description of the physics as long as $R_{10} \ll 1$ and $R_9 \gg R_{10}^{-1/2}$.

When we continue to shrink R_{10} to make it smaller than R_9^{-2}, we reach a point where the x^9 circle becomes small compared to the string scale. At this point we must switch to the T-dual picture. Recall that the $E_8 \times E_8$ heterotic string with no Wilson lines is T-dual to itself, with an enhanced gauge symmetry at the self-dual radius. Thus, the appropriate description in this regime is once again the heterotic $E_8 \times E_8$ string, compactified on a circle of radius $R_{10}^{-1} R_9^{-1}$ with string coupling $g_{h'} = R_{10} R_9^{-1}$. This description is valid for $R_{10} \ll R_9$ and $R_9 \ll R_{10}^{-1/2}$. In the low-energy effective action, the $E_8 \times E_8 \times \mathrm{U}(1)^2$ gauge group is enhanced to $E_8 \times E_8 \times \mathrm{SU}(2) \times \mathrm{U}(1)$ along the line $R_9^2 R_{10} = 1$.

This description is valid for arbitrarily small R_{10}, so next we fix R_{10} and shrink R_9. This has the effect of increasing the string coupling in the T-dual heterotic picture. For $R_9 \ll R_{10}$ we open up an extra dimension (as above) and get another region of moduli space that is also described by M theory on a cylinder. The length of the dual cylinder is $R_{10}^{1/2} R_9^{-1}$, the radius is $R_{10}^{-1} R_9^{-1}$ and the Planck length of this "other M theory" is $(l_p)_{M'} = R_9^{-1/3} R_{10}^{-1/6}$. Thus, this description is valid for $R_9 \ll R_{10} \ll R_9^{-4/5}$.

We are left with the region of $R_{10} \gg R_9^{-4/5}$ and $R_9 \ll 1$. This region is covered by the backgrounds that we get by reducing M theory on the periodic direction of the cylinder. This theory is known as type I' strings [6], and it may be viewed as an orientifold of type IIA string theory, obtained by dividing by worldsheet parity together with $\hat{x}^9 \to -\hat{x}^9$. The fixed points are now orientifold 8-planes ($O8$ planes) of type $O8^-$, which carry (-8) units of D8-brane charge. Tadpole cancellation then requires that this background must include also 16 D8-branes.

Another way to obtain the same type I' theory is as the T-dual of type I string theory. This is helpful in understanding an important feature of this background [5]. In type I strings, there are two diagrams that contribute to the dilaton tadpole – the disk and the projective plane – and these diagrams conspire to cancel, homogeneously throughout space-time. On the other hand, in type I' string theory there are two identical $O8$ planes at the boundaries, and 16 D8-branes that are free to move between the boundaries. As a result tadpole cancellation does not occur locally in this theory: the oriented disk diagram gets a contribution that is localized at the D8-branes, while the unoriented projective plane diagram gets contributions localized at the orientifold planes. Each D-brane cancels one-eighth of the contribution of an orientifold plane, and there is generally a gradient for the dilaton, whose exact form depends on the configuration of D8-branes.

The configuration of the branes between the orientifold planes also determines the low-energy gauge group of the background. One possible configuration of D-branes is to have 8 D8-branes on each $O8$ plane. In this case the dilaton is constant and there is an SO(16) gauge group at each of the boundaries. In all other cases the dilaton varies between the orientifold planes and the D-branes, and between the D-branes, with a gradient proportional to the inverse string length and to the local ten-form charge. This dilaton gradient causes the dilaton to diverge when the distance between two such planes is of order $g_{I'}^{-1} l_s$ (where $g_{I'}$ is the string coupling somewhere in the interval), imposing restrictions on the length of the interval and on the distances between the orientifold planes and the D-branes.

The last piece of information on type I' string theory that we need is that when the string coupling becomes infinite on one of the orientifold planes, there may be D0-branes that become massless there [7, 8, 9]. If there are n D8-branes on this orientifold plane, then these additional light degrees of freedom conspire to enhance the SO($2n$) gauge group to E_{n+1}. Specifically, in order to get an E_8 gauge group in this theory, we need to put 7 D8-branes on an orientifold plane, and one D8-brane away from it, precisely at a distance that will maintain the infinite string coupling at the $O8$ plane. If we do this at both ends (schematically: ($O8$+7D8)-D8-D8-($O8$+7D8)) we get an $E_8 \times E_8 \times \mathrm{U}(1) \times \mathrm{U}(1)$ gauge group (in nine dimensions), providing the vacuum of type I' string theory that is dual to the heterotic $E_8 \times E_8$ string with no Wilson line turned on. The distance in string units between the two single D8-branes will be denoted by $x_{I'}$.

After this detour on the general properties of type I' string theory, let us now return to the compactification of M theory to this background. We can reach a type I' background in two ways: starting with the original M theory and reducing on x^9, or starting from the dual M theory and reducing on the periodic direction there. Both constructions give us a type I' theory in its $E_8 \times E_8$ vacuum, with couplings:

$$g_{I'1} = R_9^{3/2} \qquad (l_s)_{I'1} = R_9^{-1/2} \qquad R_{I'1} = R_{10} \tag{2.1}$$

$$g_{I'2} = R_9^{-1} R_{10}^{-5/4} \qquad (l_s)_{I'2} = R_{10}^{1/4} \qquad R_{I'2} = R_{10}^{1/2} R_9^{-1} \tag{2.2}$$

Note that we need to be careful about what we mean by $g_{I'}$, since the string coupling varies along the interval and diverges at the boundaries; what we will mean by $g_{I'}$ (here and in other cases with a varying dilaton) is the string coupling somewhere in the interior of the interval, and the differences between different points in the interval are of higher order in $g_{I'}$ so our expression is true in the weak $g_{I'}$ limit (which is the only limit where g is well-defined anyway). Naively, the first description is valid (except near the orientifold planes where the string coupling diverges) whenever $R_9 \ll 1$ and $R_{10} \gg R_9^{-1/2}$, and the second description is valid whenever $R_{10} \gg R_9^{-4/5}$ and $R_{10} \gg R_9^4$, but this would give an overlapping range of validity to the two different descriptions (which would also overlap with some of our previous descriptions). However, requiring that the distance between each D8 and the $O8$ should be such that the orientifold plane is at infinite coupling, we find $x_{I'}$ in the two cases to be

$$1) \qquad x_{I'1} = (R_{10} R_9^2 - 1) R_9^{-3/2} \tag{2.3}$$

$$2) \qquad x_{I'2} = (1 - R_9^2 R_{10}) R_9^{-1} R_{10}^{1/4} \tag{2.4}$$

Now, the regime of validity for each of the type I' backgrounds (defined by $(g_s)_{I'} \ll 1$ and $x_{I'} \geq 0$) is distinct; the first description is valid when $R_9 \ll 1$ and $R_{10} \leq R_9^{-2}$ and the second description is valid when $R_{10} \gg R_9^{-4/5}$ and $R_{10} \geq R_9^{-2}$. Furthermore, at the line $R_{10}R_9^2 = 1$, which was the line of enhanced symmetry in the heterotic $E_8 \times E_8$ string theory, we have $x_{I'} = 0$ for both backgrounds. On this line the two D8-branes coincide, enhancing the U(1) × U(1) symmetry to SU(2) × U(1) exactly as in the heterotic case. This implies that the two type I' backgrounds are actually related by an SU(2) gauge transformation, which is consistent with the fact that $x_{I'1}/(l_s)_{I'1} = -x_{I'2}/(l_s)_{I'2}$. We see that the line $R_{10}R_9^2 = 1$ continues to be a line of enhanced symmetry throughout the moduli space.

The backgrounds described thus far cover the whole moduli space of nine dimensional $\mathcal{N} = 1$ backgrounds with an $E_8 \times E_8$ gauge group. The top graph in figure 1 displays how these backgrounds fill in all possible values of the radii of the M theory we started with. Of course, the top part of this figure (above the dashed line) is related by an SU(2) gauge transformation to the bottom part, so the true moduli space is just half of this figure (above or below the dashed line).

Next, we turn to the subspace of the rank 18 backgrounds with an unbroken SO(32) gauge symmetry.[3] This subspace includes the type I' background in its SO(32) vacuum. This vacuum is obtained by having all 16 D8-branes sit at one of the $O8^-$ planes (so that it has the same charges as an $O8^+$ plane). The dilaton then grows as we go from this $O8^-$ plane to the other one. We denote the distance between the orientifold planes as $R_{I'}$ (in string units) and the string coupling (defined, for instance, near the $O8^-$ plane with the branes) as $g_{I'}$. The regime of validity of this description is $1 \ll R_{I'} \ll g_{I'}^{-1}$.

For $R_{I'} \ll 1$, we need to switch to the T-dual of this description, which is the type I background with no Wilson lines. The string coupling in type I is $g_I = g_{I'}/R_{I'}$ and the radius of the compact circle is $R_I = R_{I'}^{-1}$. This description is thus valid for $g_{I'} \ll R_{I'} \ll 1$.

As we further decrease $R_{I'}$ we increase the coupling of the type I string theory, and eventually we should go over to the S-dual heterotic SO(32) string. The parameters of this heterotic background are $g_h = R_{I'}/g_{I'}$ and $R_h = 1/R_{I'}$, and its string length is $(l_s)_h = (g_{I'}/R_{I'})^{1/2}$, implying that it is a valid description for $R_{I'} \ll g_{I'} \ll R_{I'}^{-1}$.

On the line $g_{I'} = R_{I'}^{-1}$ we find $R_h = (l_s)_h$. This is a line of self T-duality and enhanced SU(2) gauge symmetry in the heterotic SO(32) string. We can now move to the T-dual picture, obtaining another heterotic SO(32) description. By going to the strong coupling limit of the new heterotic background, we get another type I background, and by another T-duality we get another regime described by the type

[3]Actually we will only describe one connected component of this subspace, which will turn out to be analogous to the rank 2 case. The other component has a discrete Wilson line that is only felt by spinor representations of SO(32).

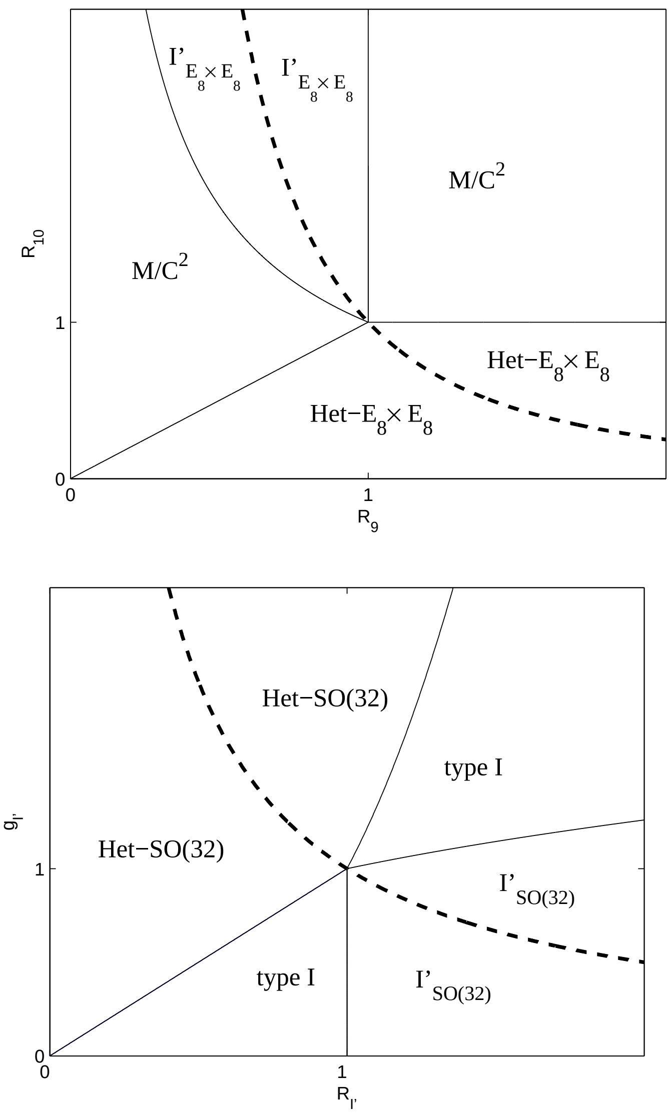

Figure 1: The $E_8 \times E_8$ and SO(32) subspaces of the moduli space of rank 18 compactifications of M theory and string theory. In the first graph the parameters are the period and length of the cylinder in Planck units as defined for the upper right compactification of M theory on a cylinder, and in the second they are the distance between the orientifold planes in units of the string length, and the string coupling on the $O8^-$ plane with the branes on it, of the type I' background appearing in the lower right corner. The dashed line in both graphs is the line of enhanced SU(2) symmetry; the regions of the graphs below and above this line are identified.

I' background. These three types of backgrounds cover the whole SO(32) subspace of the moduli space, as presented in the second graph of figure 1.

In the heterotic SO(32) description we found that along the line $R_{I'} = g_{I'}^{-1}$ a U(1) symmetry gets enhanced to SU(2), by the usual mechanism of a winding mode becoming massless at the self-dual point. As in the previous example, the same enhancement occurs on this line also in other regions of the moduli space. In the type I' description, this is the line where the $O8$ plane without the branes is at infinite coupling. The half-D0-branes that live on the orientifold plane then become massless at this line, and they are responsible for the symmetry enhancement (from $U(1)$ to $E_1 = \mathrm{SU}(2)$) in this description.

It is important to remember that the two branches depicted in figure 1 are just specific subspaces of the moduli space of rank 18 theories and that the rest can be reached by turning on Wilson lines in the heterotic or type I pictures, or equivalently moving around D8-branes in the type I' picture. Thus, the two plots in the figure are just two slices of the full moduli space. This is important to emphasize since in the next subsection a similar picture will arise, but in that case there are no Wilson lines to be turned on, so the two branches are actually two disconnected components of the 9d moduli space.

