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Neutrino oscillations in cosmological spacetime

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Abstract

Neutrino physics is one of the most intriguing and vividly discussed topics in high-energy physics. Phenomena of neutrino oscillation give rise to a large number of experiments to actually observe these oscillations among different flavors of neutrinos. Though neutrino oscillation is confirmed from the experimental scenario, however, in most situations, the effect of the expansion of the Universe in neutrino oscillation has not been considered. Expansion of the present Universe is such a generic feature that it can not be avoided in the neutrino oscillation problem which is the main theme of this article. It also provides some new insights of early-universe neutrinos. Further, the neutrino oscillation can also be used as a tool to probe the existence of torsion in the cosmology. In this article, we report the effect of the expansion of the Universe and the torsion in the neutrino oscillation phenomena.
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1. Introduction

Neutrinos which today play an important role in different branches of physics namely subatomic physics, astroparticle physics, and cosmology, were introduced almost 90 years ago by Pauli, denoted by ν. The immediate question was whether these particles are massive or not. In [1], Pontecorvo suggested one possible way to observe this which is, that if neutrino flavors are a superposition of massive eigenstates (eigenstates of non-diagonal mass-matrix in the action of theory), then they could oscillate among themselves. There are in fact several experiments looking for neutrino oscillations at that time: solar neutrino experiment, atmospheric neutrino measurements, reactor experiments, and accelerator experiments. Several experiments indeed claimed that their observation indeed shows evidence of neutrino oscillation however, the final breakthrough came in 1998 when the Super-Kamiokande collaboration [2] reported having strong evidence of such oscillations between $\nu_\mu \leftrightarrow \nu_\tau$ flavors.

In the standard quantum mechanical (QM) treatment of neutrino oscillations, neutrino mass-eigenstates are considered to be relativistic and to have the same momentum which implies that their energies differ by their masses. But QM models also suffer several difficulties in describing the neutrino oscillation process compared to quantum field theory (QFT) (see [3–6]). Pontecorvo's pioneering work is the theoretical basis of neutrino mixing, has been studied in great detail but there are still several open questions regarding the neutrino-mixing phenomena and their masses. Within the Standard Model (SM) framework, neutrinos are treated as three $(\nu_e, \nu_\mu, \nu_\tau)$ massless left-handed fermions and the leptonic number is strictly conserved. The phenomenon of neutrino mixing, and the existence of non-zero neutrino masses, have captured much attention because it opens interesting perspectives on the physics beyond SM [7–11]. Actually, in presence of mixing, the masses of the flavor neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$ are not well-defined. In fact, neutrino fields entering in charge weak currents, have definite flavor but not a definite mass. The fields (ν_1, ν_2, ν_3) with definite masses, which propagate as free fields, do not have a definite flavor, however, the latter can be obtained as a mixture of flavor fields and vice versa, depending on which fields one considers as fundamental. It is exactly this feature that leads to the observation of neutrino oscillations according to Pontecorvo. Nonetheless, neutrino absolute mass values are yet to be found.

Observation of the Hubble telescope confirms that the present Universe is expanding with non-zero acceleration [12,13]. Hence, in the neutrino oscillation over long time-interval, it must have some impact on the nature of oscillation which can not be avoided especially for early Universe neutrinos. These neutrinos went through several phases of expansion of the Universe namely cosmological constant dominated (de-Sitter Universe), radiation-dominated Universe, and matter-dominated Universe. Hence, studying neutrino oscillation in the spatially homogeneous and isotropic universe we would expect some non-trivial features coming from the scale-factor of Friedmann-Robertson-Walker (FRW) background geometry, discussed in this article.

Torsion is on the other hand a feature that is neglected in General Relativity (GR). However, the presence of torsion leads to a dramatic effect in dynamics in curved spacetime especially in the context of cosmology [14–24]. In [18], it is shown that the torsion could generate repulsive gravitational force in the early Universe which prevents the formation of singularity. However, so far no evidence or signature of torsion has been found from experimental observations [25–30]. On the other hand, the neutrino oscillation phenomenon could detect the presence of torsion in the current expanding Universe since the presence of torsion changes the equations determining the evolution of scale factor and scale factor would directly affect the nature of these oscillations. Hence, neutrino oscillation can be useful in probing the torsion in the Universe.

Our aim is to show only the effect of the expansion of the Universe in the neutrino oscillation. The matter and the radiation in the Universe essentially affect the scale factor, follows from the Einstein equation, more specifically the Friedmann and Raychaudhuri equations. The core idea of this work is that the stress-energy tensor corresponding to the Standard model particles produces the curved spacetime, follows from the Einstein equation. On this background spacetime geometry, neutrinos are propagating. Hence, the effect of the matter and radiation in the Universe is essentially taken into account through the FRW metric, more specifically through the scale factor of the Universe. The mathematical derivation provided here is mathematically consistent and rigorous than the ones valid locally [31]. This result is discussed in section 4.3 and section 4.4. Further, we also discuss the effect of torsion in the FRW Cosmology in the neutrino oscillation which is the other aim of this article. A review of the torsion in general relativity and cosmology are provided in the Appendix A. This result is discussed in the section 5 and section A.5. Considering the above possible effects of the expansion of the Universe and the Cosmological torsion in the neutrino oscillation, both the scale factor of the Universe and the torsion can be probed from the neutrino oscillation. The structure of this article is as follows. First, we briefly discuss the mathematical method to show the existence of neutrino oscillation phenomena in flat-spacetime from the Dirac field theory. Next, we briefly discuss the formulation of the Dirac-field theory in curved spacetime, followed by a detailed computation of the probability amplitude of neutrino oscillations in FRW geometry. Next, we discuss the effect of torsion consistent with the properties of FRW cosmology and how to probe it in terms of neutrino oscillations.

2. Neutrino oscillation in the Minkowski spacetime

This section is a review of the neutrino oscillation in the Minkowski spacetime which is mostly considered in high-energy phenomenologies [32–34]. This provides the preliminary of the neutrino oscillation, important for the later purposes. Dirac field theory is defined by introducing the following action

$$S = \int \bar{\psi}(x)(i\partial\!\!\!/ - m)\psi(x), \quad \partial\!\!\!/ = \gamma^\alpha \partial_\alpha, \tag{2.1}$$

which yields the following equation of motion

$$(i\partial\!\!\!/ - m)\psi = 0, \tag{2.2}$$

where γ^αs are Dirac-gamma matrices, satisfying Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}$ with $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

If neutrinos are not massless and their mass matrix will be non-diagonal (in general complex) then flavor eigenstates, denoted by $|\nu_\alpha\rangle$, can be represented as linear superposition of the mass eigenstates, denoted by $|\nu_i\rangle$:

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle, \tag{2.3}$$

where U is a unitary matrix through which we can transform mass matrix into a diagonal form. U can be parametrized like Kobyashi-Maskawa matrix [35,36] using quark-mixing angles

$$U = \begin{bmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 - s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 s_3 - c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{bmatrix}, \tag{2.4}$$

where $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$.

If at initial time $t = 0$, a beam of pure ν_α state is produced, the initial state is a superposition of the mass eigenstates

$$|\nu_\alpha(0)\rangle = \sum_i U_{\alpha i} |\nu_i\rangle. \tag{2.5}$$

The time evolution of a mass eigenstate $|\nu_i\rangle$ is determined by the Dirac equation for a propagating neutrino with definite mass m_i. From Dirac equation, we obtain

$$i\hbar\frac{\partial}{\partial t}\psi_{iL}(t,\vec{x}) = -\sqrt{\vec{p}^2c^2 + m^2c^4}\,\frac{\vec{\sigma}.\hat{\vec{p}}}{|\vec{p}|}\psi_{iL}(t,\vec{x}), \tag{2.6}$$

where $\psi_{iL}(t,\vec{x}) = \langle\vec{x}|\nu_i\rangle_t$ (phenomenological neutrinos are considered to be left-handed) and $\vec{\sigma}$ are three Pauli matrices. The Dirac equation in ultra-relativistic limit ($\frac{mc^2}{pc} \ll 1$) becomes

$$i\hbar\frac{\partial}{\partial t}\psi_{iL}(t,\vec{x}) = -\left[|\vec{p}|c + \frac{m_i c^3}{2|\vec{p}|}\right]\frac{\vec{\sigma}.\hat{\vec{p}}}{|\vec{p}|}\psi_{iL}(t,\vec{x}). \tag{2.7}$$

Considering the second term in the square-bracket to be perturbation we can write the solution of the following form

$$\psi_{iL}(t,\vec{x}) = e^{i\Phi(t)}\psi^{(0)}_{iL}(t,\vec{x}), \tag{2.8}$$

where

$$i\hbar\frac{\partial}{\partial t}\psi^{(0)}_{iL}(t,\vec{x}) = -c\vec{\sigma}.\hat{\vec{p}}\psi^{(0)}_{iL}(t,\vec{x}). \tag{2.9}$$

According to our assumption,

$$\frac{\vec{\sigma}.\hat{\vec{p}}}{|\vec{p}|}\psi^{(0)}_{iL}(t,\vec{x}) = -\psi^{(0)}_{iL}(t,\vec{x}), \tag{2.10}$$

and

$$\hat{\vec{p}}\psi_{iL}^{(0)}(t,\vec{x}) = \vec{p}\psi_{iL}^{(0)}(t,\vec{x}). \tag{2.11}$$

Under above assumption, we can write the phase factor as

$$\Phi(t) = -\frac{1}{\hbar}\int_0^t \frac{m_i c^3}{2|\vec{p}|}dt' = -\frac{m_i c^3}{2\hbar|\vec{p}|}t. \tag{2.12}$$

Hence,

$$\langle \vec{x}|\nu_i\rangle_t = \psi_{iL}(t,\vec{x}) = e^{-i\frac{m_i c^3}{2\hbar|\vec{p}|}t}\psi_{iL}^{(0)}(t,\vec{x}), \tag{2.13}$$

which is equivalent to

$$|\nu_i\rangle_t = e^{-i\frac{m_i c^3}{2\hbar|\vec{p}|}t}e^{-i\frac{pc}{\hbar}t}|\nu_i\rangle\,. \tag{2.14}$$

As a consequence of (2.3), we obtain

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha i}e^{-i\frac{m_i^2 c^3}{2p\hbar}t}e^{-i\frac{pc}{\hbar}t}|\nu_i\rangle\,. \tag{2.15}$$

Therefore, the probability amplitude of observing an initially created flavor eigenstate $|\nu_\alpha\rangle$ as the flavor eigenstate $|\nu_\beta\rangle$ at some later time t becomes

$$\langle \nu_\beta|\nu_\alpha(t)\rangle = \sum_i U_{\alpha i}U^*_{\beta i}e^{-i\frac{m_i^2 c^3}{2p\hbar}t}e^{-i\frac{pc}{\hbar}t}. \tag{2.16}$$

Hence, the probability for a transition $\nu_\alpha \to \nu_\beta$ under time evolution is

$$\mathcal{P}_{\nu_\alpha\to\nu_\beta}(t) = |\langle\nu_\beta|\nu_\alpha(t)\rangle|^2 = \sum_{i,j} U_{\alpha i}U^*_{\beta i}U^*_{\alpha j}U_{\beta j}e^{-i\frac{(m_i^2-m_j^2)c^3}{2p\hbar}t}. \tag{2.17}$$

This is the quantity we are interested in finding for large time scale t.

