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The sixth Painlevé equation as isomonodromy deformation of an irregular system: monodromy data, coalescing eigenvalues, locally holomorphic transcendents and Frobenius manifolds

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Abstract

The sixth Painlevé equation PVI is both the isomonodromy deformation condition of a 2-dimensional isomonodromic Fuchsian system and of a 3-dimensional irregular system. Only the former has been used in the literature to solve the nonlinear connection problem for PVI, through the computation of invariant quantities $p_{jk} = \mathrm{tr}(\mathcal{M}_j\mathcal{M}_k)$. We prove a new simple formula expressing the invariants p_{jk} in terms of the Stokes matrices of the irregular system, making the irregular system a concrete alternative for the nonlinear connection problem. We classify the transcendents such that the Stokes matrices and the p_{jk} can be computed in terms of special functions, providing a full non-trivial class of 3-dim. examples such that the theory of non-generic isomonodromy deformations of Cotti *et al* (2019 *Duke Math. J.* **168** 967–1108) applies. A sub-class of these transcendents realises the local structure of all the 3-dim Dubrovin–Frobenius manifolds with semisimple coalescence points of the

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type studied in Cotti *et al* (2020 *SIGMA* **16** 105). We compute all the monodromy data for these manifolds (Stokes matrix, Levelt exponents and central connection matrix).

Keywords: sixth Painlevé equation, isomonodromy deformations, irregular system, Stokes matrices, coalescing eigenvalues, Dubrovin–Frobenius manifolds, Laplace transform.

Mathematics Subject Classification numbers: 34M55

1. Introduction

The present paper proposes three main results. The first is a monodromy formula which implements the isomonodromic deformation method for the sixth Painlevé equation using a 3-dimensional irregular system. The second is the classification of the solutions of the Painlevé equation which realizes the first non-trivial class of examples satisfying the theory of non-generic isomonodromy deformations developed in [9]. The last is the computation of all the monodromy data (local moduli) of the 3-dimensional Dubrovin–Frobenius manifolds with semisimple coalescence points of the type studied in [10]. The second and third issues above provide an interesting application of sixth Painlevé equation.

The sixth Painlevé equation, hereafter denoted by PVI, is the nonlinear ordinary differential equation (ODE)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{2}\left[\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right]\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-\left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right]\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right],$$

where the coefficients can be parameterized by four complex constants $\theta_1,\theta_2,\theta_3,\theta_\infty$, with $\theta_\infty\neq 0$, as follows

$$2\beta=-\theta_1^2,\quad 2\delta=1-\theta_2^2,\quad 2\gamma=\theta_3^2,\quad 2\alpha=(\theta_\infty-1)^2. \tag{1.1}$$

PVI has three fixed singularities at $x=0,1,\infty$, called *critical points*, and its solutions are called *transcendents*, because generically are not expressible in terms of classical functions through Umemura's admissible operations [52–54].

In order to characterize a transcendent, it is important to know its behaviour, called *critical*, at the critical points. A most difficult issue is the *nonlinear connection problem*, which is to express the one or two integration constants parametrizing the critical behaviour of a branch of a transcendent at a critical point (branch cuts $|\arg x|<\pi$, $|\arg(1-x)|<\pi$ in the x-plane), in terms of the integration constants expressing its critical behaviour at another critical point. The *isomonodromy deformation method* has proved effective in solving this problem: PVI is equivalent to the isomonodromy deformation equations, i.e. the Schlesinger equations, of a 2×2 isomonodromic Fuchsian system

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}=\sum_{k=1}^{3}\frac{\mathcal{A}_k(u)}{\lambda-u_k}\Phi \tag{1.2}$$

with $u = (u_1, u_2, u_3) \in \mathbb{C}^3$ and eigenvalues of $\mathcal{A}_k = \pm\theta_k/2$, $\sum_{k=1}^{3} \mathcal{A}_k = \mathrm{diag}(-\theta_\infty/2, \theta_\infty/2)$. Equivalence means that solutions $\mathcal{A}_1(u), \mathcal{A}_2(u), \mathcal{A}_3(u)$ of the Schlesinger equations, up to constant diagonal conjugation, are in one-to-one correspondence with solutions of PVI, with

$$x = \frac{u_2 - u_1}{u_3 - u_1}.$$

The correspondence has been known since [18, 51] and is realized by the explicit formulae in appendix C of [42]. The core of the method is to *be able to explicitly compute* the three *different* pairs of integration constants for a given branch respectively at $x = 0, 1$ and ∞ in terms of *the same* traces

$$p_{jk} = \mathrm{tr}(\mathcal{M}_j \mathcal{M}_k), \quad 1 \leqslant j \neq k \leqslant 3, \tag{1.3}$$

where $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in SL(2,\mathbb{C})$ are the monodromy matrices at $\lambda = u_1, u_2, u_3$ of a fundamental matrix solution $\Phi(\lambda, u)$, defined for u varying in a sufficiently small simply connected domain of $\mathbb{C}^3 \backslash \bigcup_{j\neq k}\{u_j = u_k\}$, and λ in the plane with branch-cuts from u_1, u_2, u_3 towards infinity. This method was first used in the seminal paper [40] for a wide class of generic solutions, and then by several authors in non-generic cases (see [30] for a review).

PVI also admits an isomonodromic representation by a 3-dimensional system with Fuchsian singularity at $z = 0$ and irregular at $z = \infty$. Indeed, in [36] a class of integrable systems called *JMMS* [43] are described in the loop algebra framework of [1] and, using duality of moment maps [2], dual isomonodromic systems are obtained. In particular, the dual to (1.2) turns out to be

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = \left(U + \frac{V(u)}{z}\right) Y, \qquad U = \mathrm{diag}(u_1, u_2, u_3), \tag{1.4}$$

with a certain matrix $V(u)$, satisfying

$$\text{eigenvalues of } V = 0, \frac{\theta_\infty - \theta_1 - \theta_2 - \theta_3}{2}, \frac{-\theta_\infty - \theta_1 - \theta_2 - \theta_3}{2}, \qquad \mathrm{diag}(V) = -\mathrm{diag}(\theta_1, \theta_2, \theta_3).$$

A symmetric description of the Harnad duality [36] is provided by theorem 1.2 of [6][4]. By a Laplace transform [3], it was equivalently proved that the isomonodromy deformation equations of (1.4) reduce to PVI, in [13, 14] for $\theta_1 = \theta_2 = \theta_3 = 0$, and in [48] for the general case. This was later shown also in [4]. The one-to-one correspondence between solutions of PVI and diagonal conjugation classes of matrices V is realized in [48] by explicit formulae $V = V(x, y(x))$ (up to typing misprints).

As suggested in [48], system (1.4) may be useful for the study of symmetries, and this was later investigated in [4, 6], but (1.4) has never been used for the isomonodromic approach, which has always been based on the Fuchsian system (1.2), even in the most recent developments [21, 37, 38]. One main reason is *the lack of a formula expressing the 2-dimensional invariants* p_{jk} *in terms of the 3-dimensional Stokes matrices* of the irregular system. This issue was not considered in [48] and—to our knowledge—the formula is not available. A first result of our paper is precisely the derivation of this formula, so filling a gap in the established literature.

To state our main results, it is necessary to recall the theory of non-generic isomonodromy deformations developed in [9], and reworked in [35] by a Laplace transform. In [9] (see also [8, 32–34] for a simpler exposition), an irregular $n \times n$ system of type (1.4) is considered, with

[4] In [6], isomonodromic systems are attached to a certain class of supernova graphs. In particular, it is possible to attach isomonodromic systems of order higher than 2 to each of the six Painlevé equations, of which (1.4) is an example.

$U = \mathrm{diag}(u_1, \ldots, u_n)$ and $V(u)$ holomorphic in a sufficiently small polydisc $\mathbb{D}(u^c)$ centred at a *coalescence point* $u^c = (u_1^c, \ldots, u_n^c)$, so called because $u_j^c = u_k^c$ for some $j \neq k$. The polydisc contains a *coalescence locus* passing through u^c, where some components of $u = (u_1, \ldots, u_n)$ merge. Such n-dimensional version of (1.4) has a unique formal matrix solution

$$Y_F(z,u) = \left(I + \sum_{k \geqslant 1} F_k(u) z^{-k} \right) z^{\mathrm{diag}(V)} \exp\{zU\},$$

and 'canonical' fundamental matrix solutions $Y_\nu(z,u)$, uniquely identified by their asymptotic behaviour $Y_F(z,u)$ for $z \to \infty$ in the overlapping sectors (for small $\varepsilon > 0$)

$$\mathcal{S}_\nu : \quad \tau^* + (\nu - 2)\pi - \varepsilon < \arg z < \tau^* + (\nu - 1)\pi + \varepsilon, \qquad \nu \in \mathbb{Z}.$$

Here, $\tau^* \in \mathbb{R}$ satisfies $\tau^* \neq 3\pi/2 - \arg(u_j^c - u_k^c) \mod \pi$, for $j \neq k$ such that $u_j^c \neq u_k^c$. These solutions, their Stokes phenomenon and monodromy data are in general well defined only in a smaller subdomain of $\mathbb{D}(u^c)$ away from coalescence points. Theorem 1.1. of [9] extends the theory to the whole $\mathbb{D}(u^c)$: if the following vanishing conditions hold

$$V_{jk}(u) \longrightarrow 0 \quad \text{for } j \neq k, \text{ whenever } u_j - u_k \to 0, \tag{1.5}$$

then the matrix coefficients $F_k(u)$ and the canonical solutions are holomorphic in $\mathbb{D}(u^c)$, and the Stokes phenomenon is well posed. Moreover, the monodromy data of (1.4) are well defined and constant on the whole $\mathbb{D}(u^c)$. The *Stokes matrices*, defined by $Y_{\nu+1} = Y_\nu \mathbb{S}_\nu$, satisfy

$$(\mathbb{S}_\nu)_{jk} = (\mathbb{S}_\nu)_{kj} = 0 \quad \text{for}\, j \neq k\, \text{such that}\, u_j^c = u_k^c.$$

As a consequence, in order to compute the monodromy data, it suffices to compute the data of the restricted system

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = \left(U(u^c) + \frac{V(u^c)}{z} \right) Y. \tag{1.6}$$

The possibility of restricting at $u = u^c$ simplifies the computation of monodromy data, to the extent that sometimes it can be done in terms of classical special functions.

This theory has attracted the interest of mathematicians for its applications to Dubrovin–Frobenius manifolds [10] and quantum cohomology of some varieties [10, 11, 19], starting from dimension $n \geqslant 4$. Our paper provides the first full non trivial class of 3-dimensional irregular systems such that this theory is realized, and of 3-dimensional Frobenius manifolds where [10] applies. This may be seen as an interesting application of PVI.

1.1. Results

(1) We prove the formula expressing the p_{jk} in terms of the Stokes matrices, making the irregular system (1.4) an alternative to the Fuchsian one for the isomonodromy deformation method of PVI. There are two cases.

- If u varies in a sufficiently small polydisc of $\mathbb{C}^3 \backslash \bigcup_{j \neq k} \{u_j = u_k\}$, in theorem 3.1 we show that

$$p_{jk} = \begin{cases} 2\cos\pi(\theta_j - \theta_k) - e^{i\pi(\theta_j - \theta_k)} (\mathbb{S}_1)_{jk} (\mathbb{S}_2^{-1})_{kj}, & j \prec k, \\ 2\cos\pi(\theta_j - \theta_k) - e^{i\pi(\theta_k - \theta_j)} (\mathbb{S}_1)_{kj} (\mathbb{S}_2^{-1})_{jk},, & j \succ k, \end{cases} \tag{1.7}$$

where the ordering relation $j \prec k$ means $\Re(e^{i\tau^*}(u_j - u_k)) < 0$. Here $\tau^* \in \mathbb{R}$ satisfies $\tau^* \neq 3\pi/2 - \arg(u_j - u_k) \mod \pi$, $1 \leqslant j \neq k \leqslant 3$, for u in the polydisc. The branch cuts from u_1, u_2, u_3 to ∞ used to define the monodromies in (1.3) have direction $3\pi/2 - \tau^*$.

- If u varies in a sufficiently small polydisc $\mathbb{D}(u^c)$ centred at a coalescence point u^c, where $V(u)$ is holomorphic and satisfies (1.5), formulae (1.7) hold for $j \neq k$ such that $u_j^c \neq u_k^c$, while

$$p_{jk} = 2\cos\pi(\theta_j - \theta_k) \quad \text{for}\, j \neq k\, \text{such that}\, u_j^c = u_k^c.$$

The p_{jk} are now defined for a fundamental matrix solution $\Phi(\lambda, u)$ with u varying in a small subset of $\mathbb{D}(u^c)$ in whose interior u_1, u_2, u_3 are pairwise distinct. In this case, $\tau^* \neq 3\pi/2 - \arg(u_j^c - u_k^c) \mod \pi$, for all $j \neq k$ such that $u_j^c \neq u_k^c$.

Remark 1.1. In the paper, τ^* will be called $\tau^{(0)}$ in case there are not coalescences, and τ in case of coalescences. Notice that at $x = 0, 1, \infty$ only two out of the three u_1, u_2, u_3 coalesce, because $x = (u_2 - u_1)/(u_3 - u_1)$.

Theorem 3.1 is based on the isomonodromic Laplace transform of [35], which relates an isomonodromic irregular system to a Fuchsian one, both of dimension n, in presence of coalescences. In our case, we relate the 3-dimensional (1.4) to the 2-dimensional (1.2). A dimensional reduction of the monodromy will be studied.

(2) In section 4, we classify all the branches $y(x)$ holomorphic at $x = 0$, corresponding to the coalescence $u_2 - u_1 \to 0$, such that $V(u)$ satisfies (1.5), so that theorem 1.1 of [9] applies to system (1.4). This classification is an application of PVI providing the simplest but non-trivial full class of irregular systems realizing the theory of [9].

Moreover, for such transcendents we show that the restriction (1.6) at $u = u_c$ (namely at $x = 0$) can be solved in terms of either confluent hypergeometric functions or generalized hypergeometric of type $(p, q) = (2, 2)$. Our classification is summarized in the *table* of section 4

In section 5, we compute the Stokes matrices of (1.4) and the invariants p_{jk} associated with a selection of the classified transcendents. Being [9] applicable, the calculation will be explicitly done by means of special functions using the restriction (1.6) at $x = 0$.

Remark 1.2. It suffices to compute the critical behaviour at $x = 0$ and the corresponding integration constants in terms of the p_{jk}. The results at $x = 1, \infty$ are obtained applying the symmetries of PVI [49]. See section 3.1 of [30] for a short explanation of the procedure. For this reason, in this paper we concentrate on $x = 0$ only.

(3) In case $\beta = \gamma = \delta - 1/2 = 0$, a branch of a PVI transcendent locally encodes the structure of a 3-dimensional Dubrovin–Frobenius manifold [14, 23], whose local canonical coordinates are $u = (u_1, u_2, u_2)$. System (1.4) is related to a flat connection on the manifold. Its monodromy data are the natural moduli which locally parameterize the manifold. Given these data the manifold structure can be locally reconstructed by a Riemann–Hilbert boundary value problem [13, 14].

The coalesce of some coordinates $u_j - u_k \to 0$ satisfying theorem 1.1 of [9] corresponds to true points of the manifold, the semisimple coalescence points, whose theory is developed in [10]. This has important applications to the computation of monodromy data of quantum cohomologies in dimension $n \geqslant 4$ [11, 19].

In section 6, we compute *all* the monodromy data, namely the Stokes matrix, the Levelt exponents and the central connection matrix, parameterizing the 3-dimensional Dubrovin–Frobenius manifolds at a semisimple coalescence point. The manifold structure is in this case encoded in a transcendent holomorphic at a critical point, belonging to our classification. This provides the realization of [10] in dimension $n = 3$, as an interesting application of PVI.

(4) Preliminarily to (1)–(3) above, in section 2, theorem 2.1, we rework the results of [48] by means of the Laplace transform of Pfaffian systems studied [35], in order to obtain the correct formulae $V = V(x, y(x))$. We correct some typing misprints of [48], so making the formulae usable.

2. PVI as isomonodromy condition of an irregular system

Consider a 3-dimensional Frobenius integrable Pfaffian system

$$\mathrm{d}Y = \omega(z,u)Y, \qquad \omega(z,u) = \left(U + \frac{V - I}{z}\right)\mathrm{d}z + \sum_{k=1}^{3}(zE_k + V_k)\mathrm{d}u_k \tag{2.1}$$

where $U = \mathrm{diag}(u_1, u_2, u_3)$, $u = (u_1, u_2, u_3)$ in a domain of $\mathbb{C}^3$, and $E_k = \partial U/\partial u_k$. The set

$$\Delta_{\mathbb{C}^3} := \bigcup_{i \neq j}\{u \in \mathbb{C}^3 \mid u_i - u_j = 0\} \tag{2.2}$$

of 'diagonals' is called **coalescence locus**. $V = V(u)$ is a matrix holomorphic on a polydisc $\mathbb{D}$, that we can choose in two ways.

Case 1. $\mathbb{D} = \mathbb{D}(u^0)$ centred at u^0, such that $\mathbb{D}(u^0) \cap \Delta_{\mathbb{C}^3} = \emptyset$.

Case 2. $\mathbb{D} = \mathbb{D}(u^c)$, such that $\mathbb{D}(u^c) \cap \Delta_{\mathbb{C}^3} \neq \emptyset$, with centre at $u^c \in \Delta_{\mathbb{C}^3}$. We assume that u^c is the most coalescent point, namely if $u_j^c \neq u_k^c$ for some $j \neq k$, then $u_j \neq u_k$ for all $u \in \mathbb{D}(u^c)$. There are two possibilities: either the case with two distinct eigenvalues $\lambda_1 \neq \lambda_2$, namely

$$\lambda_1 := u_i^c = u_j^c \text{ for some } 1 \leqslant i \neq j \leqslant 3, \text{ and } \lambda_2 := u_k^c \neq u_i^c \text{ for } k \in \{1,2,3\}\backslash\{i,j\},$$

or the case $u_1^c = u_2^c = u_3^c$. The latter will not be considered. In view of the use of theorem 1.1 of [9], we assume in Case 2 that

$$\lim_{u_i - u_j \to 0} V_{ij}(u) = 0 \text{ holomorphically when } u_i - u_j \to 0 \text{ in } \mathbb{D}(u^c). \tag{2.3}$$

The integrability of the Pfaffian system (2.1) is equivalent to the strong isomonodromy of

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = \left(U + \frac{V}{z}\right)Y. \tag{2.4}$$

This means that the essential monodromy data (Stokes matrices, monodromy exponents, central connection matrix) are independent of u. In Case 1, the result is standard [41], while the isomonodromic theory in Case 2 is established in [9].

Lemma 2.1. *The integrability condition $d\omega = \omega \wedge \omega$ of (2.1) holds on a domain of $\mathbb{C}^3$ if and only if* V *and* $\mathrm{V_k}$ *satisfy on that domain the system*

$$V_k(u) = \left(\frac{V_{ij}(\delta_{ik} - \delta_{jk})}{u_i - u_j}\right)_{i,j=1}^{3}, \quad 1 \leqslant k \leqslant 3. \tag{2.5}$$

$$\partial_k V = [V_k, V], \quad k = 1, 2, 3. \tag{2.6}$$

Both (2.6) and the Pfaffian system

$$dG = \left(\sum_{j=1}^{3} V_j(u)du_j\right) G \tag{2.7}$$

are integrable. A Jordan form of V *is constant and is given by* $G^{-1}VG$ *for a suitable fundamental solution of (2.7).*

Proof. The proof of (2.5) and (2.6) and of the integrability condition $\partial_i V_j - \partial_j V_i = V_i V_j - V_j V_i$ of (2.6) and (2.7) is a computation. The last statement follows from the fact that (2.6) and (2.7) imply $\partial_j(G^{-1}VG) = 0$ for every fundamental solution of (2.7). □

Remark 2.1. In case V is analytic on either $\mathbb{D} = \mathbb{D}(u^0)$, or on $\mathbb{D} = \mathbb{D}(u^c)$ with vanishing conditions (2.3), then a fundamental solution $G(u)$ of (2.7), such that $G^{-1}VG = J$ is a Jordan form, is holomorphic and holomorphically invertible on $\mathbb{D}$. For details, we refer to [9, 32, 35] (see also [8]).

It is an exercise to prove the following

Lemma 2.2. *Any solution of the system of partial differential equations (PDEs)*

$$\sum_{k=1}^{3} \frac{\partial f}{\partial u_k} = 0, \qquad \sum_{k=1}^{3} u_k \frac{\partial f}{\partial u_k} = \alpha f, \qquad \alpha \in \mathbb{C},$$

has structure $f = (u_2 - u_1)^{\alpha} \mathcal{F}\left(\frac{u_2-u_1}{u_3-u_1}\right)$, *or* $f = (u_3 - u_1)^{\alpha} \mathcal{G}\left(\frac{u_2-u_1}{u_3-u_1}\right)$, *where* $\mathcal{F}$ *and* $\mathcal{G}$ *are some functions of their argument.*

Proposition 2.1. *The matrix*

$$\Theta = \operatorname{diag}(\theta_1, \theta_2, \theta_3) := -\operatorname{diag}(V(u))$$

is constant along the solutions of (2.5) and (2.6). Every solution of (2.5) and (2.6) admits the factorization

$$V(u) = (u_3 - u_1)^{\Theta}\ \Omega(x)\ (u_3 - u_1)^{-\Theta}, \qquad V_k(u) = (u_3 - u_1)^{\Theta}\ \Omega_k(u)\ (u_3 - u_1)^{-\Theta}, \tag{2.8}$$

where $\Omega(x)$ *depends only on* $x := \frac{u_2-u_1}{u_3-u_1}$, *and*

$$\Omega_1 = \begin{pmatrix} 0 & \frac{\Omega_{12}(x)}{u_1-u_2} & \frac{\Omega_{13}(x)}{u_1-u_3} \\ \frac{\Omega_{21}(x)}{u_1-u_2} & 0 & 0 \\ \frac{\Omega_{31}(x)}{u_1-u_3} & 0 & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0 & \frac{\Omega_{12}(x)}{u_2-u_1} & 0 \\ \frac{\Omega_{21}(x)}{u_2-u_1} & 0 & \frac{\Omega_{23}(x)}{u_2-u_3} \\ 0 & \frac{\Omega_{32}(x)}{u_2-u_3} & 0 \end{pmatrix},$$

$$\Omega_3 = \begin{pmatrix} 0 & 0 & \frac{\Omega_{13}(x)}{u_3-u_1} \\ 0 & 0 & \frac{\Omega_{23}(x)}{u_3-u_2} \\ \frac{\Omega_{31}(x)}{u_3-u_1} & \frac{\Omega_{32}(x)}{u_3-u_2} & 0 \end{pmatrix}.$$

If V *has diagonal form* $\widehat{\mu} = \operatorname{diag}(\mu_1, \mu_2, \mu_3)$, *then the general solution of (2.7) diagonalizing* V *has structure*

$$G(u) = (u_3 - u_1)^{\Theta} \cdot \widetilde{G}(x) \cdot (u_3 - u_1)^{\widehat{\mu}}, \tag{2.9}$$

where $\widetilde{G}(x)$ *diagonalizes* $\Omega(x)$, *and is defined up to* $\widetilde{G} \mapsto \widetilde{G} \cdot \operatorname{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, *where* $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{C}\backslash\{0\}$.

Proof. Substituting (2.5) into (2.6) we see that $\partial_k V_{jj} = 0$, $\forall\, j = 1,2,3$, $\forall\, k = 1,2,3$. Notice that $\sum_k V_k = 0$. This, and the conditions (2.5) and (2.6), with $\mathrm{diag}V(u) = -\Theta$, imply

$$\sum_{k=1}^{3} \partial_k V = 0, \qquad \sum_{k=1}^{3} u_k \partial_k V_{ij} = (\theta_i - \theta_j) V_{ij}.$$

Thus, (2.8) for V follows from lemma 2.2. Then, (2.8) for V_k follows from (2.5). The factorization (2.9) is also proved from (2.7) and lemma 2.2. □

Proposition 2.2. *The three equations (2.6) are equivalent to*

$$\frac{d\Omega}{dx} = \left[\widehat{\Omega}_2, \Omega\right], \tag{2.10}$$

where

$$\widehat{\Omega}_2(x) := \frac{1}{x-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Omega_{23}(x) \\ 0 & \Omega_{32}(x) & 0 \end{pmatrix} + \frac{1}{x}\begin{pmatrix} 0 & \Omega_{12}(x) & 0 \\ \Omega_{21}(x) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover, (2.7) is equivalent to

$$\frac{d\widetilde{G}}{dx} = \widehat{\Omega}_2(x)\widetilde{G}.$$

Proof. All follows from the factorizations in proposition 2.1 and the chain rule. □

In the sequel, we will be interested in the solutions of (2.5) and (2.6) satisfying:

$$\mathrm{diag}\big(V(u)\big) = \mathrm{diag}(-\theta_1, -\theta_2, -\theta_3), \tag{2.11}$$

$$V\,\text{has distinct eigenvalues} = 0,\ \frac{\theta_\infty - \theta_1 - \theta_2 - \theta_3}{2},\ \frac{-\theta_\infty - \theta_1 - \theta_2 - \theta_3}{2}. \tag{2.12}$$

V is then diagonalizable. Here, θ_∞ is just introduced to give a name to the eigenvalues. By lemma 2.1, the eigenvalues of V are constant and by proposition 2.1 the θ_j are constant, so that θ_∞ is a constant. The following statement is straightforward.

Lemma 2.3. *If* V *is a solution of (2.5) and (2.6) with constraints (2.11) and (2.12), then all the matrices*

$$K^0 \cdot V \cdot (K^0)^{-1}, \quad K^0 := \mathrm{diag}(k_1^0, k_2^0, k_3^0), \quad k_1^0, k_2^0, k_3^0 \in \mathbb{C}\backslash\{0\},$$

are solutions with the same constraints. There is no loss of generality in taking $k_3^0 = 1$.

The main result of the section is

Theorem 2.1. *The integrability condition (2.5) and (2.6), with the constraints (2.11) and (2.12), is equivalent to PVI with coefficients (1.1) given in terms of the parameters* $\theta_1, \theta_2, \theta_3, \theta_\infty$. *Equivalently, the nonlinear system (2.10) with* Ω *satisfying the same constraints (2.11) and (2.12) is equivalent to PVI.*

There is a one-to-one correspondence between transcendents y(x) *and equivalence classes*

$$\left\{K^0 \cdot V \cdot (K^0)^{-1},\ K^0 = \mathrm{diag}(k_1^0, k_2^0, 1), \quad (k_1^0, k_2^0) \in \mathbb{C}^2\backslash\{0,0\}\right\} \tag{2.13}$$

of solutions of (2.6), or the corresponding classes $\{K^0\cdot\Omega\cdot(K^0)^{-1}\}$ *of solutions of (2.10). The following explicit formulae hold.*

$$\Omega_{12}=\frac{k_1(x)}{k_2(x)}\cdot\frac{(x^2-x)\dfrac{dy}{dx}+(\theta_\infty-1)y^2+\Big(\theta_2-\theta_1+1-(\theta_\infty+\theta_2)x\Big)y+\theta_1x}{2(x-1)y},$$

$$\Omega_{21}=\frac{k_2(x)}{k_1(x)}\cdot\frac{(x^2-x)\dfrac{dy}{dx}+(\theta_\infty-1)y^2+\Big(\theta_1-\theta_2+1-(\theta_\infty-\theta_2)x\Big)y-\theta_1x}{2(x-y)},$$

$$\Omega_{13}=k_1(x)\cdot\frac{(x-x^2)\dfrac{dy}{dx}+(1-\theta_\infty)y^2+\Big((\theta_1-\theta_3)x+\theta_\infty+\theta_3-1\Big)y-\theta_1x}{2(x-1)y},$$

$$\Omega_{31}=\frac{1}{k_1(x)}\cdot\frac{(x-x^2)\dfrac{dy}{dx}+(1-\theta_\infty)y^2+\Big((\theta_3-\theta_1)x+\theta_\infty-\theta_3-1\Big)y+\theta_1x}{2x(y-1)},$$

$$\Omega_{23}=k_2(x)\cdot\frac{(x-x^2)\dfrac{dy}{dx}+(1-\theta_\infty)y^2+\Big((\theta_\infty-\theta_2)x+\theta_\infty+\theta_3-1\Big)y-x(\theta_\infty-\theta_2+\theta_3)}{2(x-y)},$$

$$\Omega_{32}=\frac{1}{k_2(x)}\cdot\frac{(x-x^2)\dfrac{dy}{dx}+(1-\theta_\infty)y^2+((\theta_\infty+\theta_2)x+\theta_\infty-\theta_3-1)y-x(\theta_\infty+\theta_2-\theta_3)}{2x(1-y)}.$$

The functions $k_j(x)$ *are obtained by the quadratures*

$$k_j(x)=k_j^0\exp\{L_j(x)\},\quad k_j^0\in\mathbb{C}\backslash\{0\},\quad L_j(x)=\int^x l_j(\xi)d\xi,\qquad j=1,2,\tag{2.14}$$

with

$$l_1(x):=\frac{x(1-x)\dfrac{dy}{dx}+(\theta_2-\theta_1-\theta_3+1)y^2+((\theta_1+\theta_3)x+\theta_1-\theta_2-1)y-\theta_1x}{2x(x-1)(y-1)y},$$

$$l_2(x):=\frac{1}{2x(1-x)(1-y)(x-y)}\left(-x(x-1)^2\frac{dy}{dx}+((\theta_1-3\theta_2+\theta_3+1)x-\theta_1+\theta_2+\theta_3-1)y^2\right.$$
$$\left.+\Big((2\theta_2-\theta_1-\theta_3)x^2+(3\theta_2-3\theta_3-1)x+\theta_1-\theta_2+1\Big)y+((\theta_1-2\theta_2+2\theta_3)x-\theta_1)x\right).$$

The proof will be given in the analytic case $\mathbb{D}=\mathbb{D}(u^0)$, or the case $\mathbb{D}=\mathbb{D}(u^c)$ with conditions (2.3), but it is based on linear algebra and calculus of derivatives. Thus, the formulae hold at every point (u_1,u_2,u_3) in a domain of $\mathbb{C}^3$ where the calculations make sense. They emend editing misprints of [48].

