

Conformal correlation functions in four dimensions from quaternionic Lauricella system

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Abstract

Correlation functions in four-dimensional Euclidean conformal field theories are expressed in terms of representations of the conformal group $SL(2, \mathbf{H})$, \mathbf{H} being the field of quaternions, on the configuration space of points. The representations are obtained in terms of a Lauricella system derived using quaternions. It generalizes the two-dimensional case, wherein the N -point correlation function is expressed in terms of solutions of Lauricella system on the configuration space of N points on the complex plane, furnishing representation of the conformal group $SL(2, \mathbf{C})$.

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1. Introduction

Correlation functions in conformal field theories in various dimensions have been studied extensively. Recent impetus to this field came from the conformal bootstrap programme [1–4]. Correlation functions of conformal fields at different points in a geometric space are obtained as equivariant quantities under the conformal group of the space. That is, correlation functions are appropriate representations of the conformal group. A representation of a group acting on a topological space is given by the lift of the group action to the space of regular functions on the topological space, or their appropriate generalizations. If the topological space is non-compact, functions on some form of completion of it is considered in order to ensure convergence

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of various functions and integrals. For conformal groups it is customary to use a conformal compactification. From now on we shall restrict our discussion to the n -dimensional Euclidean spaces, \mathbf{R}^n . In this case a popular scheme is to consider the action of the conformal group of \mathbf{R}^n isomorphic to $SO(1, n+1)$ on the light cone of \mathbf{R}^{n+2} with a metric of signature $(-, +, +, \dots)$. The light cone is stabilized by the conformal group. The Euclidean space \mathbf{R}^n is embedded into the light cone by an injective map. Its completion to include the conformal infinity is then used to construct representations of the conformal group. For example, in order to obtain the correlation functions of conformal fields on the complex plane \mathbf{C} , one first obtains the representation of the global conformal group $SO(1, 3)$ or $SL(2, \mathbf{C})$, on the conformal compactification of \mathbf{C} , namely, \mathbf{P}^1 , the complex projective line embedded into the light cone in \mathbf{R}^4 . Functions on the completion obtained by restriction from the light cone in two higher dimensions are acted on by the conformal group, thereby furnishing its representation. This picture, however, pertains to a single field in \mathbf{R}^n . Correlation functions for a multitude of fields are obtained by tensoring such representations. The correlation functions are then arranged into conformal blocks, the eigenfunctions of the quadratic Casimir, expanded in the basis of asymptotic plane waves. Since the conformal group includes scaling, construction of such a representation is often facilitated by considering the Mellin transforms [5–10]. While the two-point and three-point functions are determined by the conformal group and the structure constants, higher point correlation functions require further restrictions to be imposed. The bootstrap constraint, which has been a topic of extensive discussion recently, is one such [11,12], which restricts the correlation functions by its properties under the permutation of the points.

The representations, equivariant as they are, do not capture the nuances of various conformal field theories. These are incorporated by inserting projectors in the correlation functions such that higher point functions are expressed in terms of three-point functions. The projectors are made up of fields in a specific field theory. Hence the three point functions carry the structure constants of the operators of the same theory. We shall make extensive use of this formalism, called the shadow operator formalism [15–17].

In this article we obtain the multi-point correlation functions of conformal field theories in two- and four-dimensional Euclidean spaces in terms of representations of the corresponding conformal groups. Instead of tensoring the “single-particle” representations of the Lie algebra of the conformal group, we approach the computation of N -point correlation functions by looking at the representation of the Möbius group on the configuration space of N marked points on the Euclidean space. Among the various models of the configuration space the one we use is the Fulton-Macpherson compactification of the space of N pairwise distinct points.

In two dimensions we consider N points on the complex plane \mathbf{C} . The representation of the conformal group $SL(2, \mathbf{C})$ is then sought among the germ of functions, described by a Lauricella system, on the configuration space. The Lauricella system is given by the solutions of a system of differential equations in terms of the positions of the N points. The correlation functions are furnished by the ones equivariant under $SL(2, \mathbf{C})$. At this level, the completion of the configuration space is brought about by demanding that the functions are regular at infinity. The two-dimensional conformal group generalizes to the Möbius group $SL(2, \mathbf{H})$ in four dimensions, where \mathbf{H} denotes the field of quaternions [20–22]. We show that the Lauricella system has an appropriate generalization in terms of quaternionic variables. The correlation functions are once again given by the equivariant ones, regular at infinity. In both cases we deal with the conformal group, rather than the algebra. Higher point functions are split using the projectors and related to integrals over the N -variable Lauricella functions, dispensing with the point-wise insertion of “single-particle” Casimirs which proved to be useful too [13,17,23–26]. The integrals involved

in the correlation functions appeared earlier literature [26–31]. These are similar to Feynman integrals in higher dimensions. However, direct evaluation of the integrals is rendered difficult by their multi-valued nature and is greatly facilitated by writing them as solution to differential equations. We find that the differential equations of the Lauricella system have a close analogue in four dimensions in terms of matrix-valued quaternions. The equation for the general case with an arbitrary number of points has been written down.

In the next section we describe the Lauricella system on the configuration space of marked points in the two-dimensional case [32,33] and their appearance in the computation of chiral correlation functions through representation of the Möbius group. The projector is given by a two-point Lauricella function too. We explicitly evaluate the four and five point integrals and express the corresponding correlation functions in terms of integrals involving them, reproducing previously known results, as expected. The four-point function is expressed in terms of the Gauss hypergeometric function, while the five point function is expressed in terms of the Appell function F_2 . In the third section generalization to four dimensions is carried out. First, the complex integrals are generalized to integrals over quaternions, which generalize the field-theoretic Feynman integrals in four-vectors. By taking derivatives with respect to the matrix-valued quaternions we then obtain differential equations generalizing the Lauricella system to four dimensions. Let us stress that while the integrals appearing in the correlation functions have long been known [27] as integrals over four-vectors, Lauricella-type differential equations to evaluate them, to the best of our knowledge, have not appeared earlier. Let us also point out that the multi-valued integrals are expressed in terms of linear combinations solutions of the Lauricella system. As has been experienced in the evaluation of period integrals in the studies of mirror symmetry, obtaining them as solutions to differential equations may be more efficient for the evaluation of the integrals compared to direct computation. We then show that these integrals furnish representation of the four-dimensional Möbius group $SL(2, \mathbf{H})$ by enumerating their transformation under the group. Equations for the invariant part of the integrals, which may be related to the conformal block, expressed in terms of cross-ratios defined as determinants of a product of a quartet of quaternions and then obtained by taking traces of the matrix equations. We present the results for the case of four points, where the Lauricella system is solved with the Appell function F_4 .

2. Two dimensions

2.1. Functions on the configuration space of points

Let us begin with a description of the functions on the configuration space of N distinct points $\{z_1, z_2, \dots, z_N\}$ on the complex plane \mathbf{C} . The configuration space is

$$\mathcal{C}_N(\mathbf{C}) = \mathbf{C}^N \setminus \cup_{1 \leq i, j \leq N} \Delta_{ij}, \quad (1)$$

where

$$\Delta_{ij} = \{(z_1, z_2, \dots, z_N) \in \mathbf{C}^N; z_i = z_j\} \quad (2)$$

is called the fat diagonal. On the configuration space one considers integrals of the form

$$I_N^\mu(z) = \int \frac{dz}{(z - z_1)^{\mu_1} (z - z_2)^{\mu_2} \dots (z - z_N)^{\mu_N}}, \quad (3)$$

where vectors in boldface denote the N -tuples. The vector $\mathbf{z} = (z_1, z_2, \dots, z_N)$ collects the positions of the N points and $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_N)$ is the N -tuple of parameters, called weights.

The integral is defined over an arc in the plane connecting a pair of zeroes of the denominator of the integral, avoiding encircling any other zero and $0 < \mu_i < 1$ for each $i = 1, 2, \dots, N$. This integral defines a local system of \mathbf{C} -vector spaces over $\mathcal{C}_N(\mathbf{C})$, whose stalk at a point \mathbf{z} will also be denoted $I_N^\mu(\mathbf{z})$ by abuse of notation. Then $I_N^\mu(\mathbf{z})$ is invariant under translation of \mathbf{z} by a constant, is homogeneous of degree $1 - |\mu|$, where $|\mu| = \sum_{i=1}^N \mu_i$, and satisfies the differential equations [33]

$$z_{ij} \frac{\partial^2 I_N^\mu(\mathbf{z})}{\partial z_i \partial z_j} = \mu_j \frac{\partial I_N^\mu(\mathbf{z})}{\partial z_i} - \mu_i \frac{\partial I_N^\mu(\mathbf{z})}{\partial z_j}, \quad (4)$$

where we used $z_{ij} = z_i - z_j$. This equation is obtained by differentiating (3) with respect to the z_i under the integral sign and using the identity

$$\frac{1}{(x-y)(y-z)} + \frac{1}{(y-z)(z-x)} + \frac{1}{(z-x)(x-y)} = 0 \quad (5)$$

of three complex numbers x, y, z . The germs of $I_N^\mu(\mathbf{z})$ are expressed as the germs of the Lauricella functions [33], determined uniquely by (4). We refer to equation (4) and its solutions as the Lauricella system. In mundane terms, the solutions of equation (4) are “good functions” on the completion of the configuration space $\mathcal{C}_N(\mathbf{C})$.