Hence, all we need to know is the solution of the Dirac equation in the FRW spacetime and look its temporal behavior in order to find the effect of the expansion of the Universe in the neutrino oscillation phenomena.

3. Dirac field theory in curved spacetime

3.1. Introduction

This section is a brief review of the Dirac field theory in curved spacetime [37–40], preliminary for showing our main results in section 4.3, section 4.4 and section 5. The action for the Dirac-field theory introduced in curved spacetime is given by

$$S = \int \sqrt{-g(x)}d^4x\Big[\frac{i}{2}\big(\bar{\psi}(x)e^\mu_{\ a}(x)\gamma^a\mathcal{D}_\mu\psi(x) - \overline{\mathcal{D}_\mu\psi}(x)e^\mu_{\ a}(x)\gamma^a\psi(x)\big) - m\bar{\psi}(x)\psi(x)\Big], \tag{3.1}$$

where tetrads $e^{\mu}_{\ a}$ satisfy the following relation

$$g_{\mu\nu}e^{\mu}_{\ a}e^{\nu}_{\ b} = \eta_{ab}, \ g_{\mu\nu} = e_{\mu}^{\ a}e_{\nu}^{\ b}\eta_{ab}, \tag{3.2}$$

where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$. Further, $\mathcal{D}_\mu$ defines covariant derivative for spinors in curved spacetime with spin-connection Ω_μ, defined as follows

$$\begin{aligned} \mathcal{D}_\mu &= \partial_\mu + \Omega_\mu \\ \Omega_\mu &= \frac{1}{8}\omega_{\mu bc}\sigma^{bc}, \ \sigma^{bc} = [\gamma^b, \gamma^c] \\ \omega_{\mu ab} &= \eta_{ac}e_{\nu}^{\ c}e^{\sigma}_{\ b}\Gamma^{\nu}_{\ \sigma\mu} + \eta_{ac}e_{\nu}^{\ c}\partial_\mu e^{\nu}_{\ b} \\ \{\gamma^a, \gamma^b\} &= 2\eta^{ab}, \end{aligned} \tag{3.3}$$

where $\Gamma^{\rho}_{\ \mu\nu}$ is the metric-affine connection defined in GR.

From the action itself one can derive Dirac equation in curved spacetime which is

$$[ie^{\mu}_{\ a}\gamma^a(\partial_\mu + \Omega_\mu) - m]\Psi = 0. \tag{3.4}$$

3.2. Reduced Dirac equation

Note that

$$ie^{\mu}_{\ c}\gamma^c\Omega_\mu = i\gamma^c\Omega_c = \frac{i}{8}\omega_{cab}(\gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a), \tag{3.5}$$

where we have used the fact

$$\omega_{cab} \equiv e^{\mu}_{\ c}\omega_{\mu ab}. \tag{3.6}$$

Utilizing the following identity

$$\gamma^c\gamma^a\gamma^b = \eta^{ca}\gamma^b + \eta^{ab}\gamma^c - \eta^{cb}\gamma^a - i\epsilon^{cabd}\gamma_d\gamma^5, \tag{3.7}$$

we can write

$$\gamma^c[\gamma^a, \gamma^b] = 2\eta^{ca}\gamma^b - 2\eta^{cb}\gamma^a - 2i\epsilon^{cabd}\gamma_d\gamma^5. \tag{3.8}$$

Using above relation, we can write the Dirac equation (3.4) more explicitly in the following way

$$ie^{\mu}_{\ a}\gamma^a\partial_\mu\Psi + \frac{1}{4}i\omega_{cab}(\eta^{ca}\gamma^b - \eta^{cb}\gamma^a)\Psi - \frac{1}{4}\epsilon^{abcd}\omega_{cab}\gamma_d\gamma^5\Psi - m\Psi = 0. \tag{3.9}$$

4. Dirac equation in the FRW geometry

4.1. Reduced Dirac equation in the FRW geometry

In order to know the nature of the neutrino oscillation, the Dirac equation is required to be solved to know the time-evolution of different propagating modes. In this section, we provide a mathematical technique to solve the Dirac equation in FRW spacetime from a system of decoupled ordinary differential equations.

In order to further simplify the Dirac equation defined earlier in (3.9), we now assume here a factorizability ansatz which will work when analyzing the Dirac equation in curved spacetimes that are sufficiently symmetric. Here, we consider FRW geometry, in which line-element or metric is defined as follows

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{4.1}$$

where $a(t)$ is the scale-factor of the Universe and k is the spatial curvature of the Universe. This spacetime exhibits azimuthal symmetry and with metric elements specifically depends only on r, θ, t. This satisfies the necessary properties of the expanding Universe which is spatially homogeneous and isotropic. Further, the above metric is diagonal which also implies vierbein or tetrad is diagonal as well. Hence, the vierbeins inherit the same symmetries as the metric. Here, we choose the following ansatz

$$\Psi(x) = f(x^0, x^1, x^2)\psi(x), \tag{4.2}$$

where $x^0 = t, x^1 = r, x^2 = \theta$, such that ψ satisfies following reduced Dirac equation

$$i e^{\mu}_{\ a}\gamma^a \partial_\mu \psi - \frac{1}{4}\epsilon^{abcd}\omega_{cab}\gamma_d\gamma^5\psi - m\psi = 0. \tag{4.3}$$

In fact, it can be shown that for the FRW case the term involving the Levi-Civita symbol vanishes identically (shown later). Consequently, if there exists such an f that holds factorizability and the above equation, then we can get rid of the connection altogether.

The factorizability ansatz (4.2) with (4.3) leads to following equation

$$i e^{\mu}_{\ a}\gamma^a \partial_\mu f + \frac{i}{4}\omega_{cab}(\eta^{ca}\gamma^b - \eta^{cb}\gamma^a) f = 0. \tag{4.4}$$

The above set of partial differential equations (PDE) can be further simplified by multiplying γ^e from left and taking the trace, leading to

$$\begin{aligned} &4\eta^{ae} e^{\mu}_{\ a}\partial_\mu f + \omega_{cab}(\eta^{ca}\eta^{eb} - \eta^{cb}\eta^{ea}) f = 0 \\ &\partial_\mu \log(f) + \frac{1}{2} e_\mu^{\ c}\eta_{bc}\omega_a^{\ ab} = 0. \end{aligned} \tag{4.5}$$

From (3.3), we can write

$$e_\mu^{\ c}\eta_{bc}\omega_a^{\ ab} = \Gamma^{\nu}_{\ \mu\nu} + e_\mu^{\ a}\partial_\nu e^{\nu}_{\ a}. \tag{4.6}$$

Using the fact that $\Gamma^{\nu}_{\ \mu\nu} = \partial_\mu \log\sqrt{e}$ where $e(x)$ is the determinant of the vierbein. This leads to following formula

$$\partial_\mu \log(f) = -\partial_\mu \log\sqrt{e} - \frac{1}{2} e_\mu^{\ a}\partial_\nu e^{\nu}_{\ a}. \tag{4.7}$$

Defining $f \equiv h e^{-\frac{1}{2}}$ gives

$$\partial_\mu \log(h) = -\frac{1}{2} e_\mu^{\ a}\partial_\nu e^{\nu}_{\ a}, \tag{4.8}$$

which will determine the existence of the factorizability condition. In the case of diagonal tetrads we will have following PDEs

$$\begin{aligned} &\partial_t \log(h) = -\frac{1}{2} e_t^{\ t}\partial_t e^{t}_{\ t}, \ \partial_r \log(h) = -\frac{1}{2} e_r^{\ r}\partial_r e^{r}_{\ r}, \\ &\partial_\theta \log(h) = -\frac{1}{2} e_\theta^{\ \theta}\partial_\theta e^{\theta}_{\ \theta}, \ \partial_\varphi \log(h) = 0. \end{aligned} \tag{4.9}$$

Here, we kept in mind f in independent of φ as a consequence of azimuthal symmetry of vierbeins.

This is the point up to which we can proceed without having detailed metric but knowing its structure (diagonal) and symmetry properties. Therefore, the existence of f highly depends on the exact expressions of metric elements.

4.2. Dirac equation in FRW spacetime

Recall for FRW spacetime, metric is of the following form

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a^2(t)}{(1-kr^2)} & 0 & 0 \\ 0 & 0 & -a^2(t)r^2 & 0 \\ 0 & 0 & 0 & -a^2(t)r^2\sin^2\theta \end{bmatrix}. \tag{4.10}$$

Hence, the vierbeins and their inverse are respectively as follows

$$e^{\mu}_{\ a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{1-kr^2}}{a(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{a(t)r} & 0 \\ 0 & 0 & 0 & \frac{1}{a(t)r\sin\theta} \end{bmatrix}, \; e_{\mu}^{\ a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{a(t)}{\sqrt{1-kr^2}} & 0 & 0 \\ 0 & 0 & a(t)r & 0 \\ 0 & 0 & 0 & a(t)r\sin\theta \end{bmatrix}. \tag{4.11}$$

Imposing the factorization condition (4.9) we obtain

$$\begin{aligned} &\partial_t \log(h) = \partial_\theta \log(h) = 0 = \partial_\varphi \log(h) \\ &\partial_r \log(h) = \frac{1}{2}\frac{kr}{1-kr^2} \implies h = (1-kr^2)^{-\frac{1}{4}}, \end{aligned} \tag{4.12}$$

leading to

$$f = he^{-\frac{1}{2}} = \frac{1}{a^{\frac{3}{2}}(t)r\sin^{\frac{1}{2}}\theta}. \tag{4.13}$$

Therefore, factorizability of Dirac spinor gives

$$\Psi(x) = \frac{1}{a^{\frac{3}{2}}(t)r\sin^{\frac{1}{2}}\theta}\psi(x), \tag{4.14}$$

where $\psi(x)$ satisfies the reduced Dirac equation, mentioned earlier.

