SLAC-PUB-1460
(T)
July 1974

The Beginner's String*

Jorge F. Willemsen

Stanford Linear Accelerator Center
Stanford University, Stanford, Ca., 94305

Preface

Professor Abarbanel has asked me to prepare a set of notes on the string model to supplement the material being presented in his lectures. As a working hypothesis it was assumed that the participants know a modicum about this field. Combining this premise, the extensiveness of the work done on the model, and the constraint of space limitation, it seems to me a "survey" format is appropriate for these notes: no pretenses to completeness are maintained, and the notes are in no way a review. Rather, they are designed to introduce the vocabulary of the field, to provide a source of references to genuine reviews, and to the original literature only where reviews are not available.

(To be presented at the Summer Institute on Particle Physics, Stanford Linear Accelerator Center, Stanford, Ca., July 29 - August 10, 1974)

*Work supported by the U. S. Atomic Energy Commission.

A. What is the String Model?

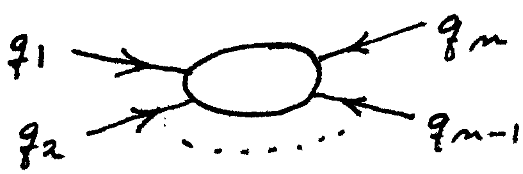
The string model is basically an attempt to understand what physical structures can underlie scattering amplitudes of the type originally written by Veneziano.[1] The hope is that by understanding the physics of these amplitudes, one may learn something about the structure of hadrons. This, in turn, should lead to a variety of new predictions.

An example of a Veneziano amplitude, for 2 to 2 scattering, is

$$A_4(s,t,u) = g\left[B(1-\alpha_s, 1-\alpha_t) + (s \leftrightarrow u) + (t \leftrightarrow u)\right]; \tag{1}$$
$$B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y) = \int_0^1 dz\, z^{x-1}(1-z)^{y-1}.$$

Veneziano proposed this simple expression for the amplitude because it incorporates many desirable features such an an amplitude should have: Regge asymptotic behaviour, crossing symmetry in the case of linearly rising trajectories, daughters with residues in fixed ratios, saturation of superconvergence relations, and duality between Regge poles and resonances. We will discuss other properties of Veneziano amplitudes later.

For now we want to focus on a different feature of Eq. (1), namely, that it is easily generalized to have an arbitrary number of external particles, and that the generalized amplitudes share the good features of the original amplitude. A very concise way to write a term in such an amplitude is the following:[2]



(all q in units of slope $\alpha' \sim (1\ \text{GeV}^2)^{-1}$.)

Figure 1

$$(2)\quad A_n = \langle 0 | V_2 \Delta_{12} V_3 \Delta_{13} \ldots\ldots V_{n-1} | 0 \rangle ,$$

where the "propagators" are

$$\Delta_{ij} = [s_{ij} + R + \alpha(0)]^{-1} = \int_0^1 dx \; x^{s_{ij} + R + \alpha(0) - 1} ,$$

with

$$s_{ij} = (q_i + q_{i+1} + \ldots\ldots + q_j)^2 \; ;$$

$$(3)\quad R \equiv \Sigma_{n=1}^{\infty} n \, a^{\dagger}_{n\mu} a_n^{\mu} \; ;$$

and the "vertices" are

$$(4)\quad V_j = \left[\exp i \Sigma_n \frac{q_{j\mu} a_n^{\dagger\mu}}{\sqrt{n/2}} \right] \left[\exp i \Sigma_n \frac{q_{j\mu} a_n^{\mu}}{\sqrt{n/2}} \right] .$$

In these equations, a_n^{μ} and $a_n^{\mu\dagger}$ are simple harmonic oscillator creation and annihilation operators,

$$[a_{n\mu} , a^{\dagger}_{m\nu}] = - \delta_{nm} g_{\mu\nu} .$$

The "ground state" $|0\rangle$ is annihilated by the a_n^{μ} : $a_n^{\mu} |0\rangle = 0$. Using simple properties of these operators, and the integral representation for the propagators, one can easily calculate an integral representation for A_n analogous to the integral representation for A_4 given in Eq. (1). The properties of the amplitude can then be studies from the integral representation. Historically, the integral representation for the amplitude was proposed first,[3] and then it was discovered that the factorized "operator" form Eq. (2) was possible. This factorization property was a major step forward in arriving at an interpretation of the physics of the model.

To see why, examine what the vertex creates from the ground state (which is interpreted as the initial, unexcited external particle):

$$(5)\quad V|0\rangle = |0\rangle + i\sum_n \frac{q a_n^{\dagger}}{\sqrt{n/2}}|0\rangle + \frac{i^2}{2!}\sum_{n,m}\frac{2}{\sqrt{nm}}(q a_n^{\dagger})(q a_m^{\dagger})|0\rangle + \cdots\cdots$$

A general term in this power series expansion of the exponential looks like:

$$(6)\quad \frac{i^n}{n!} q_{\mu_1} q_{\mu_2}\cdots q_{\mu_n} \left\{ \begin{array}{l} a^{\dagger}_{1,\mu_1} a^{\dagger}_{1,\mu_2} \cdots a^{\dagger}_{1,\mu_n} \\ a^{\dagger}_{2,\mu_1}(\qquad '' \qquad) \\ \cdots\cdots\cdots\cdots \\ a^{\dagger}_{\alpha,\mu_1} a^{\dagger}_{\beta,\mu_2}\cdots a^{\dagger}_{\omega,\mu_n} \end{array} \right\} |0\rangle$$

Any state in the bracket is an eigenstate of the "mass operator" R, with a definite eigenvalue, and we see that for a given tensor structure of n indices as above, there are an infinite number of such eigenstates. Furthermore, the index "n" also runs to infinity, and so V creates states of all possible spin from the ground state, i.e., excites the initial hadron into all possible Regge recurrences and daughters.

However, the propagators Δ only have poles for a single value of R, namely R = $s+\alpha$ (0), for fixed s. Let us suppose, for example, that ($s+\alpha(0)$) = 2. Eigenstates of R can be

$$a^{\dagger}_{1,\mu_1} a^{\dagger}_{1,\mu_2}|0\rangle \quad \& \quad a^{\dagger}_{2,\mu_1}|0\rangle$$

The first, doubly occupied, state has two tensor indices, and so maximum spin 2. The singly occupied state has spin 1. However, there is a second spin one state which is obtained by appropriately anti-symmetrizing the tensor indices μ_1 and μ_2. And so on.

Generalizing, we can readily see that at a pole, there will be a single state with maximum spin M = $\alpha(0) + s$ and a large number of states of lower spin, making up the daughters. The degeneracy of the daughter levels is very large. In fact, asymptotically one finds that the total number of states at a given pole grows exponentially (Hagedorn degeneracy.)

This is the famous requirement that dual amplitudes will be dual only if the direct channel spectrum is very rich. Most of the degeneracy of the model could be removed if we used only a <u>single</u> harmonic oscillator operator a_μ, instead of an infinite number of oscillators $a_{\mu n}$. In fact, one early attempt was made to construct dual amplitudes using only a single harmonic oscillator. The resulting amplitude was not dual.

But what physical system has just the spectrum of R? It is clearly the violin string, that is, the continuum limit of an infinite number of mass points experiencing harmonic forces between them. Eq. (2) can now be pictured as an unexcited string coming in, having momentum dumped in by a series of external potentials, and finally re-emerging as an unexcited string (see Fig. 2)

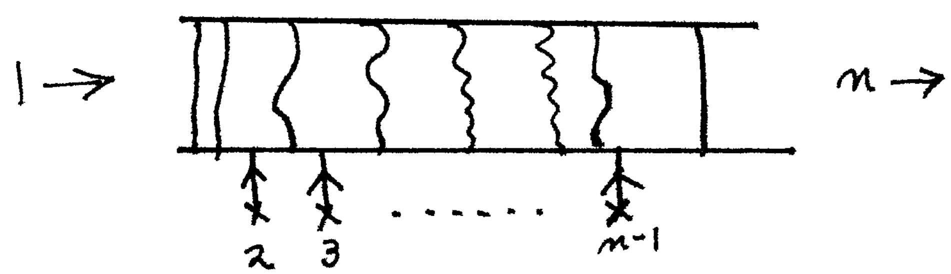


Figure 2

This is clearly a very unsymmetrical way to view a reaction whose amplitude is supposed to be crossing symmetric and dual. In the process of trying to check the crossing and duality properties of dual models directly in the operator formalism, a very interesting discovery was made.[4] It was that when the amplitude was written in one way, less intermediate states appeared than when the amplitude was written in another way. In other words, some of the states created by the vertices were spurious. (See Fig. 3)

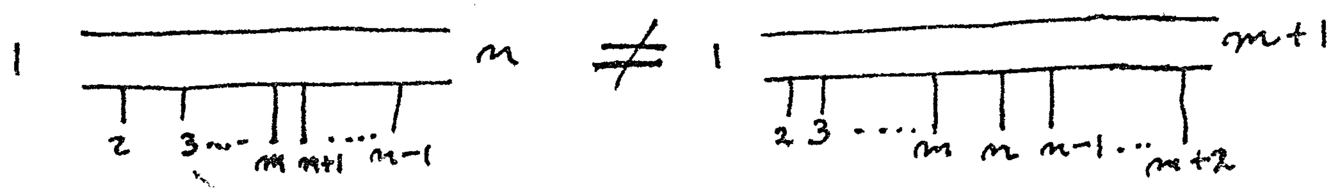


Figure 3

Now, this was very interesting indeed, because many of the states of the type exhibited in (6) are unphysical. If a timelike oscillator creates a state, that state has negative norm. This will show up in certain scattering amplitudes by having negative residues where only positive residues are allowed. Probability will not be conserved.[5]

