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7 Geometry Decides Gravity, Demanding General Relativity — it is Thus the Quantum Theory of Gravity

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Abstract. What decides the laws of physics? Geometry, at least largely. Its transformation groups (which may not be symmetry groups) greatly limit physical laws. For massless objects, electromagnetism and gravitation that can couple to massive matter, these are completely determinative [3]. Here we only outline reasons and derivations. Details, and discussions of related subjects, are elsewhere.

7.1 Transformation groups

A fundamental transformation group of our geometry is the Poincaré group ([1], sec. II.3.h, p. 45), the rotations in 3+1 space (the Lorentz group) and the translations (given by the momentum operators). Whether space is invariant under it is irrelevant. It is a transformation group, a subgroup of the complete one: the conformal group [6] of a 3+1 (locally) flat real space. This is true even if directions (simulated by the vertical) were different. Neither points nor directions need be identical. (With the earth the vertical and its center appear different — because there is a material body.)

An example of a transformation group is the rotation group (for any dimension), which we consider in a space with a direction different, simulated by a magnetic field. Functions of angles can be written as sums of basis vectors of the rotation group — spherical harmonics for our space — these forming sets called representations of the group (for 3-space labeled by the total angular momentum). A rotation changes each function — basis vector — in a sum replacing it by a sum of basis vectors. Each basis vector is replaced only by a sum of vectors of the same representation. States of a representation are mixed with states of the same representation — of the transformation group — but not with states of another. This cannot be done using states of unitary groups. Rotations are fundamental properties of real spaces, ones whose coordinates are real numbers, so more intrinsic than symmetries. It is however provocative that they are symmetries also.

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The Poincaré group is an inhomogeneous group ([1], sec. II.3.h, p. 46, one with a semisimple part — the simple Lorentz group ([1], sec. II.3.e, p. 44) — and an Abelian invariant subgroup, the translations ([1], sec. II.3.f, p. 44), this transforming under the regular representation of the semisimple part ([1], chap. VI, p. 170). Inhomogeneous groups are far richer than semisimple ones (like the rotation group) with which we are more familiar. Prejudices from the latter may be completely wrong for richer groups.

7.2 Labeling representations and states

States of a group representation are labeled by eigenvalues of a set of operators invariant under all group operations — these giving representations — plus eigenvalues labeling states. For the rotation group these are the total angular momentum, and its component along some axis. These operators are completely determined (up to isomorphism) but for inhomogeneous groups there are choices (thus richness). Which operators shall we take diagonal: semisimple ones, Abelian ones or combinations? These give representations of different forms. Here we consider only (as is usual, but not usually explicit) representations with all momentum operators diagonal (thus no others, which are all semisimple, can be).

Rotation representations have one label, those of SU(3) two, and so on. The Poincaré group requires two labels. For an object at rest these — its mass and total spin — are needed to specify the object. There can be no more (internal labels are not relevant to these transformations of geometry). For free objects there is nothing more to say.

Representations with all momentum operators diagonal break into four sets, those with real mass, $m2 > 0$ (to which we belong); imaginary mass, $m2 < 0$; zero mass $m = 0$ so $m2 = 0$; and momentum 0 representations, to which coordinates and momenta belong. Momentum has no momentum.

Here we consider just massless representations; for these we can say the most. There is then one more label, the helicity. Representations with helicity 1 give electromagnetism, with helicity 2 gravitation. (Neutrinos cannot be massless ([3], sec. 4.4.4, p. 70]).)

There is one further condition — obvious although its mathematical importance may not be. These objects must couple to massive matter else we could not know of them — they would not exist. This is very difficult, so very determinative. Massive and massless objects are really quite different.

7.3 Little groups

Representations are found using a little group ([3], sec. 1.1.3, p. 4; sec. 2.2, p. 12), a subgroup whose representations are known. We need only the action of the remaining operators on its states. Then we know all states of the full group, and the action of all operators on them.

For a massive object, which we can take at rest, the little group is the (simple) rotation group. Its representations, including explicit expressions for its states, are known. On these we calculate the action of the boosts giving the (pseudo-orthogonal) Lorentz group. On its states we find the effect of the momentum generators, and then have all representations (of this type) of the (inhomogeneous) Poincaré group.

