Integrable string models of principal chiral model type

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Abstract

We considered of string models on the Riemmann-Cartan space of string coordinates with constant torsion.

We used the invariant local chiral currents of principal chiral models for SU(n) group to construct new integrable string models on the Riemmann space of invariant chiral currents and on the dynamical Casimir operators, considered as the Hamiltonians .

1 Introduction

String model in light-cone gauge is described by Lagrangian

$$L = \frac{1}{2}\int_0^{2\pi} \eta^{\alpha\beta} g_{ab}(\phi(t,x)) \frac{\partial\phi^a(t,x)}{\partial x^\alpha}\frac{\partial\phi^b(t,x)}{\partial x^\beta} dx \qquad (1)$$

Target space local coordinates $\phi^a(x),\ a = 1,...,n$ belong certain given smooth n-dimensional manifold M^n with nongenerated metric tensor

$$g_{ab}(\phi(x)) = \eta_{\mu\nu} e^\mu_a(\phi(x)) e^\nu_b(\phi(x)), \qquad (2)$$

where $\mu, \nu = 1,...,n$ are indexes of tangent space to manifold M^n on some point $\phi^a(x)$. The veilbein $e^\mu_a(\phi)$ and its inverse $e^a_\mu(\phi)$ satisfy to the conditions:

$$e^\mu_a e^b_\mu = \delta^b_a,\ e^\mu_a e^{a\nu} = \eta^{\mu\nu}.$$

Coordinates $x^\alpha, x^0 = t,\ x^1 = x$ belong world sheet with metric tensor $g_{\alpha\beta}$ in conformal gauge. String equations of motion have the form:

$$\eta^{\alpha\beta}[\partial_{\alpha\beta}\phi^a + \Gamma^a_{bc}(\phi)\partial_\alpha\phi^b\partial_\beta\phi^c] = 0,$$

where $\Gamma^a_{bc}(\phi) = \frac{1}{2}e^a_\mu[\frac{\partial e^\mu_b}{\partial\phi^c} + \frac{\partial e^\mu_c}{\partial\phi^b}]$ is connection. In terms of canonical currents $J^\mu_\alpha(\phi) = e^\mu_a(\phi)\partial_\alpha\phi^a$, $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ equations of motion have form:

$$\eta^{\alpha\beta}\partial_\alpha J^\mu_\beta(\phi(t,x)) = 0,\ \partial_\alpha J^\mu_\beta(\phi) - \partial_\beta J^\mu_\alpha(\phi) - C^\mu_{\nu\lambda}(\phi)J^\nu_\alpha(\phi)J^\lambda_\beta(\phi) = 0,$$

where $C^\mu_{\nu\lambda}(\phi) = e^a_\nu e^b_\lambda[\frac{\partial e^\mu_a}{\partial\phi^b} - \frac{\partial e^\mu_b}{\partial\phi^a}]$ is torsion. The Hamiltonian has form:

$$H = \frac{1}{2}\int_0^{2\pi}[\eta^{\mu\nu}J_{0\mu}J_{0\nu} + \eta_{\mu\nu}J^\mu_1 J^\nu_1]dx, \qquad (3)$$

where $J_{0\mu}(\phi) = e^a_\mu(\phi)p_a$, $J^\mu_1(\phi) = e^\mu_a \frac{\partial\phi^a}{\partial x}$ and $p_a(t,x)$ is canonical momentum.

Let us introduce chiral currents:

$$U^\mu = \eta^{\mu\nu} J_{0\nu} + J^\mu_1, \; V^\mu = \eta^{\mu\nu} J_{0\nu} - J^\mu_1$$

The canonical commutation relations of chiral currents U_μ are following:

$$\{U^\mu(x), U^\nu(y)\} = C^{\mu\nu}_\lambda [\frac{3}{2}U^\lambda(x) - \frac{1}{2}V^\lambda(x)]\delta(x-y) - \eta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y), \tag{4}$$

Equations of motion of chiral currents $U_\mu(x)$ in light-cone coordinates $x^\pm = \frac{1}{2}(t \pm x)$, $\frac{\partial}{\partial x^\pm} = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}$ have form: $\partial_- U^\mu = C^\mu_{\nu\lambda}(\phi(x))U^\nu V^\lambda$.