A lot of work has gone into showing that the spurious states that were discovered are "ghosts" of this type, or else states of zero norm, and can be eliminated consistently from the theory. All of this "ghost elimination" occurs for a price, however, and we will come to that.[6]

Once one gets accustomed to the idea that the Veneziano amplitude is telling us a hadron is behaving like a string, it is natural to ask whether a formalism to deal with this physical picture exists, so that the rules for writing amplitudes can be derived. Elimination of unphysical states of excitation should also follow naturally from the formalism, as in quantum electrodynamics. In a striking generalization from the action principle describing the motion of a classical free point particle, Nambu proposed that the motion of a classical string be described by an action principle based on[7]

(7) $$\bar{I} = \iint d^2\xi \sqrt{-g}$$

As indicated in Fig. 2, a string propagating in space-time sweeps out a world sheet. In Eq. (7), ξ are the coordinates of the sheet, and g is the determinant of the metric tensor on this two-dimensional manifold. The string is actually propagating in the full four-dimensional Minkowski space, however, so the invariant interval on the sheet is

(8) $$\begin{aligned} ds^2 &= -g^{\alpha\beta}\, d\xi_\alpha\, d\xi_\beta \\ &= -g^{\mu\nu}\, d\chi_\mu\, d\chi_\nu \end{aligned}$$

where

$$g^{\mu\nu} = \begin{pmatrix} 1 & \\ & -\underset{\sim}{1} \end{pmatrix}.$$

These expressions are only compatible if

$$g^{\alpha\beta} \equiv \partial^\alpha \chi_\mu \partial^\beta \chi^\mu .$$

Given this expression for the action, one must try to proceed canonically to obtain the equations of motion of the system, any constraints that may have to be satisfied, and attempt quantization. After all the dust has settled, it turns out that the spectrum of excitations implied by (7) is indeed that of the string; to show this explicitly, it is necessary to impose certain conditions of constraint on the states of the system.[8] These conditions turn out to be just the "ghost elimination" conditions that we mentioned before.

However, we should recall that in electrodynamics the "ghost eliminating" condition $[\partial_\mu A^\mu]^{(+)}|\Psi_{ph}\rangle = 0$ is necessary only if we work in the Lorentz gauge, which is expressed classically by $(\partial A) = 0$. If we work in the radiation gauge, we can solve for one spurious photon degree of freedom explicitly, and not have to impose conditions on states. Analogous choices of gauge can be made in the string model, and the results we just mentioned are applicable in the analogue of the Lorentz gauge.

A different choice of gauge is possible.[9] We can motivate it by observing that one of the sheet coordinates, $\xi_0 \equiv \tau$ is like a "proper-time" variable; clearly one of the Minkowski variables, X_0, is a time variable in some frame, and so we expect things can be simplified if we identify these two to be the same. Working in a gauge of this type, it was found that the theory is not Lorentz invariant if the Minkowski space has only four dimensions. For the Lorentz algebra to close, it is necessary to have a 26 dimensional space-time. What is more, the ground state has to be a tachyon, $(m^2 = -1)$! (Actually, these catastrophes can be dug out of the manifestly covariant gauge as well, but we won't go into that here.)[6]

We have, then, a well-defined action and a perfectly well-behaved classical theory. Somewhere in the canonical quantization of the theory, something goes wrong, and no one quite knows what it is. I will mention some recent attempts to deal with this question as we go along.

B. How Can A Hadron Look Like a String, Anyway?

Hadrons, as opposed to leptons, are not point objects. We think they are composed of point objects, maybe quarks and gluons, which, dancing to some unknown ryhthm, give the hadron a spatial extent that can be measured experimentally. In short, we are used to visualizing a hadron as a little clump of matter - but a string?

To gain some insight into how a hadron can look like a string,[10] just remember the Feynman-Wilson interpretation of the inclusive distribution in a plot of rapidity vs. $p_\perp$. For moderate values of $p_\perp$, say $p_\perp \lesssim 400$ MeV, the "central region" is supposed to look like this:

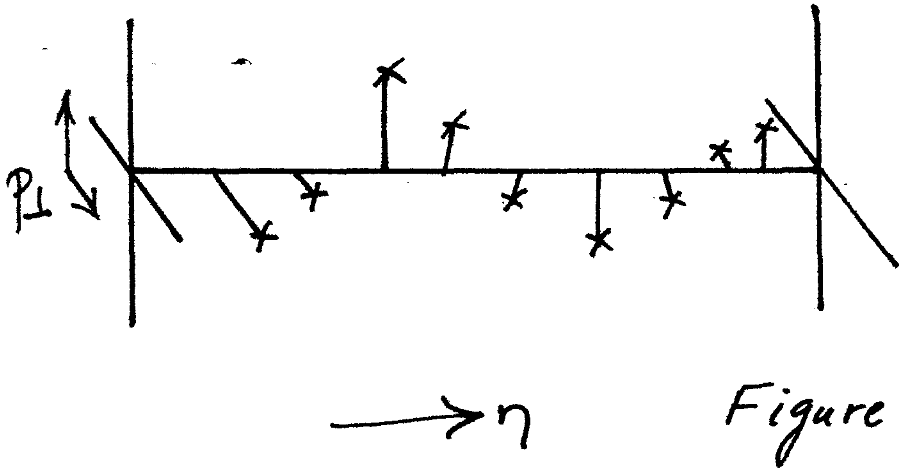


Figure 3

This figure is supposed to represent a "snapshot" of the typical, universal, interior hadronic matter distribution in terms of partons.

Imagine that at energies so large that we can see very many partons, it turns out that we can better and better interpolate between the parton-points on this plot by a smooth curve. In this case, our "snapshot" of the hadron would look like a string! If we keep the longitudinal momentum fractions as the "length" axis, and Fourier transform from $p_\perp$ -space to $x_\perp$ - space, we get just the interpretation that follows from the GGRT formulation of the string model. It is because of this precise matching of the string formalism with the parton language that we will develop the parton language to gain physical insight.

Now, what can it mean to have this kind of smooth distribution of partons? Again, recall that the Feynman-Wilson picture of scattering in the Regge region has the dominant contribution to the amplitude arising from the following steps: 1) The hadrons convert virtually into large numbers of partons, including a "wee sea" of partons with infinitesimally small fraction of the parent's longitudinal momentum; 2) the wee partons forget which hadron they belong to because all the hadrons' wee seas look pretty much alike; 3) recombination into hadrons occurs.

The probability for these things to happen can be estimated,[10] and depends on the distribution function for wee partons in the hadron. If this goes like $x^{-\alpha} dx$, α is the Regge intercept for the Regge behaved amplitude that results. Well, this is just multiperipheralism, and by looking at graphs in simple theories, one can see these dominant contributions occur when the cascade from the parents into the wee sea proceeds sequentially in the longitudinal momentum fraction x.

Multiperipheral ladder graphs, however, do not look very dual, so we must make some changes in this scheme:

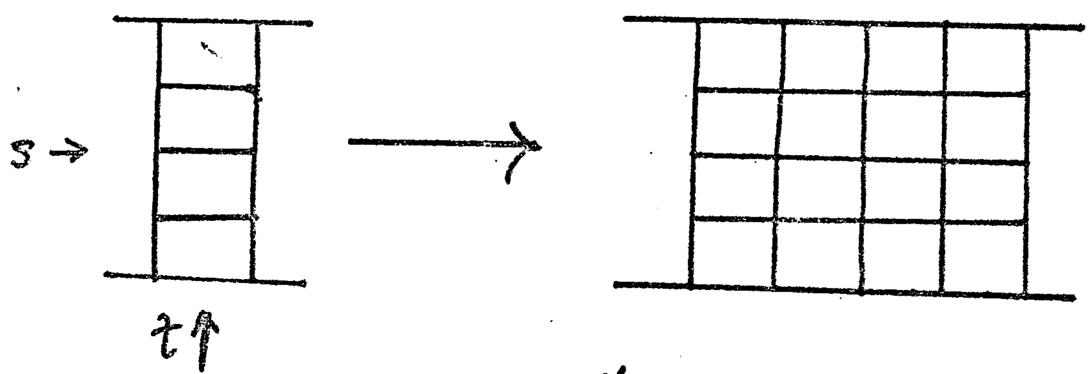
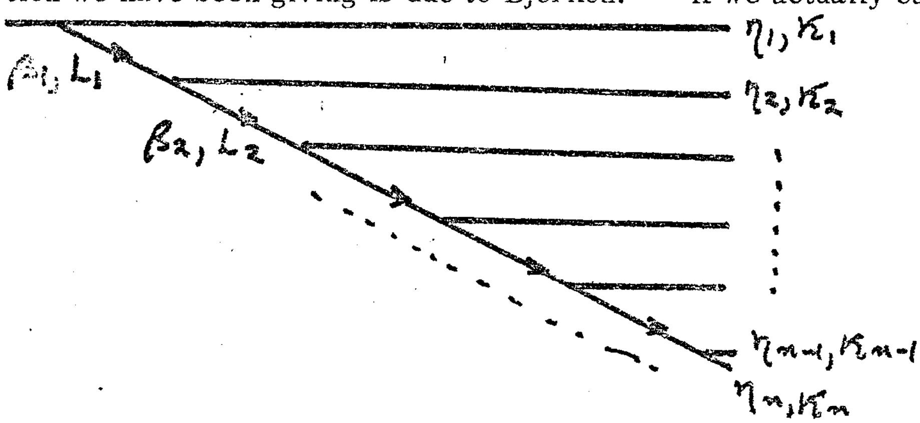


Figure 4

The first attempt to derive dual models from conventional field theory proceeded by calculating graphs of the type Fig. (6) in varying degrees of sophistication.[11] This is the "fishnet diagram" approach you may have heard of.