Invariance under translation by a constant implies that $I_N^\mu(\mathbf{z})$ depends only on the differences z_{ij} and not separately on z_i themselves. The integral is well-behaved at infinity provided $|\mu| = 2$, as can be checked by changing the integration variable z to $1/z$. The case of $N = 2$ require special treatment. Let us discuss it first. Since the integral involves only two marked points, z_1 and z_2 , we can take the path over any arc joining these two points, which is in fact homotopic to the line joining them. Thus,

$$I_2^{(\mu_1, \mu_2)}(z_1, z_2) = \int_{z_1}^{z_2} \frac{dz}{(z - z_1)^{\mu_1} (z - z_2)^{\mu_2}}. \quad (6)$$

Parametrizing the line joining the two points as $z = tz_2 + (1 - t)z_1$, such that $0 \leq t \leq 1$, the integral is evaluated to be

$$I_2^{(\mu_1, \mu_2)}(z_1, z_2) = \frac{1}{z_{12}^{\mu_1 + \mu_2 - 1}} \frac{\Gamma(1 - \mu_1) \Gamma(1 - \mu_2)}{\Gamma(2 - \mu_1 - \mu_2)}. \quad (7)$$

Here and in the following we ignore factors of powers of -1 , which can be absorbed in the normalization of the correlation functions. As mentioned before, the integral depends only on the difference z_{12} rather than on the points individually and is homogeneous of degree $1 - \mu_1 - \mu_2$. The integral is, on the other hand, not well-behaved at infinity unless $\mu_1 + \mu_2 = 2$, a feature to be called on later. When $\mu_1 + \mu_2 = 2$, it becomes

$$I_2^{(\mu_1, \mu_2)}(z_1, z_2) = \frac{1}{z_{12}} \frac{\Gamma(1 - \mu_1) \Gamma(\mu_1 - 1)}{\Gamma(0)}, \quad (8)$$

where the singular piece $\Gamma(0)$ is to be understood in a limiting sense. Demanding the integrals to be regular at infinity is equivalent to considering a completion of the configurations space. We work with the Fulton-Macpherson compactification [34,35] as discussed in section 4.

2.2. Representation of the Möbius group

Let us now obtain the representations of the conformal group $SL(2, \mathbb{C})$ on the configuration space of N points on the plane. The group acts by Möbius transformation on the space, that is as

$$z \mapsto z' = \frac{az + b}{cz + d}, \quad a, b, c, d, z \in \mathbb{C}, \quad ad - bc = 1, \quad (9)$$

with a similar action on the conjugate variable \bar{z} . In two dimensions the actions on z and \bar{z} may be treated independently. We shall display formulas for the holomorphic part only.

A holomorphic representation of the Möbius group is furnished by the regular functions on \mathbb{C}^N which transform under $SL(2, \mathbb{C})$ as

$$\begin{aligned} f(z_1, z_2, \dots, z_N) &\mapsto f(z'_1, z'_2, \dots, z'_N) \\ &= (cz_1 + d)^{\Delta_1} (cz_2 + d)^{\Delta_2} \dots (cz_N + d)^{\Delta_N} f(z_1, z_2, \dots, z_N), \end{aligned} \quad (10)$$

with $\Delta = (\Delta_1, \Delta_2, \dots, \Delta_N)$ an N -tuple of real numbers.

Let us first note that the quantities z_{ij} are equivariant under the Möbius transformation (9),

$$z_{ij} \mapsto z'_{ij} = (cz_i + d)^{-1} (cz_j + d)^{-1} z_{ij}. \quad (11)$$

From (9) we also have

$$dz' = (cz + d)^{-2} dz. \quad (12)$$

The integral (3) is equivariant with respect to (9) with degree of homogeneity -1 provided $|\mu| = 2$. In this case it transforms under the Möbius group as

$$I_N^\mu(z) \mapsto I_N^\mu(z') = (cz_1 + d)^{\mu_1} (cz_2 + d)^{\mu_2} \dots (cz_N + d)^{\mu_N} I_N^\mu(z). \quad (13)$$

Holomorphic representations of the Möbius group may thus be constructed out of z_{ij} and $I_N^\mu(z)$.

We have discussed above the form of $I_N^\mu(z)$ for $N = 2$. The expression (7) with arbitrary parameters does not transform under $SL(2, \mathbb{C})$, while (8) does. Equation (13) requires μ_1 and μ_2 to be equal. Thus, from (8)

$$I_2^{(1,1)}(z_1, z_2) = \frac{\Gamma(0)}{z_{12}}. \quad (14)$$

For the other special case $N = 3$ equation (4) is solved with

$$I_3^{(\mu_1, \mu_2, \mu_3)}(z_1, z_2, z_3) = z_{12}^{-\frac{\mu_1 + \mu_2 - \mu_3}{2}} z_{23}^{-\frac{\mu_2 + \mu_3 - \mu_1}{2}} z_{31}^{-\frac{\mu_3 + \mu_1 - \mu_2}{2}}, \quad (15)$$

for $|\mu| = \mu_1 + \mu_2 + \mu_3 = 2$ up to a multiplicative constant. This can be verified by plugging the expression into (4) and appealing to the uniqueness of its solution.

For $N > 3$ complications arise due to the fact that there exist invariants of the Möbius transformation, known as cross ratios, which may be multiplied to any function with arbitrary exponents without altering the transformation property of $I_N^\mu(z)$. This, however, may change the behavior of functions at infinity on the configuration space. A cross ratio has the form

$$\chi_{ijkl} = \frac{z_{ij} z_{kl}}{z_{ik} z_{jl}}, \quad (16)$$

its invariance under Möbius transformation follows from (11). It will turn out convenient to denote the cross ratios by

$$\xi_A = \prod_{\substack{i,j \\ 1 \leq i < j \leq N}} \alpha_{ij}^A, \quad (17)$$

with

$$\alpha_{ji}^A = \alpha_{ij}^A, \quad i < j; \quad \sum_{j=1}^N \alpha_{ij}^A = 0, \quad \forall i \quad (18)$$

for each A . This will allow treating them rather symmetrically. Then, in view of the equivariance (13), the integral $I_N^\mu(\mathbf{z})$ can be written as products of z_{ij} with appropriate indices and a function of the cross ratios as

$$I_N^\mu(\mathbf{z}) = \prod_{\substack{i,j \\ 1 \leq i < j \leq N}} z_{ij}^{\beta_{ij}} I_0(\xi), \quad (19)$$

where $I_0(\xi)$ is a function of the cross ratios $\xi = (\xi_1, \xi_2, \dots)$ and

$$\sum_{j=1}^N \beta_{ij} = -\mu_i; \quad \beta_{ji} = \beta_{ij}, \quad i < j, \quad (20)$$

for each $i = 1, 2, \dots, N$. Since $|\mu| = 2$, we also have

$$\sum_{\substack{i,j \\ 1 \leq i < j \leq N}} \beta_{ij} = -1 \quad (21)$$

Plugging in (19) with (17) and (16) in (4), we obtain a differential equation for the invariant function I_0 of the cross ratios as

$$\begin{aligned} & \sum_{A,B} \left(\sum_{\substack{k,l \\ 1 \leq k,l \leq N \\ k \neq i, l \neq j}} \alpha_{ik}^A \alpha_{jl}^B \chi_{ijkl} \right) \xi_A \xi_B \partial_A \partial_B I_0(\xi) \\ & + \sum_A \left(\alpha_{ij}^A + \sum_{\substack{k,l \\ 1 \leq k,l \leq N \\ k \neq i, l \neq j}} (\alpha_{ik}^A \alpha_{jl}^A + \alpha_{ik}^A \beta_{jl} + \alpha_{jl}^A \beta_{ik}) \chi_{ijkl} \right) \xi_A \partial_A I_0(\xi) \\ & + \left(\beta_{ij} + \sum_{\substack{k,l \\ 1 \leq k,l \leq N \\ k \neq i, j \neq l}} \beta_{ik} \beta_{jl} \chi_{ijkl} \right) I_0(\xi) = 0, \end{aligned} \quad (22)$$

where ∂_A denotes differentiation with respect to ξ_A . This equation is valid for arbitrary N .

