Inverse β-decay: a twin-model with boson fields

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Abstract. In line with a series of previous works, in this paper we discuss the inverse β-decay of accelerated protons $p \to n + e^+ + \nu_e$. To make the calculation of the transition rate easier to handle, we focus on an ideal "twin-process", assuming the counterparts of lepton fields to be scalars. The rate of $p \to n$ conversions is evaluated in both the laboratory and comoving frames, showing that the two results perfectly agree due to the Unruh effect. In spite of the minimal setting, we stress that this model is an attempt to wash technical difficulties out of the analysis of the inverse β-decay. The future goal is to investigate the concerns recently raised in literature about the compatibility of the two rates when *neutrino mixing* is taken into account.

1. Introduction

The analysis of the inverse β-decay [1, 2] has proved to be enlightening for solving a puzzling question: is the Unruh effect [3] really mandatory for the consistency of quantum field theory (QFT) in curved background? As it is reasonable to expect, the decay rate of an accelerated particle in the laboratory and comoving frames must coincide. This outcome is prompted by the requirement of general covariance for the underlying formalism, which is built so that scalar quantities, such as the mean proper lifetime (i.e. the inverse of the decay rate) are invariant, and thus equal in any reference frame. In Refs. [1, 2], it was shown that this equality does indeed hold only when considering the inertial vacuum to be a thermal bath for the accelerated observer due to the Unruh effect, solving in this way the aforementioned dilemma.

Just when the issue of the inverse β-decay appeared to have been definitively settled, it hit again the headlines, becoming even more appealing. Indeed, if on the one hand in Refs. [1, 2] it was remarked that the decay properties of particles are not a frame-independent concept (according to what discussed in Ref. [4]), on the other hand the analysis there proposed completely overlooked *flavor mixing* of neutrinos, thus providing only a partial treatment of the problem.

The ingredient of mixing in the context of the accelerated proton decay was firstly introduced in Ref. [5] and later discussed in Refs. [6, 7, 8]. In Ref. [5], the authors highlighted a disagreement between the rates in the inertial and comoving frames, concluding that the contradiction had to be solved experimentally. On the other side, in Refs. [6, 7], such a paradox was overcome on purely theoretical basis, although conflicting approaches were taken into account. In Ref. [8], however, some criticism was exposed on the statements of Ref. [7], which was shown to fail to pinpoint the peculiar aspects of mixing in the problem under consideration.

Notwithstanding the clarification of the theoretical dichotomy, it is evident that the analysis of Ref. [6] needs further improvement, since it applies only within a particular approximation. So far, however, the non-triviality of calculations has prevented any development in this direction.

Aware of these technical difficulties, in the present paper we analyze a toy model[1] in which the inverse β-decay is addressed treating the electron and neutrino as scalar fields. By use of this minimalistic approach, we evaluate the rate of $p \to n$ conversions in both the laboratory and comoving frames, showing that the agreement is not affected by our assumption at all. In spite of the simplified setting, it should be emphasized that this model must be considered as a first step toward a more subtle investigation of the decay of accelerated particles in the context of flavor mixing. In this way, we hope to lay down the foundations for definitively clarifying the ambiguities recently raised in literature [5, 7].

The paper is organized as follows: In Sec. II we review the standard quantization of the scalar field in Minkowski and Rindler spacetimes. In Sec. III we discuss an inverse β decay-like process, showing that the outcome mimics the realistic case of the decay of an accelerated proton. The last Section is devoted to conclusions and outlook. Throughout the paper, we use natural units $\hbar = c = 1$ and the metric with the conventional timelike signature.

2. Quantization of the complex scalar field in Minkowski and Rindler spacetimes

Before going into the details of calculations, it is worth reviewing the quantization of a complex scalar field $\hat{\phi}$ in Minkowski and Rindler spacetimes [9]. In $3 + 1$ dimensions, the field expansion for an inertial (Minkowski) observer reads

$$\hat{\phi}(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[\hat{a}_k e^{-ik\cdot x} + \hat{b}_k^\dagger e^{ik\cdot x} \right], \tag{1}$$

where $\omega_k = \sqrt{|\mathbf{k}|^2 + m^2}$, m is the mass of the field and $\hat{a}_k$ ($\hat{b}_k$) are the annihilators for Minkowski particles (antiparticles) with momentum $\mathbf{k}$. The plane waves in Eq. (1) satisfy the KG equation