Another very pretty way to motivate in terms of graphs the verbal description we have been giving is due to Bjorken.[12] If we actually calculate the graph



$\eta_i = (P_\parallel)_i / (P_\parallel)_{TOT}$, "$i$" is ith parton

$\kappa_i \equiv (\kappa_\perp)_i$ = TRANSVERSE MOMENTUM OF "i"

β_i = η-TRANSFER IN ith LINK

$L_i \equiv (L_\perp)_i$ = TRANS. MOM. TRANSFER IN ith LINK.

Figure 5

in the infinite momentum frame using old-fashioned perturbation theory, with the assumption that in the cascade the η - transfers are ordered ($\beta_n \to \eta_{n+1} + \beta_{n+1}$, with $\beta_{n+1} \ll \eta_{n+1}$, just as in the dominant multiperipheral scattering graphs, a very interesting qualitative picture emerges - near neighbor partons in rapidity are also close together in transverse configuration space. (It is an open question how much of this result survives in theories with vertices less trivial than ϕ^3. See Section F.)

If we want to examine rescattering corrections to this basic parton model picture of the hadron's wavefunction, it is plausible to consider that these corrections involve repeated soft interactions between near neighbors in rapidity, with the basic dynamical variables involved being the distances in transverse configuration space between the interacting partons. (This is why the string picture starts off as a first quantized theory.)

One then attempts to describe the behaviour of the wee parton sea by means of an effective Hamiltonian, which is a function of the partons' relative transverse momenta, labeled by an ordered parameter corresponding to the parton's longitudinal momentum fraction. The simplest dynamical hypothesis is that the near neighbor forces are harmonic. If, in addition, the density of partons along the longitudinal fraction axis is chosen to be constant, the string Hamiltonian is obtained in the continuum limit:

$$(9)\quad (H_{eff})_{\infty p_z} \sim \sum_{i=1}^{n} p_{i,\perp}^2 + (x_{i,\perp} - x_{i+1,\perp})^2/2\eta_i$$

$$\to \int d\theta\, \rho^{-1}(\theta)\left[\rho^2(\theta)\,\dot{x}_\perp^2 + (\partial x_\perp/\partial\theta)^2\right]$$

$$\to \int d\theta\,\left[\dot{x}_\perp^2 + x_\perp'^2\right]$$

$$\text{ff.}\quad \rho(\theta) \equiv d\eta(\theta)/d\theta = \text{const.}$$

Here θ labels the parton, $(\theta/\pi) \sim (P_{\parallel})^{-1}_{TOT} \int_0^{\theta} d\theta' P_{\parallel}(\theta')$, and $x_{\perp}(\theta)$ is the transverse coordinate of the parton labelled by θ. (Actually, instead of working in the ∞P_z frame, we can do our quantum theory off planes tangent to the light cone. Then $P_{\parallel}$ is replaced by $P^+ = P^0 + P^z$, but P^z is not necessarily approaching infinity.) The choices of relevant dynamical variables, and the interpretation of the θ label in terms of longitudinal fraction, match exactly with what emerges mathematically from the string model in the GGRT gauge.

To reiterate, physical insight into how a hadron can look like a string is gained by looking at the planar graphs in a ϕ^3 theory; observing that in a sequential ordering approximation the longitudinal and transverse dynamics decouple (see Bj's paper for details), with the longitudinal fraction serving only as a label; assuming a soft, near-neighbor residual parton-parton interaction; and finally, assuming the parts of the wavefunction with the number of partons $\to \infty$ are the most important, in some sense, so there is no $\sum_n P(n)$ in Eq. (9).

All of these assumptions are subject to questioning. We will see later that dual models fail to predict certain qualitative behaviours that we expect from hadrons. In most such instances of failure, we will be able to point to some suspicious assumption from among the above as the one that is likely at fault.

A reasonable way to proceed would be to always keep in mind that the string picture of hadrons can make sense as an approximation to some complex dynamical situation occurring within each hadron. One of the things the string model contributes to our requirements on a theory of hadrons is that it should correlate properties of the spectrum with the "soft" physics of the Regge region. However, this requirement of duality does not seem to force any of our assumptions to be strictly valid.[13]

Finally, I should mention that it is not at all clear from $H_{eff}(p^+; x_{\perp})$ that the theory can be relativistically covariant. With the string action principle, the covariance can be shown using canonical methods, albeit with the troubles that have been mentioned. However, with the strict factorization of longitudinal and transverse dynamics that occur in H_{eff}, alternate methods of analysis exist. The expressions for the Lorentz generators could have been "guessed" in advance of the string action principle if one had been clever enough. This is important for future model-building, and we will say more about it in the next section.

C. What if the Partons Have Spin?

So far we have argued the amplitudes of oscillation of the string are the transverse coordinates of the partons, and we are working in first-quantization. Experiments suggest that the valence partons have spin 1/2, and it is reasonable to assume the wee parton sea will have many spin 1/2 partons as well. We now wish to expand on Bjorken's ϕ^3 theory arguments, and suppose that, in addition to $X_{\perp}(\theta)$, it is legitimate to include the spin variables among the possible dynamical variables upon which near-neighbor parton scatterings can depend.

Working in complete analogy with the $X_{\perp}(\theta)$ arguments, we can suppose the first quantized Pauli spin matrices $\sigma_{\perp}(\theta)$ are the relevant dynamical spin variables. (Actually, Bjorken, Kogut, and Soper have shown that in the light-cone quantization of the free Dirac theory,[15] these 2 x 2 Pauli matrices are really the spin variables of the second quantized theory, even taking anti-particles into account.) As good fortune would have it, a reasonable, simple guess for the H_{eff} depending on near-neighbor spin-spin couplings,[16]

$$(H_{eff})_{SPIN} = g' \Sigma_i \, \sigma_{\perp}(\theta_i) \cdot \sigma_{\perp}(\theta_{i+1})$$

is exactly solvable in the continuum limit! It becomes just the Hamiltonian of the free, massless Dirac theory in two dimensions.[10]

Now, depending on the boundary conditions one chooses, which amounts in this case to selecting whether the string has an even or odd number of spin 1/2 partons, one obtains either the Neveu-Schwarz (NS) model[17] or the Ramond model,[18] respectively. The N-S model was originally proposed as a model for the ($\rho\pi$) trajectory system, as we will see. The Ramond model, because of its odd number of fermion constituents, is a candidate for a fermi particle and its recurrences and their daughters. This could be a nucleon, or perhaps even a quark itself. The $q\bar{q}$ amplitude in this model has poles at bosons with the same structure as the bosons of the N-S model, and in fact the emission vertices are just those of the N-S model.[19] However, let me just concentrate on the features of the N-S model, so as to get the general ideas across.

In momentum space, if $b^\dagger$ creates a fermion and $c^\dagger$ creates an anti-fermion, the H_{eff} of the N-S model is

$$[H_{eff}]_{TOT} = R + \sum_{n=1}^{\infty} (n - 1/2)\left(b_n^\dagger b_n + c_n^\dagger c_n\right).$$

Here R is just the "orbital" contribution to the energy discussed earlier. The new piece is due to the spin-spin interaction, and has a spectrum of eigenvalues of 1/2, 3/2, 5/2, ...

In the quark model we expect the ρ and π to be $q\bar{q}$ bound states in s waves, in triplet and singlet spin states respectively. Since the spin in the only difference between the ρ and π states, the only thing that can account for the mass difference between them is the spin-spin interaction. Just as we needed more than one oscillator for duality, however, we now are forced to have an infinite number of spins. We get higher and higher energy states depending upon how many of these spins are deviated from the ground state configuration (which is like the ground state of an anti-ferromagnet). On top of any one of these "spin-deviate" states we can pile on orbital excitations, with the energy spectrum given by R.