2.2 M theory on a Klein bottle and other rank 2 compactifications

We now wish to repeat the above analysis for M theory compactified on a Klein bottle (first considered in [10]) instead of a cylinder. One may start by asking if M theory makes sense at all on a Klein bottle. We will see that the answer is yes, and that this background arises as a strong coupling limit of consistent string theory backgrounds.

Let us begin by considering type IIB string theory gauged by the symmetry group $\mathbb{Z}_2 = \{1, H_9\Omega\}$, where Ω is worldsheet parity and H_9 is a shift in the 9th coordinate,

$$H_9\Omega \, : \begin{cases} X^\mu(z,\bar{z}) \simeq X^\mu(\bar{z},z), & \mu \neq 9 \\ X^9(z,\bar{z}) \simeq X^9(\bar{z},z) + 2\pi R_9. \end{cases} \tag{2.5}$$

This orientifold theory is derived by imposing $H_9\Omega = 1$ on the spectrum of the IIB theory compactified on a circle of radius $2R_9$; this breaks half of the supersymmetry (one of the gravitinos is projected out and the other one remains). In this case there are no fixed planes, and one does not need to add D-branes to cancel any tadpole, as explained in [10]. We refer to this background as the Dabholkar-Park background or the DP background.

We can compactify the DP background on an additional circle of radius $\tilde{R}_8$ in the x^8 direction, and T-dualize in this direction. Because of the orientation reversal in the original DP background, the resulting background is type IIA string theory gauged by

$$\begin{cases} X^\mu(z,\bar{z}) \simeq X^\mu(\bar{z},z), & \mu \neq 8,9 \\ X'^8(z,\bar{z}) \simeq -X'^8(\bar{z},z), & \\ X^9(z,\bar{z}) \simeq X^9(\bar{z},z) + 2\pi R_9. \end{cases} \tag{2.6}$$

Thus, this describes a type IIA compactification on a Klein bottle ($K2$) of area $(2\pi R_8)\times(2\pi R_9)$, where $R_8 = l_s^2/\tilde{R}_8$, with an orientation reversal operation (see also [11]).[4] One can lift this theory to a background of M theory in the usual way by taking the strong coupling limit (the supersymmetry of this background prohibits a potential for the dilaton). The worldsheet parity is identified in M theory with flipping the sign of all components of the 3-form field. Thus, as promised, M theory on a Klein bottle naturally arises as a strong coupling limit of a string theory.

Consider now M theory compactified on a Klein bottle of radii $K2(R_{10}, R_9)$ measured in Planck units, namely with the identification

$$(X^{10}, X^9) \simeq (-X^{10}, X^9 + 2\pi R_9), \tag{2.7}$$

including a reversal of the 3-form field, and also with X^{10} periodically identified with radius R_{10}. If we shrink R_{10} we get a background that is described in detail in [12] and [13] (additional related analysis can be found in [14]). This background can be defined as type IIA string theory with a gauging of the symmetry

$$(-)^{F_L} \times (X^9 \to X^9 + 2\pi R_9), \tag{2.8}$$

where F_L is the space-time fermion number of the left-moving fields on the worldsheet. We refer to this background as the asymmetric orbifold of type IIA or AOA. The type IIA string coupling is $g_s \simeq R_{10}^{3/2}$, and $l_s \simeq R_{10}^{-1/2}$.

If we continue to shrink R_{10}, the circle becomes small (in string units) and we need to change to the T-dual description. It was noticed in [12] that the AOA background has an enhanced SU(2) symmetry point when $R_9 = l_s/\sqrt{2}$, and in [13] it was demonstrated that (as implied by this enhanced symmetry) this background is actually self-T-dual. In appendix C we explicitly evaluate the partition function of this model, showing that it is modular invariant and respects the T-duality. Therefore, when we continue to shrink R_{10} we should switch to a T-dual AOA description, which has $(R_9)_T \simeq 1/R_9R_{10}$ and $(g_s)_T \simeq R_{10}/R_9$. This dual AOA description can in turn be lifted to a dual M theory on a Klein bottle. The regimes of validity of these different descriptions are exactly the same (up to numerical constants of order one) as those we found in the previous subsection, with M theory on the Klein bottle replacing M theory on a cylinder, and the AOA background replacing the heterotic $E_8 \times E_8$ string.

Going back to M theory on a Klein bottle, what happens if we shrink R_9? Naively, we get a theory that is an orientifold of IIA on an $S^1/\mathbb{Z}_2$, with some boundary conditions at the fixed planes. However, we have a $\mathbb{Z}_2$ symmetry that exchanges the two boundaries, and since there cannot be any tadpole for the RR 10-form, we can only

[4]The orientation reversal leads to various periodicity conditions for the p-form fields, that are explained in detail in appendix B.

have neutral orientifold planes (with respect to the 10-form charge) at the boundaries, unlike the standard $O8^-$ planes (which carry (-8) units of charge) and $O8^+$ planes (which carry $+8$ units of charge).[5] For now we conjecture that the reduction in this direction leads to a background which we call the X-background, and that the line of enhanced symmetry $R_{10}R_9^2 = 1$ is also a line of enhanced symmetry in the X-background. Thus, the two remaining regions of the moduli space are covered by the X-background and its dual (obtained by reducing the dual M theory on the periodic cycle). In §2.3 we shall propose a stringy description of this background.

The theories described above (starting with M theory on a Klein bottle) cover a complete moduli space of the form $\mathrm{SO}(1,1,\mathbb{Z})\backslash\,\mathrm{SO}(1,1)\times\mathbb{R}$, as shown on the top graph of figure 2. However, there are several additional string backgrounds with 16 supercharges and a rank 2 gauge symmetry that were not included so far, so they must be in a separate component of the moduli space. The first is the DP background described above. Next, there is the orientifold of type IIA on an interval $S^1/\mathbb{Z}_2$, with an $O8^-$ plane on one end and an $O8^+$ plane on the other [15]. We refer to this background as the $O8^{\pm}$ background, and denote the length of the interval in string units by $\pi R_{\pm}$. No D-branes are required for tadpole cancellation here. However, the dilaton does have a gradient (identical to that in the SO(32) background of the type I' string) that puts a limit on the maximum length of the interval, of the form $R_{\pm} < 1/g_{\pm}$ (where $g_{\pm}$ is again defined as the string coupling somewhere in the interval). This background was studied in [15], where it was shown to be T-dual to the DP background. An important feature of this background is that when the distance between the orientifold planes is such that the coupling on the $O8^-$ is infinite, there are half-D0-branes stuck on the orientifold that become massless. Locally, this is exactly the same as in the type I' SO(32) vacuum; in both cases the fractional branes enhance the $\mathrm{U}(1)^2$ symmetry to $\mathrm{SU}(2)\times\mathrm{U}(1)$.

Finally, one other background can be obtained by gauging type IIB string theory by the symmetry (2.8). We call this background the Asymmetric Orbifold of type IIB or AOB. Recalling the following property of IIB strings

$$(-1)^{F_L} = S\Omega S \tag{2.9}$$

(where S is the S-duality transformation of type IIB string theory) and applying the adiabatic argument of [14], we see that this background is S-dual to the DP background. Exactly like the AOA background, the AOB background is self-T-dual [13] and has an enhanced SU(2) symmetry when the radius of the circle is $l_s/\sqrt{2}$.

We can now describe the second component of the moduli space. Start with the $O8^{\pm}$ background with string coupling $g_{\pm}$ and radius $R_{\pm}$ (measured in string units).

[5] Notice that this contradicts a suggestion occasionally found in the literature, that in this limit we get a background defined by IIA on an $S^1/\mathbb{Z}_2$ with an $O8^-$ plane on one end and an $O8^+$ plane on the other.

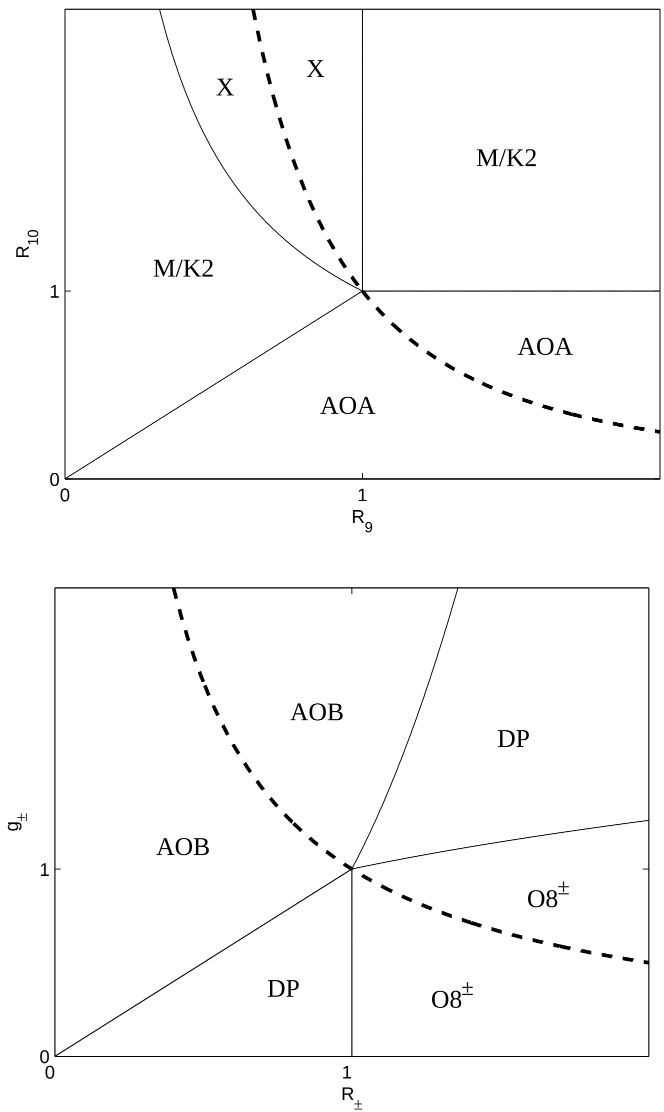

Figure 2: The two disconnected components of the moduli space including the backgrounds of subsection 2.2. In the first graph the parameters are the periods of the Klein bottle in Planck units as defined in (2.7) for the upper right M theory description. In the second graph, the parameters are the radius of the circle in string units and the string coupling (in the middle of the interval) for the $O8^{\pm}$ background on the lower right corner. The dashed line in both graphs is the line of enhanced SU(2) gauge symmetry, and the backgrounds below the line are identified with the backgrounds above the line.

This is a valid description for $g_{\pm} \ll 1/R_{\pm}$ and $R_{\pm} \gg 1$. As we decrease $R_{\pm}$, we need to T-dualize, as in [15], and we get the DP background with radius[6] $1/R_{\pm}$ and string coupling $g_{\pm}/R_{\pm}$. The DP description is valid as long as $g_{\pm} \ll R_{\pm} \ll 1$. We can further decrease $R_{\pm}$ to the point where the DP string coupling is too large.

[6]We continue to ignore numerical constants of order one.

At this point we turn to the S-dual picture, which is the asymmetric orbifold of type IIB (AOB). The string coupling is now $R_\pm/g_\pm$, and the S-dual string length is $(g_\pm/R_\pm)^{1/2}$, such that the radius $1/R_\pm$ in string units is now $(g_\pm R_\pm)^{-1/2}$. This is a good description for $g_\pm \ll 1/R_\pm$ and $R_\pm \ll g_\pm$.

We continue next by increasing $g_\pm$ to the point where the AOB radius becomes small. Then we must use the self T-duality of this background to arrive at a dual AOB description, with radius $(g_\pm R_\pm)^{1/2}$ (in units of its string length) and string coupling $g_\pm^{-1/2} R_\pm^{3/2}$. This description is good for $g_\pm \gg 1/R_\pm$ and $g_\pm \gg R_\pm^3$. The next step is to take the S-dual of the dual AOB background. This gives another DP background with radius (in units of its string length) $g_\pm^{3/4} R_\pm^{-1/4}$ and coupling $g_\pm^{1/2} R_\pm^{-3/2}$. This description is valid for $g_\pm \gg R_\pm^{1/3}$ and $g_\pm \ll R_\pm^3$. The last step is to T-dualize the new DP background to a dual $O8^\pm$ background, with radius $g_\pm^{-3/4} R_\pm^{1/4}$ (in units of its string length) and coupling $g_\pm^{-1/4} R_\pm^{-5/4}$. This description is valid for $g_\pm \ll R_\pm^{1/3}$ and $g_\pm \gg 1/R_\pm$.

These six string theories cover the entire range of values of $(g_\pm, R_\pm)$. The reader should notice that the line of enhanced SU(2) symmetry in the $O8^\pm$ background, $R_\pm \sim 1/g_\pm$, smoothly goes into the line of enhanced SU(2) symmetry of the AOB background [12], as in the previous cases we discussed.

Figure 2 summarizes the structure of the moduli space. As promised, it has two disconnected components in the nine dimensional sense. Note that the nine dimensional low-energy effective action on the two components is identical, but the massive spectrum is different. The components can be related by compactifying on an additional circle and performing a T-duality, but they are not connected as nine dimensional backgrounds. There is an obvious relation between each of these components and one of the subspaces of the moduli space of rank 18 compactifications discussed in the previous subsection and depicted in figure 1.

2.3 The X background demystified

In the previous subsection we left open the description of the limit of the Klein bottle where R_9 is small and R_{10} is large. In this limit the Klein bottle geometrically looks as in figure 3 (though the geometric description is not really valid at distances smaller than the 11d Planck scale). This limit should correspond to some 10 dimensional string theory; let us collect some features of this theory:

Figure 3: In the X region, the Klein bottle looks like a long tube between two cross-caps.

- After reducing M theory on the small circle R_9, we expect to obtain a type IIA string theory that lives on $\mathbb{R}^{8,1} \times I$ where I is an interval. On each boundary of the interval we should have an orientifold plane; however these orientifold planes are non-standard because the orientifolding is accompanied by a half-shift on the M theory circle. This half-shift modifies, among other things, the properties of D0-branes near the orientifold. One can think of this shift as a discrete RR flux characterizing the orientifold plane, which changes its charge from the usual charge of (-8) to zero [16]. We will denote such orientifold planes by $O8^0$.