$$\left(\Box + m^2 \right) \phi(x) = 0. \tag{2}$$

In order to analyze the quantization procedure in Rindler spacetime, it is useful to express the coordinates appropriate to a uniformly accelerated (Rindler) observer in terms of the Minkowski ones. Choosing to leave the coordinates $\mathbf{x}_\perp \equiv (x, y)$ unchanged, namely, allowing the acceleration to have only the z component, we have

$$t = u \sinh v, \quad z = u \cosh v, \quad \mathbf{x}_\perp = \mathbf{x}'_\perp, \tag{3}$$

where $v \in (-\infty, \infty)$ is the time of the Rindler observer and $u \in (0, \infty)$, $\mathbf{x}'_\perp \in (-\infty, \infty)$ its spatial coordinates. In this way, the Minkowski metric becomes

$$g_{\mu\nu} = \operatorname{diag}\left(u^2, -1, -1, -1 \right). \tag{4}$$

We stress at this point that the world line of a Rindler observer with proper acceleration a is obtained by requiring $u = const = a^{-1}$ [10]. The field expansion in the right Rindler wedge $(z > |t|)$ takes the form [1, 2, 10]

$$\hat{\phi}(x) = \int_0^\infty d\omega \int d^2k \left[\hat{\alpha}_\mathrm{w} \phi_\mathrm{w}^{(\omega)}(x) + \hat{\beta}_\mathrm{w}^\dagger \phi_\mathrm{w}^{(-\omega)}(x) \right], \tag{5}$$

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[1] We refer to the simplified process described by this toy model as "twin process" for the standard inverse $\beta-$decay.

where $\mathrm{w} \equiv (\omega, \mathbf{k}_\perp)$ and $\omega \in (0, \infty)$ is the Rindler proper frequency. Here $\hat{\alpha}_\mathrm{w}$ $(\hat{\beta}_\mathrm{w})$ are the annihilators for Rindler particles (antiparticles) with frequency ω and transversal momentum $\mathbf{k}_\perp \equiv (k_x, k_y)$. The modes $\phi_\mathrm{w}^{(\pm\omega)}(x)$ are solutions of the KG equation in Rindler coordinates [10]:

$$\phi_\mathrm{w}^{(\omega)}(x) \;=\; \frac{1}{2\,\pi^2} \sqrt{\frac{\sinh\frac{\pi\,\omega}{a}}{a}}\, e^{-i\frac{\omega}{a} + i\,\mathbf{k}_\perp \cdot \mathbf{x}_\perp}\, K_{\frac{i\,\omega}{a}}(l\,u)\,, \tag{6}$$

where $l = \sqrt{|\mathbf{k}_\perp|^2 + m^2}$ and $K_{i\omega/a}(l\,u)$ is the modified Bessel function of the second kind. They are normalized with respect to the scalar product in Rindler coordinates [10, 11].

3. Inverse β-decay: an ideal "twin-model"

Let us now apply the above formalism to the calculation of the inverse β-decay rate. It is known that, according to the Standard Model, protons are stable. However, this is true only in the inertial case: for non-inertial protons, the accelerating field provides the rest energy difference between initial and final states, giving rise to the following process in the laboratory frame:

$$p \;\to\; n + e^+ + \nu_e\,. \tag{7}$$

In the original treatment of this decay, a semiclassical approximation is adopted in which the proton and neutron are seen as the ground and excited states of the nucleon Hamiltonian [1], while leptons are quantized in the usual way.

To make our analysis as transparent as possible, we can define in complete analogy with the above situation an ideal "twin-process", according to

$$g \;\to\; e + \bar{a} + b\,, \tag{8}$$

where the two-level system emulating the relation between p and n is now composed by two generic particles g and e, respectively[2]:

$$\hat{H}\,|g\rangle \;=\; m_g\,|g\rangle, \quad \hat{H}\,|e\rangle \;=\; m_e\,|e\rangle, \quad m_g < m_e\,, \tag{9}$$

and a, b are the counterparts of the electron and neutrino, which we assume for simplicity to be scalar fields. For small accelerations, one can show that the interaction action for the process Eq. (8) is [11]

$$\hat{S}_I \;=\; \int d^4x\,\sqrt{-g}\,\hat{Q}\left(\hat{\phi}_a^\dagger\,\hat{\phi}_b + \hat{\phi}_b^\dagger\,\hat{\phi}_a\right), \tag{10}$$

where $\hat{Q}$ is the Hermitian monopole

$$\hat{Q} \;=\; e^{i\hat{H}\tau}\,\hat{q}\,e^{-i\hat{H}\tau}\,\delta(x)\,\delta(y)\,\delta\!\left(u - a^{-1}\right), \tag{11}$$

and $\lambda = |\langle g|\hat{q}|e\rangle|$ the effective coupling constant. Notice that the δ-functions reflect the fact that the particle g has a non-vanishing acceleration only along the z-axis.