Remark 2.2. The eigenvalues of V are distinct if and only if

$$\theta_\infty\neq 0,\pm(\theta_1+\theta_2+\theta_3).$$

This can always be fulfilled for V associated to a given PVI with coefficients $\alpha,\beta,\gamma,\delta$, because the $\theta_\nu,\nu=1,2,3,\infty$, are defined by (1.1), so the changes $\theta_\infty\mapsto 2-\theta_\infty,\theta_1\mapsto-\theta_1,\theta_2\mapsto-\theta_2$, $\theta_3\mapsto-\theta_3$ do not change the specific equation PVI under consideration.

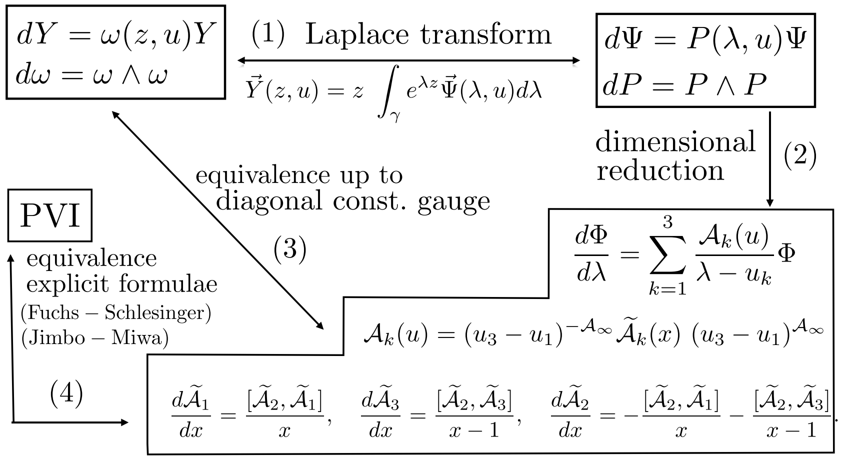

Figure 1. Proof of theorem 2.1.

Remark 2.3. In case $\theta_1 = \theta_2 = \theta_3 = 0$, then in theorem 2.1

$$V(u) \equiv \Omega(x), \qquad k_1(x) = k_1^0\frac{\sqrt{y}\sqrt{x-1}}{\sqrt{y-1}\sqrt{x}}, \qquad k_2(x) = k_2^0\frac{\sqrt{y-x}}{\sqrt{y-1}\sqrt{x}}.$$

If we choose $k_1^0 = \pm\sqrt{-1}$, $k_2^0 = \pm\sqrt{-1}$, then

$$V^T = -V$$

is associated with a Dubrovin–Frobenius manifold [14]. The expressions of theorem 2.1 reduce to the formulae in section 4 of [23] (see page 269 there for the relation $y \mapsto V$ and (48) at page 270, for the relation $V \mapsto y$). These formulae were later used in [27] and in section 22 of [9].

2.1. Proof of theorem 2.1

Since the works of Fuchs [18] and Schlesinger [51], it has been known that there is a one-to-one correspondence between solutions of PVI and equivalence classes

$$\{\mathcal{E}^{-1}\widetilde{\mathcal{A}}_1\mathcal{E},\ \mathcal{E}^{-1}\widetilde{\mathcal{A}}_2\mathcal{E},\ \mathcal{E}^{-1}\widetilde{\mathcal{A}}_3\mathcal{E}, \quad \mathcal{E} = \mathrm{diag}(\varepsilon_1,\varepsilon_3), \quad \varepsilon_1,\varepsilon_3 \in \mathbb{C}\backslash\{0\}\} \tag{2.15}$$

of solutions of the Schlesinger equations

$$\frac{\mathrm{d}\widetilde{\mathcal{A}}_1}{\mathrm{d}x} = \frac{[\widetilde{\mathcal{A}}_2,\widetilde{\mathcal{A}}_1]}{x}, \qquad \frac{\mathrm{d}\widetilde{\mathcal{A}}_3}{\mathrm{d}x} = \frac{[\widetilde{\mathcal{A}}_2,\widetilde{\mathcal{A}}_3]}{x-1}, \qquad \frac{\mathrm{d}\widetilde{\mathcal{A}}_2}{\mathrm{d}x} = -\frac{[\widetilde{\mathcal{A}}_2,\widetilde{\mathcal{A}}_1]}{x} - \frac{[\widetilde{\mathcal{A}}_2,\widetilde{\mathcal{A}}_3]}{x-1}, \tag{2.16}$$

with constraints $-(\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3) = \mathrm{diag}(\theta_\infty/2,\ -\theta_\infty/2)$ and eigenvalues of $\mathcal{A}_k = \pm\theta_k/2$. This has been summarized in appendix C of [42]. To prove theorem 2.1, we use a Laplace transform and a dimensional reduction to show that there is a one to one correspondence between equivalence classes (2.13) and (2.15). The scheme of the proof is in figure 1.

To starts, we assume that V is a solution of (2.6) with pairwise distinct eigenvalues, one being *equal to zero*. Let a diagonal form be

$$\widehat{\mu} := \mathrm{diag}(\mu_1, 0, \mu_3) = G(u)^{-1}V(u)G(u),$$

where G is a fundamental solution of (2.7), determined up to $G \mapsto G\cdot\mathrm{diag}(\varepsilon_1,\varepsilon_2,\varepsilon_3)$.

2.1.1. Step 1. Equivalent 3-dimensional Pfaffian systems—arrow (1) in figure 1. For fixed u a vector solution $\vec{Y}(z,u)$ of (2.4) is representable by a Laplace transform

$$\vec{Y}(z,u) = z\int_{\gamma} e^{\lambda z}\vec{\Psi}(\lambda,u)\mathrm{d}\lambda, \tag{2.17}$$

where γ is a suitable path such that $e^{\lambda z}(\lambda - U)\vec{\Psi}(\lambda)\Big|_{\gamma} = 0$, and $\vec{\Psi}$ is a vector solution of a Fuchsian system

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda} = \sum_{k=1}^{3}\frac{B_k(u)}{\lambda - u_k}\Psi, \qquad B_k = -E_k V. \tag{2.18}$$

This is established in [3] in generic cases, and in [31] in every case. In [35], u varies in $\mathbb{D}(u^{(0)})$ or $\mathbb{D}(u^c)$, and the Laplace transform relates (2.1) to the non-normalized Schlesinger system

$$\mathrm{d}\Psi = P(\lambda,u)\Psi, \qquad P(\lambda,u) = \sum_{k=1}^{3}\left(\frac{B_k(u)}{\lambda - u_k}\mathrm{d}(\lambda - u_k) + V_k(u)\mathrm{d}u_k\right). \tag{2.19}$$

Elementary computations show that $\mathrm{d}\omega = \omega\wedge\omega$ is equivalent to the integrability condition $\mathrm{d}P = P\wedge P$. This is also evident in [4, 5, 48], and in [15] for Dubrovin–Frobenius manifolds.

2.1.2. Step 2. Dimensional reduction—arrows (2) and (3) in figure 1. Since G satisfies (2.7), the gauge transformation $\Psi = GX$ transforms (2.19) into the normalized Schlesinger system

$$\mathrm{d}X = \widetilde{P}(\lambda,u)X, \qquad \widetilde{P}(\lambda,u) = \sum_{k=1}^{3}\frac{\widetilde{B}_k(u)}{\lambda - u_k}\mathrm{d}(\lambda - u_k), \tag{2.20}$$

where $\widetilde{B}_k := G^{-1}B_k G = -G^{-1}E_k G\widehat{\mu}$. Explicitly,

$$\widetilde{B}_1 = \frac{1}{\det G}\begin{pmatrix} -(G_{22}G_{33} - G_{23}G_{32})b_1 & 0 & -(G_{22}G_{33} - G_{23}G_{32})d_1 \\ (G_{21}G_{33} - G_{23}G_{31})b_1 & 0 & (G_{21}G_{33} - G_{23}G_{31})d_1 \\ -(G_{21}G_{32} - G_{22}G_{31})b_1 & 0 & -(G_{21}G_{32} - G_{22}G_{31})d_1 \end{pmatrix} \tag{2.21}$$

$$\widetilde{B}_2 = \frac{1}{\det G}\begin{pmatrix} (G_{12}G_{33} - G_{13}G_{32})b_2 & 0 & (G_{12}G_{33} - G_{13}G_{32})d_2 \\ -(G_{11}G_{33} - G_{31}G_{13})b_2 & 0 & -(G_{11}G_{33} - G_{31}G_{13})d_2 \\ (G_{11}G_{32} - G_{31}G_{12})b_2 & 0 & (G_{11}G_{32} - G_{31}G_{12})d_2 \end{pmatrix} \tag{2.22}$$

$$\widetilde{B}_3 = \frac{1}{\det G}\begin{pmatrix} -(G_{12}G_{23} - G_{13}G_{22})b_3 & 0 & -(G_{12}G_{23} - G_{13}G_{22})d_3 \\ (G_{11}G_{23} - G_{21}G_{13})b_3 & 0 & (G_{11}G_{23} - G_{21}G_{13})d_3 \\ -(G_{11}G_{22} - G_{21}G_{12})b_3 & 0 & -(G_{11}G_{22} - G_{21}G_{12})d_3 \end{pmatrix}, \tag{2.23}$$

where

$$b_1 = G_{11}\mu_1, \quad b_2 = G_{21}\mu_1, \quad b_3 = G_{31}\mu_1, \quad d_1 = G_{13}\mu_3, \quad d_2 = G_{23}\mu_3, \quad d_3 = G_{33}\mu_3. \tag{2.24}$$

By construction, $\mathrm{Tr}\widetilde{B}_k = \mathrm{Tr}B_k = \theta_k$, and the eigenvalues of B_k and $\widetilde{B}_k$ are $\theta_k, 0, 0$. Let a vector solution of (2.20) be denoted by

$$\vec{X} := \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \qquad \text{and let} \quad \boldsymbol{X} := \begin{pmatrix} X_1 \\ X_3 \end{pmatrix}.$$

Then, $\Phi = \prod_{j=1}^{3}(\lambda-u_j)^{-\theta_j/2}\boldsymbol{X}$ satisfies

$$d\Phi = \left(\sum_{k=1}^{3}\frac{\mathcal{A}_k}{\lambda-u_k}\,d(\lambda-u_k)\right)\Phi, \qquad \mathcal{A}_k := \begin{pmatrix}(\widetilde{B}_k)_{11} & (\widetilde{B}_k)_{13}\\ (\widetilde{B}_k)_{31} & (\widetilde{B}_k)_{33}\end{pmatrix} - \frac{\theta_k}{2}I. \tag{2.25}$$

By construction,

$$\mathcal{A}_\infty := -(\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3) = \begin{pmatrix}\theta_\infty/2 & 0\\ 0 & -\theta_\infty/2\end{pmatrix}, \qquad \mathrm{Tr}\mathcal{A}_k = 0, \quad \text{eigenvalues of } \mathcal{A}_k = \pm\frac{\theta_k}{2}, \tag{2.26}$$

where θ_∞ *is defined* by

$$\mu_1 = \frac{\theta_\infty-\theta_1-\theta_2-\theta_3}{2}, \quad \mu_3 = \frac{-\theta_\infty-\theta_1-\theta_2-\theta_3}{2}. \tag{2.27}$$

Notice that X_2 is obtained by a quadrature. The integrability condition $d\widetilde{P} = \widetilde{P}\wedge\widetilde{P}$ reduces to

$$\partial_i\mathcal{A}_k = \frac{[\mathcal{A}_i,\mathcal{A}_k]}{u_i-u_k}, \quad i\neq k; \qquad \partial_i\mathcal{A}_i = -\sum_{k\neq i}\frac{[\mathcal{A}_i,\mathcal{A}_k]}{u_i-u_k}. \tag{2.28}$$

Arrow (3) in figure 1 is contained in the following

Lemma 2.4. *If $\{\mathcal{A}_k\}_{k=1,2,3}$ is a solution of (2.28) with constraints (2.26), then all elements of the equivalence class*

$$\{\mathcal{E}^{-1}\mathcal{A}_1\mathcal{E},\ \mathcal{E}^{-1}\mathcal{A}_2\mathcal{E},\ \mathcal{E}^{-1}\mathcal{A}_3\mathcal{E}, \quad \mathcal{E} := \mathrm{diag}(\varepsilon_1,\varepsilon_3), \quad \varepsilon_1,\varepsilon_3\in\mathbb{C}\backslash\{0\}\} \tag{2.29}$$

are a solution with the same constraints. There is a one-to-one correspondence between equivalent classes (2.29) and equivalence classes

$$\{K^0\cdot V\cdot(K^0)^{-1}, \quad K^0 = \mathrm{diag}(k_1^0,k_2^0,k_3^0), \quad k_1^0,k_2^0,k_3^0\in\mathbb{C}\backslash\{0\}\} \tag{2.30}$$

of solutions of (2.5) and (2.6) satisfying the constraints (2.11) and (2.12) (see lemma 2.3).

Proof. The first assertion in straightforward. For the second, we do a proof slightly different from [48]. To every solution V of (2.5) and (2.6) satisfying the constraints (2.11) and (2.12), we associate $\mathcal{A}_k$ in (2.25), using (2.21)–(2.24), so that

$$\mathcal{A}_k = \begin{pmatrix}a_kb_k & a_kd_k\\ c_kb_k & c_kd_k\end{pmatrix} - \frac{\theta_k}{2}I, \quad k=1,2,3, \tag{2.31}$$

where

$$a_1 = \frac{-(G_{22}G_{33}-G_{23}G_{32})}{\det G}, \quad a_2 = \frac{G_{12}G_{33}-G_{13}G_{32}}{\det G}, \quad a_3 = \frac{-(G_{12}G_{23}-G_{13}G_{22})}{\det G}, \tag{2.32}$$

$$c_1 = \frac{-(G_{21}G_{32}-G_{22}G_{31})}{\det G}, \quad c_2 = \frac{G_{11}G_{32}-G_{31}G_{12}}{\det G}, \quad c_3 = \frac{-(G_{11}G_{22}-G_{21}G_{12})}{\det G}. \tag{2.33}$$

The above expressions determine a class (2.29) in terms of V. Indeed, V determines G up to $G\mapsto G\cdot\mathrm{diag}(\varepsilon_1,\varepsilon_2,\varepsilon_3)$, so that $\det G\mapsto\varepsilon_1\varepsilon_2\varepsilon_3\det G$, $a_k\mapsto\varepsilon_1^{-1}a_k$, $b_k\mapsto\varepsilon_1 b_k$, $c_k\mapsto\varepsilon_3c_k$, $d_k\mapsto\varepsilon_3d_k$, which determines $\mathcal{A}_k$ up to $\mathcal{A}_k\longmapsto\mathcal{E}^{-1}\mathcal{A}_k\mathcal{E}$. Moreover, a change $V\mapsto V' = K^0\cdot V\cdot(K^0)^{-1}$ induces $G\mapsto G' = \mathrm{diag}(k_1^0,k_2^0,k_3^0)\cdot G$. Therefore, from (2.24) and (2.32)–(2.33) we receive $a_j' = a_j/k_j^0$, $b_j' = k_j^0b_j$, $c_j' = c_j/k_j^0$, $d_j' = k_j^0d_j$. It follows that $\mathcal{A}_k$ in (2.31) is invariant.

Conversely, consider a solution of (2.28) with constraint (2.26), which can be written as in (2.31). Thus

$$V = G\widehat{\mu}G^{-1} = \begin{pmatrix} -\theta_1 & -a_2b_1 - c_2d_1 & -a_3b_1 - c_3d_1 \\ -a_1b_2 - c_1d_2 & -\theta_2 & -a_3b_2 - c_3d_2 \\ -a_1b_3 - c_1d_3 & -a_2b_3 - c_2d_3 & -\theta_3 \end{pmatrix}. \tag{2.34}$$

The above is invariant under the map $a_k \mapsto \varepsilon_1^{-1}a_k$, $b_k \mapsto \varepsilon_1 b_k$, $c_k \mapsto \varepsilon_3 c_k$, $d_k \mapsto \varepsilon_3 d_k$ and the map $\mathcal{A}_k \mapsto \mathcal{E}^{-1}\mathcal{A}_k\mathcal{E}$. Moreover, $\mathcal{A}_j$ determines a_j, b_j, c_j, d_j up to $a_j \mapsto a_j/k_j^0$, $b_j \mapsto k_j^0 b_j$, $c_j \mapsto c_j/k_j^0$, $d_j \mapsto k_j^0 d_j$, $j = 1,2,3$, so that to each $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ is associate V up to the freedom $V \mapsto K^0 \cdot V \cdot (K^0)^{-1}$. □

Lemma 2.5. *Every solution of (2.28) with constraints (2.26) admits the factorization*

$$\mathcal{A}_k(u) = (u_3 - u_1)^{-\mathcal{A}_\infty}\widetilde{\mathcal{A}}_k(x)\,(u_3 - u_1)^{\mathcal{A}_\infty}, \quad k = 1,2,3, \tag{2.35}$$

where $x = \frac{u_2 - u_1}{u_3 - u_1}$*, and the* $\widetilde{\mathcal{A}}_k(x)$ *satisfy the same constraints and solve the Schlesinger equations (2.16).*

Proof. The equations (2.28) imply that

$$\sum_{i=1}^{3} \partial_i \mathcal{A}_k = 0; \qquad \sum_{i=1}^{3} u_i \partial_i \mathcal{A}_k = \left[\mathcal{A}_k, \mathcal{A}_\infty\right], \quad k = 1,2,3.$$

In particular, $\sum_{i=1}^{3} u_i\partial_i(\mathcal{A}_k)_{12} = -\theta_\infty(\mathcal{A}_k)_{12}$, $\sum_{i=1}^{3} u_i\partial_i(\mathcal{A}_k)_{21} = \theta_\infty(\mathcal{A}_k)_{21}$, $\sum_{i=1}^{3} u_i\partial_i(\mathcal{A}_k)_{jj} = 0$. Therefore, (2.35) follows from lemma 2.2 and then (2.16) follows by the chain rule. □

The λ-component of (2.25) is the Fuchsian system (1.2), and (2.28) expresses its isomonodromy. For u away from coalescence points, (1.2) is equivalent to

$$\frac{\mathrm{d}\widetilde{\Phi}}{\mathrm{d}\widetilde{\lambda}} = \left(\frac{\widetilde{\mathcal{A}}_1(x)}{\widetilde{\lambda}} + \frac{\widetilde{\mathcal{A}}_2(x)}{\widetilde{\lambda} - x} + \frac{\widetilde{\mathcal{A}}_3(x)}{\widetilde{\lambda} - 1}\right)\widetilde{\Phi}, \tag{2.36}$$

through the gauge transformation $\widetilde{\Phi}(\widetilde{\lambda}, x) = (u_3 - u_1)^{\mathcal{A}_\infty}\Phi(\lambda, u)$ and change of variables $\widetilde{\lambda} = (\lambda - u_1)/(u_3 - u_1)$, with $x = (u_2 - u_1)/(u_3 - u_1)$. The Schlesinger equations (2.16) are the isomonodromy condition for (2.36).

Proposition 2.3. *The matrices* $\widetilde{\mathcal{A}}_k(x)$ *in (2.35) have structure*

$$\widetilde{\mathcal{A}}_k(x) = \begin{pmatrix} z_k(x) + \dfrac{\theta_k}{2} & -\widetilde{v}_k(x)z_k(x) \\ \dfrac{z_k(x) + \theta_k}{\widetilde{v}_k(x)} & -z_k(x) - \dfrac{\theta_k}{2} \end{pmatrix}, \quad k = 1,2,3, \tag{2.37}$$

for some functions $\widetilde{v}_k(x)$*,* $z_k(x)$*. The matrix* $\Omega(x)$ *in (2.8) has structure*

$$\Omega(x) = K(x)\mathbf{Z}(x)K(x)^{-1}, \qquad K(x) = \mathrm{diag}(k_1(x), k_2(x), 1) \tag{2.38}$$

for some functions $k_1(x)$*,* $k_2(x)$*, where*

$$\mathbf{Z}_{ii} = -\theta_i, \qquad \mathbf{Z}_{ij}(x) = z_j(x) - \frac{\widetilde{v}_j(x)z_j(x)}{\widetilde{v}_i(x)z_i(x)}\left(z_i(x) + \theta_i\right), \quad 1 \leqslant i \neq j \leqslant 3. \tag{2.39}$$

Proof. Conditions (2.26) imply $c_k = (\theta_k - a_k b_k)/d_k$, $k = 1,2,3$, so that (2.31) yields

$$\mathcal{A}_k = \begin{pmatrix} a_k b_k - \theta_k/2 & a_k d_k \\ (\theta_k - a_k b_k) b_k/d_k & \theta_k/2 - a_k b_k \end{pmatrix} \underset{(2.34)}{\Longrightarrow} V_{ij} = -\left(a_j b_i + \frac{d_i}{d_j}(\theta_j - a_j b_j)\right), \quad 1 \leqslant i \neq j \leqslant 3. \tag{2.40}$$

Define $z_k = z_k(u)$ and $v_k = v_k(u)$ by $z_k + \theta_k = a_k b_k$ and $v_k z_k = -a_k d_k$. Then

$$\mathcal{A}_k = \begin{pmatrix} z_k + \theta_k/2 & -v_k z_k \\ \dfrac{z_k + \theta_k}{v_k} & -z_k - \theta_k/2 \end{pmatrix}; \qquad V_{ij} = \left(z_j - \frac{v_j z_j}{v_i z_i}(z_i + \theta_i)\right)\frac{d_i}{d_j}, \quad 1 \leqslant i \neq j \leqslant 3. \tag{2.41}$$

Substituting the factorization (2.9) into (2.32) and (2.33), we receive

$$a_j(u) = \widetilde{a}_j(x)(u_3 - u_1)^{-\theta_j - \mu_1}, \qquad b_j(u) = \widetilde{b}_j(x)(u_3 - u_1)^{-\theta_j + \mu_1},$$
$$c_j(u) = \widetilde{c}_j(x)(u_3 - u_1)^{-\theta_j - \mu_3}, \qquad d_j(u) = \widetilde{d}_j(x)(u_3 - u_1)^{\theta_j + \mu_3}, \qquad j = 1,2,3.$$

Substituting into $\mathcal{A}_k$ in (2.40), recalling that $\mu_1 - \mu_3 = \theta_\infty$ and comparing with (2.35) we find

$$\widetilde{\mathcal{A}}_k(x) = \begin{pmatrix} \widetilde{a}_k(x)\widetilde{b}_k(x) - \theta_k/2 & \widetilde{a}_k(x)\widetilde{d}_k(x) \\ \left(\theta_k - \widetilde{a}_k(x)\widetilde{b}_k(x)\right)\widetilde{b}_k(x)/\widetilde{d}_k(x) & \theta_k/2 - \widetilde{a}_k(x)\widetilde{b}_k(x) \end{pmatrix}.$$

Comparison with $\mathcal{A}_k$ in (2.41) proves (2.37), with $z_k(u) \equiv z_k(x)$ and $v_k(u) = \widetilde{v}_k(x)(u_3 - u_1)^{\mu_3 - \mu_1}$. Now, V in (2.41) becomes

$$V_{ij}(u) = \left(z_j(x) - \frac{\widetilde{v}_j(x) z_j(x)}{\widetilde{v}_i(x) z_i(x)}\left(z_i(x) + \theta_i\right)\right)\frac{\widetilde{d}_i(x)}{\widetilde{d}_j(x)}(u_3 - u_1)^{\theta_i - \theta_j}, \quad 1 \leqslant i \neq j \leqslant 3.$$

The above brings back the factorization (2.8) and proves (2.38) and (2.39), with $k_1(x) := \widetilde{d}_1(x)/\widetilde{d}_3(x)$ and $k_2(x) := \widetilde{d}_2(x)/\widetilde{d}_3(x)$. □

2.1.3. Step 4. From (2.16) to PVI—arrow (4) in figure 1. As mentioned in the beginning, there is a one to one correspondence between equivalence classes (2.15) and solutions of PVI. We will use the explicit correspondence in appendix C of [42]. Our x coincides with t used in [42]. The matrices $A_0(t), A_t(t), A_1(t)$ in formula (C.47) of [42] are related to ours by the identifications

$$A_0(t) - \frac{\theta_0}{2} = \widetilde{\mathcal{A}}_1(x), \quad A_t(t) - \frac{\theta_t}{2} = \widetilde{\mathcal{A}}_2(x), \quad A_1(t) - \frac{\theta_1}{2} = \widetilde{\mathcal{A}}_3(x), \quad t = x,$$

$$(\theta_0, \theta_t, \theta_1) \text{ in } (C.47) = (\theta_1, \theta_2, \theta_3) \text{ in our notations.}$$

$$(z_0, z_t, z_1) \text{ in } (C.47) = (z_1, z_2, z_3) \text{ in our notations.}$$

$$(u, w, v) \text{ in } (C.47) = (\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3) \text{ in our notations.}$$

Following [42], the matrices $\widetilde{\mathcal{A}}_k$ are parameterized by the 7+1 independent parameters θ_1, θ_2, θ_3, θ_∞, k, y, z, x, where k, y and z are respectively defined by

$$k=(1+x)\widetilde{v}_1 z_1+\widetilde{v}_2 z_2+x\widetilde{v}_3 z_3, \qquad y=\frac{x\widetilde{v}_1 z_1}{k},$$

$$z=\frac{z_1+\theta_1}{y}+\frac{z_2+\theta_2}{y-x}+\frac{z_3+\theta_3}{y-1}\equiv\frac{(\widetilde{\mathcal{A}}_1)_{11}+\theta_1/2}{y}+\frac{(\widetilde{\mathcal{A}}_2)_{11}+\theta_2/2}{y-x}+\frac{(\widetilde{\mathcal{A}}_3)_{11}+\theta_3/2}{y-1}.$$

Do not confuse the parameter z above, taken from [42], with the independent variable z previously used in system (2.1).

Now, z_1,z_2,z_3 and $\widetilde{v}_1,\widetilde{v}_2,\widetilde{v}_3$ can be explicitly parametrized in terms of $\theta_1,\theta_2,\theta_3,\theta_\infty,k,y,z,x$ by formulae (C.51)–(C.52) (in [42], $\tilde{z}=z-\theta_1/y-\theta_2/(y-x)-\theta_3/(y-1)$ is used in place of z).

The Schlesinger equations (2.16) become a first order nonlinear system of three differential equations for y,z,k, reported in formula (C.55) of [42]. As for $k(x)$, it is computable as the exponential of a quadrature in dx involving $y(x)$ and $z(x)$, so it is determined up to a multiplicative constant k^0, which is identified with $\varepsilon_3/\varepsilon_1$ in the equivalence class (2.15). Eliminating z from the remaining first order system for y and z, we see that y solves PVI. If $y=y(x)$ is a solution, then

$$z(x)=\frac{1}{2}\left(\frac{x(x-1)}{y(x)(y(x)-1)(y(x)-x)}\frac{\mathrm{d}y(x)}{\mathrm{d}x}+\frac{\theta_1}{y(x)}+\frac{\theta_2-1}{y(x)-x}+\frac{\theta_3}{y(x)-1}\right). \tag{2.42}$$

2.1.4. Step 5. Completion of the proof of theorem 2.1: the explicit formulae. From (2.39) and (C.51) of [42], we receive

$$\mathbf{Z}_{12}=z_2-(z_1+\theta_1)\frac{x-y}{(1-x)y}, \qquad \mathbf{Z}_{13}=z_3-(z_1+\theta_1)\frac{x(y-1)}{(1-x)y},$$
$$\mathbf{Z}_{21}=z_1-(z_2+\theta_2)\frac{(1-x)y}{x-y}, \qquad \mathbf{Z}_{23}=z_3-(z_2+\theta_2)\frac{x(y-1)}{x-y},$$
$$\mathbf{Z}_{31}=z_1-(z_3+\theta_3)\frac{(1-x)y}{x(y-1)}, \qquad \mathbf{Z}_{32}=z_2-(z_3+\theta_3)\frac{x-y}{x(y-1)}.$$

Substituting (C.52) in the above, where $z(x)$ is given by (2.42), we obtain the explicit expression of $\mathbf{Z}(x)$ in terms of $y(x)$ and d$y(x)/$dx, which gives $\Omega_{ij}(x)=\mathbf{Z}_{ij}(x)k_i(x)/k_j(x)$, $1\leqslant i\neq j\leqslant 3$, $k_3:=1$, as in the statement of theorem 2.1.