- The fact that the orientifold carries no D8-brane charge is consistent with tadpole cancellation of the 10-form field and the dilaton. Note that there is a $\mathbb{Z}_2$ symmetry exchanging the two ends of the interval, so the two orientifold planes must carry the same charge (unlike the case in the $O8^{\pm}$ background described in the previous subsection).

- Naively one expects that in the limit of small R_9 the 10 dimensional string theory can be made very weakly coupled, and the description should involve the usual type IIA string theory, at least far from the boundaries of the interval (where there may or may not be large quantum corrections). We will see that there are some regions of the moduli space where this naive expectation fails.

- The discussion of the previous subsection suggests that we should have an enhanced SU(2) gauge symmetry when the interval is of size $R \sim 1/g_s$ (in type IIA string units). One may expect this enhanced symmetry to come from half-D0-branes on the orientifold planes, as in some of the examples discussed in the previous subsections. However, since we have a $\mathbb{Z}_2$ symmetry relating the two orientifolds, it is hard to imagine how the $\mathrm{U}(1)^2$ group would be enhanced to $\mathrm{U}(1) \times \mathrm{SU}(2)$ rather than to the more symmetric $\mathrm{SU}(2)^2$.

The last item above suggests some modification of the naive picture of this background. The picture that we will suggest for the correct description of this background is based on two facts :

- In our analogy between the rank 18 and the rank 2 theories, the X background plays the same role as the type I' $E_8 \times E_8$ background. In this background the two orientifold planes are always at infinite coupling, and the enhanced SU(2) symmetry arises when two D8-branes in the middle of the interval come together [9].

- In some cases, when a standard ($O8^-$) orientifold plane of charge (-8) is at infinite coupling, it can emit a D8-brane into the bulk, leaving behind an orientifold plane of charge (-9) which has no perturbative description (and which always sits at infinite coupling). This phenomenon cannot be seen in perturbation theory, but it can be deduced from an analysis of D4-brane probes [7].

Our suggestion is that each $O8^0$ plane non-perturbatively emits a D8-brane and becomes an $O8^{(-1)}$ plane which always sits at infinite coupling. The moduli space and 10-form fluxes of this system are then identical to those of the $E_8 \times E_8$ type I' background, with the $O8^{(-1)}$ plane playing the same role as the $O8^-$ plane with 7 D8-branes on it. Now, the gauge symmetry is enhanced to SU(2) in a $\mathbb{Z}_2$-symmetric manner, when the D8-branes meet in the middle of the interval, and this enhancement is perturbative (it can happen at weak coupling). Denoting the string coupling in the region between the two emitted D8-branes by g_s (it does not vary in the interval between the D8-branes), the distance of each brane from the respective $O8^{(-1)}$ plane is $\sim 1/g_s$. Thus, when the branes meet, our interval indeed has a size proportional to $1/g_s$ as required.

We suggest that this is the correct description of $O8^0$ orientifold planes. Note that such a large non-perturbative correction to the description of high-dimensional orientifold planes is not surprising; already in the case of $O7$ planes it is known [17] that they non-perturbatively split into two 7-branes, and the corrections to $O8$ planes are expected to be even larger. Our suggestion implies non-trivial corrections to compactifications of M theory involving crosscaps (similar corrections in M theory should also occur in a compactification on a Möbius strip, as will be discussed in the next subsection). These corrections are similar to the ones that occur for M theory on a cylinder with no Wilson lines. Note that when we are close to the enhanced SU(2) point, the corrections to the naive M theory picture shown in figure 3 are not just localized near the cross-caps as one may naively expect, but the string coupling actually varies along the whole interval.

In order for this proposal to be consistent, there should be no massless fractional D0-branes on the $O8^{(-1)}$ planes, which would lead to more enhanced symmetries than we need. Because of the shift in the M theory circle involved in the orientifolding, the radius of the cross-cap is actually half of the radius of the M theory circle in the bulk, which implies that only even momentum modes (in units of the minimal momentum on a standard orientifold plane) are allowed there. Hence, there are no half D0-branes in backgrounds with $O8^0$ orientifolds (or $O8^{(-1)}$ orientifolds).

2.4 M theory on a Möbius strip and other rank 10 compactifications

There is one additional disconnected component of the moduli space of nine dimensional compactifications with $\mathcal{N} = 1$ supersymmetry. Consider the heterotic $E_8 \times E_8$ string compactified to nine dimensions on a circle of radius R_9. One can consider an orbifold of this theory generated by switching the two gauge groups together with a half-period-shift of x^9. This theory is known as the CHL string; as usual one should add twisted sectors for the consistency of the orbifold (for more details see

[18, 19]). This leads to a nine dimensional compactification with a rank 10 gauge group. We will focus on the subspace of the moduli space of this theory in which the E_8 symmetry is unbroken.

An orbifold of the $E_8 \times E_8$ heterotic string on a circle can also be viewed as an orbifold of M theory compactified on a cylinder. Begin with M theory compactified on the cylinder

$$x^9 \simeq x^9 + 2\pi R_9 \quad , \quad x^{10} \in \left[-\frac{\pi}{2}R_{10}, \frac{\pi}{2}R_{10}\right]. \tag{2.10}$$

The action on the heterotic string which we described above lifts in M theory to

$$x^9 \simeq x^9 + \pi R_9 \quad , \quad x^{10} \simeq -x^{10}. \tag{2.11}$$

Upon identifying points related by this transformation we obtain a Möbius strip, with a cross-cap at $x^{10} = 0$ and a boundary at $x^{10} = \pi R_{10}/2$, as depicted in figure 4. Thus, this component of the moduli space is generated by various limits of M theory compactified on the Möbius strip [10, 20]. Notice that anomaly cancellation as in [4] tells us that the single boundary of the Möbius strip carries an E_8 gauge symmetry, which is consistent with the low-energy gauge symmetry of the CHL string.

Figure 4: M theory on a Möbius strip describes the strong coupling limit of the CHL string.

So far we have good descriptions of the regions of moduli space where both R_9 and R_{10} are large, and when R_{10} is small (leading to the CHL string). It is natural to ask what string theory backgrounds are obtained when we reduce on the other direction, $R_{10} \gg l_P \gg R_9$. In this limit, the cross-cap and the boundary (which is topologically $\mathbb{R}^{8,1} \times S^1$) of figure 4 are very far from each other. The boundary becomes a usual $O8^-$ plane of IIA string theory. Our previous description of E_8 symmetries in type IIA string theory implies that this $O8^-$ plane has 7 D8-branes on top of it, and there is an additional one displaced such that the system $O8^-$+7D8 is at infinite string coupling. The cross-cap should behave exactly as in the case of the Klein bottle, described in the previous subsection. Therefore, the naive $O8^0$ plane emits an extra D8-brane and becomes an $O8^{(-1)}$ plane at infinite string coupling.

When we approach the point $R_{10}R_9^2 \sim 1$, we find that the two D8-branes in the bulk come together, and enhance the U(1) × U(1) symmetry to SU(2) × U(1). As in our previous examples, the same symmetry enhancement arises also for small R_{10}, where it arises at the self-dual radius of the perturbative CHL string (for an exhaustive analysis of the momentum lattices in toroidal compactifications of CHL strings see [21]). In fact, it is just the same as the SU(2) enhanced symmetry of the $E_8 \times E_8$ string at the self dual radius (the gauge bosons are BPS and survive the CHL projection). This is another consistency check on our proposal for the behavior of the cross-cap in M theory.

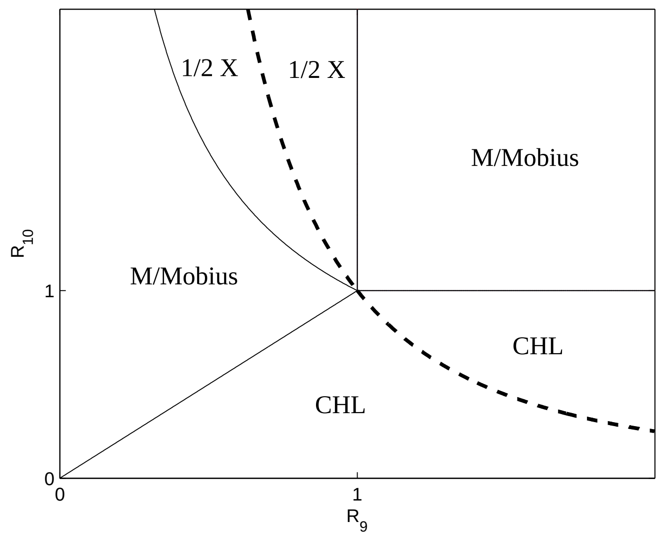

Figure 5: The moduli space of compactifications of M theory on a Möbius strip. The parameters are the period and length of the strip in Planck units as defined for the upper right M theory. The dashed line is the line of enhanced SU(2) symmetry.

We conclude that the moduli space of M theory on a Möbius strip has a line of enhanced SU(2) symmetry, and that all of its limits may be understood (with some strongly coupled physics occurring in the IIA limit). This moduli space is depicted in figure 5, where the type IIA orientifold limit is denoted by $(1/2)X$. In fact, this moduli space is identical in structure to the other moduli spaces we encountered in our survey, arising from M theory on a cylinder and on a Klein bottle.

3. The M(atrix) theory description of M theory on a cylinder and on a Klein bottle

In this section we describe the M(atrix) theory [22] which is the discrete light-cone quantization (DLCQ) of some of the theories described in the previous section – M theory compactified on a cylinder and on a Klein bottle. We begin by considering the case of a cylinder [3], and then move on to the Klein bottle. We review in detail the case of the cylinder because of the great similarity between these two compactifications (as described in the previous section), which is useful in the construction of the correct M(atrix) theory of the Klein bottle.

3.1 The M(atrix) theory of M theory on a cylinder

This subsection is based on [3], with the addition of a systematic derivation of their M(atrix) theory and of some small corrections to the identifications of parameters presented in that paper.

M(atrix) theory is the discrete light-cone quantization of M theory backgrounds [23]; it provides the Hamiltonian for these theories compactified on a light-like circle, with N units of momentum around the circle. In general, such a DLCQ description is very complicated. However, in some M theory backgrounds it simplifies, because a light-like circle may be viewed [24, 25] as a limit of a very small space-like circle, and M theory on a very small space-like circle is often very weakly coupled. This leads to a simple description of the DLCQ Hamiltonian, in which most of the degrees of freedom of M theory decouple. In particular, this is the case for the M(atrix) theory of M theory itself, which is given just by a maximally supersymmetric U(N) quantum mechanical gauge theory, and for the M(atrix) theory of M theory compactified on a two-torus, which is given by the maximally supersymmetric U(N) $2+1$ dimensional gauge theory, compactified on a dual torus.

The generic simplicity of M(atrix) theory is based on the fact that M theory compactified on a small space-like circle becomes a weakly coupled type IIA string theory. However, when boundaries are present in the M theory compactification, they usually destroy this simplicity. For instance, as is evident from figure 1, if we take M theory on $S^1/\mathbb{Z}_2$ and compactify it further on a very small space-like circle, we do not obtain a weakly coupled background (but, rather, we obtain M theory on a dual cylinder). Thus, generically the DLCQ of M theory backgrounds with boundaries is very complicated. However, there is an extra degree of freedom one can use in the DLCQ constructions, which is a Wilson line along the light-like circle; such a Wilson line becomes irrelevant in the large N limit of M(atrix) theory in which it provides a light-cone quantization of the original background (without the light-like circle), but it can have large effects for finite values of N. In the case of M theory on $S^1/\mathbb{Z}_2$, as we discussed above, the theory compactified on an additional small circle is generally strongly coupled (see [26] for a recent discussion), except when we have a Wilson line breaking the $E_8 \times E_8$ symmetry to SO(16) $\times$ SO(16). The moduli space of this subspace of compactifications on a cylinder is drawn in figure 6; as can be seen in this figure, the limit of a small space-like circle leads in this case to a weakly coupled type I string theory (with a Wilson line breaking SO(32) to SO(16) $\times$ SO(16)). The M(atrix) theory for M theory on an interval with this specific light-like Wilson line is then again a simple theory [27, 28, 29, 30, 31] – the decoupled theory of N D1-branes in this type I background, which is simply an SO(N) $1+1$ dimensional $\mathcal{N} = (0, 8)$ supersymmetric gauge theory on a circle, coupled to 32 real left-moving fermions in the fundamental representation (coming from the D1-D9 strings), half of which are periodic and half of which are anti-periodic [5].