The resulting trajectory structure is

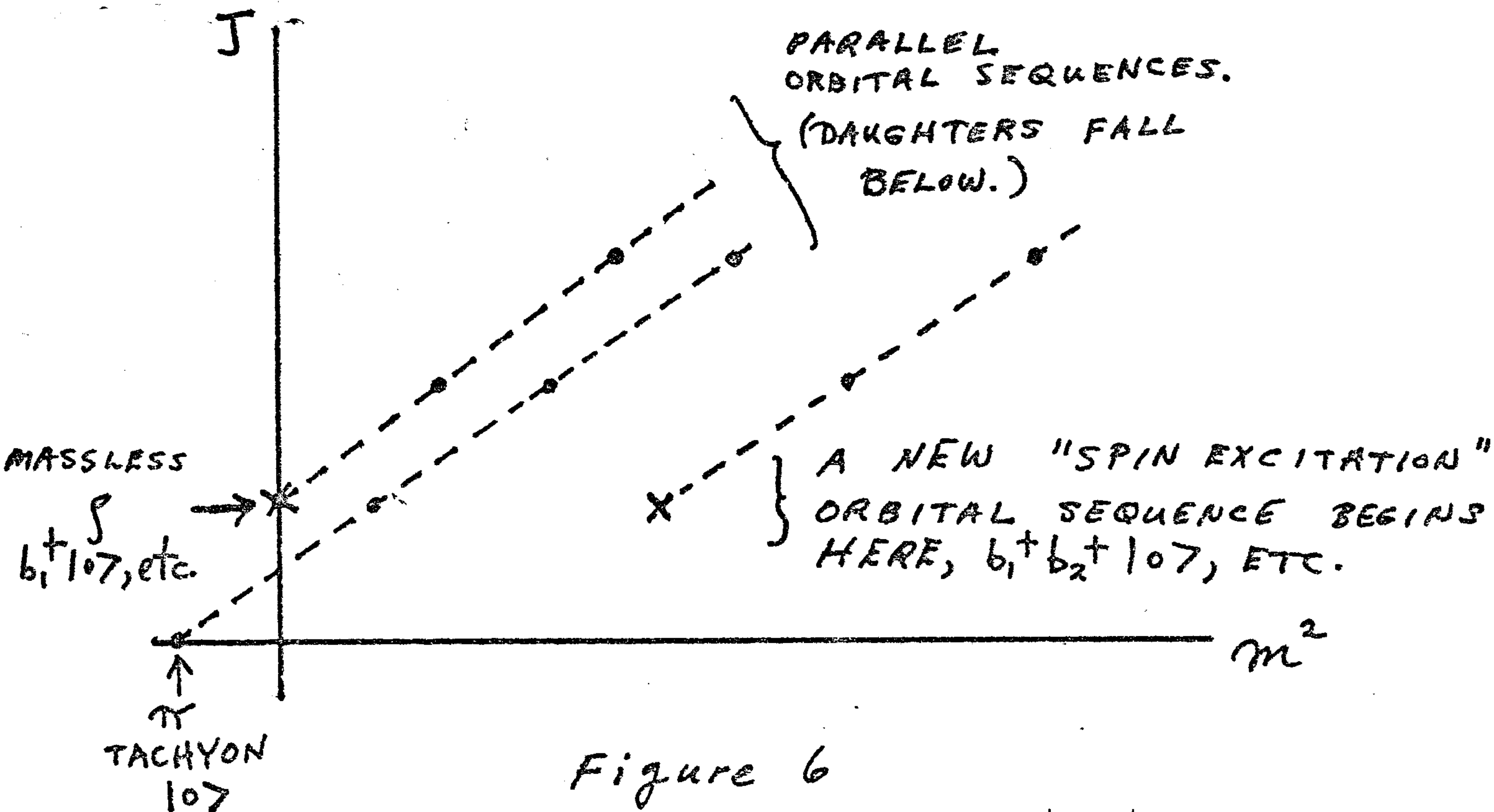


Figure 6

The reader should not be misled into believing that b^+ or c^+ are creating quarks or anti-quarks. They are creating spin deviation excitations as described above. When a spin is flipped at one point, H_{eff} moves that spin flip down the chain. A spin flip means helicity 1/2 (say) is going to helicity (-1/2) for a net helicity flip magnitude 1. That is why this system describes bosons. But at any given point there can be at most one fermion to be flipped - that, in a nutshell, is why we need operators satisfying Fermi-Dirac statistics. The helicity of an excitation is just the "charge" $Q = \sum_n (b_n^+ b_n - c_n^+ c_n)$.

Actually, the fact the interesting excitations of the system are bosons can be brought out more clearly using the Fourier decomposition of the current $j_\mu = \bar{\Psi} \gamma_\mu \Psi$. (Remember $\mu = 0, 1$ in 2 dimension, and Ψ are 2-component spinors.) It is a peculiarty of two dimensions that the Fourier coefficients

$$\rho_n \equiv 1/\pi \int_0^\pi d\theta \left[j_0 \cos n\theta + i j_1 \sin n\theta \right]$$

satisfy Bose statistics exactly,[20] $[\rho_m, \rho_n^\dagger] = \delta_{nm}$. These are, of course, "composite" operators built out of the b^+ and c^+, and one can show that[21b]

$$(H_{eff})_{TOT} = R + \frac{Q^2}{2} + \sum_{m=1}^{\infty} m \rho_m^\dagger \rho_m .$$

We could as well write $[a_{m\perp}^\dagger a_{m\perp} + \rho_m^\dagger \rho_m]$ as $\sum_{j=1}^{3} a_{mj}^\dagger a_{mj}$, where $a_{m3} \equiv \rho_m$. If this system is to have any hope of being Lorentz invariant in four-dimensions, we might expect a transformation from a transverse direction to a longitudinal direction to take $a_{m\perp}$ into ρ_m. The generator that does this would have the form

$$i \sum_n f(n) [a_{n\perp}^\dagger \rho_n - \rho_n^\dagger a_{n\perp}]$$

where f(n) can be fixed by dimensional arguments.[21a] In fact, Iwasaki and Kikkawa (IK) have discovered that the Lorentz generators indeed have terms of just this form.[22]

Unfortunately, however, there is more to the Lorentz generators than this. Remember that for the orbital part of the model, we had originally $X_\mu(\theta)$ with $\mu = 0, 1, 2, 3$. We argued $X_0 \sim \tau$, and kept $X_\perp$ as the dynamical variables. Whatever happened to X_3? Remember $\theta \sim p_{\parallel}$, so that is not it. Actually, X_3 (or X_- to be precise) could be solved for in terms of $X_\perp$, and that is why it has not been heard from. But there will be Lorentz generators of the form $\int d\theta [\chi_3 \dot{\chi}_\perp - \chi_\perp p_{\parallel}]$ which in momentum space looks like[9] $i \sum_n g(n) [L_n^\dagger a_{n\perp} - a_{n\perp}^\dagger L_n]$, where $L_n \sim \sum_{m=-\infty}^{\infty} C_{nm} : a_{n-m} a_m :$, $a_{-m} \equiv a_m^\dagger \ (m > 0)$. These Virasoro operators[4] are also composite operators as indicated, but they do not satisfy the algebra of simple harmonic oscillators. These satisfy instead

$$[L_n, L_m] = (m-n) L_{m+n} + \frac{d_\perp}{12} m(m^2-1) \delta_{m,-n}$$

Technically it is because of the algebraic properties of these L_n that we need tachyons and 26 dimensions in the orbital model.[23]

Now, there are a priori two ways to combine these two kinds of contributions to the Lorentz generators. The string formalism, properly generalized by I and K in Ref. 22 to handle the spin, seems to give naturally the result

$$M_{j+} \sim i \sum_n [f(n) \rho_n \epsilon_{jk} a^+_{nk} + g(n) a^+_{nj} L_n] + h.c.,$$

where $\epsilon_{12} = 1, \epsilon_{21} = -1.$

This turns out to be much worse for Lorentz invariance than it was before, with just the orbital part. Basically what happens is that a transverse excitation doesn't make up its mind properly whether it wants to go to an L_n or a ρ_n. To patch things up, we have to abandon our nice interpretation of the ρ_n as "longitudinal" bosons, and replace the ($a^+_n \rho_n$) generators with other objects containing 2 fermions and one boson:

$$i \sum_n \rho^+_n a_{n\perp} + h.c. \rightarrow \sum_n (b^+_n \pm i c^+_n) G_n (a,b,c) + h.c$$

$$G_n \equiv \sum_m : a^\perp_{n-m} b^\perp_m :$$

The G_n are the famous "super-gauge" operators which have come into their own recently, independent from the string or dual models.[24]

The advantage, for our purposes, of these supergauges is that for $d_\perp > 2$ the Lorentz algebra can be made to close once again. The fermions and the orbital operators a_i are assigned the <u>same</u> transverse dimensionality, and for d = 10 the theory is Lorentz invariant. The supergauge construction allows this generalization for the fermions, while the ρ_n construction is wedded to $d_\perp = 2$.[25] (Of course, we also need a tachyon at $m^2 = -1/2$, as indicated in Fig. 7)

Alternately, we might have tried to put by hand

$$M_{j-} = i \sum_n a^+_{nj} [f(n) \rho_n + g(n) L_n] + h.c.$$

This type of expression has been suggested by Fairlie, and discussed by Chodos and Thorn.[26] The net result is that you still have a tachyon but you do <u>not</u> need extra spatial dimensions. This sounds wonderful at the outset, but unfortunately, in the context of the NS model, it turns out that "Q" does not measure the helicity properly anymore. If we give up the fermions altogether and stick to ρ_n as bosons, we lose the two-trajectory structure that was so attractive.

Let me try to summarize. Partons with spin allow us to get fermionic physical particles, and a quark-model-like structure for the vector and pseudoscalar meson families. There exist reasonable physical motivations for including the effects of spin by choosing H_{eff} to be of Heisenberg type. Unfortunately, when we check Lorentz invariance, the pretty physical picture evaporates completely. I have tried to give a bit of the flavor of how the Lorentz business works so this point can be properly appreciated.

What could be some flaws in the argument?

- Why only $\sigma_\perp(i)\cdot\sigma_\perp(i+1)$? Why not put in σ_z as well? Since $2i\sigma_z = [\sigma_x, \sigma_y]$, if $\sigma_\perp \to \psi$ (fermi field, by a Klein transformation), we might try a Thirring model as a generalization[27]. What happens in the end is that the "charge = helicity" Q gets renormalized, and not much else. Nothing is gained except the useless information that in $(0 < \theta < \pi)$ the coupling constant is quantized.