From the point of view of the uniformly accelerated frame, the situation is significantly different from the one described above, being in this case the particle g at rest. With reference to the inverse β-decay, it has been shown that the $p \to n$ conversion becomes allowed due to the interaction of the proton with a flux of Rindler electrons and neutrinos popping out from the Unruh thermal bath, according to the following three processes [1, 2]:

$$(i)\quad p^+ + e^- \;\to\; n + \nu_e\,, \qquad (ii)\quad p^+ + \bar{\nu}_e \;\to\; n + e^+\,, \qquad (iii)\quad p^+ + e^- + \bar{\nu}_e \;\to\; n\,. \tag{12}$$

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[2] We remark that the labels g and e stand for *ground* and *excited* states, respectively.

Similarly, in our model we can suppose that, owing to the Unruh effect, the decay $g \to e$ occurs via the three following channels:

$$(i) \quad g + a \to e + b, \qquad (ii) \quad g + \bar{b} \to e + \bar{a}, \qquad (iii) \quad g + a + \bar{b} \to e. \qquad (13)$$

Thus, our concern is to check whether the inertial and comoving decay rates match under our assumptions, in order for General Covariance to be preserved.

3.1. Inertial frame calculation

Starting from the interaction action Eq. (10), the tree-level transition amplitude for the process Eq. (8) reads

$$\mathcal{A}_{in} = \langle e| \otimes \langle \bar{a}\, b| \, \hat{S}_I \, |0\rangle \otimes |g\rangle. \qquad (14)$$

Using the definition Eq. (11) of the Hermitian monopole, we obtain

$$\mathcal{A}_{in} = \lambda \int dt\, dz\, e^{i\Delta m \tau} \delta\left(u - a^{-1}\right) \langle \bar{a}\, b| \, \hat{\phi}_b^\dagger \, \hat{\phi}_a \, |0\rangle, \qquad (15)$$

where $\Delta m \equiv m_e - m_g$ and τ is the proper time. By means of the properties of the δ-function, Eq. (15) can be rewritten as

$$\mathcal{A}_{in} = \frac{\lambda}{2(2\pi)^3 \sqrt{\omega_a \omega_b}} \int_{-\infty}^{+\infty} d\tau\, e^{i\left[\Delta m \tau + (\omega_a + \omega_b)\sinh(a\tau)/a - \left(k_a^z + k_b^z\right)\cosh(a\tau)/a\right]}, \qquad (16)$$

where we have used the field expansion Eq. (1). The differential transition rate is now defined as

$$\frac{d^6 \mathcal{P}_{in}}{d^3 k_a d^3 k_b} \equiv |\mathcal{A}_{in}|^2 = \frac{\lambda^2}{4(2\pi)^6 \omega_a \omega_b} \int_{-\infty}^{+\infty} d\tau_1\, e^{i\left[\Delta m \tau_1 + (\omega_a + \omega_b)\sinh(a\tau_1)/a - \left(k_a^z + k_b^z\right)\cosh(a\tau_1)/a\right]}$$

$$\times \int_{-\infty}^{+\infty} d\tau_2\, e^{-i\left[\Delta m \tau_2 + (\omega_a + \omega_b)\sinh(a\tau_2)/a - \left(k_a^z + k_b^z\right)\cosh(a\tau_2)/a\right]}. \qquad (17)$$

To solve the integrals over proper time, we perform the transformation $\tau_{1/2} = s \pm \xi/2$, so that Eq. (17) becomes

$$\frac{d^6 \mathcal{P}_{in}}{d^3 k_a d^3 k_b} = \frac{\lambda^2}{2(2\pi)^6 \omega_a \omega_b} \int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} d\xi\, e^{i\left\{\Delta m \xi + 2a^{-1}\sinh(a\xi/2)\left[(\omega_a + \omega_b)\cosh(as) - \left(k_a^z + k_b^z\right)\sinh(as)\right]\right\}}. \qquad (18)$$

With the further boost-transformations along the z-axis, namely $\omega'_\alpha = \omega_\alpha \cosh(as) - k_\alpha^z \sinh(as)$ and $k'^z_\alpha = -\omega_\alpha \sinh(as) + k_\alpha^z \cosh(as)$, $\alpha = \{a, b\}$, the integral over s can be singled out, giving the total proper time T of the two-level system made up of g and e.