For FRW geometry it can be easily shown through some computations that

$$\begin{aligned} &\omega_{101} = \frac{\dot{a}}{a},\; \omega_{202} = \frac{\dot{a}}{a},\; \omega_{212} = \frac{1}{ar}\sqrt{1-kr^2} \\ &\omega_{303} = \frac{\dot{a}}{a},\; \omega_{313} = \frac{1}{ar}\sqrt{1-kr^2},\; \omega_{323} = \frac{\cot\theta}{ar}, \end{aligned} \tag{4.15}$$

which leads to following conclusion

$$\frac{1}{4}\epsilon^{abcd}\omega_{cab}\gamma_d\gamma^5 = 0. \tag{4.16}$$

Collecting above results, we can conclude that $\psi(x)$ effectively satisfies

$$ie^{\mu}_{\ a}\gamma^a\partial_\mu\psi - m\psi = 0, \tag{4.17}$$

where connection term is completely absent which makes the computation simpler. From now one consider $k = 0$ case for the FRW geometry, consistent with current observations [41–44].

4.3. Solution of reduced Dirac equation in the FRW geometry

Let us start by writing down the equation (4.17) explicitly

$$ia\frac{\partial}{\partial t}\psi = -i\gamma^t\Big[\gamma^r\frac{\partial}{\partial r}+\gamma^\theta\frac{\partial}{\partial\theta}+\gamma^\varphi\frac{\partial}{\partial\varphi}\Big]\psi + am\gamma^t\psi. \tag{4.18}$$

We have some freedom in choosing $\gamma^\mu = e^\mu_{\ a}\gamma^a$, since, vierbeins $e^\mu_{\ a}$ depends on the choice of coordinate. Since, FRW metric was written in spherical coordinates, which makes the obvious choice of vierbeins to be diagonal, we choose to work with diagonal tetrad gauge in which

$$\begin{aligned}\gamma_d^t = \gamma^0,\ \gamma_d^r = \gamma^1\\ \gamma_d^\theta = \frac{1}{r}\gamma^2,\ \gamma_d^\varphi = \frac{1}{r\sin\theta}\gamma^3,\end{aligned} \tag{4.19}$$

where $\{\gamma^a\}$s are usual flat-spacetime gamma matrices. On the other hand, in the Cartesian tetrad gauge

$$\begin{aligned}\gamma_c^t = \gamma^0,\ \gamma_c^r &= (\gamma^1\cos\varphi+\gamma^2\sin\varphi)\sin\theta+\gamma^3\cos\theta\\ \gamma_c^\theta &= \frac{1}{r}\Big[(\gamma^1\cos\varphi+\gamma^2\sin\varphi)\cos\theta-\gamma^3\sin\theta\Big]\\ \gamma_c^\varphi &= \frac{1}{r\sin\theta}(-\gamma^1\sin\varphi+\gamma^2\cos\varphi),\end{aligned} \tag{4.20}$$

in which vierbein axes point along t, x, y, z. Both of these tetrad choices are related by a similarity transformation

$$S = e^{-\frac{\varphi}{2}\gamma^1\gamma^2}e^{-\frac{\theta}{2}\gamma^3\gamma^1}\mathcal{S}, \tag{4.21}$$

where

$$\mathcal{S} = \frac{1}{2}(\gamma^1\gamma^2-\gamma^1\gamma^3+\gamma^2\gamma^3+\mathbb{I}). \tag{4.22}$$

This implies that

$$\mathcal{S}\gamma^1\mathcal{S}^{-1} = \gamma^3,\ \mathcal{S}\gamma^2\mathcal{S}^{-1} = \gamma^1,\ \mathcal{S}\gamma^3\mathcal{S}^{-1} = \gamma^2. \tag{4.23}$$

The above two different choices are hence related by

$$\gamma_c^\mu = S\gamma_d^\mu S^{-1}, \tag{4.24}$$

and $\psi_c = S\psi_d$. The real and measurable quantities are same in both choices and hence, they are equivalent.

Therefore, according to the above discussion we just consider to solve reduced Dirac equation in diagonal tetrad gauge in which Dirac equation becomes

$$ia\frac{\partial}{\partial t}\psi = T\psi, \tag{4.25}$$

where

$$T = -i\gamma^0\gamma^1\frac{\partial}{\partial r}+\frac{\gamma^1}{r}K - ia\gamma^0 m, \tag{4.26}$$

and

$$K = i\gamma^1\gamma^0\gamma^2\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\gamma^1\gamma^0\gamma^3\frac{\partial}{\partial\varphi}. \tag{4.27}$$

Through simple algebra one can easily find that $[T, K] = 0$. For our detailed computation, we choose the following representation

$$\gamma^0 = i\begin{bmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix}, \ \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix}, \tag{4.28}$$

where σ^is are the usual Pauli matrices. Hence, we are interested in the solution of the following form

$$\Psi = \frac{1}{a^{\frac{3}{2}} r \sin^{\frac{1}{2}}\theta} S\psi. \tag{4.29}$$

Using the above representation of gamma matrices, the matrix S can be expressed in the block-diagonal form

$$S = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}, \ Z = \begin{bmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}\mathcal{S}. \tag{4.30}$$

Using this choice, we can write a similar equation just like (4.25) in which $\tilde{\psi} = S\psi$ and

$$\begin{aligned} \tilde{T} &= -i\gamma^0\gamma^3\frac{\partial}{\partial r} + \frac{\gamma^3}{r}\tilde{K} - ia\gamma^2 m \\ \tilde{K} &= i\gamma^3\gamma^0\gamma^1\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\gamma^3\gamma^0\gamma^2\frac{\partial}{\partial\varphi}, \end{aligned} \tag{4.31}$$

and again $[\tilde{T}, \tilde{K}] = 0$.

We can write the general solution in following form

$$\tilde{\psi}_j = \mathcal{R}(r)\Theta(\theta)e^{im_j\varphi - i\int_{t_0}^{t}\omega(t')dt'}. \tag{4.32}$$

For the solution Ψ to be single valued we demand

$$\Psi(\varphi + 2\pi) = \Psi(\varphi), \tag{4.33}$$

and according to the expression (4.30),

$$S(\varphi + 2\pi) = -S(\varphi), \tag{4.34}$$

which means we require the solution $\tilde{\psi}_j$ to be such that it satisfies

$$\tilde{\psi}_j(\varphi + 2\pi) = -\tilde{\psi}_j(\varphi). \tag{4.35}$$

The above demands the values of $m_j = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$. Hence, in the representation (4.28) the angular part of the reduced Dirac equation becomes

$$\tilde{K}\tilde{\psi}_j = i\begin{bmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{bmatrix}\frac{\partial}{\partial\theta}\tilde{\psi}_j - \frac{m_j}{\sin\theta}\begin{bmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{bmatrix}\tilde{\psi}_j. \tag{4.36}$$

We want the eigen-solutions of the above equation of the form $\tilde{K}\tilde{\psi}_j = \kappa\tilde{\psi}_j$. To find that we demand angular part of the spinor would be of the following form

$$\begin{bmatrix} \Theta(\theta) \\ \sigma^3\Theta(\theta) \end{bmatrix} = \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_1 \\ -\Theta_2 \end{bmatrix}, \tag{4.37}$$

so that we have two free components to be determined. This shows the eigenvalue equation reduces to

$$\frac{d}{d\theta}\begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} \frac{m_j}{\sin\theta} & -\kappa \\ \kappa & \frac{m_j}{\sin\theta} \end{bmatrix}\begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}. \tag{4.38}$$

This is the set of reduced angular Dirac equation in FRW spacetime.

Now let us look at the radial part of the reduced Dirac equation. Utilizing the above equation and the ansatz (4.32) we can write

$$a\omega(t)\tilde{\psi}_{j,\kappa} = -i\gamma^0\gamma^3\frac{\partial}{\partial r}\tilde{\psi}_{j,\kappa} + \frac{\gamma^3}{r}\kappa\tilde{\psi}_{j,\kappa} - ia\gamma^0 m\tilde{\psi}_{j,\kappa}, \tag{4.39}$$

which yields

$$a\omega(t)\tilde{\psi}_{j,\kappa} = \begin{bmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{bmatrix}\frac{\partial}{\partial r}\tilde{\psi}_{j,\kappa} + \frac{\kappa}{r}\begin{bmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{bmatrix}\tilde{\psi}_{j,\kappa} + am\begin{bmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix}\tilde{\psi}_{j,\kappa}. \tag{4.40}$$

Writing the ansatz (according to [45]) explicitly in the following form

$$\tilde{\psi}_{j,\kappa} = \begin{bmatrix} P_\kappa(r)\Theta_j(\theta) \\ \sigma^3 Q_\kappa(r)\Theta_j(\theta) \end{bmatrix} e^{im_j\varphi - i\int_{t_0}^{t}\omega(t')dt'}, \tag{4.41}$$

we obtain the following set of radial equations

$$\frac{d}{dr}\begin{bmatrix} P_\kappa \\ Q_\kappa \end{bmatrix} = \begin{bmatrix} -\frac{\kappa}{r} & a(\omega+m) \\ -a(\omega-m) & \frac{\kappa}{r} \end{bmatrix}\begin{bmatrix} P_\kappa \\ Q_\kappa \end{bmatrix}, \tag{4.42}$$

which can be rewritten as

$$P_\kappa = -\frac{1}{a(\omega-m)}\left(\frac{d}{dr} - \frac{\kappa}{r}\right)Q_\kappa,\ Q_\kappa = -\frac{1}{a(\omega+m)}\left(\frac{d}{dr} + \frac{\kappa}{r}\right)P_\kappa. \tag{4.43}$$

Inserting Q_κ in first equation leads to

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \varepsilon^2\right]P_\kappa = 0, \tag{4.44}$$

where $\varepsilon = a(t)\sqrt{\omega^2(t) - m^2} \implies \omega(t) = \sqrt{\frac{\varepsilon^2}{a^2(t)} + m^2}$.