2.2.1. Four points

For four points in two dimensions there is but a single independent cross ratio which we choose to be $\xi = \chi_{1234}$. The non-vanishing exponents α for this choice are

$$\alpha_{12} = \alpha_{34} = -\alpha_{13} = -\alpha_{24} = 1, \quad (23)$$

where we have suppressed the superscript A , which is unity in this case. Equation (22) then leads to

$$f_2(\xi) \frac{d^2 I_0}{d\xi^2} + f_1(\xi) \frac{d I_0}{d\xi} + f_0(\xi) I_0 = 0, \quad (24)$$

where

$$f_2(\xi) = \xi^2(\xi - 1) \quad (25)$$

$$f_1(\xi) = \xi((\beta_{13} + \beta_{14} + \beta_{23} + \beta_{24}) + \xi(1 - \beta_{13} - \beta_{24})) \quad (26)$$

$$f_0(\xi) = -\beta_{12}\beta_{34} + \frac{\xi\beta_{14}\beta_{23}}{\xi - 1} + \xi\beta_{13}\beta_{24} \quad (27)$$

This is solved with

$$\begin{aligned} I_0(\xi) = & \xi^{-\beta_{12}}(1 - \xi)^{-\beta_{23}} C_1 F(-\beta_{12} - \beta_{13} - \beta_{23}, -\beta_{12} - \beta_{23} - \beta_{24}, 1 - \beta_{12} + \beta_{34}; \xi) \\ & + \xi^{-\beta_{34}}(1 - \xi)^{-\beta_{23}} C_2 F(1 + \beta_{12} + \beta_{13} + \beta_{14}, 1 \\ & + \beta_{12} + \beta_{14} + \beta_{24}, 1 + \beta_{12} - \beta_{34}; \xi), \end{aligned} \quad (28)$$

where F denotes the Gauss hypergeometric function and C_1 and C_2 are arbitrary constants. The six parameters β are related to the weights by the four equations (20) through

$$\begin{aligned} \beta_{12} &= 1 - \mu_1 - \mu_2 + \beta_{34}, \\ \beta_{13} &= \mu_2 - 1 - \beta_{14} - \beta_{34}, \\ \beta_{23} &= 1 - \mu_2 - \mu_3 + \beta_{14}, \\ \beta_{24} &= -\mu_4 - \beta_{14} - \beta_{34}. \end{aligned} \quad (29)$$

Plugging in these values along with (28) in (19) yields the four-point integral

$$\begin{aligned} I_4^\mu(z) = & z_{12}^{1-\mu_1-\mu_2} z_{13}^{\mu_2-1} z_{23}^{1-\mu_2-\mu_3} z_{24}^{-\mu_4} (C_1 F(1 - \mu_2, \mu_4; \mu_3 + \mu_4; \xi) \\ & + C_2 \xi^{\mu_1+\mu_2-1} F(\mu_1, 1 - \mu_3; \mu_1 + \mu_2; \xi)), \end{aligned} \quad (30)$$

with $\xi = z_{12}z_{34}/z_{13}z_{24}$, where we used $|\mu| = 2$.

2.2.2. Five points

Two independent cross ratios exist for five two-dimensional points which we choose to be $\xi_A = \chi_{A, A+1, A+2, A+3}$ for $A = 1, 2$. The non-vanishing exponents are

$$\alpha_{12}^1 = \alpha_{34}^1 = -\alpha_{13}^1 = -\alpha_{24}^1 = 1 \quad (31)$$

$$\alpha_{23}^2 = \alpha_{45}^2 = -\alpha_{24}^2 = -\alpha_{35}^2 = 1. \quad (32)$$

Equation (22) gives rise to ten equations for the ten independent choices of the pairs $\{(i, j) | i < j; i, j \in (1, 2, 3, 4, 5)\}$. Instead of solving them generally, equation (20) may be exploited to set five of the β 's to zero. We choose

$$\beta_{12} = \beta_{14} = \beta_{15} = \beta_{25} = \beta_{45} = 0. \quad (33)$$

The rest are related to the weights by (20) as

$$\begin{aligned} \beta_{13} &= -\mu_1, \quad \beta_{23} = 1 - \mu_2 - \mu_3, \quad \beta_{24} = \mu_3 - 1, \\ \beta_{34} &= 1 - \mu_3 - \mu_4, \quad \beta_{35} = -\mu_5. \end{aligned} \quad (34)$$

The equations corresponding to the choices $(i, j) = (1, 2)$ and $(i, j) = (4, 5)$ ensuing from (22) are

$$\begin{aligned} \xi_1(1 - \xi_1) \frac{\partial^2 I_0}{\partial \xi_1^2} - \xi_1 \xi_2 \frac{\partial^2 I_0}{\partial \xi_1 \partial \xi_2} + (c_1 - (1 + a + b_1) \xi_1) \frac{\partial I_0}{\partial \xi_1} - b_1 \xi_2 \frac{\partial I_0}{\partial \xi_2} - ab_1 I_0 &= 0, \\ \xi_2(1 - \xi_2) \frac{\partial^2 I_0}{\partial \xi_2^2} - \xi_1 \xi_2 \frac{\partial^2 I_0}{\partial \xi_1 \partial \xi_2} + (c_2 - (1 + a + b_2) \xi_2) \frac{\partial I_0}{\partial \xi_2} - b_2 \xi_1 \frac{\partial I_0}{\partial \xi_1} - ab_2 I_0 &= 0, \end{aligned} \quad (35)$$

where $\xi_1 = \frac{z_{12}z_{34}}{z_{13}z_{24}}$, $\xi_2 = \frac{z_{23}z_{45}}{z_{24}z_{35}}$ are the cross ratios corresponding to (31) and (32). The parameters are related to the scaling exponents

$$a = 1 - \mu_3, \quad b_1 = \mu_1, \quad b_2 = \mu_5, \quad c_1 = \mu_1 + \mu_2, \quad c_2 = \mu_4 + \mu_5, \quad (36)$$

where the sum of the scaling exponents $|\mu| = 2$. These are the equations satisfied by the second Appell hypergeometric function F_2 . The most general solution, obtained using (34) in (19) is

$$I_5^\mu(\xi) = z_{13}^{-\mu_1} z_{23}^{1-\mu_2-\mu_3} z_{24}^{\mu_3-1} z_{34}^{1-\mu_3-\mu_4} z_{35}^{-\mu_5} I_0(\xi_1, \xi_2), \quad (37)$$

where the invariant is

$$\begin{aligned} I_0(\xi_1, \xi_2) &= C_1 F_2(1 - \mu_3, \mu_1, \mu_5, \mu_1 + \mu_2, \mu_4 + \mu_5; \xi_1, \xi_2) \\ &+ C_2 \xi_1^{1-\mu_1-\mu_2} F_2(\mu_4 + \mu_5, 1 - \mu_2, \mu_5, \mu_3 + \mu_4 + \mu_5, \mu_4 + \mu_5; \xi_1, \xi_2) \\ &+ C_3 \xi_2^{1-\mu_4-\mu_5} F_2(\mu_1 + \mu_2, \mu_1, 1 - \mu_4, \mu_1 + \mu_2, \mu_1 + \mu_2 + \mu_3; \xi_1, \xi_2) \\ &+ C_4 \xi_1^{1-\mu_1-\mu_2} \xi_2^{1-\mu_4-\mu_5} F_2(1, 1 - \mu_2, 1 - \mu_4, \mu_3 + \mu_4 + \mu_5, \mu_1 + \mu_2 \\ &+ \mu_3; \xi_1, \xi_2), \end{aligned} \quad (38)$$

where C_1, C_2, C_3 and C_4 are arbitrary constants. As in the case of four points, the final result does not depend on the choice of β 's in (33). The other eight equations obtained from (22) pairwise yield the equations for the Appell function F_2 in other domains, related to the present one by analytic continuation.

Above considerations as well as all the expressions have anti-holomorphic counterparts with μ changed to μ' .

2.3. Correlation functions

Correlation functions in two-dimensional conformal field theories are well-known. We repeat some of the computations here in order to bring out the analogy with the four-dimensional counterpart. For this purpose it suffices to consider chiral primary scalar fields $\{\phi_i(z_i)\}$ with conformal dimensions Δ in line with (10). The correlation function of N chiral scalar primaries is given by a holomorphic representation (10) on the configuration space $\mathcal{C}_N(\mathbb{C})$. In particular, it is invariant under translation. The anti-holomorphic part follows suit with conjugated coordinates and primed weights. It then follows from the preceding discussion that a correlation function for chiral primaries can be expressed in terms of the differences z_{ij} and the integrals $I_N^\mu(z)$. Since

$SL(2, \mathbb{C})$ equivariance restricts the degree of homogeneity of the integrals to be -1 by constraining $|\mu| = 2$, we can write down correlation functions of a set of primary fields with given conformal dimensions by simply multiplying the integrals by powers of z_{ij} so as to satisfy (10),

$$G_N^{\Delta_1, \Delta_2, \dots, \Delta_N}(\phi_1, \phi_2, \dots, \phi_N) = \mathcal{F} \left(\prod_{\substack{i,j \\ 1 \leq i < j \leq N}} z_{ij}^{\ell_{ij}} I_N^\mu(z) \right), \quad (39)$$

where \mathcal{F} indicates a functional involving sums and integrals of I with respect to its parameters, transforming appropriately under the Möbius group. We use the shorthand ϕ_i for $\phi(z_i)$. The parameters ℓ are related to the weights and conformal dimensions of fields as

$$- \sum_{\substack{j \\ 1 \leq j \leq N; j \neq i}} \ell_{ij} + \mu_i = \Delta_i, \quad \sum_{i=1}^N \mu_i = 2 \quad (40)$$

for each $i = 1, 2, \dots, N$ and we have defined $\ell_{ji} = \ell_{ij}$ if $j > i$. The product in front of the integral in (39) is referred to as the leg factor. We shall suppress the superscripts in G_N if the conformal dimensions involved are clear from the context.