Relabeling $\omega'_\alpha \to \omega_\alpha$, $k'^z_\alpha \to k_\alpha^z$, from Eq. (18) we now obtain

$$\frac{1}{T}\frac{d^6 \mathcal{P}_{in}}{d^3 k_a d^3 k_b} = \frac{\lambda^2}{2(2\pi)^6 \omega_a \omega_b} \int_{-\infty}^{+\infty} d\xi\, e^{i\left[\Delta m \xi + 2a^{-1}(\omega_a + \omega_b)\sinh(a\xi/2)\right]}, \qquad (19)$$

which can be solved in terms of the modified Bessel function of the second kind [12]. By taking into account the result of the above integral and the general definition of decay rate

$$\Gamma = \frac{\mathcal{P}}{T}, \qquad (20)$$

we are finally able to write the following integral expression for the inertial decay rate

$$\Gamma_{in} = \frac{2\,\lambda^2\, e^{-\frac{\pi \Delta m}{a}}}{a(2\pi)^6} \int \frac{d^3 k_a}{\omega_a} \int \frac{d^3 k_b}{\omega_b} \, K_{2i\Delta m/a}\left(2\,\frac{\omega_a + \omega_b}{a}\right). \qquad (21)$$

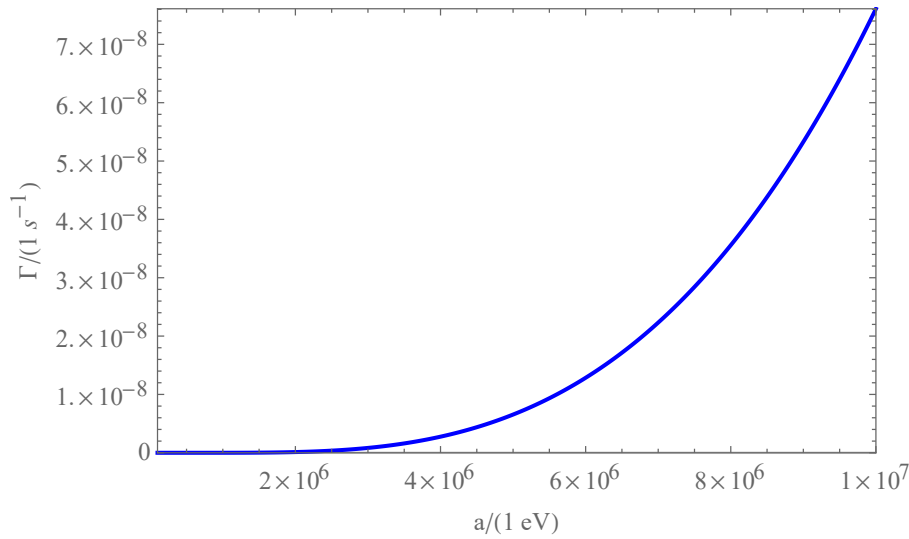The plot of Γ_{in} versus the acceleration is shown in the figure below.

Figure 1. Plot of the decay rate as a function of the proper acceleration, for $m_a = 0.5\,\mathrm{MeV}$, $m_b = 0.1\,\mathrm{eV}$, $\Delta m = 1\,\mathrm{MeV}$ and $\lambda = 10^{-12}\,\mathrm{eV}^{-1}$. For $a \to 0$, one has $\Gamma \to 0$, which means that the particle g would be stable in absence of acceleration, as expected.

3.2. Comoving frame calculation

We now turn to the calculation of the decay rate in the comoving frame. For simplicity, we only deal with the process (i) in Eq. (13): similar considerations can be straightforwardly extended to the other two transitions.

In line with Eq. (14), the tree-level transition amplitude for the process (i) can be written as

$$\mathcal{A}_{(i)} \equiv \langle e| \otimes \langle b| \, \hat{S}_I \, |a\rangle \otimes |g\rangle = \frac{\lambda}{a} \int dv \, du \, e^{i\Delta m \tau} \delta\left(u - a^{-1}\right) \langle b| \, \hat{\phi}_b^\dagger \, \hat{\phi}_a \, |a\rangle. \tag{22}$$

Using the Rindler expansion Eq. (5) for the fields $\hat{\phi}_a$ and $\hat{\phi}_b$, we get

$$\mathcal{A}_{(i)} = \frac{\lambda \sqrt{\sinh\left(\frac{\pi \omega_a}{a}\right) \sinh\left(\frac{\pi \omega_b}{a}\right)}}{2\pi^3 a} \delta\left(\omega_a - \omega_b - \Delta m\right) K_{i\omega_a/a}\left(\frac{l_a}{a}\right) K_{i\omega_b/a}\left(\frac{l_b}{a}\right), \tag{23}$$

with l_α defined as in the previous section for each of the two fields.