Now we can move on to the case we are interested in, which is the M(atrix) theory of M theory on $\mathbb{R}^9 \times S^1 \times (S^1/\mathbb{Z}_2)$, with an arbitrary Wilson line W for the $E_8 \times E_8$ gauge group on the S^1. To construct the M(atrix) theory we should again consider the limit of this theory on a very small space-like circle [24, 25], with a particular scaling of the size of the cylinder as the size of this extra circle goes

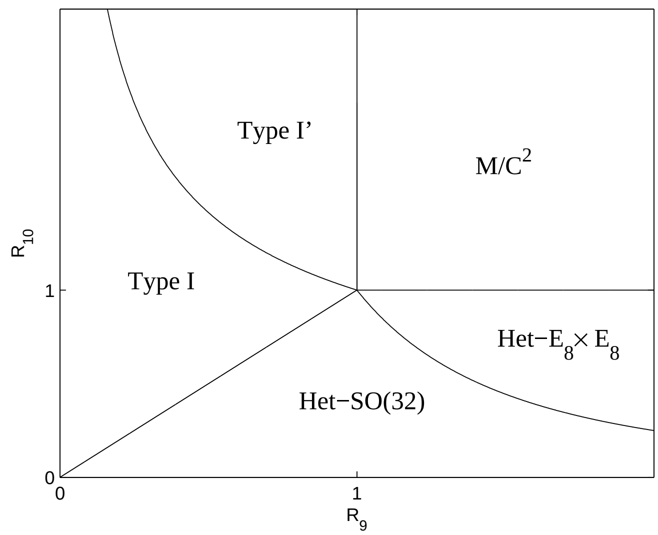

Figure 6: The subspace of the moduli space of M theory on a cylinder, where all backgrounds include a Wilson line breaking the gauge group to SO(16) $\times$ SO(16). The parameters in the plot are the period R_9 and length R_{10} of the cylinder in Planck units.

to zero. Again, in general this limit gives a strongly coupled theory, except in the case where we have an additional Wilson line breaking the $E_8 \times E_8$ gauge group to SO(16) $\times$ SO(16) on the additional circle (the original Wilson line W must commute with this Wilson line in order to obtain a weakly coupled description).[7] In such a case we obtain precisely the theory described in the previous paragraph, compactified on an additional very small circle with a Wilson line $\tilde{W}$ (which is the translation of the original Wilson line from the $E_8 \times E_8$ variables of the original M theory to the SO(32) variables of the dual type I background). Since the additional circle is very small we need to perform a T-duality on this circle. We then obtain a type I' theory of the type described in the previous section, still compactified on a circle with the SO(16) $\times$ SO(16) Wilson line, and with the positions of the D8-branes determined by the eigenvalues of the Wilson line $\tilde{W}$. The D1-branes we had before now become D2-branes which are stretched both along the interval between the orientifold planes and along the additional circle.

The usual derivation of M(atrix) theory [24, 25] shows that the M(atrix) theory is precisely the decoupled theory living on these D2-branes, in the limit that the string mass scale goes to infinity keeping the Yang-Mills coupling constant on the D2-branes, which is proportional to

$$g_{YM}^2 \propto g_s/l_s, \tag{3.1}$$

[7]In principle we could also have a light-like Wilson line for the other two U(1) gauge fields appearing in the low-energy nine dimensional effective action, but we will not discuss this here.

fixed. The D2-brane lives on a cylinder, with a circle of radius R_1 (related to the parameters of the original M theory background by $R_1 = l_p^3/R_{10}R$, where $1/R$ is the energy scale associated with the light-like circle) and an interval of length πR_2 (given by $R_2 = l_p^3/R_9R$). In the standard case of toroidal compactifications, only the disk contributions to the D2-brane action survive in this limit, giving a standard supersymmetric Yang-Mills theory, with Yang-Mills coupling $g_{YM}^2 = R/R_9R_{10}$. However, in our case it turns out that some contributions to the D2-brane action from Möbius strip diagrams also survive; this is evident from the fact that g_s in the type I' background is generally not a constant, leading through (3.1) to a non-constant Yang-Mills coupling. This was taken into account in [3], where the Lagrangian for any distribution of D8-branes was obtained, and it was shown that the Möbius strip contributions are crucial to cancel anomalies in the gauge theory.

There is one special case when the Möbius contributions are absent; this is the case when the type I' background has a constant dilaton, with eight D8-branes on each orientifold plane. According to the discussion above, this case provides the DLCQ description of M theory on a cylinder with a Wilson line breaking the gauge symmetry to SO(16) $\times$ SO(16) (so that we are at some point on the moduli space of figure 6), and with an additional light-like Wilson line which breaks the gauge symmetry in the same way. We begin by describing this special case. In this case the theory on the D2-branes away from the orientifold planes is just the standard maximally supersymmetric 2+1 dimensional U(N) Yang-Mills theory, with a gauge coupling related to the parameters of the original M theory background by $g_{YM}^2 = R/R_9R_{10}$ (which is the same relation as in toroidal compactifications). The field content of this theory includes a gauge field A_μ, seven scalar fields X^j and eight Majorana fermions ψ_A. The boundary conditions project the U(N) gauge group to SO(N). In addition, the D2-D8 strings give rise to 8 complex chiral fermions in the fundamental representation χ_k ($k = 1, \cdots, 8$) at the boundary $x^2 = 0$ and 8 additional fermions $\tilde{\chi}_k$ ($k = 1, \cdots, 8$) at the other boundary $x^2 = \pi R_2$. The action is given by

$$S = \int dt \int_0^{2\pi R_1} dx^1 \Bigg[\int_0^{\pi R_2} dx^2 \frac{1}{2g_{\rm YM}^2} \,{\rm Tr} \bigg(-\frac{1}{2} F_{\mu\nu}F^{\mu\nu} + (D_\mu X^j)^2 + \\ + \frac{1}{2}[X^j, X^i][X^j, X^i] + i\bar{\psi}_A \gamma^\alpha D_\alpha \psi_A - i\bar{\psi}_A \gamma^i_{AB}[X_i, \psi_B] \bigg) + \\ + i \sum_{k=1}^{8} \bar{\chi}_k (\partial_- + iA_-|_{x^2=0})\chi_k + i \sum_{k=1}^{8} \bar{\tilde{\chi}}_k (\partial_- + iA_-|_{x^2=\pi R_2})\tilde{\chi}_k \Bigg], \quad (3.2)$$

where D_μ is the covariant derivative for the adjoint representation, $\partial_- \equiv \partial_t - \partial_1$ and similarly for A_-. Our conventions for fermions and spinor algebra are summarized in appendix A. Due to the light-like Wilson line described above, the fermions χ_k are periodic around the x^1 circle, while the fermions $\tilde{\chi}_k$ are anti-periodic; this can alternatively be described by adding a term $\frac{1}{4\pi R_1}$ to the kinetic term of the $\tilde{\chi}_k$ in (3.2).

The boundary conditions can be determined by consistency conditions for D2-branes ending on an $O8^-$ plane. For the bosonic fields, at both boundaries, the boundary conditions take the form

$$\begin{aligned} X^j &= (X^j)^T, & \partial_2 X^j &= -(\partial_2 X^j)^T, \\ A^{0,1} &= -(A^{0,1})^T, & \partial_2 A^{0,1} &= (\partial_2 A^{0,1})^T, \\ A^2 &= (A^2)^T, & \partial_2 A^2 &= -(\partial_2 A^2)^T. \end{aligned} \tag{3.3}$$

These boundary conditions break the U(N) bulk gauge group to SO(N). The zero modes along the interval are SO(N) gauge fields $A^{0,1}$ and eight scalars in the symmetric representation of SO(N) coming from A^2 and X^j. For the fermions, the boundary conditions take the form

$$\psi_A = -i\gamma^2\psi_A^T, \qquad \partial_2\psi_A = i\gamma^2\partial_2\psi_A^T. \tag{3.4}$$

The zero modes for the right-moving fermions are in the adjoint representation of SO(N), and those of the left-moving fermions are in the symmetric representation. This leads to an anomaly in the low-energy $1+1$ dimensional SO(N) gauge theory, which is precisely cancelled by the 16 additional chiral fermions in the fundamental representation; this cancellation occurs locally at each boundary.

The bulk theory has eight $2+1$ dimensional supersymmetries (16 real supercharges), but the boundary conditions and the existence of the D2-D8 fermions break this to a $\mathcal{N} = (0,8)$ chiral supersymmetry (SUSY) in $1+1$ dimensions. The supersymmetry transformation rules are given by

$$\begin{aligned} \delta_\epsilon A_\alpha &= \frac{i}{2}\bar{\epsilon}_A\gamma_\alpha\psi_A, \\ \delta_\epsilon X^i &= -\frac{1}{2}\bar{\epsilon}_A\gamma^i_{AB}\psi_B, \\ \delta_\epsilon\psi_A &= -\frac{1}{4}F_{\alpha\beta}\gamma^{\alpha\beta}\epsilon_A - \frac{i}{2}D_\alpha X_i\gamma^\alpha\gamma^i_{AB}\epsilon_B - \frac{i}{4}[X_i, X_j]\gamma^{ij}_{AB}\epsilon_B, \\ \delta_\epsilon\chi_k &= 0\,, \qquad \delta_\epsilon\tilde{\chi}_k = 0\,. \end{aligned} \tag{3.5}$$

These transformation rules are consistent with the boundary conditions (3.3), (3.4) only for

$$\epsilon_A = i\gamma^2\epsilon_A\,, \tag{3.6}$$

and thus indeed the boundary conditions preserve only 8 of the original 16 supercharges. Decomposing the fields by their γ^2 eigenvalues $\pm i$:

$$\epsilon_A = \begin{pmatrix}\epsilon_A^+ \\ \epsilon_A^-\end{pmatrix}, \qquad \psi_A = \begin{pmatrix}\psi_A^+ \\ \psi_A^-\end{pmatrix}, \tag{3.7}$$

it follows that the unbroken SUSY is for ϵ_A^-.

In the more general case, as described above, we consider a similar background but with the D8-branes at arbitrary positions in the bulk. One obvious change is then that the chiral fermions χ and $\tilde{\chi}$ are no longer localized at the boundaries but rather at the positions of the D8-branes. More significant changes are that the varying dilaton leads to a varying gauge coupling constant, and the background 10-form field in the type I' background leads to a Chern-Simons term, which is piece-wise constant along the interval. The most general Lagrangian was written in [3], where it was also verified that it is supersymmetric and anomaly-free. The relation between the positions of the D8-branes and the varying coupling and 10-form field, which in the bulk string theory comes from the equations of motion, is reproduced in the gauge theory by requiring that there is no anomaly in arbitrary $2+1$ dimensional gauge transformations.

For the purposes of comparison with the Klein bottle case that we will discuss in the next subsection, it is useful to consider the special case where there is no Wilson loop on the cylinder. This gives, in particular, the M(atrix) theory of the $E_8 \times E_8$ heterotic string compactified on a circle with no Wilson line. In this case we have a configuration where all D8-branes are on the same orientifold plane, say the one at $x^2 = \pi R_2$. The action in this special case may be written in the form (now denoting the scalar fields by Y^i and the adjoint fermions by Ψ_A)

$$S = \int dt \int_0^{2\pi R_1} dx^1 \Bigg[\int_0^{\pi R_2} dx^2 \frac{1}{4g_{\rm YM}^2} \,\mathrm{Tr}\bigg(-z(x^2) F_{\mu\nu}F^{\mu\nu} + 2z^{1/3}(x^2)(D_\mu Y^j)^2 + \\ + z^{-1/3}(x^2)[Y^j,Y^i][Y^j,Y^i] + 2iz^{1/3}(x^2)\bar{\Psi}_A\gamma^\alpha D_\alpha \Psi_A + \frac{dz^{1/3}(x^2)}{dx^2}\bar{\Psi}_A\Psi_A - \\ - 2i\bar{\Psi}_A\gamma^i_{AB}[Y_i,\Psi_B] + \frac{4}{3}\frac{dz(x^2)}{dx^2}\epsilon^{\alpha\beta\gamma}(A_\alpha\partial_\beta A_\gamma + i\frac{2}{3}A_\alpha A_\beta A_\gamma)\bigg) + \\ + i\sum_{k=1}^{8}\bar{\chi}_k(\partial_- + iA_-|_{x^2=\pi R_2})\chi_k + i\sum_{k=1}^{8}\bar{\tilde{\chi}}_k(\partial_- + iA_-|_{x^2=\pi R_2})\tilde{\chi}_k\Bigg]. \quad (3.8)$$

The varying coupling constant is given by

$$z(x^2) = 1 + \frac{6g_{\rm YM}^2}{\pi}(x^2 - \frac{\pi R_2}{2}), \quad (3.9)$$

and the coupling grows as we approach the $O8^-$ plane with no D8-branes on it. Here we arbitrarily defined $g_{\rm YM}$ to be the effective coupling constant at the middle of the interval (other choices would modify the constant term in (3.9)). The linear term in (3.9) is related to the background 10-form field. The effective Yang-Mills coupling constant is

$$(g_{\rm YM}^{eff})^2 = \frac{g_{\rm YM}^2}{1+6g_{\rm YM}^2(x^2-\pi R_2/2)/\pi} = \frac{1}{1/g_{\rm YM}^2 + 6(x^2/\pi - R_2/2)}. \quad (3.10)$$

This description is valid as long as the Yang-Mills coupling constant does not diverge anywhere, namely for $g_{YM}^2 < 1/(3R_2)$. This is the same condition as the string coupling not diverging in the type I' string theory which we used for deriving this action. The boundary conditions on the fields are essentially the same as before, but there as some modifications in the boundary conditions which involve derivatives and in the SUSY transformation laws. The same modifications will appear in the Klein bottle case that we will discuss in the next subsection, and we will discuss them explicitly there. The fermions χ_k are still periodic and the fermions $\tilde{\chi}_k$ anti-periodic due to the light-like Wilson line.

Note that in the two special cases that we described, (3.2) and (3.8), the theory is exactly free for $N = 1$ (as was the original BFSS M(atrix) theory), since the gauge fields $A_{0,1}$ vanish at both boundaries; however, this is not the case at more general points on the moduli space. The usual argument that $N = 1$ DLCQ theories should be free is that $N = 1$ is the minimal amount of possible light-like momentum, so the theory must contain a single particle with this momentum and no interactions. However, this is no longer true in the presence of generic light-like Wilson lines, which modify the quantization of the light-like momentum for charged states.

3.2 The M(atrix) theory of the Klein bottle compactification

In this subsection we describe the M(atrix) theory of M theory on a Klein bottle, and we will see that it is very similar to the case described in the previous subsection.[8] Again, to derive the M(atrix) theory we need to consider M theory on a Klein bottle times a very small space-like circle. Now we do not need to add any Wilson lines to get a weakly coupled theory; instead we directly obtain N D0-branes in the weakly coupled type IIA string theory on a Klein bottle that was mentioned in the previous section, in the limit in which the Klein bottle has a very small size. We then need to perform two T-dualities to go back to a finite-size compact manifold. The relevant T-dualities were already described in section 2.2: one T-duality (which is straightforward) leads to the DP background, and the next leads to the $O8^{\pm}$ background. Thus, the M(atrix) theory is the decoupled theory on $2N$ D2-branes stretched between an $O8^-$ plane and an $O8^+$ plane[9] (we obtain $2N$ D2-branes due to the presence of D0-branes as well as their images on the original Klein bottle). This theory is very similar to the theory (3.8) we wrote down in the previous subsection for D2-branes stretched between an $O8^-$ plane with no D8-branes and another $O8^-$ plane with 16 D8-branes, since the dilaton and 10-form field are identical in both of these configurations; the only difference is that the D2-D8 fermions are not present, and the boundary conditions on the $O8^+$ plane are different from those on the $O8^-$

[8]The M(atrix) theory of M theory on a Klein bottle was also discussed in [32, 33, 34] but our results are different. Perhaps some of these other theories arise from different choices of light-like Wilson lines.