There are two linearly independent solution of the above equation which are

$$\begin{aligned} P_1^{(\kappa)}(r) &= e^{i\varepsilon r}\sum_{l=0}^{\infty}\frac{\mathcal{C}_l}{r^l},\ \mathcal{C}_{l+1} = \frac{l(l+1) - \kappa(\kappa+1)}{2i\varepsilon(l+1)}\mathcal{C}_l,\ \mathcal{C}_0 = 1 \\ P_2^{(\kappa)}(r) &= e^{i\varepsilon r}\sum_{l=0}^{\infty}\mathcal{D}_l r^l,\ \mathcal{D}_{l+1} = -\frac{l(l-1) - \kappa(\kappa+1)}{2i\varepsilon(l-1)}\mathcal{D}_l,\ \mathcal{D}_0 = 1. \end{aligned} \tag{4.45}$$

To have a regular solution at the origin and converging solution at infinity, we choose the general solution of the reduced radial Dirac equation to be

$$P_\kappa(r) = P_1^{(\kappa)}(r_>) + P_2^{(\kappa)}(r_<), \tag{4.46}$$

where $r_< = \min(r, R),\ r_> = \max(r, R)$ where $r = R$ is the surface on which boundary condition is imposed.

For $Q_\kappa(r)$ we just need to reverse the sign of κ in the above solution, leading to following general solution

$$Q_\kappa(r) = P_1^{(-\kappa)}(r_>) + P_2^{(-\kappa)}(r_<). \tag{4.47}$$

In order to convergent series solution, the series defined in equation (4.45) must be truncated for each value of κ which shows that $\kappa \in \mathbb{Z}$ (integers).

Now we move back to angular part of reduced radial Dirac equation which are

$$\Theta_1 = -\frac{1}{\kappa}\left(\frac{d}{d\theta} + \frac{m_j}{\sin\theta}\right)\Theta_2,\ \Theta_2 = \frac{1}{\kappa}\left(\frac{d}{d\theta} - \frac{m_j}{\sin\theta}\right)\Theta_1. \tag{4.48}$$

In the small θ limit we can write the solutions in following way

$$\begin{aligned} \Theta_1^{(j)} &= e^{i\kappa\theta}\sum_{l=0}^{\infty} d_l \theta^{\frac{l}{2}},\ d_{l=1} = -\frac{\frac{l}{2}(\frac{l}{2}+1) - m_j(m_j-1)}{i\kappa(\frac{l}{2}-1)} d_l,\ d_0 = 1 \\ \Theta_2^{(j)} &= e^{i\kappa\theta}\sum_{l=0}^{\infty} d_l \theta^{\frac{l}{2}},\ d_{l=1} = -\frac{\frac{l}{2}(\frac{l}{2}+1) - m_j(m_j+1)}{i\kappa(\frac{l}{2}-1)} d_l,\ d_0 = 1. \end{aligned} \tag{4.49}$$

Using all the above information, we can write the solution of the Dirac equation in the FRW geometry as follows

$$\begin{aligned} \Psi_{\varepsilon,m_j,\kappa} &= \frac{1}{a^{\frac{3}{2}}(t) r \theta^{\frac{1}{2}}} \begin{bmatrix} e^{-i\frac{\varphi}{2}} & -\frac{\theta}{2}e^{-i\frac{\varphi}{2}} \\ \frac{\theta}{2}e^{i\frac{\varphi}{2}} & e^{i\frac{\varphi}{2}} \end{bmatrix} \begin{bmatrix} P_\kappa \Theta^{(j)} \\ \sigma^3 Q_\kappa \Theta^{(j)} \end{bmatrix} e^{i m_j \varphi - i\int_{t_0}^{t}\omega(t')dt'} \\ &= \frac{1}{a^{\frac{3}{2}}(t) r \theta^{\frac{1}{2}}} e^{-i\int_{t_0}^{t}\omega(t')dt'} \begin{bmatrix} e^{i(m_j-\frac{1}{2})\varphi}(P_\kappa - \frac{\theta}{2}\sigma^3 Q_\kappa)\Theta^{(j)} \\ e^{i(m_j+\frac{1}{2})\varphi}(\frac{\theta}{2}P_\kappa + \sigma^3 Q_\kappa)\Theta^{(j)} \end{bmatrix} \\ &\equiv \frac{1}{a^{\frac{3}{2}}(t)} e^{-i\int_{t_0}^{t}\omega(t')dt'} \Phi_{m_j,\kappa}(r,\theta,\varphi). \end{aligned} \tag{4.50}$$

4.4. Neutrino oscillation in the FRW geometry

Inserting (4.50) in (2.14) we obtain

$$|\nu_i(t)\rangle = \frac{1}{a^{\frac{3}{2}}(t)} e^{-i\int_0^t \omega_i(t')dt'} |\nu_i\rangle\,, \tag{4.51}$$

in the FRW cosmological background where $\omega_i(t) = \sqrt{\frac{\varepsilon^2}{a^2(t)} + m_i^2}$ where $\omega_i(t)$ is the analogous to energy quanta (but not exactly since FRW geometry does not have time like Killing vector field) and m_i is the mass of the ith mass eigenstate of neutrino.

Hence, the probability for a transition $\nu_\alpha \to \nu_\beta$ under time evolution in FRW geometry is

$$\mathcal{P}_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 \sqrt{-g(t)} = \sum_{i,j} U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} e^{-i \int_0^t dt' (\omega_i(t') - \omega_j(t'))}. \tag{4.52}$$

The above expression shows that in the expanding Universe, the probability for a transition oscillates with time through a non-trivial dependence of $\omega_i(t)$ on scale factor $a(t)$ of the expansing Universe.

Further, the above expression also carries the information about the different phases of the Universe. For instance, in radiation- and matter-dominated era of the Universe, the scale factor behaves as $a(t) = \beta t^\alpha$ where $\alpha = \frac{1}{2}, \frac{2}{3}$ respectively. For $\alpha = \frac{1}{2}$, we obtain the following expression

$$\begin{aligned} \int_0^t dt' \omega_i(t') &= \left[\frac{\tilde{\varepsilon}^2}{2m_i} \log \left| \frac{\sqrt{\frac{m_i^2 t + \tilde{\varepsilon}^2}{t}} + m_i}{\sqrt{\frac{m_i^2 t + \tilde{\varepsilon}^2}{t}} - m_i} \right| + \sqrt{t(m_i^2 t + \tilde{\varepsilon}^2)} \right], \ \tilde{\varepsilon} = \frac{\varepsilon}{\beta} \\ &= m_i t \left[\frac{\zeta_i}{2} \log \left| \frac{\sqrt{1+\zeta_i} + 1}{\sqrt{1+\zeta_i} - 1} \right| + \sqrt{1+\zeta_i} \right], \ \zeta_i = \frac{\tilde{\varepsilon}^2}{m_i^2 t}. \end{aligned} \tag{4.53}$$

In $\zeta_i \to 0$ limit, the expression inside the parenthesis behaves as

$$1 + \frac{\zeta_i}{2} \left[\log\left(\frac{4}{\zeta_i} \right) + 1 \right] + \frac{\zeta_i^2}{8} + \ldots \tag{4.54}$$

whereas in $\zeta_i \to \infty$, it behaves as $2\sqrt{\zeta_i} + \frac{1}{3}\sqrt{\frac{1}{\zeta_i}}$. This shows that the transition probability among two soft neutrinos behaves as

$$\begin{aligned} \mathcal{P}^{(\text{soft})}_{\nu_\alpha \to \nu_\beta}(t) &\to \sum_{i,j} U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} e^{-i \frac{\varepsilon^2}{2\beta^2} \left(\frac{1}{m_i} - \frac{1}{m_j} \right)} e^{-i(m_i - m_j)t} \\ &\times \left(\frac{4m_i^2 t \beta^2}{\varepsilon^2} \right)^{i \frac{\varepsilon^2}{2m_i^2 t \beta^2}} \left(\frac{4m_j^2 t \beta^2}{\varepsilon^2} \right)^{-i \frac{\varepsilon^2}{2m_j^2 t \beta^2}}, \end{aligned} \tag{4.55}$$

in leading order. The above expression also holds in $t \gg \beta^{-2}$ region. This is very well expected since β^{-2} is the only non-trivial time-scale in neutrino oscillation in radiation-dominated Universe apart. On the other hand, for high energy neutrinos ($\varepsilon \to \infty$), the transition probability behaves as

$$\mathcal{P}_{\nu_\alpha \to \nu_\beta}(t) \to \sum_{i,j} U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} e^{-\frac{\beta}{3\varepsilon}(m_i^2 - m_j^2) t^{\frac{3}{2}}}, \tag{4.56}$$

in the leading order of the exponential term. This also holds for finite-energy neutrinos for $t \ll \beta^{-2}$ domain. The series expansion in (4.54) is valid provided $\beta \neq 0$.

