Non Perturbative Phenomena In Field Theory

Thesis for the Degree of
Doctor of Philosophy

by

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to My Parents
and to Ida

Tyger! Tyger! burning bright
In the forests of the night,
What immortal hand or eye
Could frame thy fearful symmetry?

—William Blake

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Introduction

The work done during my Ph.D. studies is mainly concerned with the investigation of certain non perturbative phenomena in field theories. Two main such phenomena were investigated. These are the fractional charge acquired by topological excitations when they are coupled to fermions, and the strong coupling limit of gauge theories in two dimensions using various bosonization schemes.

The fractional charge phenomenon was first discovered by Jackiw and Rebbi (1976). They noticed that both a monopole or a soliton acquire a half integral charge when they are coupled to fermions. It was rather shocking that a theory containing only integral fermion charges, would predict fractional charges for some particles. This was particularly interesting because such monopoles are a necessary consequence of the various grand unification schemes. It was later observed (Goldstone et al. 1981) that for a massive fermion we may get a transcendental fermion number. For

$$\mathcal{L} = \bar{\psi}(i\, \partial\!\!\!/ - m - ig\gamma_5\phi)\psi, \qquad \text{where } \phi(\infty) = -\phi(-\infty) = \nu,$$

we get for the vacuum charge:

$$q = \frac{1}{\pi}\arctan\frac{g\nu}{m}.$$

The above lagrangian is for the two dimensional system. In the 4–dimensional case, one gets the same value for the charge. This peculiar equality was explained by us from two different points of view (Frishman et al. 1983; 1984). In the first work a connection between fractional charge and anomalous commutators was found. Analogous expressions for the anomalous commutators of the two theories led to similar results for the charges. Thus some of the enigma was explained. Also an important connection between the anomaly and the fractional charge was made, thus relating these two important aspects of field theory.

In the second paper (Frishman et al. 1984) the connection between two and four dimensional theories was directly explored. It was proven that the $j = L+S+I = 0$ sector of the monopole theory is equivalent to the soliton theory, and that the $j = 0$ sector dominates the whole contribution to the charge for a heavy monopole. We have also treated the resulting "half fermions" separately showing that different fractional charges arise for them due to different boundary conditions.

Chapter one consists of a review of the fractional charge phenomena, including our results in the area.

In the subject of bosonization and gauge theories, two works have been done. In the first work (Cohen et al. 1983) we have performed bosonization of QCD_2 in a theory with many flavors, all in the fundamental representation of color $SU(N)$. Bosonization is a technique in which a fermion theory is mapped into a bosonic one (Coleman 1976). Its importance follows from the fact that in many cases the resulting theory is simpler. Also, the weak coupling regime of the bosonic theory in many cases is mapped into the strong one of the fermionic theory. Examples are the massive Schwinger model (Coleman et al. 1975, Coleman 1976), and QCD_2 with one flavor (Baluni 1980, Steinhardt 1980).

Using the abelian bosonization we obtained the bosonic form of multi–flavor QCD (Cohen et al. 1983). The application of the abelian bosonization to gauge theories is described in chapter two. It is shown that the bosonic hamiltonian contains a nonlocal term $V(\phi_i, \pi_i)$ of high degree of complexity (it has integrals of $\pi_i - \pi_j$). We then apply the strong coupling limit of $g \to \infty$ in order to get the low lying spectrum. In view of the nonlocal interaction, we also took the static approximation, $\pi \to 0$. Then the states obtained are not in multiplets of isospin. Hence we conclude that the static approximation is not justified.

These problems were solved in the second paper (Gepner 1984), which is the subject of the third chapter. Using Witten's recently proposed non abelian bosonization (Witten 1984), a bosonic form of QCD_2 is obtained for two flavors. It is shown that the low lying theory is a sigma model with a Wess–Zumino term the coefficient of which is the number of colors. This result is closely analogous to current algebra theories proposed to describe the QCD spectrum in four dimensions, based on semi–phenomenological grounds (Witten 1983). Unlike the four dimensional case, here, however, we are able to prove exactly this result starting directly from the QCD lagrangian.

In order to analyze the bosonic theory we have also to discuss the following two issues: 1) Regularization, normal–ordering and dimension of operators. This is needed in order to get the scale of the resulting theory. 2) Semi–classical analysis, needed to get the spectrum and masses of particles in the theory. These subjects are treated in two appendices.

We then find baryons with isospin $N_c/2$ or higher. There is also a baryonium family with arbitrary integer isospin. Mass formulae are given for these particles.

As an interesting application of these methods we treat the multi–flavor Schwinger model. We obtain the low lying lagrangian and are able to extend to an arbitrary number of flavors, Coleman's results for the two flavor case (Coleman 1976). It is proven that the theory contains one multiplet in the adjoint and one iso–singlet. Semi–classical masses are given for them, reasonably agreeing with Coleman's exact results for the two flavors theory. As a by–product we are able to show that an $SU(2)$ Wess–Zumino theory with mass term is equivalent to a non trivial sine gordon theory ($\beta = \sqrt{2\pi}$).

Chapter 1

Fractional Charge

§1.1 Introduction

An interesting phenomenon was discovered in field theory by Jackiw and Rebbi (1976). They have considered a fermion soliton system in two dimensions. It was found that the the soliton acquired a fermionic charge equal to one half; namely, the soliton has a charge of half a fermion! This phenomenon became known as fractional charge. The authors have also shown that a monopole coupled to a fermion acquires the same half fermionic charge.

Later, Goldstone and Wilczek (1981) have considered giving the fermion a mass. It was then shown that the topological excitation (a soliton or a monopole) acquires a transcendental charge! The values of the charges of the monopole and of the soliton were found to be given by identical expressions.

This peculiar equality was explained by us from two different points of view (Frishman, Gepner and Yankielowicz 1983, 1984). In the first work a connection between fractional charge and anomalous commutators was found. Analogous expressions for the anomalous commutators in the two theories led to similar results for the charges. Thus some of the enigma was explained. Also, an important connection between the anomaly and the fractional charge was established, thus relating these two important aspects of field theory.

In the second paper the connection between the four and the two dimensional theories was investigated. It was shown that the $J = L + S + I = 0$ sector of the monopole theory is equivalent to the soliton theory. This fact directly explains why their vacuum charges are identical.

The prospects for the phenomena of fractional charge to be realized in particle physics are not very clear. The magnetic monopole, although predicted by grand unified theories, has not yet been found. In condensed matter systems, however, this phenomenon is very likely to arise in one–dimensional molecules like polyacetylen. (For a discussion of experimental and theoretical aspects of polyacetylen see A. Heeger 1981.)

§1.2 The fractional charge

We shall follow the method of Jackiw and Rebbi to show that the soliton has a half integral fermion number. This charge is seen to arise due to a zero mode of the fermion in presence of the soliton.

For the sake of convenience let us take a kink solution. The lagrangian of the kink–fermion system is then given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}(\varphi^2 - 1)^2 + \bar{\psi} i \not\partial \psi + g \bar{\psi} \psi \varphi \qquad (1.1)$$

The bosonic part of this lagrangian is known to have a classical solution which is the kink solution of the φ^4 theory (Dashen et al. 1974b, Goldstone and Jackiw 1975, Polyakov 1974). The equation of motion is

$$\ddot{\varphi} - \varphi'' = 2\varphi^3 - 2\varphi \qquad (1.2)$$

The kink is a time independent solution with the boundary condition $\varphi(\infty) = -\varphi(-\infty) = 1$. The kink is stable, since it is the lowest energy solution in the sector of solutions obeying that boundary condition, and because this sector is protected from decay by the finiteness of energy demand. This property of the solution suggests that it describes the classical limit of an actual particle. The kink solution is given by

$$\varphi(x) = \tanh x \qquad (1.3)$$

The quantization of the classical solution is needed for exhibiting that it describes a real particle appearing in the spectrum of the corresponding field theory. This was done by several authors using different methods (Dashen et al. 1974b, Cahill 1974, Goldstone and Jackiw 1975, Polyakov 1975a,b). The semi-classical quantization enables the computation of masses, quantum numbers and various matrix elements, in analogy to the WKB approximation in quantum mechanics.

In order to treat the kink-fermion system, we apparently need to solve the quantum field theory (1). This is a very difficult task. Fortunately, we can use again semi-classical methods as generalized to fermionic systems (Dashen et al. 1974b). It can then be seen that in the leading semi-classical approximation one may ignore altogether the quantum nature of the soliton, retaining only the fermion as a quantum field. This much simplified approach is what we employ in the following.

Consider then a fermion in a classical external kink field. We shall show that the fractional charge is due to a zero energy solution of the Dirac equation—a zero mode. The equation of motion of ψ is

$$\left[-i\alpha_1 \frac{d}{dx} - g\beta\varphi(x)\right]\psi = \epsilon\psi, \tag{1.4}$$

in which ϵ is the energy; we use the representation $\beta = \sigma_1$ and $\alpha_1 = \sigma_2$. In component notation ψ is

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} \tag{1.5}$$

We get after a little manipulation

$$\left(-\frac{d^2}{dx^2} + g^2\varphi^2 + g(1-\varphi^2)\right)u = \epsilon^2 u, \tag{1.6}$$

and

$$\left(-\frac{d^2}{dx^2} + g^2\varphi^2 - g(1-\varphi^2)\right)v = \epsilon^2 v. \tag{1.7}$$

The solution to this differential equation is well known. Equations (6) and (7) are simply the Schrödinger equation of a particle moving in the potential

$$V(x) = g^2\varphi^2 \pm g(1-\varphi^2) = g^2\tanh^2 x \pm g\,\mathrm{sech}^2 x \tag{1.8}$$

Far from the origin $V(x)$ is equal to g^2. Hence there will be a continuum of energy levels given by $\epsilon(k)^2 = k^2 + g^2$. Below g^2 there are discrete energy levels, the energies of which are exactly known (Morse and Feshbach 1953, Dashen et al. 1974b)

$$\epsilon_r^2 = 2ng - n^2 \qquad n = 0, 1, \ldots < g \tag{1.9}$$

The actual form of the wave function of this state is not necessary, in fact, for the purpose of computing the fractional charge. All that is needed is the way the wave function transforms under charge conjugation. The charge conjugation operator is σ_3. Then from (4) we easily see that if ψ_ϵ is a solution with energy ϵ, then $\sigma_3\psi_\epsilon$ is a solution having the energy $-\epsilon$. Hence all the states are paired as a fermion at energy ϵ with an anti-fermion at energy $-\epsilon$. The only possible exception to this rule is the states of zero energy—fermionic vacua. These states can be their own anti-states, obeying $\sigma_3\psi_0 = \pm\psi_0$.

Intuitively, the states of nonzero energy can not contribute to the fractional charge, since they cancel each other in pairs connected by charge conjugation. So only the fermion number self conjugate states at zero energy are of importance. We have exactly one such state in (8) obtained by taking $n = 0$. Its wave function is

$$\psi_0 = \nu_0 \begin{pmatrix} 0 \\ (\cosh x)^{-g} \end{pmatrix} \tag{1.10}$$

This state is self charge conjugate, since we have $\sigma_3\psi_0 = -\psi_0$.

We turn now to the second quantization formulation of our system. We have to expand Ψ in the wave functions, where the coefficients are creation and annihilation operators

$$\Psi(x,t) = b_0\psi_0 + \sum_{r\geq 1} b_r e^{-i\epsilon_r t}\psi_r^+(x) + d_r^\dagger e^{i\epsilon_r t}\psi_r^-(x) \tag{1.11}$$

Here $\psi_r^\pm$ are the positive and negative energy solutions respectively, having the energy $\pm\epsilon_r$. (It includes the nonzero discrete spectrum and the continuum.) Now the charge operator is given by

$$Q = \frac{1}{2}\int dx(\Psi^\dagger\Psi - \Psi\Psi^\dagger) = b_0^\dagger b_0 - \frac{1}{2} + \sum_{r\geq 1} b_r^\dagger b_r - d_r^\dagger d_r \tag{1.12}$$

The fermionic vacuum will be a state with only the kink in it. Denote this state by $|\text{kink}-\rangle$. Then it obeys

$$b_0|\text{kink}-\rangle = b_r|\text{kink}-\rangle = d_r|\text{kink}-\rangle = 0 \tag{1.13}$$

Since no fermion mode is excited, this state will have all the properties of a free kink like the mass, but the charge of it will be $-1/2$; from (12) and (11)

$$Q|\text{kink}-\rangle = -\frac{1}{2}|\text{kink}-\rangle \tag{1.14}$$

The soliton has a half integral fermion number.

There will be another state with the same energy—the kink with the fermion zero mode excited:

$$|\text{kink}+\rangle = b_0^\dagger|\text{kink}-\rangle \tag{1.15}$$

The charge of this state is obtained by applying Q and using the anti–commutators relations for b and d

$$Q|\text{kink}+\rangle = \frac{1}{2}|\text{kink}+\rangle \tag{1.16}$$

So we see that we have two degenerate states in the kink sector having as charge $\pm\frac{1}{2}$.

Notice that the only condition for having this result is the existence of a self charge conjugate zero energy state. Hence the value of the fractional charge is a sort of an index. We have chosen the kink solution (3) only for convenience. Suppose we had a general external field $\varphi(x)$ such that $\varphi(\infty) > 0$ and $\varphi(-\infty) < 0$. Then nothing would have been changed. The only condition for the vacuum fermion number to be 1/2 is the existence of exactly one self conjugate zero mode with $C = -1$. This would still be true. Solving (4) with $\epsilon = 0$ we get

$$\psi_0(x) = \begin{pmatrix} 0 \\ \exp\left(-\int_0^x \varphi(y)dy\right) \end{pmatrix} \tag{1.17}$$

This is the desired state and it is the only zero mode. Hence we see that the fractional charge is a topological quantity, independent of the particular form of the solitonic field. For example, we get the same value of half for the sine gordon soliton.

Jackiw and Rebbi have treated also the case of a monopole coupled to a massless isodoublet fermion. Again it was seen that the monopole acquires a half integral charge. The proof is in total analogy with the two dimensional case. Again the states come in charge conjugation pairs canceling each other's charges, with the exception of one zero mode. The zero mode, just as before, is fermion number self conjugate, with $C = -1$, and is responsible for giving the monopole a fermionic charge.

§1.3 The Massive Fermion Case

Goldstone and Wilczek (1981) considered a soliton coupled to a massive fermion. Then not only the fermion acquires a non integer fermion number, but the value of it turns out to be an arbitrary transcendental number. As a first case they considered the model (1), now with a fermion mass term

$$\mathcal{L}_F = \bar{\Psi}(i\not\partial + \varphi + iM\gamma_5)\Psi \tag{1.18}$$

where $\varphi(\infty) = -\varphi(-\infty) = \nu$. In order to compute the fermion number these authors switched on adiabatically the soliton field φ and then used perturbation theory. The result which they found for the vacuum charge is

$$Q = \frac{1}{\pi}\arctan\frac{\nu}{M}. \tag{1.19}$$

Notice that, again, the value of the charge is independent of the details of the external field. Only the values of φ at infinity enter into (19).

Here we shall offer a different way of computing this charge. The computation will be in analogy to the massless case carried out in section 2. The first step is to solve (18) with a particular external field. For the sake of convenience we choose

$$\varphi(x) = \nu\frac{x}{|x|} \tag{1.20}$$

The equation of motion for ψ as derived from (18) is

$$\left[-i\alpha_1\frac{d}{dx} - \beta\varphi - i\beta\alpha M\right]\eta(x) = \epsilon\eta(x) \tag{1.21}$$

Also, in order to make our treatment more rigorous, we shall consider the system to be in a box with periodic boundary conditions. Namely we assume

$$\eta\left(-\frac{L}{2}\right) = \eta\left(\frac{L}{2}\right) \tag{1.22}$$

L is the box length.

The solutions of this equation are seen to be in three classes.

1) Positive energy solutions. Their energy is given by

$$\epsilon_n = \sqrt{k_n^2 + \nu^2 + m^2}, \qquad \text{and } Lk_n = 2n\pi \tag{1.23}$$

They are given by

$$\eta_n^1 = \frac{1}{\sqrt{L}}\begin{pmatrix} \sqrt{\frac{\epsilon-M}{\epsilon}}\cos(k_n|x| + \delta_n) \\ -\sqrt{\frac{\epsilon+M}{\epsilon}}\sin k_n x \end{pmatrix} \tag{1.24}$$

and

$$\eta_n^2 = \frac{1}{\sqrt{L}}\begin{pmatrix} -\sqrt{\frac{\epsilon-M}{\epsilon}}\sin k_n x \\ \sqrt{\frac{\epsilon+M}{\epsilon}}\cos(k_n|x| - \delta_n) \end{pmatrix} \tag{1.25}$$

δ_n is defined by

$$e^{i\delta_n} = \frac{k_n + i\nu}{|k_n + i\nu|} \tag{1.26}$$

The solutions are normalized to have probability one. We denote these solutions collectively as η_p where $p = 1, 2, \ldots$

2) Negative energy solutions. Their energy is $\epsilon_n = -\sqrt{k_n^2 + \nu^2 + M^2}$. They can be obtained by taking $\eta \to \sigma_3 \eta$ and $M \to -M$. This is simply a charge conjugation. We denote them collectively as $\bar{\eta}_p$ where $p = 1, 2, \ldots$

3) The discrete solutions. For $M = 0$, those are the zero modes of section (2) now lifted to the energy $\epsilon = \pm M$. These are

$$\eta_0 = \left(\frac{\nu}{1 - e^{-\nu L}}\right)\begin{pmatrix} e^{-\nu|x|} \\ 0 \end{pmatrix} \qquad \text{for } \epsilon = -M \tag{1.27}$$

$$\bar{\eta}_0 = \left(\frac{\nu}{1 - e^{-\nu L}}\right)\begin{pmatrix} 0 \\ e^{\nu(|x| - L/2)} \end{pmatrix} \qquad \text{for } \epsilon = M \tag{1.28}$$

In order to compute the fractional charge, we shall employ

$$J_0 = \frac{1}{2}[\Psi^\dagger, \Psi] \tag{1.29}$$

Here Ψ is the second quantized field defined in analogy to (10).

$$\Psi(x,t) = \sum_{p=0}^{\infty} b_p e^{-i\epsilon_p t}\eta_p(x) + d_p^\dagger e^{i\epsilon_p t}\bar{\eta}_p(x) \tag{1.30}$$

(b_p, d_p obey canonical anti-commutation relation.) Then we can compute the density of the vacuum charge

$$\rho(x) = \langle 0|J_0(x)|0\rangle = \frac{1}{2}\sum_{p=0}^{\infty} \bar{\eta}_p(x)\bar{\eta}_p(x) - \eta_p(x)\eta_p(x) \tag{1.31}$$

Suppose now we were to integrate $\rho(x)$ over the box in order to get the vacuum charge. Then we would get no fractional charge! (This is due to the orthonormality the wave functions.)

$$\int_{-L/2}^{L/2} \rho(x) = 0 \tag{1.32}$$

So apparently the vacuum has no charge. This was first found for $M = 0$ by Bell and Rajaraman (1982). In order to get the vacuum charge one has to be more careful. Let us introduce a second scale l, such that

$$\frac{1}{\sqrt{\nu^2 + M^2}} \ll l \ll L \tag{1.33}$$

We shall now compute the vacuum charge that is contained in the region $(-\frac{l}{2}, \frac{l}{2})$ and show that, in fact, it gives the correct value for the charge (19)

$$Q(l) = \int_{-l/2}^{l/2} \rho(x)\,dx \tag{1.34}$$

The physical interpretation of this result needs some explanation. If we preformed an experiment to measure the fractional charge, naturally we would do it in a small region around the soliton center. Hence it is justified to compute the charge not for the entire box, but for a small region around the soliton (given between $-\frac{l}{2}$ and $\frac{l}{2}$). However, it could have been claimed now that we are actually computing not the eigenvalue of the charge operator, but an expectation value in a mixed state. As such it would not be of no surprising to find the non integral value (19). It can be shown, however, that the fluctuations of the charge when $l \to \infty$ are decreasing to zero. (The walls of the box must then be made "softer".) Thus our vacuum asymptotically becomes an eigenstate of the charge operator with a fractional eigenvalue. For detailed computations of the fluctuations for $M = 0$, see Bell and Rajaraman (1982) and also, Frishman and Horovitz (1982).