[9]The spectrum of D-branes in the $O8^{\pm}$ background was analyzed in [35].

plane. In particular, they project the U($2N$) gauge group to USp($2N$) instead of to SO($2N$) (which is another way to see that the rank of the gauge group must be even).

Another naive way to derive this M(atrix) theory would be to start from the effective action of N D0-branes on the Klein bottle, and to perform two Fourier transforms of this action, along the lines of the original derivations of M(atrix) theory compactifications [22, 36]. This analysis is performed in appendix D; it gives the correct boundary conditions, but it only gives the terms in the action coming from the disk and it does not include the effects related to the variation of $z(x^2)$ which come from Möbius strip diagrams, so it leads to an anomalous gauge theory.

The complete action for the M(atrix) theory of M theory on a Klein bottle is

$$S = \frac{1}{4g_{\rm YM}^2}\int dt \int_0^{2\pi R_1} dx^1 \int_0^{\pi R_2} dx^2 \,{\rm Tr}\Big(-z(x^2)F_{\mu\nu}F^{\mu\nu} + 2z^{1/3}(x^2)(D_\mu Y^j)^2 + \\ + z^{-1/3}(x^2)[Y^j,Y^i][Y^j,Y^i] + 2iz^{1/3}(x^2)\bar{\Psi}_A\gamma^\alpha D_\alpha\Psi_A + \frac{dz^{1/3}(x^2)}{dx^2}\bar{\Psi}_A\Psi_A - \\ - 2i\bar{\Psi}_A\gamma^i_{AB}[Y_i,\Psi_B] + \frac{4}{3}\frac{dz(x^2)}{dx^2}\epsilon^{\alpha\beta\gamma}(A_\alpha\partial_\beta A_\gamma + i\frac{2}{3}A_\alpha A_\beta A_\gamma)\Big) \quad (3.11)$$

where, as in the previous subsection,

$$z(x^2) = 1 + \frac{6g_{\rm YM}^2}{\pi}(x^2 - \frac{\pi R_2}{2}). \quad (3.12)$$

The boundary conditions could be derived from the open string theory of D2-branes ending on orientifold planes, but they can also be derived directly in the gauge theory by requiring the absence of boundary terms and consistency with the SUSY transformations which are described below. It will be convenient in this section to think of the U($2N$) matrices as made of four $N\times N$ blocks, and to use Pauli matrices that are constant within these blocks. In this notation the scalars Y^i satisfy the boundary conditions[10]

$$\underline{x^2=0}: \qquad Y^j = \sigma^1(Y^j)^T\sigma^1, \qquad \partial_2 Y^j = -\sigma^1(\partial_2 Y^j)^T\sigma^1, \quad (3.13)$$
$$\underline{x^2=\pi R_2}: \qquad Y^j = \sigma^2(Y^j)^T\sigma^2, \qquad \partial_2 Y^j = -\sigma^2(\partial_2 Y^j)^T\sigma^2. \quad (3.14)$$

The SUSY transformations of the action (3.11) are

$$\delta_\epsilon Y^i = -\frac{1}{2}\bar{\epsilon}_A\gamma^i_{AB}\psi_B, \quad (3.15)$$
$$\delta_\epsilon\psi_A = -\frac{i}{4}z^{-1/3}[Y_i,Y_j]\gamma^{ij}_{AB}\epsilon_B - \frac{1}{4}z^{1/3}F_{\alpha\beta}\gamma^{\alpha\beta}\epsilon_A - \frac{i}{2}D_\alpha Y_i\gamma^\alpha\gamma^i_{AB}\epsilon_B, \quad (3.16)$$
$$\delta_\epsilon A_\alpha = \frac{i}{2}z^{-1/3}\bar{\epsilon}_A\gamma_\alpha\psi_A. \quad (3.17)$$

[10]Note that our boundary conditions at $x^2=0$ seem different from those of the previous subsection, but the two are simply related by multiplying all adjoint fields by σ^1.

Unbroken SUSY transformations are those with $\epsilon_A = i\gamma^2\epsilon_A$. Notice that the SUSY transformations now include the function $z(x^2)$. We can use these transformations to determine the boundary conditions for the fermions. The non-derivative boundary conditions are the naive ones related to (3.13),(3.14), namely

$$\underline{x^2 = 0} \;:\; \psi_A = -i\sigma^1\gamma^2\psi_A^T\sigma^1, \qquad \underline{x^2 = \pi R_2} \;:\; \psi_A = -i\sigma^2\gamma^2\psi_A^T\sigma^2. \tag{3.18}$$

The derivative boundary condition for the upper component of the spinor follows immediately from (3.15),

$$\underline{x^2 = 0} \;:\; \partial_2\psi_A^+ = -\sigma^1(\partial_2\psi_A^+)^T\sigma^1, \qquad \underline{x^2 = \pi R_2} \;:\; \partial_2\psi_A^+ = -\sigma^2(\partial_2\psi_A^+)^T\sigma^2. \tag{3.19}$$

For the lower component, we have to use (3.16) to obtain

$$\begin{aligned} \underline{x^2 = 0} \;:\; \quad & \partial_2\psi_A^- + \frac{1}{3}\frac{z'}{z}\psi_A^- = \sigma^1(\partial_2\psi_A^- + \frac{1}{3}\frac{z'}{z}\psi_A^-)^T\sigma^1, \\ \underline{x^2 = \pi R_2} \;:\; \quad & \partial_2\psi_A^- + \frac{1}{3}\frac{z'}{z}\psi_A^- = \sigma^2(\partial_2\psi_A^- + \frac{1}{3}\frac{z'}{z}\psi_A^-)^T\sigma^2. \end{aligned} \tag{3.20}$$

The deviations from the naive boundary conditions are proportional to z' which is related to the varying string coupling.

Finally, we will also need to know the boundary conditions on A_2. Note that (3.17) implies (using $\bar{\epsilon}_A = -i\bar{\epsilon}_A\gamma^2$)

$$\delta_\epsilon A_2 = -\frac{1}{2}z^{-1/3}(y)\bar{\epsilon}\psi \propto z^{-1/3}(y)\epsilon^-\psi^+\,. \tag{3.21}$$

Thus, the boundary conditions on $(z^{1/3}A_2)$ are the same as those we wrote above for the scalar fields, with no additional terms. Finally, by further investigation of (3.17) one can see that $(z^{1/3}A_{0,1})$ satisfy boundary conditions of exactly the same form (3.20) as ψ^-.

3.3 The AOA limit of the M(atrix) theory

As we described in the previous section, there is a limit of M theory on a Klein bottle, corresponding to small R_{10}, which gives a weakly coupled string theory – the theory which we called the AOA background. In this limit we should be able to see that the M(atrix) theory we constructed becomes a second quantized theory of strings in this background.[11] Recall that the standard M(atrix) theory for weakly coupled type IIA strings is given by a maximally supersymmetric U(N) $1+1$ dimensional gauge theory; at low energies this flows to a sigma model on $\mathbb{R}^{8N}/S_N$, which describes free type IIA strings (written in Green-Schwarz light-cone gauge) [37, 38, 39, 40]. The string interactions arise from a twist operator which is the leading, dimension 3,

[11] A similar limit for the theory described in section 3.1 should lead to a second quantized theory of $E_8 \times E_8$ heterotic strings, but we will not discuss this in detail here.

correction to the sigma model action [39]. Similarly, in our case we expect that in the limit where we should obtain a weakly coupled string theory, the low-energy effective action should be a symmetric product of the sigma model of the AOA strings, again written in a Green-Schwarz light-cone gauge (in this gauge the sigma model action is identical to that of AOB strings in a static gauge, just like in the type IIA case we get the action of type IIB strings in a static gauge).

The mapping of parameters described in the previous subsection implies that the limit of small R_{10} corresponds to small R_2 compared to the other scales in the gauge theory. Thus, in this limit we obtain the (strongly coupled limit of the) $1+1$ dimensional theory of the zero modes along the interval. We will analyze this theory in detail for the case of $N=1$, in which the bulk gauge group is U(2); it is easy (by a similar analysis to that of [39]) to see that for higher values of N we obtain (at low energies) the N'th symmetric product of the $N=1$ theories (deformed by higher dimensional operators giving the string interactions).

The zero modes for the scalars Y^i are easily determined by noting that the boundary conditions (3.13),(3.14) are satisfied by the identity matrix

$$Y^i(t,x^1,x^2)_{ab} = Y^i(t,x^1)\mathbb{I}_{ab}, \tag{3.22}$$

where a,b are U(2) indices which we suppress henceforth. The matrices proportional to the identity matrix are actually a completely decoupled sector of the theory (for any value of N), with no interactions. The zero mode analysis for the fermions is a little more involved. The subtlety here is that one should keep in mind that the fermions are Majorana when deriving the equations of motion.[12] To obtain the zero modes we need to solve the equations

$$z^{1/3}\partial_2\psi_A^+ = 0, \qquad\qquad z^{1/3}\partial_2\psi_A^- + (z^{1/3})'\psi_A^- = 0, \tag{3.23}$$

subject to the boundary conditions described in the previous subsection. For the upper component of the spinor the solution is

$$\psi_A^+(t,x^1,x^2) = \psi_A^+(t,x^1)\mathbb{I}\,,$$

which manifestly satisfies the equations of motion and the boundary conditions.[13] For the lower component ψ^-, the equation of motion (3.23) guarantees that the derivative boundary conditions (3.20) are satisfied. In order to satisfy the non-derivative boundary conditions (3.18), we simply choose the direction in the gauge group to be σ^3. Hence, the solution is

$$\psi_A^-(t,x^1,x^2) = \psi_A^-(t,x^1)z^{-1/3}(x^2)\sigma^3\,. \tag{3.24}$$

[12] The relevant part of the Lagrangian in components is $\mathcal{L} \supseteq -z^{1/3}(\psi^+\partial_2\psi^- + \psi^-\partial_2\psi^+) + (z^{1/3})'\psi^-\psi^+$.

[13] The existence of this mode is guaranteed by the fact that it is actually the Goldstino for the 8 supercharges broken by the D2-branes.

Similar zero modes arise for the A_0 and A_1 component of the gauge field. These lead to a U(1) gauge field in the low-energy effective action, but since there are no charged fields, this does not lead to any physical states. Finally, there is a scalar field coming from the zero mode of A_2,

$$A_2(t,x^1,x^2) = A_2(t,x^1)z^{-1/3}(x^2)\mathbb{I}\,. \tag{3.25}$$

This scalar is actually compact due to large gauge transformations, as we describe below.

Of course, all these fields fill out $\mathcal{N}=(0,8)$ supersymmetry representations in 1+1 dimensions; the vector multiplet contains (A_0, A_1, ψ^-), and the matter multiplet contains (A_2, Y^i, ψ^+). For a detailed description of this kind of SUSY see [27]. For general values of N we find both types of multiplet in the adjoint representation of U(N), the same field content as in the M(atrix) theory of type IIA strings. The low-energy spectrum turns out to be non-chiral (for any value of N), guaranteeing that there are no anomalies.

In order to identify our theory with the AOA background we need to show that the theory is invariant under the transformation $(-1)^{F_L}$ together with a half-shift on the scalar field coming from A_2. Consider a U(2) gauge transformation of the form

$$g(x) = e^{if(x^2)}\sigma_1\,. \tag{3.26}$$

Note that our theory is not invariant under generic U(2) gauge transformations since these are broken by the boundary conditions. The transformation (3.26) preserves all the boundary conditions. In order for it to leave the theory in the same topological sector (the simplest way to verify this is to regard the cylinder as an orbifold of the torus and use the usual classification of sectors on the torus) we require that $f(\pi R_2) - f(0) = (2n+1)\pi/2$ for some integer n. In general the transformation (3.26) mixes the zero mode (3.25) with other modes of A_2; however, all other modes can be gauged away so this mixing is not really physical. We can work in a gauge where all the non-zero modes are set to zero, and an appropriate choice of a large gauge transformation which preserves this gauge is

$$f(x^2) = \frac{z^{2/3}(x^2)\pi/2}{z^{2/3}(\pi R_2) - z^{2/3}(0)} \quad \to \quad \partial_2 f(x^2) = \frac{z'}{3}\frac{z^{-1/3}(x^2)\pi}{z^{2/3}(\pi R_2) - z^{2/3}(0)}\,. \tag{3.27}$$

The action of this transformation on the zero mode (3.25) implies that we should identify

$$A_2(t,x^1) \simeq A_2(t,x^1) + \frac{1}{3}\frac{z'\pi}{z^{2/3}(\pi R_2) - z^{2/3}(0)} = A_2 + \frac{2g_{YM}^2}{z^{2/3}(\pi R_2) - z^{2/3}(0)}. \tag{3.28}$$

In the low-energy effective action, the large gauge transformation (3.26) acts also on the vector multiplet, implying that the identification (3.28) is accompanied by

$$\psi^-(t,x^1) \to -\psi^-(t,x^1), \qquad A_{0,1}(t,x^1) \to -A_{0,1}(t,x^1). \tag{3.29}$$

This establishes that our low-energy $1+1$ dimensional sigma model is gauged by $((-)^{F_L} \times \text{shift})$, as expected.