To complete the proof of theorem 2.1, we find the differential equations for $k_1(x)$ and $k_2(x)$. The factorization (2.38) implies that

$$\widehat{\Omega}_2=K\mathbf{Z}_2K^{-1}, \qquad \mathbf{Z}_2=\begin{pmatrix}0 & \mathbf{Z}_{12}/x & 0\\ \mathbf{Z}_{21}/x & 0 & \mathbf{Z}_{23}/(x-1)\\ 0 & \mathbf{Z}_{32}/(x-1) & 0\end{pmatrix}.$$

Substituting the above and (2.38) into (2.10) we find

$$\left[K^{-1}\frac{\mathrm{d}K}{\mathrm{d}x}\,,\,\mathbf{Z}\right]=[\mathbf{Z}_2,\mathbf{Z}]-\frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}x},$$

namely

$$\frac{\mathrm{d}\ln k_1}{\mathrm{d}x}=\begin{cases}\dfrac{1}{\mathbf{Z}_{13}}\left([\mathbf{Z}_2,\mathbf{Z}]_{13}-\dfrac{\mathrm{d}\mathbf{Z}_{13}}{\mathrm{d}x}\right)\\[2mm] \dfrac{1}{\mathbf{Z}_{31}}\left(\dfrac{\mathrm{d}\mathbf{Z}_{31}}{\mathrm{d}x}-[\mathbf{Z}_2,\mathbf{Z}]_{31}\right)\end{cases}, \qquad \frac{\mathrm{d}\ln k_2}{\mathrm{d}x}=\begin{cases}\dfrac{1}{\mathbf{Z}_{23}}\left([\mathbf{Z}_2,\mathbf{Z}]_{23}-\dfrac{\mathrm{d}\mathbf{Z}_{23}}{\mathrm{d}x}\right)\\[2mm] \dfrac{1}{\mathbf{Z}_{32}}\left(\dfrac{\mathrm{d}\mathbf{Z}_{32}}{\mathrm{d}x}-[\mathbf{Z}_2,\mathbf{Z}]_{32}\right)\end{cases}.$$

Substituting the expressions of the entries of $\mathbf{Z}$ and $\mathbf{Z}_2$ in terms of y and $\mathrm{d}y/\mathrm{d}x$, the r.h.s. of the above expressions respectively become $l_1(x)$ and $l_2(x)$ in the statement of theorem 2.1. Notice that $k_1(x)$ and $k_2(x)$ contain multiplicative integration constants k_1^0 and k_2^0 responsible for the correspondence between y and the equivalence class $\{K^0\cdot V\cdot(K^0)^{-1},\ K_0=\mathrm{diag}(k_1^0,k_2^0,1)\}$. The proof of theorem 2.1 is complete. □

3. Monodromy data p_{jk} in terms of Stokes matrices

As mentioned in the introduction, the monodromy data p_{jk} of the 2×2 Fuchsian system (1.2) parameterize Painlevé VI transcendents, allowing us to solve the nonlinear connection problem. We prove a formula expressing the p_{jk} *in terms of the Stokes matrices* of the 3×3 system (2.4), so making system (2.4) a *concrete alternative* to (1.2) in the isomonodromy deformation method for PVI. The formula is stated in theorem 3.1 below and, to our knowledge, does not appear in the literature[5].

Preliminarily, we recall when the p_{jk} can be used to parametrize univocally the branch of a transcendent. Consider $\boldsymbol{\mathcal{M}}:=SL(2,\mathbb{C})^3/(\mathcal{M}_j\mapsto C^{-1}\mathcal{M}_jC,\ \ \det C\neq 0)\equiv SL(2,\mathbb{C})^3/SL(2,\mathbb{C})$, the space of conjugacy classes of triples $\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3$. The ring of its invariant polynomials is generated by the traces [17]

$$p_{jk}=\mathrm{tr}\,(\mathcal{M}_j\mathcal{M}_k),\quad j\neq k\in\{1,2,3\},$$

$$p_\infty=\mathrm{tr}\,(\mathcal{M}_{j_3}\mathcal{M}_{j_2}\mathcal{M}_{j_1})=2\cos(\pi\theta_\infty),\quad j_1\prec j_2\prec j_3;\qquad p_j=\mathrm{Tr}\mathcal{M}_j=2\cos(\pi\theta_j).$$

The ordering relation $\prec$ will be explained in (3.3). Two facts play a crucial role. First, a conjugacy class belonging to a ‘big’ open subset[6] of $\boldsymbol{\mathcal{M}}$ can be explicitly parameterized by the p_{jk}, p_j, p_∞, according to tables 1 and 2 of [39] (generalized to the Garnier system with two times in [7]). The second fact is that there is a one-to-one correspondence between monodromy data and branches of Painlevé transcendents, if none of the $\mathcal{M}_j, j=1,2,3$ and $\mathcal{M}_{j_3}\mathcal{M}_{j_2}\mathcal{M}_{j_1}$ is equal to $\pm I$, where I is the identity matrix [26]. In this case the integration constants expressing the critical behaviours can be univocally written in terms of the p_{jk}, provided the triple $\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3$ is in the big open of $\boldsymbol{\mathcal{M}}$. Such explicit formulae can be found in [4, 16, 22, 24–26, 28, 40, 44, 46]. An example of a point not in the big open is a triple generating a reducible group, another example is $(p_{12},p_{23},p_{31})=(\pm2,\pm2,\pm2)$, discussed in section 6.2.1. The analytic continuation of a branch is obtained by a completely explicit action of the braid group on the monodromy data p_{jk} [4, 16, 24, 39].

[5] What we can find in the literature is theorem I (theorem 1) at page 389 of [31], where the relation is given between Stokes matrices of the irregular system of dimension n and traces of products of monodromy matrices of certain selected solutions of the Laplace-transformed Fuchsian system *of the same dimension*. Moreover, in theorem 2 of [4], and in [13, 14] in case of Frobenius manifolds, a Killing–Coxeter identity is given relating a product of Stokes matrices and a product of monodromy matrices (pseudo-reflections) of the Laplace-transformed Fuchsian system of the same dimension.

[6] Let (i,j,k) be a cyclic permutation of (j_1,j_2,j_3). According to [39], the big open is the complement in $\boldsymbol{\mathcal{M}}$ of the set where the following six algebraic equations are satisfied:

$$\begin{cases}(p_{jk}^2-4)(p_{jk}^2+p_i^2+p_\infty^2-p_{jk}p_ip_\infty-4)=0,\\ (p_{jk}^2-4)(p_{jk}^2+p_j^2+p_k^2-p_{jk}p_jp_k-4)=0,\end{cases}\qquad \text{for all } i=1,2,3.$$

.

Definition 3.1. The **Stokes rays** of system (2.4) associated with $U(u) = \mathrm{diag}(u_1, u_2, u_3)$ are the infinitely many half-lines in the universal covering of the punctured z-plane $\mathbb{C}\backslash\{0\}$, issuing from $z = 0$ towards ∞, defined by $\Re((u_j - u_k)z) = 0$, $\Im((u_j - u_k)z) < 0$, for $u_j \neq u_k$.

The following refinement of the size of the polydisc $\mathbb{D}$ must be applied.

In Case 1, $\mathbb{D} = \mathbb{D}(u^0)$. Let $\tau^{(0)}, \eta^{(0)} \in \mathbb{R}$ satisfy

$$\tau^{(0)} = 3\pi/2 - \eta^{(0)}, \qquad \eta^{(0)} \neq \arg(u_i^0 - u_j^0) \bmod \pi,\ \forall\, i \neq j. \tag{3.1}$$

$\tau^{(0)}$ is an *admissible direction* in the z-plane for the Stokes rays of $U(u^0)$, that is no such rays have directions $\tau^{(0)} + h\pi$, $h \in \mathbb{Z}$. The size of $\mathbb{D}(u^0)$ is so small that the Stokes rays of $U(u)$ in the z-plane do not cross the directions $\tau^{(0)} + h\pi$, as u varies in $\mathbb{D}(u^0)$.
In Case 2, $\mathbb{D} = \mathbb{D}(u^c)$. The components of $u^c = (u_1^c, u_2^c, u_3^c)$ only have two distinct values λ_1, λ_2. Let $\tau, \eta \in \mathbb{R}$ satisfy

$$\tau = 3\pi/2 - \eta, \qquad \eta \neq \arg(\lambda_1 - \lambda_2) \bmod \pi. \tag{3.2}$$

τ is an *admissible direction* in the z-plane for the Stokes rays of $U(u^c)$. The size of $\mathbb{D}(u^c)$ is so small that no Stokes rays associated with pairs (u_j, u_k) such that $u_j^c \neq u_k^c$ cross the admissible directions $\arg z = \tau + h\pi$, $h \in \mathbb{Z}$, as u varies in $\mathbb{D}(u^c)$. For this to occur, it is necessary and sufficient [9, 35] that $\mathbb{D}(u^c) = \{u \in \mathbb{C}^3 \mid \max_{1 \leqslant l \leqslant 3} |u_j - u_j^c| \leqslant \epsilon_0\}$ has size $\epsilon_0 < \delta/2$, where $\delta = \min_{\rho > 0} |\lambda_1 - \lambda_2 + \rho \exp\{\eta\sqrt{-1}\}|$ = distance in the λ-plane between two half-lines respectively issuing from λ_1 and λ_2 towards infinity in direction η.

If $\mathbb{D}$ is as small as specified above, an ordering relation $\prec$ is well defined in $\{1, 2, 3\}$:

- In Case 1,

$$j \prec k \quad \Longleftrightarrow \quad \Re(e^{i\tau^{(0)}}(u_j^0 - u_k^0)) < 0, \quad j \neq k. \tag{3.3}$$

- In Case 2, there is no ordering relation for j, k such that $u_j^c = u_k^c$, while

$$j \prec k \quad \Longleftrightarrow \quad \Re(e^{i\tau}(u_j^c - u_k^c)) < 0, \quad j \neq k \text{ and } u_j^c \neq u_k^c. \tag{3.4}$$

With the above assumptions on $\mathbb{D}$, according to [9] system (2.4) admits a unique formal solution

$$Y_F(z, u) = \left(I + \sum_{k \geqslant 1} F_k(u) z^{-k} \right) z^{\mathrm{diag}(V)} e^{zU}, \tag{3.5}$$

with matrix coefficients $F_k(u)$ holomorphic on $\mathbb{D}$, and unique canonical fundamental matrix solutions $Y_\nu(z, u)$, $\nu \in \mathbb{Z}$, such that

$$Y_\nu(z, u) \sim Y_F(z, u), \qquad z \to \infty, \tag{3.6}$$

in the sector (for ε small enough)

$$\mathcal{S}_\nu: \quad \tau^* + (\nu-2)\pi - \varepsilon < \arg z < \tau^* + (\nu-1)\pi + \varepsilon,$$
$$\tau^* = \tau^{(0)} \text{ or } \tau, \text{ depending on Case 1 or Case 2.}$$

This is standard in Case 1, while in Case 2 it holds as a result of the vanishing conditions (2.3), as established in [9].

Remark 3.1 (Uniqueness of the formal solution). In the case $\mathbb{D} = \mathbb{D}(u^c)$, the notion of *partial resonance* for $V(u^c)$ is introduced in corollary 4.1 of [9] (the name is first used in [50]). In our case with vanishing conditions (2.3), partial resonance occurs if and only if

$$\theta_i - \theta_j \in \mathbb{Z}\backslash\{0\} \quad \text{for } i \neq j \text{ such that } u_i^c = u_j^c.$$

For example, for $x = (u_2-u_1)/(u_3-u_1) \to 0$, partial resonance means $\theta_1 - \theta_2 \in \mathbb{Z}\backslash\{0\}$. According[7] to corollaries 1.1 and 4.1 of [9], if there is no partial resonance system (2.4) *restricted at* fixed $u = u^c$ has a unique formal solution equal to $Y_F(z, u^c)$.

Definition 3.2. The **Stokes matrices** of system (2.4) are the connection matrices such that

$$Y_{\nu+1}(z,u) = Y_\nu(z,u)\mathbb{S}_\nu. \tag{3.7}$$

It will suffice to only consider $\mathbb{S}_1$ and $\mathbb{S}_2$ because $\mathbb{S}_{\nu+2} = e^{-2\pi i \,\mathrm{diag}(V)}\, \mathbb{S}_\nu\, e^{2\pi i \mathrm{diag}(V)}$.

The Stokes matrices are constant on $\mathbb{D}$ by the integrability of (2.1). This is standard [41] in case of $\mathbb{D}(u^0)$, while it follows from[8] theorem 1.1 of [9] in case of $\mathbb{D}(u^c)$. In the latter case, for each $\nu \in \mathbb{Z}$,

$$(\mathbb{S}_\nu)_{ij} = (\mathbb{S}_\nu)_{ji} = 0 \quad \text{if } u_i^c = u_j^c.$$

Let for short call

$$\eta^* := \eta^{(0)} \text{ or } \eta, \qquad \text{and } u^* = u^0 \text{ or } u^c,$$

depending on case (3.1) or (3.2). Recall that $\eta^* = 3\pi/2 - \tau^*$. In the λ-plane, consider branch cuts $L_1(\eta^*), L_2(\eta^*), L_3(\eta^*)$ oriented from u_1, u_2, u_3 respective to infinity in direction η^*, see figure 2. Let $\mathcal{P}_{\eta^*}(u)$ be the λ-plane with these cuts and with the determinations

$$\eta^* - 2\pi < \arg(\lambda - u_k) < \eta^*.$$

The domain of definition of the solutions of (2.18), (2.25) and (1.2) is the set

$$\mathcal{P}_{\eta^*}(u)\hat{\times}\mathbb{D}(u^*) := \{(\lambda, u) \mid u \in \mathbb{D},\ \lambda \in \mathcal{P}_{\eta^*}(u)\} \equiv \bigcup_{u\in\mathbb{D}} \Big(\mathcal{P}_{\eta^*}(u) \times \{u\}\Big).$$

We are ready to state the main result of the section.

Theorem 3.1. *Let $\mathbb{D}$ be $\mathbb{D}(u^0)$, or $\mathbb{D}(u^c)$, and η^*, τ^* as above. For every point $u^\bullet$ in $\mathbb{D}(u^0)$, or in $\mathbb{D}(u^c)\backslash\Delta_{\mathbb{C}^3}$, there is a neighbourhood $\mathcal{U}$ of $u^\bullet$ in $\mathbb{D}$ and a fundamental matrix solution $\Phi_{hol}(\lambda, u)$ of system (1.2), holomorphic of $(\lambda, u) \in \mathcal{P}_{\eta^*}(u)\hat{\times}\,\mathcal{U}$, whose monodromy invariants*

$$p_{jk} := \mathrm{tr}\,(\mathcal{M}_j(u)\mathcal{M}_k(u)),$$

[7] The statement of corollary 1.1. of [9] is imprecise: it is not that 'the diagonal entries of $\widehat{A}_1(0)$ do not differ by non-zero integers', but the elements of each sequence $(\widehat{A}_1(0))_{j_1j_1}, (\widehat{A}_1(0))_{j_2j_2}, \ldots, (\widehat{A}_1(0))_{j_\ell j_\ell}$ corresponding to $u_{j_1}(0) = u_{j_2}(0) = \ldots = u_{j_\ell}(0)$, which precisely is the partial resonance.

[8] The matrices $\mathbb{S}_1$ and $\mathbb{S}_2$ correspond to $\mathbb{S}_\nu$ and $\mathbb{S}_{\nu+\mu}$ in [9, 35]. The sectors in (3.6) correspond to the sectors $\mathcal{S}_\nu(\mathbb{D}(u^*))$, $\mathcal{S}_{\nu+\mu}(\mathbb{D}(u^*))$ and $\mathcal{S}_{\nu+2\mu}(\mathbb{D}(u^*))$ of [35], where $u^* := u^0$ or u^c, depending on the type of polydisc considered.

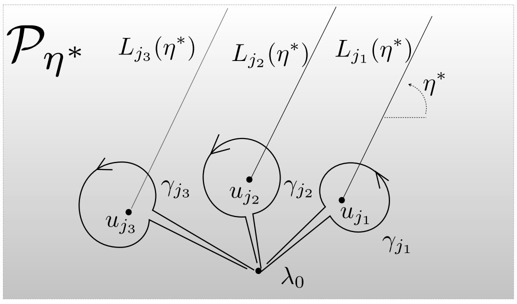

Figure 2. $\mathcal{P}_{\eta^*}(u)$, branch-cuts and basic loops. The base point λ_0 for the loops γ_j belongs to $\bigcap_{u\in\mathbb{D}}\mathcal{P}_{\eta^*}(u)$.

are independent of u. *Here,* $\mathcal{M}_j(u)$ *is the monodromy matrix at* $\lambda = u_j$ *of* $\Phi_{hol}(\lambda,u)$. *They are expressed in terms of the Stokes matrices of system (2.4):*

$$p_{jk} = \begin{cases} 2\cos\pi(\theta_j-\theta_k) - e^{i\pi(\theta_j-\theta_k)}(\mathbb{S}_1)_{jk}(\mathbb{S}_2^{-1})_{kj}, & j\prec k, \\ 2\cos\pi(\theta_j-\theta_k) - e^{i\pi(\theta_k-\theta_j)}(\mathbb{S}_1)_{kj}(\mathbb{S}_2^{-1})_{jk},, & j\succ k, \end{cases} \tag{3.8}$$

with ordering (3.3) or (3.4) according to the polydisc being either $\mathbb{D}(u^0)$ *or* $\mathbb{D}(u^c)$. *In the latter case,*

$$p_{jk} = 2\cos\pi(\theta_j-\theta_k) \quad \textit{for } j\neq k \textit{ such that } u_j^c = u_k^c. \tag{3.9}$$

To appreciate the general validity of (3.8), notice that every fundamental solution $\Phi'(\lambda,u)$ of system (1.2), defined at $u^\bullet$, is $\Phi'(\lambda,u) = \Phi_{\mathrm{hol}}(\lambda,u^\bullet)C'(u)$, with $\det C'(u)\neq 0$, so that its $\mathrm{tr}(\mathcal{M}_j'\mathcal{M}_k')$ coincide with p_{jk} in (3.8). The following proposition is proved in appendix B.1.

Proposition 3.1. *Consider either Case 1 or Case 2. In Case 2, let the coalescence be* $u_1^c = u_2^c$ *or* $u_2^c = u_3^c$, *while* $u_3 - u_1 \neq 0$. *Then, in order to compute the monodromy data of (2.4), it suffices to compute the data of*

$$\frac{dY}{dz} = \left(U(x) + \frac{\Omega(x)}{z}\right)Y, \qquad U(x) = \begin{pmatrix} 0 & 0 & 0\\ 0 & x & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{3.10}$$

where Ω *is given in theorem 2.1. The monodromy data for the system (2.4) relative to a prefixed admissible direction* $\arg z = \tau^{(0)}$ *(Case 1) or* τ *(Case 2) are the same data of system (3.10) relative to the admissible direction* $\tau^{(0)} + \arg(u_3^0 - u_1^0)$, *or* $\tau + \arg(u_3^c - u_1^c)$.

The case $u_3 - u_1 \to 0$, $x\to\infty$, can be recovered from the proposition above by the symmetry

$$\theta_2' = \theta_3, \quad \theta_3' = \theta_2; \quad \theta_1' = \theta_1, \quad \theta_\infty' = \theta_\infty; \quad y'(x') = \frac{1}{x}y(x), \quad x = \frac{1}{x'}.$$

Replacing (2.4) by (3.10) is equivalent to assuming that $u_1 = 0$, $u_2 = x$, $u_3 = 1$ in (2.4).

Corollary 3.1. *The results of theorem 3.1 hold for the monodromy invariants* $\mathrm{p_{jk}}$ *of system (2.36), in terms of the Stokes matrices of system (3.10).*

3.1. Proof of theorem 3.1

The Stokes matrices are related to the monodromy of certain *selected column vector solutions* $\vec{\Psi}_1(\lambda,u)$, $\vec{\Psi}_2(\lambda,u)$, $\vec{\Psi}_3(\lambda,u)$ of system (2.18). These are uniquely determined in theorem 5.1 of [35], to which we refer (with $n=3$ and the identification $\theta_k=-\lambda_k'-1$, being λ_k' used in [35]). They are holomorphic on $\mathcal{P}_\eta(u)\hat{\times}\mathbb{D}(u^*)$. A solution $\vec{\Psi}_k$ has a branching point at $\lambda=u_k$ in case $\theta_k\notin\mathbb{Z}$, and for $\lambda\in\mathcal{P}_{\eta^*}(u)$ its monodromy corresponding to a loop $\gamma_j:(\lambda-u_j)\longmapsto(\lambda-u_j)e^{2\pi i}$ in figure 2 is given in [35] by:

$$\vec{\Psi}_k\longmapsto\begin{cases}e^{2\pi i\theta_k}\vec{\Psi}_k, & j=k,\\ \vec{\Psi}_k+\alpha_j c_{jk}\vec{\Psi}_j, & j\neq k,\end{cases}\qquad \alpha_j:=\left\{\begin{array}{ll} e^{2\pi i\theta_j}-1, & \theta_j\notin\mathbb{Z}\\ 2\pi i, & \theta_j\in\mathbb{Z}\end{array}\right., \qquad j,k\in\{1,2,3\} \tag{3.11}$$

with certain connection coefficients c_{jk}. The above formulae imply that $c_{kk}=1$ for $\theta_k\notin\mathbb{Z}$, and $c_{kk}=0$ for $\theta_k\in\mathbb{Z}$. If $\theta_k\in\mathbb{Z}_-:=\{-1,-2,-3,\dots\}$, in some cases depending on the specific V it may happen that $\vec{\Psi}_k\equiv 0$, then $c_{jk}=0$ for every j.

It is proved in [35] that the c_{jk} are *isomonodromic connection coefficients*, i.e. they do not depend on $u\in\mathbb{D}$, so that the transformation (3.11) holds for every u in the polydisc. In case of coalescences,

$$c_{jk}=0\quad\text{for } j\neq k \text{ such that } u_j^c=u_k^c.$$

In this case, as u varies in $\mathbb{D}(u^c)$ the branch cuts $L_j(\eta)$ and $L_k(\eta)$ can overlap, but this causes no difficulties because the corresponding $c_{jk}=0$, so that $\vec{\Psi}_k$ has trivial monodromy at $\lambda=u_j$. Hence, (3.11) makes sense also in case of coalescences in $\mathbb{D}=\mathbb{D}(u^c)$.

A matrix solution of system (2.18) is constructed with the selected solutions:

$$\Psi(\lambda,u):=\left(\vec{\Psi}_1(\lambda,u)\;\middle|\;\vec{\Psi}_2(\lambda,u)\;\middle|\;\vec{\Psi}_3(\lambda,u)\right). \tag{3.12}$$

It has constant monodromy, but it is not necessarily fundamental.

Remark 3.2. A sufficient condition to be fundamental is that V has no integer eigenvalues, which is not our case. If V has some integer eigenvalues and $\Psi(\lambda,u)$ is fundamental, necessarily at least one $\theta_k\in\mathbb{Z}$ (see [3] for the generic case and [31, 35] for the most general case).

By the Laplace transform (2.17), [31, 35] prove that

$$c_{jk}=\begin{cases}\dfrac{e^{2\pi i\theta_k}}{\alpha_k}(\mathbb{S}_1)_{jk}, & j\prec k,\\[2ex] -\dfrac{(\mathbb{S}_2^{-1})_{jk}}{e^{2\pi i(\theta_j-\theta_k)}\alpha_k}, & j\succ k,\end{cases} \tag{3.13}$$

where the Stokes matrices are those of system (2.4). This is true without any assumptions on the matrix V, and holds also in case of coalescences with vanishing conditions (2.3).

Let

$$X(\lambda,u):=G^{-1}\Psi(\lambda,u)=\begin{pmatrix}X_1^1 & X_2^1 & X_3^1\\ X_1^2 & X_2^2 & X_3^2\\ X_1^3 & X_2^3 & X_3^3\end{pmatrix} \tag{3.14}$$

be the matrix solution of system (2.20) corresponding to (3.12), and consider the associated matrix solutions $\Phi^{[jk]}=\prod_{j=1}^{3}(\lambda-u_j)^{-\theta_j/2}X^{[jk]}$ of system (2.25), where

$$X^{[jk]}(\lambda,u)=\Big(\boldsymbol{X}_j(\lambda,u)\,\Big|\,\boldsymbol{X}_k(\lambda,u)\Big),\qquad \boldsymbol{X}_j:=\begin{pmatrix}X_j^1\\ X_j^3\end{pmatrix},\qquad 1\leqslant j<k\leqslant 3.$$

The monodromy (3.11) induces the transformations:

$$\gamma_1:\quad \boldsymbol{X}_1\longmapsto e^{2\pi i\theta_1}\boldsymbol{X}_1,\qquad \boldsymbol{X}_2\longmapsto \boldsymbol{X}_2+\alpha_1c_{12}\boldsymbol{X}_1,\quad \boldsymbol{X}_3\longmapsto \boldsymbol{X}_3+\alpha_1c_{13}\boldsymbol{X}_1;\tag{3.15}$$

$$\gamma_2:\quad \boldsymbol{X}_1\longmapsto \boldsymbol{X}_1+\alpha_2c_{21}\boldsymbol{X}_2,\quad \boldsymbol{X}_2\longmapsto e^{2\pi i\theta_2}\boldsymbol{X}_2,\qquad \boldsymbol{X}_3\longmapsto \boldsymbol{X}_3+\alpha_2c_{23}\boldsymbol{X}_2;\tag{3.16}$$

$$\gamma_3:\quad \boldsymbol{X}_1\longmapsto \boldsymbol{X}_1+\alpha_3c_{31}\boldsymbol{X}_3,\quad \boldsymbol{X}_2\longmapsto \boldsymbol{X}_2+\alpha_3c_{32}\boldsymbol{X}_3,\quad \boldsymbol{X}_3\longmapsto e^{2\pi i\theta_3}\boldsymbol{X}_3;\tag{3.17}$$

We distinguish two cases: (1) there are two linearly independent $\boldsymbol{X}_j$, $\boldsymbol{X}_k$, $j\neq k$; (2) all the $\boldsymbol{X}_j$ are linearly dependent.