On the other hand, for the early Universe, the scale factor behaves as $a(t) = e^{Ht}$ where H is the Hubble constant, hence,

$$
\begin{aligned}
\int_0^t dt'\omega_i(t') &= \int_0^t dt'\sqrt{\frac{\varepsilon^2}{e^{2Ht'}} + m_i^2} \\
&= \left[\frac{m_i}{2H}\log\left|\frac{\sqrt{\varepsilon^2 e^{-2Ht'} + m_i^2} + m_i}{\sqrt{\varepsilon^2 e^{-2Ht'} + m_i^2} - m_i}\right| - \frac{\sqrt{\varepsilon^2 e^{-2Ht'} + m_i^2}}{H}\right]_0^t, \ \varepsilon > 0 \\
&= \frac{m_i}{H}\left[\frac{1}{2}\log\left|\frac{\sqrt{\chi_i^2+1}+1}{\sqrt{\chi_i^2+1}-1}\right| - \sqrt{\chi_i^2+1} - \frac{1}{2}\log\left|\frac{\sqrt{\bar{\chi}_i^2+1}+1}{\sqrt{\bar{\chi}_i^2+1}-1}\right| \right. \\
&\quad \left. + \sqrt{\bar{\chi}_i^2+1}\right], \ \chi_i = \frac{\varepsilon}{m_i}e^{-Ht}, \ \bar{\chi}_i = \frac{\varepsilon}{m_i},
\end{aligned}
\tag{4.57}
$$

which carries the information of H. On the other hand, for soft particles, we obtain

$$
\int_0^t dt'\omega_i(t') = \lim_{\varepsilon\to 0}\int_0^t dt'\sqrt{\frac{\varepsilon^2}{e^{2Ht'}} + m_i^2} = m_i t. \tag{4.58}
$$

For high-energy neutrinos in $t \ll \frac{1}{H}$ domain, the expression inside the parenthesis in (4.57) behaves as

$$
\frac{\varepsilon}{H}\left[[1-e^{-Ht}] + \frac{1}{2\bar{\chi}_i^2}[e^{Ht}-1] - \frac{1}{24\bar{\chi}_i^4}[e^{3Ht}-1] + \dots\right], \tag{4.59}
$$

and as a result of that in leading order, the transition probability behaves as

$$
\mathcal{P}_{\nu_\alpha\to\nu_\beta}(t) \to \sum_{i,j} U_{\alpha i}U_{\beta i}^* U_{\alpha j}^* U_{\beta j} e^{-i\left(\frac{e^{Ht}-1}{2H\varepsilon}\right)(m_i^2-m_j^2) + i\left(\frac{e^{3Ht}-1}{24H\varepsilon^3}\right)(m_i^4-m_j^4)+\dots}. \tag{4.60}
$$

On the other hand, for $t \gg \frac{1}{H}$, the expression inside the parenthesis in (4.57) behaves as

$$
\frac{m_i}{H}\left[Ht - \frac{\bar{\chi}_i^2}{4}e^{-2Ht} + \dots\right] \tag{4.61}
$$

and as a result of that in leading order, the transition probability behaves as

$$
\mathcal{P}_{\nu_\alpha\to\nu_\beta}(t) \to \sum_{i,j} U_{\alpha i}U_{\beta i}^* U_{\alpha j}^* U_{\beta j} e^{-i(m_i-m_j)t + i\frac{e^{-2Ht}\varepsilon^2}{4H}\left(\frac{1}{m_i}-\frac{1}{m_j}\right)+\dots}. \tag{4.62}
$$

Thus, we obtain the qualitative corrections due to the expansion of the Universe in the neutrino oscillation with non-trivial inverse time-scales β, H in various domains of coordinate time t.

Hence, in the FRW geometry, the transition probability in the neutrino flavor oscillation carries the information about the expansion of the Universe with a great number of details of the scale factor of the background geometry.

5. Probing torsion in the neutrino oscillation

In order to probe torsion in the neutrino oscillation, we need to look at the effect of torsion in the Dirac equation in the FRW cosmology in the presence of torsion. A detailed discussion on general relativity with the torsion is given in the Appendix A. Further, an ansatz for torsion compatible with the FRW geometry is also discussed in the Appendix A.

5.1. Effect of torsion in the Dirac equation

Recall that the Dirac equation in curved spacetime is (with metric sign being $(1,-1,-1,-1)$)

$$[ie^{\mu}_{\ a}\gamma^a(\partial_\mu + \Omega_\mu) - m]\Psi = 0, \tag{5.1}$$

with

$$\Omega_\mu = \frac{1}{8}\omega_{\mu bc}[\gamma^b, \gamma^c], \tag{5.2}$$

where effect of torsion is taken into account by including spin-tensor (see appendix for details) compatible with symmetries of FRW geometry

$$\begin{aligned}\omega_{\mu ab} &= \eta_{ac}e_\nu^{\ c}e^\sigma_{\ b}\Gamma^\nu_{\ \sigma\mu} + \eta_{ac}e_\nu^{\ c}\partial_\mu e^\nu_{\ b} \\ &= \tilde{\omega}_{\mu ab} + \eta_{ac}e_\nu^{\ c}e^\sigma_{\ b}K^\nu_{\ \sigma\mu}.\end{aligned} \tag{5.3}$$

Note that using the ansatz in [28]

$$\begin{aligned}\eta_{ac}e_\nu^{\ c}e^\sigma_{\ b}K^\nu_{\ \sigma\mu} &= 2\eta_{ac}e_\nu^{\ c}e^\sigma_{\ b}\phi(h^\nu_{\ \mu}u_\sigma - h_{\sigma\mu}u^\nu) \\ &= 2\eta_{ac}e_\nu^{\ c}e^\sigma_{\ b}\phi(g^\nu_{\ \mu}u_\sigma - g_{\sigma\mu}u^\nu) \\ &= 2\phi\eta_{ac}(e_\mu^{\ c}e^\sigma_{\ b}u_\sigma - u^\nu e_\nu^{\ c}e_{\mu b}) \\ &= 2\phi\delta^c_a(e_{\mu c}u^\sigma e_{\sigma b} - u^\nu e_{\nu c}e_{\mu b}) \\ &= 4\phi u^\nu e_{\mu[a}e_{\nu b]},\end{aligned} \tag{5.4}$$

where $\phi(t)$ is a function of time-coordinate only. In co-moving frame with velocity 1-form $u_\mu = (1,0,0,0)$, we obtain

$$\omega_{\mu ab} = \tilde{\omega}_{\mu ab} + 4\phi e_{\mu[a}e^0_{\ b]}, \tag{5.5}$$

which leads to

$$\begin{aligned}ie^\mu_{\ c}\gamma^c\Omega_\mu &= \frac{i}{8}\omega_{cab}(\gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a) = \frac{i}{8}\tilde{\omega}_{cab}(\gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a) \\ &\quad + \frac{i\phi}{2}e^\mu_{\ c}e_{\mu[a}e^0_{\ b]}(\gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a) \\ &= \frac{i}{8}\tilde{\omega}_{cab}(\gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a) + \frac{i\phi}{2}\eta_{c[a}e^0_{\ b]}(\gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a) \\ &= \frac{i}{8}\tilde{\omega}_{cab}(\gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a) + \frac{i\phi}{4}[(\gamma_a\gamma^a\gamma^b - \gamma_a\gamma^b\gamma^a)e^0_{\ b} \\ &\quad - (\gamma_b\gamma^a\gamma^b - \gamma_b\gamma^b\gamma^a)e^0_{\ a}] \\ &= \frac{i}{8}\tilde{\omega}_{cab}(\gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a) + \frac{i\phi}{2}(4\gamma^b - 2\gamma^b + 4\gamma^b)e^0_{\ b} \\ &= \frac{i}{8}\tilde{\omega}_{cab}(\gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a) + 3i\phi\gamma^t.\end{aligned} \tag{5.6}$$

Effect of $3i\phi\gamma^t$ can be thought of as a transformation of fields in the form $\Psi \to \Psi\, e^{-3\int^t dt'\phi(t')}$, $\bar{\Psi} \to \bar{\Psi}\, e^{3\int^t dt'\phi(t')}$ which modifies the exponent in (4.50).

5.2. Effect of torsion in the neutrino oscillation

Though structurally the Dirac equation remains the same even in the presence of torsion in FRW cosmology with modified $\omega_{\mu ab}$ (shown in (5.5)), however, the time evolution of the states will be affected because of the modified exponent shown in the previous section. Since the scale factor itself is a solution of equations (A.62), (A.63) and (A.64) (shown in the appendix), hence, the scale factor will be affected by torsion whose signature can be observed from the time evolution factor $\int_0^t \omega_i(t')dt'$ integral in the exponent in equation (4.51) and so does in equation (4.52) for the probability of the transition of neutrino flavors. Hence, it is concluded that the presence of torsion in FRW cosmology can be probed based on the observation of neutrino oscillation phenomena. It is shown that solution of the Dirac equation in FRW geometry with the torsion, compatible with symmetries of FRW geometry can be obtained through the transformation of fields from its solution without torsion in the form $\Psi \to \Psi\, e^{-3\int^t dt'\phi(t')}$, $\bar{\Psi} \to \bar{\Psi}\, e^{3\int^t dt'\phi(t')}$. This follows from the fact that the covariant derivative contains $-\Omega_\mu$ when it acts on $\bar{\psi}$ since $\gamma^0\Omega_\mu^\dagger\gamma^0 = -\Omega_\mu$. These non-oscillating factors $e^{\pm 3\int^t dt'\phi(t')}$ due to the coupling with the torsion does not violate the unitarity since $\bar{\psi}[\ldots]\psi$ is invariant under the above-mentioned transformation where $[\ldots]$ contains a non-trivial matrix depending on the symmetry group of the theory. This also means the norm of the spinor remains invariant as the expression of norm requires the product of ψ and $\bar{\psi}$ which remains the same under the transformation $\Psi \to \Psi\, e^{-3\int^t dt'\phi(t')}$, $\bar{\Psi} \to \bar{\Psi}\, e^{3\int^t dt'\phi(t')}$ due to the presence of torsion.

6. Discussion

It is shown that the expansion of the Universe in the FRW cosmology affects the neutrino oscillation phenomena in a significant manner due to the cosmological scale factor. Further, the nature of the expansion of the Universe can be found out from the oscillating phase of the probability of transition among different neutrino flavor states. We have also considered an FRW cosmological model with the presence of torsion which is compatible with the symmetries of the FRW cosmology (suggested in [28]). The effect of torsion is non-trivial which can be probed from the observation of neutrino oscillation phenomena, shown through the detailed study of the Dirac equation in FRW spacetime. Our approach of finding the nature of neutrino oscillation in FRW geometry is much more general and covariant in nature since we have not used any sort of assumption unlike in [46–48]. Further, our result shows explicitly that torsion can indeed induce neutrino oscillation which is consistent with the result in [49] though the derivation in [49] is only valid locally. Furthermore, our derivation is exact unlike the approximated one, considered in [50].