Let us now actually compute (34), using the wave functions (23–28). For the continuum states we find from (31),

$$\rho(x) = \frac{1}{2}\sum_{n=1}^{\infty} \bar{\eta}_n^1 \bar{\eta}_n^1 + \bar{\eta}_n^2 \bar{\eta}_n^2 - \eta_n^1 \eta_n^1 - \eta_n^1 \eta_n^1 = \frac{M}{\epsilon L}\sum_n \sin(2k_n|x|)\sin 2\delta_n . \tag{1.35}$$

Computing the integral (34) we then get

$$Q(l) = \sum_n \frac{2\nu M}{L\epsilon_n(\epsilon_n^2 - M^2)}(1 - \cos k_n l) \tag{1.36}$$

We can now take the box to infinity. Then $L \to \infty$, and the sum becomes an integral.

$$Q(l) = \int_{-\infty}^{\infty} f(k)\, dk, \tag{1.37}$$

and

$$f(k) = \frac{\nu M}{2\pi}\frac{1 - e^{ikl}}{\epsilon(\epsilon^2 - M^2)} \tag{1.38}$$

To evaluate this integral we deform the real line into the closed curve C (see fig. 1). We now use the residue theorem for evaluating (37). The only pole of $f(k)$ inside the contour C is at $k = -i\nu$. There is also a branch line ("cut") on $(-i\infty, -i\sqrt{\nu^2 + M^2})$. The value of the residue is given by

$$2\pi i \operatorname*{Res}_{-i\nu} f(k) = -\frac{1}{2}(1 - e^{-\nu l}) \tag{1.39}$$

Using then the residue theorem, the value of the integral is seen to be

$$Q(l) = -\frac{1}{2}(1 - e^{-\nu l}) + \frac{\nu M}{\pi}\int_{\sqrt{\nu^2+M^2}}^{\infty} \frac{1 - e^{-kl}}{\sqrt{k^2 - \nu^2 - M^2}\,(k^2 - \nu^2)}\, dk, \tag{1.40}$$

where the integral on the l.h.s of (40) represents the integral of $f(k)$ along the branch line. Now l can be safely taken to infinity, and we get for the continuum contribution to the

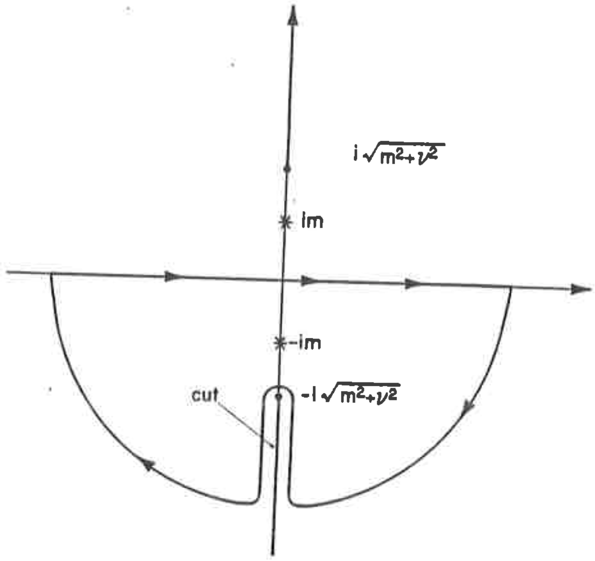


Fig 1

fractional charge:

$$\lim_{l\to\infty} Q(l) = -\frac{1}{2}+\frac{\nu M}{\pi}\int_{\sqrt{\nu^2+M^2}}^{\infty}\frac{dk}{\sqrt{k^2-\nu^2-M^2}\,(k^2-\nu^2)} = -\frac{1}{2}+\frac{1}{\pi}\arctan\frac{M}{\nu}. \quad (1.41)$$

The contribution from the discrete states is 1/2, since from (27–28) we can compute,

$$Q_0 = \lim_{l\to\infty}\int_{-l/2}^{l/2}\frac{1}{2}(\eta_0(x)\eta_0(x) - \bar{\eta}_0(x)\,\bar{\eta}_0(x))\,dx = \frac{1}{2}. \quad (1.42)$$

Summing (41) and (42), we get the value of the fractional charge stated in (19).

Two remarks are in order:

1) The value of the charge is independent of the particular solution we have chosen, but it depends only on the asymptotic values of it. Let's say that $\varphi(x)$ is some general external field. Also replace the constant mass term M in (18) by arbitrary function $M(x)$. Then the fractional charge is given by

$$Q = \Big(\lim_{x\to\infty} - \lim_{x\to-\infty}\Big)\frac{1}{2\pi}\arctan\frac{M(x)}{\varphi(x)} \quad (1.43)$$

This may be seen either from the perturbative computation of Goldstone and Wilczek or from arguments given by Y. Frishman (1983). The latter showed that the fractional charge depends only on the asymptotic values of the external fields, provided fermion current conserving regularization can be found for any external field configuration.

2) We have two types of continuum states given by (24) and (25). It can be easily seen from our computation that the contributions to the vacuum charge coming from each of them are equal. Their respective vacuum charges account for exactly one half of (41). So let us separate all our solutions into the two types:

$$\text{Type 1}: \qquad \eta_1 = \begin{pmatrix} \text{symmetric} \\ \text{antisymmetric} \end{pmatrix}, \tag{1.44}$$

$$\text{Type 2}: \qquad \eta_2 = \begin{pmatrix} \text{antisymmetric} \\ \text{symmetric} \end{pmatrix}. \tag{1.45}$$

The state (27) will contribute half to the fractional charge of the type 1 solutions and nothing to the type 2. (for $\nu < 0$, it is the opposite). Hence the total charge of each of them is,

$$Q_1 = \frac{1}{2\pi}\arctan\frac{M}{\nu} + \frac{1}{4}, \tag{1.46}$$

$$Q_2 = \frac{1}{2\pi}\arctan\frac{M}{\nu} - \frac{1}{4}. \tag{1.47}$$

Of course, jointly, the two types give the total fractional charge of the system: $Q = Q_1 + Q_2$. The importance of this two subsets of solutions will become apparent in the next chapter. It will be shown that they correspond to Callan's "half" fermions.

A simple way to get the fractional charge of (18) is made possible via the use of the so called bosonization technique. This method will be described in detail in the next section. For the time being it will suffice to mention that a fermion theory in two dimensions may be translated into a bosonic one. For that purpose, one uses the following "dictionary",

$$\bar{\psi}\, i\, \partial\!\!\!/ \psi \rightarrow \frac{1}{2}(\partial_\mu \chi)^2, \tag{1.48}$$

$$\bar{\psi}\psi \to M\cos(2\sqrt{\pi}\chi), \tag{1.49}$$

$$\bar{\psi}i\gamma_5\psi \to M\sin(2\sqrt{\pi}\chi), \tag{1.50}$$

$$\bar{\psi}\gamma_\mu\psi \to \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial_\nu\chi. \tag{1.51}$$

(χ is a canonical bosonic field.)

Then (18) may be translated into the purely bosonic theory,

$$\mathcal{L}_F = \frac{1}{2}(\partial_\mu\chi)^2 + \varphi\cos(2\sqrt{\pi}\chi) + M\sin(2\sqrt{\pi}\chi). \tag{1.52}$$

We can write (52) as,

$$\mathcal{L}_F = \frac{1}{2}(\partial_\mu\chi)^2 + A(x)\cos(2\sqrt{\pi}\chi - B(x)), \tag{1.53}$$

where $B(x) = -\arctan(M(x)/\varphi(x))$.

Now, the vacuum of the theory (53) is described by a certain classical solution of it, obeying the boundary condition,

$$2\sqrt{\pi}\chi(x) = B(x), \qquad \text{for } x \to \pm\infty. \tag{1.54}$$

Then from (51) we can obtain the fractional charge of the vacuum using semi-classical methods for χ:

$$Q = \int_{-\infty}^{\infty} \bar{\psi}\gamma_0\psi\, dx = \int \frac{1}{\sqrt{\pi}}\partial_1\chi\, dx = \frac{1}{\sqrt{\pi}}(\chi(\infty) - \chi(-\infty)) \tag{1.55}$$

and using (54) we again see,

$$Q = \frac{1}{\pi}\arctan\frac{M}{\nu}. \tag{1.56}$$

This is an example of the power of the bosonization technique in analyzing non-perturbative phenomena. We shall see in chapter 2 more uses of this kind.

Goldstone and Wilczek have also treated a monopole coupled to a massive fermion. Again using their method of adiabatic computation, they obtained for the fractional charge an expression identical to (19).

In the next section we shall not only prove this result, but also explain why the charge is the same for the two and four dimensional theories.

§1.4 The fractional charge of the magnetic monopole

In this section we shall consider a magnetic monopole coupled to a fermion. It was found by Jackiw and Rebbi (1976) and later Goldstone and Wilczek (1981) that the monopole then develops a fractional fermionic charge. The method of computing the charge that we shall employ here, has the advantage of explaining why the monopole and the soliton develop the same charges. Namely, we shall prove that the $j = 0$ sector of the monopole is actually equivalent to the soliton theory (18). This section is based on the work of Frishman, Gepner and Yankielowicz (1984).

The magnetic monopole was found by Dirac as a consistent extension to electromagnetism. Some time ago 't Hooft (1976) and Polyakov (1976) showed that in a large class of spontaneously broken gauge theories, there exists a classical solution, that when quantized, becomes a particle carrying a magnetic charge. Thus the magnetic monopole became no more a luxury but a particle predicted by certain theories, in particular by grand unification schemes.

For simplicity we consider the original 't Hooft Polyakov monopole found in an $SU(2)$ theory spontaneously broken to $U(1)$. The monopole of other gauge groups carry fractional charges similarly. The lagrangian is then given by,

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + (D_\mu \varphi^a)^2 + V(\varphi^a). \tag{1.57}$$

where φ^a is the Higgs field ($a = 1, 2, 3$) in the adjoint of $SU(2)$. D_μ is the covariant derivative given by,

$$D_\mu = \partial_\mu + ieA^a_\mu T^a, \tag{1.58}$$

(T^a are generators of $SU(2)$ taken here in the adjoint.)

The lagrangian (57) has a classical solution protected from decay by a topological conservation law, i.e., the second homotopy group of $SU(2)/U(1)$. This classical solution can be seen to represent a particle carrying a magnetic charge, where the topological conservation law becomes the conservation of magnetic charge. (For a review on monopoles see for example Coleman 1975.) The fields of the monopole solution obey

$$A_0^a(x) = 0,$$

$$eA_i^a(x) = \epsilon_{aij} x_j A(r), \qquad (1.59)$$

$$\pi_a(x) = x^a \varphi(r).$$

At large r the fields behave as $A(r) = -1/r + O[\exp(-M_W r)]$, and $\varphi(\infty) = \nu$. Also $A(0) = \varphi(0) = 0$.

We shall consider now a theory in which a monopole is coupled to a fermion. The fermion's isospin is taken to be one half. The fermionic lagrangian is then

$$\mathcal{L}_F = \bar{\psi} i \not{D}\psi - \bar{\psi}(m + ig\gamma_5 T^a \varphi^a)\psi. \qquad (1.60)$$

The fractional charge of the monopole is (Goldstone and Wilczek):

$$q = \pi^{-1} \arctan\left(\frac{g\nu}{m}\right), \qquad (1.61)$$

the same as for the two dimensional theory (18), now written slightly differently as

$$\mathcal{L}_F = \bar{\psi} i \not{\partial}\psi - \bar{\psi}(m + ig\gamma_5 \varphi)\psi. \qquad (1.62)$$

Here $\varphi(\infty) = -\varphi(-\infty) = \nu$.

We shall prove that (60) and (62) are equivalent theories. Thus we shall also explain the identity of their respective vacuum charges.

In order to prove that assertion we shall treat the monopole fermion system using methods originated by Goldstone and Jackiw (1976) and elaborated by Callan (1982 a,b). It was shown by them that due to the spherical symmetry of the monopole solution (59), the lagrangian (60) conserves a peculiar quantum number which is the sum of isospin, spin and angular momentum. We denote this quantum number by j, where $j = L + S + I$.

For m=0 the complete analysis of the classical equation of motion coming from (60) was done by Jackiw and Rebbi. They found that a zero mode appears in the $j = 0$ partial wave, hence giving rise to one half fractional charge as discussed in section 2. The higher partial wave have also been analyzed and it was shown that no zero mode arises in them and consequently they do not contribute to the fractional charge. This result can actually be understood without explicitly solving the equation of motion of (60). For the higher partial waves there is a centrifugal barrier, preventing the fermion from getting close to the monopole. In the $M_W \to \infty$ limit the monopole is pointlike and since the fractional charge arises from the monopole–fermion interaction we expect the fractional charge to be saturated by the $j = 0$ partial wave. This argument holds also when m is not zero. Hence we can concentrate on the $j = 0$ sector and the treatment of (60) is simplified.

Following Callan we decompose the $j = 0$ Fermi fields as

$$\psi_j^{(\pm)} = \begin{pmatrix} X_\pm \\ \pm X_\pm \end{pmatrix}, \tag{1.63}$$

where the $(\pm)$ refers to helicity. X is a 2×2 matrix where one index is spin and the other is isospin. For $j = 0$, X may be written as

$$X_\pm = (\sqrt{8\pi}\, r)^{-1}(g_\pm + p_\pm \vec{x} \cdot \vec{\tau})\tau_2, \tag{1.64}$$

where $g_\pm$ and $p_\pm$ are functions of r only. This is the most general form of a solution which is j–symmetric. It was shown by Callan that g and p may be grouped into a two dimensional Fermi fields,

$$\chi_\pm = \begin{pmatrix} g_\pm \\ \pm i p_\pm \end{pmatrix}. \tag{1.65}$$

Then it can be seen that the fields $\chi_\pm$ obey canonical anticommutation relations. Thus we arrive at an effective two dimensional field theory for χ by inserting (63–65) into the lagrangian (60). We then get the lagrangian (not the lagrangian density),

$$L(t) = \int_0^\infty dr \left\{ \bar{\chi}_+ i\gamma_\alpha \partial^\alpha \chi^+ + \bar{\chi}_- i\gamma_\alpha \partial^\alpha \chi^- + m(\bar{\chi}_+ \chi^- + \bar{\chi}_- \chi^+) \right.$$
$$\left. + i\left(A - \frac{1}{r}\right)(\bar{\chi}_+ \gamma_5 \chi^+ + \bar{\chi}_- \gamma_5 \chi^-) + ig\varphi(\bar{\chi}_+ \gamma_5 \chi^- + \bar{\chi}_- \gamma_5 \chi^+) \right\}, \tag{1.66}$$

where the γ_μ's are the two dimensional ones: $\gamma_0 = \tau_3$, $\gamma_1 = -i\tau_1$, $\gamma_5 = \tau_2$.

$(A - 1/r)$ vanishes exponentially rapidly outside the monopole core. Inside, it goes like $(-1/r)$. Hence in the large M_W limit, the term containing $(A - 1/r)$ may be eliminated altogether from the lagrangian (65), provided we impose the boundary condition $\tau_+ \chi_\pm = 0$. This condition ensures that $\chi_\pm$ will not have a non integrable singularity at $r = 0$ (Callan (1982) and Besson (1981)).

It would be convenient now to make the transformation,

$$\lambda_{1,2} = \frac{1}{\sqrt{2}}(\chi_+ \pm \chi_-), \tag{1.67}$$

which produces a flavor diagonal lagrangian,

$$L(t) = \int_0^\infty dr \sum_{n=1}^{2} [\bar{\lambda}_n i \not{\partial} \lambda_n - (-)^n \bar{\lambda}_n (m + ig\gamma_5 \varphi) \lambda_n], \tag{1.68}$$

with the boundary condition $\tau_+ \lambda_n(0) = 0$ for $n = 1, 2$.

The lagrangian (68) is a half line lagrangian of two flavors. We shall show that it is actually equivalent to a full line lagrangian of a one flavor, which is our two dimensional soliton lagrangian (62). One further transformation must be made:

$$\rho(x) = \begin{cases} 2^{-1/2} [\lambda_2(x) + \bar{\gamma}_5 \lambda_1(x)] & \text{for } x > 0 \\ 2^{-1/2} \gamma_0 [\lambda_2(-x) - \gamma_5 \lambda_1(-x)] & \text{for } x < 0 \end{cases} \tag{1.69}$$

The boundary condition $\tau_+ \lambda_n(0) = 0$ gurantees the continuity of $\rho(x)$ at $x = 0$. It can be seen that ρ is a canonical Fermi field. Inserting (69) into the lagrangian (68), we get

$$L(t) = \int_{-\infty}^\infty dx\, [\bar{\rho}\, i \not{\partial} \rho - \bar{\rho}\,(m + ig\gamma_5 \tilde{\varphi})\rho], \tag{1.70}$$

where $\tilde{\varphi}$ is defined as

$$\tilde{\varphi} = \begin{cases} \varphi(x) & \text{for } x > 0 \\ -\varphi(-x) & \text{for } x < 0 \end{cases} \tag{1.71}$$

The lagrangian (70) is identical to the two dimensional theory (62).

