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COHERENT RADIATION OF A MICROBUNCHED BEAM IN A SHORT UNDULATOR

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Abstract

We calculate the coherent radiation of a modulated beam in a short resonantly tuned undulator taking into account the finite transverse size and the angular spread of the beam. The result allows one to optimize the radiation by controlling the beam Twiss parameters at the entrance to the undulator.

INTRODUCTION

Coherent undulator radiation of a microbunched beam is the fundamental mechanism that enables the operation of Free Electron Lasers (FELs). This process is typically examined within the broader context of FEL dynamics, which integrates the evolution of microbunching within the beam with the impact of the radiation reaction force inside the FEL undulator. Nonetheless, for certain applications, it proves beneficial to decouple these phenomena and investigate the undulator radiation of a pre-modulated beam assuming that the modulation does not change as the beam propagates through the undulator. Such a problem, for a sinusoidally modulated beam, has been studied in Ref. [1], the results of which we summarize below.

Ref. [1] considers an electron bunch with the sinusoidally modulated current,

$$I(z) = I_0[1 + 2b\cos(k_0 z)], \tag{1}$$

where b is the bunching factor, I_0 is the average current through the bunch, $k_0 = 2\pi/\lambda$, with λ the wavelength of the modulation, and z the longitudinal coordinate in the bunch. The bunch is assumed to have a Gaussian radial distribution of the density, $\propto e^{-r^2/2\sigma^2}$, with the rms transverse size σ. This radial profile remains the same when the bunch travels through the undulator.

When the beam defined by Eq. (1) is sent through a planar undulator of length L_u that is resonantly tuned to the wavelength λ_1 of the fundamental undulator frequency, its radiation power P is given by the following formula:

$$P = P_0 F(s), \tag{2}$$

where

$$P_0 = \frac{\pi}{4}[JJ]^2 \frac{K^2 k_0}{\gamma^2 c} I_0^2 b^2 L_u, \tag{3}$$

with $[JJ] = J_0(\xi) - J_1(\xi)$, where $\xi = K^2/[2(2+K^2)]$. Here J_0 and J_1 are the Bessel functions of the zero and first order, K is the undulator parameter and γ is the Lorentz factor. The function $F(s)$ is given by

$$F(s) = \frac{2}{\pi}\left[\arctan\left(\frac{1}{2s}\right) + s\ln\left(\frac{4s^2}{4s^2+1}\right)\right], \tag{4}$$

with $s = k_0\sigma^2/L_u$. This function reaches its maximum value at $s = 0$, $F(0) = 1$, and monotonically decreases with s.

The above analysis shows that in order to increase the radiation power, one has to focus the beam inside the undulator to the minimal transverse size. However, a strong focusing in the undulator would inevitably lead to the variation of the transverse beam size over the length of the undulator which would invalidate the analysis of Ref. [1]. This leads us to the need to study a more general case of the beams that are focused inside the undulator.

A CONVERGING BEAM IN THE UNDULATOR

We consider the beam which at the entrance to the undulator has the following distribution function:

$$f(\boldsymbol{\xi}) = \frac{1}{2\pi\epsilon_x}\exp\left[-\frac{1}{2\beta_x\epsilon_x}\left(x^2 + (\beta_x\theta_x + \alpha_x x)^2\right)\right] \times \frac{1}{2\pi\epsilon_y}\exp\left[-\frac{1}{2\beta_y\epsilon_y}\left(y^2 + (\beta_y\theta_y + \alpha_y y)^2\right)\right], \tag{5}$$

where x, y are the transverse coordinates, θ_x, θ_y are the corresponding angles for the particles, and we use the abbreviated notation $\boldsymbol{\xi} = (x, \theta_x, y, \theta_y)$. Nonzero positive values of α_x and α_y mean that the beam converges when it propagates through the undulator with the focal point in x and y planes located at the coordinates $z_x = \alpha_x\beta_x/(1+\alpha_x^2)$ and $z_y = \alpha_y\beta_y/(1+\alpha_y^2)$, correspondingly. In what follows, we neglect the undulator focusing and approximate particle orbits inside the undulator by straight lines (apart from the wiggling motion due to the undulator magnetic field).