The correlation functions for $N = 2$ and $N = 3$, the two-point and three-point functions, respectively, are fixed up to a constant by their $SL(2, \mathbb{C})$ equivariance. For example, by (39),

$$G_2^{\Delta_1, \Delta_2}(\phi_1, \phi_2) = z_{12}^{\ell_{12}} I_2^{(1,1)}(z_1, z_2), \quad (41)$$

and we have, by (40),

$$\ell_{12} = 1 - \Delta_1 = 1 - \Delta_2. \quad (42)$$

It follows, in accordance with (10), that $\Delta_1 = \Delta_2$. Using (14) we thus obtain

$$G_2^{\Delta_1, \Delta_2}(\phi_1, \phi_2) = \frac{C_{\Delta_1} \Gamma(0)}{z_{12}^{\Delta_1}} \delta_{\Delta_1, \Delta_2}, \quad (43)$$

where C_{Δ_1} is an arbitrary constant for each field of conformal dimension Δ . Similarly, for the three-point function

$$G_3^{\Delta_1, \Delta_2, \Delta_3}(\phi_1, \phi_2, \phi_3) = C_{\Delta_1, \Delta_2, \Delta_3} z_{12}^{\ell_{12}} z_{13}^{\ell_{13}} z_{23}^{\ell_{23}} I_3^{(\mu_1, \mu_2, \mu_3)}(z_1, z_2, z_3). \quad (44)$$

Then by (40) the exponents of the leg factor satisfy the three equations

$$\begin{aligned} \ell_{12} + \ell_{13} &= \mu_1 - \Delta_1, \\ \ell_{12} + \ell_{23} &= \mu_2 - \Delta_2, \\ \ell_{13} + \ell_{23} &= \mu_3 - \Delta_3, \end{aligned} \quad (45)$$

which are solved to obtain

$$\begin{aligned} \ell_{12} &= -\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3) + \frac{1}{2}(\mu_1 + \mu_2 - \mu_3) \\ \ell_{13} &= -\frac{1}{2}(\Delta_1 + \Delta_3 - \Delta_2) + \frac{1}{2}(\mu_1 + \mu_3 - \mu_2) \\ \ell_{23} &= -\frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1) + \frac{1}{2}(\mu_2 + \mu_3 - \mu_1). \end{aligned} \quad (46)$$

Using (15) for I_3 then yields the three-point function

$$G_3^{\Delta_1, \Delta_2, \Delta_3}(\phi_1, \phi_2, \phi_3) = C_{\Delta_1 \Delta_2 \Delta_3} z_{12}^{-\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3)} z_{13}^{-\frac{1}{2}(\Delta_1 + \Delta_3 - \Delta_2)} z_{23}^{-\frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1)}. \quad (47)$$

Let us remark that in these two cases the integrals did not have a role to play. The leg factor in both cases were so arranged as to obviate the μ 's, thereby effacing the trace of the integrals. Thus, the two- and three-point correlation function of primaries are completely determined by their equivariance under the Möbius group and the given conformal dimensions. This does not generalize to higher point functions, however. While the leg factors could be so arranged as to annul the contributions of β 's in (19), the cross ratios introduce arbitrariness in the leg factors. This calls for further restrictions on the correlation functions. One such stipulation arises from requiring that higher point functions can be pared down to products of three-point functions, which we now proceed to discuss.

2.4. Projectors

Parsing of higher point correlation function in terms of the three-point function is effected by using projectors [15,16,26]. There is an appropriate set of projectors $\{\Pi_\Delta\}$ summing up to the identity operator \mathcal{I}

$$\mathcal{I} = \sum_{\Delta} \Pi_{\Delta}, \quad (48)$$

such that, the N -point function can be parsed as

$$\begin{aligned} G_N(\phi_1, \phi_2, \dots, \phi_N) &= \langle \phi_1 \phi_2 \dots \phi_N \rangle \\ &= \langle \phi_1 \phi_2 \mathcal{I} \phi_3 \mathcal{I} \phi_4 \dots \phi_{N-2} \mathcal{I} \phi_{N-1} \phi_N \rangle \\ &= \sum_{\Delta, \Delta', \dots, \Delta''} \langle \phi_1 \phi_2 \Pi_{\Delta} \phi_3 \Pi_{\Delta'} \phi_4 \dots \phi_{N-2} \Pi_{\Delta''} \phi_{N-1} \phi_N \rangle, \end{aligned} \quad (49)$$

where $\phi_{\Delta}(x)$ denotes a primary field of conformal dimension Δ at $x \in \mathbb{C}$. The operator Π_{Δ} is defined as

$$\Pi_{\Delta} = \frac{1}{N_{\Delta}} \int \frac{\phi_{\Delta}(x) \phi_{\Delta}(y)}{(x-y)^{2-\Delta}} dx dy, \quad (50)$$

where N_{Δ} is a constant and the path of integration, written formally in this expression, is fixed only when used in conjunction with a correlation function. By (11) and (12), Π_{Δ} is invariant under the Möbius group. The constant is determined by requiring the projector to be consistent with two point functions. The composition of the projectors is defined as

$$\Pi_{\Delta} \circ \Pi_{\Delta'} = \frac{1}{N_{\Delta} N_{\Delta'}} \int \frac{\phi_{\Delta}(x) G_2(\phi_{\Delta}(y) \phi_{\Delta'}(x')) \phi_{\Delta'}(y')}{(x-y)^{2-\Delta} (x'-y')^{2-\Delta'}} dx dy dx' dy', \quad (51)$$

which also defines their action on fields in parsing the correlation function. Using (43) this yields

$$\Pi_{\Delta} \circ \Pi_{\Delta'} = \delta_{\Delta, \Delta'} \frac{C_{\Delta} \Gamma(0)}{N_{\Delta}^2} \int \frac{\phi_{\Delta}(x) \phi_{\Delta}(y')}{(x-y)^{2-\Delta} (x'-y')^{2-\Delta} (y-x')^{\Delta}} dx dy dx' dy'. \quad (52)$$

The integral over x' can be performed using (8). This leads to

$$\Pi_{\Delta} \circ \Pi_{\Delta'} = \delta_{\Delta, \Delta'} \frac{C_{\Delta} \Gamma(1-\Delta) \Gamma(\Delta-1)}{N_{\Delta}^2} \int \phi_{\Delta}(x) \phi_{\Delta}(y') dx dy' I_2^{(2-\Delta, 1)}(x, y') \quad (53)$$

Let us note that the integral $I_2^{(2-\Delta,1)}(x, y')$ appearing in this expression does not have $|\mu| = 2$. Hence it is not well-behaved at infinity. The final result is indeed conformal invariant. Using (7), we evaluate the above product to be

$$\Pi_\Delta \circ \Pi_{\Delta'} = \delta_{\Delta, \Delta'} \frac{C_\Delta \Gamma(0) \Gamma(1 - \Delta) \Gamma(\Delta - 1)}{N_\Delta^2} \int \frac{\phi_\Delta(x) \phi_\Delta(y')}{(x - y')^{2-\Delta}} dx dy' \quad (54)$$

Since a projector is idempotent, equating to (50) we obtain

$$N_\Delta = C_\Delta \Gamma(0) \Gamma(1 - \Delta) \Gamma(\Delta - 1). \quad (55)$$

We have absorbed factors of powers of (-1) in the constant C_Δ . The apparent lack of convergence of the projector is due to the unspecified nature of the sum over Δ in (48).

2.5. Higher points correlation functions

Let us now use the projectors to express higher point correlation functions in terms of the Lauricella functions. We shall demonstrate this for $N = 4$ and $N = 5$.

The four point correlation function $G_4(z_1, z_2, z_3, z_4) = \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$ is written by inserting the projector (48) with (50) as

$$\begin{aligned} G_4(\phi_1, \phi_2, \phi_3, \phi_4) &= \sum_{\Delta} \frac{1}{N_\Delta} \int \frac{dx dy}{(x - y)^{2-\Delta}} \langle \phi_1 \phi_2 \phi_\Delta(x) \rangle \langle \phi_\Delta(y) \phi_3 \phi_4 \rangle \\ &= \sum_{\Delta} \frac{1}{N_\Delta} \int \frac{dx dy}{(x - y)^{2-\Delta}} G_3^{\Delta_1, \Delta_2, \Delta}(z_1, z_2, x) G_3^{\Delta, \Delta_3, \Delta_4}(y, z_3, z_4). \end{aligned} \quad (56)$$

We have thus expressed the four point function in terms of the three point functions. Expanding the latter using (47) we first collect all the terms containing the integration variable x . They combine into $I_3^{(2-\Delta, \frac{\Delta+\Delta_1-\Delta_2}{2}, \frac{\Delta_2+\Delta-\Delta_1}{2})}(y, z_1, z_2)$, whose weights add up to 2. Using (15) this furnishes powers of two linear forms in y , which combined with the two more from the second G_3 factor in the last integral leaves us with an integral in y with a total of four factors of powers of linear forms in y in the integrand. Collecting all yields

$$\begin{aligned} G_4(\phi_1, \phi_2, \phi_3, \phi_4) &= \sum_{\Delta} \frac{C_{\Delta_1 \Delta_2 \Delta} C_{\Delta, \Delta_3, \Delta_4}}{N_\Delta} z_{12}^{-\frac{\Delta_1+\Delta_2+\Delta-2}{2}} z_{34}^{-\frac{\Delta_3+\Delta_4-\Delta}{2}} \\ &\quad \times I_4^{\left(\frac{2-\Delta+\Delta_1-\Delta_2}{2}, \frac{2-\Delta+\Delta_2-\Delta_1}{2}, \frac{\Delta+\Delta_3-\Delta_4}{2}, \frac{\Delta+\Delta_4-\Delta_3}{2}\right)}(z_1, z_2, z_3, z_4). \end{aligned} \quad (57)$$