Next, we wish to compute the physical radius of the scalar arising from A_2 to verify that it agrees with the physical radius we expect. Carrying out the dimensional reduction explicitly (setting all other fields except the zero mode of A_2 to zero) we get

$$
\begin{aligned}
S = \frac{1}{4g_{\rm YM}^2} \int d^2x \int_0^{\pi R_2} dx^2 \mathrm{Tr}_f \Big(-z(x^2) F_{\mu\nu}F^{\mu\nu} + ... \Big) = \\
= \frac{1}{g_{\rm YM}^2} \int d^2x \int_0^{\pi R_2} dx^2 (z^{1/3}(x^2) \partial_\mu A_2 \partial^\mu A_2 + ...) = \\
= \frac{1}{g_{\rm YM}^2} \int d^2x (\partial_\mu A_2 \partial^\mu A_2) \int_0^{\pi R_2} dx^2 z^{1/3}(x^2) = \\
= \frac{\pi}{8g_{\rm YM}^4} \int d^2x (\partial_\mu A_2 \partial^\mu A_2)(z^{4/3}(\pi R_2) - z^{4/3}(0)). \quad (3.30)
\end{aligned}
$$

Thus, the physical, dimensionless radius of the scalar A_2 is

$$
\begin{aligned}
\frac{1}{2\pi} \cdot \frac{2g_{\rm YM}^2}{z^{2/3}(\pi R_2) - z^{2/3}(0)} \cdot \sqrt{\frac{\pi}{8g_{\rm YM}^4}(z^{4/3}(\pi R_2) - z^{4/3}(0))} = \\
= \frac{1}{z^{2/3}(\pi R_2) - z^{2/3}(0)} \sqrt{\frac{1}{8\pi}(z^{4/3}(\pi R_2) - z^{4/3}(0))} = \\
= \sqrt{\frac{1}{8\pi}} \sqrt{\frac{z^{2/3}(\pi R_2) + z^{2/3}(0)}{z^{2/3}(\pi R_2) - z^{2/3}(0)}}. \quad (3.31)
\end{aligned}
$$

Recall that, as discussed in the previous section, the AOB sigma model has a self-T-duality at a physical radius of $(8\pi)^{-1/2}$. We see from (3.31) that this corresponds to $z(0) = 0$, which is exactly the case where the Yang-Mills coupling diverges at one side of the interval (due to a diverging coupling on the $O8^-$ plane in the $O8^\pm$ background). As discussed in the previous section, at this point of diverging coupling the $O8^\pm$ background has an enhanced SU(2) gauge symmetry in space-time, which should correspond to an enhanced SU(2) global symmetry in our gauge theory; we see that in the low-energy effective action this enhanced global symmetry is precisely the one associated with the AOB sigma model at the self-dual radius.

In the M(atrix) theory interpretation of our gauge theory, the line $z(0) = 0$ precisely maps to the line of enhanced SU(2) symmetry of the compactification of M theory on a Klein bottle. Note that our gauge theory only makes sense for $z(0) \geq 0$, since otherwise we obtain negative kinetic terms for some fields. Thus, our M(atrix) theory description only makes sense above the self-dual line in figure 2. Of course, the theories below the line are identified by a duality with the theories above the line, so we do have a valid description for the full moduli space of Klein bottle compactifications. A similar analysis for the case of a cylinder (with no Wilson lines) again shows that infinite gauge coupling is obtained precisely on the line of enhanced SU(2) symmetry in space-time shown in figure 1.

4. Conclusions and open questions

In this paper we analyzed in detail the moduli space of nine dimensional compactifications of M theory with $\mathcal{N}=1$ supersymmetry and their M(atrix) theory descriptions. We found several surprises : the moduli space of theories with rank 2 turned out to have two disconnected components, and in order to obtain a consistent description of theories with cross-caps we had to conjecture a non-perturbative splitting of $O8^0$ planes into a D8-brane and an infinitely coupled $O8^{(-1)}$ plane. The M(atrix) theories we found are $2+1$ dimensional gauge theories on a cylinder, but generically they are rather complicated theories with a varying gauge coupling. The only case where we obtained a standard gauge theory is the case of M theory on a cylinder with a Wilson line breaking the gauge theory to SO(16) $\times$ SO(16); the simplicity of the M(atrix) theory in this case is related to the fact that (unlike all other backgrounds we discussed) this background does not have a subspace with enhanced gauge symmetry in space-time. In all other cases, the manifold with enhanced gauge symmetry in space-time is mapped in M(atrix) theory to having infinite gauge coupling on one side of the interval.

There are several interesting directions for further research. We provided some circumstantial evidence for our description of the X and $1/2X$ backgrounds, but it would be nice to obtain more evidence for this. One way to obtain such evidence would be to compactify these backgrounds on an additional circle; these backgrounds then have an F theory description, with the $O8^+$ plane becoming an unresolvable D_8 singularity [15]. One can then consider the nine dimensional limit of this background, as discussed in [41, 42], and hope to recover our picture with the D8-brane emitted into the bulk. Another possible way to study the X background is by its M(atrix) theory dual; this is given by the limit of the M(atrix) theory for the Klein bottle that we constructed in section 3 in which the circle is much smaller than the interval. It would be nice to understand the dynamics of the theory in this limit in detail in order to understand the X background better; it may be necessary for this to analyze the regime of large N with energies of order $1/N$, which is most directly related to the space-time physics.

Another possible way to study these backgrounds is by brane probes, such as D2-branes stretched between the two orientifold planes. These D2-branes are interesting also for another reason, since (at least naively) they provide the M(atrix) theory for some of the additional nine dimensional backgrounds that we did not discuss in the previous section, with the M(atrix) theory for the DP background (and the other backgrounds in the same moduli space) related to D2-branes in the X background, and the M(atrix) theory for M theory on a Möbius strip related to D2-branes in the $1/2X$ background. It would be interesting to understand these theories better; naively one obtains an anomaly from the D2-D8 fermions, and it is not clear how this is cancelled.

Another natural question involves the realization of the non-perturbative duality symmetries in the M(atrix) theory. In toroidal compactifications of M theory, such dualities are related [43] to non-trivial dualities relating electric and magnetic fields in the M(atrix) theory gauge theories. We mentioned above that at the enhanced SU(2) symmetry points in space-time the M(atrix) theory is supposed to have a non-trivial enhanced SU(2) global symmetry (which can also be seen by viewing this theory as the theory of a D2-brane in a type I' background with an enhanced symmetry); the realization of this symmetry will be discussed in detail in [44]. More generally, there is a $\mathbb{Z}_2$ duality symmetry relating the two sides of each graph in figures 1 and 2, which should map to a duality between two gauge theories in M(atrix) theory. Unfortunately, so far we have only been able to find the M(atrix) description for the large radius region of the moduli space (as described above), and we could not yet find an independent (dual) description for the small radius region. It would be interesting to investigate this further [44]; it requires continuing the moduli space of backgrounds with $O8^-$ planes beyond the point where the string coupling diverges at one of the orientifold planes, but without changing to the dual variables.

Finally, it would be interesting to generalize our results concerning the classification of backgrounds with 16 supercharges to lower dimensions (for a partial classification see [1]), and to see if there are any new components or unexplored corners of the moduli space there (as we found in nine dimensions).

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A. Spinor conventions

We summarize our conventions for spinors in $2+1$ dimensions. We choose the metric to be mostly minus, $\eta^{\alpha\beta} = \mathrm{diag}(1, -1, -1)$. The gamma matrices of SO(2, 1) satisfy

$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$, where α, β are vector indices of SO(2, 1), and the spinor indices are denoted by $a, b = 1, 2$ (but are usually suppressed). A convenient basis for SO(2, 1) gamma matrices is

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{A.1}$$

Our fermions lie in spinor representations of the global $Spin(7)_R$ symmetry. The gamma matrices of $Spin(7)_R$ satisfy $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$, where i,j are vector indices of $Spin(7)_R$. We shall denote spinor indices of $Spin(7)_R$ by A, B. Finally, for a Majorana fermion ψ, we define $\bar{\psi} = \psi^T \gamma^0$.

B. Periodicities of p-form fields in some 9d compactifications

In this appendix we discuss the periodicities of the p-form fields in some of the backgrounds described in section 2.2. We being with the AOA background [12, 13], in which one gauges the following symmetry of type IIA string theory :

$$(-)^{F_L} \times (X^9 \to X^9 + 2\pi R_9). \tag{B.1}$$

The NS-NS 2-form $B_{\mu\nu}$ is periodic along the circle in the 9'th direction:

$$B_{\mu\nu}(x^9 + 2\pi R_9) = B_{\mu\nu}(x^9) \tag{B.2}$$

and leads to a massless mode. On the other hand, the RR 3-form field $A^{(3)}$ gets a minus sign from $(-)^{F_L}$, hence it has to be antiperiodic in the direction x^9. In components,

$$A^{(3)}_{\mu\nu\rho}(x^9 + 2\pi R_9) = -A^{(3)}_{\mu\nu\rho}(x^9). \tag{B.3}$$

As discussed in section 2.2, the target space of the M theory lift of this background is $\mathbb{R}^{8,1} \times K2$, with $x^{10} \to -x^{10}$ under the involution. In eleven dimensions the involution must take $C^{M(3)} \to -C^{M(3)}$ in order to be a symmetry of M theory. We can also verify that this is consistent with the periodicities we wrote above. For $C^{M(3)}_{\mu\nu(10)}$, taking into account the coordinate transformation we get that

$$C^{M(3)}_{\mu\nu(10)}(x^9 + 2\pi R_9, -x^{10}) = C^{M(3)}_{\mu\nu(10)}(x^9, x^{10}). \tag{B.4}$$

Reducing this on x^{10} we find agreement with (B.2). The other components are all anti-periodic, in agreement with (B.3).

Next, we compactify this M theory on an additional circle $\mathbb{R}^{7,1} \times K2 \times S^1$. We can reduce on this S^1 to obtain a different IIA theory, which is the one described by (2.6). Now the fact that $C^{M(3)} \to -C^{M(3)}$ implies different periodicities in the type IIA theory. For the NS-NS 2-form

$$B^{(2)} \to -B^{(2)}, \tag{B.5}$$

which in components implies for instance

$$B^{(2)}_{\mu 8}(-x^8, x^9 + 2\pi R_9) = B^{(2)}_{\mu 8}(x^8, x^9). \tag{B.6}$$

Similarly, for the RR 3-form,

$$A^{(3)} \to -A^{(3)}. \tag{B.7}$$

These results can also be derived directly from the IIA perspective. We first note that $\Omega \times (X^8 \to -X^8)$ is a symmetry of IIA because it changes the chirality of fermions and the RR forms twice.[14] Since Ω flips its sign the B field has to be odd under the involution, in agreement with (B.5). We can check the consistency on the RR sector in the Green-Schwarz formalism. In this formalism type IIA superstrings have space-time fermions $\theta_L(z)$ on the left and $\theta_R(\bar z)$ on the right. The transformation (2.6) that produces the Klein bottle acts on the space-time fermions as

$$\theta_L(z) \to \Gamma^8 \theta_R(\bar z), \tag{B.8}$$

$$\theta_R(\bar z) \to \Gamma^8 \theta_L(z). \tag{B.9}$$

The RR vertex operator is

$$V = \theta_R(\bar z)^\dagger \Gamma_{\mu_1 \cdots \mu_p} \theta_L(z) + \text{c.c.}. \tag{B.10}$$

Under the transformation the fermionic part becomes

$$\begin{aligned} \theta_L(z)^\dagger \Gamma^8 \Gamma_{\mu_1\cdots\mu_p} \Gamma^8 \theta_R(\bar z) + \text{c.c.} &= -\theta_R(\bar z)^T (\Gamma^8)^T (\Gamma_{\mu_1\cdots\mu_p})^T (\Gamma^8)^T \theta_L(z)^* + \text{c.c.} = \\ = -\theta_R(\bar z)^\dagger \Gamma^8 (\Gamma_{\mu_1\cdots\mu_p})^\dagger \Gamma^8 \theta_L(z) + \text{c.c.} &= (-1)^{[p/2]+1} \theta_R(\bar z)^\dagger \Gamma^8 \Gamma_{\mu_1\cdots\mu_p} \Gamma^8 \theta_L(z) + \text{c.c.}. \end{aligned} \tag{B.11}$$

Thus, the 1-form transforms as a 1-form (changing sign only for the component in the 8 direction) and the 3-form transforms as a pseudo-3-form (changing sign for components not in the 8 direction). In other words, only the twisted 3-form cohomology survives, and only the untwisted cohomology of 1-forms survives (see [45] for more details[15]).

C. Modular invariance and T-duality in the AOA partition function

In this appendix we compute the partition function of the AOA theory (defined in section 2) and verify that it is modular invariant and T-duality invariant. Throughout

[14] By super-conformal invariance of the worldsheet we get that the worldsheet fermions ψ and $\widetilde{\psi}$ flip sign under this involution and hence change the chirality in the R sectors.

[15] Loosely speaking, the untwisted cohomology is the subset of the covering space cohomology which contains only the forms which are even. The twisted cohomology contains the odd forms.

this appendix we set $l_s = 1$. The subtle point is that the definition of the theory includes special phases in the twisted sectors (which are required for the consistency of the theory).

The sectors of the theory and their matter GSO projections (including $(-1)^w$) are [13]:[16]

A: $NS - NS$ sector with $n \in \mathbb{Z}, w \in 2\mathbb{Z}$ and GSO projection $-/-$.

B: $NS - R$ sector with $n \in \mathbb{Z}, w \in 2\mathbb{Z}$ and GSO projection $-/+$.

C: $R - NS$ sector with $n \in \mathbb{Z} + \frac{1}{2}, w \in 2\mathbb{Z}$ and GSO projection $-/-$.

D: $R - R$ sector with $n \in \mathbb{Z} + \frac{1}{2}, w \in 2\mathbb{Z}$ and GSO projection $-/+$.

E: $R - NS$ sector with $n \in \mathbb{Z}, w \in 2\mathbb{Z} + 1$ and GSO projection $+/-$.

F: $R - R$ sector with $n \in \mathbb{Z}, w \in 2\mathbb{Z} + 1$ and GSO projection $+/+$.