For spacetimes with no time-like Killing vector fields like in FRW cosmology, the Hamiltonian operator is an integral of the Hamiltonian density over spacelike hypersurfaces Σ_t and as a result of it, for spacetimes with no time-like Killing vector fields, the Hamiltonian operator must be time-dependent. That is why we obtain the explicit time dependence in the exponential factor of the transition probability in terms of the scale factor. It leads to a time-dependent phase factor rather than an operator of the form $e^{i\int_\gamma P_\mu(x)dx^\mu}$, used in [31,51–53] in order to find the solution of the Dirac-equation in a curved spacetime where $\gamma : \mathbb{R} \mapsto \mathcal{M}$ is a curve on the spacetime manifold connecting two given points. The above-mentioned operator is used in literature in order to solve the Dirac-equation in a curved spacetime through the Hamiltonian-Jacobi method (also known as an eikonal approximation and WKB approximation) with local approximations. Further, in general, $e^{i\int_{\gamma_1} P_\mu(x)dx^\mu} \neq e^{i\int_{\gamma_2} P_\mu(x)dx^\mu}$ where γ_1, γ_2 are two different curves connecting the same two points in spacetime manifold since $P_\mu(x)dx^\mu$ is not a closed-form or in other words, $\nabla_\mu P_\nu - \nabla_\nu P_\mu \neq 0$ in general. This is consistent with the fact that parallel transports of a spinor along two different curves connecting the same two points are different in curved spacetime in general. However, the transition probability is a physical observable which must be unique and well-defined that can not be obtained from these approaches as a specific path can not be singled out in quantum theory. Moreover, these approaches also require considering certain local approximations in order to show the neutrino oscillation which is certainly ambiguous. In many cases, the trajectories of neutrinos are considered to be null trajectories to show the effect of curved spacetime in the neutrino oscillation which is also mathematically inconsistent since the neutrino oscillation itself suggests neutrinos are massive fermions. These points are often ignored in the literature while making the local approximations which we do not consider here.

The direct effect of the scale factor of our expanding Universe in the neutrino oscillation can be used as a tool to probe various features of the Universe including the torsion. Since the scale factor of the Universe itself depends on the equation of state of the matter and the radiation contained in the Universe, hence, these features can also be probed through the neutrino oscillation by considering our results. Hence, our results are important for future prospects and observations in Cosmology. Though our discussion does not consider the interactions between neutrinos and other particles, allowed by the Standard model of particle physics, however, those interactions can be added in a covariant manner by extending our mathematical techniques to the interacting field theories in curved spacetime.

CRediT authorship contribution statement

There is only one author of this manuscript. This work is entirely done by him.

Declaration of competing interest

The Author declares that there is no conflict of interest.

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Appendix A

A.1. Inclusion of torsion in GR

Covariant derivative or connection in differential geometry is introduced in following way (with metric signature being $(-1, 1, 1, 1)$)

$$\nabla_b v_a = \partial_b v_a - \Gamma^c{}_{ab} v_c. \tag{A.1}$$

In GR connection $\Gamma^c{}_{ab}$ is considered to be symmetric but this may not be the case in general. In that case, torsion tensor is defined as

$$S^a{}_{bc} \equiv \Gamma^a{}_{[bc]}, \tag{A.2}$$

which is anti-symmetric in lower two indices.

Considering the metric-compatibility we can write

$$\Gamma^a{}_{bc} = \tilde{\Gamma}^a{}_{bc} + S^a{}_{bc} + S_{bc}{}^a + S_{cb}{}^a, \tag{A.3}$$

where $\tilde{\Gamma}^a{}_{bc}$ are the Christoffel symbols, the metric-affine connection defined in GR. Torsion can also equivalently introduced in terms of defined contorsion tensor [54]

$$K^a{}_{bc} = S^a{}_{bc} + S_{bc}{}^a + S_{cb}{}^a \implies K_{abc} = -K_{bac}. \tag{A.4}$$

The possible contractions of contorsion tensors are

$$K^b{}_{ab} = 2S_a = 2S^b{}_{ab}, \;\; K^b{}_{ba} = 0. \tag{A.5}$$

Locally, it is always possible to split spacetime into the 3-dimensional instantaneous rest space of the observer and the world-lines of the observer, perpendicular to 3-dim hypersurface. Consider the 4-velocity of the observer to be u^a which satisfies $u^a u_a = -1$. To project any geometrical quantity on 3-dim hypersurface we will use the following metric

$$h_{ab} = g_{ab} + u_a u_b, \tag{A.6}$$

where g_{ab} is the metric of the spacetime. Observer's motion can be described by using $1+3$ splitting of the covariant derivative of the 4-velocity:

$$\begin{aligned} \nabla_b u_a &= \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab} - A_a u_b \\ &\equiv \mathcal{D}_b u_a - A_a u_b, \end{aligned} \tag{A.7}$$

where $\mathcal{D}_b u_a$ is the 3-dim covariant derivative of 4-velocity and $\sigma_{ab} = \mathcal{D}_{(b}u_{a)} - \frac{1}{3}\Theta h_{ab}, \;\omega_{ab} = \mathcal{D}_{[b}u_{a]}, \;A_a \equiv u^b \nabla_b u_a$. One can easily show following features using the definitions

$$h_{ab}u^b = 0 = \sigma_{ab}u^b = A_a u^a. \tag{A.8}$$

Since, in a general spacetime with torsion

$$\nabla_a u_a = \tilde{\nabla}_b u_a - S_{cab}u^c - 2S_{(ab)c}u^c, \tag{A.9}$$

where $\tilde{\nabla}$ denotes covariant derivative *w.r.t.* Christoffel symbols. Additionally, the spatial covariant derivative breaks into

$$\mathcal{D}_b u_a = \tilde{\mathcal{D}}_b u_a - h_a{}^c h_b{}^d S_{ecd}u^e - 2h_a{}^c h_b{}^d S_{(cd)e}u^e, \tag{A.10}$$

from which it can be shown in few simple mathematical steps that

$$\begin{aligned} \Theta &= \tilde{\Theta} + 2S_a u^a \\ \omega_{ab} &= \tilde{\omega}_{ab} - h_{[a}^c h_{b]}^d S_{ecd}u^e \\ \sigma_{ab} &= \tilde{\sigma}_{ab} - 2h_{(a}{}^c h_{b)}{}^d S_{(cd)e}u^e - \frac{2}{3}S_c u^c h_{ab} \\ A_a &= \tilde{A}_a + 2S_{(bc)a}u^b u^c, \end{aligned} \tag{A.11}$$

where $\tilde{}$ quantities are defined without torsion.

Similar way, from the definition it follows that [55–57]

$$\begin{aligned}\mathcal{R}^a{}_{bcd} &= \tilde{\mathcal{R}}^a{}_{bcd} + \partial_c K^a{}_{bd} - \partial_d K^a{}_{bc} + \tilde{\Gamma}^a{}_{ec} K^e{}_{bd} - \tilde{\Gamma}^e{}_{bc} K^a{}_{ed} \\ &\quad - \tilde{\Gamma}^a{}_{ed} K^e{}_{bc} + \tilde{\Gamma}^e{}_{bd} K^a{}_{ec} + K^e{}_{bd} K^a{}_{ec} - K^e{}_{bc} K^a{}_{ed} \\ \Longrightarrow \mathcal{R}_{ab} &= \tilde{\mathcal{R}}_{ab} + \partial_c K^c{}_{ab} - \partial_b K^c{}_{ac} + \tilde{\Gamma}^d{}_{cd} K^c{}_{ab} - \tilde{\Gamma}^c{}_{ad} K^d{}_{cb} \\ &\quad - \tilde{\Gamma}^d{}_{cb} K^c{}_{ad} + \tilde{\Gamma}^c{}_{ab} K^d{}_{cd} + K^c{}_{ab} K^d{}_{cd} - K^c{}_{ad} K^d{}_{cb} \\ \mathcal{R} &= \tilde{\mathcal{R}} + g^{ab}\partial_c K^c{}_{ab} - g^{ab}\partial_b K^c{}_{ac} + \tilde{\Gamma}^c{}_{ac} K^{ab}{}_b - \tilde{\Gamma}^a{}_{bc} K^c{}_a{}^b \\ &\quad - \tilde{\Gamma}^c{}_{ab} K^{ab}{}_c + g^{ab}\tilde{\Gamma}^c{}_{ab} K^d{}_{cd} + K^{ab}{}_b K^c{}_{ac} - K^{ab}{}_c K^c{}_{ab}.\end{aligned} \tag{A.12}$$

Note that,

$$\mathcal{R}_{abcd} = -\mathcal{R}_{bacd},\ \mathcal{R}_{abcd} = -\mathcal{R}_{abdc}, \tag{A.13}$$

however,

$$\mathcal{R}_{abcd} \neq \mathcal{R}_{cdab},\ \mathcal{R}_{ab} \neq \mathcal{R}_{ba}. \tag{A.14}$$

Einstein-Hilbert action with presence of cosmological constant is

$$S = \int \left[\frac{1}{2\kappa}(\mathcal{R} - 2\Lambda) + \mathcal{L}_M\right]\sqrt{-g}d^4x, \tag{A.15}$$

where $\kappa = \frac{8\pi G}{c^4}$. Variation of action *w.r.t.* metric tensor and contorsion tensor leads to Einstein-Cartan equations [58,59]

$$\begin{aligned} G_{ab} &\equiv \mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}g_{ab} = \kappa T_{ab} - \Lambda g_{ab},\ T_{ab} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{ab}} \\ S_{abc} &= -\frac{\kappa}{4}(2s_{bca} + g_{ca}s_b - g_{ab}s_c),\ s_a = s^b{}_{ab},\ s_{ab}{}^c \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta K^{ab}{}_c}, \end{aligned} \tag{A.16}$$

where s_{abc} is known as spin-tensor.