This does not yet guarantee that the monopole and the soliton have the same fractional charge. We need to see what happens to the fermion number current through the transformations we have preformed. The four dimensional fermion charge density for a $j = 0$ fermion may be seen to be

$$4\pi r^2 \psi^\dagger \psi = \chi_+^\dagger \chi_+ + \chi_-^\dagger \chi_- = j_0(\chi_+) + j_0(\chi_-) \tag{1.72}$$

We can also see that after the transformation (67) and (69), j_0 will obey,

$$j_0(\rho, x{=}r) + j_0(\rho, x{=}-r) = j_0(\chi_-, x{=}r) + j_0(\chi_-, x{=}r) \tag{1.73}$$

Hence, combining (73) and (72), we see that the four dimensional fermion number density is identical to the that of the equivalent two dimensional theory as summed on $\pm r$. Also the vacuum is translated into a vacuum all the way. Hence the fractional charges of (61) and (59) are equal:

$$q(2) = q(4) = q. \tag{1.74}$$

We thus have shown that the two theories are equivalent and in such a way that that their respective vacuum charges are identical. In fact this holds equally well for their charge densities, as elucidated in eq. 73. This is, however, particular to the $M_W \to \infty$ limit as can be seen, for example, from the explicit expressions given by Goldstone and Wilczek. The above discussion consists also of an independent proof for the value of the monopole's fermionic charge (eq. 61).

We shall now present a variation of this proof that also allows us to compute the fractional charge of each of the two "half" fermions, λ. (By half fermions we mean fermions living on a half line.) This computation may also be interesting due to the importance of the half–fermions in Callan's work.

Suppose λ_2^ϵ is a classical solution with energy ϵ. From (68) we have

$$(-i\gamma_5\partial_1 - m\gamma_0 - ig\varphi\gamma_1)\lambda_2^\epsilon = \epsilon\lambda_2^\epsilon, \tag{1.75}$$

and the boundary condition $\tau_+\lambda_2^\epsilon = 0$. Define then

$$\rho_2^\epsilon = \begin{cases} \lambda_2^\epsilon(x) & \text{for } x > 0 \\ \gamma_0\lambda_2^\epsilon(x) & \text{for } x < 0 \end{cases} \tag{1.76}$$

Then $\rho_2^\epsilon(x)$ is a solution of the equation of motion derived from (70). It is not the most general solution, since it obeys $\tau_+\rho_2^\epsilon = 0$. In fact, we can see that it corresponds exactly to the type 2 solutions described in section 3 (eq. 45). Moreover, all the type 2 solutions may be obtained using the transformation (76). The fractional charge of λ is then given by an expression analogous to (42),

$$q_2 = \lim_{r\to\infty}\int_{-r}^{r} dx\,\frac{1}{2}\sum_{\epsilon>0}[\lambda_2^{\epsilon\dagger}\lambda_2^\epsilon - \lambda_2^{-\epsilon\dagger}\lambda_2^{-\epsilon}] = \lim_{r\to\infty}\int_{-r}^{r} dx\,\frac{1}{2}\sum_{n=\text{type }2}[\rho^{(n)\dagger}\rho^{(n)} - \tilde{\rho}^{(n)\dagger}\tilde{\rho}^{(n)}] \tag{1.77}$$

Here $\rho^{(n)}$ are the negative energy solutions. Similarly we can make the transformation $\lambda_1^\epsilon \to \gamma_5\lambda_1^\epsilon$ and use $-\gamma_0$ for $x < 0$ in (76), and then the sum extends over the type 1 solutions. Now, the fractional charge of each of the two types has been computed in section 3 (eq. 46–47). Hence we get

$$\begin{aligned} q_1 &= \frac{1}{2\pi}\arctan\left(\frac{gv}{m}\right) + \frac{1}{4}, \\ q_2 &= \frac{1}{2\pi}\arctan\left(\frac{gv}{m}\right) - \frac{1}{4}, \end{aligned} \tag{1.78}$$

for the fractional charges of the two "half" fermions. Note that the different boundary conditions caused the charges to be different from one another, and not just half of the total charge. In particular, for $m = 0$ we get $q_1 = \frac{1}{2}$ and $q_2 = 0$. Again, this can be seen from the general arguments of section 2. For $m = 0$, charge conjugation is a good symmetry, also since the boundary conditions do not spoil it. Hence the fractional charge must come in halves related to the zero modes. In this case, λ_1 has a zero mode and λ_2 does not. Hence the result follows.

§1.5 Fractional charge and anomalous commutators

We shall now investigate the connection between the fractional charge and the anomaly. It will be shown with the help of a chiral rotation (advocated by Bardeen et al. (1983)), that the value of the fractional charge may be directly obtained from anomalous commutators. We shall treat both the monopole and the soliton cases. This section is based on the work of Frishman, Gepner and Yankielowicz (1984).

We shall start by analyzing the soliton theory (62). Define a chiral angle by

$$\tan\theta(x) = \frac{g\varphi}{m} \tag{1.79}$$

Then

$$m + ig\gamma_5\varphi = \rho(x)\exp(i\gamma_5\theta(x)) \tag{1.80}$$

It can be seen that making a chiral rotation on the lagrangian (62) with the angle $\theta(x)$ leaves us with a lagrangian with no fractional charge. The operator that generates the chiral rotation is

$$U(\theta) = \exp\left[\frac{i}{2}\int j_{05}(x)\theta(x)\,dx\right], \tag{1.81}$$

where $j_{\mu 5} = \bar{\psi}\gamma_5\gamma_\mu\psi$ is the fermion axial vector current. Then the effect of $U(\theta)$ on ψ is a chiral rotation:

$$U(\theta)\psi(x)U(\theta)^{-1} = \left\{\exp\left[-\frac{i}{2}\gamma_5\theta(x)\right]\right\}\psi(x) \tag{1.82}$$

Hence the lagrangian (62) transforms as

$$U(\theta)^{-1}\mathcal{L}U(\theta) = \mathcal{L}(\theta) = \bar{\psi}i\not{\partial}\psi + \bar{\psi}\gamma_\mu\gamma_5(\partial^\mu\theta)\psi - g\rho\bar{\psi}\psi \tag{1.83}$$

There is also a term coming from the jacobian of the transformation (an anomaly term) but since it does not involve the fermion field we can ignore it.

We have now to determine how the fermion number current transforms under chiral rotation. Then it is seen that

$$U(\theta)j_0(x)U(\theta) = j_0(x) + \frac{1}{2\pi}\theta'(x) \tag{1.84}$$

where we used the Schwinger term in two dimensions,

$$[j_0(x), j_{05}(y)] = \frac{i}{\pi}\delta'(x-y) \tag{1.85}$$

In order to get the fractional charge, we can compute the value of the transformed current (84) in the presence of the transformed lagrangian (83).

We claim now that the fractional charge of (83) is zero—

$$\int dx\, \langle j_0(x)\rangle = 0. \tag{1.86}$$

The reason is that the fractional charge depends only on the asymptotic values of the bosonic fields (see Frishman 1983). The theory (83) is asymptotically a free fermion theory and hence (86) follows. That the theory (83) possesses no vacuum charge can be also seen more explicitly by dimensional arguments (Bardeen et al. 1983).

We thus obtain for the fermionic charge,

$$Q_F = \int dx \left[j_0(x) + \frac{1}{2\pi}\theta'(x)\right] = \frac{1}{2\pi}(\theta(\infty) - \theta(-\infty)), \tag{1.87}$$

which gives the known value of the charge eq. 61.

Let us now turn to the magnetic monopole case. The treatment will be analogous to the two dimensional case. The theory that we treat is given by (60), where the monopole fields obey eq. 59. We then define a chiral rotation by

$$U(\theta) = \exp\left(i\int j_{05}^a\theta^a\, d^3x\right), \tag{1.88}$$

where the chiral angles θ_a are defined by

$$m + ig\gamma_5\varphi_a = \rho(x)\, e^{i\gamma_5\tau^a\theta^a}. \tag{1.89}$$

The effect of $U(\theta)$ on ψ is a chiral rotation:

$$U(\theta)\psi(x)U(\theta)^{-1} = \exp\left[-\frac{i}{2}\gamma_5\tau^a\theta^a(x)\right]\psi(x) = S(x)\psi(x). \tag{1.90}$$

The next step is to see how the fermionic lagrangian (59) gets transformed:

$$U(\theta)\mathcal{L}_F U(\theta)^{-1} = \mathcal{L}_F(\theta) = \bar{\psi}(i\not{D} - e\not{B})\psi - g\bar{\psi}\rho\psi, \tag{1.91}$$

where B contains a vector and an axial vector parts,

$$\begin{gathered} B_k = S^{-1}A_k S - \frac{i}{e}S^{-1}\partial_k S \\ A_k \overset{\text{def}}{=} \frac{1}{2}\tau^a A_k^a, \qquad B_k \overset{\text{def}}{=} \frac{1}{2}\tau^a B_k^a \end{gathered} \tag{1.92}$$

Now for large r, B_k is the same as the original potential A_k,

$$B_k \rightarrow \frac{1}{er}\epsilon_{kab}\,\hat{x}_b \tag{1.93}$$

So the lagrangian (91) possesses no fractional charge, since it asymptotically corresponds to a theory in which there is only a vector and scalar interactions.

Now, under the chiral rotation j_0 transforms as

$$U(\theta)j_0U(\theta)^{-1} = j_0 + \frac{e}{4\pi^2}\partial_k\left[H_k^a(x)\theta(x)\right] \tag{1.94}$$

We have used here the anomalous commutators for an $SU(2)$ theory (Adler 1970),

$$[j_0(x,t), j_{05}^a(y,t)] = \frac{i}{4\pi^2}eH_k^a(y,t)\partial_k\delta^3(x-y) \tag{1.95}$$

Here h_k^a are the magnetic fields of A_k^a. Now, as before, the fermion number of the monopole theory (59) can be obtained by computing the value of (94) for the theory (91). Then we get

$$Q_F = \frac{e}{4\pi^2}\int d^3x\,\partial_k\left[H_k^a(x)\theta(x)\right] \tag{1.96}$$

As $r \rightarrow \infty$ we have $\theta^a(x) \rightarrow \hat{x}\,\theta(\infty)$ and $H_k^a(x) \rightarrow \hat{x}_a\,\hat{x}_r\,/(er^2)$. Hence,

$$Q_F = \frac{e}{4\pi^2}(4\pi)\lim_{r\rightarrow\infty}(r^2)\frac{1}{er^2}\theta(\infty) = \pi^{-1}\theta(\infty) = \pi^{-1}\arctan(g\nu/m), \tag{1.97}$$

and we, again, get the result for the fractional charge eq. (61).

Chapter 2

Bosonization and gauge theories in two dimensions

§2.1 Introduction

Bosonization has proved itself to be a very useful tool in the investigation of two dimensional field theories. This technique enables one to map a fermionic theory into a bosonic one. Then the analysis of the resulting bosonic theory is in many examples much simplified. A very important characteristic is that, often enough, perturbative or semi–classical expansion of the of the bosonic theory corresponds to the strong coupling regime of the fermionic theory. Hence one can, that way, arrive at highly non–perturbative results with very little cost. We have already seen one such "miracle" happen in section 1.3, where we got the fractional charge of a soliton in a straightforward way. Another problem ,where bosonization plays an important rôle, is the Callan–Rubakov effect (Callan 1982ab, Rubakov 1982). Although this effect involves a monopole in four dimensions, the application of bosonization to the $j = 0$ sector , which is basically a two dimensional system, is very useful.

In this chapter, we shall concentrate on the application of bosonization to gauge theories in two dimensions. In some examples, bosonization completely solves the theory (e.g. the Schwinger model). In others, we can immediately read the low lying theory.

We shall be mainly interested here in QCD_2 and massive QED_2 with one or many flavors. It will be shown how to derive the low lying spectrum of these theories. In this chapter the "old" abelian bosonization is used, while we postpone to the next chapter the application of Witten's non abelian bosonization, which will resolve several serious problems to be discussed here.

§2.2 Abelian bosonization

In 1975 an interesting correspondence between two field theories was found by Coleman. The theories are the quantum sine–gordon model and the massive Thirring model. This caused a great deal of surprise, since they were two apparently unconnected much investigated models. This correspondence forms the basis of the bosonization technique. The lagrangians are

$$\mathcal{L} = \bar{\psi}\, i\, \partial\!\!\!/ \psi - \frac{g}{2}(\bar{\psi}\,\gamma_\mu \psi)^2 - m\,\bar{\psi}\,\psi, \tag{2.1}$$

and

$$\mathcal{L}_{SG} = \frac{1}{2}(\partial_\mu \phi)^2 - M\mu N_\mu \cos\beta\phi. \tag{2.2}$$

We use here a normal ordering prescription with respect to an arbitrary mass μ, denoted by N_μ. To be more specific, separate the ϕ field into positive and negative frequency parts according to the mass μ:

$$\begin{aligned} \phi &= \phi_- + \phi_+ \\ \phi_- &= \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega(k,\mu)}} a(k,\mu)\, e^{-ikx} \\ \phi_+ &= \phi_-^\dagger \end{aligned} \tag{2.3}$$

where $\omega(k,m) = (k^2 + \mu^2)^{1/2}$, and $a(k,m)$, $a^\dagger(k,m)$ are annihilation and creation operators with the mass μ. Then the vacuum $|0_\mu\rangle$ is the state annihilated by all $a(k,m)$,

$$a(k,\mu)|0_\mu\rangle = 0 \tag{2.4}$$

Our normal-ordering prescription then consists of arranging all the $a(k,\mu)$ to the left of the operator in question.

An important formula relating to this normal-ordering prescription was derived by Coleman,

$$N_m \exp\beta\phi = \left(\frac{\mu^2}{m^2}\right)^{\beta^2/8\pi} N_\mu \exp\beta\phi \tag{2.5}$$

The theories (1) and (2) are equivalent by the following identification of the parameters:

$$\begin{aligned}\frac{1}{1+g/\pi} &= \frac{\beta^2}{4\pi}\\ M^2 &= cm^2\end{aligned} \tag{2.6}$$

Most importantly, this discovery enables a mapping of any two dimensional Fermi theory into a bosonic one. Then one can also use the correspondence of operators found by Coleman,

$$\begin{aligned}\bar{\psi}\,\gamma_\mu\psi &= -\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi,\\ \bar{\psi}\,(1\pm\gamma_5)\psi &= cM\mu N_\mu e^{i\beta\phi},\end{aligned} \tag{2.7}$$

to map any one–fermion model.

Shortly later Mandelstam discovered that this equivalence can be written in a more explicit way. He did so by finding the bosonic operators that create the soliton out of the vacuum. The soliton is then seen to be the basic fermion of the Thirring model. Denote by $\psi_{1,2}$ the upper and lower components of the fermion. Then they are given by

$$\psi_1(x) = \left(\frac{c\mu}{2\pi}\right)^{1/2} e^{\mu/8\epsilon}\exp\left[-2\pi i\beta^{-1}\int_{-\infty}^{x} d\varsigma\;\dot{\pi}(\varsigma) - \frac{1}{2}i\beta\phi(x)\right],$$

$$\psi_2(x) = -i\left(\frac{c\mu}{2\pi}\right)^{1/2} e^{\mu/8\epsilon}\exp\left[-2\pi i\beta^{-1}\int_{-\infty}^{x} d\varsigma\;\dot{\pi}(\varsigma) + \frac{1}{2}i\beta\phi(x)\right]. \tag{2.8}$$

(The integral is regularized by multiplying it with $\exp(-\epsilon\varsigma)$.)

It was shown by Mandelstam that with this definition, ψ is a canonical Fermi field obeying the field equations of the massive Thirring model when the ϕ field is taken to be the sine–gordon quantum.

The extension of Mandelstam's methods to more than one fermion flavor was done by Halpern (1975). He showed that eq. (8) holds for a system of fermions by simply taking one bosonic field for every Fermi field, and by adding a so called Klein factor needed to ensure the anti–symmetric commutation relations between different fermions:

$$\psi_1^n(x) = \left(\frac{c\mu}{2\pi}\right)^{1/2} K_n\exp\left[-2\pi i\beta^{-1}\int_{-\infty}^{x} d\varsigma\;\dot{\pi}_n(\varsigma) - \frac{1}{2}i\beta\phi_n(x)\right],$$

$$\psi_2^n(x) = -i\left(\frac{c\mu}{2\pi}\right)^{1/2} K_n \exp\left[-2\pi i\beta^{-1}\int_{-\infty}^{x} d\varsigma\ \dot{\pi}_n(\varsigma) - \frac{1}{2}i\beta\phi_n(x)\right]. \tag{2.9}$$

The Klein factors are defined by

$$K_n = \prod_{n'<n} (-1)^{N_{n'}}, \tag{2.10}$$

where $N_{n'}$ is the number operator for the n'th Fermi field.

Using (9) one may bosonize any many–fermion theory. For example, the non diagonal fermion bi-linears, not given in (7), may be easily computed to be

$$\bar{\psi}^n(1 \pm \gamma_5)\psi^m = \left(\frac{c\mu}{\pi}\right) F_{nm} N_\mu \exp\left[i\sqrt{\pi}\left(\int_{-\infty}^{x} d\varsigma\,(\pi_n - \pi_m) \pm (\phi_n + \phi_m)\right)\right], \tag{2.11}$$

where F_{nm} are the remnant of the Klein factors:

$$F_n^m = \begin{cases} 1 & \text{for } n = m \\ \prod_{n<\delta<m} (-1)^{N_\delta} & \text{for } n < m \\ \prod_{n>\delta>m} (-1)^{N_\delta + 1} & \text{for } n > m \end{cases} \tag{2.12}$$

An expression for the non–diagonal currents can be likewise written.

We shall be applying this formulae in the sequel to obtain the bosonized version of multi–flavor QCD_2.