The five point function is similarly parsed as

$$\begin{aligned}
G_5(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) &= \sum_{\Delta, \Delta'} \frac{1}{N_\Delta N_{\Delta'}} \int \frac{\langle \phi_1 \phi_2 \phi_\Delta(x) \rangle \langle \phi_\Delta(y) \phi_3 \phi_{\Delta'}(x') \rangle \langle \phi_{\Delta'}(y') \phi_4 \phi_5 \rangle}{(x-y)^{2-\Delta} (x'-y')^{2-\Delta'}} dx dy dx' dy' \\
&= \sum_{\Delta, \Delta'} \frac{1}{N_\Delta N_{\Delta'}} \\
&\quad \times \int \frac{G_3^{\Delta_1, \Delta_2, \Delta}(z_1, z_2, x) G_3^{\Delta_3, \Delta_4, \Delta'}(y, z_3, x') G_3^{\Delta', \Delta_4, \Delta_5}(y', z_4, z_5)}{(x-y)^{2-\Delta} (x'-y')^{2-\Delta'}} dx dy dx' dy'.
\end{aligned} \tag{58}$$

Using (15) repeatedly and performing integrals in turn until the integration over only a single variable is left, this is finally written in terms of I_5 as

$$\begin{aligned}
G_5(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) &= \left(\frac{1}{2\pi i} \right)^3 \sum_{\Delta, \Delta'} \frac{C_{\Delta_1 \Delta_2 \Delta} C_{\Delta_3 \Delta_4 \Delta'} C_{\Delta' \Delta_4 \Delta_5}}{N_\Delta N_{\Delta'}} z_{12}^{-\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta)} z_{45}^{-\frac{1}{2}(\Delta_4 + \Delta_5 + \Delta' - 2)} \\
&\quad \times \int ds_1 ds_2 ds_3 \frac{\Gamma(-s_1) \Gamma(-s_2) \Gamma(-s_3) \Gamma(m+s_1) \Gamma(n+s_2) \Gamma(p+s_3)}{\Gamma(m) \Gamma(n) \Gamma(p)} \\
&\quad \times \int d\tau (-\tau)^{-(s_1+s_2+s_3+2)} \\
&\quad \times I_5^{\left(\frac{1}{2}(\Delta_1 - \Delta_2 + \Delta), \frac{1}{2}(\Delta_2 - \Delta_1 + \Delta), \frac{1}{2}(2 + \Delta_3 - \Delta - \Delta') - s_1, -s_2, -s_3\right)}(z_1, z_2, z_3, z_4, z_5),
\end{aligned} \tag{59}$$

where the quantities

$$m = \frac{1}{2}(\Delta_3 + \Delta + \Delta' - 2), \quad n = \frac{1}{2}(\Delta_4 - \Delta_5 + 2 - \Delta'), \quad p = \frac{1}{2}(\Delta_5 - \Delta_4 + 2 - \Delta') \tag{60}$$

have been defined and repeated use of the integral

$$\frac{1}{(1-x)^n} = \frac{1}{2\pi i} \frac{1}{\Gamma(n)} \int_{-i\infty}^{i\infty} ds (-x)^s \Gamma(-s) \Gamma(n+s) \tag{61}$$

has been made.

Correlation functions with more number of points can be similarly written down in terms of the Lauricella functions I_N . We have thus related the conformal correlation functions of scalar primaries to the Lauricella system, defined on the configuration space of points in two dimensions.

3. Four dimensions

The conformal or Möbius group of the compactified four-dimensional Euclidean space $M = \mathbf{R}^4 \cup \{\infty\}$ is $SL(2, \mathbf{H})$ [21,22]. The correlation functions of scalar primaries of a four-dimensional conformal field theory are obtained as representations of $SL(2, \mathbf{H})$ on the configuration space of N points in M . In this section we show that the considerations of the previous section carry over *mutatis mutandis* to the four-dimensional Euclidean conformal field theories. In order to fix notations let us begin by recalling some facts about quaternions and the Möbius transformations [18–20].

3.1. Quaternions

A quaternion can be written as a 2×2 matrix with a pair of complex numbers U and V and their respective complex conjugates \bar{U} and \bar{V} as

$$Q = \begin{pmatrix} U & V \\ -\bar{V} & \bar{U} \end{pmatrix} \in \mathbf{H}, \quad U, V \in \mathbf{C}. \quad (62)$$

The norm squared of a quaternion is

$$\|Q\|^2 = QQ^\dagger = |Q| = U\bar{U} + V\bar{V}, \quad (63)$$

where Q^\dagger denotes the Hermitian conjugate and $|Q|$ denotes the determinant of the matrix (62). The inverse of the matrix Q is

$$Q^{-1} = \frac{1}{|Q|} Q^\dagger. \quad (64)$$

A quaternion can also be looked upon as a Euclidean real four-vector $\mathbf{q} = (q_0, q_1, q_2, q_3)$ by writing $U = q_0 + iq_3$ and $V = q_1 + iq_2$. From (62), then,

$$Q = \begin{pmatrix} q_0 + iq_3 & q_1 + iq_2 \\ -q_1 + iq_2 & q_0 - iq_3 \end{pmatrix} \quad (65)$$

The norm-squared of the quaternion Q is the Euclidean norm-squared of the four-vector,

$$\|Q\|^2 = |Q| = q^2 = \mathbf{q} \cdot \mathbf{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2. \quad (66)$$

The volume form of the four-dimensional Euclidean space is then written as the wedge product of the column vectors of the differential of Q divided by 2^4 ,

$$d^4Q = d^4q = dq_0 \wedge dq_1 \wedge dq_2 \wedge dq_3. \quad (67)$$

This generalizes the two-dimensional volume form $dz \wedge d\bar{z}$. In the previous section we chose to only write the holomorphic parts to leave provision for spin. In four dimensions we need to consider four-dimensional integrals. We consider integrals similar to (3) in four dimensions. We shall denote these by the same symbol as in (3). Let us define

$$I_N^\mu(Q) = \int \frac{d^4Q}{|Q - Q_1|^{\mu_1} |Q - Q_2|^{\mu_2} \cdots |Q - Q_N|^{\mu_N}}, \quad (68)$$

where Q denotes the N -tuple of quaternions, $Q = (Q_1, Q_2, \dots, Q_N)$. From (64) we have

$$\frac{\partial |Q|}{\partial Q} = Q^\dagger = |Q|Q^{-1}. \quad (69)$$

For the following it is useful to indicate the matrix indices of the quaternions, $Q = (Q)_{ab}$ and $Q^{-1} = (Q)^{ba}$, $1 \leq a, b \leq 2$. Then $(Q)_{ab}(Q)^{bc} = \delta_a^c$ and the last equation becomes

$$\frac{\partial |Q|}{\partial (Q)_{ab}} = |Q|(Q)^{ba}. \quad (70)$$

Using this expression for the derivative of the determinant and the identity

$$(Q - Q_i)^{-1}(Q_i - Q_j)(Q - Q_j)^{-1} = (Q - Q_i)^{-1} - (Q - Q_j)^{-1}, \quad (71)$$

which generalizes (5), we obtain, by differentiating under the integral sign in (68) a differential equation

$$\sum_{b,c=1}^2 (Q_{ij})_{bc} \frac{\partial}{\partial (Q_i)_{ba}} \frac{\partial I_N^\mu(Q)}{\partial (Q_j)_{dc}} = \mu_j \frac{\partial I_N^\mu(Q)}{\partial (Q_i)_{da}} - \mu_i \frac{\partial I_N^\mu(Q)}{\partial (Q_j)_{da}}, \quad (72)$$

where $i, j = 1, 2, \dots, N$ and we used the abbreviation $Q_{ij} = Q_i - Q_j$. This equation generalizes (4). We refer to this as the quaternionic Lauricella system. Let us stress that the order of quaternions are important in these formulas, since they are non-commutative and represented here as complex matrices.

As in two dimensions, $N = 2$ and $N = 3$ are special. Let us discuss them first. We have, using (66) in (68)

$$I_2^{(\mu_1, \mu_2)}(Q_1, Q_2) = \int \frac{d^4 q}{(q - q_1)^{2\mu_1} (q - q_2)^{2\mu_2}}, \quad (73)$$

which is evaluated using Feynman parametrization of the integrand to be

$$I_2^{(\mu_1, \mu_2)}(Q_1, Q_2) = \frac{\pi^2 \Gamma(2 - \mu_1) \Gamma(2 - \mu_2) \Gamma(\mu_1 + \mu_2 - 2)}{\Gamma(\mu_1) \Gamma(\mu_2) \Gamma(4 - \mu_1 - \mu_2)} \frac{1}{|Q_{12}|^{\mu_1 + \mu_2 - 2}}. \quad (74)$$

It can be verified that this satisfies (72). Let us note that it is translation invariant and homogeneous with degree $2 - |\mu|$. This expression generalizes (7) with doubled numbers reflecting the doubling of dimension from two to four.