- If we have a Nambu-Goldstone pion, how do we reconcile it with the quark picture ? In principle the string-with-spin model should be capable of shedding new light on this old question. The reason is that each individual hadron has all the complications of many-body theory. The net quantum numbers have to be given by the quark model, but we have a fermion sea to play with. The pion occupies a special position because it is the ground state meson. Bardacki has been working on this kind of approach.[28]

— How about spin-orbit couplings? This has also been studied by Bardacki, and Halpern.[29] Not much has been done to study the relativistic properties of these models. Also, they suffer from a much larger degeneracy than the uncoupled theories. A more modern approach would be to Melosh transform the N-S model.[30]

— Perhaps the whole picture of how spin is to be incorporated is totally wrong.

See Section F for a concrete way this could be so.

D. What good is the String Model?

The string model is only as good as the amplitudes it predicts. These are, unfortunately, not well-suited for phenomenological analysis at all. In recent years, dual phenomenologists seeking to fit data have turned increasingly to non-Veneziano, non-factorizable dual amplitudes that have nothing to do with strings.[31]

Nevertheless, generalized Veneziano amplitudes do maintain the qualitative features we mentioned at the outset that motivated Veneziano in the first place. This makes them valuable tods, satisfying many desirable prerequisites on hadronic amplitudes, for studying questions of consistency among the assumptions. They are, in other words, a valuable theoretical laboratory. Extensive references to studies into high energy limits of dual amplitudes and their discontinuities, relating to multiparticle production, inclusive reactions, and the role of the Pomeron, can be found in Veneziano's review paper, Ref. 5.

We mustn't try to get off the hook that easily, though. The string model, and the possible "variations on a string" models, do consistently come up with features that must be dealt with as predictions, even if they are unpleasant. It is, for the time being, excusable if the spectrum is not fully correct; it is a serious defect that we have a persistent tachyon. It is satisfying to have a mathematical realization of Muellerism; but immensely disturbing that "deep scattering"[32] cannot be dealt with even qualitatively.[1] It is stated one needs a Hagedorn spectrum to accomodate duality; but I know of only one unpublished paper (by Koba) where the decay patterns of high mass, high spin resonances of dual models are analyzed, to give experimentalists an idea of what to look for.

Let me say a bit more about the "deep scattering" qualitative failure of the Veneziano model. For both s and t large, where the CM scattering angle is held fixed, the elastic ($2 \rightarrow 2$) amplitude Eq. (1) behaves[1] like $\exp(-s \ln 2)$ at 90°, as opposed to the power-law behaviour observed experimentally. This is interesting, because (apart from an overall ∞ !) a straight forward calculation of the elastic form factor[33] of the ground state hadron also behaves as $\exp(-q^2 \ln 2)$. The origin of the factor ($\ln 2$) in the latter case is that the mean-square distance in transverse configuration space between the partons at opposite ends of the string is

$$\left[\langle (x_\perp(\pi) - x_\perp(0))^2 \rangle - \langle x_\perp^2(\pi)\rangle - \langle x_\perp^2(0)\rangle\right]$$

$\sim \ln 2$. We see the obscene constant ($\ln 2$) is something like the size of the hadron, and even in deep-scattering the hadron behaves according to this characteristic size.[34]

This is important, because current theoretical explanations for power-law fall-offs in these kinds of experiments invariably start from the assumption it is the behaviour of the pointlike constituents that is responsible. In the string picture, the extreme view is taken that the important part of the hadron wavefunction is the one that is maximally occupied by partons. We tend to lose touch of the "valence" partons, of the part of the wavefunction measured by extremely short wavelength probes.

Another interesting physical point should be noted. Dual models tend to give form factors[33] (forgetting the $2-q^2$ for now) that look like

$$F_{elastic}(q^2) \sim \int d\theta\, (\sin\theta)^{\alpha - q^2}.$$

Ideally, one would like to get form factors like $\int d\theta\, (\sin\theta)^{\alpha - q^2} \cos^\beta \theta$ which fit data well,[35] but this does not emerge naturally from any model. (Remember that θ is the longitudinal momentum fraction of the parton with coordinate $X_\perp(\theta)$).

Now, BBG also get formulae for how form factors behave, asymptotically, from a more direct parton approach. They find $F(q^2) \sim (q^2)^{-n}\, I$, where I is a definite integral over the longitudinal fraction.[32] The significant point to note is that the string model result <u>cannot be written in this form</u>. The asymptotic behaviours in the two models are coming from different regions of phase space.

These brief discussions are intended to illustrate significant ways in which the assumptions that go into the string model can be inadequate. A straight forward look at how deep-inelastic e p scattering works for strings supports these views as to how the string model fails: the partons never behave point-like;[36] and probably it would be helpful to relax the strict adherence to a one-dimensionally extended object.[37]

It is possible that more recent developments in the string model can overcome the second of these problems. Let's see what these developments are.

E. Are Strings Alive and Well?

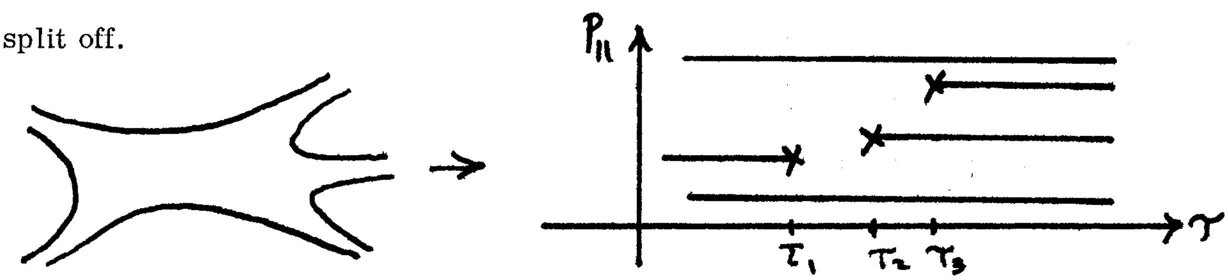
There are several directions in which recent progress has been made. One of these directions is in addressing the nagging problems of dimensions, and of tachyons. Recent approaches to these problems share a belief that there is no strict requirement that "canonical quantization" has to give a consistent quantum theory. In one view, the string picture is not required to make sense except in the large occupation number limit. The low lying levels of the spectrum can be totally different, and in fact the ground state particle is not anymore a simple mechanical object.[14]

This is important because suppose (classically) the "ground state" is simply a collection of particles moving together at the speed of light in the z-direction. Going over to a quantum picture, the "springs" between these particles cannot simply be at equilibrium, but rather there must be ground state random oscillations. The string cannot be well-localized, but must have a spread. But if a portion of the string is spending time moving in a transverse direction while neighboring portions are proceeding in the z-direction, the string will not hold together and move at $V = C$ - unless the transverse moving portion exceeds the speed of light. While this argument is incomplete, it suggests we really do not want the ground state to be a mechanical object. Anyway, a simple model has been constructed which illustrates a non-mechanical ground state.[14]

Other interesting ways to deal with the problem of dimension which have been proposed rely on giving the operators in the theory a "color" index. In one method, new colorful gluons are required,[38] and the interesting result is obtained that (d = dim. of space-time) d = 10 -2N, where SU(N) is the color symmetry group. In another approach, basic ambiguities inherent in the canonical quantization prescription are exploited to introduce a "color" - like index in a very natural fashion, without requiring extra fields.[39] (There are still tachyons in these models.) These developments are technical because the questions they are trying to answer arise from technical points, and I will not be able to supply any details. It should be realized, however, that the modifications in Refs. 38, 39 are not merely relabellings of the redundant transverse degrees of freedom.

By and large, the most interesting recent progress achieved has been in completing the formal theory of the interacting string. So far, after all, we have been talking about free strings, while the great virtue of the whole approach was that scattering amplitudes exist. How do we derive the rules for obtaining Eq. (2)?

To eliminate the asymmetry already noted in Fig. 2, it would be nice to say that two strings come together, fuse into a single string, and then other strings split off.



(The τ_i are the "times" at which string fusions and fissions occur.) Mandelstam succeeded in inventing the clever tricks needed to make this plausible picture into a mathematical reality.[40a] To do it, he first re-drew the Rosner-Harari structure in the figure as indicated. Recognizing that in terms of these drawings the length of the string should represent its longitudinal momentum, rather than its spatial "size" (which, recall, had nothing to do with $0 < \theta < \pi$), is a very important point. The constant width of the strip is simply an expression of $P_{\parallel}$ momentum conservation.