There is now a subtle point to be considered: as mentioned above, in the comoving frame g interacts with a thermal bath of Rindler particles, giving rise to the processes Eq. (13). Hence, we must take account of this effect including the proper bosonic thermal factor of absorption/emission in the evaluation of the decay rate. For the process (i), we have

$$\frac{d^6 \mathcal{P}_{(i)}}{d\omega_a d\omega_b d^2 k_a d^2 k_b} = |\mathcal{A}_{(i)}|^2 \, n_B\left(\omega_a\right) \left[1 + n_B\left(\omega_b\right)\right], \tag{24}$$

where $n_B(\omega)$ is the Unruh thermal distribution for bosonic fields

$$n_B(\omega) = \frac{1}{e^{2\pi\omega/a} - 1}. \tag{25}$$

By inserting Eqs. (23) and (25) into Eq. (24) and recalling that the total proper lifetime of the two-level system is $T = 2\pi\delta(0)$, it follows that

$$\frac{1}{T} \frac{d^6 \mathcal{P}_{(i)}}{d\omega_a d\omega_b d^2 k_a d^2 k_b} = \frac{\lambda^2 \, e^{-\frac{\pi \Delta m}{a}}}{32\pi^7 a^2} \delta\left(\omega_a - \omega_b - \Delta m\right) K_{i\omega_a/a}^2\left(\frac{l_a}{a}\right) K_{i\omega_b/a}^2\left(\frac{l_b}{a}\right). \tag{26}$$

The decay rate can thus be obtained using the definition Eq. (20)

$$\Gamma_{(i)} = \frac{\lambda^2 e^{-\frac{\pi \Delta m}{a}}}{32\pi^7 a^2} \int_{\Delta m}^{+\infty} d\omega \int d^2 k_a \int d^2 k_b \, K^2_{i\omega/a}\left(\frac{l_a}{a}\right) K^2_{i(\omega-\Delta m)/a}\left(\frac{l_b}{a}\right). \tag{27}$$

If we now repeat similar calculations for the processes (ii) and (iii) and add up the three contributions, we finally get the following integral expression for the total (tree-level) decay rate:

$$\Gamma_{acc} \equiv \Gamma_{(i)} + \Gamma_{(ii)} + \Gamma_{(iii)} = \frac{\lambda^2 e^{-\frac{\pi \Delta m}{a}}}{32\pi^7 a^2} \int_{-\infty}^{+\infty} d\omega \int d^2 k_a \int d^2 k_b \, K^2_{i\omega/a}\left(\frac{l_a}{a}\right) K^2_{i(\omega-\Delta m)/a}\left(\frac{l_b}{a}\right). \tag{28}$$

The plot of Γ_{acc} as a function of the acceleration is identical to the one obtained in the inertial frame[3], as it can be seen from Fig. 1. Thus, the obtained result confirms the internal consistency of our theoretical model, in spite of its minimal setting.

4. Conclusions

In this paper, we have introduced a toy model for the investigation of the inverse β-decay of accelerated protons. To make the calculation of the transition rate as streamlined as possible, we have treated the electron and neutrino as scalar fields. Retracing the original steps of Ref. [1], we have shown that the General Covariance of QFT in curved background is preserved also within our simplified framework, provided that the Unruh effect is taken into account.

Clearly, this is just a preliminary study: as a next step, we plan to examine our formalism in the context of neutrino *flavor mixing*. Indeed, as discussed in Ref. [6], there is still an open debate on how to accommodate mixing effects in the analysis of the inverse β-decay without spoiling the agreement of the two rates, also in the light of a possible non-thermality of Unruh radiation arising in that case [13]. In particular, to clarify the ambiguities recently raised in literature [5, 7], the aim is to go beyond the approximation considered in Ref. [6], an effort that has not been successful so far due to the complexity of calculations. Further study in this direction is in progress [14].

Acknowledgements

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[3] Besides the numerical evaluation, it should be noted that the agreement between the two decay rates can also be proved analytically (see, for example, Ref. [2] for the case of the accelerated proton decay).