3.2. Representation of the Möbius group

The conformal group of $\mathbf{R}^4 \cup \{\infty\}$ is isomorphic to the group of 2×2 matrices whose blocks are quaternions, namely, $SL(2, \mathbf{H})$ [20,21]. We have,

$$SL(2, \mathbf{H}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| |AC^{-1}DC - BC| = 1; A, B, C, D \in \mathbf{H} \right\}. \quad (75)$$

The matrix whose determinant is set to unity in this definition can be written in seven alternative forms [20]. We shall have occasion to use only the present one. The Möbius group acts on a quaternion Q similarly as the fractional linear transformation (9),

$$Q' = (AQ + B)(CQ + D)^{-1}. \quad (76)$$

Representation of the Möbius group is furnished by complex-valued functions of quaternions transforming as,

$$\begin{aligned} f(Q_1, Q_2, \dots, Q_N) &\mapsto f(Q'_1, Q'_2, \dots, Q'_N) \\ &= |CQ_1 + D|^{\Delta_1} |CQ_2 + D|^{\Delta_2} \dots |CQ_N + D|^{\Delta_N} f(Q_1, Q_2, \dots, Q_N), \end{aligned} \quad (77)$$

where $\Delta = (\Delta_1, \Delta_2, \dots, \Delta_N)$ denotes the N -tuple of weights, as before. Generalizing the transformation (11) of z_{ij} , the difference of two quaternions transforms under the Möbius group as

$$\begin{aligned} Q'_{ij} &= (AQ_i + B)(CQ_i + D)^{-1} - (AQ_j + B)(CQ_j + D)^{-1} \\ &= \left((AQ_i + B) - AC^{-1}(CQ_i + D) \right) (CQ_i + D)^{-1} \\ &\quad - \left((AQ_j + B) - AC^{-1}(CQ_j + D) \right) (CQ_j + D)^{-1} \\ &= (AC^{-1}D - B)(CQ_j + D)^{-1} CQ_{ij} (CQ_i + D)^{-1}, \end{aligned} \quad (78)$$

where we used the identity (71) in the last step. Taking the determinant of the matrices on both sides and using the fact that the determinant in (75) is unity, we obtain [20]

$$|Q'_{ij}| = |CQ_i + D|^{-1} |CQ_j + D|^{-1} |Q_{ij}|. \quad (79)$$

Let us derive the transformation of the volume element, generalizing (12). The differential of Q' , obtained from (76) is

$$dQ' = AdQ(CQ + D)^{-1} + (AQ + B)d(CQ + D)^{-1} \quad (80)$$

Since $dM^{-1} = -M^{-1}dMM^{-1}$ for any matrix M , we obtain

$$\begin{aligned} dQ' &= \left(A - (AQ + B)(CQ + D)^{-1}C \right) dQ(CQ + D)^{-1} \\ &= \left(AC^{-1}(CQ + D) - (AQ + B) \right) (CQ + D)^{-1} CdQ(CQ + D)^{-1} \\ &= (AC^{-1}D - B)(CQ + D)^{-1} CdQ(CQ + D)^{-1} \\ &= (AC^{-1}DC - BC)C^{-1}(CQ + D)^{-1} CdQ(CQ + D)^{-1}. \end{aligned} \quad (81)$$

We have thus a relation between the quaternion differentials as

$$dQ' = XdQY, \quad X, Y \in \mathbf{H}, \quad (82)$$

where the quaternions are expressed as 2×2 matrices. In order to obtain the transformation of the volume form (67) it is convenient to go over to the four-vector \mathbf{q} , written as a column matrix. A transformation of a quaternion by another $dQ \mapsto XdQ$ given in the 2×2 form can be written as a transformation of a four-vector as

$$\begin{pmatrix} dq_0 \\ dq_1 \\ dq_2 \\ dq_3 \end{pmatrix} \mapsto \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_3 & -x_2 & x_1 & x_0 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \end{pmatrix} \begin{pmatrix} dq_0 \\ dq_1 \\ dq_2 \\ dq_3 \end{pmatrix}. \quad (83)$$

The determinant of the 4×4 transformation matrix equals $|X|^2$. The volume form (67) obtained by taking wedge product of the components, transforms under this as

$$d^4q' = |X|^2 d^4q. \quad (84)$$

Similarly, a transformation of a quaternion by another $dQ \mapsto dQY$ from the right, given in the 2×2 form can be written as a transformation of the four-vector as

$$\begin{pmatrix} dq_0 \\ dq_1 \\ dq_2 \\ dq_3 \end{pmatrix} \mapsto \begin{pmatrix} y_0 & -y_1 & -y_2 & -y_3 \\ y_3 & y_2 & -y_1 & y_0 \\ y_1 & y_0 & y_3 & -y_2 \\ y_2 & -y_3 & y_0 & y_1 \end{pmatrix} \begin{pmatrix} dq_0 \\ dq_1 \\ dq_2 \\ dq_3 \end{pmatrix}. \quad (85)$$

The determinant of the 4×4 transformation matrix equals $|Y|^2$. Hence the volume form (67) transforms as

$$d^4q' = |Y|^2 d^4q. \quad (86)$$

Thus, under a transformation (82) the volume form transforms as

$$d^4q' = |X|^2 |Y|^2 d^4q. \quad (87)$$

Using this for the transformation (81) along with the unity of the determinant of the first factor as in the definition (75), we obtain

$$d^4 q' = |CQ + D|^{-4} d^4 q. \quad (88)$$

The exponent 4 is the dimension of the space, as did was 2 in (12). Using (79) and (88) we conclude that the integral (68) is equivariant with degree of homogeneity -2 , equal to the dimension of the space, provided $|\mu| = 4$, as can be verified by transforming the Q_i as well as the variable of integration Q in (68) according to (76), yielding

$$I_N^\mu(Q') = |CQ_1 + D|^{\mu_1} |CQ_2 + D|^{\mu_2} \cdots |CQ_N + D|^{\mu_N} I_N^\mu(Q), \quad (89)$$

with $|\mu| = 4$. Representations of the Möbius group $SL(2, \mathbf{H})$ may thus be constructed out of $|Q_{ij}|$ and $I_N^\mu(Q)$.

As in the two-dimensional case, (89) requires equality of μ_1 and μ_2 for $N = 2$, along with $\mu_1 + \mu_2 = 4$, to be equivariant. Thus, from (74) we derive the equivariant expression

$$I_2^{(1,1)}(Q_1, Q_2) = \frac{\pi^2 \Gamma(0)}{|Q_{12}|^2}. \quad (90)$$

For the other special case $N = 3$, the equation (72) is solved with

$$I_3^{(\mu_1, \mu_2, \mu_3)}(z_1, z_2, z_3) = |Q_{12}|^{-\frac{\mu_1 + \mu_2 - \mu_3}{2}} |Q_{23}|^{-\frac{\mu_2 + \mu_3 - \mu_1}{2}} |Q_{31}|^{-\frac{\mu_3 + \mu_1 - \mu_2}{2}}, \quad (91)$$

up to a multiplicative constant and $\mu_1 + \mu_2 + \mu_3 = 4$. As in the two-dimensional case, this can be verified by plugging the solution into (72).

For $N > 3$ complications as in two dimensions arise due to the existence of cross-ratios. These are invariants of the $SL(2, \mathbf{H})$ transformation. Considering a product of the determinants of the quaternions $\prod_{i,j=1}^N |Q_{ij}|$, we recall that it transforms according to (79). Writing a matrix with entries showing the order of transformation of Q_{ij} in Q_i along the rows, the invariants are given by the vectors in its kernel. For example, for $N = 4$ the matrix of exponents is

$$\mathcal{M} = \begin{matrix} & \begin{matrix} Q_{12} & Q_{13} & Q_{14} & Q_{23} & Q_{24} & Q_{34} \end{matrix} \\ \begin{matrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{matrix}. \quad (92)$$

The kernel of this matrix is two-dimensional. We can choose the basis vectors of the kernel as the transpose of

$$\begin{pmatrix} Q_{12} & Q_{13} & Q_{14} & Q_{23} & Q_{24} & Q_{34} \\ 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 & -1 & 0 \end{pmatrix}, \quad (93)$$

where we indicated the quaternions. Two invariants are correspondingly given by $\frac{|Q_{12}||Q_{34}|}{|Q_{13}||Q_{24}|}$ and $\frac{|Q_{14}||Q_{23}|}{|Q_{24}||Q_{13}|}$. Generally, for N quaternions, the matrix of exponents is $N \times \frac{N(N-1)}{2}$. Its kernel has dimension $\frac{N(N-3)}{2}$, which is the number of independent invariants that can be constructed from the determinant of the quaternions. The counting in two dimensions was similar, but the Plücker relations among the invariants further reduced their number. Thus, for $N = 4$ there was but a

single invariant, as we dealt with before, but in four dimensions there are two invariants for $N = 4$. Let us first define another set of quaternions

$$\chi_{ijkl} = Q_{ij} Q_{ik}^{-1} Q_{kl} Q_{jl}^{-1}. \quad (94)$$

The determinants of these matrices are invariant under $SL(2, \mathbf{H})$ thanks to (79). Determinants of all the χ 's are, however, not independent. A choice for the independent ones is to be made, thereby fixing the asymptotic behavior of the integrals. These are taken to be the cross ratios, the rest being functions of them. We shall denote the cross ratios as before

$$\xi_A = \prod_{\substack{i,j \\ 1 \leq i < j \leq N}} |Q_{ij}|^{\alpha_{ij}^A}, \quad (95)$$

where α^A for each A designates a basis vector in the kernel of the matrix \mathcal{M} , as the rows in (93), for example. These satisfy (18) as before.