1) Linearly independent case. Without loss of generality, assume that $\boldsymbol{X}_1$ and $\boldsymbol{X}_3$ are linearly independent (the discussion is analogous for another independent pair $\boldsymbol{X}_j,\boldsymbol{X}_k$). We compute the monodromy matrices $\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3$, corresponding to the loops $\gamma_1,\gamma_2,\gamma_3$, of the following fundamental matrix solution of system (1.2)

$$\Phi_{\text{hol}}(\lambda,u):=\Phi^{[13]}(\lambda,u)=\prod_{j=1}^{3}(\lambda-u_j)^{-\theta_j/2}X^{[13]}(\lambda,u).\tag{3.18}$$

From (3.15) and (3.17) we receive

$$\mathcal{M}_1=\begin{pmatrix}e^{\pi i\theta_1} & e^{-\pi i\theta_1}\alpha_1c_{13}\\ 0 & e^{-\pi i\theta_1}\end{pmatrix},\qquad \mathcal{M}_3=\begin{pmatrix}e^{-\pi i\theta_3} & 0\\ e^{-\pi i\theta_3}\alpha_3c_{31} & e^{\pi i\theta_3}\end{pmatrix},$$

so that

$$p_{13}=e^{-i\pi(\theta_1+\theta_3)}\alpha_1\alpha_3c_{13}c_{31}+2\cos(\pi(\theta_1-\theta_3)).$$

Now, $\boldsymbol{X}_2=a\boldsymbol{X}_1+b\boldsymbol{X}_3$ for some $a,b\in\mathbb{C}$, so that for the loop γ_2 the transformation (3.16) yields

$$\mathcal{M}_2=e^{-i\pi\theta_2}\begin{pmatrix}1+\alpha_2c_{21}a & \alpha_2c_{23}a\\ \alpha_2c_{21}b & 1+\alpha_2c_{23}b\end{pmatrix}.$$

In order to find a,b, we recall that

$$\operatorname{Tr}\mathcal{M}_2=2\cos\pi\theta_j,\qquad \det\mathcal{M}_2=1.\tag{3.19}$$

Since $\alpha_2=e^{2i\pi\theta_2}-1$ for $\theta_2\notin\mathbb{Z}$ and $\alpha_2=2\pi i$ for $\theta_2\in\mathbb{Z}$, both (3.19) are equivalent to

$$c_{21}a+c_{23}b=\begin{cases}1, & \text{if } \theta_2\notin\mathbb{Z};\\ 0, & \text{if } \theta_2\in\mathbb{Z}.\end{cases}\tag{3.20}$$

In order to find other conditions on a and b, we consider the transformation of $\boldsymbol{X}_2$ in (3.15) along the loop γ_1:

$$\begin{aligned}\boldsymbol{X}_2 &\longmapsto \boldsymbol{X}_2+\alpha_1c_{12}\boldsymbol{X}_1\equiv a\boldsymbol{X}_1+b\boldsymbol{X}_3+\alpha_1c_{12}\boldsymbol{X}_1,\\ \boldsymbol{X}_2=a\boldsymbol{X}_1+b\boldsymbol{X}_3 &\longmapsto ae^{2i\pi\theta_1}\boldsymbol{X}_1+b(\boldsymbol{X}_3+\alpha_1c_{13}\boldsymbol{X}_1).\end{aligned}$$

The above holds if and only if $a(e^{2i\pi\theta_1}-1)+\alpha_1(bc_{13}-c_{12})=0$. Thus

$$c_{13}b=\begin{cases} c_{12}-a, & \text{if } \theta_2\notin\mathbb{Z};\\ c_{12}, & \text{if } \theta_2\in\mathbb{Z}.\end{cases} \tag{3.21}$$

For the loop γ_3 in (3.17):

$$\begin{aligned} \boldsymbol{X}_2 &\longmapsto \boldsymbol{X}_2+\alpha_3c_{32}\boldsymbol{X}_3\equiv a\boldsymbol{X}_1+b\boldsymbol{X}_3+\alpha_3c_{32}\boldsymbol{X}_3,\\ \boldsymbol{X}_2=a\boldsymbol{X}_1+b\boldsymbol{X}_3 &\longmapsto a(\boldsymbol{X}_1+\alpha_3c_{31}\boldsymbol{X}_3)+be^{2\pi i\theta_3}\boldsymbol{X}_3.\end{aligned}$$

The above is true if and only if $b(e^{2i\pi\theta_3}-1)+\alpha_3(ac_{31}-c_{32})=0$, namely

$$c_{31}a=\begin{cases} c_{32}-b, & \text{if } \theta_2\notin\mathbb{Z};\\ c_{32}, & \text{if } \theta_2\in\mathbb{Z}.\end{cases} \tag{3.22}$$

We are ready to compute $\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2)$. In case $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are also independent (i.e. $b\neq 0$), since $\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2)$ is invariant by conjugation, we can compute it using $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ as a basis, and this is done as above for the case $\boldsymbol{X}_1,\boldsymbol{X}_3$, yielding

$$p_{12}=e^{-i\pi(\theta_1+\theta_2)}\alpha_1\alpha_2c_{12}c_{21}+2\cos(\pi(\theta_1-\theta_2)).$$

In case $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are not independent, then $b=0$ and

$$\mathcal{M}_2=\begin{pmatrix} e^{-i\pi\theta_2}(1+\alpha_2c_{21}a) & e^{-i\pi\theta_2}\alpha_2c_{23}a\\ 0 & e^{-i\pi\theta_2}\end{pmatrix}.$$

Using (3.20)–(3.22) we receive

$$\mathcal{M}_2=\begin{cases}\begin{pmatrix} e^{i\pi\theta_2} & 2i\sin(\pi\theta_2)c_{12}c_{23}\\ 0 & e^{-i\pi\theta_2}\end{pmatrix}, \quad c_{12}c_{21}=1, \quad c_{32}=c_{31}c_{12}, & \text{for } \theta_2\notin\mathbb{Z};\\[2ex] (-1)^{\theta_2}\begin{pmatrix}1 & 2\pi i c_{23}a\\ 0 & 1\end{pmatrix}, \quad c_{12}=0, \quad c_{31}a=c_{32}, \quad c_{21}a=0, & \text{for } \theta_2\in\mathbb{Z}.\end{cases}$$

Therefore,

$$p_{12}=2\cos(\pi(\theta_1+\theta_2)) \underset{c_{12}c_{21}=1}{\equiv} 2c_{12}c_{21}(\cos\pi(\theta_1+\theta_2)-\cos\pi(\theta_1-\theta_2))+2\cos\pi(\theta_1-\theta_2),$$
$$\theta_2\notin\mathbb{Z},$$

$$p_{12}=(-1)^{\theta_2}2\cos(\pi\theta_1) \underset{c_{12}=0}{\equiv} 2c_{12}c_{21}(\cos\pi(\theta_1+\theta_2)-\cos\pi(\theta_1-\theta_2))+2\cos\pi(\theta_1-\theta_2),$$
$$\theta_2\in\mathbb{Z}.$$

The computation of $p_{23}=\mathrm{Tr}(\mathcal{M}_2\mathcal{M}_3)$ can be done in an analogous way. In conclusion, all the possibilities considered lead to the formula

$$p_{jk}=e^{-i\pi(\theta_j+\theta_k)}\alpha_j\alpha_kc_{jk}c_{kj}+2\cos(\pi(\theta_j-\theta_k)). \tag{3.23}$$

Finally, substituting (3.13) we receive (3.8) in full generality.

2) The linearly-dependent case. The gauge ${}_\gamma Y:=z^{-\gamma}Y$, $\gamma\in\mathbb{C}$, transforms (2.1) into

$$\mathrm{d}({}_\gamma Y)=\omega(z,u;\gamma)\,{}_\gamma Y, \qquad \omega(z,u;\gamma)=\left(U+\frac{V-(1+\gamma)I}{z}\right)\mathrm{d}z+\sum_{k=1}^{3}(zE_k+V_k)\mathrm{d}u_k.$$

This changes $\theta_k\mapsto\theta_k+\gamma$, while θ_∞ is unchanged, and (2.27) changes to $\mu_1[\gamma]=(\theta_\infty-\theta_1-\theta_2-\theta_3-3\gamma)/2$, $\mu_3[\gamma]=(-\theta_\infty-\theta_1-\theta_2-\theta_3-3\gamma)/2$.

There exists $\gamma_0 > 0$ sufficiently small such that $V - \gamma$ has non-integer eigenvalues and non-integer diagonal entries for $0 < |\gamma| < \gamma_0$. Hence, the analogous of the matrix (3.12), here called ${}_\gamma\Psi(\lambda,u) = \left({}_\gamma\vec{\Psi}_1(\lambda,u) \mid {}_\gamma\vec{\Psi}_2(\lambda,u) \mid {}_\gamma\vec{\Psi}_3(\lambda,u)\right)$, is fundamental, so that there are two independent column vectors in the triple ${}_\gamma\boldsymbol{X}_1(\lambda,u), {}_\gamma\boldsymbol{X}_2(\lambda,u), {}_\gamma\boldsymbol{X}_3(\lambda,u)$. The connection coefficients in (3.11), depending on γ, will be called $c_{jk}[\gamma]$, with

$$\alpha_k[\gamma] = e^{2\pi i(\theta_k+\gamma)} - 1.$$

The discussion of the independent case can be repeated, with the $c_{jk}[\gamma]$ and $\alpha_k[\gamma]$ in the transformations (3.15)–(3.17). We can assume that ${}_\gamma\boldsymbol{X}_1$ and ${}_\gamma\boldsymbol{X}_3$ are independent, otherwise the discussion is analogous for another independent pair. After the gauge

$${}_\gamma\Phi = \prod_{j=1}^{3}(\lambda-u_j)^{-(\theta_j+\gamma)/2}\,{}_\gamma\boldsymbol{X},$$

we obtain the analogous of system (1.2):

$$\frac{\mathrm{d}({}_\gamma\Phi)}{\mathrm{d}\lambda} = \sum_{k=1}^{3}\frac{\mathcal{A}_k[\gamma]}{\lambda-u_k}\,{}_\gamma\Phi, \qquad \mathcal{A}_k := A_k[\gamma] - \frac{\theta_k+\gamma}{2}. \tag{3.24}$$

Let

$${}_\gamma\Phi^{[1,3]}(\lambda,u) = \prod_{j=1}^{3}(\lambda-u_j)^{-\theta_j/2}\left({}_\gamma\boldsymbol{X}_1(\lambda,u) \,\middle|\, {}_\gamma\boldsymbol{X}_3(\lambda,u)\right)$$

be the analogous of (3.18), and let $\mathcal{M}_j[\gamma]$ be its monodromy matrix at u_j. The same procedure leading to (3.23) yields

$$\begin{aligned} p_{jk}[\gamma] &:= \mathrm{tr}\left(\mathcal{M}_j[\gamma]\mathcal{M}_k[\gamma]\right) \\ &= e^{-i\pi(\theta_j+\theta_k+2\gamma)}\alpha_j[\gamma]\alpha_k[\gamma]c_{jk}[\gamma]c_{kj}[\gamma] + 2\cos(\pi(\theta_j-\theta_k)), \quad j\neq k. \end{aligned} \tag{3.25}$$

In general, $c_{jk}[\gamma]$, ${}_\gamma\Psi$, ${}_\gamma\boldsymbol{X}_k$ and ${}_\gamma\Phi^{[1,3]}$ diverge for $\gamma \to 0$. Therefore, the monodromy matrices $\mathcal{M}_1[\gamma]$, $\mathcal{M}_2[\gamma]$, $\mathcal{M}_3[\gamma]$ generate the monodromy group for $0<|\gamma|<\gamma_0$, but may be not defined at $\gamma=0$. To overcome the problem, we use a relation proved in full generality in [31], and in [3] in a generic case. In case $\mathbb{D}=\mathbb{D}(u^0)$, the relation says that at any $u\in\mathbb{D}(u^0)$

$$\alpha_k c_{jk} = \begin{cases} e^{-2\pi i\gamma}\alpha_k[\gamma]\; c_{jk}[\gamma], & \text{if } k\succ j, \\ \alpha_k[\gamma]\; c_{jk}[\gamma], & \text{if } k\prec j, \end{cases} \qquad \text{for real } 0<\gamma<\gamma_0. \tag{3.26}$$

The ordering $\prec$ is (3.3). In case $\mathbb{D}=\mathbb{D}(u^c)$, the same relation holds at any $u\in\mathbb{D}(u^c)\backslash\Delta_{\mathbb{C}^3}$ for $j\neq k$ such that $u_j^c\neq u_k^c$, the ordering relation being (3.4). For $j\neq k$ such that $u_j^c=u_k^c$ the ordering relation is not defined, but $c_{jk}=c_{jk}[\gamma]=0$, so that we can state that (3.26) still holds. Using (3.26), (3.25) becomes

$$p_{jk}[\gamma] = e^{i\pi\gamma}\, e^{-i\pi(\theta_j+\theta_k)}\alpha_j\alpha_k c_{jk}c_{kj} + 2\cos(\pi(\theta_j-\theta_k)), \qquad 0<\gamma<\gamma_0\text{real}, \tag{3.27}$$

for both $u\in\mathbb{D}(u^0)$ and $u\in\mathbb{D}(u^c)\backslash\Delta_{\mathbb{C}^3}$ (in the latter case, (3.27) is true also for $j\neq k$ such that $u_j^c=u_k^c$, because it just reduces to the identity $2\cos(\pi(\theta_j-\theta_k)) = 2\cos(\pi(\theta_j-\theta_k))$). Since both the c_{jk} and $c_{jk}[\gamma]$ are constant, (3.27) extends analytically at $\Delta_{\mathbb{C}^3}$.

Now, (3.27) holds for $0<\gamma<\gamma_0$, the r.h.s. depends holomorphically on $\gamma\in\mathbb{C}$, while the l.h.s. $p_{jk}[\gamma]$ has been defined for $0<|\gamma|<\gamma_0$. We show that $p_{jk}[\gamma]$ can also be obtained from a fundamental matrix solution of (3.24) which is holomorphic at $\gamma=0$, so is well defined at $\gamma=0$. To do that, recall from [9] that if $\mathbb{D}=\mathbb{D}(u^c)$, the choice of an admissible direction τ determines a cell decomposition of $\mathbb{D}(u^c)$ into topological cells, called $\boldsymbol{\tau}$**-cells**. They are the connected components of $\mathbb{D}(u^c)\backslash(\Delta_{\mathbb{C}^3}\cup X(\tau))$, where $X(\tau)$ is the locus of points $u=(u_1,u_2,u_3)\in\mathbb{D}(u^c)$ such that $\Re(e^{i\tau}(u_j-u_k))=0$.

For $\mathbb{D}=\mathbb{D}(u^0)$, let $u^\bullet\in\mathbb{D}(u^0)$. For $\mathbb{D}=\mathbb{D}(u^c)$, let $u^\bullet$ belong to a τ-cell of $\mathbb{D}(u^c)$. Then, there is a sufficiently small neighbourhood $\mathcal{U}$ of $u^\bullet$ such that, as u varies in $\mathcal{U}$, the point u_k represented in the λ-plane remains inside a closed disc D_k centred at $u_k^\bullet$, with $D_j\cap D_k=\emptyset$ for $1\leqslant j\neq k\leqslant 3$. Consider the simply connected domain

$$\mathcal{B}_{u^\bullet}:=\mathcal{P}_{\eta^*}(u^\bullet)\backslash(D_1\cup D_2\cup D_3).$$

Since system (3.24) holomorphically depends on the parameters $(u,\gamma)\in\mathcal{U}\times\{\gamma\in\mathbb{C}\text{s.t. }|\gamma|<\gamma_0\}$, according to a general result (see for example [55], theorem 24.1) it has a fundamental matrix solution

$$\Phi_{\text{hol}}^{(u^\bullet)}(\lambda,u,\gamma)$$

holomorphic of $(\lambda,u,\gamma)\in\mathcal{B}_{u^\bullet}\times\mathcal{U}\times\{\gamma\in\mathbb{C}\text{s.t. }|\gamma|<\gamma_0\}$. If $\mathcal{U}$ is sufficiently small, it holomorphically extends to $\left(\mathcal{P}_{\eta^*}(u)\widehat{\times}\,\mathcal{U}\right)\times\{\gamma\in\mathbb{C}\text{s.t. }|\gamma|<\gamma_0\}$. For some invertible connection matrix $C(u,\gamma)$ we have

$$\Phi_{\text{hol}}^{(u^\bullet)}(\lambda,u,\gamma)={}_\gamma\Phi^{[13]}(\lambda,u)\cdot C(u,\gamma),\quad 0<|\gamma|<\gamma_0.$$

Now, $C(u,\gamma)$ is holomorphic of $u\in\mathcal{U}$, for any $0<|\gamma|<\gamma_0$, but may diverge as $\gamma\to0$. On the other hand, the monodromy matrix $\mathcal{M}_k^{\text{hol}}(u,\gamma)$ of $\Phi_{\text{hol}}^{(u^\bullet)}(\lambda,u,\gamma)$ at $\lambda=u_k$ (the loop going around ∂D_k) is holomorphic of $(u,\gamma)\in\mathcal{U}\times\{\gamma\in\mathbb{C}\text{s.t. }|\gamma|<\gamma_0\}$, including $\gamma=0$. Moreover,

$$\mathcal{M}_k[\gamma]=C(u,\gamma)\cdot\mathcal{M}_k^{\text{hol}}(u,\gamma)\cdot C(u,\gamma)^{-1}.$$

Since $\text{Tr}(\mathcal{M}_j^{\text{hol}}(u,\gamma)\mathcal{M}_k^{\text{hol}}(u,\gamma))=\text{Tr}(\mathcal{M}_j[\gamma]\mathcal{M}_k[\gamma])\equiv p_{jk}[\gamma]$, we see from (3.27) that

$$\text{Tr}\left(\mathcal{M}_j^{\text{hol}}(u,\gamma)\mathcal{M}_k^{\text{hol}}(u,\gamma)\right)=e^{i\pi\gamma}\,e^{-i\pi(\theta_j+\theta_k)}\alpha_j\alpha_k c_{jk}c_{kj}+2\cos(\pi(\theta_j-\theta_k)),\qquad 0<\gamma<\gamma_0.$$

Now, both the l.h.s. and the r.h.s. are continuous of γ in a neighbourhood of $\gamma=0$. Therefore, taking the limit $\gamma\to0_+$ we obtain

$$p_{jk}:=\text{tr}\left(\mathcal{M}_j^{\text{hol}}(u,0)\mathcal{M}_k^{\text{hol}}(u,0)\right)=\,e^{-i\pi(\theta_j+\theta_k)}\alpha_j\alpha_k c_{jk}c_{kj}+2\cos(\pi(\theta_j-\theta_k)).\tag{3.28}$$

In case $\mathbb{D}=\mathbb{D}(u^c)$, we can repeat the above discussion also for $u^\bullet$ on the boundary of one τ-cell (provided that $u^\bullet\notin\Delta_{\mathbb{C}^3}$), because $\tau=3\pi/2-\eta$ can be slightly changed without affecting the properties of fundamental solutions.

Therefore, we have proved that for every $u^\bullet\in\mathbb{D}(u^0)$ or $u^\bullet\in\mathbb{D}(u^c)\backslash\Delta_{\mathbb{C}^3}$, we can find a fundamental solution $\Phi_{\text{hol}}^{(u^\bullet)}(\lambda,u,0)$ of system (1.2), holomorphically depending on $(\lambda,u)\in\mathcal{P}_{\eta^*}(u)\widehat{\times}\,\mathcal{U}$, with $\mathcal{U}$ small enough, such that its monodromy invariants are the p_{jk} in (3.28). Thus, we conclude that the formulae (3.8) always hold. In the linearly dependent case, $\Phi_{\text{hol}}(\lambda,u)$ in the statement of the theorem is precisely $\Phi_{\text{hol}}^{(u^\bullet)}(\lambda,u,0)$ above, while in the linearly independent case it is a fundamental matrix like (3.18). □

4. Classification of transcendents satisfying vanishing conditions (4.1). Reduction to special functions

By proposition 3.1, in order to compute the monodromy data useful to solve the nonlinear connection problem for PVI, it suffices to take system (3.10).

We classify all branches of transcendents holomorphic at $x=0$ (behaviour $y(x)=\sum_{n=0}^{\infty} b_n x^n$), such that the associated $\Omega(x)$ of theorem 2.1 satisfies

$$\lim_{x\to 0}\Omega_{12}(x)=\lim_{x\to 0}\Omega_{21}(x)=0 \quad \text{holomorphically} \tag{4.1}$$

(by the symmetries of PVI, it is enough to study $x=0$). The reason why we study this case is for the importance of the conditions (4.1): according to theorem 1.1 of [9], they allow to compute all the monodromy data of the isomonodromically x-dependent system (3.10) with coalescing eigenvalues 0 and $x\to 0$ using the system reduced at the coalescence $x=0$:

$$\frac{\mathrm{d}Y}{\mathrm{d}z}=\left(U_0+\frac{\Omega_0}{z}\right)Y, \qquad U_0:=\mathrm{diag}(0,0,1), \quad \Omega_0:=\Omega(0). \tag{4.2}$$

The latter is simpler than (3.10), because $\Omega_{12}(0)=\Omega_{21}(0)=0$. Hence, for the nonlinear connection problem, theorem 3.1 and Corollary 3.1 allow us to obtain the p_{jk}'s in terms of the Stokes matrices of the simpler system (4.2).

When (4.1) holds, we will show that system (4.2) is equivalent to a scalar confluent hypergeometric equation or to a generalized hypergeometric equation of type $(q,p)=(2,2)$, so that the Stokes matrices, and consequently the p_{jk}, can be concretely computed. This computation will be explicitly done in section 5 for a selection of transcendents, being all other cases carried out analogously.

The reduction to a scalar ODE follows two steps. First, (4.2) is changed by a gauge transformation $Y=G_0\widetilde{Y}$, where $G_0^{-1}\Omega_0 G_0=\mathrm{diag}(\mu_1,0,\mu_3)$. A column $(\widetilde{y}_1,\widetilde{y}_2,\widetilde{y}_3)^{\mathrm{T}}$ of $\widetilde{Y}$ satisfies a first order system of ODEs. Then, this system is reduced by elimination[9] to a single scalar ODE for one of the components $\widetilde{y}_j$. The latter is in general of the third order (generalized hypergeometric), but depending on the structure of Ω_0 we may receive a second order equation (confluent hypergeometric).

The classification and reduction to special functions is technically performed in appendix A and summarized in the table below. The first column refines the part of the table of [29] concerning transcendents with Taylor series at $x=0$. The parameters $\theta_1,\theta_2,\theta_3,\theta_\infty$ and the free parameter (the integration constant) y_0 or $y_0^{(|N|)}$ in $y(x)$, if any, must satisfy the conditions in the second column. In the third column, we give our classification according to the fulfilment of the vanishing conditions (4.1) and indicate the classical special functions in terms of which (4.2) can be solved. For the classical special functions appearing, we refer to appendices B.2 and B.3. For $N\in\mathbb{Z}\backslash\{0\}$ we also define

$$\mathcal{N}_N:=\begin{cases}\{0,2,4,\ldots,|N|-1\}\cup\{-2,-4,\ldots,-(|N|-1)\}, & \text{if } N \text{ is odd}\\ \{1,3,\ldots,|N|-1\}\cup\{-1,-3,\ldots,-(|N|-1)\}, & \text{if } N \text{ is even .}\end{cases} \tag{4.3}$$

[9] G_0 is defined with the freedom $G_0\mapsto G_0\Delta$, where Δ is a constant diagonal matrix. The transformation $(\widetilde{y}_1,\widetilde{y}_2,\widetilde{y}_3)^{\mathrm{T}}\mapsto\Delta\cdot(\widetilde{y}_1,\widetilde{y}_2,\widetilde{y}_3)^{\mathrm{T}}$ does not change the scalar *linear* ODE obtained after elimination of two components.

Taylor series	Conditions on parameters	Classical special functions
$y(x) = y_0 + \sum_{n\geqslant 1} b_n(y_0)\, x^n$ Solution (A.1).	$y_0 = \dfrac{\theta_\infty - 1 + \theta_3}{\theta_\infty - 1}$, $\theta_\infty + \theta_3 \notin \mathbb{Z}$, $\theta_\infty \neq 1$; or $y_0 = \dfrac{\theta_\infty - 1 - \theta_3}{\theta_\infty}$, $\theta_\infty - \theta_3 \notin \mathbb{Z}$, $\theta_\infty \neq 1$.	The vanishing condition (4.1) does not hold
$y(x) = y_0 + \sum_{n\geqslant 1} b_n(y_0)\, x^n$ Solution (A.2).	$\theta_3 = 0, \theta_\infty = 1$, y_0 free parameter.	
$y(x) = \sum_{n=0}^{\lvert N\rvert-1} b_n x^n + \dfrac{y_0^{(\lvert N\rvert)}}{(\lvert N\rvert)!} x^{\lvert N\rvert} + \sum_{n\geqslant \lvert N\rvert+1} b_n(y_0^{(\lvert N\rvert)})\, x^n$ (T1) Solution (A.3).	$b_0 = \dfrac{N}{\theta_\infty - 1}$, $\theta_\infty \neq 1$. $y_0^{(\lvert N\rvert)}$ free parameter. $\theta_\infty - 1 + \theta_3 = N \in \mathbb{Z}\setminus\{0\}$ (section A.1.2) or $\theta_\infty - 1 - \theta_3 = N \in \mathbb{Z}\setminus\{0\}$, and $\theta_1 - \theta_2 \in \mathcal{N}_N$ (section A.1.1)	**Confluent hypergeometric** equation (A.8) if $\theta_1 = \theta_2$; equations (A.10), (A.13), $\theta_1 > \theta_2$; equations (A.11), (A.14), $\theta_1 < \theta_2$.
	$b_0 = \dfrac{N}{\theta_\infty - 1}$, $\theta_\infty \neq 1$. $y_0^{(\lvert N\rvert)}$ free parameter. $\theta_\infty - 1 + \theta_3 = N \in \mathbb{Z}\setminus\{0\}$ or $\theta_\infty - 1 - \theta_3 = N \in \mathbb{Z}\setminus\{0\}$, and either $\theta_\infty - 1 \in \begin{cases}\{-1,\ldots,-\lvert N\rvert+1\}\\ \{1,\ldots,\lvert N\rvert-1\}\end{cases}$ or $\theta_1 + \theta_2 \in \mathcal{N}_N$	The vanishing condition (4.1) does not hold
$y(x) = y_0' x + \sum_{n\geqslant 2} b_n(y_0')\, x^n$ (T2) Solution (A.15)	$y_0' = \dfrac{\theta_1}{\theta_1 - \theta_2}$, $\theta_1 - \theta_2 \notin \mathbb{Z}$ $\theta_1 \neq 0,\ \theta_2 \neq 0$ (section A.2 and A.2.1)	(2,2)—**Generalized hypergeometric** equation (A.22)
	$y_0' = \dfrac{\theta_1}{\theta_1 + \theta_2}$, $\theta_1 + \theta_2 \notin \mathbb{Z}$, $\theta_1 \neq 0$, $\theta_2 \neq 0$ (section A.2 and A.2.2)	The vanishing condition (4.1) does not hold

(Continued.)

$y(x) = y_0' x + \sum_{n\geqslant 2} b_n(y_0')\, x^n$ (T3) Solution (A.16)	$\theta_1 = \theta_2 = 0$ $y_0' \neq 0, 1$ free parameter (section A.2.3)	**Confluent hypergeometric** equation (A.24)
$y(x) = \sum_{n=1}^{\lvert N\rvert} b_n x^n + \frac{y_0^{(\lvert N\rvert+1)}}{(\lvert N\rvert+1)!} x^{\lvert N\rvert+1} + \sum_{n\geqslant \lvert N\rvert+2} b_n(y_0^{(\lvert N\rvert+1)})\, x^n$ (T4) Solution (A.17)	$\theta_1 - \theta_2 = N \in \mathbb{Z}\setminus\{0\}$ $\theta_1 \in \begin{cases}\{-1,-2,\ldots,-\lvert N\rvert+1\}\\ \{1,2,\ldots,\lvert N\rvert-1\}\end{cases}$ $y_0^{(\lvert N\rvert+1)}$ free parameter (section A.2.4)	(2, 2)—**Generalized hypergeometric** equation (A.25)
	$\theta_1 - \theta_2 = N \in \mathbb{Z}\setminus\{0\}$ or $\theta_1 + \theta_2 = N \in \mathbb{Z}\setminus\{0\}$ and $\theta_1 = 0, N,$ $y_0^{(\lvert N\rvert+1)}$ free parameter (section A.2.4)	**Confluent hypergeometric** equation (A.27) (either $\theta_1 = N \geqslant 1$, or $\theta_1 = 0, N \leqslant -1$) equation (A.29) (either $\theta_1 = N \leqslant -1$, or $\theta_1 = 0, N \geqslant 1$)
	$\theta_1 - \theta_2 = N \in \mathbb{Z}\setminus\{0\}$ $\{(\theta_3 + \theta_\infty - 1), (-\theta_3 + \theta_\infty - 1)\} \cap \mathcal{N}_N \neq \emptyset$ $y_0^{(\lvert N\rvert+1)}$ free parameter (section A.2.5)	(2, 2)—**Generalized hypergeometric** equation (A.31)
	$\theta_1 + \theta_2 = N \in \mathbb{Z}\setminus\{0\}$ $\theta_1 \in \begin{cases}\{-1,-2,\ldots,-\lvert N\rvert+1\}\\ \{1,2,\ldots,\lvert N\rvert-1\}\end{cases}$ or $\{(\theta_3 + \theta_\infty - 1), (-\theta_3 + \theta_\infty - 1)\} \cap \mathcal{N}_N \neq \emptyset$ $y_0^{(\lvert N\rvert+1)}$ free parameter (section A.2.6)	The vanishing condition (4.1) does not hold

Before going into the details of the construction of this table, we would like to highlight two facts:

i) in cases (T1), (T3) and (T4), even though the parameters $\theta_1, \theta_2, \theta_3, \theta_\infty$ are highly degenerate, the solutions for which conditions (4.1) are satisfied are still higher order transcendental functions;
ii) For generic parameters θ's, there exists precisely one solution for which conditions (4.1) are satisfied, that is case (T2).

Remark 4.1. In the table of [29] there is a misprint. Corresponding to the branches (45), the correct condition is $\sqrt{-2\beta} \in \{-1,-2,\dots,N+1\}$ for $N<0$, and $\sqrt{-2\beta} \in \{1,2,\dots,N-1\}$ for $N>0$. In (61), the correct condition is $\sqrt{2\alpha} \in \{-1,-2,\dots,N+1\}$ for $N<0$, and $\sqrt{2\alpha} \in \{1,2,\dots,N-1\}$ for $N>0$. Solutions to PVI with Taylorexpansions have been studied also in [45]

5. Monodromy data—Examples

For a selection of the tabulated transcendents satisfying the vanishing (4.1), we compute the Stokes matrices of system (3.10), and by formulae (3.8) the corresponding monodromy invariants p_{jk} of the 2×2 Fuchsian system (2.36).

Thanks to theorem 1.1 in [9], it suffices to consider the simplified system (4.2) at $x=0$ and its fundamental solutions $Y_1(z), Y_2(z), Y_3(z)$ with canonical asymptotics

$$Y_j(z) \sim Y_F(z,0), \quad z\to\infty \text{ in } \mathcal{S}_j, \quad j=1,2,3, \tag{5.1}$$

where $Y_F(z,0)$ is the value at $x=0$ of the unique formal solution

$$Y_F(z,x) = \left(I+\sum_{k\geqslant 1} F_k(x) z^{-k}\right) z^{-\Theta} e^{zU(x)},$$

of system (3.10), whose coefficients $F_k(x)$ are also holomorphic at $x=0$. Then, the Stokes matrices of (3.10) are just obtained from the solutions (5.1) through the relations

$$Y_2(z) = Y_1(z)\mathbb{S}_1, \qquad Y_3(z) = Y_2(z)\mathbb{S}_2.$$

We will choose the basic Stokes sectors to be

$$\mathcal{S}_1: -\frac{3\pi}{2} < \arg(z) < \frac{\pi}{2}, \qquad \mathcal{S}_2: -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}, \qquad \mathcal{S}_1: \frac{\pi}{2} < \arg(z) < \frac{5\pi}{2},$$

so that $\mathcal{S}_1$ contains the admissible ray in direction $\tau=0$, corresponding to $\eta = 3\pi/2$ in the λ-plane. We will do the gauge transformation $Y_j = G_0\widetilde{Y}_j$, diagonalizing Ω_0, which does not affect the Stokes matrices, so that the canonical asymptotics will be

$$\widetilde{Y}_j(z) \sim \widetilde{Y}_F(z) := G_0^{-1} Y_F(z,0), \quad z\to\infty \text{ in } \mathcal{S}_j, \quad j=1,2,3.$$

In the examples below, no partial resonance occurs, so that it is not necessary to compute $Y_F(z,x)$ and evaluate it at $x=0$, being sufficient to compute directly the formal solution of (4.2), with behaviour $Y_F(z) = (I+\sum_{k\geqslant 1}\mathring{F}_k z^{-k}) z^{-\Theta} e^{zU_0}$, which is unique (remark 3.1) and coincides with $Y_F(z,0)$.