Massless objects, like the photon, cannot be at rest. Their little groups are the subgroups leaving invariant a momentum component. Such little groups are not semisimple ([1], sec. IV.9.a, p. 144), but solvable ([1], sec. XIII.3.a, p. 376). These types of groups are quite different. Hence it is almost impossible to construct interactions between (massive) semisimple objects and (massless) solvable ones. Thus electromagnetism and gravitation are fully determined. Restrictions are so great that there is no choice. We might expect that coupling two such different objects is impossible. Fortunately it is in two cases, helicity 1 and 2 (perhaps 0).

7.4 All terms must transform the same

In an equation all terms (in the sum) must transform as (perhaps different realizations ([1], sec. V.3.c, p. 157) of the same state of the same representation else it would be different in different systems — inconsistent. Dirac's equation is a sum of terms one the mass (a scalar) times the solution, the statefunction. Hence all terms must transform as the solution (a bispinor). These include interactions between the massive object and electromagnetic and gravitational fields. For coupling such interactions have to transform as the solution. They are products of semisimple terms and solvable ones (actually functions found from these by the remaining Poincaré transformations, these different for different types of objects).

This is actually not difficult for electromagnetism. It requires minimal coupling, the reason that the photon couples this way ([3], sec. 5.3, p. 81). For helicity-2 gravitation it is much harder, almost impossible, to couple.

Helicity-2 has five states. Products of it with (massive) Lorentz group states must transform properly under all groups. We need scalars formed from products of interaction terms with a Lorentz basis vector (which solutions of Dirac's equation, massive statefunctions, are). However there is no irreducible Lorentz representation with five states ([3], sec. 4.4, p. 67). There can be no such scalar, just as there cannot be one constructed from angular momentum 1 and 2 representations.

The number of components must be reduced requiring relations between them, nonlinear ones. Fortunately the helicity-2 representation has such: the Bianchi identities. Hence massive objects and gravitation can interact.

A gravitational field is produced by energy, and has energy. Thus a gravitational field produces a gravitational field — it is nonlinear. This argument is correct but it hides the underlying mathematics. Gravitation must be nonlinear since only that non- linear representation can couple to matter. Fortunately both arguments give the same condition.

Also the gravitational field is attractive ([3], sec. 4.2.5, p. 60) while the electromagnetic charge can have either sign.

7.5 Objects for massless representations

Before outlining the derivation of Einstein's equation we need certain aspects of massless representations: connections ([3], sec. 3.2 p. 33), why they are the basis states of massless representations (and only these), what gauge transformations are ([3], sec. 3.4 p. 43), why massless representations (only) have them, and what the fundamental fields are ([3], sec. 3.3 p. 37).

That gauge transformations are Poincaré transformations for massless objects, and only these, is clear. Take an electron and photon with momentum and spin parallel (so both spins are parallel to the momentum). Lorentz transform to a system in which both momenta remain the same but the electron's spin is changed. Its spin and momentum are no longer along the same line, but those of the photon must be — it is transverse. Thus there are transformations that act on the electron but not on the photon (?). This cannot be. Poincaré transformations do not depend on the object acted on. What are these extra transformations? Of course gauge transformations. Their properties are given by the Poincaré group.

Gauge transformations are neither rotations nor boosts but products. Go to the electron's rest frame, rotate its spin, then reversing the first transformation go the frame with the original momenta. The momenta remain the same but the spin direction of the electron is different. Gauge transformations for the photon are given by this product acting on it.

It is important to understand that the electromagnetic field is not transverse because of gauge invariance. The form of group operators is determined by its structure. Generators of the rotation group are fixed by their commutation relations, not by the hydrogen atom for example. Thus the commutation relations of the Poincaré group require that the basis states of its massless representations be connections and undergo gauge transformations. And clearly these are possible only for massless representations.