§2.3 QED_2 with abelian bosonization

In this section we shall describe the bosonization of electro–dynamics in two dimensions. It will be shown how bosonization can, in quite a straightforward manner, help us understand this model. This section is based on the work of Coleman et al. (1975) and Coleman (1976).

The massive Schwinger model is quantum electro dynamics in two dimensional space time. The lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\,\partial\!\!\!/ - e\,A\!\!\!/ - m)\psi \tag{2.13}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

In order to bosonize this model it is convenient to cast the theory into a hamiltonian form. For that purpose, we first need to impose a gauge condition. We shall choose the gauge:

$$A_1 = 0 \tag{2.14}$$

Then the equation of motion of A_0 becomes a constraint to be imposed on the hamiltonian:

$$\partial_1^2 A_0 = -eJ_0 = -e : \psi^\dagger \psi : \tag{2.15}$$

The solution of this equation is

$$A_0 = -e\partial_1^{-2} - Fx - G \tag{2.16}$$

(F and G are some constants of integration.) The electric field is

$$F_{01} = e\partial_1^{-1} + F \tag{2.17}$$

F has a physical significance, as it represents a background electric field. It is convenient to express F in terms of a θ–angle,

$$\theta = 2\pi F/e. \tag{2.18}$$

Then physics can be seen to be a periodic function of θ with a period of 2π. The reason for that is as follows; since F represents a c–number background electric field, it can be imagined as if created by charges of $\pm F$ sitting at positive and negative infinity. Then if $F > e$, an electron positron pair will be created from the vacuum. The pair will then be attracted to infinity, reducing the value of the background field by e, to an energetically favorable state. Hence, F will always be brought to a value between $-e$ and e, and the periodicity follows.

Now, the hamiltonian corresponding to the lagrangian (13) is

$$\mathcal{H} = \bar{\psi}(i\gamma_1\partial_1 + m)\psi + \frac{1}{2}F_{01}^2 \tag{2.19}$$

Let us now use the bosonization formulae to translate (19) into a bosonic language. Using (7), we can rewrite (17) as

$$F_{01} = e\partial_1^{-1} + \frac{e\theta}{2\pi} = \frac{e}{\sqrt{\pi}}\left(\phi + \frac{\theta}{2\sqrt{\pi}}\right). \tag{2.20}$$

Inserting this constraint into the hamiltonian and also using bosonization for the free part, we get

$$\mathcal{H} = \frac{1}{2}\Pi_\phi^2 + \frac{1}{2}(\partial_1\phi)^2 - mc\mu N_\mu \cos\phi + \frac{e^2}{2\pi}\left(\phi + \frac{\theta}{2\sqrt{\pi}}\right)^2, \tag{2.21}$$

which is the bosonic form of the massive Schwinger model.

Some very non trivial results can be immediately read from the bosonic hamiltonian (21). First, let us assume that $m = 0$ (the Schwinger model). Then we see that making the transformation $\phi \to \phi - \theta/(2\sqrt{\pi})$ eliminates altogether the θ dependance from the hamiltonian and we are left with a free massive bose theory. This is the well known solution of the Schwinger model. We can then also easily compute any expectation values we wish. For example,

$$\langle\bar{\psi}\,\psi\rangle = \langle 0_\mu | N_\mu \cos 2\sqrt{\pi}\,\phi | 0_\mu\rangle = \frac{ec}{\sqrt{\pi}}, \tag{2.22}$$

where $\mu = e/\sqrt{\pi}$ is the photon mass. This is a known result originally obtained with much more labor by other methods (Casher et al. 1973).

Let us turn to the massive theory ($m \neq 0$). Then the mass term can be regarded as a small perturbation in a free massive boson theory, if $m/e \ll 1$. Then a non relativistic aproximation can be used (Coleman 1975), showing a $\delta(x)$–force coming from the mass term. Coleman showed that for $\cos\theta < 0$ there are no bound states; otherwise there is exactly one n-body weakly bound state.

We wish to disscuss now electro dynamics with two flavors:

$$\mathcal{L} = \sum_{i=1,2} \bar{\psi}_i (i\,\partial\!\!\!/ - m - e\,A\!\!\!/)\psi_i - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.23}$$

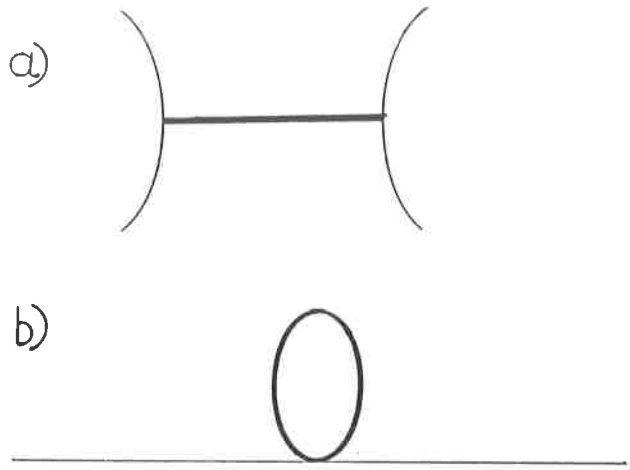


Fig. 2a represents a typical diagram involving the exchange of a heavy field. Fig. 2b is a diagram with emission and absorption of a heavy field at the same vertex. Heavy particles are represented by heavy lines.

In order to study the strong coupling limit of this theory, we need to write it in a bose form. We use two Bose fields as explained in section 2. The electric charge density is

$$J_0 =: \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 = \pi^{-1/2} \partial_1 (\phi_1 + \phi_2) \tag{2.24}$$

Thus, similarly to the one flavor case, the hamiltonian takes the form

$$\begin{aligned} \mathcal{H} = N_m \Big[&\frac{1}{2}\Pi_1^2 + \frac{1}{2}\Pi_2^2 + \frac{1}{2}(\partial_1\phi_1)^2 + \frac{1}{2}(\partial_1\phi_2)^2 \\ &+ cm^2 \cos(2\sqrt{\pi}\,\phi_1) + cm^2 \cos(2\sqrt{\pi}\,\phi_2) + \frac{e^2}{2\pi}\Big(\phi_1 + \phi_2 + \frac{\theta}{2\sqrt{\pi}}\Big)^2 \Big] \end{aligned} \tag{2.25}$$

It is convenient to define

$$\begin{aligned} \phi_+ &= 2^{-1/2} \Big(\phi_1 + \phi_2 + \frac{\theta}{2\sqrt{\pi}} \Big), \\ \phi_- &= 2^{-1/2} (\phi_1 - \phi_2), \end{aligned} \tag{2.26}$$

and,

$$\mu^2 = \frac{2e^2}{\pi}, \tag{2.27}$$

and the hamiltonian takes the form,

$$\mathcal{H} = N_m \left\{ \frac{1}{2}\Pi_+^2 + \frac{1}{2}(\partial_1\phi_+)^2 + \frac{\mu^2}{2}\phi_+^2 + \frac{1}{2}\Pi_-^2 + \frac{1}{2}(\partial_1\phi_-)^2 \right.$$
$$\left. -2cm^2 \cos\left[\sqrt{2\pi}\,\phi_+ - \frac{1}{2}\theta\right] \cos\left[2\sqrt{\pi}\,\phi_-\right] \right\} \tag{2.28}$$

The bosonic hamiltonian (28) is suitable for investigating the strong coupling limit when $e/m \gg 1$. Note that the manifest isospin invariance of the fermionic theory is concealed in the bosonic language. The diagonal isospin current may be written as

$$J_\mu^3 = \frac{1}{2}\bar{\psi}\,\gamma_\mu\psi_1 - \frac{1}{2}\bar{\psi}_2\,\gamma_\mu\psi_2 = \sqrt{2\pi}\,\epsilon^{\mu\nu}\partial_\nu\phi_-. \tag{2.29}$$

The other two currents become the complex nonlocal expressions given in section 2.

In the strong coupling limit the ϕ_+ field represents a very heavy field, its mass is approximately μ. The ϕ_- field remains in a much smaller energy scale. We shall not try to analyze the high lying spectrum of the theory, distributed around μ, because of its complexity. On the other hand, the analysis of the low lying spectrum is rather straightforward. In that case we can simply insert the value of the ϕ_+ field,

$$\phi_+ = \frac{\theta}{2\sqrt{2\pi}}, \tag{2.30}$$

into the hamiltonian (28), to obtain a low lying effective hamiltonian. This can also be seen diagrammatically. Suppose we limit ourselves to Green functions involving only the light ϕ_- field in external legs. Then all the diagrams containing an exchange of the heavy field will be suppressed by powers of m/μ (see fig 2a). The only exception to this rule involves diagrams with emission and absorption of the heavy fields on the same vertex (fig. 2b). Those diagrams would contain a factor of $\ln(m/\mu)$. Hence we can eliminate the heavy field altogether if we can get rid of diagrams like fig. 2b. This we do by simply setting the normal-ordering scale of the theory to the heavy fields mass:

$$m = \mu = \sqrt{\frac{2e}{\pi}} \tag{2.31}$$

We then get for the low lying hamiltonian

$$\mathcal{H} = N_m \left[\frac{1}{2}\Pi_-^2 + \frac{1}{2}(\partial_1\phi_-)^2 + 2cm^{2/2}\mu^{1/2}\cos(\sqrt{2\pi}\,\phi_-) \right] \tag{2.32}$$

The hamiltonian (32) apparently involves two mass scales m and μ. This can however be remedied with the help of the renormal-ordering formula (5). Then

$$\mathcal{H} = N_{m'} \left\{ \frac{1}{2}\Pi_-^2 + \frac{1}{2}(\partial_1\phi_-)^2 + m'^2\cos(\sqrt{2\pi}\,\phi_-) \right\}, \tag{2.33}$$

where

$$m' = \left(2cm\mu^{1/2}\cos\frac{1}{2}\theta \right)^{2/3} . \tag{2.34}$$

So we see that the low lying theory obeys the sine-gordon dynamics with $\beta = \sqrt{2\pi}$. Unlike the one flavor case, the influence of θ is rather trivial—a change in the overall mass scale.

The original isospin invariance we had is now buried inside the dynamics of this sine-gordon. The approximation we made by imposing the strong coupling limit cannot spoil it, since the eliminated ϕ_+ field is an iso-singlet.

The spectrum of the sine-gordon theory is exactly known (e.g., Coleman 1975). The theory has, as stable particles, a soliton and an anti-soliton and a breather family. Denoting the soliton mass by M, the breather mass is given by (Dashen et al. 1975b)

$$M_n = 2M\sin\left(\frac{\beta'^2}{16}\right) \qquad \text{for } \beta'^2 = \frac{\beta^2}{1-\beta^2/8\pi} \tag{2.35}$$

and $n = 1, 2, \ldots < 8\pi/\beta'^2$.
Hence, for the value of β that we have here ($\beta = \sqrt{2\pi}$), we get two breathers. The mass of the lighter one is

$$M_1 = 2M\sin(\pi/6) = M. \tag{2.36}$$

The diagonal isospin of this particle, I_3, can be computed using the classical solution and the expression for the charge eq. (29). Then we see that the soliton, anti–soliton and the lightest breather have the same mass and an isospin of $I_3 = 1, -1, 0$ respectively. Hence we conclude that they form an iso–triplet. The heavier breather has a mass of

$$M_2 = \sqrt{3}\, M. \tag{2.37}$$

and, since its I_3 is zero, we conclude that it is an iso–singlet.

In conclusion, we see that we have an iso–triplet and an iso–singlet. The original isospin symmetry, obscured by the bosonization, reappeared in the spectrum as a result of the dynamics of the sine–gordon.

In the next chapter we shall return to this system using the non–abelian bosonization. Then the isospin symmetry will be manifest also in the bose theory. We shall then be also able to generalize this results to more than two flavors.

§2.4 QCD_2 with one flavor

We wish to turn our attention to QCD in two dimensions. This will serve us as a toy model to the real QCD in four dimensions. We shall start by deriving the low lying theory for the one–flavor case. In the next section the many flavor theory will be addressed. Our treatment here of the one flavor case is based on the work of Baluni (1980) and Steinhardt (1980) and is in close analogy to that of electro dynamics, done in section 3. We shall see that the low lying spectrum of the theory obeys a sine gordon dynamics. Hence, we have solitons and breathers accordingly. The soliton is shown to be a baryon and the breather—a meson, interpreted as a stable bound state of baryons—a baryonium.

The model we consider is an $SU(2)$ gauge theory with one fermion in the fundamental representation. The hamiltonian is

$$\mathcal{H} = \sum_{i,j=1}^{N} \{g^2(E^i_j)^2 + \bar{\psi}^i \gamma_1(\delta^j_i \partial_1 - iA^j_i) + m\delta^j_i \bar{\psi}^i_j \psi^j_i\} \tag{2.38}$$

The canonical variables $\{E^i_j, A^i_j\}$ and $\{\psi^{*i}, \psi_i\}$ are constrained by Gauss law as in electro–dynamics,

$$\partial E^i_j = i[A, E]^i_j + \frac{1}{2}\left[\psi^{*i}\,\psi_j - \frac{1}{N}\delta^i_j\,\psi^{\dagger^k}\,\psi_k\right] \tag{2.39}$$

We use here a matrix notation for the gauge fields defined by $2E^i_j = (\lambda^a)^i_j E^a$, where the λ^a are the Gell-Mann matrixes normalized as $\mathrm{Tr}(\lambda^a\lambda^b) = 2\delta^{ab}$.

The hamiltonian (38) represents a many fermion system. So we apparently ought to use the many–fermion formulation of section 2. In the one–flavor case, we can however avoid this complication by choosing a special gauge. In this, so called, Baluni gauge, the fermionic hamiltonian, after the elimination of the gauge fields, involves only diagonal fermion bi–linears. Hence the one fermion bosonization formulae may be used.

The Baluni gauge is

$$A^i_i = 0, \qquad E^k_i = 0, \qquad \text{for } i \neq k \tag{2.40}$$

Then Gauss law (39) can be written as

$$\begin{aligned} \partial_1 e_i &= \sqrt{\pi}\,\psi^{\dagger^i}\,\psi_i = \sqrt{\pi}\,J^i_0 \\ i(e_i - e_j)A^k_i &= \sqrt{\pi}\,\psi^{\dagger^k}\,\psi_i = \sqrt{\pi}\,J^k_{0i}, \qquad i \neq k \end{aligned} \tag{2.41}$$

where the e_i's are defined by

$$2\sqrt{\pi}\,E^i_i = -\left(e_i - \frac{1}{N}\sum_k e_k\right), \qquad i = 1, 2, \ldots, N. \tag{2.42}$$

Inserting (41) into the hamiltonian enables us to eliminate the gauge fields from the hamiltonian. Doing also a Fierz transformation on the Fermi fields we see that the hamiltonian contains only diagonal fermion bi–linears,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$$

$$\mathcal{H}_0 = \sum_i i\,(\bar{\psi}^i\,\gamma_1\partial_1\psi_1 + m\,\bar{\psi}_i\,\psi_i)$$

$$\mathcal{H}_1 = \frac{g^2}{8\pi N}\sum_{i,j}(e_i - e_j)^2 + \frac{1}{2}\sqrt{\pi}\, i \sum_{i\neq j} \bar{\psi}^i(1+\gamma_5)\psi^j\ \bar{\psi}^j(1-\gamma_5)\psi^i/(e_i - e_j) \tag{2.43}$$

We can now easily bosonize the hamiltonian (43) using our correspondence formulae (7). Defining a bosonic field ϕ_i for each fermion ψ_i we get from (7) and (41),

$$\partial_1 e_i = \sqrt{\pi}\,\psi^{\dagger^i}\,\psi_i = \partial_1\phi_i \tag{2.44}$$

Integrating we get

$$e_i = \phi_i + c_i \tag{2.45}$$

The c_i's are constants of integration representing, as for QED_2, a background electric field. Here, however, since our fermions are in the fundamental representation, the constants can be set to zero (see Witten 1979).

Using then the bosonization formulae (7) for the fermion bi-linears, we finally get the Bose version of QCD_2 with one flavor,

$$\mathcal{H} = N_\Lambda\left\{\sum_i \frac{1}{2}\Pi_i^2 + \frac{1}{2}(\partial_1\phi_i)^2 + 2m\Lambda(1-\cos 2\sqrt{\pi}\,\phi_i)\right.$$
$$\left. + \frac{g^2}{8\pi N}\sum_{i\neq j}(\phi_i - \phi_j)^2 + \frac{\Lambda^2}{2}\frac{\sin(\phi_i-\phi_j)}{\phi_i - \phi_j}\right\} \tag{2.46}$$

Like the QED_2 case, the analysis of the hamiltonian (46) is rather difficult. However, deriving the strong coupling limit is quite straightforward. It is done in analogy to the treatment of QED_2 with two flavors (Steinhardt 1980). We first notice that the hamiltonian (46) contains a mass term for all the fields except of the combination:

$$\chi = \frac{1}{\sqrt{N}}\sum_i \phi_i. \tag{2.47}$$

The rest of the fields can be taken to form a unitary matrix together with (47). Denote them by ψ_i, $i = 1, 2, \ldots, N-1$. We can then read the mass of the ψ_i fields,

$$M_\psi = \frac{g}{2\sqrt{\pi}}\sqrt{1-\frac{1}{N}} \tag{2.48}$$

We assume now the strong coupling limit, $g/m \gg 1$, so the ψ fields become very heavy and decouple from the light field χ. In order to eliminate the heavy fields we employ a similar argument to that used in section 3 for QED_2 with two flavors. We then see that we just have to set the ψ_i fields to zero, and to set the normal-ordering scale Λ to the mass of the heavy fields,

$$\Lambda = M_\psi \tag{2.49}$$

We then get the low-lying theory,

$$\mathcal{H} = N_\Lambda \left\{ \frac{1}{2}\Pi_\chi^2 + \frac{1}{2}(\partial_1\chi)^2 + 2m\Lambda N \cos\left(\frac{2\sqrt{\pi}\chi}{\sqrt{N}}\right)\right\} \tag{2.50}$$

which is a sine gordon theory with $\beta = 2\sqrt{\pi/N}$.