5.1. Case (T3) of the table (i.e. solutions (A.16))

In this case, the gauge $\widetilde{Y} = G_0 Y$ transforms system (4.2) into

$$\frac{\mathrm{d}\widetilde{Y}}{\mathrm{d}z} = \left[\frac{1}{2\theta_\infty}\begin{pmatrix} -\theta_3+\theta_\infty & 0 & -\theta_3+\theta_\infty \\ 0 & 0 & 0 \\ \theta_3+\theta_\infty & 0 & \theta_3+\theta_\infty \end{pmatrix} + \frac{1}{2z}\begin{pmatrix} -\theta_3+\theta_\infty & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\theta_3-\theta_\infty \end{pmatrix}\right]\widetilde{Y}, \tag{5.2}$$

where

$$G_0 = \begin{pmatrix} -\tilde{k}_1^0 \dfrac{\theta_3+\theta_\infty}{\theta_3-\theta_\infty} & -\dfrac{\tilde{k}_1^0}{\tilde{k}_2^0}\dfrac{1-y_0'}{y_0'} & -\tilde{k}_1^0 \\ -\tilde{k}_2^0 \dfrac{\theta_3+\theta_\infty}{\theta_3-\theta_\infty} & 1 & -\tilde{k}_2^0 \\ 1 & 0 & 1 \end{pmatrix} \tag{5.3}$$

is a diagonalizing matrix of (A.23). A column vector $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)^T$ of $\widetilde{Y}$ satisfies the 1st order system

$$\begin{cases} \dfrac{\mathrm{d}\tilde{y}_1}{\mathrm{d}z} = \dfrac{\theta_\infty-\theta_3}{2\theta_\infty}(\tilde{y}_1+\tilde{y}_3) + \dfrac{\theta_\infty-\theta_3}{2z}\tilde{y}_1, \\ \dfrac{\mathrm{d}\tilde{y}_2}{\mathrm{d}z} = 0, \\ \dfrac{\mathrm{d}\tilde{y}_3}{\mathrm{d}z} = \dfrac{\theta_\infty+\theta_3}{2\theta_\infty}(\tilde{y}_1+\tilde{y}_3) - \dfrac{\theta_\infty+\theta_3}{2z}\tilde{y}_3. \end{cases}$$

Substituting the first equation into the third to eliminate $\tilde{y}_3$ and setting $\tilde{y}_1 = w z^{(\theta_\infty-\theta_3)/2}$ we receive the confluent hypergeometric equation (A.24).

In this section we consider the case with θ_3, θ_∞ not integers, while the case $\theta_3 = 0$ and $\theta_\infty \in \mathbb{Z}\setminus\{0\}$ will be considered in section 6. Since $\theta_1 - \theta_2 \notin \mathbb{Z}\setminus\{0\}$ there is no partial resonance, so that the formal solution of system (5.2) is unique. It is computed following section 4 of [9], receiving

$$\widetilde{Y}_F(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} -y_0' \dfrac{\theta_3-\theta_\infty}{2\tilde{k}_1^0\theta_\infty} & -(1-y_0')\dfrac{\theta_3-\theta_\infty}{2\tilde{k}_2^0\theta_\infty} & -\dfrac{\theta_3-\theta_\infty}{2\theta_\infty} z^{-\theta_3} e^z \\ -y_0' \dfrac{\tilde{k}_2^0}{\tilde{k}_1^0} & y_0' & 0 \\ y_0' \dfrac{\theta_3-\theta_\infty}{2\tilde{k}_1^0\theta_\infty} & (1-y_0')\dfrac{\theta_3-\theta_\infty}{2\tilde{k}_2^0\theta_\infty} & \dfrac{\theta_3+\theta_\infty}{2\theta_\infty} z^{-\theta_3} e^z \end{pmatrix},$$

where $\tilde{k}_1^0, \tilde{k}_2^0$ are in (A.23). The elements of the second row of the solutions of (5.2) are constants, while from (A.24) the elements of the first row have the general form

$$\tilde{y}_1(z; a, b, m, n) = \left[aM\left(ze^{2m\pi i}; \frac{\theta_\infty-\theta_3}{2}, \theta_\infty\right) + bU\left(ze^{2n\pi i}; \frac{\theta_\infty-\theta_3}{2}, \theta_\infty\right)\right] z^{(\theta_\infty-\theta_3)/2},$$
$$a, b \in \mathbb{C}, n, m \in \mathbb{Z},$$

where M and U are the confluent hypergeometric functions of appendix B.2. It is sufficient to compute the first two rows of the fundamental solutions $\widetilde{Y}_1, \widetilde{Y}_2, \widetilde{Y}_3$ with asymptotics $\widetilde{Y}_F$ in the Stokes sectors $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, respectively. The computation of the second row is immediate by comparison with the second row of the leading term of $\widetilde{Y}_F$. The first row is obtained by comparison of the first row of the leading term of $\widetilde{Y}_F$ and the leading coefficients of the asymptotics of M and U. For the fundamental solution $\widetilde{Y}_1$ we use formulas (B.4) and (B.6) with $\epsilon = -1$:

$$\widetilde{Y}_1(z) = \begin{pmatrix} \tilde{y}_1(z; 0, b_1, 0, 0) & \tilde{y}_1(z; 0, b_2, 0, 0) & \tilde{y}_1(z; a_3, b_3, 0, 0) \\ -y_0' \dfrac{\tilde{k}_2^0}{\tilde{k}_1^0} & y_0' & 0 \\ (\widetilde{Y}_1)_{31} & (\widetilde{Y}_1)_{32} & (\widetilde{Y}_1)_{33} \end{pmatrix},$$

where

$$a_3 = -\Gamma\left(\frac{\theta_\infty - \theta_3}{2}\right)\frac{\theta_3 - \theta_\infty}{2\theta_\infty},$$

$$b_1 = -y_0'\frac{\theta_3 - \theta_\infty}{2\tilde{k}_1^0\theta_\infty}, \quad b_2 = -(1-y_0')\frac{\theta_3 - \theta_\infty}{2\tilde{k}_2^0\theta_\infty}, \quad b_3 = \frac{\Gamma((\theta_\infty - \theta_3)/2)}{\Gamma((\theta_\infty + \theta_3)/2)}\frac{\theta_3 - \theta_\infty}{2\theta_\infty}e^{-i\pi(\theta_\infty - \theta_3)/2}.$$

For the fundamental solution $\widetilde{Y}_2$ we use formulas (B.4) and (B.6) with $\varepsilon = 1$:

$$\widetilde{Y}_2(z) = \begin{pmatrix} \tilde{y}_2(z;0,b_1,0,0) & \tilde{y}_1(z;0,b_2,0,0) & \tilde{y}_1(z;a,b_3e^{i\pi(\theta_\infty-\theta_3)/2},0,0) \\ -y_0'\dfrac{\tilde{k}_2^0}{\tilde{k}_1^0} & y_0' & 0 \\ (\widetilde{Y}_2)_{31} & (\widetilde{Y}_2)_{32} & (\widetilde{Y}_2)_{33} \end{pmatrix}.$$

For the fundamental solution $\widetilde{Y}_3$ we have to use the cyclic relation (B.5) with $n = -1$ and again formulas (B.4) and (B.6) with $\epsilon = -1$ to obtain the asymptotics of the function U in the sector $\mathcal{S}_3$:

$$\widetilde{Y}_3(z) = \begin{pmatrix} \tilde{y}_1(z;a_1,\tilde{b}_1,0,0) & \tilde{y}_1(z;a_2,\tilde{b}_2,0,0) & \tilde{y}_1(z;a_3e^{-i\pi(\theta_\infty+\theta_3)},\tilde{b}_3,-1,-1) \\ -y_0'\dfrac{\tilde{k}_2^0}{\tilde{k}_1^0} & y_0' & 0 \\ (\widetilde{Y}_3(z))_{31} & (\widetilde{Y}_3(z))_{32} & (\widetilde{Y}_3(z))_{33} \end{pmatrix},$$

where

$$a_1 = 2iy_0'\frac{\theta_3 - \theta_\infty}{2\tilde{k}_1^0\theta_\infty}\Gamma\left(\frac{\theta_\infty + \theta_3}{2}\right)\sin\left[\frac{\pi}{2}(\theta_\infty + \theta_3)\right]e^{i\pi\theta_3},$$
$$a_2 = 2i(1-y_0')\frac{\theta_3 - \theta_\infty}{2\tilde{k}_2^0\theta_\infty}\Gamma\left(\frac{\theta_\infty + \theta_3}{2}\right)\sin\left[\frac{\pi}{2}(\theta_\infty + \theta_3)\right]e^{i\pi\theta_3},$$
$$\tilde{b}_1 = b_1e^{i\pi(\theta_\infty+\theta_3)}, \quad \tilde{b}_2 = b_2e^{i\pi(\theta_\infty+\theta_3)}, \quad \tilde{b}_3 = b_3e^{-i\pi(\theta_\infty+\theta_3)}.$$

The non trivial entries $(\mathbb{S}_1)_{13}$ and $(\mathbb{S}_1)_{23}$ can be computed from the entries $(1,3)$ and $(2,3)$ of the equation $\widetilde{Y}_2(z) = \widetilde{Y}_1(z)\mathbb{S}_1$, while the non trivial entries $(\mathbb{S}_2)_{31}$ and $(\mathbb{S}_2)_{32}$ can be computed from the entries $(1,1)$ and $(1,2)$ of the equation $\widetilde{Y}_3(z) = \widetilde{Y}_2(z)\mathbb{S}_2$, obtaining

$$\mathbb{S}_1 = \begin{pmatrix} 1 & 0 & -2i\tilde{k}_1^0\dfrac{\Gamma((\theta_\infty-\theta_3)/2)}{\Gamma((\theta_\infty+\theta_3)/2)}\sin\left[\dfrac{\pi}{2}(\theta_\infty-\theta_3)\right] \\ 0 & 1 & -2i\tilde{k}_2^0\dfrac{\Gamma((\theta_\infty-\theta_3)/2)}{\Gamma((\theta_\infty+\theta_3)/2)}\sin\left[\dfrac{\pi}{2}(\theta_\infty-\theta_3)\right] \\ 0 & 0 & 1 \end{pmatrix} \tag{5.4}$$

and

$$\mathbb{S}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{2i\pi y_0'e^{i\pi\theta_3}}{\tilde{k}_1^0\Gamma(1-(\theta_\infty+\theta_3)/2)\Gamma((\theta_\infty-\theta_3)/2)} & -\dfrac{2i\pi(1-y_0')e^{i\pi\theta_3}}{\tilde{k}_2^0\Gamma(1-(\theta_\infty+\theta_3)/2)\Gamma((\theta_\infty-\theta_3)/2)} & 1 \end{pmatrix}. \tag{5.5}$$

With the choice of the admissible direction $\tau = 0$, corresponding to $\eta = 3\pi/2$, formulas (3.8) become

$$p_{12} = 2, \quad p_{13} = 2\cos(\pi\theta_3) - 4y_0' \sin\left[\frac{\pi}{2}(\theta_\infty - \theta_3)\right] \sin\left[\frac{\pi}{2}(\theta_\infty + \theta_3)\right],$$
$$p_{23} = 2\cos(\pi\theta_3) - 4(1 - y_0') \sin\left[\frac{\pi}{2}(\theta_\infty - \theta_3)\right] \sin\left[\frac{\pi}{2}(\theta_\infty + \theta_3)\right].$$

The above expressions confirm the results of [29], page 3260, Case (46).

5.2. *Case (T1) of the table with* $N = -1$, $\theta_1 = \theta_2$, $\theta_3 = \theta_\infty$

We consider the transcendent in case (T1) for $\theta_2 = \theta_1$, $\theta_3 = \theta_\infty$:

$$y(z) = \frac{1}{1 - \theta_\infty} + y_0' x + O(x^2), \quad x \to 0. \tag{5.6}$$

This solution appears in formula (16) of [25], whose monodromy data $\{p_{jk},\ 1 \leqslant j \neq k \leqslant 3\}$ for the 2×2 Fuchsian system are given in theorem 3 of [25], for θ_1, θ_∞ not integers. It also coincides with (63) of [29], with monodromy data given at page 3258, especially in the footnote 4. Here, we compute the Stokes matrices and apply the formulae of theorem 3.1, so re-obtain and verifying the p_{jk} of [25, 29].

If G_0 diagonalizes Ω_0 in (A.6), system (4.2) is transformed by the gauge $Y = G_0 \widetilde{Y}$ into

$$\frac{d\widetilde{Y}}{dz} = \left[\frac{1}{\theta_2 + \theta_3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \theta_2 & \theta_2 \\ 0 & \theta_3 & \theta_3 \end{pmatrix} - \frac{1}{z}\begin{pmatrix} \theta_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \theta_2 + \theta_3 \end{pmatrix}\right]\widetilde{Y}. \tag{5.7}$$

For a column $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)^{\mathrm{T}}$ of $\widetilde{Y}$, the first and second equations of the system yield

$$\tilde{y}_1(z) = Cz^{-\theta_2}, \quad C \in \mathbb{C}, \qquad \tilde{y}_3 = \frac{\theta_2 + \theta_3}{\theta_2}\frac{d\tilde{y}_2}{dz} - \tilde{y}_2, \tag{5.8}$$

and plugging this expression into the third equation we get the confluent hypergeometric equation (A.8) for $\tilde{y}_2$. Since $\theta_1 - \theta_2 \notin \mathbb{Z}\backslash\{0\}$, there is no partial resonance, so that the formal solution of (5.7) is unique and computed following [9], section 4:

$$\widetilde{Y}_F(z) = \left(1 + O\left(\frac{1}{z}\right)\right) \cdot \begin{pmatrix} \frac{\tilde{k}_2^0}{\tilde{k}_1^0}\frac{2y_0'(\theta_\infty - 1) + \theta_\infty(\theta_1 - 1)}{2\theta_1\theta_\infty} z^{-\theta_1} & \frac{2y_0'(\theta_\infty - 1) + \theta_\infty(\theta_1 - 1)}{2\theta_1\theta_\infty} z^{-\theta_1} & 0 \\ -\frac{2y_0'(\theta_\infty - 1) + \theta_\infty(\theta_1 - 1)}{2\tilde{k}_1^0\sqrt{\theta_\infty}(\theta_1 + \theta_\infty)} z^{-\theta_1} & -\frac{2y_0'(\theta_\infty - 1) - \theta_\infty(\theta_1 + 1)}{2\tilde{k}_2^0\sqrt{\theta_\infty}(\theta_1 + \theta_\infty)} z^{-\theta_1} & \frac{\theta_1}{\theta_1 + \theta_\infty} z^{-\theta_\infty} e^z \\ \frac{2y_0'(\theta_\infty - 1) + \theta_\infty(\theta_1 - 1)}{2\tilde{k}_1^0\sqrt{\theta_\infty}(\theta_1 + \theta_\infty)} z^{-\theta_1} & \frac{2y_0'(\theta_\infty - 1) - \theta_\infty(\theta_1 + 1)}{2\tilde{k}_2^0\sqrt{\theta_\infty}(\theta_1 + \theta_\infty)} z^{-\theta_1} & \frac{\theta_\infty}{\theta_1 + \theta_\infty} z^{-\theta_\infty} e^z \end{pmatrix},$$

where $\tilde{k}_1^0, \tilde{k}_2^0$ are given in (A.7). As in [25], we consider the case θ_1, θ_∞ not integers. For the sake of the computation of the Stokes matrices, it is sufficient to compute the first two rows of the fundamental solutions $\widetilde{Y}_1, \widetilde{Y}_2, \widetilde{Y}_3$ with asymptotics $\widetilde{Y}_F$ in the Stokes sectors $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, respectively. The computation of the first row is immediate, by comparison with the first row of the leading term of $\widetilde{Y}_F$. The second row is obtained by comparison of the second row of the leading term of $\widetilde{Y}_F$ and the leading coefficients of the asymptotics of the confluent hypergeometric functions M and U appearing in the general solution of (A.8):

$$\tilde{y}_2(z; a, b, m, n) = aM(ze^{2\pi i m}; \theta_1, \theta_1 + \theta_\infty) + bU(ze^{2\pi i n}; \theta_1, \theta_1 + \theta_\infty), \quad a, b \in \mathbb{C}, n, m \in \mathbb{Z}.$$

Following the same procedure as in the previous section 5.1, we find that the Stokes matrices are

$$\mathbb{S}_1 = \begin{pmatrix} 1 & 0 & 2i\tilde{k}_1^0 \dfrac{\sin(\pi\theta_1)}{\sqrt{\theta_\infty}} \dfrac{\Gamma(\theta_1)}{\Gamma(\theta_\infty)} \\ 0 & 1 & -2i\tilde{k}_2^0 \dfrac{\sin(\pi\theta_1)}{\sqrt{\theta_\infty}} \dfrac{\Gamma(\theta_1)}{\Gamma(\theta_\infty)} \\ 0 & 0 & 1 \end{pmatrix} \tag{5.9}$$

and

$$\mathbb{S}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{i\pi\, e^{-i\pi(\theta_1-\theta_\infty)}}{\tilde{k}_1^0\sqrt{\theta_\infty}} \dfrac{2y_0'(\theta_\infty-1)+\theta_\infty(\theta_1-1)}{\Gamma(1+\theta_1)\Gamma(1-\theta_\infty)} & \dfrac{i\pi\, e^{-i\pi(\theta_1-\theta_\infty)}}{\tilde{k}_2^0\sqrt{\theta_\infty}} \dfrac{2y_0'(\theta_\infty-1)-\theta_\infty(\theta_1+1)}{\Gamma(1+\theta_1)\Gamma(1-\theta_\infty)} & 1 \end{pmatrix}. \tag{5.10}$$

We apply now formulae (3.8). For $\tau = 0$ and $\eta = 3\pi/2$, the ordering relation is $1 \prec 3$ and $2 \prec 3$, while there is no ordering $1 \leftrightarrow 2$, because $u_1 - u_2 \to 0$. So we have $p_{12} = 2$ and

$$p_{13} = 2\cos\pi(\theta_1-\theta_3) - e^{i\pi(\theta_1-\theta_3)}(\mathbb{S}_1)_{13}(\mathbb{S}_2^{-1})_{31}, \quad 1 \prec 3,$$
$$p_{23} = 2\cos\pi(\theta_2-\theta_3) - e^{i\pi(\theta_2-\theta_3)}(\mathbb{S}_1)_{23}(\mathbb{S}_2^{-1})_{32}, \quad 2 \prec 3.$$

Substituting the entries of the Stokes matrices, we exactly obtain (and confirm!) the known result of theorem 3 of [25] or formulas in the footnote 4 at page 3258 of [29], namely $p_{12} = 2$ and

$$p_{13} = \frac{-4s\sin(\pi\theta_1)\sin(\pi\theta_\infty)}{\theta_1} + 2\cos(\pi(\theta_1+\theta_\infty)),$$
$$p_{23} = \frac{4s\sin(\pi\theta_1)\sin(\pi\theta_\infty)}{\theta_1} + 2\cos(\pi(\theta_1-\theta_\infty)),$$

where the parameter s is equivalent to y_0' through

$$y_0' = \frac{\theta_\infty(2s+\theta_1+1)}{2(\theta_\infty-1)}.$$

We receive the same result also if we choose $\tau = \pi$ and $\eta = \pi/2$. In this case, the ordering relation is $3 \prec 1$ and $3 \prec 2$. Then,

$$p_{13} = 2\cos\pi(\theta_1-\theta_3) - e^{i\pi(\theta_3-\theta_1)}(\mathbb{S}_2)_{31}(\mathbb{S}_3^{-1})_{13} \quad 1 \succ 3,$$

$$p_{23} = 2\cos\pi(\theta_2-\theta_3) - e^{i\pi(\theta_3-\theta_2)}(\mathbb{S}_2)_{32}(\mathbb{S}_3^{-1})_{23} \quad 2 \succ 3.$$

We need in this case the Stokes matrices $\mathbb{S}_2$ and $\mathbb{S}_3$, the latter being obtainable from $\mathbb{S}_1$ by the formulae

$$\mathbb{S}_{2\nu+1} = e^{2\pi i\nu\Theta}\mathbb{S}_1 e^{-2\pi i\nu\Theta}, \quad \mathbb{S}_{2\nu} = e^{2\pi i(\nu-1)\Theta}\mathbb{S}_1 e^{-2\pi i(\nu-1)\Theta}, \quad \nu \in \mathbb{Z},$$

where $\Theta := \mathrm{diag}(\theta_1, \theta_2, \theta_3) \equiv \mathrm{diag}(\theta_1, \theta_1, -\theta_\infty)$.

Remark 5.1. The Stokes matrices (5.9) and (5.10) are the same as those for case (T1) of the table with $N=-1$, $\theta_1=\theta_2$, $\theta_3=-\theta_\infty$, since the corresponding system is equivalent to the one for the transcendents considered in this section via the gauge transformation

$$\widetilde{Y}\mapsto P\widetilde{Y},\quad P=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.$$

6. Monodromy data of 3-dimensional Dubrovin–Frobenius manifolds at a semisimple coalescence point

A semisimple Dubrovin–Frobenius manifold M of dimension n is a complex analytic manifold whose tangent bundle is equipped with a Frobenius algebra structure, semisimple on an open dense subset $M_{ss}\subset M$, and with a z-deformed flat connection [10, 13, 14]. In suitable local coordinates $u=(u^1,\dots,u^n)$, called *canonical*, the zero-curvature condition is equivalent to the Frobenius integrability of an n-dimensional analogue of the Pfaffian system (2.1):

$$\mathrm{d}Y=\omega(z,u)Y,\quad \omega(z,u)=\left(U(u)+\frac{V(u)}{z}\right)\mathrm{d}z+\sum_{k=1}^{n}(zE_k+V_k(u))\,\mathrm{d}u_k,\quad (6.1)$$

where $U(u)=\mathrm{diag}(u^1,\dots,u^n)$, $E_k=\partial U/\partial u_k$, and $V(u)$ is skew-symmetric. $V(u)$ is holomorphically diagonalized on M_{ss} by a matrix $G(u)$, which Dubrovin calls $\Psi(u)$, so that (2.7) in dimension n reads

$$\frac{\partial\Psi}{\partial u_k}=V_k(u)\Psi,\quad k=1,\dots,n.\quad (6.2)$$

$\Psi^{\mathrm{T}}\cdot\Psi$ is constant, satisfying a normalization condition fixed by the (non-positive definite) metric of the manifold.

When some coordinates $u_i-u_j\to 0$ coalesce, this corresponds to a true point of the manifold if and only if $V(u)$ is holomorphic and $\lim_{u_i-u_j\to 0}V_{ij}(u)=0$. Such points are called *semisimple coalescence points* and their theory was established in [10] by a geometric application of [9].

The monodromy data of the z-component of (6.1)

$$\frac{\mathrm{d}Y}{\mathrm{d}z}=\left(U(u)+\frac{V(u)}{z}\right)Y,\quad (6.3)$$

locally parametrize the manifold at semisimple points [13, 14], including the semisimple coalescent ones [10]. This means that given the manifold, so that $V(u)$ is locally given in coordinates, the monodromy data of the z-component are determined. Conversely, given the data, $V(u)$ is obtained at semisimple points by solving a Riemann–Hilbert boundary value problem, and then the manifold structure can be locally constructed from certain fundamental matrix solutions of (6.3). This is also true for data give only at a semisimple coalescence point [12, 50]. The global structure of the manifold is related to the analytic continuation of $V(u)$, and encoded in an explicit action of the braid group on the monodromy data (see [14] and [10] for details).

The matrix $V(u)$ has a precise definition in terms of the metric and an Euler vector field defined on the manifold. Here, it suffices to say that for $n=3$, (6.1) is a special sub-case of (2.1), with

$$\theta_1=\theta_2=\theta_3=0,\qquad \text{eigenvalues of } V=\mu,0,-\mu,\qquad \mu:=\theta_\infty/2\neq 0.\quad (6.4)$$

For $n = 3$, (2.8) implies

$$V(u) = \Omega(x),$$

and the choice of remark 2.3 must be made to obtain $V^{\mathrm{T}} = -V$.

The structure of a 3-dimensional Dubrovin–Frobenius manifold can be written in terms of the transcendent associated with V, PVI having parameters (6.4). More precisely, a branch of the transcendent encodes the structure of a region called *chamber* of the manifold (similar to a local chart: see definition 2.35 in [10]), through the explicit formulae appearing in [23]. The monodromy data of (6.3) associated with the branch, i.e. the data used for the nonlinear connection problem!, are the data which locally parameterize the manifold, as explained above.

Consider a Dubrovin–Frobenius manifold M at a semisimple coalescence point $p \in M$, corresponding to canonical coordinates u^c, such that $u_1^c = u_2^c$. As u varies in a sufficiently small $\mathbb{D}(u^c)$, the corresponding point of M varies in a domain intersecting two chambers of M, the semisimple coalescence point p being on their boundary. Semisimplicity implies $V_{12} \to 0$ as $u_1 - u_2 \to 0$ in $\mathbb{D}(u^c)$. Hence, the monodromy data of both chambers are the data associated with a Painlevé transcendent $y(x)$ such that *the vanishing condition (4.1) holds*. Therefore, it necessarily has Taylor expansion in the class (T3) of our table, that for parameters (6.4) becomes

$$y(x) = y_0' x + y_0'(1 - y_0')(2\mu^2 - 2\mu + 1)x^2 + O(x^3), \qquad y_0' \neq 0, 1. \tag{6.5}$$

The result of this section is the computation of *all* the monodromy data of the chambers at the coalescence point p, namely the data of (6.3) associated with (6.5). By proposition 3.1, they are the data of system (3.10), which can be computed from the restricted system (4.2).

Our computations provide all the cases when the theory of [10] applies in dimension $n = 3$. The results are in propositions 6.1 and 6.2 below.

6.1. The restricted system

The matrix $\Omega(0) = \Omega_0$ in (4.2) is (A.23) of section A.2.3, skew-symmetric only for (up to the sign) $\tilde{k}_1^0 = i\sqrt{y_0'}$ and $\tilde{k}_2^0 = i\sqrt{1 - y_0'}$, namely

$$\Omega_0 = \begin{pmatrix} 0 & 0 & i\mu\sqrt{y_0'} \\ 0 & 0 & i\mu\sqrt{1-y_0'} \\ -i\mu\sqrt{y_0'} & -i\mu\sqrt{1-y_0'} & 0 \end{pmatrix}.$$

The diagonalizing matrix G_0, given in (5.3), with the parameters $\tilde{k}_1^0, \tilde{k}_2^0$ fixed above, must be renormalized by

$$G_0 \longmapsto \Psi_0 := G_0 \cdot \mathrm{diag}(1/\sqrt{2}, \sqrt{y_0'}, 1/\sqrt{2}) = \begin{pmatrix} i\sqrt{y_0'/2} & -\sqrt{1-y_0'} & -i\sqrt{y_0'/2} \\ i\sqrt{(1-y_0')/2} & \sqrt{y_0'} & -i\sqrt{(1-y_0')/2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}. \tag{6.6}$$

In this way

$$\Psi_0^{-1}\Omega_0\Psi_0 = \hat{\mu} = \mathrm{diag}(\mu, 0, -\mu),$$

and the following normalization, prescribed by the metric on the manifold, holds

$$\Psi_0^{\mathrm{T}} \cdot \Psi_0 = \eta, \qquad \eta := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.7}$$

A change of the normalization (6.7) does not affect the computation of Stokes matrices, but changes the central connection matrix $\mathcal{C}^{(0)}$ (defined in (6.14) below). In conclusion, system (5.2) for $\widetilde{Y} = \Psi_0^{-1} Y$ becomes

$$\frac{\mathrm{d}\widetilde{Y}}{\mathrm{d}z} = \left[\mathcal{U}_0 + \frac{\mu}{z}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\right]\widetilde{Y}, \qquad \mathcal{U}_0 := \frac{1}{2}\begin{pmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 1\end{pmatrix}. \tag{6.8}$$

Remark 6.1. (6.6) can be obtained by linear algebra, but must also come from the solution $\Psi(u)$ of (6.2) at $u_1 = u_2$. The general solution when $n = 3$ has structure (2.9)

$$\Psi(u) = \widetilde{\Psi}(x) H^{\widehat{\mu}}, \qquad H^{\widehat{\mu}} = \mathrm{diag}(H^{\mu}, 1, H^{-\mu}), \qquad H := u_3 - u_1.$$

It is given by explicit formulae in [23] in terms of a PVI transcendent $y(x)$. In our case, these formulae yield

$$\Psi(u)\big|_{u_1=u_2} = \widetilde{\Psi}(0) H^{\widehat{\mu}}, \quad \widetilde{\Psi}(0) = \begin{pmatrix} -i\sqrt{y_0'}/2 & -\sqrt{1-y_0'} & -i\sqrt{y_0'}\\ -i\sqrt{1-y_0'}/2 & \sqrt{y_0'} & i\sqrt{1-y_0'}\\ -1/2 & 0 & -1\end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{w_0} & & \\ & 1 & \\ & & \sqrt{w_0}\end{pmatrix},$$

where $w_0 \in \mathbb{C}\backslash\{0\}$. When considering system (3.10), H is replaced by 1. Our matrix Ψ_0 is $\widetilde{\Psi}(0)$ for $\sqrt{w_0} = -1/\sqrt{2}$. It follows that (6.6) can always be substituted by any $\Psi_0' = \Psi_0 \cdot \Delta$, where $\Delta := \mathrm{diag}(1/\sqrt{w_0}, 1, \sqrt{w_0})$. This changes the central connection matrix by $\mathcal{C}^{(0)\prime} = \Delta^{-1}\mathcal{C}^{(0)}$, and corresponds to a rescaling of the flat coordinates of the manifold.