A.2. Cosmology with torsion

Here, we first present two most important kinematic equations in the cosmological model which are Friedmann and Raychaudhuri equations in the presence of torsion. One can derive the Friedmann equation from the line element of FRW metric with consecutive contractions of 3-dim curvature tensor, which is presented as follows

$$R_{abcd} = h_a^{\ q} h_b^{\ s} h_c^{\ d} h_d^{\ p} \mathcal{R}_{qsfp} - \mathcal{D}_c u_a \mathcal{D}_d u_b + \mathcal{D}_d u_a \mathcal{D}_c u_b, \tag{A.17}$$

where $\mathcal{D}_b u_a = \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}$. In addition, using the fact that the curvature tensor in 4-dim spacetime satisfies

$$\mathcal{R}_{abcd} = \mathcal{C}_{abcd} + \frac{1}{2}(g_{ac}\mathcal{R}_{bd} + g_{bd}\mathcal{R}_{ac} - g_{bc}\mathcal{R}_{ad} - g_{ad}\mathcal{R}_{bc}) - \frac{\mathcal{R}}{6}(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{A.18}$$

it can be shown that

$$R = 2\left(\kappa T_{(ab)}u^a u^b + \Lambda - \frac{1}{3}\Theta^2 + \sigma^{ab}\sigma_{ab} - \omega_{ab}\omega^{ab}\right), \tag{A.19}$$

where R is the spatial curvature. In FRW geometry $R = \frac{6k}{a^2}$ where $k = 0, \pm 1$ depending on flat, open and closed Universe. Considering $H = \frac{\Theta}{3}$ and the fact that for maximally symmetric spacetime (FRW geometry is an example of that) $\sigma = 0 = \omega$ implies that

$$H^2 = \frac{\kappa}{3} T_{(ab)} u^a u^b - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{A.20}$$

Raychaudhuri equation turns out to be [60]

$$\begin{aligned}\dot{\Theta} = &-\frac{1}{3}\Theta^2 - \mathcal{R}_{(ab)} u^a u^b - 2(\sigma^{ab}\sigma_{ab} - \omega_{ab}\omega^{ab}) + \mathcal{D}_a A^a + A_a A^a \\ &+ \frac{2}{3} S_a u^a - 2S_{\langle ab\rangle c}\sigma^{ab} u^c + 2S_{[ab]c}\omega^{ab} u^c - 2S_{(ab)c} u^a u^b A^c \end{aligned} \tag{A.21}$$

$$S_{\langle ab\rangle c}\sigma^{ab} = S_{abc}\sigma^{ab} + \frac{1}{3} u^a u^b \Theta S_{abc}.$$

Since, in FRW type models $\sigma = 0 = \omega = A$ which implies the above equation reduces to the following form

$$\begin{aligned} &\dot{\Theta} = -\frac{1}{3}\Theta^2 - \mathcal{R}_{(ab)} u^a u^b + \frac{2}{3}\Theta S_a u^a \\ \implies &\dot{H} = -H^2 - \frac{1}{3}\kappa T_{ab} u^a u^b - \frac{1}{6} T + \frac{1}{3}\Lambda + \frac{2}{3} H S_a u^a. \end{aligned} \tag{A.22}$$

Using Bianchi identities

$$\begin{aligned} \nabla_{[e}\mathcal{R}^{ab}{}_{cd]} &= 2\mathcal{R}^{ab}{}_{f[e} S^f{}_{cd]} \\ \mathcal{R}^a{}_{[bcd]} &= -2\nabla_{[b} S^a{}_{cd]} + 4S^a{}_{e[b} S^e{}_{cd]}, \end{aligned} \tag{A.23}$$

and the symmetries of curvature tensor, it can be show that

$$\begin{aligned} &\nabla^b G_{ba} = 2\mathcal{R}_{bc} S^{cb}{}_a + \mathcal{R}_{bcda} S^{dcb} \\ \implies &\nabla^b T_{ba} = \frac{1}{\kappa}(2\mathcal{R}_{bc} S^{cb}{}_a + \mathcal{R}_{bcda} S^{dcb}). \end{aligned} \tag{A.24}$$

Further, it can also be shown that

$$\begin{aligned} &G_{[ab]} = \nabla_a S_b - \nabla_b S_a + \nabla^c S_{cab} - 2S^c S_{cab} \\ \implies &\nabla^b G_{ab} = \nabla^b G_{ba} + 2(\nabla^b\nabla_a S_b - \nabla^b\nabla_b S_a + \nabla^b\nabla^c S_{cab}) - 4\nabla^b(S^c S_{cab}) \\ \implies &\nabla^b T_{ab} = \nabla^b T_{ba} + \frac{1}{\kappa}\Big[2(\nabla^b\nabla_a S_b - \nabla^b\nabla_b S_a + \nabla^b\nabla^c S_{cab}) - 4\nabla^b(S^c S_{cab})\Big]. \end{aligned} \tag{A.25}$$

Assuming isotropic cosmology, if we replace torsion tensor by spin-tensor we obtain Raychaudhuri equation in the following form

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \kappa T_{(ab)} u^a u^b - \frac{1}{2}\kappa T - \frac{1}{6}\kappa\Theta s_a u^a, \tag{A.26}$$

and

$$\begin{aligned} \nabla^b T_{ba} &= \mathcal{R}_{bc} s_a{}^{bc} + \frac{1}{2}\mathcal{R} s_a - \frac{1}{4}\mathcal{R}_{bcda} s^{cbd} - \mathcal{R}_{bc} s^b \\ \nabla^b T_{ab} &= \nabla^b T_{ba} - \nabla^b\nabla^c s_{abc} - \frac{\kappa}{2}\nabla^b s^c s_{abc}. \end{aligned} \tag{A.27}$$

Using Cartan field equations, we can write

$$T_{[ab]} = -\frac{1}{2}\nabla^c s_{abc} - \frac{\kappa}{4} s^c s_{abc}. \tag{A.28}$$

On the other hand, symmetric part of T_{ab} can be decomposed in following way

$$\begin{aligned} T_{(ab)} &= \rho u_a u_b + p h_{ab} + 2u_{(a}q_{b)} + \pi_{ab} \\ \implies \rho &= T_{ab}u^a u^b \text{ (effective matter energy density)} \\ p &= \frac{1}{3} T_{ab}h^{ab} \text{ (effective isotropic pressure)} \\ q_a &= -h_a^{\ b} T_{ac} u^c \text{ (effective total-energy flux vector)} \\ \pi_{ab} &= h^{\ c}_{\langle a} h^{\ d}_{b\rangle} T_{cd} \text{ (effective anisotropic stress-tensor).} \end{aligned} \tag{A.29}$$

A.3. Torsion in FRW cosmology

In the previous subsection, Friedmann and Raychaudhuri equations were presented without any restriction on the form of the torsion. Here, we have considered an ansatz, proposed based on the compatibility with the isotropic FRW cosmological model.

In order to preserve homogeneity and isotropy in a maximally symmetric 3-dim space, we consider the following ansatz [28] for the torsion tensor:

$$S_{abc} = \phi(h_{ab}u_c - h_{ac}u_b) = \phi(g_{ab}u_c - g_{ac}u_b). \tag{A.30}$$

Based on the homogeneity requirement, ϕ can only be a scalar function depending on time. The fact that the choice of our ansatz respects the isotropy of 3-dim space becomes clear by taking the contraction to produce the torsion vector:

$$S_a = -3\phi u_a, \tag{A.31}$$

which shows $\phi > 0$ implies S^a and u^a are anti-parallel and for $\phi < 0$, S^a and u^a are parallel.

$$|S| = \sqrt{S_a S^a} = 3\phi. \tag{A.32}$$

Using the ansatz, spin-tensor and vector can be written as

$$s_{abc} = \frac{4}{\kappa}\phi(h_{ac}u_b - h_{bc}u_a) \implies s_a = \frac{12}{\kappa}\phi u_a. \tag{A.33}$$

Next step is to formulate the continuity equation for the scalar function $\phi(t)$:

$$\begin{aligned} \nabla^a S_a &= -3\nabla^a(\phi u_a) \\ &= -3u^a\nabla_a\phi - 3\phi\nabla^a u_a \\ &= -3\frac{d}{d\tau}\phi - 3\phi\nabla^a u_a. \end{aligned} \tag{A.34}$$

Further, we demand that

$$\nabla^a S_a \propto \nabla^a u_a \implies \nabla^a S_a = n\phi\nabla^a u_a, \tag{A.35}$$

where

$$n = -\left(3 + \frac{1}{\phi}\frac{d}{d\tau}\phi \times \frac{1}{\nabla^a u_a}\right). \tag{A.36}$$

The physical motivation behind this demand is that the torsion vector to remain with its world-line, which consistent with the relation (A.31) which shows S_a lies in the same world-line with 4-velocity vector. Hence, imposing this restriction, we try to ensure that these two vectors will continue to belong in the same world-line.

Combining (A.34) and (A.35), we obtain

$$\begin{aligned} \nabla^a S_a &= n\phi\nabla^a u_a \\ \implies -3\dot{\phi} &= (n+3)\phi\nabla^a u_a \\ \dot{\phi} &= -(n+3)\phi\frac{\nabla^a u_a}{3} \\ &= -(n+3)\phi H \equiv -\nu\phi H, \end{aligned} \tag{A.37}$$

where we have used the fact that $\nabla^a u_a = \Theta$. This can also be rewritten as

$$\dot{\phi} = -\nu\phi\frac{\dot{a}}{a} \implies \phi(t) = \phi_i\frac{a_i^\nu}{a^\nu(t)}, \tag{A.38}$$

where ϕ_i and a_i are the initial values.