As for QED_2 we can show that the the theory (50) contains only one mass scale by performing a renormal-ordering. Using eq. (5) we get

$$\mathcal{H} = N_{m'}\left[\frac{1}{2}\Pi_\chi^2 + \frac{1}{2}(\partial_1\chi)^2 + (m')^2 \cos\left(\frac{2\sqrt{\pi}}{\sqrt{N}}\chi\right)\right], \tag{2.51}$$

with

$$m' = \left[Nm\left(\frac{g}{\sqrt{\pi}}\right)^{1-1/N}\right]^{N/(2N-1)}, \tag{2.52}$$

which is the scale of the theory. The sine-gordon system predicts a soliton and a breather. The soliton solution obeys $\chi(-\infty) = 0$, $\chi(\infty) = 2\pi/\beta$. Then we can compute the quark number current using (7),

$$J_\mu = \frac{1}{\sqrt{\pi}}\sum_i \epsilon_{\mu\nu}\partial_\nu\phi_i = \frac{\sqrt{N}}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial_\nu\chi_1 \tag{2.53}$$

and the quark number,

$$q = \int_{-\infty}^{\infty} j_0\, dx = \int \frac{\sqrt{N}}{\sqrt{\pi}}\partial_1\chi\, dx = \frac{2\pi}{\beta}\frac{\sqrt{N}}{\sqrt{\pi}} = N \tag{2.54}$$

So we see that the soliton has a quark number of N. Defining the baryon number of a quark as $1/N$, we see that it has a baryon number one—the soliton is a baryon. Similarly the anti-soliton is an anti-baryon.

The theory (51) predicts also a family of stable breathers. The breather solution $\beta(x,t)$ obeys $\beta(-\infty,t)=\beta(\infty,t)=0$. Hence we can compute its baryon number using (53) and we see that it is zero. The breather of the sine–gordon theory may be interpreted as a bound state of soliton and an anti–soliton. Hence we see that the breather is a baryonium—a baryon anti–baryon bound state. The mass of the baryonium can be obtained from (35),

$$M_{\bar{\sigma}\sigma} = 2M_B \sin\left(\frac{\pi}{2}\frac{n}{2N-1}\right) \underset{N\to\infty}{\longrightarrow} 4\sqrt{\pi}\sqrt{m\left(\frac{g^2}{\pi}\right)^{1/2}}\; n \tag{2.55}$$

and $n=1,2,\ldots,2N-1$. altogether there are $2N-1$ baryoniums.

§2.5 QCD_2 with many flavors

In this section we shall treat QCD_2 with many flavors using the abelian bosonization. The bosonized version of the theory will be obtained. It is shown that there is a basic difference between the one–flavor theory and the many–flavor case. An extra nonlocal interaction arises which complicates the analysis. This section is based on the work of Cohen, Frishman and Gepner (1982).

Our hamiltonian is a generalization of (38) to many flavors. Denote the flavor index by Greek letters. Then,

$$\mathcal{H} = \sum_{i,j=1}^{N_c}\left[g^2 E_j^{i\,2} + \sum_{\alpha}\bar{\psi}^{i\alpha}\gamma_1(\delta_i^j\partial_1 - iA_i^j)\psi^{j\alpha} + m\delta_j^i\bar{\psi}^{i\alpha}\psi^{j\alpha}\right] \tag{2.56}$$

N_c and N_f are the number of colors and flavors respectively.

The first step, which is the elimination of the gauge fields, is a simple extension of the one–flavor treatment done in section 4. Choosing the Baluni gauge (40), Gauss law becomes

$$\begin{aligned} \partial_1 e_i &= \sqrt{\pi}\sum_{\alpha}\psi^{\dagger i}\psi_i = \sqrt{\pi}J_0^i \\ i(e_i - e_j)A_i^k &= \sqrt{\pi}\sum_{\alpha}\psi^{\dagger k}\psi_i = \sqrt{\pi}J_{0i}^k, \qquad i\neq k \end{aligned} \tag{2.57}$$

where the e_i's are defined as before (eq. 42). We now wish to insert (57) into the hamiltonian (56) for eliminating the gauge fields. Now, however, unlike the one flavor case, we cannot use the Fierz transformation for getting only diagonal fermion bi-linears. We can have the bi-linears diagonal in either color or flavor, but not in both. We choose the former. Thus, we need to use Halpern's formulae for many fermion bosonization. Substituting a bosonic field $\phi_{i\alpha}$ for every fermion $\psi^{i\alpha}$ we get in particular from (11),

$$\bar{\psi}^{\beta i}(1 \pm \gamma_5)\psi_{\alpha i} = \frac{c\mu}{\pi} F\, N_\mu \exp\left[i\sqrt{\pi}\Big(\int d\varsigma\,(\pi_{\beta i} - \pi_{\alpha i}) \pm (\phi_{\beta i} + \phi_{\alpha i})\Big)\right], \tag{2.58}$$

where the F's are defined by (12).

Using this expression we arrive at the bosonic form of QCD_2 with flavor,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$$

$$\mathcal{H}_0 = N_\mu \sum_{\alpha i} \frac{1}{2}(\Pi_{\alpha i})^2 + \frac{1}{2}(\partial_1 \phi_{\alpha i})^2 + \left(\frac{c}{\pi}\right) m\mu (1 - \cos 2\sqrt{\pi}\,\phi_{\alpha i}) \tag{2.59}$$

$$\mathcal{H}_1 = \frac{g^2}{8\pi N_c} N_\mu \sum_{i,j}\left(\sum_\alpha (\phi_{\alpha i} - \phi_{\alpha j})\right)^2 - \left(2c^2 \frac{\sqrt{\pi}}{\pi^2}\right)\mu^2 N_\mu \sum_{i \neq j}\sum_{\alpha\beta} K_{\alpha\beta,ij}$$

$$\times \sin\sqrt{\pi}\left(\int_{-\infty}^{x} (\pi_{\alpha i} - \pi_{\beta i} + \pi_{\beta j} - \pi_{\alpha j})\,d\varsigma - (\phi_{\alpha i} + \phi_{\beta j} - \phi_{\alpha j} - \phi_{\beta j})\right) \Big/ \sum_\alpha (\phi_{\alpha i} - \phi_{\alpha j})$$

The important contrast to the one flavor result (46) lies in the second term of $\mathcal{H}_1$. This cumbersome looking interaction term is further complicated by its nonlocality.

We shall now try to apply the strong coupling limit to the hamiltonian (59). We have here a mass term for the fields $\sum_\alpha \phi_{i\alpha} - \phi_{j\alpha}$, which makes them very heavy. Hence, as before, we can set these fields to zero provided that the normal-ordering scale is taken to be the mass of these fields. So we set

$$\mu = \frac{g\sqrt{n_f}}{\sqrt{2\pi}}\sqrt{1 - \frac{1}{N}}. \tag{2.60}$$

Let us concentrate on the two flavor case. Then it is convenient to define the field $\chi_i^- = 2^{-1/2}(\phi_{1i} - \phi_{2i})$, and $\chi = \frac{1}{\sqrt{N_c}}\sum \phi_{\alpha i}$. The interaction term then assumes the form,

$$I = \frac{4c^2}{\pi} G^2 N_G \sum_{i \neq j}\left(\cos\sqrt{2\pi}\,(\chi_i^- - \chi_j^-) - (-1)^{N(i)+N(j)} \cos\sqrt{2\pi}\int_{-\infty}^{x} (\pi_i^- - \pi_j^-)\right), \tag{2.61}$$

where $N(i)$ is the number of fields with color i and flavor 1.

The analysis is further complicated by the presence of momenta in the interaction. Hence we resort to the static approximation which means taking π to be zero. Then the interaction (61) will further force some fields to be zero. To see exactly which, we must find the minimum of I and substitute the fields which give the minimum, into the hamiltonian.

We can write (61) as,

$$I = V_0 + V_1 \sum_{i,j} \cos\sqrt{2\pi}\,(\chi_i^- - \chi_j^-) = v_0 + v_1 \left| \sum_i e^{i\sqrt{2\pi}\chi_i^-} \right|^2 \qquad (2.62)$$

Hence the minimum is obtained when

$$\sum_i e^{i\sqrt{2\pi}\chi_i^-} = 0, \qquad (2.63)$$

and the interaction term becomes zero under this condition. The mass term (the last term of $\mathcal{H}_0$ in eq. (59)) also becomes zero, since

$$\sum_{i,\alpha} \cos(2\sqrt{\pi}\,\phi_{\alpha i}) = \sum_i \cos\left(\sqrt{\frac{2\pi}{N_c}}\chi + \sqrt{2\pi}\,\chi_i^-\right) = \mathrm{Re}\, e^{i\sqrt{2\pi/N_c}\chi} \sum_i e^{i\sqrt{2\pi}\chi_i^-} = 0 \quad (2.64)$$

Hence we are left with a massless hamiltonian for the fields solving (62). This is an unacceptable result since those fields are not physical. Our result means that there are **no** light particles in the spectrum. This is simply wrong.

Hence we conclude that the approximations we have carried are not justified. The blame, we believe, is on the static approximation that was needed due to the nonlocal nature of the interaction. This nonlocality is intimately connected with the isospin invariance. namely, the isospin transformations which are manifest and simple in the fermionic theory, become complex and nonlocal in the bosonic theory. Once we made the static approximation, we broke by hand the isospin symmetry.

Thus we see that there is a need for an improved bosonization scheme, one in which the isospin symmetry remains manifest and local. precisely these properties are obeyed by the non–abelian bosonization discovered by Witten (1984). In the next chapter we address this question again, showing that the low–lying theory of multi flavor QCD_2 contains a rich spectrum.

Chapter 3

Non Abelian Bosonization and Gauge Theories

§3.1 Introduction

Recently, Witten has suggested a new bosonization procedure (1984) that consists of an improvement over the usual bosonization (Coleman 1975) by being manifestly isospin invariant. Here we shall return to the gauge models that were treated in chapter 2, applying on them this technique. We shall see how the non abelian bosonization resolves the problems previously encountered. This chapter is based on my work (Gepner 1984).

Recently, there was also quite an interest in QCD effective low energy theories in four dimensions. It was suggested by Witten (1983) that baryons can be solitons of such a theory. The analogous problem in two dimensions is treated here. We start from the lagrangian of QCD_2 with many fermion flavors in the fundamental representation of color. Then we prove that indeed such an effective theory is the outcome of the analysis and that it describes baryons as solitons of that theory.

In order to analyze the resulting bosonic theory, we need to discuss two main issues. First we need to perform renormal-ordering of operators, which is necessary for showing that the theory has one mass-scale and for computing this mass scale. Hence we discuss the regularization and the dimension of operators of this WZ non-linear sigma model. We also need to preform a semi-classical quantization in order to get the isospin content and approximate masses. We show that the representations that can possibly appear are all those having a fixed set of quantum numbers determined by the classical solution. We show also that those quantum numbers can be obtained from Witten's expression for the currents. Since these two issues are outside of our main course, they are treated in the appendices.

We apply our methods to QED_2 with mass as an interesting example. The case of two flavors was previously discussed in section (2.3), using the old bosonization scheme, following Coleman (1975). We reach an agreement with his results and this provides an important check for our methods. In addition, we are able to generalize his results to more than two flavors.

The abelian bosonization was shown in chapter 2 to be a very useful tool for analyzing gauge models in two dimensions. In particular, the massive Schwinger model (section 1.3) and QCD_2 with one flavor (section 1.4) were treated and the spectrum of the theories was found in the strong coupling limit.

On the other hand, bosonization has been less successful when more than one flavor was included (Amati et al. 1981, Cohen et al. 1983). In section (2.5) QCD_2 with flavor was bosonized, resulting in a non local hamiltonian. The isospin invariance was hidden inside a complicated interaction and an attempt to analyze it using static approximation ended with producing unacceptable results. Here we shall address this problem again using Witten's non abelian bosonization scheme.

This chapter is organized in analogy to chapter 2. In Section 1 we discuss generalities of the non abelian bosonization.

In section 2 we treat the multi-flavor massive Schwinger model and it is shown that there are two low-lying multiplets. One is an iso-singlet, and the other is in the adjoint of $SU(N)$. Their masses are given and for two flavors they reasonably agree with the results of Coleman (section 2.3), who treated the two flavor case using the abelian bosonization. As a side result we are able to show that an $SU(2)$ WZ theory with mass term is equivalent to a sine-gordon theory with $\beta = \sqrt{2\pi}$.

In Section 3 QCD with many flavors is analyzed and we find the spectrum in the strong coupling limit for two flavors. We show that the low lying theory is exactly described by a WZ effective lagrangian, analogous to the four dimensional one that was proposed to describe QCD_4 on semi-phenomenological grounds (Witten 1983). Like the four dimensional case, the baryons appear as a soliton of the theory. The spectrum that we find in the low lying theory, for two flavors, is closely parallel to the results of Steinhardt (1980), who investigated the one flavor theory. Like him, we show that the spectrum consists of a baryon family (soliton) and a baryonium family (breather). However, here, unlike the one flavor case, an interesting isospin structure is revealed.

Using semi-classical methods, the isospin of the baryon was shown to be $N_c/2$—the totally symmetric isospin representation. Namely, only the decuplet exists for $SU(3)_f$, as we expect in two dimensions from naive quark model arguments. In addition, we expect iso-rotationally excited baryons, having isospin higher than $N_c/2$ by some integer.

A breather solution shown to exist in our theory, describes a baryonium—a baryon anti-baryon state, (as the breather of the sine-gordon theory describes a soliton anti-soliton state). The lowest baryonium is an iso-singlet. It is excited to give more particles, in two ways. 1) "radial" excitations, found by the Bohr-Sommerfeld quantization condition. They have the same isospin but higher masses. 2) Iso-rotational excitations. They give higher isospin baryoniums, having an arbitrary integer isospin. The hierarchy of the baryoniums is, in fact, analogous to the levels of the hydrogen atom, having a principal "radial" quantum number and isospin quantum numbers (angular momentum). Approximate masses were given for these particles.

We postpone to the appendices two issues concerned with the analysis of the resulting bosonic theories. In appendix A questions of regularization and renormal-ordering are addressed. In appendix B the semi classical quantization of the resulting bosonic lagrangians is treated.

The importance of the work presented in this chapter may lie in various directions. First, many of the methods developed in the appendices will enable further application of non abelian bosonization for the investigation of two dimensional models. Those can also describe four dimensional phenomena, e.g. the Callan-Rubakov effect (Callan 1982, Rubakov 1981; 1982). We should mention that the study of two dimensional WZ theories is not yet completed, and our results may assist it; in particular—the equivalence we have found between an $SU(2)$ WZ theory and a sine–gordon theory.

The spectrum of QCD_2, found by us, may be instructive concerning the validity of composite models of quarks and leptons. It is vital for these models to find massless fermions in a confined strongly interacting gauge theory. Exactly such fermions were found here. In this context, it may be interesting to study how 't Hooft consistency condition is satisfied here, as it may teach us more about this important model building tool.

Finally, our results on QCD_2 provide an important support for Wess–Zumino current algebra theories as candidates to describe the dynamics of QCD. We have found a direct link between these theories and QCD_2. It may encourage us to think that such a link exists in four dimensions as well; or, at least, that these theories should be taken even more seriously, than they had already been, for the purpose of understanding the strong interaction.

§3.2 The non abelian bosonization

In section 2.6 we have treated QCD_2 with flavor using the abelian bosonization. We than had to use Halpern's expressions for the fermion bi–linears. Those are cumbersome and nonlocal and hence prevented us from analysing the strong coupling regime of the theory. This is a general drawback of the abelian bosonization. The other intimately related problem is the spoiling of the manifest isospin invariance. This invariance, which is a clear and simple $U(N)$ group for the fermion lagrangian, becomes complex and non local for the equivalent bosonic theory.

This was seen in section (2.3) where QED_2 with two flavors was treated. Luckily, in the two–flavor case we did not need to use non diagonal fermion bi–linears. Still, the isospin symmetry was buried inside the dynamics instead of being manifest. For more than two flavors we again encounter the problem of nonlocal expressions.

Thus comes the need for an improved bosonization scheme, one in which a many fermion theory would be mapped into a many boson theory that would have a manifest isospin symmetry and would be simple enough to be useful.

This problem was solved by Witten (1984). In the following we describe the bosonization for Dirac fermions, instead of Majorana, since this is what we shall later need. Take a theory of n Dirac fermions (massless and free)

$$\mathcal{L} = \bar{\psi}_k i \not{\partial} \psi_k \tag{3.1}$$

The internal symmetry group of this lagrangian is $G = SU(N)_L \times SU(N)_R \times U(1)_V$ (We disregard the $U(1)_A$.) Let us define a bosonic multiplet g that takes its values in $U(N)$ and transforms under $SU(N) \times SU(N)$ in the so called non linear realization and is a singlet of $U(1)_V$

$$g \to A g B^{-1} \qquad A \in SU(N)_L\ , \quad B \in SU(N)_R \tag{3.2}$$

Then Witten shows that the theory (1) is equivalent to a bosonic non linear sigma model given by the lagrangian

$$\mathcal{L} = \frac{1}{2\lambda^2} Tr(\partial_\mu g)(\partial_\mu g^{-1}) + n\Gamma(g) \tag{3.3}$$

with $\lambda = \sqrt{\frac{4\pi}{n}}$ and $n = 1$. Γ is a two dimensional Wess–Zumino term

$$\Gamma = \frac{1}{12\pi} \int_B d^3y \epsilon^{ijk} Tr\, \bar{g}^{-1} \frac{\partial \bar{g}}{\partial y_i} \bar{g}^{-1} \frac{\partial \bar{g}}{\partial y_j} \bar{g}^{-1} \frac{\partial \bar{g}}{\partial y_k} \tag{3.4}$$

B is an extension of our space time to a three dimensional ball whose boundary is space time. $\bar{g}$ is an extension of g to that space. Γ is a local lagrangian in the usual meaning since the integrand can be locally written as a total divergence and then as a surface integral using Gauss theorem.

Then the following dictionary of bosonization is given

$$J_+^{ij} = \psi_+^{i\dagger} \psi_+^j = \frac{in}{2\pi} (g^{-1} \partial_+ g)_{ij} \tag{3.5}$$

$$J_-^{ij} = \psi_-^{i\dagger} \psi_-^j = -\frac{in}{2\pi} (\partial_- g \cdot g^{-1})_{ij} \tag{3.6}$$

Where we use the light cone variables, $\partial_\pm = \partial_0 \pm \partial_1$; $J_\pm = J_0 \pm J_1$.

A mass bi-linear is identified by

$$\psi_+^{i\dagger}\psi_-^k = Kg_{ik} \tag{3.7}$$

K is a regularization dependant constant. For Majorana fermions one must take g in $O(N)$ and a factor of half in (3).