6.2. *Stokes matrices and data* p_{jk}

Proposition 6.1. *The Stokes matrices for the chambers of a semisimple Dubrovin–Frobenius manifold associated with the branch (6.5), with $\mu \neq 0$, are*

$$\mathbb{S}_1 = \begin{pmatrix} 1 & 0 & 2\sqrt{y_0'}\sin(\mu\pi)\\ 0 & 1 & 2\sqrt{1-y_0'}\sin(\mu\pi)\\ 0 & 0 & 1\end{pmatrix}, \quad \mathbb{S}_2 = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -2\sqrt{y_0'}\sin(\mu\pi) & -2\sqrt{1-y_0'}\sin(\mu\pi) & 1\end{pmatrix}, \tag{6.9}$$

and the invariants are

$$p_{12} = 2, \quad p_{13} = 2 - [(\mathbb{S}_1)_{13}]^2 = 2 - 4y_0'\sin^2(\pi\mu), \quad p_{23} = 2 - [(\mathbb{S}_1)_{23}]^2 = 2 - 4(1-y_0')\sin^2(\pi\mu).$$

Proof. If $2\mu \notin \mathbb{Z}$, this is a particular case of the Stokes matrices (5.4) and (5.5), with $\theta_1 = \theta_2 = \theta_3 = 0$, $\theta_\infty = 2\mu$. If $2\mu \in \mathbb{Z}$, we need two facts. The first is that the symmetries of the Pfaffian system imply [13]

$$\mathbb{S}_2 = \mathbb{S}_1^{-\mathrm{T}}. \tag{6.10}$$

The second fact is that for a Dubrovin–Frobenius manifold, in order to compute $\mathbb{S}_1$, it suffices to have $\hat{\mu}$, the nilpotent exponent R (defined below) and the central connection matrix $\mathcal{C}^{(0)}$ (also defined below). Indeed, the following relations hold (see [10, 13])

$$\mathbb{S}_1 = (\mathcal{C}^{(0)})^{-1} e^{-i\pi R} e^{-i\pi\hat{\mu}} \eta^{-1} (\mathcal{C}^{(0)})^{-\mathrm{T}}, \quad \text{or} \quad \mathbb{S}_1^{\mathrm{T}} = (\mathcal{C}^{(0)})^{-1} e^{i\pi R} e^{i\pi\hat{\mu}} \eta^{-1} (\mathcal{C}^{(0)})^{-\mathrm{T}}. \tag{6.11}$$

We will compute R and $\mathcal{C}^{(0)}$ for all values of $\mu \neq 0$ in section 6.3 below. The result, substituted in (6.11), gives (6.9) in all cases. The invariant p_{jk} follow from theorem 3.1. □

The invariants p_{12}, p_{13}, p_{23} in proposition 6.1 coincide with the data obtained by the direct analysis of the 2×2 Fuchsian system, at the bottom of page 1335 of [24] (see the formulae for $\sigma = 0$ there, where there is a misprint, to be corrected by the replacement: $x_1^2 \mapsto x_1, x_\infty^2 \mapsto x_\infty$).

6.2.1. *A remark on the transcendents (6.5) with integer μ.*

If $\mu \in \mathbb{Z} \backslash \{0\}$,

$$\mathbb{S}_1 = \mathbb{S}_2 = I, \quad \Longrightarrow \quad p_{12} = p_{13} = p_{23} = 2.$$

This corresponds to a degenerate case mentioned in section 3, when the triple $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ and the integration constant in the transcendent cannot be parametrized by the p_{jk}. For example, if $\mu = 1$, the series (6.5) is the Taylor expansion at $x = 0$ of the one-parameter family of solutions

$$y(x) = \frac{ax}{1-(1-a)x}, \qquad a = y_0' \in \mathbb{C} \backslash \{0\}, \qquad \mu = 1. \tag{6.12}$$

The Fuchsian system (2.36) has an isomonodromic fundamental solution

$$\widetilde{\Phi}(\lambda, x) = \begin{pmatrix} 1 & L(\lambda, x) \\ r(\lambda, x) & \dfrac{(1+(a-1)x)^2}{k_0} + r(\lambda, x) L(\lambda, x) \end{pmatrix}, \qquad k_0 \neq 0,$$

where

$$L(\lambda, x) = -a \ln(\lambda) + (a-1) \ln(\lambda - x) + \ln(\lambda - 1), \quad r(\lambda, x) = \frac{2((a-1)x+1)\lambda + (1-a)x^2 - 1}{2k_0},$$

with monodromy matrices

$$\mathcal{M}_1 = \begin{pmatrix} 1 & -2i\pi a \\ 0 & 1 \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} 1 & 2i\pi(a-1) \\ 0 & 1 \end{pmatrix}, \quad \mathcal{M}_3 = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix},$$

generating a reducible monodromy group. Notice that a appears in $y(x)$ and the matrices, but not in the traces p_{jk}. For other integer values of μ, we obtain the corresponding solutions with expansion (6.5), applying to (6.12) the symmetry of PVI given in lemma 1.7 of [16], which transforms $y(x)$ for a PVI with given μ, to a solution $\tilde{y}(x)$ of a PVI with $-\mu$ or equivalently $1 + \mu$. In this way, all values $\mu + N$, $N \in \mathbb{Z}$, are obtained. For example, from (6.12), we obtain

$$\tilde{y}(x) = \frac{ax(1+(a-1)x^2)^2}{(1+x(a-1))(1+x(a-1)(ax^3 - x^3 + 4x^2 - 6x + 4))}, \qquad \mu = -1, 2.$$

6.3. *Fundamental solution at z = 0 and central connection matrix*

System (6.8) has a fundamental solution at the Fuchsian singularity $z = 0$ with the Levelt form

$$\widetilde{Y}^{(0)}(z) = \mathcal{G}(z) z^{\hat{\mu}} z^R, \qquad \mathcal{G}(z) = I + \sum_{k=0}^{\infty} G_k z^k \tag{6.13}$$

where the series is convergent and R and the G_k's will be constructed recursively in propositions 6.3 and 6.4. The full monodromy data of the chambers of the Dubrovin–Frobenius manifold associated to the branch (6.5) include the constant **central connection matrix** $\mathcal{C}^{(0)}$, defined by

$$\widetilde{Y}_1(z) = \widetilde{Y}^{(0)}(z) \mathcal{C}^{(0)}, \tag{6.14}$$

where $\widetilde{Y}_1(z)$ is the fundamental solution in $\mathcal{S}_1$, related to the one of section 5.1 by a change of normalization

$$\widetilde{Y}_1(z) = \mathrm{diag}\left(\sqrt{2},\ 1/\sqrt{y_0'},\ \sqrt{2}\right)\cdot \widetilde{Y}_1(z)^{\mathrm{section5.1}},$$

with $\tilde{k}_1^0 = i\sqrt{y_0'}$, $\tilde{k}_2^0 = i\sqrt{1-y_0'}$, corresponding to $\Psi_0 = G_0^{\mathrm{section5.1}}\cdot \mathrm{diag}(1/\sqrt{2},\sqrt{y_0'},1/\sqrt{2})$.

Proposition 6.2. *The central connection matrix defined in (6.14) has the following entries,*

$$\mathcal{C}_{21}^{(0)} = -\sqrt{1-y_0'}, \qquad \mathcal{C}_{22}^{(0)} = \sqrt{y_0'}, \qquad \mathcal{C}_{23}^{(0)} = 0. \tag{6.15}$$

If μ is not a half-integer, the remaining entries are:

$$\mathcal{C}_{11}^{(0)} = -i\frac{\sqrt{2\pi y_0'}}{2^{1+2\mu}}\frac{\sec(\mu\pi)}{\Gamma(1/2+\mu)}, \quad \mathcal{C}_{31}^{(0)} = i(1-2\mu)\frac{\sqrt{2\pi y_0'}}{2^{2(1-\mu)}}\frac{\sec(\mu\pi)}{\Gamma(3/2-\mu)}, \tag{6.16}$$

$$\mathcal{C}_{12}^{(0)} = -i\frac{\sqrt{2\pi(1-y_0')}}{2^{1+2\mu}}\frac{\sec(\mu\pi)}{\Gamma(1/2+\mu)}, \quad \mathcal{C}_{32}^{(0)} = i(1-2\mu)\frac{\sqrt{2\pi(1-y_0')}}{2^{2(1-\mu)}}\frac{\sec(\mu\pi)}{\Gamma(3/2-\mu)}, \tag{6.17}$$

$$\mathcal{C}_{13}^{(0)} = \frac{\sqrt{2\pi}}{2^{1+2\mu}}\frac{\sec(\mu\pi)}{\Gamma(1/2+\mu)}e^{i\mu\pi}, \quad \mathcal{C}_{33}^{(0)} = (1-2\mu)\frac{\sqrt{2\pi}}{2^{2(1-\mu)}}\frac{\sec(\mu\pi)}{\Gamma(3/2-\mu)}e^{-i\mu\pi}. \tag{6.18}$$

For $\mu >$ half-integer, the remaining entries are:

$$\mathcal{C}_{11}^{(0)} = -\frac{e^{i\pi\mu}\sqrt{2y_0'}}{2^{2\mu+1}\sqrt{\pi}(\mu-1/2)!}\left(4\log 2+\psi(\mu+1/2)-\gamma\right), \quad \mathcal{C}_{31}^{(0)} = \frac{e^{i\pi\mu}\sqrt{2y_0'}}{2^{2\mu}R_{13}\sqrt{\pi}(\mu-1/2)!}, \tag{6.19}$$

$$\mathcal{C}_{12}^{(0)} = -\frac{e^{i\pi\mu}\sqrt{2(1-y_0')}}{2^{2\mu+1}\sqrt{\pi}(\mu-1/2)!}\left(4\log 2+\psi(\mu+1/2)-\gamma\right), \quad \mathcal{C}_{32}^{(0)} = \frac{e^{i\pi\mu}\sqrt{2(1-y_0')}}{2^{2\mu}R_{13}\sqrt{\pi}(\mu-1/2)!}, \tag{6.20}$$

$$\mathcal{C}_{13}^{(0)} = \frac{\sqrt{2\pi}}{2^{2\mu}(\mu-1/2)!} + \frac{i\sqrt{2}(4\log 2+\psi(\mu+1/2)-\gamma)}{2^{2\mu+1}\sqrt{\pi}(\mu-1/2)!}, \quad \mathcal{C}_{33}^{(0)} = -\frac{i\sqrt{2}}{2^{2\mu}R_{13}\sqrt{\pi}(\mu-1/2)!}, \tag{6.21}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$.

For $\mu < 0$ half integer, the above results apply exchanging the first and third rows of $\mathcal{C}^{(0)}$, with R_{31} in place of R_{13} and $-\mu$ in place of μ.

The proposition is proved as follows. $\widetilde{Y}_1(z)$ is explicitly obtained from the reduction to the hypergeometric equation (A.24) (with $\theta_3 = 0$, $\theta_\infty = 2\mu$), and its entries are chosen as linear combinations of hypergeometric functions that match with the asymptotics $\widetilde{Y}_F(z)$, as done in section 5.1. The behaviour of $\widetilde{Y}_1(z)$ at $z = 0$ can also be exactly computed form the expansion of the hypergeometric functions at $z = 0$. It must then be compared with (6.13) in order to extract $\mathcal{C}^{(0)}$. Hence, to concretize the computation, we need to write (6.13) in more details.

6.3.1. Computation of Levelt form (6.13). From the general theory of Fuchsian singularities [14, 55], the matrices G_k and R are recursively computed by formal substitution into the system, obtaining the following formulae.

- For $k=1$, if $\mu_i-\mu_j\neq 1$ then

$$\begin{cases} \text{choose } (R_1)_{ij}=0 \\ (G_1)_{ij}=\dfrac{(\mathcal{U}_0)_{ij}}{\mu_j-\mu_i+1}, \end{cases}$$

else if $\mu_i-\mu_j=1$ then

$$\begin{cases} \text{necessarily } (R_1)_{ij}=(\mathcal{U}_0)_{ij} \\ (G_1)_{ij}\text{is arbitrary;} \end{cases}$$

- For $k\geqslant 2$, if $\mu_i-\mu_j\neq k$, then

$$\begin{cases} \text{choose } (R_k)_{ij}=0 \\ (G_k)_{ij}=\dfrac{1}{\mu_j-\mu_i+k}\left(\mathcal{U}_0G_{k-1}-\displaystyle\sum_{p=1}^{k-1}G_pR_{k-p}\right)_{ij}, \end{cases}$$

else if $\mu_i-\mu_j=k$ then

$$\begin{cases} \text{necessarily } (R_k)_{ij}=\left(\mathcal{U}_0G_{k-1}-\sum_{p=1}^{k-1}G_pR_{k-p}\right)_{ij} \\ (G_k)_{ij}\text{is arbitrary;} \end{cases}$$

here $\mu_1=\mu$, $\mu_2=0$, $\mu_3=-\mu$ are the diagonal elements of $\hat{\mu}$. The nilpotent matrix R (which depends on μ) is

$$R=\sum_{k=1}^{\infty}R_k \quad \text{finite sum.}$$

Proposition 6.3. *If system (6.8) is non-resonant (i.e. $2\mu\notin\mathbb{Z}\backslash\{0\}$), then it has a fundamental matrix solution (6.13) with*

$$R=0 \tag{6.22}$$

$$(G_1)_{ij}=\frac{1}{\mu_j-\mu_i+1}(\mathcal{U}_0)_{ij}, \tag{6.23}$$

$$(G_k)_{ij}=\frac{1}{\mu_j-\mu_i+k}\sum_{l_{k-1},\dots,l_1=1}^{3}\frac{(\mathcal{U}_0)_{il_1}\dots(\mathcal{U}_0)_{l_{k-1}j}}{(\mu_j-\mu_{l_1}+k-1)\dots(\mu_j-\mu_{l_{k-1}}+1)},\quad k\geqslant 2, \tag{6.24}$$

where $i,j=1,2,3$ and $\mu_1=\mu$, $\mu_2=0$, $\mu_3=-\mu$.

Proof. From the general theory sketched above we can choose $R=0$ and G_1 is (6.23). To prove formula (6.24) we proceed by induction: with the choice $R=0$, the recursive relations for $(G_k)_{ij}$, $k\geqslant 2$, reduce to

$$(G_k)_{ij} = \frac{1}{\mu_j - \mu_i + k} (\mathcal{U}_0 G_{k-1})_{ij}. \tag{6.25}$$

Then, (6.24) is easily verified for $k = 2$. Let us suppose (6.24) holds for $k-1$, then substituting it into (6.25) we receive (6.24). □

Remark 6.2. Due to the particular form of $\mathcal{U}_0$, the only non zero entries of G_k, $k \geqslant 1$, are those at positions $(1,1)$, $(1,3)$, $(3,1)$ and $(3,3)$.

Consider now the resonant case, $2\mu \in \mathbb{Z} \setminus \{0\}$. For $\mu > 0$, the resonance $\mu_i - \mu_j \in \mathbb{Z} \setminus \{0\}$ occurs only for $i = 1$ and $j = 3$ at step 2μ of the recursive construction if μ is half-integer, and also for $i = 1, j = 2$ and $i = 2, j = 3$ at step μ when μ is an integer. For $\mu < 0$, the above applies with i, j exchanged.

Proposition 6.4. *If system (6.8) with $\mu > 0$ is resonant, then it has a fundamental matrix solution (6.13) with the following properties. For μ half-integer we can choose the matrix* R *in such a way that the only (possibly) non zero entry is*

$$R_{13} = (\mathcal{U}_0)_{13}, \quad \text{for } \mu = \frac{1}{2}, \tag{6.26}$$

$$R_{13} = \sum_{m=1}^{3} \frac{(\mathcal{U}_0)_{1m}}{\mu_3 - \mu_m + 2\mu - 1} \sum_{l_{2\mu-2},\ldots,l_1=1}^{3} \frac{(\mathcal{U}_0)_{ml_1} \ldots (\mathcal{U}_0)_{l_{2\mu-2}3}}{(\mu_3 - \mu_{l_1} + 2\mu - 2) \ldots (\mu_3 - \mu_{l_{2\mu-2}} + 1)}, \quad \mu \geqslant \frac{3}{2}. \tag{6.27}$$

For μ positive integer

$$R = 0.$$

If μ is half-integer, the matrix coefficient $G_{2\mu}$ has entries $(1,1)$, $(3,1)$, $(3,3)$ fully determined and the entry $(1,3)$ is an arbitrary parameter; if μ is an integer the matrix coefficient G_μ *has free parameters at entries $(1,2)$ and $(2,3)$ and another free parameter occurs in $G_{2\mu}$ at position $(1,3)$.*

For $\mu < 0$, the above results apply exchanging the entry $(1,3)$ with the entry $(3,1)$ for μ half-integer or exchanging the entries $(1,2)$, $(2,3)$, $(1,3)$ with the entries $(2,1)$, $(3,2)$, $(3,1)$, respectively, for μ integer.

Proof. From the general theory of Fuchsian singularities, we can choose R such that only R_{13} is possibly non zero. Let us start considering half-integer values of μ. For the case $\mu = 1/2$ we have

$$(R_1)_{ij} = (\mathcal{U}_0)_{ij} \quad \text{and} \quad (G_1)_{ij} \text{ arbitrary}.$$

Let $\mu \geqslant 3/2$, then $(R)_{13} = (\mathcal{U}_0 G_{2\mu-1})_{13}$. Now, for $k = 1, \ldots, 2\mu - 1$, formula (6.24) holds, hence we easily obtain the sought expression of $(R)_{13}$. Again, the fact that $(G_{2\mu})_{13}$ is a free parameter is just a consequence of the general theory. Consider now the case of integer μ. For $k = 1, \ldots, \mu - 1$ formula (6.24) holds and we have $(R_\mu)_{12} = (\mathcal{U}_0 G_{\mu-1})_{12}$, but $(\mathcal{U}_0)_{i2} = (\mathcal{U}_0)_{2j} = 0$, for each $i, j = 1, 2, 3$, hence $(R_\mu)_{12} = 0$. Similarly we get $(R_\mu)_{23} = 0$. From the general theory, $(G_\mu)_{12} = g_1$ and $(G_\mu)_{23} = g_2$ are free parameters. Let us use the inductive definition to compute

$$(G_{\mu+1})_{ij} = \frac{1}{\mu_j - \mu_i + \mu + 1} (\mathcal{U}_0 G_\mu)_{ij},$$

where the product $\mathcal{U}_0 G_\mu$ has structure

$$2\mathcal{U}_0 G_\mu = \begin{pmatrix} B & g_1 & C \\ 0 & 0 & 0 \\ B & g_1 & C \end{pmatrix},$$

with $B = (G_\mu)_{11} + (G_\mu)_{31}$ and $C = (G_\mu)_{13} + (G_\mu)_{33}$. Now, from (6.24) we see that $B = C = 0$, thus $(\mathcal{U}_0 G_k)_{11} = (\mathcal{U}_0 G_k)_{31} = (\mathcal{U}_0 G_k)_{13} = (\mathcal{U}_0 G_k)_{33} = 0$ for all $k \geqslant \mu$ and therefore $(R_{2\mu})_{13} = 0$.

The last statement for $\mu < 0$ is obvious. □

Summing up, the fundamental solution (6.13) has the following structure. For the non-resonant case

$$R = 0, \qquad \mathcal{G}(z) = I + \frac{z}{2}\begin{pmatrix} 1 & 0 & 1/(1-2\mu) \\ 0 & 0 & 0 \\ 1/(1+2\mu) & 0 & 1 \end{pmatrix} + O(z^2). \qquad (6.28)$$

For the resonant case

$$z^R = \begin{pmatrix} 1 & 0 & R_{13}\log(z) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mu \geqslant \frac{1}{2}; \quad z^R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ R_{13}\log(z) & 0 & 1 \end{pmatrix}, \quad \mu \leqslant -\frac{1}{2}.$$

If $\mu \neq \pm 1/2, \pm 1$, the expansion of $\mathcal{G}(z)$ up to the first order is the same as in (6.28), whereas if $\mu = \pm 1/2, \pm 1$ then $\mathcal{G}(z)$ has the form

$$\mathcal{G}(z) = I + \frac{z}{2}\begin{pmatrix} 1 & 0 & g \\ 0 & 0 & 0 \\ 1/2 & 0 & 1 \end{pmatrix} + O(z^2), \quad \mu = \frac{1}{2};$$

$$\mathcal{G}(z) = I + \frac{z}{2}\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 0 & 0 \\ g & 0 & 1 \end{pmatrix} + O(z^2), \quad \mu = -\frac{1}{2},$$

$$\mathcal{G}(z) = I + \frac{z}{2}\begin{pmatrix} 1 & g_1 & -1 \\ 0 & 0 & g_2 \\ 1/3 & 0 & 1 \end{pmatrix} + O(z^2), \quad \mu = 1;$$

$$\mathcal{G}(z) = I + \frac{z}{2}\begin{pmatrix} 1 & 0 & 1/3 \\ g_1 & 0 & 0 \\ -1 & g_2 & 1 \end{pmatrix} + O(z^2), \quad \mu = -1,$$

being $g, g_1, g_2 \in \mathbb{C}$ free parameters.

6.3.2. Computation of the Central connection matrix. The parameters of the confluent hypergeometric functions $U(z;a,b), M(z;a,b)$ solving the hypergeometric equation (A.24) (with $\theta_3 = 0$, $\theta_\infty = 2\mu$) are $a = \mu$ and $b = 2\mu$. According to lemma B.1 of appendix B, we can express $\widetilde{Y}_1(z)$ in terms of the **Hankel functions** $H_\nu^{(1)}(z), H_\nu^{(2)}(z)$, with

$$\nu = \mu - \frac{1}{2}.$$

We notice immediately that the second row of the connection matrix is (6.15). In order to compute the other rows, we have to distinguish between the non-resonant and resonant cases.

Let us start with the non-resonant case. We need the series expansion of the Hankel functions in a neighbourhood of $z = 0$ when $\nu \notin \mathbb{Z}$:

$$H_\nu^{(1)}(z) = \sum_{k=0}^{\infty} \left(a_k^{(1)}(\nu) \left(\frac{z}{2}\right)^{\nu} - b_k^{(1)}(\nu) \left(\frac{z}{2}\right)^{-\nu} \right) z^{2k}, \tag{6.29}$$

where

$$a_k^{(1)}(\nu) = i\csc(\nu\pi) \frac{(-1)^k}{4^k k!} \frac{e^{-i\nu\pi}}{\Gamma(\nu+k+1)}, \quad b_k^{(1)}(\nu) = i\csc(\nu\pi) \frac{(-1)^k}{4^k k!} \frac{1}{\Gamma(-\nu+k+1)},$$

and

$$H_\nu^{(2)}(z) = \sum_{k=0}^{\infty} \left(a_k^{(2)}(\nu) \left(\frac{z}{2}\right)^{-\nu} - b_k^{(2)}(\nu) \left(\frac{z}{2}\right)^{\nu} \right) z^{2k}, \tag{6.30}$$

where

$$a_k^{(2)}(\nu) = i\csc(\nu\pi) \frac{(-1)^k}{4^k k!} \frac{1}{\Gamma(-\nu+k+1)}, \quad b_k^{(2)}(\nu) = i\csc(\nu\pi) \frac{(-1)^k}{4^k k!} \frac{e^{i\nu\pi}}{\Gamma(\nu+k+1)}.$$

Let us consider the entry $(1,1)$ of (6.14):

$$\sqrt{\pi}\sqrt{y_0'}e^{i\nu\pi/2} \left[\frac{i^{\nu} a_0^{(1)}(\nu)}{2^{\nu+\mu+1}} z^{\mu} - \frac{i^{-\nu} b_0^{(1)}(\nu)}{2^{-\nu-\mu+1}} \frac{z}{2} z^{-\mu} \right] (1+O(z))$$
$$= \left(\left(1+\frac{z}{2}\right) z^{\mu} \mathcal{C}_{11}^{(0)} + \frac{z^{-\mu}}{1-2\mu} \frac{z}{2} \mathcal{C}_{31}^{(0)} \right) (1+O(z^2))$$

thus we can read the entries $(1,1)$ and $(3,1)$ of the connection matrix and see they are (6.16). The entries $(1,2)$ and $(3,2)$ have the same form with the substitution $y_0' \to 1-y_0'$, and are (6.17). The computation of the entries $(1,3)$ and $(3,3)$ is carried out with the same procedure using the expansion of $H_\nu^{(2)}(z)$, obtaining (6.18).

Next, we consider the resonant case $2\mu \in \mathbb{Z}\backslash\{0\}$. For integer μ the computations are the same as those of the previous non-resonant case. Through the relations (6.11) with $R = 0$, we obtain the Stokes matrices (6.9), and more specifically $\mathbb{S}_1 = \mathbb{S}_2 = I$. For half-integer values of μ,

$$\nu = \mu - 1/2 = n \in \mathbb{Z},$$

hence the local representations of the Hankel functions in a neighbourhood of $z = 0$ are

$$H_n^{(1)}(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} f_k^{(1)}(z) \left(\frac{z}{2}\right)^{2k} - \frac{i}{\pi} \left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k}, \tag{6.31}$$

where

$$f_k^{(1)}(z) = \frac{(-1)^k}{k!} \left[\frac{1+(2i/\pi)\log(z/2)}{\Gamma(n+k+1)} - \frac{i}{\pi} \frac{\psi(k+1)+\psi(n+k+1)}{(n+k)!} \right],$$

being $\psi(x) = \Gamma'(x)/\Gamma(x)$. For the second Hankel function we have:

$$H_n^{(2)}(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} f_k^{(2)}(z) \left(\frac{z}{2}\right)^{2k} + \frac{i}{\pi}\left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k}, \tag{6.32}$$

where

$$f_k^{(2)}(z) = \frac{(-1)^k}{k!}\left[\frac{1-(2i/\pi)\log(z/2)}{\Gamma(n+k+1)} + \frac{i}{\pi}\frac{\psi(k+1)+\psi(n+k+1)}{(n+k)!}\right].$$

Formulae (6.31) and (6.32) are valid for $n \in \mathbb{N}$. Nevertheless, we can use them also for negative integer n recalling that

$$H_{-n}^{(1)}(z) = (-1)^n H_n^{(1)}(z), \quad H_{-n}^{(2)}(z) = (-1)^n H_n^{(2)}(z), \qquad n \in \mathbb{N}.$$

Suppose that $\mu > 0$. In this case $R_{13} \neq 0$. We start studying the entry $(1,1)$ of (6.14): the left hand side is

$$-\sqrt{\pi}\sqrt{y_0'}\frac{e^{in\pi}}{2^{2\mu+1/2}(\mu-1/2)!}\left(2\log 2 + \frac{i}{\pi}\left(\psi\left(\mu+1/2\right)-\gamma\right)\right)(1+O(z))z^{\mu} +$$

$$+i\sqrt{\pi}\sqrt{y_0'}\frac{e^{in\pi}}{2^{2\mu-1/2}\pi(\mu-1/2)!}(1+O(z))z^{\mu}\log z + O(z^2)(1+O(\log z))z^{\mu} + O(z)z^{-\mu},$$

while the right hand side is

$$(1+O(z))(\mathcal{C}^{(0)})_{11}z^{\mu} + (\mathcal{C}^{(0)})_{31}(R)_{13}z^{\mu}\log z.$$

Equating the two last relations and exploiting the dependence on μ and y_0' we receive (6.19). Similarly, we compute (6.20) and (6.21).