Finally employing the ansatz, we can write the Raychaudhuri equation as

$$\dot{H} = -H^2 - \frac{1}{3}\kappa T_{ab}u^a u^b - \frac{1}{6}\kappa T + \frac{1}{3}\Lambda + 2\phi H. \tag{A.39}$$

Using following facts

$$\begin{aligned} \nabla_b S_a &= 3\dot{\phi}u_a u_b - \phi\Theta h_{ab} = \nabla_a S_b \\ S^c S_{cab} &= -3\phi^2(u^c h_{ca}u_b - u^c h_{cb}u_a) = 0 \\ \nabla^c S_{cab} &= \nabla^c[\phi(g_{ca}u_b - g_{cb}u_a)] \\ &= (\nabla_a\phi u_b - \nabla_b\phi u_a) = -\dot{\phi}(u_a u_b - u_b u_a) = 0, \end{aligned} \tag{A.40}$$

we can deduce that

$$G_{[ab]} = 0 = R_{[ab]} = T_{[ab]}. \tag{A.41}$$

Therefore, in FRW type cosmological model the Einstein, Ricci and the energy-momentum tensor is symmetric, however, torsion will affect the conservation law in following way

$$\begin{aligned} \nabla^b T_{ab} &= -4\phi(T_{ab}u^b - \frac{\Lambda}{\kappa}u_a) \\ u^a\nabla^b T_{ab} &= -4\phi(T_{ab}u^a u^b + \frac{\Lambda}{\kappa}). \end{aligned} \tag{A.42}$$

To maintain the isotropy and homogeneity of 3-dim space, we will restrict energy-momentum tensor to be that of perfect fluid:

$$T_{ab} = \rho u_a u_b + p h_{ab}. \tag{A.43}$$

Substituting above expression in (A.42) leads to

$$\dot{\rho} = -3H(\rho + P) + 4\phi(\rho + \frac{\Lambda}{\kappa}). \tag{A.44}$$

Further, the Friedmann and Raychaudhuri equations will be reduced to following form

$$\begin{aligned} H^2 &= \frac{\kappa}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \\ \dot{H} &= -H^2 - \frac{\kappa}{6}(\rho + 3P) + \frac{\Lambda}{3} + 2\phi H. \end{aligned} \tag{A.45}$$

Combining above two equations leads to

$$\begin{aligned} 2H\dot{H} &= -2H^3 - \frac{\kappa}{3}H(\rho + 3P) + \frac{2\Lambda H}{3} + 4\phi H^2 \\ \frac{\kappa}{3}\dot{\rho} + \frac{2k}{a^2}H &= -2H^3 - \frac{\kappa}{3}H(\rho + 3P) + \frac{2\Lambda H}{3} + 4\phi H^2 \\ \Longrightarrow \dot{\rho} &= -3H(\rho + P) + \frac{12}{\kappa}\phi H^2. \end{aligned} \tag{A.46}$$

Although $\dot{\rho}$ has been calculated in (A.44) and (A.46) in two different ways, they must be consistent with each other which leads to the following conclusion that

$$\frac{\phi k}{\kappa a^2} = 0, \tag{A.47}$$

which shows that isotropy, 3-dim curvature, and torsion are incompatible with each other in the sense, that presence of torsion demands the spatial curvature of the FRW cosmological model to be zero exactly. This is consistent with the observation that the Universe is spatially Euclidean flat at a significant level.

A.4. Decomposition of energy-momentum tensor

As it has been discussed earlier that

$$\begin{aligned} \mathcal{R}_{ab} &= \tilde{\mathcal{R}}_{ab} + \tilde{\nabla}_c K^c{}_{ab} - \tilde{\nabla}_b K^c{}_{ac} + K^d{}_{ab}K^c{}_{dc} - K^d{}_{ac}K^c{}_{db} \\ \mathcal{R} &= \tilde{\mathcal{R}} - 2\tilde{\nabla}_a K^{ba}{}_b - K^{ba}{}_b K^c{}_{ac} + K^{ab}{}_c K^c{}_{ba}, \end{aligned} \tag{A.48}$$

but we also want to know what is the affect of torsion in energy-momentum tensor which can be found from following decomposition

$$T_{ab} = \tilde{T}_{ab} + \Delta_{ab}. \tag{A.49}$$

Incorporating this into Einstein-Cartan field equation

$$\mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}g_{ab} = \kappa T_{ab} - \Lambda g_{ab}, \tag{A.50}$$

we obtain

$$\tilde{\mathcal{R}}_{ab} - \frac{1}{2}\tilde{\mathcal{R}}g_{ab} + Q_{ab} - \frac{1}{2}Qg_{ab} = \kappa(\tilde{T}_{ab} + \Delta_{ab}) - \Lambda g_{ab}, \tag{A.51}$$

where

$$\begin{aligned} Q_{ab} &= \tilde{\nabla}_c K^c{}_{ab} - \tilde{\nabla}_b K^c{}_{ac} + K^d{}_{ab}K^c{}_{dc} - K^d{}_{ac}K^c{}_{db} \\ Q &= -2\tilde{\nabla}_a K^{ba}{}_b - K^{ba}{}_b K^c{}_{ac} + K^{ab}{}_c K^c{}_{ba}. \end{aligned} \tag{A.52}$$

However, '$\sim$' quantities satisfy the Einstein field equations

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{R}g_{ab} = \kappa \tilde{T}_{ab} - \Lambda g_{ab}, \tag{A.53}$$

which leads to

$$\kappa \Delta_{ab} = Q_{ab} - \frac{1}{2}Qg_{ab}. \tag{A.54}$$

According to our ansatz,

$$\begin{aligned} K_{abc} &= 2\phi(h_{bc}u_a - h_{ac}u_b) \\ &= 2\phi(g_{bc}u_a - g_{ac}u_b) \\ \implies K_a &= K^b{}_{ab} = -6\phi u_a, \end{aligned} \tag{A.55}$$

from which it can be derived that

$$\begin{aligned} Q_{ab} &= -(6\dot{\phi} - 12\phi^2 + 2\phi\Theta)u_a u_b + (2\dot{\phi} - 12\phi^2 + \frac{10}{3}\phi\Theta)h_{ab} \\ Q &= 12(\dot{\phi} - 4\phi^2 + \phi\Theta) \\ \implies \kappa\Delta_{ab} &= -(12\phi^2 - 12\phi H)u_a u_b - (4\dot{\phi} - 12\phi^2 + 8\phi H)h_{ab}. \end{aligned} \tag{A.56}$$

Considering matter as a perfect fluid, we can write

$$\tilde{T}_{ab} = \tilde{\rho}u_a u_b + \tilde{p}h_{ab}, \tag{A.57}$$

which makes

$$T_{ab} = \left(\tilde{\rho} - 12\frac{12\phi^2}{\kappa} + 12\frac{\phi H}{\kappa}\right)u_a u_b + \left(\tilde{p} - 4\frac{\dot{\phi}}{\kappa} + 12\frac{\phi^2}{\kappa} - 8\frac{\phi H}{\kappa}\right)h_{ab}. \tag{A.58}$$

This essentially makes the effective matter energy density and isotropic pressure to be

$$\begin{aligned} \rho &= \tilde{\rho} - 12\frac{\phi^2}{\kappa} + 12\frac{\phi H}{\kappa} \\ p &= \tilde{p} + 12\frac{\phi^2}{\kappa} - 8\frac{\phi H}{\kappa}. \end{aligned} \tag{A.59}$$

Hence, the hidden contributions of torsion in the Friedmann and Raychaudhuri equations are

$$\begin{aligned} H^2 &= \frac{\kappa}{3}\tilde{\rho} - \frac{k}{a^2} + \frac{\Lambda}{3} + 4\phi H - 4\phi^2 \\ \dot{H} &= -H^2 - \frac{\kappa}{6}(\tilde{\rho} + 3\tilde{p}) + \frac{\Lambda}{3} + 4\phi H - 4\phi^2 + 2\dot{\phi}. \end{aligned} \tag{A.60}$$

Combining above two equations, we obtain

$$\begin{aligned} \dot{H} &= -\frac{\kappa}{2}(\tilde{\rho} + \tilde{p}) + 2\dot{\phi} \\ \implies H &= \tilde{H} + 2\phi. \end{aligned} \tag{A.61}$$

Above relation states that positive value of ϕ enhances the effective Hubble parameter and negative value weakens it.

A.5. Solution of the Friedmann equation

For $k = 0$ (spatially flat 3-dim space) and $\tilde{\rho} = \tilde{w}\tilde{p}$, we obtain

$$\begin{aligned} \tilde{H} &= \frac{2}{3(1+\tilde{w})t} \\ \implies \tilde{a}(t) &\propto t^{\frac{2}{3(1+\tilde{w})}}, \end{aligned} \tag{A.62}$$

where the barotropic index $\tilde{w}$ determines the type of matter.

We now find the time evolution of H in the presence of torsion in the case of flat Universe where cosmological constant dominates in which case

$$\begin{aligned} \tilde{H} &= \sqrt{\frac{\Lambda}{3}} \\ H &= 2\phi + \sqrt{\frac{\Lambda}{3}} \\ \implies \dot{H} &= 2\dot{\phi}. \end{aligned} \tag{A.63}$$

Using (A.37), we can also write

$$\begin{aligned} \dot{H} &= -2\nu\phi H \\ \implies \dot{\phi} &= -\nu\phi\left(2\phi + \sqrt{\frac{\Lambda}{3}}\right). \end{aligned} \tag{A.64}$$

Solving the above equation, we obtain

$$\begin{aligned} \phi(t) &= \frac{\nu\sqrt{\frac{\Lambda}{3}}}{\left(\frac{\nu}{\phi_i}\sqrt{\frac{\Lambda}{3}} + 2\nu\right)e^{\nu\sqrt{\frac{\Lambda}{3}}t} - 2\nu} \\ \implies H(t) &= \sqrt{\frac{\Lambda}{3}} + \frac{\nu\sqrt{\frac{\Lambda}{3}}}{\left(\frac{\nu}{H_i - \sqrt{\frac{\Lambda}{3}}}\sqrt{\frac{\Lambda}{3}} + \nu\right)e^{\nu\sqrt{\frac{\Lambda}{3}}t} - \nu}, \end{aligned} \tag{A.65}$$

where $H_i = \phi_i + \sqrt{\frac{\Lambda}{3}}$. For $\nu > 0$, we obtain

$$\lim_{t\to\infty}\phi(t) = 0, \quad \lim_{t\to\infty} H(t) = \sqrt{\frac{\Lambda}{3}}, \tag{A.66}$$

and for $\nu < 0$, we obtain

$$\lim_{t\to\infty}\phi(t) = -\frac{1}{2}\sqrt{\frac{\Lambda}{3}}, \quad \lim_{t\to\infty} H(t) = 0. \tag{A.67}$$

The case of positive value of ν states that after a long time evolution of Universe, torsion vanishes with a non-zero Hubble parameter which is positive definite. This is consistent with the fact that current observations fail to detect the presence of torsion.

Further, it can also be shown that if the initial torsion is very high such that it dominates then equation (A.60) becomes

$$
\begin{aligned}
&(H-2\phi)^2=0\\
&\implies \frac{\dot{a}(t)}{a(t)}=2\phi(t)\\
&\implies a(t)=a(t_0)e^{2\int_{t_0}^{t}\phi(t')dt'}.
\end{aligned}
\tag{A.68}
$$

Hence, the evolution of torsion is fully encapsulated inside the scale factor which can easily be probed by neutrino oscillation. Further, expansion of Universe demands that scale factor be monotonically increasing which means that $\phi(t)$ needs to be positive definite. This also shows S_a and u_a are anti-parallel according to (A.31).

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