We wish to emphasize one major difference between the $U(N)$ and the $O(N)$ bosonizations. This arises because $U(N)$ is not a simple group. Hence the $U(1)$ part should be treated separately e.g. when we wish to compute various commutators as Witten does for $O(N)$. Then one can repeat Witten's computation of the commutators for the $SU(N)$ part, and for the $U(1)$ part use canonical commutators, and by that prove the equivalence of the Dirac fermions theory and the $U(N)$ WZ model. It is evident that one should then disregard the $U(1)_A$ symmetry. Hence the Green functions of the bosonic theory would correspond to the Green functions of the fermionic theory regularized as to conserve $SU(N)_L \times SU(N)_R \times U(1)_V$. This was originally discussed using path integral methods by Di–Vecchia et al. and Gonzales et al. (1984). They show that the $SU(N)$ vector part is not conserved in Green's functions involving two or more currents. It was argued by Y. Frishman (1984) that one can still classify states using this symmetry, because than we need only to use the current bracketed between the states in question.

We shall now return to the models treated in chapter 2. It will be shown how using non–abelian bosonization cures the previously mentioned problems. The low lying spectrum of this theories will be thus derived.

§3.3 Electro Dynamics With Flavor

We shall return here to the electro dynamics model with flavor, using the non abelian bosonization. The lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{N} \bar{\psi}_i(i \not\partial - e \not A - m)\psi_i \tag{3.8}$$

This model has a $SU(N)$ symmetry. The case of $N = 2$ was treated in section (2.4) (Coleman 1975). Using the abelian bosonization scheme it was shown that the low lying spectrum of the theory consists of one iso–singlet and one iso–triplet of particles. Their masses were also given, and they obey

$$M_{I=0} = \sqrt{3}\, M_{I=1} \tag{3.9}$$

We shall analyze the lagrangian (8) using the non abelian bosonization. We show that for general N the spectrum contains one singlet particle and one particle multiplet in the adjoint of $SU(N)$. We shall also give the semi–classical masses. In the case of $N = 2$ we get an agreement with Coleman. This consists of an important check of our formalism.

In order to bosonize the lagrangian we must first cast it into a hamiltonian form and eliminate the gauge fields. This is done exactly as in section (2.4). For the sake of convenience we repeat the main steps. We choose the gauge $A_1 = 0$, then the equation of motion for A_0 becomes a constraint:

$$\partial_1^2 A_0 = -e\sum_i : \bar{\psi}_i\psi_i := -eJ_0 \tag{3.10}$$

The solution is

$$A_0 = -e\partial_1^{-2}J_0 - Fx - G \tag{3.11}$$

(F and G are some constants of integration). or

$$F_{01} = e\partial_1^{-1}J_0 + F \tag{3.12}$$

F is physical and should be interpreted as background electric field. The hamiltonian is

$$\mathcal{H} = \sum_i \bar{\psi}_i(i\gamma_1\partial^1 + m)\psi_i + \frac{1}{2}F_{01}^2 \tag{3.13}$$

We are now ready to proceed with the non abelian bosonization. Taking g in $U(N)$, we have for J_0

$$J_0 = \frac{i}{2\pi}\partial_1 ln\ det\ g \tag{3.14}$$

Substituting that into (12) we get

$$F_{01} = \frac{ie}{2\pi} ln\ det\ g + F \tag{3.15}$$

For convenience let us separate g into $U(1) \times SU(N)$

$$g = e^{i\varphi\sqrt{4\pi/N}}s \qquad s \in SU(N) \tag{3.16}$$

and φ is a scalar field. Also define a θ–angle by $\theta = 2\pi F/e$. Then, substituting equation (15) into the hamiltonian, we get the bosonized form¶

$$\mathcal{H} = \frac{1}{2}\pi_\varphi^2 + \frac{1}{2}(\partial_1\varphi)^2 + \mathcal{H}_{free}(s) + \frac{1}{2}\frac{e^2N}{\pi}\left(\varphi - \frac{\theta}{2\sqrt{\pi N}}\right)^2 + mc\mu\sqrt{N}\, N_\mu\left(Re\ Tr\ e^{i\varphi\sqrt{4\pi/N}}s\right) \tag{3.17}$$

Let us now assume the strong coupling limit $e \gg m$. Then φ becomes a heavy field with the mass

$$m_\varphi \approx \frac{e\sqrt{N}}{\sqrt{\pi}} \tag{3.18}$$

and $m_\varphi \gg m$. All the diagrams that involve an exchange of the heavy φ field (fig. 2a, in page 32) would contain a propagator and hence would be smaller by powers of m/m_φ. The only exception to that are diagrams in which the φ field is emitted and absorbed at the same vertex (fig. 2b, in page 32). This diagrams would contain a factor of $ln(m/m_\varphi)$.

¶ It can be seen that the hamiltonian that corresponds to the WZ theory (A.1) is given by

$$\mathcal{H} = \frac{\lambda^2}{16}(J_+J_+ + J_-J_-) - m \cdot Re\ Tr(g)$$

Then it is also seen that computing the classical energy for a diagonal classical solution can be done either using this hamiltonian, or by naively inserting it into the lagrangian (A.1) and computing as for an abelian group.

Diagrams like fig. 3b can be eliminated by normal ordering with respect to the mass of the heavy fields. Actually, since we are dealing with an interacting theory, normal-ordering cannot be defined in the usual sense. For a discussion of the regularization and related matters see appendix A. Hence we can eliminate φ from the hamiltonian by setting $\varphi = \theta/2\sqrt{\pi N}$, and $\mu = m_\varphi$. Then we get for the low lying hamiltonian,

$$\mathcal{H} = \mathcal{H}_{free}(s) + mc\mu\sqrt{N}\, N_\mu Re\; e^{i\theta/N} Tr\; s \tag{3.19}$$

and $\mu = e\sqrt{\frac{n}{\pi}}$. This looks like $U(N)$ fermion theory with mass and the extra constraint $det(U) = 1$. In order to get the mass scale of this theory we have to do renormal-ordering. This is done using equations (A.27) and (A.33)

$$\mu^{1-\frac{1}{N}} N_\mu s = \bar{m}^{1-\frac{1}{N}}\, N_{\bar{m}} Re\; Tr\; s \tag{3.20}$$

then writing

$$\frac{mc}{\sqrt{N}}\mu N_\mu (Re\; Tr\; s) = \bar{m}^2\, N_{\bar{m}}(Re\; Tr\; s) \tag{3.21}$$

and solving for $\bar{m}$ we get:

$$\bar{m} = \left(mc\sqrt{N}\,\mu^{1/N}\right)^{N/N+1} \tag{3.22}$$

Then (19) can be written with only one mass scale $\bar{m}$

$$\mathcal{H} = \mathcal{H}_{free}(s) + \bar{m}^2\sqrt{2}\, N_{\bar{m}}(Re\; e^{i\theta/N} Tr\; s) \tag{3.23}$$

In the case of $N = 2$, which is the case that Coleman treats, we can further simplify (23). This is because the trace of an $SU(N)$ matrix is always real. Then we get

$$\mathcal{H} = \mathcal{H}_{free}(s) + m'^2 N_{m'} Re\; Tr\; s \tag{3.24}$$

and

$$m' = \left(\sqrt{2}\, mc\, \mu^{1/2} cos\frac{\theta}{2}\right)^{2/3} \tag{3.25}$$

Note, the important fact that the only way the θ influences the two flavor theory, is by changing the overall mass scale—θ can, in fact, be ignored. This is in radical contrast to the one flavor theory (Coleman et al. 1975, Coleman 1976). We also expect θ to be important for more than two flavors.

For the same theory (8) with $N = 2$, we got in section (2.4) the following hamiltonian for the low lying spectrum

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\varphi)^2 + (m_1)^2 N_{m_1} cos\,(\sqrt{2\pi}\,\varphi) \tag{3.26}$$

$$m_1 = (2cm\mu^{\frac{1}{2}} cos\,\frac{1}{2}\theta)^{\frac{2}{3}} \tag{3.27}$$

and now we see that the mass scale m_1 and m' coincide apart from a constant factor

$$m_1 = m' 2^{\frac{1}{3}} \tag{3.28}$$

So we agree with Coleman about the mass scales.

Now we would like to know what particles are found in the lagrangian (23), and what are their quantum numbers. This will be done using semi-classical methods. For a detailed discussion of the semi-classical methods we use here see appendix B.

First we observe that all the particles must have zero quark number (mesons, if you wish). From semi-classical arguments the quark number is

$$B = \frac{i}{2\pi} ln\ det(s)\Big|_{-\infty}^{\infty} = 0 \tag{3.29}$$

The next step is to carry out the semi-classical quantization. The first thing we need to know is the classical minima of the hamiltonian (23). Say s_0 minimizes V. Then we can diagonalize s_0 .

$$s_0 = diag(e^{i\alpha_1}, \ldots, e^{i\alpha_n}) \qquad \sum \alpha_i = 2\pi r \tag{3.30}$$

(r is some integer.) The condition for extremum then becomes— $\frac{\partial V}{\partial \alpha_i} = \frac{\partial V}{\partial \lambda} = 0$; where V is given by

$$V(\alpha_i, \lambda) = \sum_i cos\left(\alpha_i + \frac{\theta}{N}\right) + \lambda(\alpha_i - 2\pi r) \tag{3.31}$$

(λ is a Lagrange multiplier). The global minimum is then found to be for $\alpha_i = 2\pi m_i$ (the m_i's are integers), for θ between $-\pi$ and π. (If θ is outside the range we can bring it in by simultaneously shifting $\theta \to \theta - 2\pi k$ and $s \to s \cdot e^{2i\pi k/N}$, which leaves the hamiltonian unchanged. This shows that physics is 2π–periodic in θ). Now, due to the constraint $det(s) = 1$, the simplest solitonic solution we can have is

$$s(x) = diag(e^{i\alpha(x)}, e^{-i\alpha(x)}, 1, \ldots 1) \tag{3.32}$$

and $\alpha(-\infty) = 0$; $\alpha(\infty) = 2\pi$.

Interestingly, for this particular solution, θ is again irrelevant even for more than two flavors. θ can be absorbed in the overall mass scale $\bar{m}$ as in the two flavor case—define $m' = \bar{m}\,(cos\,\frac{\theta}{N})^{N/(N+1)}$. However, there are other classical solutions, for which the influence of θ is far less trivial (for $N > 3$).

The quantum numbers of this state can be obtained using (B.7) (with $n = 1$). They agree with the quantum numbers of a quark–antiquark state, $\bar{q}_1 q_2$. Hence the possible representations coming from semi-classical quantization are in the adjoint of $SU(N)$ or higher representations that contains states with the adjoint quantum numbers. We believe those higher representations to be unstable since here $n = 1$, and apart from different groups, this is the free fermion case. (This can also be seen from Coleman's work, where there are no more stable particles for the sine–gordon theory he obtains.) There would be a breather solution if in (32) we replace $\alpha(x)$ by $\beta(x,t)$, which is a breather solution of the sine–gordon. Then it is easily seen that this solution is a singlet. For $N = 2$ we get isospin zero and one, in agreement with Coleman.

We would now like to obtain an estimate for the masses of these states. This is done by simply computing the classical energy of those solutions. The result for the soliton mass is (for any N)

$$M_0 = \frac{8m'}{\sqrt{\pi}} \tag{3.33}$$

Computing the classical energy of the soliton of the theory (26), which for $N = 2$ is equivalent to our theory (24), we get

$$M_s^{(Col)} = \frac{8m_1}{\sqrt{2\pi}} = 2^{-1/6} M_s \tag{3.34}$$

This is a very reasonable agreement.

In order to get the mass of the breather we need to use Bohr–Sommerfeld quantization formula, much in analogy to the treatment of the sine–gordon breather. (See, for example, Coleman 1975.) We get (for any N)

$$M_{breather} = 2M_s sin\left(\frac{k\pi}{8}\right) \qquad k = 1, 2, 3 \tag{3.35}$$

Since the $k = 1$ breather is supposed to be the $I = 0$ state of Coleman, we see that we have

$$\frac{M_{I=0}}{M_{I=1}} = 2sin\left(\frac{\pi}{8}\right) \approx 0.766 \tag{3.36}$$

Comparing this with equation (9), we see that we have about half of the expected ratio. We attribute this discrepancy to higher order quantum effects that we have not included in our computation.

Finally, we would like to include the iso–rotational energy as a correction to the classical masses. We shall do that in the case of $N = 2$. Then equation (B.6) gives us the iso–rotational energy of the soliton if we take $I = eg = 1$. Computing the moment of inertia M we get

$$M = \frac{4}{\lambda^2} \int_{-\infty}^{\infty} sin^2\alpha \, dx = \frac{32}{3\pi^2} \cdot \frac{1}{M_s} \tag{3.37}$$

Hence, the iso–rotational energy of the soliton is

$$E_{(iso)} = \frac{1}{2M} = \frac{3\pi^2}{64} M_s \approx 0.46 M_s \tag{3.38}$$

Including this energy in the soliton mass (33) will give us a discrepancy of about 30% in (34), which is reasonable. Also it will further decrease the mass ratio (36), worsening our agreement with Coleman by about 30%.

§3.4 QCD_2 with flavor

We now wish to discuss QCD in two dimensions. The one flavor theory was treated using the abelian bosonization in section (2.4). For one flavor, the low lying spectrum of the theory was obtained and it was shown to include a baryon and baryoniums. Analyzing the multi-flavor theory using these methods (section 2.5) ran into difficulties due to the non locality and complexity of the resulting bosonized hamiltonian. Also the manifest isospin invariance was hidden in the bosonic language. These are exactly the problems that the non abelian bosonization comes to solve.

Applying it here we shall obtain the low lying spectrum of QCD_2 with flavor. Our main result is showing that a WZ type effective lagrangian arises dynamically for QCD in two dimensions. We shall further analyze the spectrum of this WZ lagrangian using the methods described in the appendices. We show that a baryon exists as a soliton of this theory much in the same way as was phenomenologically suggested for QCD_4 (Witten 1983).

The first step in the analysis is to choose a gauge and to eliminate the gauge fields. This is done exactly as in section (2.5). For convenience we repeat it here. The hamiltonian of QCD_2 with flavor is

$$\mathcal{H} = \sum_{i,j=1}^{N_c} \left\{ g^2 (E^i_j)^2 + \sum_{\alpha=1}^{N_f} \bar{\psi}^{\alpha i} \gamma_1 (i\delta^j_i \partial_1 - A^j_i) \psi_{\alpha j} + m \bar{\psi}^{\alpha i} \psi_{\alpha j} \delta^j_i \right\} \tag{3.39}$$

(i,j are color indices; α is a flavor index; N_c and N_f are the numbers of colors and flavors respectively.)

We assume that all the fermions have the same mass. Matrix notation for the gauge fields is also used

$$E^i_j = \frac{1}{2} (\lambda^a)^i_j E^a \tag{3.40}$$

We have to supplement (39) with Gauss law as a constraint

$$\partial_1 E^i_j = i[A, E]^i_j + \frac{1}{2} \left(\psi^{\dagger \alpha i} \psi_{\alpha j} - \sum_k \frac{1}{N_c} \delta^i_j \psi^{\dagger \alpha k} \psi_{\alpha k} \right) \tag{3.41}$$

(We assume no summation over repeated indices in this section.) As a gauge condition we take Baluni's gauge

$$A_i^i = 0 \;;\quad E_k^i = 0 \qquad for \quad i \neq k \tag{3.42}$$

Using Gauss low and the gauge condition we can eliminate the gauge fields. Then

$$\partial_1 e_i = \sqrt{\pi} \sum_\alpha \psi^{\dagger \alpha i} \psi_{\alpha i} \tag{3.43}$$

$$-i(e_i - e_k) A_i^k = \sqrt{\pi} \sum \alpha \psi^{\dagger \alpha k} \psi_{\alpha i} \qquad for \quad i \neq k \tag{3.44}$$

with the e's defined by

$$2\sqrt{\pi} E_i^i = e_i - \frac{1}{N_c} \sum_k e_k \tag{3.45}$$

The next step is to bosonize the hamiltonian. We have in total $N_c \times N_f$ fermions so we apparently need to take $U(N_c \times N_f)$. In fact we need a much smaller group, namely, $U(N_f)^{N_c}$. Specifically, the only currents necessary are those that are diagonal in color. So we choose N_c matrices of $U(N_f)$.

$$g_i \in U(N_f) \qquad i = 1, 2, \ldots, N_c \tag{3.46}$$

Then

$$(J_+^i)_{\alpha\beta} = \psi_+^{\dagger \alpha i} \psi_{+\beta i} = \frac{i}{2\pi} (g_i^{-1} \partial_+ g_i)_{\alpha\beta} \tag{3.47}$$

$$(J_-^i)_{\alpha\beta} = \psi_-^{\dagger \alpha i} \psi_{-\beta i} = -\frac{i}{2\pi} (\partial_- g_i \, g_i^{-1})_{\alpha\beta} \tag{3.48}$$

In order to translate (43) we need

$$\sum_\alpha \psi^{\dagger \alpha i} \psi_{\alpha i} = Tr(J_0^i) = \frac{i}{2\pi} Tr \; (g_i^{-1} \partial_1 g_i) \tag{3.49}$$

We can then write (43) as

$$\partial_1 e_i = \frac{i}{2\sqrt{\pi}} \partial_1 Tr \; ln \; g_i \tag{3.50}$$

or

$$e_i = \frac{i}{2\sqrt{\pi}} Tr\ ln\ g_i + C_i \tag{3.51}$$

The C_i's are constants of integration which express background color electric fields. However, unlike the case of $U(1)$, here they can be set to zero (Witten 1979).

From (44) we get

$$A_i^k = 2\pi \frac{\psi^{\dagger ak}\psi_{ai}}{Tr\ ln\ g_i g_k^{-1}} \qquad for\ \ i \neq k \tag{3.52}$$

Inserting (52) into (39), making a Fierz transformation and, using the bosonization formula for the fermion bi-linears, we arrive at

$$\mathcal{H} = \mathcal{H}_{free}(g_i) + \mathcal{H}_{int}(g_i) \tag{3.53}$$

$$\mathcal{H}_{int}(g_i) = \sum_{i,j} -\frac{g^2}{32\pi^2 N_c}(Tr\ ln\ g_i g_j^{-1})^2 - \pi\Lambda^2 \frac{Tr\ g_i g_j^{-1}}{Tr\ ln\ g_i g_j^{-1}} + \sum_i mc\Lambda\sqrt{N_f}\,Tr\ g_i \tag{3.54}$$

This is the bosonized version of QCD_2 with flavor.