Suppose now that $\mu < 0$. In this case $R_{31} \neq 0$. The gauge $\widetilde{Y} = P\widetilde{Y}'$ brings back to the case $-\mu > 0$, where $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. We have $\widetilde{Y}_1 = P\widetilde{Y}_1'$, $\widetilde{Y}^{(0)} = P\widetilde{Y}'^{(0)}P^{-1}$, so that $\mathcal{C}^{(0)} = P\mathcal{C}'^{(0)}$, with R_{31} in place of R_{13} and $-\mu$ in place of μ.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. The classification

A.1. Transcendents with Taylor expansion $y(x) = y_0 + O(x)$

We study the behaviour of $\Omega(x)$ of theorem 2.1 when $y(x) = y_0 + O(x)$, where $O(x) = \sum_{n\geqslant 1} b_n x^n$ is a Taylor series in a neighbourhood of $x = 0$. The functions k_1, k_2 can be written as

$$k_1(x) = k_1^0(1 + O(x))\frac{\sqrt{y}(x-1)^{(1+\theta_2)/2}}{\sqrt{y-1}x^{(1+\theta_2)/2}}x^{(\theta_1+\theta_3 y_0/(y_0-1))/2},$$

$$k_2(x) = k_2^0(1 + O(x))\frac{\sqrt{y-x}(x-1)^{\theta_2-\theta_3}}{\sqrt{y-1}x^{(1+\theta_1)/2}}x^{(\theta_2+\theta_3 y_0/(y_0-1))/2},\ k_1^0, k_2^0 \in \mathbb{C}\backslash\{0\}.$$

Hence, the off diagonal elements of $\Omega(x)$ have the structure

$$\Omega_{12}(x) = \omega_{12}(x)x^{\theta_1-\theta_2}, \qquad \Omega_{13}(x) = \omega_{13}(x)x^{(\theta_1+\theta_3 y_0/(y_0-1)-\theta_2-1)/2},$$

$$\Omega_{21}(x) = \omega_{21}(x)x^{\theta_2-\theta_1}, \qquad \Omega_{23}(x) = \omega_{23}(x)x^{(\theta_2+\theta_3 y_0/(y_0-1)-\theta_1-1)/2},$$

$$\Omega_{31}(x) = \omega_{31}(x)x^{-(\theta_1+\theta_3 y_0/(y_0-1)-\theta_2-1)/2-1}, \quad \Omega_{32}(x) = \omega_{32}(x)x^{-(\theta_2+\theta_3 y_0/(y_0-1)-\theta_1-1)/2-1},$$

where $\omega_{ij}(x)$ are holomorphic functions at $x = 0$, explicitly computed from the formulae of theorem 2.1.

Remark A.1. From the previous formulas, we see that holomorphicity of $\Omega(x)$ at $x = 0$ requires

$$\theta_1 - \theta_2 \in \mathbb{Z} \quad \text{and} \quad \frac{\theta_3\ y_0}{y_0 - 1} \in \mathbb{Z}.$$

There are *three classes* of solutions $y(x) = y_0 + O(x)$, obtained in [25] and classified in the tables of [29]. The generic one is

$$y(x) = y_0(\theta_\infty, \theta_3) + \sum_{n=1}^{\infty} b_n(\vec{\theta})x^n, \qquad \theta_\infty \neq 1, \quad \vec{\theta} = (\theta_1, \theta_2, \theta_3, \theta_\infty), \tag{A.1}$$

with two possibilities

$$y_0 = \frac{\theta_\infty - 1 + \theta_3}{\theta_\infty - 1}, \quad \theta_\infty + \theta_3 \notin \mathbb{Z}; \qquad \text{or} \qquad y_0 = \frac{\theta_\infty - 1 - \theta_3}{\theta_\infty - 1}, \quad \theta_\infty - \theta_3 \notin \mathbb{Z}.$$

By remark A.1, for this class of solutions $\Omega(x)$ is not holomorphic at $x = 0$. The other two classes consist of the following one-parameter families of solutions: the family

$$y(x) = y_0 + \frac{(1 - y_0)(1 + \theta_1^2 - \theta_2^2)}{2}x + \sum_{n=2}^{\infty} b_n(y_0, \theta_1, \theta_2)x^n, \quad \theta_3 = 0, \quad \theta_\infty = 1, \tag{A.2}$$

where y_0 is a free parameter, and the family

$$y(x) = \sum_{n=0}^{|N|-1} b_n(\vec{\theta})x^n + \frac{y_0^{(|N|)}}{(|N|)!}x^{|N|} + \sum_{n=|N|+1}^{\infty} b_n(y_0^{(|N|)}, \vec{\theta})x^n, \tag{A.3}$$

where $y_0^{(|N|)}$ is a free parameter, $N \in \mathbb{Z} \setminus \{0\}$, the leading order coefficient is $b_0 = N/(\theta_\infty - 1)$, $\theta_\infty \neq 1$, and

$$\theta_\infty - 1 + \theta_3 = N \quad \text{or} \quad \theta_\infty - 1 - \theta_3 = N,$$

with either

$$\theta_\infty - 1 \in \begin{cases} \{-1,-2,\ldots,N+1\} & \text{if } N<0 \\ \{1,2,\ldots,N-1\} & \text{if } N>0 \end{cases} \tag{A.4}$$

or (here, $\mathcal{N}_N$ is (4.3))

$$\{(\theta_1+\theta_2),\, (\theta_1-\theta_2)\} \cap \mathcal{N}_N \neq \emptyset. \tag{A.5}$$

For the family (A.2), a direct inspection of the explicit formulae for the ω_{ij}'s shows that the vanishing conditions (4.1) cannot hold.

A.1.1. Solutions (A.3)—Case $\theta_\infty = \theta_3 + N + 1$. Let us start with the case $\theta_1 = \theta_2$. From the explicit formulae, $\omega_{12}(0) = \omega_{21}(0) = 0$ hold if and only if $N = -1$, that is $\theta_3 = \theta_\infty$. With this condition $\Omega(x)$ has holomorphic limit at $x = 0$:

$$\Omega_0 = \begin{pmatrix} -\theta_2 & 0 & -\tilde{k}_1^0\sqrt{\theta_3} \\ 0 & -\theta_2 & \tilde{k}_2^0\sqrt{\theta_3} \\ \frac{1}{\tilde{k}_1^0\sqrt{\theta_3}}\left(\frac{\theta_3(1-\theta_2)}{2} + (1-\theta_3)y_0'\right) & \frac{1}{\tilde{k}_2^0\sqrt{\theta_3}}\left(\frac{\theta_3(1+\theta_2)}{2} + (1-\theta_3)y_0'\right) & -\theta_3 \end{pmatrix}, \tag{A.6}$$

where

$$\tilde{k}_1^0 = ik_1^0 e^{i\pi\theta_2/2} \quad \tilde{k}_2^0 = -k_2^0 e^{i\pi(\theta_2-\theta_3)}. \tag{A.7}$$

Denoting by G_0 a diagonalizing matrix of Ω_0, with the gauge $Y = G_0\widetilde{Y}$, system (4.2) for a column $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)^{\mathrm{T}}$ of $\widetilde{Y}$ is reduced by elimination to

$$z\frac{\mathrm{d}^2\tilde{y}_2}{\mathrm{d}z^2} + (\theta_2+\theta_3-z)\frac{\mathrm{d}\tilde{y}_2}{\mathrm{d}z} - \theta_2\tilde{y}_2 = 0, \tag{A.8}$$

which is a **confluent hypergeometric equation** with parameters $a = \theta_2$ and $b = \theta_2 + \theta_3$.

If $\theta_1 - \theta_2 \in \mathbb{Z}_{>0}$, then for the requirement $\lim_{x\to 0}\Omega_{21}(x) = 0$ holomorphically to be fulfilled it is necessary that $\omega_{21}(0) = 0$, which is equivalent to $N \leqslant -2$. Furthermore, from the explicit formulae, we have

$$\lim_{x\to 0}\Omega_{23}(x) = \omega_{23}(0) = \tilde{k}_2^0\sqrt{|N|\theta_3}, \quad \text{and} \quad \omega_{32}(0) = 0, \quad \omega_{31}(0) = 0.$$

Now $\Omega_{32} = \omega_{32}/x$, and so

$$\lim_{x\to 0}\Omega_{32}(x) = \frac{d}{dx}\omega_{32}(x)\Big|_{x=0} = \frac{\theta_2}{\tilde{k}_2^0}\sqrt{\frac{\theta_3}{|N|}}.$$

The entries $(2,1)$ and $(3,1)$ of the isomonodromic deformation equations (2.10) give

$$\begin{cases} \frac{\mathrm{d}\Omega_{21}}{\mathrm{d}x} = (\theta_2-\theta_1)\frac{\Omega_{21}}{x} + \frac{\Omega_{31}\Omega_{23}}{x-1} \\ \frac{\mathrm{d}\Omega_{31}}{\mathrm{d}x} = \frac{\Omega_{21}\Omega_{32}}{x(x-1)} \end{cases} \implies \begin{cases} \frac{\mathrm{d}\omega_{21}}{\mathrm{d}x} = \frac{\omega_{31}\omega_{23}}{x(x-1)} \\ \frac{\mathrm{d}\omega_{31}}{\mathrm{d}x} + N\frac{\omega_{31}}{x} = \frac{\omega_{21}\omega_{32}}{x(x-1)}. \end{cases} \tag{A.9}$$

From the second equation we compute $\omega_{31}'(0) = \lim_{x\to 0}\omega_{31}(x)/x$ by taking its limit for $x\to 0$:

$$\begin{aligned}\omega_{31}'(0) - |N|\lim_{x\to 0}\frac{\omega_{31}(x)}{x} &= \lim_{x\to 0}\frac{\omega_{21}(x)\omega_{32}(x)}{x(x-1)}\\ &= \lim_{x\to 0}\frac{\left(\omega_{21}'(0)x+O(x^2)\right)\left(\omega_{32}'(0)x+O(x^2)\right)}{x} = 0,\end{aligned}$$

so that we have (recall that $N\le -2$)

$$\omega_{31}'(0)(1-|N|) = 0 \quad\Longrightarrow\quad \omega_{31}'(0) = 0.$$

From the above and first equation we obtain $\omega_{21}'(0) = 0$, because

$$\begin{aligned}\omega_{21}'(0) &= \lim_{x\to 0}\frac{\omega_{31}(x)\omega_{23}(x)}{x(x-1)} = \lim_{x\to 0}\frac{\left(\omega_{31}'(0)x+O(x^2)\right)\left(\omega_{23}(0)+O(x)\right)}{x(x-1)}\\ &= -\omega_{31}'(0)\,\tilde{k}_2^0\sqrt{|N|\theta_3} = 0.\end{aligned}$$

By differentiating n times equations (A.9), it follows that

$$\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}(\omega_{21}) = \sum_{j=0}^{n}\binom{n}{j}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left(\frac{\omega_{31}}{x}\right)\frac{\mathrm{d}^{n-j}}{\mathrm{d}x^{n-j}}\left(\frac{\omega_{23}(x)}{x-1}\right)$$

and

$$\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}(\omega_{31})\left(1-\frac{|N|}{n+1}\right) = \sum_{j=0}^{n}\binom{n}{j}\frac{\mathrm{d}^j}{\mathrm{d}x^j}(\omega_{21}(x))\frac{\mathrm{d}^{n-j}}{\mathrm{d}x^{n-j}}\left(\frac{\omega_{32}(x)}{x(x-1)}\right).$$

Thus, taking the limit $x\to 0$, we get

$$\omega_{31}^{(n+1)}(0)\left(1-\frac{|N|}{n+1}\right) = \lim_{x\to 0}\sum_{j=0}^{n}\binom{n}{j}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left(\frac{\omega_{21}(x)}{x}\right)\frac{\mathrm{d}^{n-j}}{\mathrm{d}x^{n-j}}\left(\frac{\omega_{32}(x)}{x-1}\right),$$

so that we can repeat the computations till $1-|N|/(n+1) = 0$, that is up to step $n = |N|-2$ included, and show that all the derivatives of $\omega_{21}(x)$ and $\omega_{31}(x)$ at $x = 0$ up to the order $|N|-1$ vanish. In conclusion,

$$\omega_{21}(x) = x^{|N|}\left(\frac{\omega_{21}^{(|N|)}(0)}{(|N|)!}+O(x)\right),\quad \omega_{31}(x) = x^{|N|}\left(\frac{\omega_{31}^{(|N|)}(0)}{(|N|)!}+O(x)\right)$$

and

$$\lim_{x\to 0}\Omega_{21}(x) = 0,\quad \lim_{x\to 0}\Omega_{31}(x) = \sqrt{\frac{\theta_3}{|N|}}\frac{K(\theta_2,\theta_3,N,A)}{\tilde{k}_1^0}\quad \text{holomorphically,}$$

where

$$K(\theta_2,\theta_3,N,A) = \frac{\mathrm{d}^{|N|}}{\mathrm{d}x^{|N|}}\left[\frac{x(x-1)\mathrm{d}y/\mathrm{d}x+\theta_3 y(y-x)}{2(1-y)}\right]\Bigg|_{x=0} + |N|A,\qquad A := \frac{y_0^{(|N|)}}{2}.$$

Here A is the free parameter. The matrix $\Omega(x)$ at the coalescence point $x=0$ is then

$$\Omega_0 = \begin{pmatrix} 1-|N|-\theta_2 & 0 & 0 \\ 0 & -\theta_2 & \tilde{k}_2^0\sqrt{|N|\theta_3} \\ \sqrt{\frac{\theta_3}{|N|}}\frac{K(\theta_2,\theta_3,N,A)}{\tilde{k}_1^0} & \frac{\theta_2}{\tilde{k}_2^0}\sqrt{\frac{\theta_3}{|N|}} & -\theta_3 \end{pmatrix},$$

where $\tilde{k}_1^0, \tilde{k}_2^0$ are as in (A.7). Following the same procedure as in the previous case, with a diagonalizing matrix G_0 for Ω_0, system (4.2) for $\widetilde{Y}=(G_0)^{-1}Y$ is reduced to

$$\tilde{y}_1(z) = Cz^{1-|N|-\theta_2}, \quad C\in\mathbb{C}, \qquad \tilde{y}_3 = \frac{\theta_2+\theta_3}{\tilde{k}_2^0\sqrt{|N|\theta_3}}\frac{\mathrm{d}\tilde{y}_2}{\mathrm{d}z} - \frac{\theta_2}{\tilde{k}_2^0\sqrt{|N|\theta_3}}\tilde{y}_2 - Cz^{1-|N|-\theta_2}$$

and

$$z\frac{\mathrm{d}^2\tilde{y}_2}{\mathrm{d}z^2} + (\theta_2+\theta_3-z)\frac{\mathrm{d}\tilde{y}_2}{\mathrm{d}z} - \theta_2\tilde{y}_2 = Lz^{1-(|N|+\theta_2)}, \quad L := \frac{C\tilde{k}_2^0\sqrt{|N|\theta_3}(1-|N|+\theta_3)}{\theta_2+\theta_3}, \tag{A.10}$$

which is an inhomogeneous **confluent hypergeometric equation** with parameters $a=\theta_2$ and $b=\theta_2+\theta_3$.

If $\theta_1-\theta_2<0$, then for the requirement $\lim_{x\to 0}\Omega_{12}(x)=0$ holomorphically to be fulfilled it is necessary that $\omega_{12}(0)=0$, which is equivalent to $N\leqslant -2$. Proceeding as before, we obtain

$$\Omega_0 = \begin{pmatrix} -\theta_2+|N|-1 & 0 & -\tilde{k}_1^0\sqrt{|N|\theta_3} \\ 0 & -\theta_2 & 0 \\ -\sqrt{\frac{\theta_3}{|N|}}\frac{\theta_2-|N|+1}{\tilde{k}_1^0} & -\sqrt{\frac{\theta_3}{|N|}}\frac{H(\theta_2,\theta_3,N,A)}{\tilde{k}_2^0} & -\theta_3 \end{pmatrix},$$

where $\tilde{k}_1^0, \tilde{k}_2^0$ are as in (A.7) and

$$H(\theta_2,\theta_3,N,A) = \frac{1}{2}\frac{\mathrm{d}^{|N|}}{\mathrm{d}x^{|N|}}\left[\frac{x(x-1)\mathrm{d}y/\mathrm{d}x+\theta_3 y(y-x)}{y-1}\right]\Bigg|_{x=0} - |N|A, \qquad A := \frac{y_0^{(|N|)}}{2}.$$

After a diagonalizing gauge G_0, system (4.2) reduces by elimination to an inhomogeneous **confluent hypergeometric equation** with parameters $a=\theta_2-|N|+1$ and $b=\theta_2+\theta_3-|N|+1$:

$$z\frac{\mathrm{d}^2\tilde{y}_2}{\mathrm{d}z^2} + (\theta_2+\theta_3-|N|+1-z)\frac{\mathrm{d}\tilde{y}_2}{\mathrm{d}z} - (\theta_2-|N|+1)\tilde{y}_2 = Lz^{-\theta_2}, \quad L := \frac{C\tilde{k}_1^0\sqrt{|N|\theta_3}\,(|N|-1-\theta_3)}{\theta_2+\theta_3-|N|+1}. \tag{A.11}$$

A.1.2. Solutions (A.3)—Case $\theta_\infty = -\theta_3+N+1$. For this case, the techniques to find the solutions for which condition (4.1) is satisfied are the same as those used in the previous sections, so we will just report the main results.

For $\theta_1-\theta_2=0$, the matrix $\Omega(x)$ holomorphically has limit Ω_0 as $x\to 0$ with vanishing $\Omega_{12}(0)=\Omega_{21}(0)=0$ if and only if $N=-1$, that is $\theta_\infty=-\theta_3$. We have

$$\Omega_0=\begin{pmatrix} -\theta_2 & 0 & \dfrac{\tilde{k}_1^0}{\sqrt{\theta_3}}\left(\dfrac{\theta_3(1+\theta_2)}{2}+(1-\theta_3)y_0'\right) \\ 0 & -\theta_2 & \dfrac{\tilde{k}_2^0}{\sqrt{\theta_3}}\left(\dfrac{\theta_3(1-\theta_2)}{2}+(1-\theta_3)y_0'\right) \\ \dfrac{\sqrt{\theta_3}}{\tilde{k}_1^0} & -\dfrac{\sqrt{\theta_3}}{\tilde{k}_2^0} & -\theta_3 \end{pmatrix}, \qquad \begin{array}{l}\tilde{k}_1^0=k_1^0e^{i\pi\theta_2/2},\\ \tilde{k}_2^0=-ik_2^0e^{i\pi(\theta_2-\theta_3)}\end{array} \tag{A.12}$$

and system (4.2) is again integrable in terms of **confluent hypergeometric** functions (A.8) with parameters $a=\theta_2$ and $b=\theta_2+\theta_3$.

If $\theta_1-\theta_2>0$, condition (4.1) holds if and only if $N\leqslant -2$, and the matrix $\Omega(x)$ has holomorphic limit

$$\Omega_0=\begin{pmatrix} 1-|N|-\theta_2 & 0 & -\tilde{k}_1^0\sqrt{\dfrac{\theta_3}{|N|}}(\theta_2+|N|-1) \\ 0 & -\theta_2 & -\tilde{k}_2^0\sqrt{\dfrac{|N|}{\theta_3}}\widetilde{H}(\theta_2,\theta_3,N,A) \\ -\dfrac{\sqrt{|N|\theta_3}}{\tilde{k}_1^0} & 0 & -\theta_3 \end{pmatrix},$$

where

$$\widetilde{H}(\theta_2,\theta_3,N,A)=\frac{\mathrm{d}^{|N|}}{\mathrm{d}x^{|N|}}\left[\frac{x(x-1)\frac{\mathrm{d}y}{\mathrm{d}x}-(1-\theta_2)y(y-1)}{2(y-x)}\right]\Bigg|_{x=0}-(\theta_2+\theta_3+|N|-1)A,\quad A=\frac{y_0^{(|N|)}}{2},$$

and $\tilde{k}_1^0,\tilde{k}_2^0$ are as in (A.12). System (4.2) is again reduced to a **confluent hypergeometric** equation with parameters $a=\theta_2+|N|-1$ and $b=\theta_2+\theta_3+|N|-1$:

$$z\frac{\mathrm{d}^2\tilde{y}_2}{\mathrm{d}z^2}+(\theta_2+\theta_3+|N|-1-z)\frac{\mathrm{d}\tilde{y}_2}{\mathrm{d}z}-(\theta_2+|N|-1)\tilde{y}_2=0. \tag{A.13}$$

If $\theta_1-\theta_2<0$, condition (4.1) holds if and only if $N\leqslant -2$ and we have

$$\Omega_0=\begin{pmatrix} -\theta_2+|N|-1 & 0 & -\tilde{k}_1^0\sqrt{\dfrac{|N|}{\theta_3}}\widetilde{K}(\theta_2,\theta_3,N,A) \\ 0 & -\theta_2 & \tilde{k}_2^0\theta_2\sqrt{\dfrac{\theta_3}{|N|}} \\ 0 & \dfrac{\sqrt{|N|\theta_3}}{\tilde{k}_2^0} & -\theta_3 \end{pmatrix},$$

where

$$\widetilde{K}(\theta_2,\theta_3,N,A)=\frac{\mathrm{d}^{|N|}}{\mathrm{d}x^{|N|}}\left[\frac{x((x-1)\mathrm{d}y/\mathrm{d}x+\theta_2-|N|+1)}{2y}\right]\Bigg|_{x=0}-(\theta_3+|N|)A,\quad A=\frac{y_0^{(|N|)}}{2},$$

and $\tilde{k}_1^0, \tilde{k}_2^0$ as in (A.12). System (4.2) reduces to a **confluent hypergeometric equation** with parameters $a = \theta_2$ and $b = \theta_2 + \theta_3$:

$$\tilde{z}\frac{\mathrm{d}^2\tilde{y}_2}{\mathrm{d}\tilde{z}^2} + (\theta_2 + \theta_3 - \tilde{z})\frac{\mathrm{d}\tilde{y}_2}{\mathrm{d}\tilde{z}} - \theta_2\tilde{y}_2 = 0. \tag{A.14}$$

A.2. Transcendents with Taylor expansion $y(x) = y_0'x + O(x^2)$

There are three classes of solutions. The generic one is

$$y(x) = y_0'(\theta_1, \theta_2)x + \sum_{n=2}^{\infty} b_n(\vec{\theta})x^n, \quad \vec{\theta} = (\theta_1, \theta_2, \theta_3, \theta_\infty), \quad \theta_1, \theta_2 \neq 0, \tag{A.15}$$

where

$$y_0' = \frac{\theta_1}{\theta_1 - \theta_2}, \quad \theta_1 - \theta_2 \notin \mathbb{Z}; \qquad \text{or} \qquad y_0' = \frac{\theta_1}{\theta_1 + \theta_2}, \quad \theta_1 + \theta_2 \notin \mathbb{Z}.$$

Within this class, if $\theta_1 = 0$ or $\theta_2 = 0$, all the coefficients $b_n(\vec{\theta}) = 0$ and this gives respectively the 'singular solutions'

$$y(x) \equiv 0, \quad \text{or} \quad y(x) \equiv 1.$$

The other two classes are one-parameter families of solutions. One family is

$$y(x) = y_0'x + y_0'(y_0' - 1)\frac{\theta_3^2 - (\theta_\infty - 1)^2 - 1}{2}x^2 + \sum_{n=3}^{\infty} b_n(y_0', \theta_3, \theta_\infty)x^n, \quad \theta_1 = \theta_2 = 0, \tag{A.16}$$

where y_0' is a free parameter. The expression (A.16) for $y_0' = 0$ or $y_0' = 1$ reduces to the singular solution

$$y(x) \equiv 0 \quad \text{or} \quad y(x) \equiv 1 \quad \text{respectively.}$$

The other family is

$$y(x) = \sum_{n=1}^{|N|} b_n(\vec{\theta})x^n + \frac{y_0^{(|N|+1)}}{(|N|+1)!}x^{|N|+1} + \sum_{n=|N|+2}^{\infty} b_n(y_0^{(|N|+1)}, \vec{\theta})x^n, \quad b_1 = \frac{\theta_1}{N}, \tag{A.17}$$

where $y_0^{(|N|+1)}$ is a free parameter, $N \in \mathbb{Z} \setminus \{0\}$, the relation

$$\theta_1 - \theta_2 = N \quad \text{or} \quad \theta_1 + \theta_2 = N,$$

holds, and either

$$\theta_1 \in \begin{cases} \{0, -1, -2, \ldots, N\}, & \text{if } N < 0 \\ \{0, 1, 2, \ldots, N\}, & \text{if } N > 0 \end{cases} \tag{A.18}$$

or ($\mathcal{N}_N$ is the set (4.3))

$$\{(\theta_3 + \theta_\infty - 1), (-\theta_3 + \theta_\infty - 1)\} \cap \mathcal{N}_N \neq \emptyset. \tag{A.19}$$

If $y_0' = \theta_1 = 0$, (A.17) becomes

$$y(x) = Ax^{|N|+1} + \sum_{n=|N|+2}^{\infty} b_n(A, \theta_3, \theta_\infty)x^n, \qquad A := \frac{y_0^{(|N|+1)}}{(|N|+1)!} \tag{A.20}$$

Here, A is the free parameter. If $y_0' = 1$ (and $\theta_1 = N$), it becomes

$$y(x) = x + Ax^{|N|+1} + \sum_{n=|N|+2}^{\infty} b_n(A,\theta_3,\theta_\infty)x^n. \tag{A.21}$$

As seen above, $y_0' \neq 0,1$ for (A.15) and (A.16) and the functions k_1, k_2 can be written in a neighbourhood of $x = 0$ as

$$k_1(x) = k_1^0(1+O(x))\sqrt{-y_0'+O(x)}(x-1)^{(1+\theta_2)/2}x^{(\theta_1(y_0'-1)/y_0'-\theta_2)/2},$$
$$k_2(x) = k_2^0(1+O(x))\sqrt{1-y_0'+O(x)}(x-1)^{\theta_2-\theta_3}x^{(\theta_2 y_0'/(y_0'-1)-\theta_1)/2}, \quad k_1^0, k_2^0 \in \mathbb{C}\backslash\{0\}.$$

Hence, the structure of the off-diagonal elements of the matrix $\Omega(x)$ is the following:

$$\Omega_{12}(x) = \omega_{12}(x)x^{(2y_0'-1)(\theta_1/y_0'-\theta_2/(y_0'-1))}, \qquad \Omega_{13}(x) = \omega_{13}(x)x^{(\theta_1(y_0'-1)/y_0'-\theta_2)/2},$$
$$\Omega_{21}(x) = \omega_{21}(x)x^{-(2y_0'-1)(\theta_1/y_0'-\theta_2/(y_0'-1))}, \quad \Omega_{23}(x) = \omega_{23}(x)x^{(\theta_2 y_0'/(y_0'-1)-\theta_1)/2},$$
$$\Omega_{31}(x) = \omega_{31}(x)x^{-(\theta_1(y_0'-1)/y_0'-\theta_2)/2}, \qquad \Omega_{32}(x) = \omega_{32}(x)x^{-(\theta_2 y_0'/(y_0'-1)-\theta_1)/2},$$

where $\omega_{ij}(x)$ are holomorphic functions at $x = 0$.

A.2.1. Generic solution (A.15)—Case $y_0' = \theta_1/(\theta_1-\theta_2)$. The matrix $\Omega(x)$ is holomorphic at $x = 0$ with $\Omega_1(0) = \Omega_{21}(0) = 0$, and we have

$$\Omega_0 = \begin{pmatrix} -\theta_1 & 0 & \frac{\tilde{k}_1^0}{2}(\theta_1-\theta_2-\theta_3-\theta_\infty) \\ 0 & -\theta_2 & -\frac{\tilde{k}_2^0}{2}(\theta_1-\theta_2+\theta_3+\theta_\infty) \\ -\frac{1}{2\tilde{k}_1^0}(\theta_1-\theta_2-\theta_3+\theta_\infty)\frac{\theta_1}{\theta_1-\theta_2} & -\frac{1}{2\tilde{k}_2^0}(\theta_1-\theta_2+\theta_3-\theta_\infty)\frac{\theta_2}{\theta_1-\theta_2} & -\theta_3 \end{pmatrix},$$

where

$$\tilde{k}_1^0 = k_1^0\sqrt{\frac{\theta_1}{\theta_1-\theta_2}}e^{i\pi\theta_2/2} \quad \text{and} \quad \tilde{k}_2^0 = ik_2^0\sqrt{\frac{\theta_2}{\theta_1-\theta_2}}e^{i\pi(\theta_2-\theta_3)}.$$

System (4.2) can be reduced to the **generalized hypergeometric equation**

$$z^2\frac{d^3w}{dz^3} + z(b_2+a_2z)\frac{d^2w}{dz^2} + (b_1+a_1z)\frac{dw}{dz} + a_0w = 0, \tag{A.22}$$

where $w = z^{(\theta_1+\theta_2+\theta_3-\theta_\infty)/2}\tilde{y}_1$ and the parameters are

$$a_0 = \frac{1}{4}(\theta_1+\theta_2+\theta_3-\theta_\infty)(\theta_1+\theta_2-\theta_3+\theta_\infty) - \theta_1\theta_2, \quad a_1 = \theta_3-\theta_\infty-1, \quad a_2 = -1,$$

$$b_1 = -\frac{\theta_\infty}{2}(\theta_1+\theta_2+\theta_3-\theta_\infty), \quad b_2 = \frac{2+3\theta_\infty-\theta_1-\theta_2-\theta_3}{2}.$$

A.2.2. Generic solution (A.15)—Case $y_0' = \theta_1/(\theta_1+\theta_2)$. For the solution (A.15) with $y_0' = \theta_1/(\theta_1+\theta_2)$ the condition (4.1) does not hold. Indeed, $\Omega(x)$ is holomorphic if and only if $\theta_1 = \theta_2 = 0$, which is not admissible.