What is the difference between a connection and a tensor? Transformations of a tensor are homogeneous (with an a for each index), schematically,

$$T'_{i\ldots} = a_{ij}\ldots T_{j\ldots} \tag{7.1}$$

Transformations of a connection are inhomogeneous,

$$\Gamma'_{i\ldots} = a_{ij}\ldots \Gamma_{j\ldots} + \Lambda_{i\ldots}; \tag{7.2}$$

Λ does not depend on Γ but is a function of the transformation (the a's). A tensor transformation changes two components simultaneously, for a connection only one need be. An example of a tensor transformation is

$$x' = x\cos\theta + y\sin\theta, \;\; y' = -x\sin\theta + y\cos\theta, \tag{7.3}$$

while for a connection (here the electromagnetic vector potential),

$$A'_x = A_x + \Lambda_x, \tag{7.4}$$

where Λ is arbitrary, and similarly for the gravitational Γ.

The finite-dimensional representations of the Euclidean group SE(2), the little group for massless representations with rotation operator M diagonal, are not unitary as the algebra matrices are not hermitian. They consist of blocks of the form, say, with the N's the other two generators,

$$M = \begin{pmatrix} m & 0 \\ 0 & m-1 \end{pmatrix}, \tag{7.5}$$

and

$$N_1 = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, \; N_2 = \begin{pmatrix} 0 & iu \\ 0 & 0 \end{pmatrix}, \; \text{or } N_1 = \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}, \; N_2 = \begin{pmatrix} 0 & 0 \\ -iv & 0 \end{pmatrix}; \tag{7.6}$$

u, v arbitrary (N's give arbitrary gauge transformations — these also depending on group parameters, add arbitrary functions to representation basis vectors). There are two representation forms, upper and lower.

For semisimple algebras for each off-diagonal entry there is a corresponding one across the diagonal, but with solvable algebras this is not true for all entries. Thus, as easily seen from this simple example, there are transformations of a solvable group that add terms to basis states, as we are quite familiar for electromagnetism and gravitation. That is why their states are connections, and only their states. For other classes of (momentum-diagonal) representations the little group is semisimple.

We thus see what connections and gauge transformations are, how they are related, are required by the little group being solvable, and why they are properties of massless representations, and possible only for them.

7.6 Fundamental fields

Equations to determine the electromagnetic and gravitational fields are needed. But which fields? They are connections, the gravitational connection (not the metric which transforms under a momentum-zero representation) and the electromagnetic connection, the potential A. These are massless objects, and connections are the massless states.

Electromagnetic fields E and B are not physical objects, not gauge invariant ([3], sec. 3.3.1, p. 37) and do not transform under a proper representation. They are products of states of massless and of momentum-zero representations. For rotation groups a product of representations can be written as a sum. However this product is of representations of different types (there is only one type for the rotation group). Such products have not been characterized, and perhaps cannot be. There may be nothing further to be said about them. This has to be looked at.

That the electromagnetic field is not a physical object and can- not be measured is trivial. How do we measure a field? We observe the behavior of a charged object in it. In elementary physics we use pith balls. But there are no such things. These are merely collections of electrons, protons and neutrons (and so on), as are we. Fields act on these.

What acts on a electron? From Dirac's equation clearly the potential. The behavior of the electron then gives that, and that is what is measured. Other fields are merely functions of it, unmeasurable thus without basic significance (for this reason also).

We can now outline derivations of the equations for gravitation and electromagnetism. These are standard. What is different is the context. The theories are derived from the Poincaré group thus are unique. Electromagnetism and gravitation are what they are because that is what geometry wants them to be. They are not guesses that happened to be correct. And gravitation is not determined by the equivalence principle — that is a consequence. Too many believe that clues leading to the discovery of a theory are the reason for it. But how we discover does not underlie physics. There are reasons for the way physics is. We generally do not know them (geometry is highly suspect) but for massless representations we do.