Let us denote the trace of the 2×2 matrices χ by

$$\tau_{ijkl} = \text{Tr } \chi_{ijkl}. \quad (96)$$

Then, in view of the equivariance (89) of $I_N^\mu(Q)$, it can be written as products of $|Q_{ij}|$ with appropriate indices and a function of the cross ratios as

$$I_N^\mu(Q) = \prod_{\substack{i,j \\ 1 \leq i < j \leq N}} |Q_{ij}|^{\beta_{ij}} I_0(\xi), \quad (97)$$

where $I_0(\xi)$ is a function of the $N(N-3)/2$ cross ratios $\xi = (\xi_1, \xi_2, \dots, \xi_{\frac{N(N-3)}{2}})$ and the β 's satisfy (20), while (21) is replaced with

$$\sum_{\substack{i,j \\ 1 \leq i < j \leq N}} \beta_{ij} = -2. \quad (98)$$

Plugging in (97) with (95) and (94) in (72), we obtain an equation for $I_N^\mu(\xi)$ similar to (22) in terms of the quaternions χ . It is equivariant under $SL(2, \mathbf{H})$. An invariant set of equations is obtained by taking trace of the matrices involved. Taking trace on both sides the equations are expressed in terms of the quantities (96). We have, for each pair (i, j) ,

$$\begin{aligned} & \sum_{A,B} \left(\sum_{\substack{k,l \\ 1 \leq k,l \leq N \\ k \neq i, l \neq j}} \alpha_{ik}^A \alpha_{jl}^B \tau_{ijkl} \right) \xi_A \xi_B \partial_A \partial_B I_0(\xi) \\ & + \sum_A \left(4\alpha_{ij}^A + \sum_{\substack{k,l \\ 1 \leq k,l \leq N \\ k \neq i, l \neq j}} (\alpha_{ik}^A \alpha_{jl}^A + \alpha_{ik}^A \beta_{jl} + \alpha_{jl}^A \beta_{ik}) \tau_{ijkl} \right) \xi_A \partial_A I_0(\xi) \end{aligned} \quad (99)$$

$$+ \left(4\beta_{ij} + \sum_{\substack{k,l \\ 1 \leq k,l \leq N \\ k \neq i, l \neq j}} \beta_{ik} \beta_{jl} \tau_{ijkl} \right) I_0(\xi) = 0,$$

which generalizes (22). In order to write the equations in terms of cross ratios we need to relate the trace and determinant of χ_{ijkl} . To this end let us first note that

$$\begin{aligned} \chi_{ijkl} \chi_{ijlk} &= Q_{ij} Q_{ik}^{-1} Q_{kl} Q_{jl}^{-1} \underline{Q_{ij} Q_{il}^{-1} Q_{lk} Q_{jk}^{-1}} \\ &= Q_{ij} Q_{ik}^{-1} Q_{kl} Q_{jl}^{-1} \underline{(Q_{il} - Q_{jl})} Q_{il}^{-1} Q_{lk} Q_{jk}^{-1} \\ &= Q_{ij} Q_{ik}^{-1} Q_{kl} Q_{jl}^{-1} \underline{Q_{lk} Q_{jk}^{-1}} - Q_{ij} Q_{ik}^{-1} \underline{Q_{kl} Q_{il}^{-1}} Q_{lk} Q_{jk}^{-1} \\ &= Q_{ij} Q_{ik}^{-1} Q_{kl} Q_{jl}^{-1} \underline{(Q_{jk} - Q_{jl})} Q_{jk}^{-1} - Q_{ij} Q_{ik}^{-1} \underline{(Q_{il} - Q_{ik})} Q_{il}^{-1} Q_{lk} Q_{jk}^{-1} \\ &= \chi_{ijkl} + \chi_{ijlk}, \end{aligned} \quad (100)$$

where the underlined terms indicate the replacements made at various intermediate stages. Since χ_{ijkl} defined in (94) is a quaternion, this is an equation of 2×2 complex matrices. Let us rewrite it as

$$\chi_{ijkl} = -(\mathbf{I}_2 - \chi_{ijkl}) \chi_{ijlk}, \quad (101)$$

where \mathbf{I}_2 denotes the 2×2 identity matrix. We further note that for any 2×2 matrix M the identity

$$\det(\mathbf{I}_2 + M) = 1 + \text{Tr } M + \det M \quad (102)$$

holds. Taking determinant of both sides of (101) and using this identity we derive

$$\tau_{ijkl} = 1 - |\chi_{lijk}| + |\chi_{ijkl}|. \quad (103)$$

This relation will be used to express τ_{ijkl} in terms of the cross ratios in equation (99).

3.3. Four points

Let us write down the invariant case of four points $N = 4$ explicitly. Since the equations are rather cumbersome, we present the forms obtained by choosing $\beta_{14} = \beta_{34} = 0$, a freedom allowed by (20). We choose the independent cross ratios as in (93), namely,

$$x = |\chi_{1234}|, \quad y = |\chi_{4123}|. \quad (104)$$

In terms of these the equation (99) yields two equations for the invariant I_0 , for $(i, j) = (1, 2)$ and $(1, 3)$, namely,

$$\begin{aligned} (x + y - 1) \partial_x^2 I_0(x, y) + 2x \partial_{xy} I_0(x, y) - (x(\beta_{13} + \beta_{24}) + (1 - y)\beta_{12}) \partial_x I_0(x, y) \\ + x(2 + \beta_{12}) \partial_y I_0(x, y) + x\beta_{13}\beta_{24} I_0(x, y) = 0, \\ (x + y - 1) \partial_y^2 I_0(x, y) + 2y \partial_{xy} I_0(x, y) - (y(\beta_{13} + \beta_{24}) + (1 - x)\beta_{23}) \partial_y I_0(x, y) \\ + y(2 + \beta_{23}) \partial_x I_0(x, y) + y\beta_{13}\beta_{24} I_0(x, y) = 0, \end{aligned} \quad (105)$$

where $\vartheta_x = x \frac{\partial}{\partial x}$ denotes the logarithmic derivative. These are solved by the fourth Appell function [14,36], F_4 . The general solution is

$$\begin{aligned} I_0(x, y) = & C_1 F_4(2 - \mu_2, \mu_4, \mu_3 + \mu_4 - 1, \mu_1 + \mu_4 - 1; x, y) + \\ & C_2 x^{2-\mu_3-\mu_4} F_4(\mu_1, 2 - \mu_3, \mu_1 + \mu_2 - 1, \mu_1 + \mu_4 - 1; x, y) + \\ & C_3 y^{2-\mu_1-\mu_4} F_4(\mu_3, 2 - \mu_1, \mu_3 + \mu_4 - 1, \mu_2 + \mu_3 - 1; x, y) + \\ & C_4 x^{2-\mu_3-\mu_4} y^{2-\mu_1-\mu_4} F_4(2 - \mu_4, \mu_2, \mu_1 + \mu_2 - 1, \mu_2 + \mu_3 - 1; x, y), \end{aligned} \quad (106)$$

where C_1, C_2, C_3, C_4 are arbitrary constants and we used solutions of (20) with $\beta_{14} = \beta_{34} = 0$ and $|\mu| = 4$. Plugging in the four solutions in terms of this for $I_4(x, y)$, (97) gives the complete expression for $I_4^\mu(Q)$. Equations ensuing from the other choices of the indices are either not independent, as for $(i, j) = (1, 4)$, for example, or related to it by analytic continuation.

3.4. Correlation functions

The correlation functions are related to the integrals $I_N^\mu(Q)$ exactly as in the two-dimensional case, (39), namely,

$$G_N^{\Delta_1, \Delta_2, \dots, \Delta_N}(\phi_1, \phi_2, \dots, \phi_N) = \mathcal{F} \left(\prod_{\substack{i,j \\ 1 \leq i < j \leq N}} z_{ij}^{\ell_{ij}} I_N^\mu(Q) \right), \quad (107)$$

satisfying (40). Here we use $\phi_i = \phi(Q_i)$. Considerations same as before lead to the two and three point functions,

$$G_2^{\Delta_1, \Delta_2}(\phi_1, \phi_2) = \pi^2 C_{\Delta_1} \delta_{\Delta_1, \Delta_2} \Gamma(0) \frac{1}{|Q_{12}|^{\Delta_1}}, \quad (108)$$

$$\begin{aligned} G_3^{(\Delta_1, \Delta_2, \Delta_3)}(\phi_1, \phi_2, \phi_3) \\ = C_{\Delta_1 \Delta_2 \Delta_3} |Q_{12}|^{-\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3)} |Q_{13}|^{-\frac{1}{2}(\Delta_1 + \Delta_3 - \Delta_2)} |Q_{23}|^{-\frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1)}. \end{aligned} \quad (109)$$

Let us point out that while in the two-dimensional case we considered only chiral fields, in here we consider a general scalar field although we retain the same notation for the constants as in the two-dimensional case. The integrations are thus over the four-dimensional space rather than on contours now.