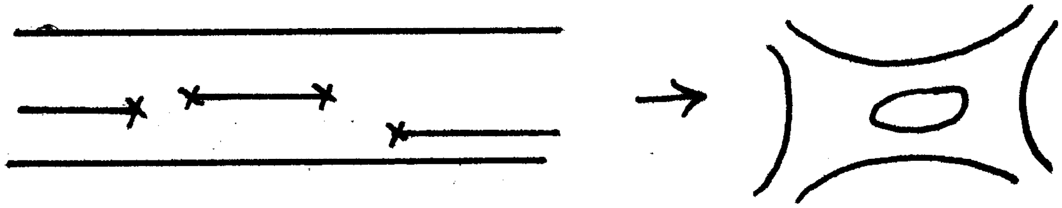
The mathematical problem of calculating the amplitude is complicated, but only once. As in the Feynman-Dyson theory, once the rules for calculation are justified, we can forget the derivation if we like, and use the rules with ease. Also as in Feynman-Dyson theory, the underlying physical picture is as elegant as the rules are simple. The Feynman particle path integral is the relevant formal tool, and one has

$$A_n = \int \cdots \int dz_2 \ldots dz_{n-2} \, H(z_1, \ldots, z_n) \; ;$$

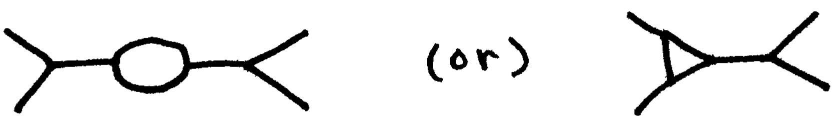
$$H = V(z_1 \ldots z_n) \prod_{r=1}^{m} \prod_{n,i} dp^{i}_{n,r} \, \psi_r(p^{i}_{n,r}) \, W(p^{i}_{n,r}, P_r^{TOT}).$$

One must ask for the probability that n strings can come in, merge, and become m strings, over all possible things they were and could be. Here the Z_i are the "times" associated with points on the graph where splittings or recombinations occur, indicated by X's on the graph. The integrand H contains a normalization V; the products of the wavefunctions associated with the N interacting strings (r labels the string), $\psi_r(p^{i}_{n,r})$; and a weight factor W. The $(p^{i}_{n,r})$ are the momenta in the i th transverse direction carried in the normal mode of excitation n of the r th string. W is a complicated factor containing three pieces of information: a) A Fourier transform to relate this path integral to the standard one in configuration space; b) the statistical weight $\exp i \iint d\theta \, d\tau \, \mathcal{L}^{eff}$ where $\mathcal{L}^{eff}$ is the string Lagrangian, $(\dot{x}^2 + x'^2)$; and c) (Neumann) functions that assert, essentially, that one is interested in an amplitude with a given topology, such as that of the figure. This picture has been extended to include fermions, and allows calculation of fermion-fermion and meson-fermion couplings,[40b] as discussed earlier.

Now among the things that could happen, if we are really to count them all, is the following possibility:



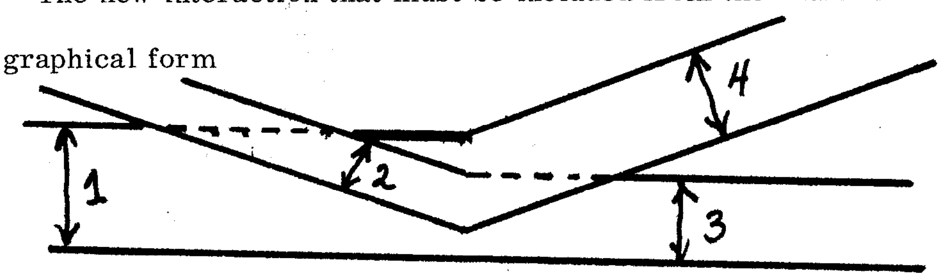
The string decides it isn't time to split yet, so it recombines for awhile. You can see in the drawing how that would look as a Harari-Rosner diagram. If we had point particles instead of strings, this diagram would be a radiative correction to a Feynman graph:



We can have, then, virtual string states, strings off-shell.

The desire to describe these processes using conventional second quantization techniques rather than particle path integrals has recently led Kaku and Kikkawa to develop a field theory of strings.[41] The bookkeeping for the various possibilities is generally simpler this way, and these authors have enumerated the basic vertices of the theory, the possible ways strings interact. They have found that the triple coupling used in the above drawings must be supplemented by a direct four-string interaction.

The new interaction that must be included from the outset in the theory has the graphical form



I want to stress that this observation is not merely a curiosity, but is intimately connected with the possibility of resolving some of the basic qualitative problems the model has had up to now. For example, I have not harped on the point that the resonances of the Veneziano model have zero width. This is alright in Born approximation, provided "Born approximation" has meaning within a complete theory. The field theory of strings accomodates this approximation, and justifies formally the hope perturbative unitarity may be implemented. Note that this field theory is a field theory of a totally new kind, involving multilocal rather than local dynamical variables. Many questions regarding whether such a theory satisfies general requirements that locality insures in ordinary field theories are discussed in Ref. 41, but many problems remain that require investigation. There are other potentialities in the field theory, and this brings us to our final topic.

F. What do Strings Have to do with Anything Else?

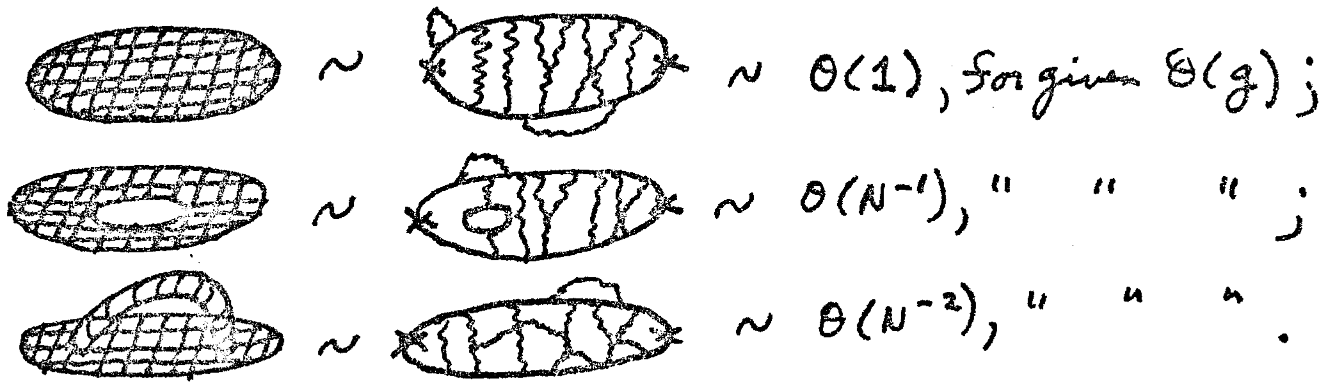
So far we have tried to motivate why the string model is interesting, and to explain in what ways it might succeed, in what ways it might fail. We now want to try to view this model in a broader context, asking its relation to other recent developments in strong interaction physics, and, in aswering this question, attempt to assess its remaining potentialities.

The string model is actually one of a class of models which try to recognize at the outset that hadrons are composite objects, although perhaps of a very different kind than other bound states such as atoms or nuclei. It has long been suspected that string models are "infinite component wave equation" (ICWE) models, for example, although only with the recent formulation of the field theory of strings could this connection be firmly established. I mention this because, even though I have stressed the "parton" school's views about the meaning of the string, there is no logical necessity for this point of view. Any approach which succeeds by whatever means to incorporate relativity, quantum mechanics, and reality as revealed by experiment into a consistent synthesis is surely logically acceptable. String models in light-cone quantization and in noncovariant gauges bring to ICWE a fresh approach, unencumbered by manifest covariance or manifest locality. This can perhaps help in evading the premises of no-go theorems that plague the ICWE approach, and its cousin, saturation of the current algebra.

Even in the conventional field theory, however, there has been a recent resurgence of interest in obtaining non-conventional solutions. One possible way to view a composite hadron is as a three-dimensionally extended volume in which fields are contained. To actually do this, however, the boundary of the domain aquires the status of an independent entity - the bag.[42] But other possibilities exist. Approximate solutions to the classical Yang-Mills isospin theory in the static approximation exist, e.g., that tend to be localized in a finite region of space.[43] The fields just sit there and feed on each other. In addition, field theories of fermions coupled to scalar mesons have been discussed using various methods to display at least approximate "confinement" of the fermions.[44] And what is of interest to us, solutions exist to the electrodynamics of scalar mesons that in the strong coupling limit, tend to look like strings.[45]

This last result is exceedingly important if we want to know whether some of the features mentioned in the parton interpretation of the string model are due only to the simplicity of the ϕ^3 theory used to discuss them, or whether they can be present in a large class of field theories. The derivation of stringlike solutions from a gauge invariant theory (which are like vortex circulations about trapped magnetic fields in Type II superconductors) encourages the belief that the relevant features may be quite general.[46]

Additional support for this point of view comes from calculations in Yang-Mills gauge theories utilizing a new kind of approximation scheme. Assume, for example, that there are not just three colors for the quarks, but N, where N is very large. In the limit where N is infinte, all the possible Feynman graphs of the theory collapse into a small subset of graphs.[47] In the case that an external colorless source current creates a $q\bar{q}$ pair and subsequently another current annihilates it, e.g., the surviving graphs are those with a single quark loop on the periphery, and with vector gluons filling in the loop, but only in planar configurations. If N is not infinite, say N = 3, other graphs survive, but are suppressed by powers of 1/N:



In this application, therefore, the 1/N expansion is a topology-selecting expansion.[48] There are three points I want to discuss regarding this:

1) Perturbation expansion of string model is similar.

Kaku and Kikkawa have gone on to study higher order effects in their field theory of strings. "Hole" and "wormhole" graphs of the kind drawn in the figure have long been proposed as candidates for radiative corrections to the basic string amplitudes,[5] and this come out systematically in the string field theory approach. There are other, more exotic, topologies possible in both the string and the Yang-Mills gauge theories which I have not written down. I am not trying to argue that there is an exact matching, graph for graph, between these theories. (Indeed, so far we have said nothing about isospin, SU(3), color, etc., in the K and K theory.)

What I am trying to suggest is that since the Yang-Mills theory has dynamics in all three spatial directions, but can be made to look like a planar theory in a well-defined approximation, the inverse process may be possible for string models. There, the "two-dimensional" structure was constructed first, but by calculating higher order corrections, latent higher-dimensional dynamical structure may emerge (i.e., the longitudinal and transverse variables may no longer decouple as they did in Bj's illustrative model.) Some support for this point of view already exists: more sensible results for amplitudes involving currents are found if the currents are "tubelike" probes.[33b]

The idea, then, is to perturb away from the planar approximation. A natural question is whether this perturbation expansion converges rapidly enough to be useful. It is important to do calculations in higher orders to see whether any of the gross qualitative failures of the string picture get rectified by this procedure, in manageable orders of the perturbation.[49]

2) But there are big differences !