7.7 Electromagnetism and what it must be

Details for the electromagnetic case can easily be worked out ([3], sec. 7.2, p. 124) so we just summarize. One Maxwell equation is the Bianchi identity, the other is the trace of the electromagnetic tensor; this is equal to the current. Since the electromagnetic tensor is not a physical object we need an expression for the potential. This is given by its covariant derivative, the momentum operator acting on the electromagnetic statefunction,

$$A_{\mu;\nu} = A_{\mu,\nu} - \frac{2ie}{m2}(\psi^{+}\gamma_{\mu}\psi_{,\nu} + \psi^{+}_{,\nu}\gamma_{\mu}\psi). \tag{7.7}$$

The second covariant derivative is zero; the momentum belongs to the momentum-zero class, so its momentum — its covariant derivative — is zero. The covariant derivative of a spinor gives minimal coupling.

The equations are for the potential giving it in terms of the statefunctions of the charged objects; their equations include it. These equations govern. Maxwell's equations are classical, and for a nonphysical object: the electromagnetic field. Thus they are only of calculational use, they are not fundamental. Since neither the electric nor the magnetic field really exists that Maxwell's equations seem to distinguish between them is meaningless. It is purely a matter of notation. Hence for example a magnetic monopole cannot exist ([3], sec. 7.3, p. 131). There is no way of putting one in.

7.8 Gravitation

These objects are determined by the Poincaré group so we need expressions for its generators, here for the Abelian part, the momentum operators, the covariant derivatives ([3], sec. 1.2.2, p. 8). Thus we have to find the covariant derivative of, the momentum acting on, the connection — the gravitational statefunction. There are two ways of finding it ([3], sec. 8.2.1, p. 146), using the covariant derivative of the covariant derivative, and the ordinary derivative of a vector whose a covariant derivative is given by the usual rules for that of a tensor. These must be the same. So we get the covariant derivative of the gravitational statefunction, the connection. Thus

$$ip_\kappa \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu;\kappa} = \Gamma^\lambda_{\mu\nu,\kappa} + \Gamma^\lambda_{\phi\kappa}\Gamma^\phi_{\mu\nu} - \Gamma^\phi_{\mu\kappa}\Gamma^\lambda_{\phi\nu} - \Gamma^\phi_{\nu\kappa}\Gamma^\lambda_{\mu\phi}. \tag{7.8}$$

From this we get the second covariant derivative, then the commutator of momentum operators (0 since momenta form an Abelian subgroup), and the expressions for the Casimir invariants and curvature tensor. Einstein's equation for free gravitation then follows using any standard derivation.

With matter present a term has to be added to the covariant derivative. This gives the energy-momentum tensor. It is here that there may be some freedom. What is the energy-momentum tensor? There are expressions for scalars, spin-$\frac{1}{2}$ objects and the electromagnetic field ([3], sec. 9.2, p. 153). (There is no such thing as dust.) While these are reasonable it is not clear there are no other possibilities. This remains to be looked at. These give the equations for the fields as determined by the sources.

This raises a problem for scalar objects ([3], sec. 4.2.7, p. 61).Do they interact with gravity? There is no reason to believe so. It is an article of faith that gravitation is universal, interacting with all objects, and in the same way. Actually it has only been tested in two cases, collections of spin-$\frac{1}{2}$ objects and the electromagnetic field. An open mind can be useful.

7.9 Trajectories are geodesics

What determines the behavior of objects in fields? Interaction terms ([3], sec. 5.3, p. 81). For a scalar, trajectories are geodesics ([3], sec. 5.2, p. 74). Coordinates, velocities, momenta and the metric all belong to the momentum-zero class of representations. Their momenta — covariant derivatives — are thus 0. Setting the covariant derivative of the velocity to 0 gives the geodesic. This can also be found quantum mechanically.

7.10 A cosmological constant must be 0

It is traditional to include in Einstein equation a cosmological constant. Clearly that must be 0 ([3], sec. 8.1.4, p. 139). It sets a constant equal to a function of space and time, and a real number, the cosmological constant, equal to a complex one G, obviously wrong. Why is a gravitational field complex? Regarding it as due to curvature of space can be misleading. Mathematically it is possible to write metric g and Γ as spacetime functions giving the geometry of the entire 3+1- dimensional space. But fields depends on matter and its behavior. These are arbitrary and can be varied at will (unless the statefunction of the entire universe for all time is known). Thus the field is a function of time. From the field at one time we get it at all times using (as usual) the equation for the statefunction (schematically)