G: $NS - NS$ sector with $n \in \mathbb{Z} + \frac{1}{2}, w \in 2\mathbb{Z} + 1$ and GSO projection $+/-$.

H: $NS - R$ sector with $n \in \mathbb{Z} + \frac{1}{2}, w \in 2\mathbb{Z} + 1$ and GSO projection $+/+$.

Let Z^α_β be a path integral on the torus over fermion fields ψ with periodicities

$$\begin{aligned} \psi(w + 2\pi) &= -e^{i\pi\alpha}\psi(w), \\ \psi(w + 2\pi\tau) &= -e^{i\pi\beta}\psi(w). \end{aligned} \tag{C.1}$$

The fermionic partition sums are then ($0, v, s, c$ stand for the four possible combinations of NS and R sectors with GSO projections)

$$\begin{aligned} \chi_0 &= \frac{1}{2}\big[Z^0_0(\tau)^4 + Z^0_1(\tau)^4\big], \\ \chi_v &= \frac{1}{2}\big[Z^0_0(\tau)^4 - Z^0_1(\tau)^4\big], \\ \chi_s &= \frac{1}{2}\big[Z^1_0(\tau)^4 + Z^1_1(\tau)^4\big], \\ \chi_c &= \frac{1}{2}\big[Z^1_0(\tau)^4 - Z^1_1(\tau)^4\big]. \end{aligned} \tag{C.2}$$

[16]Note that the conventions in [13] for right/left-moving sectors are different from the usual.

The partition function on the torus that we obtain by summing over all the sectors described above is :

$$Z(\tau) = iV_{10} \int_F \frac{d^2\tau}{16\pi^2\tau_2^2} Z_X^7 |\eta(\tau)|^{-2} (\chi_v - \chi_s) \times$$
$$\times \Bigg(\bar{\chi}_v \sum_{n\in\mathbb{Z}, w\in 2\mathbb{Z}} \exp\big[-\pi\tau_2(\frac{n^2}{R^2} + w^2R^2) + 2\pi i\tau_1 nw\big]$$
$$+ \bar{\chi}_0 \sum_{n\in\mathbb{Z}+1/2, w\in 2\mathbb{Z}+1} \exp\big[-\pi\tau_2(\frac{n^2}{R^2} + w^2R^2) + 2\pi i\tau_1 nw\big]$$
$$- \bar{\chi}_s \sum_{n\in\mathbb{Z}, w\in 2\mathbb{Z}+1} \exp\big[-\pi\tau_2(\frac{n^2}{R^2} + w^2R^2) + 2\pi i\tau_1 nw\big]$$
$$- \bar{\chi}_c \sum_{n\in\mathbb{Z}+1/2, w\in 2\mathbb{Z}} \exp\big[-\pi\tau_2(\frac{n^2}{R^2} + w^2R^2) + 2\pi i\tau_1 nw\big] \Bigg), \quad \text{(C.3)}$$

where Z_X^7 is the partition sum for the 7 transverse bosons. After resummation we get

$$Z(\tau) = iV_{10} \int_F \frac{d^2\tau}{16\pi^2\tau_2^2} Z_X^8 (\chi_v - \chi_s) \sum_{m,w\in\mathbb{Z}} \Bigg(\bar{\chi}_v \exp\left[-\frac{\pi R^2 |m - 2w\tau|^2}{\tau_2}\right]$$
$$+ \bar{\chi}_0 \exp\left[-\frac{\pi R^2 |m - (2w+1)\tau|^2}{\tau_2} + \pi i m\right]$$
$$- \bar{\chi}_s \exp\left[-\frac{\pi R^2 |m - (2w+1)\tau|^2}{\tau_2}\right]$$
$$- \bar{\chi}_c \exp\left[-\frac{\pi R^2 |m - 2w\tau|^2}{\tau_2} + \pi i m\right] \Bigg). \quad \text{(C.4)}$$

From the modular transformations of Z_β^α :

$$Z_\beta^\alpha(\tau+1) = \epsilon^{\pi i(3\alpha^2-1)/12} Z_{\beta-\alpha+1}^\alpha(\tau),$$
$$Z_\beta^\alpha(-1/\tau) = Z_\alpha^{-\beta}, \quad \text{(C.5)}$$

we get the transformations of the χ's :

$$\chi_v(\tau+1) = \epsilon^{2\pi i/3} \chi_v(\tau),$$
$$\chi_0(\tau+1) = \epsilon^{-\pi i/3} \chi_0(\tau),$$
$$\chi_{s/c}(\tau+1) = \epsilon^{2\pi i/3} \chi_{s/c}(\tau),$$
$$\chi_{v/0}(-1/\tau) = \frac{1}{2}\big[Z_0^0(\tau)^4 \mp Z_0^1(\tau)^4\big],$$
$$\chi_{s/c}(-1/\tau) = \frac{1}{2}\big[Z_1^0(\tau)^4 \pm Z_1^1(\tau)^4\big]. \quad \text{(C.6)}$$

Modular invariance of (C.4) under $\tau \to \tau + 1$ is straightforward : the phase from $(\chi_v - \chi_s)$ is cancelled by the phase coming from the left-moving sector (the phase of $\bar{\chi}_0$ gets another (-1) from the transformation of the sum over momenta and windings). The modular invariance under $\tau \to -\frac{1}{\tau}$ follows from the fact that if we take $m \to -w$ and $w \to m$, the bosonic sums that multiply $(\bar{Z}_0^0)^4$ and $(\bar{Z}_1^1)^4$ do not change and the bosonic sums that multiply $(\bar{Z}_1^0)^4$ and $(\bar{Z}_0^1)^4$ are switched.

If we look at the partition function (C.3), we see that it is invariant under T-duality with $R \to \frac{1}{2R}$, $n \to \frac{w}{2}$, and $w \to 2n$. Recall that in order to verify T-duality one has to flip the GSO projection of the right-moving modes of the Ramond sector, and then the partition function is manifestly invariant (as usual, to verify this one needs to use $Z_1^1 = 0$).

D. Quantum mechanics of D0-branes on the Klein bottle

In this appendix we take a naive approach to the derivation of M(atrix) theory for M theory on a Klein bottle, by starting with D0-branes on a Klein bottle, imposing the appropriate identifications on the Chan-Paton indices, and performing a Fourier transform. Our D0-branes are moving in the background with the identification (2.6), but we will rename the directions of the Klein bottle x^1 and x^2, and denote the periodicities of these variables by $2\pi R_1'$ and $4\pi R_2'$. Our identifications of the Chan-Paton indices will differ from those in [33] (which describe a non-commutative version, as pointed out in [46]), but they are similar to those of [34]. We construct the identifications by simply moving strings with the transformation

$$(x^1, x^2) \sim (-x^1, x^2 + 2\pi R_2') \tag{D.1}$$

and then reversing their orientation. Figure 7 is an example of how we obtain the various relations.

For convenience, we should now have $2N$ D0-branes in each fundamental domain of the torus. Let X^i stand for all the transverse Euclidean directions to the Klein bottle ($i = 3, \cdots, 9$). We will write everything in components first and then try to find a more elegant formalism. In general, the indices are $\Lambda_{k,l;a,b;i,j}$ where k and l are two dimensional vectors of integers (corresponding to the periodic images of the D0-branes), a and b are in $\{1, 2\}$ and i and j define an $N \times N$ matrix. The identifications coming from the torus periodicities are the standard ones (see, e.g., [36]). For our additional identifications, define $\hat{k} = (-k_1, k_2)$, and then :

$$\begin{aligned} X^i_{k,l;1,1;i,j} &= X^i_{\hat{l},\hat{k};2,2;j,i}, \\ X^i_{k,l;2,2;i,j} = X^i_{\hat{l},\hat{k};1,1;j,i} &= X^i_{\hat{l}+e_2,\hat{k}+e_2;1,1;j,i}, \\ X^i_{k,l;1,2;i,j} &= X^i_{\hat{l}+e_2,\hat{k};1,2;j,i}, \\ X^i_{k,l;2,1;i,j} &= X^i_{\hat{l},\hat{k}+e_2;2,1;j,i}. \end{aligned} \tag{D.2}$$

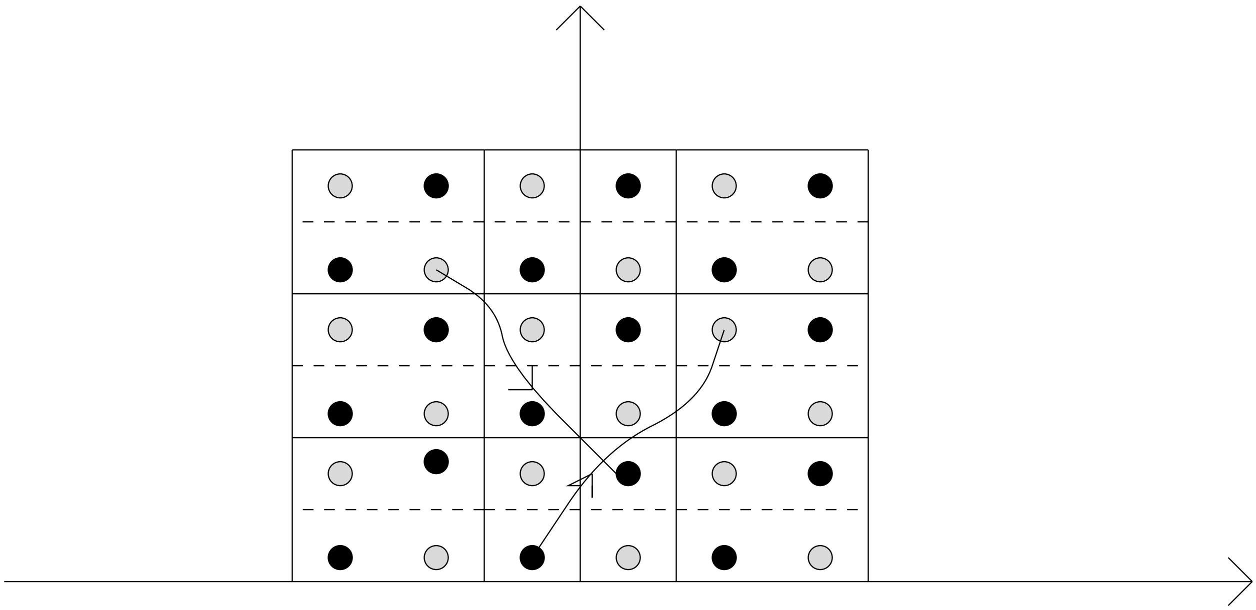

Figure 7: To demonstrate the identifications we propose, we show an explicit example of how it works. In this figure, the middle cell in the lowest row is the (0,0) cell. We act with the orientifold transformation to demonstrate the following equality $X_{(0,0),(1,1);1,2;i,j} = X_{(-1,2),(0,0);1,2;j,i} = X_{(0,0),(1,-2);1,2;j,i}$ where X is any of the orthogonal coordinates. The dashed line separates each torus fundamental domain to two copies of the Klein bottle. The filled circles are D0-branes (and the two colors correspond to two different D0-branes). The two depicted strings are identified under the transformation.

These are completely geometric identifications, similar to the particular one exhibited in figure 7. The purpose of writing the second row as it is will become clear below. One should also notice that there are no identifications on the orthogonal coordinates besides (D.2) and the usual torus identifications.

On the A^0 matrices similar identifications hold up to an additional minus sign because of the action on the vertex operator (which contains a normal derivative) :

$$\begin{gathered} A^0_{k,l;1,1;i,j} = -A^0_{\hat{l},\hat{k};2,2;j,i}, \\ A^0_{k,l;2,2;i,j} = -A^0_{\hat{l},\hat{k};1,1;j,i} = -A^0_{\hat{l}+e_2,\hat{k}+e_2;1,1;j,i}, \\ A^0_{k,l;1,2;i,j} = -A^0_{\hat{l}+e_2,\hat{k};1,2;j,i}, \\ A^0_{k,l;2,1;i,j} = -A^0_{\hat{l},\hat{k}+e_2;2,1;j,i}. \end{gathered} \tag{D.3}$$

The constraints on the coordinates X^1 and X^2 are also very easy to find by following the fate of a string after the symmetry operation acts on it. The results are summarized in the following list :

$$
\begin{aligned}
X^1_{k,l;1,1;i,j} &= -X^1_{\hat{l},\hat{k};2,2;j,i}, \\
X^1_{k,l;2,2;i,j} &= -X^1_{\hat{l},\hat{k};1,1;j,i} = -X^1_{\hat{l}+e_2,\hat{k}+e_2;1,1;j,i}, \\
X^1_{k,l;1,2;i,j} &= -X^1_{\hat{l}+e_2,\hat{k};1,2;j,i}, \\
X^1_{k,l;2,1;i,j} &= -X^1_{\hat{l},\hat{k}+e_2;2,1;j,i}, \\
X^2_{k,l;1,1;i,j} &= X^2_{\hat{l},\hat{k};2,2;j,i} - \delta_{l,k}2\pi R'_2\delta_{i,j}, \\
X^2_{k,l;2,2;i,j} &= X^2_{\hat{l},\hat{k};1,1;j,i} + \delta_{l,k}2\pi R'_2\delta_{i,j} = X^2_{\hat{l}+e_2,\hat{k}+e_2;1,1;j,i} - \delta_{l,k}2\pi R'_2\delta_{i,j}, \\
X^2_{k,l;1,2;i,j} &= X^2_{\hat{l}+e_2,\hat{k};1,2;j,i}, \\
X^2_{k,l;2,1;i,j} &= X^2_{\hat{l},\hat{k}+e_2;2,1;j,i}.
\end{aligned}
\tag{D.4}
$$

As was explained in section 3.2, we know there exists a nice T-dual description of this theory as the worldvolume of a D2-brane. It should be possible, therefore, to rearrange the very inconvenient set of identifications on the infinite matrices that we wrote above as a $2+1$ dimensional gauge theory of finite matrices.