We now consider the radiation of a single electron of the beam in the direction of the unit vector $\boldsymbol{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ where θ and ϕ are the polar and the azimuthal angles measured relative to the undulator axis. For the electron moving along the undulator axis (that is $x = y = \theta_x = \theta_y = 0$), the radiation field in the far zone, e_ω, at the frequency ω, can be found in Ref. [2]. Omitting factors that are not important for our calculations, for small angles θ, e_ω does not depend on angle ϕ and is given by the following expression:

$$e_\omega(\theta) \propto \frac{\exp[2\pi i N_u(\omega/\omega_1(\theta) - 1)] - 1}{\omega/\omega_1(\theta) - 1}, \tag{6}$$

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with $\omega_1(\theta) = \omega_1(0)/(1+\gamma^2\theta^2/(1+K^2/2))$ and $\omega_1(0) = 2\gamma^2 ck_w/(1+K^2/2)$, where $k_w = 2\pi/\lambda_w$ with λ_w the undulator period. In what follows, we will assume that the modulation frequency of the beam, $\omega = ck$, is close to the fundamental frequency of the undulator,

$$\omega = \omega_1(0)(1+\Delta), \tag{7}$$

with $|\Delta| \ll 1$. We then have

$$e_\omega(\theta) \propto \frac{\exp\left[2\pi i N_u\left(\Delta+\gamma^2\theta^2/(1+K^2/2)\right)\right]-1}{\Delta+\gamma^2\theta^2/(1+K^2/2)}. \tag{8}$$

A particle that is launched with offsets x, y parallel to the undulator axis (that is, $\theta_x = \theta_y = 0$) emits radiation with an extra phase factor $e^{-ik\boldsymbol{n}\cdot\boldsymbol{q}}$ where $\boldsymbol{q} = (x, y)$. For a particle that travels at angles θ_x, θ_y, the direction defined by the vector $\boldsymbol{n}$ has a different angle $\tilde{\theta}$ relative to its direction of its motion. To find $\tilde{\theta}$ we introduce the unit vector in the direction of motion, $\boldsymbol{m} \approx (\theta_x, \theta_y, 1-\theta_x^2-\theta_y^2)$, and write $\boldsymbol{n} = (\theta\cos\phi, \theta\sin\phi, 1-\theta^2/2)$ (here we have used the smallness of θ_x, θ_y and θ). We then have

$$\boldsymbol{m}\cdot\boldsymbol{n} \approx 1-\frac{1}{2}\tilde{\theta}^2 = \theta\theta_x\cos\phi+\theta\theta_y\sin\phi+1-\frac{1}{2}\theta^2, \tag{9}$$

from which we find

$$\tilde{\theta}^2 = \theta^2 - 2\theta(\theta_x\cos\phi+\theta_y\sin\phi). \tag{10}$$

Let us denote by $E_\omega(\boldsymbol{n},\boldsymbol{\xi})$ the electric field in the far zone radiated by a particle that is characterized by the four-dimesional vector $\boldsymbol{\xi}$. The field (6) corresponds to $\boldsymbol{\xi}=\boldsymbol{0}$, that is $E_\omega(\boldsymbol{n},\boldsymbol{\xi}=\boldsymbol{0}) = e_\omega(\theta)$. As explained above, a non-zero $\boldsymbol{\xi}$ adds a phase factor to e_ω and changes the angle θ to $\tilde{\theta}$ in the argument of the field:

$$E_\omega(\boldsymbol{n},\boldsymbol{\xi}) = e^{-ik(n_x x+n_y y)} \times e_\omega\left(\sqrt{\theta^2+\theta_x^2+\theta_y^2-2\theta(\theta_x\cos\phi+\theta_y\sin\phi)}\right). \tag{11}$$

The electric field of the whole beam emitted in the direction $\boldsymbol{n}$, $E_\omega^{(\text{beam})}(\boldsymbol{n})$, is obtained by summing the fields of all electrons,

$$E_\omega^{(\text{beam})}(\boldsymbol{n}) = \int dx\, d\theta_x\, dy\, d\theta_y\, f(\boldsymbol{\xi})E_\omega(\boldsymbol{n},\boldsymbol{\xi}), \tag{12}$$

where f is the distribution function (5). We note here that the coordinates x and y in Eq. (11) enter only in the exponential factor, and with the distribution function (5), the integration over x, y can be carried out analytically. The spectral energy per unit solid angle is proportional to $|E_\omega^{(\text{beam})}(\boldsymbol{n})|^2$ and integrated over the angle gives the power of the radiation, $P(\omega)$, of the modulated beam,

$$P(\omega) \propto \int_0^\infty \theta d\theta \int_0^{2\pi} d\phi |E_\omega^{(\text{beam})}(\boldsymbol{n})|^2. \tag{13}$$