For the higher ones we need, once again, a projector. The projector in four dimensions is given by (48) with

$$\Pi_\Delta = \frac{1}{N_\Delta} \int \frac{\phi_\Delta(Q) \phi_\Delta(Q')}{|Q - Q'|^{4-\Delta}} d^4 Q d^4 Q', \quad (110)$$

where the constant of normalization is given by

$$N_\Delta = \frac{\pi^6 C_\Delta \Gamma(0) \Gamma(\Delta - 2) \Gamma(2 - \Delta)}{\Gamma(\Delta) \Gamma(4 - \Delta)} \quad (111)$$

The expressions for the correlation functions assume exactly the same form as in two dimensions, with quaternions in the integrals in lieu of complex variables and the values of N_Δ changed to (111) and I_0 taken to be a solution of (99). For example, the four-dimensional four-point function is given with such changes from (57) by

$$\begin{aligned}
& G_4^{\Delta_1, \Delta_2, \Delta_3, \Delta_4}(\phi_1, \phi_2, \phi_3, \phi_4) \\
&= \sum_{\Delta} \frac{C_{\Delta_1 \Delta_2 \Delta} C_{\Delta, \Delta_3, \Delta_4}}{N_{\Delta}} |Q_{12}|^{-\frac{\Delta_1 + \Delta_2 - \Delta}{2}} |Q_{34}|^{-\frac{\Delta_3 + \Delta_4 + \Delta - 4}{2}} \\
&\quad \times I_4 \left(\frac{\Delta_1 + \Delta - \Delta_2}{2}, \frac{\Delta_2 + \Delta - \Delta_1}{2}, \frac{4 + \Delta_3 - \Delta_4 - \Delta}{2}, \frac{4 + \Delta_4 - \Delta_3 - \Delta}{2} \right) (Q_1, Q_2, Q_3, Q_4),
\end{aligned} \tag{112}$$

where N_{Δ} is given by (111) and I_4 is given by (97) with $\beta_{14} = \beta_{34} = 0$, $\beta_{12} = 2 - \Delta$, $\beta_{13} = \frac{\Delta_2 - \Delta_1 + \Delta - 4}{2}$, $\beta_{23} = \frac{\Delta_1 + \Delta_4 - \Delta_2 - \Delta_3}{2}$, $\beta_{24} = \frac{\Delta_3 - \Delta_4 + \Delta - 4}{2}$ and (105) with $\xi_1 = x$ and $\xi_2 = y$.

4. Discussion and summary

To summarize, in this article we study N -point correlation functions of conformal primaries of conformal field theories in two- and four-dimensional Euclidean spaces. In the former case the conformal group is $SL(2, \mathbf{C})$, while in the latter case it is $SL(2, \mathbf{H})$. We demonstrate the semblance of the computations in the two cases.

Instead of copies of the conformal compactification of the Euclidean space within the light cone in two higher dimensions, we choose to work directly with the Fulton-Macpherson compactification of the N -point configuration space. For the four-dimensional Euclidean space with infinity adjoined, $M = \mathbf{R}^4 \cup \{\infty\}$, the configuration space of N points is

$$C_N(M) = M^N \setminus \{q_i \in M, q_i \neq q_j; i, j = 1, 2, \dots, N\} \tag{113}$$

The Fulton-Macpherson completion is achieved by considering the embedding [34,35]

$$\begin{aligned}
& \gamma : C_N(M) \longrightarrow M^N \times (S^3)^{\binom{N}{2}} \times [0, \infty]^{\binom{p}{3}}, \\
& (q_1, q_2, \dots, q_N) \longmapsto (q_1, q_2, \dots, q_N, v_{12}, \dots, v_{(N-1)N}, a_{123}, \dots, a_{(N-2)(N-1)N}),
\end{aligned} \tag{114}$$

where each of

$$v_{ij} = (q_i - q_j)/|q_i - q_j| \tag{115}$$

describes a three-sphere S^3 and the scalars

$$a_{ijk} = \frac{|q_{ij}|}{|q_{ik}|} \tag{116}$$

assume values in the non-negative real line. Representations of the conformal group, in particular, the integral I_N^{μ} , is to be chosen from among the functions of these variables. Invariance under translation bars a representation to depend on q_i alone and rotational invariance keeps it from having dependence on v_{ij} . The difference $|q_i - q_j|$, however, is allowed. Let us note that v_{ij} will appear in the correlation functions of higher rank tensor fields. The expression (97) is thus a regular function on the Fulton-Macpherson compactification of the configuration space with the cross-ratios expressed as

$$|\chi_{ijkl}| = a_{ijk} a_{lkj}. \tag{117}$$

Correlation functions are given by representations of the conformal groups on the configuration space. We obtain the representations of the groups directly without recourse to the corresponding Lie algebras. Consistency of the expressions can be verified by writing down the generators of the groups as differential operators. The integrals then get related to conformal blocks. While

using the conformal algebra is effective in two dimensions, non-commutativity of the quaternions render the computations difficult in the four dimensional case. This approach also avoids building the N -point functions from the “single-particle” representations by tensoring and eschews the insertion of “single-particle” Casimirs.

In two dimensions, the representation of the conformal or the Möbius group is obtained in terms of a Lauricella system. A differential equation for the invariant part is derived for $N > 3$ from the Lauricella system. We present solutions for $N = 4, 5$, the former in terms of Gauss Hypergeometric function and the latter in terms of the Appell function F_2 . Parsing the correlation functions into three-point functions by inserting projectors we write integral formulas for the correlation functions from the representations. The projectors themselves are expressed in terms of the two-point Lauricella system.

These considerations directly generalize to the four-dimensional case. We define integrals in terms of determinants of quaternions. Differentiating with the complex 2×2 matrices representing quaternions we then set up a generalized Lauricella system of differential equations for the integrals. Representations of the conformal group $SL(2, \mathbf{H})$ are then obtained from the solutions of the differential equation. The invariant cross-ratios are given by the determinant of quaternions. In order to write the equations for the invariant part we use the relation between the trace and determinant of 2×2 matrices. While the equations are obtained for an arbitrary N , we present the computation for $N = 4$, in which case the integral is given by the Appell function F_4 . As in the two dimensional case, the correlation functions are parsed using projectors obtained as solutions to the Lauricella system for $N = 2$, without requiring it to transform under the conformal group. Let us stress that the correlations functions in the two-dimensional case have been known for decades. The four-dimensional four-point function in the comb channel has been worked out earlier [23] and our results match these expressions. It is their direct connection with the Fulton-Macpherson compactification of configuration spaces of N marked points and the quaternionic Lauricella system that governs them in four dimensions that is novel in here.

Let us also point out that the projectors (50) and (110) can be expressed in terms of the so-called *shadow operator* by choosing to perform the integration over y first [16]. We have chosen to postpone it to a later stage of the computation in order to relate to the integrals I_N^μ . Further, we have presented the most general expressions for the solutions of the Lauricella systems. However, the correlation functions were parsed in terms of three-point functions. In order to be concomitant with the operator product expansion some of the terms must be discarded in the final expressions for the correlation functions by the monodromy projection [16]. For example, only one of the two terms in (28) is to be retained in (57), namely,

$$I_4 \left(\frac{2-\Delta+\Delta_1-\Delta_2}{2}, \frac{2-\Delta+\Delta_2-\Delta_1}{2}, \frac{\Delta+\Delta_3-\Delta_4}{2}, \frac{\Delta+\Delta_4-\Delta_3}{2} \right) (z_1, z_2, z_3, z_4) \Bigg|_{\text{Projected}} \quad (118)$$

$$= z_{12}^{\Delta-1} z_{13}^{\frac{\Delta_2-\Delta_1-\Delta}{2}} z_{23}^{\frac{\Delta_1-\Delta_2-\Delta_3+\Delta_4}{2}} z_{24}^{\frac{\Delta_3-\Delta_4-\Delta}{2}} F \left(\frac{\Delta+\Delta_1-\Delta_2}{2}, \frac{\Delta+\Delta_4-\Delta_3}{2}, \Delta; \xi \right).$$

Similarly, only two of the four terms in (106) survive the monodromy projection. The integral I_4 to be used in (112) is

$$\begin{aligned}
& I_4 \left(\frac{\Delta_1 + \Delta - \Delta_2}{2}, \frac{\Delta_2 + \Delta - \Delta_1}{2}, \frac{4 + \Delta_3 - \Delta_4 - \Delta}{2}, \frac{4 + \Delta_4 - \Delta_3 - \Delta}{2} \right) (Q_1, Q_2, Q_3, Q_4) \Big|_{\text{Projected}} \\
&= x^{\Delta-2} |Q_{12}|^{2-\Delta} |Q_{13}|^{\frac{\Delta_2 - \Delta_1 + \Delta - 2}{2}} |Q_{23}|^{\frac{\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4}{2}} |Q_{24}|^{\frac{\Delta_3 - \Delta_4 + \Delta - 4}{2}} \times \\
&\quad \left(C_2 F_4 \left(\frac{\Delta + \Delta_1 - \Delta_2}{2}, \frac{\Delta + \Delta_4 - \Delta_3}{2}, \Delta - 1, \frac{\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4}{2} + 1; x, y \right) \right. \\
&\quad \left. + C_4 y^{\frac{-\Delta_1 + \Delta_2 + \Delta_3 - \Delta_4}{2}} F_4 \left(\frac{\Delta + \Delta_3 - \Delta_4}{2}, \frac{\Delta + \Delta_2 - \Delta_1}{2}, \Delta - 1, \frac{\Delta_2 - \Delta_1 + \Delta_3 - \Delta_4}{2} + 1; x, y \right) \right). \tag{119}
\end{aligned}$$

In all the cases the monodromy considerations project out part of the basis of the Lauricella system. The integrals entering the expressions for correlation functions are generically multi-valued, rendering their direct evaluation complicated. Expressing these as solutions to differential equations may be very useful in this regard. The situation is similar to the evaluation of periods of algebraic varieties, whose evaluation in various domains of convergence is substantially facilitated by expressing them as solutions Picard-Fuchs differential equations. The Lauricella system developed here in terms of quaternions are quite general. We expect this formalism to be useful in computing the correlation functions in four dimensions as well as in computing Feynman integrals in quantum field theories.

CRedit authorship contribution statement

All authors contributed equally.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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