To see this, define

$$
M_{k,k';a,a'} = \delta_{k',\hat{k}+\tau_{a,a'}e_2}\sigma^1_{a,a'}, \tag{D.5}
$$

where we have defined

$$
\tau = \frac{1}{2}\left(\sigma^1 - i\sigma^2\right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{D.6}
$$

One can easily show that

$$
(M^{-1})_{k',l;a',a} = \delta_{k',\hat{l}+\tau_{a,a'}e_2}\sigma^1_{a',a}. \tag{D.7}
$$

Now, equations (D.2),(D.3),(D.4) can be neatly rewritten in the following form :

$$
\begin{aligned}
X^i &= M(X^i)^T M^{-1}, \\
A^0 &= -M(A^0)^T M^{-1}, \\
X^1 &= -M(X^1)^T M^{-1}, \\
X^2 &= M(X^2)^T M^{-1} - 2\pi R'_2\delta_{k,l}\delta_{a,b}\delta_{i,j}.
\end{aligned}
\tag{D.8}
$$

Of course, these identifications are imposed together with the torus identifications. The equivalence of (D.8) with (D.2),(D.3),(D.4) can be demonstrated by simple calculations.

Next, we rearrange the fields as Fourier components in the k,l indices :

$$\begin{aligned}
A^0_{a,b;i,j}(x,t) &= \sum_k (A^0)_{0,k;a,b;i,j} e^{i\left(\frac{k_1x_1}{R_1}+\frac{k_2x_2}{R_2}\right)},\\
A^1_{a,b;i,j}(x,t) &= \sum_k X^1_{0,k;a,b;i,j} e^{i\left(\frac{k_1x_1}{R_1}+\frac{k_2x_2}{R_2}\right)},\\
A^2_{a,b;i,j}(x,t) &= \sum_k X^2_{0,k;a,b;i,j} e^{i\left(\frac{k_1x_1}{R_1}+\frac{k_2x_2}{R_2}\right)},\\
\forall l = 3...9 \quad X^l_{a,b;i,j}(x,t) &= \sum_k X^l_{0,k;a,b;i,j} e^{i\left(\frac{k_1x_1}{R_1}+\frac{k_2x_2}{R_2}\right)},
\end{aligned} \tag{D.9}$$

where $R_1 \equiv l_s^2/R_1'$ and $R_2 \equiv l_s^2/2R_2'$. The Lagrangian governing these fields is given by a maximally supersymmetric (after including the fermions) Yang-Mills theory in $2+1$ dimensions (16 real supercharges), which is the dimensional reduction of the ten dimensional $\mathcal{N}=1$ SYM theory. However, there are some inter-relations among the fields which we now derive. For convenience we suppress the i,j indices. The inter-relations comprise the following set of additional relations on the theory with 16 supercharges ($j=3,\cdots,9$) :

$$\begin{aligned}
X^j_{1,1}(x,t) &= (X^j_{2,2}(-\hat{x},t))^T,\\
X^j_{1,2}(x,t) &= e^{-i\frac{x_2}{R_2}}(X^j_{1,2}(-\hat{x},t))^T,\\
X^j_{2,1}(x,t) &= e^{i\frac{x_2}{R_2}}(X^j_{2,1}(-\hat{x},t))^T,\\
A^0_{1,1}(x,t) &= -(A^0_{2,2}(-\hat{x},t))^T,\\
A^0_{1,2}(x,t) &= -e^{-i\frac{x_2}{R_2}}(A^0_{1,2}(-\hat{x},t))^T,\\
A^0_{2,1}(x,t) &= -e^{i\frac{x_2}{R_2}}(A^0_{2,1}(-\hat{x},t))^T,\\
A^1_{1,1}(x,t) &= -(A^1_{2,2}(-\hat{x},t))^T,\\
A^1_{1,2}(x,t) &= -e^{-i\frac{x_2}{R_2}}(A^1_{1,2}(-\hat{x},t))^T,\\
A^1_{2,1}(x,t) &= -e^{i\frac{x_2}{R_2}}(A^1_{2,1}(-\hat{x},t))^T,\\
A^2_{1,1}(x,t) &= (A^2_{2,2}(-\hat{x},t))^T - 2\pi R_2'\delta_{i,j},\\
A^2_{1,2}(x,t) &= e^{-i\frac{x_2}{R_2}}(A^2_{1,2}(-\hat{x},t))^T,\\
A^2_{2,1}(x,t) &= e^{i\frac{x_2}{R_2}}(A^2_{2,1}(-\hat{x},t))^T.
\end{aligned} \tag{D.10}$$

We see that our theory can be written on a cylinder of volume $(2\pi R_1)\times(\pi R_2)$ with orientifold planes at the boundaries. We can read the relevant boundary conditions from (D.10). As the action appears to be an orbifold of the maximally supersymmetric action we write it explicitly :

$$S = \frac{1}{2g^2_{\rm YM}} \int_{\mathbb{T}^2} dt d^2x \mathrm{Tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + (D_\mu X^j)^2 + \frac{1}{2}[X^j,X^i][X^j,X^i] + \text{fermions}\right). \tag{D.11}$$

The size of $\mathbb{T}^2$ is $(2\pi R_1)\times(2\pi R_2)$. All the fields are periodic and in our case take the form

$$X=\begin{pmatrix}X_{1,1;i,j} & X_{1,2;i,j}\\ X_{2,1;i,j} & X_{2,2;i,j}\end{pmatrix},\qquad A=\begin{pmatrix}A_{1,1;i,j} & A_{1,2;i,j}\\ A_{2,1;i,j} & A_{2,2;i,j}\end{pmatrix},\qquad i,j=1,\cdots,N. \tag{D.12}$$

We split the action :

$$S=\frac{1}{2g_{\rm YM}^2}\int dt\int_0^{2\pi R_1}dx^1\int_{-\pi R_2}^{0}dx^2\,{\rm Tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+(D_\mu X^j)^2+\frac{1}{2}[X^j,X^i][X^j,X^i]\right)+$$
$$+\frac{1}{2g_{\rm YM}^2}\int dt\int_0^{2\pi R_1}dx^1\int_0^{\pi R_2}dx^2\,{\rm Tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+(D_\mu X^j)^2+\frac{1}{2}[X^j,X^i][X^j,X^i]\right) \tag{D.13}$$

Using (D.10) we can reexpress all the fields in the domain $[-\pi R_2,0]$ as fields in the domain $[0,\pi R_2]$. The computation becomes very easy if one notices that the set of equations (D.10) is actually simple, $\forall j=3,\cdots,9$:

$$\begin{aligned}X^j(x,t)&=e^{-\frac{ix^2}{2R_2}\sigma^3}\sigma^1(X^j(-\hat{x},t))^T\sigma^1e^{\frac{ix^2}{2R_2}\sigma^3},\\ A^{0(1)}(x,t)&=-e^{-\frac{ix^2}{2R_2}\sigma^3}\sigma^1(A^{0(1)}(-\hat{x},t))^T\sigma^1e^{\frac{ix^2}{2R_2}\sigma^3},\\ A^2(x,t)&=e^{-\frac{ix^2}{2R_2}\sigma^3}\sigma^1(A^2(-\hat{x},t))^T\sigma^1e^{\frac{ix^2}{2R_2}\sigma^3}-2\pi R_2'\sigma^3,\end{aligned} \tag{D.14}$$

where the relations are on the full $2N\times 2N$ matrices. The sigma matrices are always tensored with the unit matrix of dimension $N\times N$. One can see that this gives a nice Lagrangian, as most of the commutators are invariant, or change sign. One easy way to proceed is to rewrite the above lagrangian with new fields in the following way :

$$\begin{aligned}\widetilde{X}^j(x,t)&=e^{\frac{ix^2\sigma^3}{4R_2}}X^j(x,t)e^{-\frac{ix^2\sigma^3}{4R_2}},\\ \widetilde{A}^{0(1)}(x,t)&=e^{\frac{ix^2\sigma^3}{4R_2}}A^{0(1)}(x,t)e^{-\frac{ix^2\sigma^3}{4R_2}},\\ \widetilde{A}^2(x,t)&=e^{\frac{ix^2\sigma^3}{4R_2}}A^2(x,t)e^{-\frac{ix^2\sigma^3}{4R_2}}+\sigma^3\pi R_2'.\end{aligned} \tag{D.15}$$

This transformation is a U($2N$) gauge transformation. Hence the Lagrangian of the tilded fields is actually the same Lagrangian as (D.13). We will still name those fields in the usual notation omitting the tildes. The identification of the tilded fields[17] is very suggestive:

$$\begin{aligned}X^j(x,t)&=\sigma^1(X^j(-\hat{x},t))^T\sigma^1,\\ A^{0(1)}(x,t)&=-\sigma^1(A^{0(1)}(-\hat{x},t))^T\sigma^1,\\ A^2(x,t)&=\sigma^1(A^2(-\hat{x},t))^T\sigma^1.\end{aligned} \tag{D.16}$$

[17]As mentioned, we won't use the notation of the tildes anymore.

Using this transformations we transform the $[-\pi R_2, 0]$ interval of the action to $[0, \pi R_2]$, and we find that it precisely agrees with the original action for this interval. The relative minus signs in the gauge fields exactly compensate for the minus signs that arise because $[A^T, B^T] = -([A,B])^T$ and because $\partial_2 \to -\partial_2$. So, our final result is

$$S = \frac{1}{2g_{\rm YM}^2} \int dt \int_0^{2\pi R_1} dx^1 \int_0^{\pi R_2} dx^2 \, \mathrm{Tr} \Big(-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + (D_\mu X^j)^2 + \\ + \frac{1}{2} [X^j, X^i][X^j, X^i] + \text{fermions} \Big). \quad \text{(D.17)}$$

We will determine $g_{\rm YM}$ soon, so we omitted the factor 2 for now.[18] The boundary conditions are interesting. Equations (D.15) tell us that

$$X^j(x^1, x^2 + 2\pi R_2, t) = \sigma^3 X^j(x^1, x^2, t)\sigma^3, \\ A^{0,1,2}(x^1, x^2 + 2\pi R_2, t) = \sigma^3 A^{0,1,2}(x^1, x^2, t)\sigma^3. \quad \text{(D.18)}$$

Combined with equations (D.16) we obtain

$$X^j(x^1, -x^2 + 2\pi R_2, t) = \sigma^2 (X^j(x,t))^T \sigma^2, \\ A^{0,1}(x^1, -x^2 + 2\pi R_2, t) = -\sigma^2 (A^{0,1}(x^1, x^2, t))^T \sigma^2, \\ A^2(x^1, -x^2 + 2\pi R_2, t) = \sigma^2 (A^2(x^1, x^2, t))^T \sigma^2. \quad \text{(D.19)}$$

The boundary conditions at the two orientifold planes are, hence, different.[19] For $x^2 = 0$ the matrices satisfy

$$X^j = \sigma^1 (X^j)^T \sigma^1, \qquad \partial_2 X^j = -\sigma^1 (\partial_2 X^j)^T \sigma^1, \quad \text{(D.20)}$$

$$A^{0(1)} = -\sigma^1 (A^{0(1)})^T \sigma^1, \qquad \partial_2 A^{0(1)} = \sigma^1 (\partial_2 A^{0(1)})^T \sigma^1, \quad \text{(D.21)}$$

$$A^2 = \sigma^1 (A^2)^T \sigma^1, \qquad \partial_2 A^2 = -\sigma^1 (\partial_2 A^2)^T \sigma^1, \quad \text{(D.22)}$$

so they are of the form

$$A^2\ ,\ X^j = \begin{pmatrix} N & S \\ \tilde{S} & N^T \end{pmatrix} \qquad N^+ = N,\ S^T = S,\ \tilde{S}^T = \tilde{S},\ S^+ = \tilde{S}, \\ A^0\ ,\ A^1 = \begin{pmatrix} M & R \\ \tilde{R} & -M^T \end{pmatrix} \qquad M^+ = M,\ R^T = -R,\ \tilde{R}^T = -\tilde{R},\ R^+ = \tilde{R}. \quad \text{(D.23)}$$

Collecting all these facts together we get that the gauge group in the bulk is U$(2N)$ while on the $x^2 = 0$ boundary the gauge group is SO$(2N)$. On this boundary the fields A^2, X^j are in the symmetric representation of the gauge group.

[18] The Lagrangian (D.17) on the cylinder defines what we mean by $g_{\rm YM}$ so there is no ambiguity.
[19] These are the anticipated O^- and O^+ planes.

The second boundary $x^2 = \pi R_2$ is substantially different. We write the boundary conditions there in a familiar form

$$\sigma^2 X^j \sigma^2 = (X^j)^T, \qquad \sigma^2 \partial_2 X^j \sigma^2 = -(\partial_2 X^j)^T, \tag{D.24}$$

$$\sigma^2 A^{0,1} \sigma^2 = -(A^{0,1})^T, \qquad \sigma^2 \partial_2 A^{0,1} \sigma^2 = (\partial_2 A^{0,1})^T, \tag{D.25}$$

$$\sigma^2 A^2(x^1,t) \sigma^2 = (A^2(x^1,t))^T, \qquad \sigma^2 \partial_2 A^2 \sigma^2 = -(\partial_2 A^2)^T. \tag{D.26}$$

Consequently, the gauge group at this boundary is USp($2N$). The most general matrices are of the form

$$A^2\ ,\ X^j = \begin{pmatrix} N & S \\ \tilde{S} & N^T \end{pmatrix} \qquad N^+ = N,\ S^T = -S,\ \tilde{S}^T = -\tilde{S},\ S^+ = \tilde{S},$$

$$A^0\ ,\ A^1 = \begin{pmatrix} M & R \\ \tilde{R} & -M^T \end{pmatrix} \qquad M^+ = M,\ R^T = R,\ \tilde{R}^T = \tilde{R},\ R^+ = \tilde{R}. \tag{D.27}$$

It is easy to verify that this precisely agrees with the action and boundary conditions that we wrote down in the main text for the D2-brane between two orientifold planes, if we only include the disk contributions to this action and not the Möbius strip contributions.

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