The combination of Eqs. (11), (12) and (13) expresses the radiation power $P(\omega)$ in terms of several integrals, and can be carried out with standard mathematical packages such as Mathematica or Matlab. The general expression for the beam with unequal emittances ϵ_x and ϵ_y and also different β and α functions in x and y directions is cumbersome. In the interest of space, in this paper, we will present the final expression for $P(\omega)$ for an axisymmetric case, when $\epsilon_x=\epsilon_y\equiv\epsilon$, $\beta_x=\beta_y\equiv\beta$ and $\alpha_x=\alpha_y\equiv\alpha$. In this case, the radiation field of the beam is also axisymmetric and the function $E_\omega^{(\text{beam})}(\boldsymbol{n})$ depends only on the angle θ between the vector $\boldsymbol{n}$ and the axis of the undulator. This eliminates the integration over ϕ in Eq. (13), $\int_0^{2\pi} d\phi \to 2\pi$. The final result for $P(\omega)$ takes the following form,

$$P(\omega) = \frac{b^2 I_0^2 N_u}{8\pi c}\frac{K^2}{1+\frac{1}{2}K^2}[JJ]^2\int_0^\infty d\xi\,|A(\xi)|^2, \tag{14}$$

where A is given by

$$\begin{aligned} A(\xi) = {} & \frac{1}{1+\alpha^2}\int_0^\infty \psi d\psi \int_0^{2\pi} d\mu \exp\left(-\frac{\psi^2}{2}\right) \\ & \times \frac{\exp(2i[\pi N_u\epsilon+\xi+a^2\psi^2-2a\sqrt{\xi}\psi\cos\mu])-1}{\pi N_u\Delta+\xi+a^2\psi^2-2a\sqrt{\xi}\psi\cos\mu} \\ & \times \exp\left[\frac{1}{2(1+\alpha^2)}[4i\sqrt{s\xi}\alpha\psi\cos\mu-4s\xi+\alpha^2\psi^2]\right], \end{aligned} \tag{15}$$

with $a = \sqrt{\pi\epsilon N_u\gamma_z^2/\beta}$, $s = k^2\epsilon\beta/4\pi N_u\gamma_z^2$ and $\gamma_z = \gamma/\sqrt{1+K^2/2}$. Since our equations (6) and (13) are missing dimensional factors, they do not allow to find the factor in front of the integral in Eq. (14). This factor was established requiring that Eq. (14) reduces to Eq. (4) in the limit when $\alpha = 0$ and the angular spread of the beam can be neglected.

Eq. (1) assumes an infinitely long bunch with constant modulation amplitude $2b$. In reality, for a bunch of finite length, the current I_0 in the bunch, as well as the amplitude of the modulation b, vary with the longitudinal coordinate z inside the bunch on the scale that is much larger than λ[1], $I_0 \to I_0(z)$, $b \to b(z)$. In this case, one needs to integrate Eq. (14) over time to obtain the energy E of the undulator radiation, $E(\omega) = \int P(\omega)dz/c$. For a bunch with a longitudinal Gaussian distribution,

$$I_0(z) = \frac{Qc}{\sqrt{2\pi}\sigma_z}e^{-z^2/2\sigma_z^2}, \tag{16}$$

where σ_z is the rms bunch length and Q is the bunch charge, and a constant value of the bunching factor $b = b_0$, the integration over z can be carried out analytically with the following result,

$$E(\omega) = \frac{b_0^2 Q^2 N_u}{16\pi^{3/2}\sigma_z}\frac{K^2}{1+\frac{1}{2}K^2}[JJ]^2\int_0^\infty d\xi\,|A(\xi)|^2. \tag{17}$$

[1] A more accurate analysis show that we need to require that they vary on the scale larger than $N_u\lambda$, where N_u is the number of periods in the undulator.

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COMPUTER SIMULATIONS

To confirm the theoretical analysis of the previous section, we carried out simulations with the computer code Genesis. The undulator section had $N_u = 50$ periods with the period length of $\lambda_u = 3$ cm. A pre-bunched Gaussian beam with a small charge and a small energy spread, had the rms bunch length of 6 μm, the beam energy of 1 GeV and the normalized beam emittance of 0.8 μm. The bunch had a current modulation with the period of $\lambda = 13.5$ nm and the bunching factor $b = 0.2$. The bunch was launched into the undulator with the beta functions $\beta_x = \beta_y = 5$ m and variable values of the alpha functions, $\alpha_x = \alpha_y = \alpha$. The total energy of the radiation E was simulated in Genesis and compared with Eq. (17). The small charge was chosen in order to avoid the FEL amplification of the signal over the length of the undulator $N_u\lambda_u = 1.5$ m. The small energy spread prevented the beam from debunching inside the undulator due to chromatic effects. Note, that for these parameters, the resonant value of the undulator parameter K is equal to 2.2121. The figures below show the radiated energy normalized by the charge squared, E/Q^2, which, according to Eq. (17) does not depend on the charge.