A.2.3. Solutions (A.16). In this case $\Omega(x)$ is holomorphic at $x=0$ with $\Omega_{12}(0)=\Omega_{21}(0)=0$, we have

$$\Omega_0=\begin{pmatrix} 0 & 0 & \frac{\tilde{k}_1^0}{2}(\theta_3+\theta_\infty) \\ 0 & 0 & \frac{\tilde{k}_2^0}{2}(\theta_3+\theta_\infty) \\ \frac{1}{2\tilde{k}_1^0}(\theta_\infty-\theta_3)y_0' & \frac{1}{2\tilde{k}_2^0}(\theta_\infty-\theta_3)(1-y_0') & -\theta_3 \end{pmatrix}, \qquad \begin{array}{c} \tilde{k}_1^0=k_1^0\sqrt{y_0'}, \\ \tilde{k}_2^0=-k_2^0\sqrt{1-y_0'}e^{-i\pi\theta_3}, \end{array} \tag{A.23}$$

and system (4.2) reduces to a **confluent hypergeometric equation** with parameters $a=(\theta_\infty-\theta_3)/2$ and $b=\theta_\infty$:

$$z\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+(\theta_\infty-z)\frac{\mathrm{d}w}{\mathrm{d}z}-\frac{\theta_\infty-\theta_3}{2}w=0. \tag{A.24}$$

A.2.4. Solution (A.17)—Case $\theta_1-\theta_2=N$ with condition (A.18). Let $\theta_1=k$ integer, in the set (A.18), so that $\theta_2=k-N$. We can divide the problem in three cases.

(i) If $k\neq 0,N$, so that $y_0'\neq 0,1$, then the matrix $\Omega(x)$ is holomorphic at $x=0$, $\Omega_{12}(0)=\Omega_{21}(0)=0$, and

$$\Omega_0=\begin{pmatrix} -k & 0 & \frac{\tilde{k}_1^0}{2}(\theta_3+\theta_\infty-N) \\ 0 & N-k & \frac{\tilde{k}_2^0}{2}(\theta_3+\theta_\infty+N) \\ \frac{1}{2\tilde{k}_1^0}\frac{k}{N}(\theta_\infty-\theta_3+N) & \frac{1}{2\tilde{k}_2^0}\frac{N-k}{N}(\theta_\infty-\theta_3-N) & -\theta_3 \end{pmatrix}, \qquad \begin{array}{c} \tilde{k}_1^0=k_1^0\sqrt{\frac{k}{N}}e^{i\pi\theta_2/2}, \\ \tilde{k}_2^0=-k_2^0\sqrt{\frac{N-k}{N}}e^{i\pi(\theta_2-\theta_3)}. \end{array}$$

System (4.2) reduces to the **generalized hypergeometric equation**

$$z^2\frac{\mathrm{d}^3w}{\mathrm{d}z}+z(a_2+b_2z)\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+(a_1+b_1z)\frac{\mathrm{d}w}{\mathrm{d}z}+a_0w=0 \tag{A.25}$$

with parameters

$$a_0=\frac{1}{2}(N^2+2k(-2+k-N+\theta_3-\theta_\infty)+N(2-\theta_3+\theta_\infty)),$$

$$a_1=\frac{1}{2}(-4-N+2k+3\theta_3-3\theta_\infty),\quad a_2=-1,$$

$$b_1=\frac{1}{2}(2+\theta_\infty)(2+N-2k-\theta_3+\theta_\infty),\quad b_2=\frac{1}{2}(8+N-2k-\theta_3+3\theta_\infty).$$

(ii) If $\theta_1\equiv k=0$, that is $\theta_1=0$ and $\theta_2=-N$, then the transcendent is (A.20) and

$$k_1(x)=k_1^0(1+O(x))x^{(N+|N|)/2}(x-1)^{(1-N)/2},\quad k_2(x)=k_2^0(1+O(x))(x-1)^{-N-\theta_3},\quad k_1^0,k_2^0\in\mathbb{C}.$$

We need to distinguish the cases with negative and positive N. First, let $N \leq -1$, then $\Omega(x)$ is holomorphic at $x=0$ and

$$\Omega_0 = \begin{pmatrix} 0 & 0 & -\frac{\tilde{k}_1^0}{2}(\theta_3+\theta_\infty+|N|) \\ 0 & -|N| & -\frac{\tilde{k}_2^0}{2}(\theta_3+\theta_\infty-|N|) \\ 0 & \frac{1}{2\tilde{k}_2^0}(\theta_3-\theta_\infty-|N|) & -\theta_3 \end{pmatrix}, \qquad \tilde{k}_1^0 = ik_1^0 e^{i\pi|N|/2} \quad \tilde{k}_2^0 = k_2^0 e^{i\pi(|N|-\theta_3)}, \tag{A.26}$$

after the diagonalizing gauge, system (4.2) is reduced to the **confluent hypergeometric equation** with parameters $a=(\theta_\infty-\theta_3+|N|)/2$ and $b=\theta_\infty$ (here $\tilde{y}_1 = wz^{(\theta_\infty-\theta_3-|N|)/2}$)

$$z\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + (\theta_\infty - z)\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{\theta_\infty-\theta_3+|N|}{2}w = 0. \tag{A.27}$$

If $N \geqslant 1$, then

$$\Omega_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & |N| & -\frac{\tilde{k}_2^0(\theta_3+\theta_\infty+|N|)}{2} \\ \frac{A(\theta_3-\theta_\infty-|N|)}{2\tilde{k}_1^0} & \frac{\theta_3-\theta_\infty+|N|}{2\tilde{k}_2^0} & -\theta_3 \end{pmatrix}, \qquad \begin{aligned} \tilde{k}_1^0 &= ik_1^0 e^{-i\pi|N|/2}, \\ \tilde{k}_2^0 &= k_2^0 e^{i\pi(|N|+\theta_3)}. \end{aligned} \tag{A.28}$$

System (4.2) reduces to the inhomogeneous **confluent hypergeometric equation** with parameters $a=(\theta_\infty-\theta_3+|N|)/2$ and $b=\theta_\infty$:

$$z\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + (\theta_\infty - z)\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{\theta_\infty-\theta_3-|N|}{2}w + C\frac{(\theta_3+\theta_\infty-|N|)(\theta_3-\theta_\infty+|N|)}{2} = 0, \tag{A.29}$$

with $\tilde{y}_1 = wz^{(\theta_\infty-\theta_3+|N|)/2}$ and $C=\tilde{y}_2$ is constant.

(iii) If $\theta_1 \equiv k = N (\neq 0)$, then the transcendent is (A.21), and

$$k_1(x) = k_1^0(1+O(x))\sqrt{x-1}, \qquad k_2(x) = k_2^0(1+O(x))(x-1)^{-\theta_3}x^{(|N|-N)/2}.$$

We have to distinguish between positive and negative N.

If $N \geqslant 1$, then

$$\Omega_0 = \begin{pmatrix} -|N| & 0 & -\frac{\tilde{k}_1^0(\theta_3+\theta_\infty-|N|)}{2} \\ 0 & 0 & -\frac{\tilde{k}_2^0(\theta_3+\theta_\infty+|N|)}{2} \\ \frac{\theta_3-\theta_\infty-|N|}{2\tilde{k}_1^0} & 0 & -\theta_3 \end{pmatrix}, \qquad \tilde{k}_1^0 = ik_1^0, \quad \tilde{k}_2^0 = k_2^0 e^{-i\pi\theta_3}. \tag{A.30}$$

We do gauge transformation $\widehat{Y} = PY$ by the permutation matrix P below

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad P^{-1}U_0P = U_0,$$

$$P^{-1}\Omega_0 P = \begin{pmatrix} 0 & 0 & -\dfrac{\tilde{k}_2^0(\theta_3+\theta_\infty+|N|)}{2} \\ 0 & -|N| & -\dfrac{\tilde{k}_1^0(\theta_3+\theta_\infty-|N|)}{2} \\ 0 & \dfrac{\theta_3-\theta_\infty-|N|}{2\tilde{k}_1^0} & -\theta_3 \end{pmatrix}.$$

This transforms system (4.2) with Ω_0 in (A.30) into a system with the same structure as that for the previous case $k = 0, N \leqslant -1$, with the replacements $\tilde{k}_1^0 \to \tilde{k}_2^0$ and $\tilde{k}_2^0 \to \tilde{k}_1^0$ (see (A.26)). We conclude that also on this case the computation of the fundamental matrix solution is reduced to the **confluent hypergeometric equation** (A.27) with parameters $a = (\theta_\infty - \theta_3 + |N|)/2$ and $b = \theta_\infty$.

If $N \le -1$, then

$$\Omega_0 = \begin{pmatrix} |N| & 0 & -\dfrac{\tilde{k}_1^0}{2}(\theta_3+\theta_\infty+|N|) \\ 0 & 0 & 0 \\ \dfrac{1}{2\tilde{k}_1^0}(\theta_3-\theta_\infty+|N|) & -\dfrac{A}{2\tilde{k}_2^0}(\theta_3-\theta_\infty-|N|) & -\theta_3 \end{pmatrix},$$

where $\tilde{k}_1^0, \tilde{k}_2^0$ are as in (A.30). Again, if P is the same permutation matrix as before, the gauge $\widehat{Y} = PY$ leads to a system with the same form (A.28) of the previous case with $k = 0, N \geqslant 1$, exchanging names of $\tilde{k}_1^0$ and $\tilde{k}_2^0$. The computation of the fundamental matrix solution is then reduced to the same inhomogeneous **confluent hypergeometric equation** (A.29).

A.2.5. Solutions (A.17)—Case $\theta_1 - \theta_2 = N$ with condition (A.19). The parameters θ_3, θ_∞ satisfy condition (A.19): $\{(\theta_3+\theta_\infty-1), (-\theta_3+\theta_\infty-1)\} \cap \mathcal{N}_N \neq \emptyset$. We have $\Omega_{ij}(x) = \omega_{ij}(x)$, $i,j = 1,2,3, i \neq j$, thus the matrix $\Omega(x)$ is holomorphic at $x = 0$, $\omega_{12}(0) = \omega_{21}(0) = 0$ and

$$\Omega_0 = \begin{pmatrix} -(N+\theta_2) & 0 & \dfrac{\tilde{k}_1^0}{2}(\theta_3+\theta_\infty-N) \\ 0 & -\theta_2 & -\dfrac{\tilde{k}_2^0}{2}(\theta_3+\theta_\infty+N) \\ \dfrac{1}{2\tilde{k}_1^0}\dfrac{N+\theta_2}{N}(\theta_\infty-\theta_3+N) & \dfrac{1}{2\tilde{k}_2^0}\dfrac{\theta_2}{N}(\theta_\infty-\theta_3-N) & -\theta_3 \end{pmatrix}, \quad \begin{aligned} \tilde{k}_1^0 &= k_1^0\sqrt{\tfrac{N+\theta_2}{N}}e^{i\pi\theta_2/2}, \\ \tilde{k}_2^0 &= ik_2^0\sqrt{\tfrac{\theta_2}{N}}e^{i\pi(\theta_2-\theta_3)}. \end{aligned}$$

System (4.2) is integrable in terms of solutions of the **generalized hypergeometric equation**

$$z^2\frac{d^3w}{dz} + z(a_2+b_2z)\frac{d^2w}{dz^2} + (a_1+b_1z)\frac{dw}{dz} + a_0w = 0 \tag{A.31}$$

(here $\tilde{y}_3 = wz^{1-l}$, $l = \theta_2 + \frac{\theta_3-\theta_\infty+N}{2}$) with parameters

$$a_0 = \frac{1}{2}(N^2 + 2\theta_2(-2+\theta_2+\theta_3-\theta_\infty) + N(-2+2\theta_2+\theta_3-\theta_\infty)),$$

$$a_1 = \frac{1}{2}(-4+N+2\theta_2+3\theta_3-3\theta_\infty),$$

$$a_2 = -1, \quad b_1 = \frac{1}{2}(2+\theta_\infty)(2-N-2\theta_2-\theta_3+\theta_\infty), \quad b_2 = \frac{1}{2}(8-N-2\theta_2-\theta_3+3\theta_\infty).$$

A.2.6. Solution (A.17)—Case $\theta_1+\theta_2 = N$. Let us start assuming condition (A.19) for θ_3, θ_∞ holds. For this family of solutions, (4.1) does not hold. Indeed, the entries $(1,2)$ and $(2,1)$ of $\Omega(x)$ are

$$\Omega_{12}(x) = \omega_{12}(x)x^{2(N-2\theta_2)} \quad \text{and} \quad \Omega_{21}(x) = \omega_{21}(x)x^{-2(N-2\theta_2)}.$$

If $\mathrm{Re}(\theta_2) = N/2$, $N \in \mathbb{Z}\backslash\{0\}$, then the conditions $\omega_{12}(0) = \omega_{21}(0) = 0$ imply $N = 0$ and $\theta_2 = 0$, a contradiction. If $\mathrm{Re}(\theta_2) < N/2$, then condition $\omega_{21}(0) = 0$ is equivalent to $\theta_2 = N$, an absurd. If $\mathrm{Re}(\theta_2) > N/2$, then $\omega_{12}(0)$ imply $\theta_2 = 0$, which is again a contradiction.

If condition (A.18) hold, by the same arguments, $\Omega(x)$ is not holomorphic at $x = 0$ for $k \neq 0, N$, while for $k = 0$ or $k = N$ the problem can be traced back to the previous section.

A.3. A remark on the analytic τ-function of PVI

Some of the branches of PVI-transcendents analytic at $x = 0$, behaving as

$$y(x) = y_0'x + O(x^2),$$

may possibly be obtained from the τ-function with hypergeometric kernel appearing in section 5.1 of [20], in cases when $\theta_2 = 0$ and θ_1 is integer. The parameters θ's in [20] are related to ours by $\theta_1 = 2\theta_0^{[20]}$, $\theta_2 = 2\theta_t^{[20]}$, $\theta_3 = 2\theta_1^{[20]}$, $\theta_\infty = 2\theta_\infty^{[20]} + 2$ (up to $\theta_j \mapsto -\theta_j$, $j = 1,2,3$, and $\theta_\infty \mapsto 2-\theta_\infty$, $\theta_\infty^{[20]} \mapsto -1-\theta_\infty^{[20]}$). Also, our $\alpha,\beta,\gamma,\delta$ are equal to twice the same symbol of [20]. Keeping this into account, the τ-function of section 5.1 of [20] requires

$$\theta_2 = 0, \quad (\theta_1,\theta_3,\theta_\infty) = (\nu+\nu'+\eta+\eta',\ \nu-\nu',\ \eta-\eta'+2), \qquad \nu,\nu',\eta,\eta' \in \mathbb{C},$$

(the $+2$ in $\eta-\eta'+2$ in unessential, since θ_∞ appear in expressions like $\cos\pi\theta_\infty$).
If $\theta_1 = \nu+\nu'+\eta+\eta' \in \mathbb{N}$, we expect the τ-function (5.6) of [20] to be analytic at $x = 0$ (x is denoted by t in [20]), and consequently formulae (2.22) and (2.27) of [20] give

$$y(x) = \frac{(\theta_\infty-2)^2+\theta_1^2-\theta_3^2-\kappa_G}{(\theta_\infty-2)^2+\theta_1^2-\theta_3^2}\, x \;+\; O(x^2).$$

Here κ_G is the integration constant of [20]. For example, if

$$\theta_1 = \nu+\nu'+\eta+\eta' = 0,$$

we should receive our class of Taylor series (T3). For it, both in our paper and in [20], the monodromy data are

$$p_1 = p_2 = p_{12} = 2, \qquad p_{13}+p_{23} = 2\big(\cos\pi\theta_3+\cos\pi\theta_\infty\big),$$

while

$$p_{23} = \begin{cases} 2\big[(1-y_0')\cos\pi\theta_\infty + y_0'\cos\pi\theta_3\big], & \text{from our computation in section 5.1;} \\ 2\cos\pi(\nu+\nu'), & \text{from [20].} \end{cases}$$

Our free parameter is y_0', while in [20] it can be taken to be κ_G or any of the ν, ν', η, η', for example $\nu+\nu'$. Identifying the two expressions of p_{23}, we receive

$$y_0' = \frac{\cos\pi\theta_\infty - \cos\pi(\nu+\nu')}{\cos\pi\theta_\infty - \cos\pi\theta_3}.$$

PVI τ-functions analytic at a critical point may be further investigated, possibly providing an alternative classification of the analytic branches of transcendents.

Appendix B

B.1. Proof of proposition 3.1

In (2.4), consider the gauge and change of variable

$$Y = (u_3-u_1)^{\Theta} e^{zu_1}\widehat{Y}, \qquad z = \frac{\widehat{z}}{u_3-u_1}, \tag{B.1}$$

Here $(u_3-u_1)^{\Theta}e^{zu_1}$ is analytic if $u_3-u_1\neq 0$ and u varies in a sufficiently small polydisc such that u_3-u_1 does not make a loop around zero (this occurs in both cases of $\mathbb{D}(u^0)$ and $\mathbb{D}(u^c)$). By the chain rule, (2.4) and its isomonodromy conditions $\partial_{u_k}Y = (zE_k+V_k)Y$, $k=1,2,3$, become

$$\frac{\mathrm{d}\widehat{Y}}{\mathrm{d}\widehat{z}} = \left(U(x)+\frac{\Omega(x)}{\widehat{z}}\right)\widehat{Y}, \tag{B.2}$$

$$\frac{\partial\widehat{Y}}{\partial x} = \left(\widehat{z}E_2+\widehat{\Omega}_2(x)\right)\widehat{Y}, \tag{B.3}$$

so that (B.2) is strongly isomonodromic. Note that (B.2) is (3.10). The Stokes rays in the z-plane and $\widehat{z}$-plane are

$$\Re(z(u_3-u_1)) = \Re\widehat{z} = 0, \quad \Re(z(u_2-u_1)) = \Re(x\widehat{z}) = 0, \quad \Re(z(u_3-u_2)) = \Re((1-x)\widehat{z}) = 0.$$

The admissible directions in the z-plane and $\widehat{z}$-plane respectively are $\arg z = \tau^*$ and $\arg\widehat{z} = \widehat{\tau}^* := \tau^* + \arg(u_3-u_1)$ (with $\tau^* = \tau^{(0)}$ or τ), and the sectors are in correspondence by

$$\mathcal{S}_\nu:\ \tau^*+(\nu-2)\pi-\varepsilon < \arg z < \tau^*+(\nu-1)\pi-\varepsilon,$$

$$\widehat{\mathcal{S}}_\nu:\ \widehat{\tau}^*+(\nu-2)\pi-\varepsilon < \arg \widehat{z} < \widehat{\tau}^*+(\nu-1)\pi-\varepsilon,$$

$\nu\in\mathbb{Z}$. If u varies in $\mathbb{D}=\mathbb{D}(u^0)$ as small as assumed, then x varies in a ball around a $x_0\neq 0$, so small that Stokes rays do not cross the admissible directions $\arg\widehat{z} = \widehat{\tau}^{(0)}$ mod π. If u varies in a polydisc $\mathbb{D}=\mathbb{D}(u^c)$, where either $u_1^c = u_2^c$ or $u_2^c = u_3^c$, with vanishing condition (2.3), the Stokes rays corresponding to the coalescence play no role, and the results of [9] can be applied to both (2.4) and (B.2).

We show that the transformation (B.1) preserves the Levelt form at $z=0$ and the formal and canonical solutions at $z=\infty$. This will imply that the monodromy data for the system (2.4) relative to the admissible direction $\tau^{(0)}$ or τ are the same data for (B.2) relative to the direction $\widehat{\tau}^{(0)}$ or $\widehat{\tau}$. Since $u_3-u_1\neq 0$ and the Stokes rays $\Re(u_3-u_1)z$ do not cross the admissible directions in the z-plane, we can equally use $\widehat{\tau}^{(0)} := \tau^{(0)} + \arg(u_3^{(0)}-u_1^{(0)})$, or $\widehat{\tau} := \tau + \arg(u_3^c-u_1^c)$ for the computation of monodromy data.

The Levelt forms are preserved. Consider a Levelt forms at $\widehat{z}=0$ for (B.2)

$$\widehat{Y}^{(0)}(\widehat{z},x)=\widetilde{G}(x)(I+\sum_{\ell=1}^{\infty}\widehat{\phi}_\ell(x)\widehat{z}^\ell)\widehat{z}^{\widehat{\mu}}.$$

From the standard formal computation [55] of the coefficients, we receive

$$(\widehat{\phi}_\ell(x))_{ij}=\frac{(\mathcal{U}(x)\widehat{\phi}_{\ell-1}(x))_{ij}}{\mu_j-\mu_i+\ell},\quad \ell\geqslant 1,\qquad \mathcal{U}(x):=\widetilde{G}(x)^{-1}U(x)\widetilde{G}(x),\qquad \widehat{\phi}_0=I.$$

We can always represent a Levelt forms at $z=0$ for (2.4) as

$$Y^{(0)}(z,u)=e^{zu_1}G(u)(I+\sum_{\ell=1}^{\infty}\phi_\ell(u)z^\ell)z^{\widehat{\mu}}.$$

The standard formal computation of the ϕ_ℓ yields

$$\phi_\ell(u)=(u_3-u_1)^\ell\cdot(u_3-u_1)^{-\widehat{\mu}}\widehat{\phi}_\ell(x)(u_3-u_1)^{\widehat{\mu}}.$$

For G and $\widetilde{G}$ related by (2.9), it is easy to check that

$$Y^{(0)}(z,u)=(u_3-u_1)^{\Theta}e^{zu_1}\widehat{Y}^{(0)}(\widehat{z},x).$$

Canonical solutions at infinity are preserved. Let $Y_\nu(z,u)$, $\nu=1,2,3$, be canonical solutions at $z=\infty$ for (2.4) and $\widehat{Y}_\nu(\widehat{z},x)$ at $\widehat{z}=\infty$ for (B.2), having asymptotic behaviours respectively given by *the unique* formal solutions

$$Y_F(z,u)=\Big(I+\sum_{\ell=1}^{\infty}F_\ell(u)z^{-\ell}\Big)z^{-\Theta}e^{zU(u)},\qquad \widehat{Y}_F(\widehat{z},x)=\Big(I+\sum_{\ell=1}^{\infty}\widehat{F}_\ell(x)\widehat{z}^{-\ell}\Big)\widehat{z}^{-\Theta}e^{\widehat{z}U(x)},$$

with $U(u)=\mathrm{diag}(u_1,u_2,u_3)$ and $U(x)=\mathrm{diag}(0,x,1)$, respectively in the sectors $\mathcal{S}_\nu$ or $\widehat{\mathcal{S}}_\nu$, in the cases of $\mathbb{D}(u^{(0)})$ or $\mathbb{D}(u^c)$. By the formal computation of the coefficients we see that

$$F_\ell(u)=(u_3-u_1)^{-\ell}\,(u_3-u_1)^{\Theta}\widehat{F}_\ell(x)(u_3-u_1)^{-\Theta}.$$

This implies

$$Y_F(z,u)=(u_3-u_1)^{\Theta}e^{zu_1}\,\widehat{Y}_F(\widehat{z},x),$$

and, by *uniqueness* of the Y_ν and $\widehat{Y}_\nu$,

$$Y_\nu(z,u)=(u_3-u_1)^{\Theta}e^{zu_1}\,\widehat{Y}_\nu(\widehat{z},x).$$

B.2. Confluent hypergeometric functions

The confluent hypergeometric equation is

$$zw''+(b-z)w'+aw=0,\quad a,b\in\mathbb{C}.$$

Two linearly independent solutions are

$$M(z;a,b):=\sum_{s=0}^{\infty}\frac{(a)_s}{s!\Gamma(b+s)}z^s,$$

where $(a)_s$ denotes the Pochhammer symbol, and the function $U(z;a,b)$, which is uniquely determined by the asymptotic condition

$$U(z;a,b)\sim z^{-a},\quad z\to\infty,\quad -\frac{3}{2}\pi<\arg(z)<\frac{3}{2}\pi. \tag{B.4}$$

The function $M(z;a,b)$ is an entire functions of z,a,b, while $U(z;a,b)$ has a branch point at $z=0$, all its branches being entire in a,b. The analytic continuation of $U(z;a,b)$ is given by the cyclic relation

$$U(ze^{2\pi i n}) = \frac{2\pi i e^{-\pi i b n}\sin(\pi b n)}{\Gamma(1+a-b)\sin(\pi b)} M(z;a,b) + e^{-2\pi i b n} U(z;a,b), \quad -\pi < \arg(z) < \pi, \quad n\in\mathbb{Z}. \tag{B.5}$$

The function $M(z;a,b)$ admits an asymptotic expansion as $z\to\infty$, $-\pi/2 < \epsilon\arg(z) < 3\pi/2$ given by

$$M(z;a,b) \sim \frac{e^z z^{a-b}}{\Gamma(a)} \sum_{s=0}^{\infty} \frac{(1-a)_s (b-a)_s}{s!} z^{-s} + \frac{e^{\epsilon\pi i a} z^{-a}}{\Gamma(b-a)} \sum_{s=0}^{\infty} \frac{(a)_s (a-b+1)_s}{s!} (-z)^{-s}, \tag{B.6}$$

where $\epsilon = -1, 1$ and $a, b-a$ are not zero or a negative integer.

Lemma B.1. *If $b = 2a$, we can write the general solution of the hypergeometric equation in terms of the Hankel functions $H^{(1)}_\nu(z), H^{(2)}_\nu(z)$, with $\nu = (b-1)/2$, through the following relations:*

$$U\left(-2iz; \nu+\frac{1}{2}, 2\nu+1\right) = \frac{i\sqrt{\pi}}{2} e^{i\pi\nu} (2z)^{-\nu} e^{-iz} H^{(1)}_\nu(z),$$
$$U\left(2iz; \nu+\frac{1}{2}, 2\nu+1\right) = -\frac{i\sqrt{\pi}}{2} e^{-i\pi\nu} (2z)^{-\nu} e^{iz} H^{(2)}_\nu(z).$$

The Hankel functions have the asymptotics

$$H^{(1)}_\nu(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z-\nu\pi/2-\pi/4)}, \quad z\to\infty, \quad -\pi < \arg(z) < 2\pi,$$
$$H^{(2)}_\nu(z) \sim \sqrt{\frac{2}{\pi z}} e^{-i(z-\nu\pi/2-\pi/4)}, \quad z\to\infty, \quad -2\pi < \arg(z) < \pi.$$

B.3. $(2,2)$*—Generalized hypergeometric functions*

The generalized hypergeometric equation of kind $(2,2)$ is

$$z^2 w''' + z(b_2 + a_2 z) w'' + (b_1 + a_1 z) w + a_0 w = 0.$$

If b_1, b_2 are not negative integers and $b_1 - b_2$ is not an integer, then a fundamental set of solutions is

$$w_0(z;\mathbf{a},\mathbf{b}) = {}_2F_2\left(\left.\begin{matrix} a_1, a_2 \\ b_1, b_2 \end{matrix}\right| z\right), \; w_1(z;\mathbf{a},\mathbf{b}) = z^{1-b_1} {}_2F_2\left(\left.\begin{matrix} 1+a_1-b_1, 1+a_2-b_1 \\ 2-b_1, 1+b_2-b_1 \end{matrix}\right| z\right),$$

$$w_2(z;\mathbf{a},\mathbf{b}) = z^{1-b_2} {}_2F_2\left(\left.\begin{matrix} 1+a_1-b_2, 1+a_2-b_2 \\ 1+b_1-b_2, 2-b_2 \end{matrix}\right| z\right),$$

where

$${}_2F_2\left(\left.\begin{matrix} a_1, a_2 \\ b_1, b_2 \end{matrix}\right| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{\Gamma(b_1+k)\Gamma(b_2+k)} \frac{z^k}{k!}$$

is an entire function of z and of the parameters a_1, a_2, b_1, b_2. The following asymptotics hold:

$$ {}_2F_2\left(\begin{matrix} a_1, a_2 \\ b_1, b_2 \end{matrix}\middle| z\right) \sim \frac{1}{\Gamma(a_1)\Gamma(a_2)}\left[K_{2,2}(z) + L_{2,2}(ze^{i\epsilon\pi})\right], \quad z \to \infty, $$
$$ -(2+\epsilon)\frac{\pi}{2} < \arg(z) < (2-\epsilon)\frac{\pi}{2}, $$

where $\epsilon = -1, 1$,

$$ K_{2,2}(z) = e^z z^\gamma \sum_{k=0}^{\infty} d_k z^{-k}, \quad d_0 = 1, \quad \gamma = \sum_{h=1,2}(a_h - b_h) $$

(the recursive formulas for d_k, $k \geqslant 1$ can be found in [47], formula (6) of section 5.11.3) and

$$ L_{2,2}(z) = \sum_{m=1,2} z^{-a_m} \sum_{k=0}^{\infty} c_{m,k} \frac{(-1)^k z^{-k}}{k!}, \quad c_{m,k} = \Gamma(a_m + k) \frac{\Gamma(a_l - a_m - k)}{\prod_{n=1,2}\Gamma(b_n - a_m - k)}, \, l \neq m. $$

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