The analogy between the second quantized string perturbation theory and the $1/N$ expansion for gauge fields should not be taken too literally. One simple difference, e.g., is that in the latter case, the $q\bar{q}$ loop is filled with gluons, and there is no fermi sea. If the physical arguments we gave really have anything to do with the Neveu-Schwarz and Ramond models, the fermi sea should be present in the hadron wavefunction to "leading order." On the other hand, the Yang-Mills approach readily picks out valence particles for us.

Another important point has been discussed in detail by 't Hooft.[41] It is that while the $1/N$ perturbation procedure selects out planar graphs as the leading order, there is no reason to conclude from this fact that the "effective" theory obtained in this approximation looks anything at all like the string theory.[50] Recall that to say that the planar ϕ^3 theory could look like a string, it was necessary to assume more than just that the relevant graphs were planar. Assumptions had to be made as to how momentum flows through the graphs. It is important to study what sensible approximations to the momentum flow problem will lead to in gauge theories. There is no reason to expect they will be identical to what happens in ϕ^3 (but see point 3 below.)

One should not get the impression, however, that it is necessary to "derive" the string theory in full from conventional field theories. The string theory may well prove to be a new and fully consistent approach to hadron physics which is not equivalent to conventional field theory. Still, it is necessary to understand the general features the string theory may share with other theories, since the string theory is incomplete.

3) Other Surprising Similarities.

I will conclude by mentioning two further points of similarity between the string theory and Yang-Mills gauge theories that are surprising and tantalizing.

The first has to do with a remarkable property of Neveu-Schwarz amplitudes in the limit that the slope of the Regge trajectories goes to zero.

Each amplitude is assigned an isospin factor in such a manner that correct SU(2) values are assigned to graphs, and such that amplitudes with poles in exotic channels (such as $\pi^+ \pi^+$) receive coefficient zero.[51] It is then found that the tree graphs of the theory have the coupling structure of the analogous SU(2) Yang-Mills theory with vectors and pseudoscalars.[52] However, in the usual NS model, there are only trilinear couplings of strings, and care is required in taking the $\alpha' \to 0$ limit to pick up all the required terms. (Recall that in YM theory, there are also quadrilinear couplings of the gauge fields.)

In the K and K field theory, a number of simplification occur.[41] One does not have to put the isospin factors by hand, but can assign quantum numbers directly to the string field variables. Also, the new four-string interaction leads to just the isospin and helicity structure of the four field interaction in the Yang-Mills theory in an appropriate gauge. One may conjecture from this that the connection between strings and Yang-Mills theory is deep-seated.

Finally, I want to mention a different kind of calculation that has been done recently. In the study of phase transitions in bulk matter, it seems to be the case that detailed knowledge of the microscopic interaction in the particular species of matter is totally irrelevant to understanding certain features of the phase transition. An atomistic point of view is not relevant for a study of these systems- rather than seeking what differences arise in systems as we probe deeper into them, the relevant question is something like "what is it the deeper 'layers' share in common in their response to certain kinds of probes?"[53] K. Wilson has developed a theoretical formalism to deal with this kind of question.[54] One of the features of this formalism is the sensible point that if the microscopic details are really irrelevant, we are better off if we "integrate" these details out at the outset.

This kind of procedure may be reasonable for the study of the planar graphs of the YM theory. If we want to study those graphs in which momentum flows more or less uniformly throughout, we might do it by first lumping subgraphs in which "hot" lines occur into new effective vertices among soft lines.

In any case, K. Wilson himself has recently studied spinor electrodynamics in a spatial lattice, and in the strong coupling limit.[55] The first device cuts off the magnitudes of the momenta that can flow. The strong coupling requirement intuitively suggests dominance of graphs rich in vertices. The result of his calculation is that, unlike weak coupled electrodynamics, the current-current correlation function in this theory can be described using an effective action which is proportional to the area of a fermion loop. It is difficult to pin down precisely what the connection of this result with the Nambu action principle, Eq. (7) actually is.

So, again: Are strings a separate contribution, or are they an extrapolation from existing theories? It is clear that the physical picture itself points to inadequacies, and the string theorist has much to gain by studying how conventional theories deal with these problems. On the other hand, the manner in which string theory actually realizes the underlying physical assumptions can still be viewed as a promising and stimulating approach.[56]

References and Footnotes

(As noted in the preface, this article is not a comprehensive review. References are representative rather than exhaustive, and I apologize for many omissions. The interested reader should, however, be able to trace the relevant literature from this sample. PRT means "plus references therein" that are essential.)

1. G. Veneziano, N.C. 57A, 190 (1968)
   C. Lovelace, Phys. Lett. 28B, 265 (1968)
   J. A. Shapiro, Phys. Rev. 179, 1345 (1969)

2. S. Fubini, P. Gordon, G. Veneziano, Phys. Lett. 29B, 679 (1969)
   S. Mandelstam, Phys. Rev. D1, 1734 (1969), (PRT)
   Y. Nambu, Proc. Intern. Conf. on Symmetries and Quark Models, Wayne University (1969), p. 269 (Gordon and Breach, 1970).
   L. Susskind, Phys. Rev. Lett. 23, 545 (1969)

3. Z. Koba, H. B. Nielsen, Nucl. Phys. B12, 517 (1969), (PRT).

4. M. A. Virasoro, Phys. Rev. D1, 2933 (1970), (PRT)

5. We have little space for calculations. Almost all needed details, as well as different points of view, can be found in one or more of the recent extensive reviews by:
   G. Veneziano, Physics Reports v.9C, No. 4 (1974);
   J. Schwarz, Physics Reports v.8C, No. 4 (1973);
   C. Rebbi, CERN TH-1785, to be published in Physics Reports;
   P.H. Frampton, Syracuse Lectures;
   J. Scherk, NYU-TR3-74.

6. R. C. Brower, Phys. Rev. D6, 1655 (1972)
   P. Goddard and C. B. Thorn, Phys. Lett. 40B, 235 (1972)
   My prejudice is that if unphysical features are required to show ghost elimination at the free theory level, the theory is already sick. An alternate view is that only some gauges are consistent for quantization of the free theory, and these gauges turn out to be disease-free. See, e.g., A. Patrascioiu, C00-2220-23 (1974). The unphysical features may then show up only in scattering amplitudes, so that 26 dim. is no problem at all, since the amplitudes are alright in the physical subspace. The tachyon would be the only real trouble. Those who like puzzles, however, may find relevant comments on quantization in Schwinger functional and Feynman path integral formalisms in, resp.,
   R. Arnowitt and S. Fickler, Phys. Rev. 127, 1821 (1962);
   K. S. Cheng, J. Math. Phys. 13, 1723 (1972).

7. Y. Nambu, Copenhagen Summer Symposium (1970).

8. L. N. Chang and F. Mansouri, Phys. Rev. D5, 2535 (1972)
T. Goto, Prog. Theor. Phys. 46, 1560 (1971)
O. Hara, Prog. Theor. Phys. 46, 1549 (1971)
F. Mansouri and Y. Nambu, Phys. Lett. B39, 375 (1972)

9. P. Goddard, J. Goldstone, C. Rebbi, C. B. Thorn, Nucl. Phys. B56, 109 (1973), hereafter referred to as GGRT.

10. Complete discussions along these lines are in J. Kogut, L. Susskind, Physics Reports v. 8C, No. 2 (1973), (PRT).

11. A. B. Kraemmer, H. B. Nielsen, L. Susskind, Nucl. Phys. B28, 34 (1971), (PRT).

12. J. D. Bjorken, Proc. of Intern. Conf. on Duality and Symmetry in Hadron Physics, Tel-Aviv, (1971), ed. by E. Gotsman (Weizmann Science Press, Jerusalem, 1971).

13. H. B. Nielsen, Nucl. Phys. B54, 477 (1973) argued the "softness" of the interparton force led to one-dimensionally extended hadrons.
J. Dethlefsen, H. B. Nielsen, and H. C. Tze, Phys. Lett. B48, 48 (1974), (PRT) maintain that the Hagedorn spectrum is independent of the dimensionality in a "jelly" of n-dimensions.
A. Kraemmer, H. B. Nielsen, and H. C. Tze, NBI-HE-74-10 more recently advocate the notion of a "dynamical dimensionality" for hadronic matter, i.e., what we see depends on how we look. This is our point of view as well.

14. A reasonably complete set of Refs. may be found in C. E. Carlson, L. N. Chang, F. Mansouri, and J. F. Willemsen, SLAC-PUB 1418 (to be publ. in Phys. Rev. D.)

15. J. D. Bjorken, J. Kogut, and D. Soper, Phys. Rev. D3, 1382 (1971), (PRT).

16. Y. Aharonov, A. Casher, L. Susskind, Phys. Rev. D5, 988 (1972), (PRT).

17. A. Neveu, J. Schwarz, Nucl. Phys. B31, 86 (1971),
A. Neveu, J. Schwarz, C. B. Thorn, Phys. Lett. 35B, 529 (1971).

18. P. Ramond, Phys. Rev. D3, 2415 (1972).

19. S. Mandelstam, Nucl. Phys. B69, 77 (1974)

20. D. Mattis, E. Lieb, J. Math. Phys. 6, 304 (1965) See also Ref. 14.

21. a. J. F. Willemsen, Phys. Rev. D9, 507 (1974)
b. J. F. Willemsen, Phys. Rev. D, to be published (SLAC-PUB-1397), (PRT).