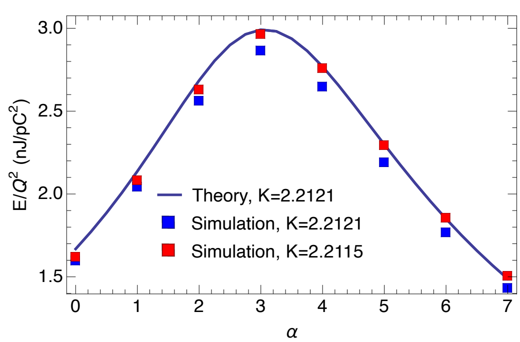

Figure 1: The dependence of the radiation energy E (normalized by Q^2) as a function of the Twiss parameter α. The blue symbols are the Genesis simulations with the nominal undulator parameter $K = 2.2121$ and the solid curve is the theoretical formula Eq. (17). Note that the theory systematically overestimates the energy by a few percents. The red symbols are the simulations with a slightly de-tuned undulator parameter, $K = 2.2115$; they show a better agreement with the theoretical curve (computed for $K = 2.2121$).

Figure 1 shows the dependence of the radiation energy versus the Twiss parameter α. As expected, the radiation energy increases when the beam gets focused in the undulator, $\alpha > 0$, until it reaches the maximal value $\alpha = 3$ after which the radiation energy goes down. Note that with the optimal choice of the parameter $\alpha = 3$, the radiated energy increases by about a factor of two when compared with the parallel beam propagation corresponding to $\alpha = 0$. Figure 2 shows the rms transverse size of the beam as a function of the coordinate s along the axis of the undulator for $\alpha = 3$. Note that the bunch size decreases due to non-zero initial α

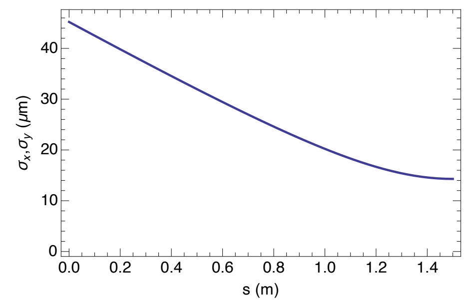

Figure 2: Transverse size of the beam as a function of coordinate s along the axis of the undulator.

by more than two times reaching a minimum at the exit of the undulator.

For a given modulation wavelength of the beam λ (see Eq. (1)), the optimal tuning of the undulator does not necessarily mean that the undulator frequency $\omega_1(0)$ should be equal to $2\pi c/\lambda$. To explore how undulator detuning affects the radiated energy E, we scanned the undulator parameter decreasing it from the nominal value $K = K_0 = 2.2121$ to $K = K_0 - \Delta K$. Figure 3 shows the dependence of E/Q^2 versus the de-tuning parameter ΔK for the optimal value $\alpha = 3$. Note that the theory systematically gives a somewhat larger value than the simulations. Interestingly, choosing $N_u = 51$ in the simulations gives a better agreement with the theoretical curve computed for $N_u = 50$.

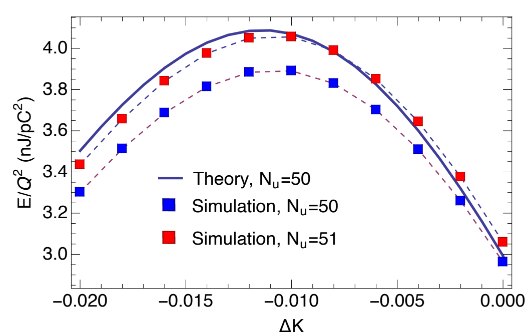

Figure 3: The dependence of E/Q^2 versus the de-tuning parameter ΔK for the optimal value $\alpha = 3$. The blue symbols show the simulations with the nominal number of undulator periods $N_u = 50$, and the solid curve shows the theoretical prediction.

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