To get the low lying spectrum, let us look on the strong coupling limit where $g/m \gg 1$. Then define the potential V by

$$V(g_i) = \sum_{i,j} -\frac{g^2}{32\pi^2 N_c}(Tr\ ln\ g_i g_j^{-1})^2 - \pi\Lambda^2 \frac{Tr\ g_i g_j^{-1}}{Tr\ ln\ g_i g_j^{-1}} \tag{3.55}$$

It is convenient to separate g_i from its $U(1)$ part

$$g_i = e^{i2\sqrt{\pi/N_f}\varphi_i} u_i \qquad u_i \in SU(N_f) \qquad e^{i2\sqrt{\pi/N_f}\varphi_i} \in U(1) \tag{3.56}$$

Now V will take the more transparent form

$$V(\varphi_i, u_i) = \sum_{i,j} \frac{g^2 N_f}{8\pi N_c}(\varphi_i - \varphi_j)^2 + \frac{i\sqrt{\pi}\Lambda^2}{2\sqrt{N_f}} \frac{e^{i2\sqrt{\pi/N_f}(\varphi_i - \varphi_j)} K_{ij}}{\varphi_i - \varphi_j} \tag{3.57}$$

where $K_{ij} = Tr(u_i u_j^{-1})$. It can be seen that K_{ij} is hermitian, $K^\dagger = K$. For $N_f = 2$, K_{ij} is real as it is the trace of an $SU(2)$ matrix. In order for the potential V to be free of singularity we need exactly this property of K_{ij}. So from now on we shall concentrate on the case of $N_f = 2$. Then we can write V as

$$V = \sum_{i,j} \frac{g^2 N_f}{8\pi N_c}(\varphi_i - \varphi_j)^2 - \frac{\sqrt{\pi}\,\Lambda^2}{2\sqrt{N_f}} \frac{sin\ 2\sqrt{\pi/N_f}\,(\varphi_i - \varphi_j)}{\varphi_i - \varphi_j} K_{ij} \tag{3.58}$$

The absolute minimum of this potential when varied on φ_i is at $\varphi_i = \varphi_j$ for all i and j. Then V becomes,

$$V = -\frac{\sqrt{\pi}\,\Lambda^2}{2\sqrt{N_f}} \sum_{i,j} K_{ij} \tag{3.59}$$

Now we should minimize V with respect to the u_i's. Remembering then that $Tr(u_i u_j^{-1}) \leq N_f$, where equality holds only for $u_i = u_j$, we get for the minimum of V,

$$\begin{cases} u_i = u_j = u \\ \varphi_i = \varphi_j = \varphi \end{cases} \tag{3.60}$$

Then we can insert (60) into the hamiltonian (54) and V drops out. The result is

$$\mathcal{H} = N_c \mathcal{H}_{free}(g) + N_c mc\sqrt{N_f}\ \Lambda N_\Lambda Tr\ g \tag{3.61}$$

and, $g \in U(N_c)$. The lagrangian corresponding to (61) is

$$\mathcal{L} = \frac{N_c}{8\pi} Tr(\partial_\mu g)(\partial_\mu g^{-1}) + Nc\Gamma - N_c m\ \Lambda\sqrt{N_f}\, N_\Lambda Tr\ g \tag{3.62}$$

By reasons similar to those in section 2 we also have to take Λ equal to the the mass of the heavy fields. Their mass is given by

$$M_\varphi \approx \frac{g\sqrt{N_f}}{2\pi N_c} \tag{3.63}$$

Only then can we ignore a diagram like fig. 3b. We see that (62) is a WZ like effective low energy lagrangian of the kind proposed for QCD in four dimensions (Witten 1983). The rest of this section will be devoted to the analysis of this lagrangian. We can rewrite (62) using our renormal–ordering prescription with one mass scale by computing the anomalous dimension of g. From (A.23) we can can compute for the $SU(N_f)$ part

$$\Delta(u) = \frac{2C_1}{C_2 + n} = \frac{N_f^2 - 1}{N_f(N_c + N_f)} \tag{3.64}$$

For the $U(1)$ part we get from (A.18)

$$\Delta\left(e^{i\varphi\sqrt{4\pi/N_cN_f}}\right) = \frac{1}{N_cN_f} \tag{3.65}$$

Summing up we get

$$\Delta(g) = \frac{N_f^2 - 1}{N_f(N_c + N_f)} + \frac{1}{N_cN_f} \tag{3.66}$$

Then solving,

$$\mu^2 N_\mu Trg = N_c m\Lambda\sqrt{N_f}\, N_\Lambda Tr\ g = N_c m\Gamma\sqrt{N_f}\left(\frac{\mu}{\Gamma}\right)^\Delta N_\mu Tr\ g \tag{3.67}$$

(Eq. A.17 was used), the mass scale of the theory, μ is seen to be

$$\mu = \left(\sqrt{N_f}\, N_c m\Lambda^{1-\Delta}\right)^{1/(2-\Delta)} \tag{3.68}$$

Then the lagrangian (62) becomes

$$L = \frac{N_c}{8\pi}(\partial_\mu g)(\partial_\mu g^{-1}) + N_c\Gamma - \mu^2 N_\mu Re\ Tr\ g \tag{3.69}$$

We see that μ is the only mass scale of the low lying theory.

From the semi–classical analysis of section 1 we know what to expect for the spectrum of this lagrangian. A soliton that appears in the theory is given by (B.1). Using (B.3) we realize that it has a quark number equal N_c. Defining the baryon number of a quark as $1/N_c$, we get for the soliton a baryon number one—the soliton is a baryon. Using (B.5) we see that this baryon has an isospin $N_c/2$. It is fully symmetric in the isospin wave function, as we would expect from quark model arguments (see section 1). The soliton mass can be obtained by computing the classical energy and the iso–rotational energy. The result is

$$M_s = \frac{4}{\sqrt{\pi}}\mu\sqrt{N_c} \tag{3.70}$$

This result becomes exact for N_c large enough. There would also be a baryoniums family given by the breather solution (B.29). These particles are mesons (no baryonic charge) and have isospin $0, 1, 2, \ldots$.

To get the masses of the breathers we can use the Bohr–Sommerfeld quantization formula as in section 2. Then we get in fact the same result as for the sine gordon breather (Coleman 1975),

$$M_{(breather)} = 2M_s sin\left(\frac{\beta^2 k}{16}\right) = 2M_s sin\left(\frac{\pi k}{4N_c}\right) \tag{3.71}$$

and $k = 1, 2, 3, \ldots, 2N_c - 1$.

It is surprising that we get here the same number of breathers and an identical mass formula for them (except of the mass parameter) as Steinhardt who treated the one flavor case (see section 2.4).

We can compute the iso–rotational energy of the soliton using (B.21) and (B.26), and by identifying $eg = N_c/2$ and $n = N_c$. We get for the moment of inertia M

$$M = \frac{N_c}{2\pi}\int_{-\infty}^{\infty}(1 - cos\alpha)dx = \frac{4N_c^2}{\pi^2 M_s} \tag{3.72}$$

And the iso–rotational energy for the lowest soliton ($I = N_c/2$) is

$$E^{(iso)} = \frac{1}{2M}(I(I+1) - e^2g^2) = \frac{N_c}{4M} = \left(\frac{\pi^2}{16N_c}\right)M_s \tag{3.73}$$

For $N_c = 3$ we get $E^{(iso)} \approx 0.21M_s$, or about 20% increase in the soliton mass. The iso–rotational energy of the particles with higher isospin, coming from the iso–rotationally excited soliton, is given by (B.26). Part of this particles may be stable as they will be energetically forbidden to decay.

Appendices

Appendix A: Regularization and Renormal–Ordering

We now wish to consider the questions relating to the regularization of our sigma lagrangian with a mass term.

$$\mathcal{L} = \frac{1}{2\lambda^2} Tr(\partial_\mu g)(\partial_\mu g^{-1}) + n\Gamma(g) + \frac{1}{2} m \cdot Tr(g + g^{-1}) \tag{A.1}$$

We assume that $\lambda^2 = \frac{4\pi}{n}$, but n is not necessarily 1. Let us assume first that m is equal to zero. For $n = 1$ this theory describes massless free fermions and thus it is conformally invariant and the beta function must also vanish. It is also known that, for $n \neq 1$, this theory remains conformally invariant (Witten 1984), even though the theory does not describe massless free fermions. The case $m = 0$, $n \neq 1$, describes a theory of n massless fermions constrained to move together. But if $m \neq 0$, then this is a non trivial theory, the spectrum of which occupy us in appendix B.

For $m = 0$ we are interested in computing Green functions such as

$$G(x_1, x_2, \ldots, x_n) = \langle g(x_1) g(x_2) \ldots g(x_r) g(x_{r+1})^\dagger \ldots g(x_n)^\dagger \rangle \tag{A.2}$$

by using perturbation theory. First, we define $g = exp(i\lambda\varphi^a T^a)$, where T^a are generators of $U(N)$, so that $Tr(T^a T^b) = 2\delta^{ab}$. Now we can compute in perturbation theory the Green functions and divert the infrared divergencies by assuming that φ^a has a small mass m and in the end let m go to zero. We expand the lagrangian in powers of λ and get

$$\mathcal{L} = \sum_a \frac{1}{2}(\partial_\mu \varphi^a)^2 + O(\lambda^2) \tag{A.3}$$

Those higher orders in λ we regard as a perturbation, and compute in the framework of perturbation theory. This procedure is analogous to that used by Coleman (1975) for computing

$$G(x_1, x_2, \ldots , x_n) = \langle : e^{i\beta_1\varphi(x_1)} : : e^{i\beta_2\varphi(x_2)} : \ldots : e^{i\beta_n\varphi(x_n)} : \rangle \tag{A.4}$$

for a free massless scalar theory (namely, assuming a mass m and letting it go to zero in the end of the computation).

Now, for $m \neq 0$ we are interested in how m gets renormalized. An equivalent question is how the operator g gets renormalized in the massless theory. We shall propose the following renormalization prescription. Denote by $|0_m\rangle$ the vacuum which is annihilated by the negative frequency part of the fields φ^a decomposed according to the mass m

$$\varphi^a_- = \int \frac{dk}{\sqrt{2\pi}\sqrt{2k_0}} a^a_k e^{-ikx} \tag{A.5}$$

And $\varphi^a_- |0_m\rangle = 0$, defines this state. Now we compute with momentum cutoff

$$\langle 0_m | g_{ij} | 0_m \rangle = Z\left(\frac{\Lambda}{m}\right)\delta_{ij} \tag{A.6}$$

(This is the form since g is naively dimensionless.) Define the regularized g by

$$\langle 0_m | N_m(g) | 0_m \rangle = 1 \tag{A.7}$$

or

$$N_m(g) = Z^{-1}\left(\frac{\Lambda}{m}\right) g \tag{A.8}$$

The anomalous dimension of g is defined as usual by

$$\Delta(g) = \lim_{\Lambda \to \infty} - m \frac{\partial}{\partial m} ln(Z) \tag{A.9}$$

Thus we can use the usual renormalization group reasoning to get the dependance of $N_m(g)$ on m. Define

$$\langle 0_\mu | N_m(g) | 0_\mu \rangle = f(\mu, m, \Lambda) \tag{A.10}$$

Then we know that

$$f(\mu, m, \Lambda) Z\left(\frac{\Lambda}{m}\right) = \langle 0_\mu | g | 0_\mu \rangle \tag{A.11}$$

and is independent of m. Thus applying $m\frac{\partial}{\partial m}$ on it we get

$$m\frac{\partial}{\partial m} f(\mu, m, \Lambda) + m\frac{\partial}{\partial m} ln\ Z\left(\frac{\Lambda}{m}\right) = 0 \tag{A.12}$$

Taking $\Lambda \to \infty$ and defining,

$$g\left(\frac{\mu}{m}\right) = \lim_{\Lambda \to \infty} f(\mu, m, \Lambda) \tag{A.13}$$

we reach

$$\left[\frac{m}{\partial m} - \Delta\right] g(\mu/m) = 0 \tag{A.14}$$

The solution to this equation is

$$g(\mu/m) = \left(\frac{\mu}{m}\right)^{\Delta} \tag{A.15}$$

or

$$\langle 0_\mu | N_m g | 0_\mu \rangle = \left(\frac{\mu}{m}\right)^{\Delta} \tag{A.16}$$

thus

$$\mu^{\Delta} N_\mu g = m^{\Delta} N_m g \tag{A.17}$$

This is the desired "renormal-order" formula, to be compared with Coleman's formula for scalar field.

$$N_m e^{i\beta\varphi} = \left(\frac{\mu}{m}\right)^{\beta^2/4\pi} N_\mu e^{i\beta\varphi} \tag{A.18}$$

Here we have tacitly assumed that the operator g does not mix with other operators upon regularization, or that the theory (A.1) is renormalizable. For $n \neq 1$ this is not obvious but can be proved using the conformal invariance of the theory with $m = 0$ (Zamolodchikov 1984).

In what follows we shall need the anomalous dimension of g in order to preform renormal ordering. For $n = 1$ and $G = SU(N)$ we know that the dimension is

$$\Delta(g) = 1 - \frac{1}{N} \tag{A.19}$$

This is because for $g \in SU(N)$ and ψ a Dirac fermion we have

$$\psi^{\dagger}_{+i}\psi_{-j} = c\ e^{i\varphi\sqrt{4\pi/N}}g_{ij} \tag{A.20}$$

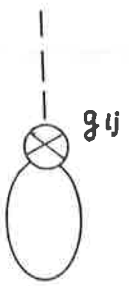
Using Coleman's formula for the dimension of the scalar operator gives us

$$\Delta\left(e^{i\varphi\sqrt{4\pi/N}}\right) = \frac{4\pi}{N}\cdot\frac{1}{4\pi} = \frac{1}{N} \tag{A.21}$$

and since φ is decoupled from g the anomalous dimension is simply additive, and we get (A.19).

We can also quite easily compute the dimension of g for n large enough, by simply computing the lowest order diagram for the vacuum expectation of g, which is given by fig.3.

Fig. 3



The result is

$$\Delta(g) = \frac{C_1}{2n} \tag{A.22}$$

Where C_1 is defined by $C_1 \cdot I = T^a T^a$.

For $SU(N)$ we have $C_1 = 2(N^2-1)/N$. While this value must be exact for $n \to \infty$, for $n = 1$ it does not give the value expected from (A.19). A dimension that would be consistent with both (A.22) and (A.19), could look like

$$\Delta(g) = \frac{(N^2-1)/N}{n+N} \tag{A.23}$$

In fact, it was shown by Al. Zamolodchikov (1984) that this value is exact, using the conformal invariance of the theory. It was also shown by him that the general expression for the dimension of g, for any group G, is

$$\Delta(g) = \frac{C_1}{C_2+n} \tag{A.24}$$

C_2 is the second Casimir of the adjoint, $C_2\delta_{ad} = f_{abc}f_{dbc}$. (For $O(N)$ we need to take $n = 1/2$ to get $\Delta(g) = 1$, due to the extra factor of 1/2 that we have in the lagrangian (3) for Majorana fermions).

Now, in (A.19) both sides must be regularized. We use our previously defined renormalization prescription (A.7). From dimensional arguments, the constant c appearing in (A.20) must be proportional to the renormalization mass. Then we get

$$\psi_+^{i\dagger}\psi_-^j = Km : e^{i\varphi\sqrt{4\pi/N}} : N_m(g) \tag{A.25}$$

Here K is a constant which can depend on N. Using the definition of normal-ordering made above, we wish to determine this constant. For this purpose it is convenient to determine first the value of the following Green function of the massless theory.

$$G_{ik}(x) = \lim_{m \to 0} m^{2\Delta}\langle 0_m | T\ N_m g_{ij}(x)\ N_m g_{jk}^\dagger(0) | 0_m \rangle \tag{A.26}$$

From renormalization group considerations similar to those we had before, we know that G(x) behaves like

$$G_{ik}(x) = k\delta_{ik}x^{-2\Delta(g)} \tag{A.27}$$

Computing the constant K is not a simple task and we shall have to resort to perturbation theory and some amount of guess work. Since we have here $\lambda^2 = 4\pi/n$, a perturbative expansion in λ is sensible when n is large enough. Obviously we are mainly interested in the case n=1. The lowest order diagram for $G(x)$ is given by fig.4 .