22. Y. Iwasaki, K. Kikkawa, Phys. Rev. D8, 440 (1973), (PRT).

23. If the "$a_{\perp,n}$" operators have ($d_\perp$) transverse dimensions, one finds that the (null-plane quantization) "boosts" M_{i+} (see Ref. 14, 15) do not commute as they should, but rather give in the orbital model

$$[M_{i+}, M_{j+}] \propto \Sigma_n \{ [n \, (1 - d_\perp/24) + n^{-1} \left(\frac{d_\perp}{24} - \alpha_0 \right) [a^i_{-n} a^j_n - a^j_{-n} a^i_n] \}$$

The operators on the right-hand side have coefficient zero term by term if $d_\perp = 24$, and if the Regge intercept is $\alpha_0 = 1$, implying the presence of a tachyon. Notice this is a purely quantum effect.

24. A recent contribution with earlier reference is S. Ferrara, J. Wess, B. Zumino, CERN-TH 1863 (1974).

25. Historically, supergauges were invented in the first paper of Ref. 17, and were incorporated into the Lorentz algebra in Ref. 22. However, the particular combination of fermions and supergauge operators that appears in the Lorentz generators collapse to the orbital operator and "plasmon" (ρ_n) form when $d_\perp = 2$.

26. A. Chodos, C. B. Thorn, MIT-CTP-389, (1973), submitted to Nucl. Phys. B. These authors also discuss a modification of the basic action, so the string is massive.

27. I. Andric, J. L. Gervais, Nucl. Phys. B42, 240 (1972)
K. Kikkawa, Prog. Theor. Phys. 48, 625 (1972)
(R. Pettorino and I studied whether fermi fields of the forms given in these papers (PRT) can actually be solutions of the equations of motion. With some minor modifications, this turns out to be so provided the coupling constant is quantized.) One reason the Thirring model was appealing was the belief the Regge intercept depended upon the field theoretic dimension of the dynamical variables. In Thirring model this dimension depends upon the strength of the coupling. However, the dimensions of the currents do not change, and it is these that ultimately affect the value of the intercept. A very different use of the Thirring model has been proposed by H. Tye, MIT-CTP-383, to be published in Phys. Rev.

28. K. Bardakci, Nucl. Phys. B68, 331 (1974).

29. K. Bardakci, M. Halpern, Phys. Rev. D3, 2493 (1971).

30. H. J. Melosh, Phys. Rev. D.9, 1095 (1974)
Comments relevant in the present context may be found in R. Dashen, M. Gell-Mann, Coral Gables Conf. on Symm. Principles at High Energy, (1966); E. Eichten, F. Feinberg, J. F. Willemsen, Phys. Rev. D8, 1204 (1973); A. Casher, L. Susskind, Phys. Rev. D9, 436 (1974) ; (PRT).

31. Examples: G. Schierholz, M. Schmidt, Phys. Lett. B48, 341 (1974); L. Gonzales Mestres, R. Hong-Tuan, Orsay preprints (1973, 1974); G. Cohen-Tannoudji, F. Henyey, G. L. Kane, W. J. Zakrzewski, Phys. Rev. Lett. 26, 112 (1971);
P. Olesen, Nucl. Phys. B32, 609 (1971). See also Veneziano's review, Ref. 5.

32. J. Gunion, S. J. Brodsky, R. Blankenbecler, Phys. Rev. D8, 287 (1973), (PRT). (hereafter referred to as BBG.)

33. Elastic and Inelastic form factors in dual models have an extensive literature of their own. There are no real reviews on this subject. Samples:

a. Strings
Y. Nambu and A. Hacinliyan, EFI-70-67 (1970) unpublished

Y. Nambu, Phys. Rev. D4, 1193 (1971)
I. T. Drummond, Nucl. Phys. B35, 269 (1971)
Kraemmer et al, Ref. 11
A. Hacinliyan, Nuovo Cim. 8A, 541 (1972)
K. Kikkawa, B. Sakita, Phys. Rev. D6, 1734 (1972)
R. C. Brower, L. Susskind, Phys. Rev. D7, 1032 (1973)
C. Rebbi, Lett. Nuovo Cim. 1, 967 (1971)
H. C. Tze, Phys. Rev. D7, 1675 (1973)
L. Susskind, A. Casher, J. Kogut, Phys. Rev. D8, 4449 (1974)
J. F. Willemsen, ibid., p. 4457.

b. Non-planar Approaches:
A. Neveu, J. Scherk, Nucl. Phys. B41, 365 (1972)
E. Cremmer, J. Scherk, Nucl. Phys. B48, 29 (1972)
J. Schwarz, Nucl. Phys. B65, 131 (1973), and CALT-68-419, 424.

34. Actually, $\langle (x_\perp(\pi) - x_\perp(0))^2 \rangle \to \infty$. This may be viewed as a statement that the hadron fills all $\perp$ space. As an exercise, the reader may wish to calculate at what harmonic the string should be cut off to give a size of 1 fermi. The result is physically indistinguishable from ∞, even considering ultraviolet scales such as those associated with gravitation.

35. P. H. Frampton, Phys. Rev. D1, 3141 (1970)

36. This may not be bad after all: M. Chanowitz, S. D. Drell, Phys. Rev. D9, 2078 (1974).

37. In addition to Ref. 13, See F. Mansouri, Phys. Rev. D8, 1159 (1973)

38. J. Schwarz, Nucl. Phys. B68, 221 (1974)

39. F. Ardalan, F. Mansouri, Yale 3075-72 (1974), (PRT).

40. S. Mandelstam, a) Nucl. Phys. B64, 205 (1973), b) Ref. 19.

41. M. Kaku, K. Kikkawa, CC of the CUNY preprints (1973, 1974)

42. A. Chodos, R. Jaffe, K. Johnson, C. Thorn, V. Weisskopf, to be publ. in Phys. Rev. D. (See also M. Creuz, BNL 18789 (1974)).

43. C. N. Yang, T. T. Wu, "Properties of Matter under unusual Conditions", ed. Fernbach and Mark, p. 349.

44. E. P. Solodovnikova, A. N. Tavkhelidze, O. A. Khrustalev, Theo. and Math. Phys. 10, 162 (1972) (a lot of physics is already in this non-relativistic model.)
T. D. Lee and G. C. Wick, Phys. Ref. D9, 2291 (1974)
W. Bardeen, M. Chanowitz, S. Drell, M. Weinstein, T. M. Yan, to be publ.

45. H. B. Nielsen, P. Olesen, Nucl. Phys. B61, 45 (1973)

46. Strings appear elsewhere in mathematical physics, associated with the theory magnetic poles. P. A. M. Dirac, Phys. Rev. 74, 817 (1948); J. Schwinger,

Phys. Rev. 144, 1087 (1966). Since a trapped magnetic field is required in Ref. 45, an open string should have magnetic poles at the ends. The possible relevance of magnetic poles to the string model has been discussed previously in Ref. 7, and Mansouri and Nambu, Ref. 8, e.g. More recent physical discussions, with Ref. 45 in mind, have been given by Y. Nambu, Proc. of the Johns Hopkins Workshop on Current Problems in High Energy Particle Theory (1974);
P. Olesen, H. C. Tze, Niels Bohr Inst. preprint (1974);(PRT);
G. Parisi, CO-2271-29 (1974).

47. Case of O(N): G. 't Hooft, CERN-TH-1786; SU(N): G. Canning, Niels Bohr Inst. preprint (1974).

48. References to (N^{-1}) expansion in other applications are given in S. Ma, Rev. Mod. Phys. 45, 589 (1973). Recent in particle physics: H. Schnitzer, Brandeis preprint; S. Coleman, R. Jackiw, H. Politzer, MIT-CTP-413, (PRT).

49. D. Amati, V. A. Alessandrini, Nuovo Cim. 4A, 793 (1971). With one loop, these authors obtain a negative result for deep scattering. An interesting physical discussion is in P. G. O. Freund, S. Nandi, EFI-74/33 (1974).

50. G. 't Hooft constructs an "effective Hamiltonian" for ϕ^3 that generates the planar graphs, and discusses the problem of having the eigenvalues of H_{eff} "crystallize" from a continuous to a discrete spectrum. This may occur as a function of the coupling constant - see Ref. 44.

51. H. M. Chan, J. E. Paton, Nucl. Phys. B10, 516 (1969).

52. J. Scherk, Nucl. Phys. B31, 222 (1971)
A. Neveu, J. Scherk, Nucl. Phys. B36, 155 (1972).

53. Kadanoff's hypothesis, Physics 2, 263 (1966), is that in a spin system, as $T \rightarrow T_c$, small clusters of spins start to act as though they were a single big spin; next, these clusters form larger ones which behave as a unit, etc., until finally the entire sample is correlated. See also J. Kogut, L. Susskind, Phys. Rev. D9, 697 (1974).

54. J. Kogut, K. Wilson, Physics Reports (to be published).

55. K. Wilson, preprint CLNS-262 (1974). A special feature of Wilson's methods is that gauge invariance (though not Lorentz invariance) is maintained. The values of planar graphs in the N^{-1} method, Ref. 47, on the other hand, depend on the choice of gauge.

56. In the field theory of strings, Ref. 41, the master field is a functional of the configuration space coordinates $X_\mu(\theta, \tau)$, $\Psi = \Psi(X(\theta, \tau))$. These X_μ in turn may be viewed as fields on the manifold (θ, τ). A generalization could be $\Psi = \Psi(\phi(X))$, where $\phi(X)$ are quantum fields in the four-dimensional Minkowski manifold.