2 String model of principal chiral model type

Let torsion $C_{\mu\nu\lambda} = f_{\mu\nu\lambda}$ are structure constant of simple Lie algebra. We will consider string model with constant torsion in light-cone gauge in target space. This model is coincide to principal chiral model on compact simple Lie group. We can not divide motion on right and left mover because of chiral currents $\partial_- U^\mu = f^\mu_{\nu\lambda}U^\nu V^\lambda$, are not conserve. Correspondent charges are not Casimirs. Evans, Hassan, MacKay, Mountain ([1] and other references therein) constructed local invariant chiral currents as polynomials of initial chiral currents of $SU(n)$, $SO(n)$, $SP(n)$ principal chiral models and they found commutative combination of them. Correspondent charges are Casimir operators of this dynamical systems. This paper was based on the paper of de Azcaraga, Macfarlane, MacKay, Perez Buena ([2] and other references therein) about invariant tensors for simple Lie algebras. Polynomials of local chiral currents was considered by Goldshmidt and Witten [3] (see also [4]). Local conserved chiral charges in principal chiral models was considered by Evans, Hassan, MacKay, Mountain [1]. Integrable string models of hydrodynamic type was considered by author [5], [6].

Let t_μ are $n \otimes n$ traceless hermitian matrix representations of generators Lie algebra:

$$[t_\mu, t_\nu] = 2if_{\mu\nu\lambda}t_\lambda, Tr(t_\mu t_\nu) = 2\delta_{\mu\nu}. \tag{5}$$

There is additional relation for $SU(n)$ algebra:

$$\{t_\mu, t_\nu\} = \frac{4}{n}\delta_{\mu\nu} + 2d_{\mu\nu\lambda}t_\lambda, \; \mu = 1, ..., n^2 - 1. \tag{6}$$

De Azcarraga et. al. gave some examples of invariant tensors of simple Lie algebras and they gave general method to calculate them. Invariant tensors may to construct as invariant symmetric polinomials on $SU(n)$:

$$d_{(\mu_1 ... \mu_M)} = \frac{1}{M!} STr(t_{\mu_1} ... t_{\mu_M}), \tag{7}$$

where STr means of completely symmetrized product of matrices and $d_{(\mu_1 ... \mu_M)}$ is totally symmetric tensor and $M = 2, 3, ..., \infty$. Another family of invariant symmetric tensors [7], [8], such named d-family , based on the product of the symmetric structure constant $d_{\mu\nu\lambda}$ of $SU(n)$ algebra:

$$d_{(\mu_1 ... \mu_M)} = d^{k_1}_{(\mu_1\mu_2} d^{k_2}_{\mu_3 k_1} ... d^{k_{M-3}}_{\mu_{M-1}\mu_M)}, \tag{8}$$

There are $n-1$ primitive invariant tensors on $SU(n)$. The invariant tensors for $M \geq n$ are functions of primitive tensors. De Azcarraga et. al. gave some examples these functions and they gave general method to calculate them. The Casimir operators on $SU(n)$ have form:

$$C_M = d_{(\mu_1 ... \mu_M)} t_{\mu_1} ... t_{\mu_M}.$$

Evans et.al. introduced local chiral currents based on the invariant symmetric polinomials on simple Lie algebra:

$$J_M(U) = STr(U...U) \equiv STrU^m = d_{\mu_1 ... \mu_M} U^{\mu_1} ... U^{\mu_M}, \tag{9}$$

where $U = t_\mu U^\mu$ and $\mu = 1, ..., n^2-1$. We obtained following expression for local invariant chiral currents J_M:

$$J_2 = 2U^2, J_3 = 2d^3U^3, J_4 = 2d^4U^4 + \frac{4}{n}(U^2)^2, J_5 = 2d^5U^5 + \frac{8}{n}U^2(d^3U^3),$$

$$J_6 = 2d^6U^6 + \frac{4}{n}(d^3U^3)^2 + \frac{8}{n}U^2(d^4U^4) + \frac{8}{n^2}(U^2)^3, \tag{10}$$

$$J_7 = 2d^7U^7 + \frac{8}{n}(d^3U^3)(d^4U^4) + \frac{8}{n}U^2(d^5U^5) + \frac{24}{n^2}(U^2)^2(d^3U^3),$$

$$J_8 = 2d^8U^8 + \frac{4}{n}(d^4U^4)^2 + \frac{8}{n}(d^3U^3)(d^5U^5) + \frac{8}{n}U^2(D^6U^6) +$$

$$+ \frac{24}{n^2}U^2(d^3U^3)^2 + \frac{24}{n^2}(U^2)^2(d^4U^4) + \frac{16}{n^3}(U^2)^4.$$

For simplicity, we introduced short notations $U^2 = U_\mu U_\mu$ and

$$d^MU^M = d^{k_1}_{\mu_1\mu_2} d^{k_1k_2}_{\mu_3} ... d^{k_{M-3}}_{\mu_{M-1}\mu_M} U_{\mu_1} U_{\mu_2} ... U_{\mu_M}. \tag{11}$$