$$i\frac{d\Gamma}{dt} = H\Gamma; \tag{7.9}$$

H is the gravitational Hamiltonian which includes arbitrary matter. Thus Γ is a gravitational wave. Solving we get that Γ, the gravitational field, is complex. A physical field, a physical wave, must be complex. More fundamentally G is a function of massless basis vectors, while the cosmological constant belongs to the momentum-zero representation. Setting them equal is like equating a vector and a scalar. And with a cosmological constant gravitational waves would have the fascinating property that the metric, thus detectors, react to them, not only an infinitely long time before they arrive, but even an infinitely long time before they are emitted. The argument is the same, but here stronger, as that showing classical physics is inconsistent and quantum mechanics necessary ([2], chap. 1, p. 1).

Taking the gravitational field as a purely geometric object is not fully useful, and likely not fully possible because it is determined by physics and physical objects have arbitrariness. This is not surprising since the gravitational field is massless and that has no meaning for geometrical objects.

Geometry then, through its transformation group — the Poincaré group — determines what gravitation is: knowing that it is a massless helicity-2 object. What is left? First the functions of the matter statefunctions that gives it — energy-momentum tensors might still have some freedom, although perhaps not. This should be looked at. More mysterious is the value of the coupling constant, the gravitational constant (perhaps more than one ([3], sec. 9.3.4, p. 162)). Yet the most fundamental question is why gravitation exists. If it does, and fortunately it does, it is determined. But the Poincaré group does not require its existence. What does?

7.11 General relativity is quantum gravity

Can there then be a quantum theory of gravity? There is: general relativity. It is the first complete, consistent quantum theory. What is a quantum theory and why do people dislike general relativity?

A quantum theory ([2], chap. 1, p. 1; [5], chap. II, p. 54) is a consistent theory that includes (at least) proper definitions of the Poincaré generators ([1], sec. XIII.4.b, p. 382), those of the Lorentz transformations and of translations (thus momentum operators). This is necessary else it would be impossible to transform to different systems, but transformations are possible and necessary. Without these physics cannot be. Beyond that what else is there to require? What else is possible? General relativity satisfies these requirements, there are strong reasons to believe it is consistent, and is unique. Thus it is the quantum theory of gravity. There is nothing that can be done to "quantize" it.

There is confusion abut quantum mechanics [5]. Both this name and wavefunction are unfortunate. Discreteness is neither universal nor fundamental. For angular momentum it is a property of the rotation group (and forms of representations of semisimple groups in general). For atoms it comes from requirements such as that there be no infinities. But there is no quantitization for a free particle, one tunneling through a barrier, or for a huge number of other cases.

Nor does quantum mechanics lead to objects fluctuating. The gravitational field does not fluctuate. If an experiment is redone many times the results are different (usually only a little) for each repetition. For a box of neutrons the number decaying in each second varies. But the neutrons do not vary, they do not oscillate.

There are also fears of infinities. But these occur for intermediate steps in a calculational procedure: perturbation theory. They are regarded as due to point particles. But there are no such objects as point particles, even classically. There is nothing in any fundamental equation of physics that even hints of them. Overemphasis on infinities is based of a belief that the universe is carefully designed to make physicists' favorite approximation method work. That is not likely. What they show is that perturbation theory has problems, not electromagnetic theory or gravitation.

There is no reason to think either is inconsistent, but rather there are strong reasons to believe both are consistent.

If general relativity is a quantum theory does it have uncertainty relations? As with any field theory it does. But they are different from those we are more familiar with — for which the product of uncertainties is greater than some number. But gravitation is, necessarily, nonlinear. Thus for it the product of uncertainties is greater than a function of the statefunction (the connection). They are far more complicated and depend on the physical situation. It would be interesting find these for some cases.

There is much understood but still much to be learned about gravitation. Perhaps these comments can stimulate further thought about such questions. We see again that geometry imposes its will on physics [1,2,3,4,5,6]. For massless representations this is particularly clear, for others less so — but perhaps only for now.

Some of these topics will be discussed elsewhere [7], for some in greater depth and in a more elementary manner.

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