Fig. 4

The value of which is

$$G_0(x) = -\frac{(T^aT^a)_{ik}}{n} ln(mcx) = -\frac{C_2\delta_{ik}}{n} ln(mcx) \qquad (A.28)$$

(c is a mathematical constant related to Euler constant.) The coefficient looks like the lowest order of

$$2\Delta(g) = \frac{C_2}{C_1 + n} \qquad (A.29)$$

Thus the full answer is likely to be

$$e^{-2\Delta(g) ln(mcx)} = (mcx)^{-2\Delta(g)} \qquad (A.30)$$

or

$$G_{ik} = (cx)^{-2\Delta(g)}\delta_{ik} \qquad (A.31)$$

This argument is not fully rigorous even for a very large n, since we have avoided the question of interchanging the small m and the large n limits. The fermion equivalent to G(x) for $U(N)$ and $n = 1$ is

$$G(x) = \left\langle T : \bar{\psi}_i\left(\frac{1+\gamma_5}{2}\right)\psi_j : (x) \ : \bar{\psi}_j\left(\frac{1-\gamma_5}{2}\right)\psi_k : (0)\right\rangle = \frac{\delta_{ik}N}{x^2} \qquad (A.32)$$

Thus we get

$$K = c\sqrt{N} \tag{A.33}$$

Appendix B: Quantum Numbers and Semi–Classical Quantization

As mentioned earlier the lagrangian (A.1) describes a free massive fermion theory for $n = 1$ and $G = O(N)$ or $U(N)$. However, it becomes non trivial for $n \neq 1$, or for different groups. Then it describes an n fermion theory with non trivial interaction. In what follows we need the spectrum of that theory. We suspect that this is an exactly integrable model, but this is yet to be proved. (For $U(1)$ this theory reduces to a non trivial ($\beta \neq 2\sqrt{\pi}$) sine–gordon which is known as integrable.) So we are forced to make the analysis in the framework of semi–classical quantization. This will give us the possible representations of the particles of the theory for any n and G, and estimates for their masses which are exact for n large enough. The first step is to pick up an anzats for a classical solution. We shall start by limiting ourselves to $U(N)$. Then a possible classical solution is

$$Q(x) = diag\left(e^{-i\alpha(x)}, 1, 1, \ldots, 1\right) \tag{B.1}$$

(By "diag" we mean the diagonal matrix having these values in the diagonal .) $\alpha(x)$ obeys the classical sine–gordon equation

$$\frac{n}{4\pi}\alpha'' + m \cdot sin\alpha = 0 \tag{B.2}$$

The finite energy condition for the hamiltonian, derived from the lagrangian (A.1), implies that $Tr(g)$ is maximal or that $\alpha(x) \doteq 2\pi m$ and $m \in Z$. Because of our experience with the sine–gordon equation, we choose $\alpha(-\infty) = 0$, $\alpha(\infty) = 2\pi$. We expect this to be a stable particle, the soliton of this theory. It is simple to compute some of the quantum numbers of this soliton using the expressions for the currents. The quark number current is

$$B_\mu = \frac{in}{2\pi}\epsilon_{\mu\nu}Tr(g^{-1}\partial_\nu g) \tag{B.3}$$

So the quark number is

$$B = \int_{-\infty}^{\infty} B_0 dx = \frac{in}{2\pi} \int_{-\infty}^{\infty} \partial_1 (Tr\ ln\ g) dx = \frac{in}{2\pi} Tr\ ln\ g \Big|_{-\infty}^{\infty} \qquad (B.4)$$

Computing for $Q(x)$ we get a quark number n for the soliton. Take $U(2)$ for simplicity. Then we can compute in a similar fashion the I_3 of $Q(x)$.

$$I_3 = \int_{-\infty}^{\infty} Tr(\sigma_3 Q^{-1} \partial_1 Q) dx = \frac{n}{2} \qquad (B.5)$$

So the isospin of the soliton must be at least $n/2$.

$$I = \frac{n}{2} + k \qquad k \geq 0 \quad integer \qquad (B.6)$$

The reason that I_1, I_2 can not be computed reliably this way, is that using the semi-classical method we compute only expectation values of operators bracketed between the states in question. So we can get the eigenvalues only for operators, for which the classical solution corresponds to an eigenvector. This method can also be applied quite simply to other groups. We see that the general rule is that only representations, which contain a state with the appropriate set of quantum numbers, can appear. For $U(N)$ we can get in this way all the diagonal charges by simply computing

$$I^a = \frac{in}{2\pi} Tr\ T^a lnQ(x) \Big|_{-\infty}^{\infty} \qquad (B.7)$$

T^a is diagonal. Then the possible representations that can appear when quantizing $Q(x)$ are those that contain a state with the set of quantum numbers I^a.

Now we will approach this problem in a totally different way. We shall make a semi-classical quantization of the zero modes. Our reason for this is two fold. First, this will consist of a check of our formalism, and it will also give us semi-classical mass corrections. We wish to carry out a semi-classical quantization around the solution $Q(x)$ for the lagrangian (A.1).

The usual procedure for semi-classical quantization is to define

$$g = A(t)Q(x)A(t)^{-1} \qquad (B.8)$$

and to compute a lagrangian for $A(t)$ by substituting (B.8) into the lagrangian (A.1). For the sake of simplicity let us limit ourselves to the case of $U(2)$. Then the soliton solution is

$$Q(x) = diag\left(e^{i\alpha(x)}, 1\right) \tag{B.9}$$

and, $\alpha(-\infty) = 0$; $\alpha(\infty) = 2\pi$. We can now show that the lagrangian for $A(t)$ describes a particle moving on a two-sphere S^2 with a monopole in the center of it. First, it is immediately seen that the lagrangian must possess the following symmetries

$$A(t) \to A(t)B(t) \quad ; \; B(t) \in H \tag{B.10}$$

$$A(t) \to C \cdot A(t) \quad ; \; C \in G \tag{B.11}$$

The symmetry (B.10) arises because it leaves g unchanged in (B.8). It is a gauge like symmetry and it means that $A(t)$ takes its values in G/H , or in this case in $SU(2)/U(1) \approx S^2$, a two sphere. The symmetry (B.11) is due to the $U(2)$ vector symmetry of the lagrangian (A.1), $g \to CgC^{-1}$. It means that the lagrangian for $A(t)$ has a global G symmetry. In the $U(2)$ case this is a particle moving on a two sphere with a spherical symmetry. We represent $A(t) \in SU(2)$ by

$$A(t) = a_0(t) + i\vec{\sigma} \cdot \vec{a}\,(t) \tag{B.12}$$

and $a_0^2 + \vec{a}^2 = 1$.

Then making the rotation $A(t) \to A(0)^{-1}A(t)$, and considering t to be small, we can assume that $a_0 \approx 1$, $\vec{a} \approx 0$. Then we easily substitute (B.8) into (A.1). Computing for the kinetic term is straightforward and gives

$$Tr(\partial_\mu g)(\partial_\mu g^{-1}) = Tr[A^{-1}\dot{A}, Q][A^{-1}\dot{A}, Q^{-1}] \tag{B.13}$$

Then substituting $A^{-1}\dot{A} = \dot{\vec{a}} \cdot \vec{\sigma}$ into (B.13), we get

$$L_1 = \frac{1}{2}M(\dot{a}_1^2 + \dot{a}_2^2) \tag{B.14}$$

with

$$M = \frac{n}{2\pi}\int_{-\infty}^{\infty}(1 - cos\alpha)dx \tag{B.15}$$

For the Wess–Zumino term it is convenient to use the polar coordinates form, available for $SU(2)$ (Witten 1984),

$$\Gamma = \frac{1}{\pi}\int d^2x\ \phi(x)sin^2\psi(x)sin\theta(x)\epsilon^{\mu\nu}(\partial_\mu\psi)(\partial_\nu\theta) \tag{B.16}$$

where (ψ, θ, ϕ) are polar angles describing the three sphere which is $SU(2)$, its Cartesian coordinates are (a_0, a_1, a_2, a_3). Then Q would correspond to the curve $(\alpha(x)/2, \pi/2, \pi/2)$ and g would be

$$g = \left(\frac{\alpha(x)}{2}, \frac{\pi}{2} - a_2, \frac{\pi}{2} + a_1\right) \tag{B.17}$$

(We need to keep only the first order in a since we assumed it is small.) Then we get for Γ

$$\Gamma = \frac{1}{\pi}\int d^2x\left(\frac{\pi}{2} + a_1\right)sin^2(\alpha/2)\cdot\epsilon^{10}\left(\frac{1}{2}\frac{d\alpha}{dx}\right)\left(-\frac{da_2}{dt}\right) = \int dtL_2 \tag{B.18}$$

Where L_2 is (up to total time derivatives)

$$L_2 = \frac{1}{4}(a_1\,\dot{a}_2 - a_2\,\dot{a}_1) \tag{B.19}$$

Hence the full lagrangian is

$$L = \frac{1}{2}M(\dot{a}_1^2 + \dot{a}_2^2) + \frac{1}{4}(a_1\,\dot{a}_2 - a_2\,\dot{a}_1) \tag{B.20}$$

$$L = \frac{1}{2}M(\dot{a}_1^2 + \dot{a}_2^2) + \frac{n}{4}(a_1\,\dot{a}_2 - a_2\,\dot{a}_1) \tag{B.21}$$

where

$$M = \frac{n}{2\pi}\int_{-\infty}^{\infty}(1 - cos\alpha)dx \tag{B.22}$$

The lagrangian (B.20) is mathematically identical to the lagrangian of a particle moving on a plane with magnetic field perpendicular to the plane. We can now use the spherical symmetry to reconstruct the full lagrangian. We then see that the full equation of motion (without assuming t is small) must be

$$M\,\dot{a}_i = \frac{n}{2}\epsilon_{ijk}a_j\,\dot{a}_k \tag{B.23}$$

This describes a free particle moving freely on the sphere with a mass M and a monopole sitting at the origin (radial magnetic field) with a strength $eg = -n/2$. Dirac quantization condition is equivalent to n being an integer, which is a result of the quantization of the WZ term. The quantization of this quantum mechanical lagrangian involves monopole harmonics (Coleman 1981). The correct isospin operator is

$$\vec{I} = m\vec{a}\times\dot{\vec{a}} - eg\frac{\vec{a}}{a} \tag{B.24}$$

which obeys the usual angular momentum algebra. We can also compute

$$I_3|\theta = 0\rangle = -eg|\theta = 0\rangle \tag{B.25}$$

(This is analogous to our computation of I_3 of $Q(x)$ using Witten's current formula.)

Thus the allowed values for the isospin I are

$$I = |eg|, |eg|+1, \ldots \tag{B.26}$$

It can be seen that we get here a value for the monopole strength, eg, that is identical to the value of I_3 as computed using (B.5). This would be true for any classical solution that we would choose. Hence, we again see¶¶ that the allowed isospin representations are those that contain a state whose quantum numbers are the same as the ones given by equation (B.7).

¶¶ We can assume that our lagrangian for A is a WZ theory in a quotient space G/H in one dimension, since it possesses the right symmetries. Then our rule for the allowed representations can be seen also from the work of Rabinovici et al. (1984) where they treat such models in odd dimensions. The semi-classical quantization was done in four dimensions for $SU(3)$ by Guadagnini (1984).

The iso-rotational energy for definite isospin takes the form

$$E_{ir}^{I} = \frac{1}{2M}[I(I+1) - e^2g^2] \tag{B.27}$$

Hence the mass of the soliton with isospin I is (to this approximation)

$$M_I = M_s + E_{ir}^{I} \tag{B.28}$$

M_s is the classical energy of the soliton. For $n = 1$ we know that the theory contains one stable fermion of isospin 1/2, an elementary free fermion in the fundamental representation. This is also seen from the analysis above. The higher isospin states are non elementary in this case. We see that we have a family of particles with isospin starting with $n/2$. Which of those particles is stable can only be answered by further analysis

We can interpret the lagrangian (A.1) as describing an effective low energy theory for QCD with n colors. In fact, in section 3 we prove that this is the result of the dynamics of QCD_2. Then the soliton has quark number n from equation (B.3), or baryon number one. The soliton is a baryon. The fact that the lowest isospin for the soliton is $n/2$ should not surprise us. This can be seen by applying quark model arguments. Since the color wave function is anti-symmetric, and the total wave function is anti-symmetric as well, the isospin wave function must be totally symmetric. So we expect to find the totally symmetric isospin representation which is $I = n/2$. (Unlike four dimensions we have no spin, so we do not get the mixed symmetry representations.) In addition, we have higher isospin representations, but none lower.

The most general classical abelian solution we could have chosen is

$$Q(x) = diag\left(e^{i\alpha_1(x)}, e^{i\alpha_2(x)}, \ldots, e^{i\alpha_n(x)}\right) \tag{B.29}$$

and $\alpha_i(-\infty) = 0$; $\alpha_i(\infty) = 2\pi m_i$.

Most of these particles are not going to be stable, but will decay into others. To answer rigorously which are the ones would require further analysis. By analogy with the sine-gordon system, we believe in the stability of the following breather solution

$$Q(x) = diag\left(e^{i\beta(x,t)}, 1, 1, \ldots, 1\right) \tag{B.30}$$

where $\beta(x,t)$ is the breather solution of the sine–gordon. This solution has a zero baryon number and it represent a baryonium. Moreover, for this solution the "monopole strength", eg, equals zero. So the allowed isospin values are $I = 0, 1, 2, \ldots$

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בעיות אלה נפתרו במאמר השני (Gepner, 1984), אשר מתואר בפרק מספר שלוש. בהשתמשנו בבוזוניזציה הלא אבלית שהוצעה לאחרונה על ידי Witten (1984), קיבלנו את הצורה הבוזונית של QCD_2 עם שני טעמים. הוכח שהתורה המתקבלת באנרגיות נמוכות מהווה מודל סיגמה עם איבר Wess-Zumino אשר המקדם שלו הוא מספר הצבעים. תוצאה זאת היא אנלוגית למודלים של אלגברת הזרמים, אשר הוצעו לתאר את הספקטרום של QCD בארבעה מימדים מתוך שיקולים חצי-פנומנולוגיים (Witten, 1983). בניגוד למקרה הארבעה-מימדי, כאן אנו מסוגלים להוכיח תוצאה זאת במדוייק ישירות מתוך הלגרגיאן של QCD.

על מנת לנתח את התורה הבוזונית אנו נאלצים לטפל גם בשתי הנקודות הבאות: 1) רגולריזציה, סידור נורמלי ומימדים של אופרטורים. זה נחוץ על מנת לקבל את הסקלה של התיאוריה המתקבלת. 2) ניתוח סמי-קלסי הנחוץ לשם קבלת הספקטרום והמסות של החלקיקים בתורה. נושאים אלה מטופלים בשני נספחים.

אזי אנו מוצאים בריונים עם איזו-ספין Nc/2 או יותר. ישנה גם משפחה של בריוניומים (Baryoniums) עם איזו-ספין שלם כלשהוא. נוסחאות מסה ניתנות עבור חלקיקים אלה.

כשימוש מעניין לשיטות אלה אנו מטפלים במודל שווינגר מרובה הטעמים אנו מקבלים את הספקטרום הנמוך ובכך מכלילים למספר כלשהוא של טעמים את תוצאותיו של קולמן (Coleman, 1976) עבור המקרה של שני טעמים. מוכח כי התורה מכילה מולטיפלט אחד בהצגת ה- adjoint וחלקיק אחד שהוא סינגלט של איזו-ספין. מסות סמי-קלסיות ניתנות עבורם, בהתאמה סבירה עם התוצאות המדויקות של Coleman במקרה של שני טעמים. כתוצר לוואי אנו מסוגלים להראות שתורת Wess-Zumino של SU(2) עם איבר מסה היא אקוויולנטית לתורת Sine-Gordon לא טריויאלית ($\beta = \sqrt{2\pi}$).

במאמר השני (Frishman et. al. ,1984) הקשר בין התגרות הדו והארבע מימדיות נחקר ישירות. הוכח שהסקטור בו $J=L+S+I=0$ של התורה המונופולית, אקויולנטי לתורה הסוליטונ וכן שהמטען מתקבל כולו וסקטור זה. טיפלנו גם בנפרד ב-"חצי פרמיונים" המתקבלים והראינו שמטענים שבורים שונים נובעים עבורם כתוצאה מתנאי השפה השונים.

הפרק הראשון בעבודה כולל סקירה של תופעת המטען השבור, כמו גם תיאור התוצאות השונות שקיבלנו בנושא זה.

בשטח של בוזוניזציה ותורות כיול, נעשו שתי עבודות. בראשונה שבהן (Cohen et. al. ,1983) בצענו בוזוניזציה של QCD_2 בתורה המכילה מספר טעמים (flavors) כולם בהצגה היסודית של $SU(N)$ צבע (color). בוזוניזציה היא טכניקה בה תורה פרמיונית ממופה לתורה בוזונית (Coleman 1976). החשיבות שלה נובעת מזה שבמקרים רבים התורה המתקבלת היא פשוטה יותר. כמו כן, התחום של קבוע צימוד חלש בתורה הבוזונית במקרים רבים ממופה לזה החזק בתורה הפרמיונית. דוגמאות הם מודל שווינגר עם מסה (Coleman et. al. ,1975 , Coleman ,1976), וכן QCD_2 עם טעם אחד (Steinhardt ,1980 ,Baluni,1980).

תוך שימוש בבוזוניזציה האבלית קיבלנו את הצורה הבוזונית של QCD עם כמה טעמים (Cohen et. al. ,1983). השימוש בבוזוניזציה האבלית לתורות כיול מתואר בפרק השני.

אנו מראים שההמילטוניאן הבוזוני מכיל איבר לא לוקלי $V(\phi_i,\pi_i)$ מדרגת סיבוכיות גבוהה (הוא מכיל אינטגרלים של $\pi_i-\pi_j$). אזי אנו בודקים את הגבול של קבוע צימוד חזק על מנת לקבל את הספקטרום הנמוך. משום האינטראקציה הלא-לוקלית לקחנו גם את הקירוב הסטטי $\pi\to 0$. אזי המצבים המתקבלים אינם בהצגות של איזו-ספין. מכאן שהקירוב הסטטי אינו מוצדק.

הקדמה

העבודה אשר עשיתי במשך הדוקטורט שלי עוסקת בעיקר בחקר של תופעות לא הפרעתיות בתורות שדות. שתי תופעות עיקריות כאלו נחקרו, והם המטען השבור הנרכש ע"י עירורים טופולוגיים כאשר הם מצומדים לפרמיונים, וכן חקר הגבול של קבוע צימוד חזק בתורות כיול בשני מימדים בעזרת השימוש בסכימות בוזוניזציה שונות.

תופעת המטען השבור נתגלתה לראשונה ע"י Jackiw and Rebbi (1976). הם שמו לב שמונופול או סוליטון רוכשים מטען חצי שלם כאשר הם מצומדים לפרמיון. זה היה די מדהים שבתורה המכילה רק מטענים שלמים עבור הפרמיונים, ימצאו חלקיקים הנושאים מטען לא שלם. זה מעניין במיוחד משום שמונופולים כאלה הם מסקנה המתבקשת מסכימות "האיחוד הכביר" (Grand Unification). מאוחר יותר הסתבר (Goldstone and Wilczek, 1981) שבמקרה בו הפרמיון הוא בעל מסה ניתן לקבל מספר פרמיוני טרנסנדנטלי.

עבור

$$\mathcal{L} = \bar{\psi}(i\,\not\partial - m - i\,g\gamma_5\phi)\psi,$$

כאשר $\phi(\infty) = -\phi(-\infty) = \nu$, אנו מקבלים את המטען הטרנסנדנטלי

$$g = \frac{1}{\pi}\,\mathrm{arctg}\,\frac{g\nu}{m}$$

הלגרגיאן למעלה הוא עבור התורה הדו-מימדית. במקרה ה-4-מימדי מתקבל אותו ערך עבור המטען. שוויון מוזר זה הוסבר על ידינו משתי נקודות מבט שונות (Frishman et. al., 1983, 1984). בעבודה הראשונה נמצא הקשר בין המטען השבור לבין הקומוטטורים האנומליים. ביטויים אנאלוגיים עבור הקומוטטורים האנומליים בשתי התיאוריות הובילו לתוצאות דומות עבור המטענים. כך הוסברה חלק מחידה זאת, כמו גם מציאת קשר חשוב בין האנומליה לבין המטען השבור, שהם שני אספקטים חשובים של תורת השדות.