The commutation relations of invariant chiral currents show, that these currents are not densities of Casimir operators. Evans et. al postulated simple form of commutation relations for some basis of invariant chiral currents:

$$\{J_M(x), J_N(y)\} = -[MN\, J_{M+N-2}(x) - \frac{MN}{n} J_{M-1}(x) J_{N-1}(x)] \frac{\partial}{\partial x}\delta(x-y) -$$

$$-[\frac{MN(N-1)}{M+N-2}\frac{\partial J_{M+N-2}(x)}{\partial x} - \frac{MN}{n} J_{M-1}(x) \frac{\partial J_{N-1}(x)}{\partial x}]\delta(x-y). \tag{12}$$

Let us note that these PB's are PB's of hydrodynamic type. The ultralocal term with antisymmetric structure constant $f_{\mu\nu\lambda}$ in commutation relation of chiral currents U^μ (4) does not contribution to commutation relations of invariant chiral currents because of totally symmetric invariant tensors $d_{\mu_1 ... \mu_M}$. Therefore chiral currents $J_M(U(x))$ form closed algebra under canonical PB. Evans et.al. found the functions of invariant chiral currents which will yield commuting charges for the $SU(n)$:

$$K_2(U) = J_2(U),\ K_3(U) = J_3(U),\ K_4(U) = J_4(U) - \frac{3}{2n}J_2^2(U),$$

$$K_5(U) = J_5(U) - \frac{10}{3n}J_2(U)J_3(U), \tag{13}$$

$$K_6(U) = J_6(U) - \frac{5}{3n}J_3^2(U) - \frac{15}{4n}J_4(U)J_2(U) + \frac{25}{8n^2}J_2^3(U).$$

Corresponding charges of chiral currents $K_M(U)$ are Casimir operators.

Another family of invariant chiral currents C_M forms closed algebra under canonical PB and corresponding charges are dynamical Casimir operators. They have form:

$$C_M(U(x)) = d^{k_1}_{\mu_1\mu_2} d^{k_1k_2}_{\mu_3} \ldots d^{k_{M-3}}_{\mu_{M-1}\mu_M} U_{\mu_1} U_{\mu_2} \ldots U_{\mu_M}. \tag{14}$$

The commutation relations are following:

$$\{C_M(U(x)), C_N(U(y))\} = -MNC_{M+N-2}(x)\frac{\partial}{\partial x}\delta(x-y) - \frac{MN(N-1)}{M+N-2}\frac{\partial C_{M+N-2}(x)}{\partial x}\delta(x-y). \tag{15}$$

Both families of invariant chiral currents $J_M(U(x))$ and $C_M(U(x))$ satisfy to conservation equations $\partial_- J_M(U(x)) = 0$,$\partial_- C_M(U(x)) = 0$.

Let us apply hydrodynamic approach to integrable string models with constant torsion. In this case we must to consider the conserved chiral currents $K_M(U(x))$ or currents $C_M(U(x))$, $M = 2, 3, \ldots, n-1$ as local fields of Riemmann manifold. Non-primitive local charges of invariant chiral currents with $M \geq n$ form the hierarchy of new Hamiltonians in bi-Hamiltonian approach to integrable systems. The commutation relations of invariant chiral currents are local PBs of hydrodynamic type. Metric tensor $g_{MN}(C(U(x)))$ for the $SU(n)$ group has following form:

$$g_{MN}(C(U(x))) = -MNC_{M+N-2} \tag{16}$$

The invariant chiral currents C_M with $M \geq 3$ for the $SU(3)$ group can be obtained from following relation:

$$d_{kln}d_{kmp} + d_{klm}d_{knp} + d_{klp}d_{knm} = \frac{1}{3}(\delta_{ln}\delta_{mp} + \delta_{lm}\delta_{np} + \delta_{lp}\delta_{nm})$$

The corresponding currents have form:

$$C_{2M} = (U_\mu U_\mu)^M = (U^2)^M = (C_2)^M, C_{2M+1} = (U_\mu U_\mu)^{M-1} d_{kln} U_k U_l U_n = (C_2)^{M-1} C_3 \tag{17}$$

Invariant chiral currents C_2, C_3 are local coordinates of Riemann manifold M^2. The local charges C_{2M}, $M \geq 2$ form hierarchy of hamiltonians. New nonlinear equations of motion for chiral currents are following:

$$\frac{\partial C_k(U(x))}{\partial t_M} = \{C_k(U(x)), \int_0^{2\pi} (C_2)^M (U(y))dy\}, k = 2, 3.$$

$$\frac{\partial C_2(U(x))}{\partial t_M} = -2(2M-1)\frac{\partial (C_2)^M(U(x))}{\partial x}, \tag{18}$$

$$\frac{\partial C_3(U(x))}{\partial t_M} = -6MC_3(U(x))\frac{\partial C_2^{M-1}(U(x))}{\partial x} - 2M(C_2)^{M-1}(U(x))\frac{\partial C_3(U(x))}{\partial x}. \tag{19}$$

The construction of integrable models with $SU(n)$ $n \geq 4$ symmetries has difficulties of reduction non-primitive invariant currents on primitive currents.

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