Thermodynamic analysis of Kerr-Newman black holes

O Ruiz[1], U Molina[1], and P Viloria[2]

[1] Departamento de Física, Universidad del Atlántico, Puerto Colombia, Colombia
[2] Departamento de Ciencias Naturales y Exactas, Universidad de la Costa, Barranquilla, Colombia

E-mail: ubaldomolina@mail.uniatlantico.edu.co, pviloria@cuc.edu.co

Abstract. In this paper we calculate the Hawking temperature of a black hole described by the Kerr-Newman metric, starting from the surface gravity, the area of the event horizon and the angular velocity of the black hole. To do this we apply the laws of black hole thermodynamics: we first set the energy conservation through a relationship between the mass M, the charge Q and the angular momentum J, then we implement the Hawking's theorem of areas by setting an upper bound to the energy and we get finally the surface gravity of the black hole. In addition, we study the relationship between the black hole parameters (mass M, angular momentum J, electric charge Q) and the Hawking temperature.

1. Introduction

In 1974 Hawking [1,2] predicted that the curvature of space-time at the event horizon of a black hole is sufficient to excite photons from vacuum and cause a continuous flow of them, known as Hawking radiation. The continuous process causes the black hole to lose energy with the consequent decrease of its mass, until after a while and the hole disappears completely. Hawking predicted that this radiation has a well-defined temperature proportional to the superficial gravity in its horizon of events.

In 2009 V. Pankovic [3], presents a simplified method for describing and calculating the basic characteristics, dynamics (horizons) and thermodynamics of a Kerr-Newman black hole. His method was based on principles of classical mechanics, electrodynamics, thermodynamics, statistics, non-relativistic quantum mechanics and on the elementary form of the general principle of relativistic equivalence; which represented results already proposed in the theory of quantum gravity.

In 2010 F. Belgiorno and his colleagues [4], created an optical analogue of the event horizon of a black hole, the results of which coincide with Stephen Hawking's quantum predictions for radiation emitted by a black hole evaporating. If the result of Franco Belgiorno and his colleagues is confirmed, it would be the first observation of Hawking radiation.

This paper presents a simplified method for estimating Hawking temperature and evaporation time of Kerr-Newman black holes. The relationship between the hole parameters (mass M, angular momentum J, electric charge Q) and this time is also studied.

2. Special relativity

The Kerr-Newman metric is identified by three parameters (M; a; Q). The general line element for this family in Boyer-Lindquist [5] coordinates is expressed by Equation (1) as,

$$ds^2 = \frac{\rho^2 \Delta}{\Sigma^2} dt^2 - \frac{\rho^2}{\Sigma^2}\left(d\emptyset - \frac{2aMr}{\Sigma^2}dt\right)^2 \sin^2\theta - \frac{\rho^2 \Delta}{\Sigma^2}dr^2 - \rho^2 d\theta^2, \tag{1}$$

where M is the mass, Q is the electrical charge, $M = \frac{J}{M}$ is the specific angular momentum of the black hole, J being the total angular momentum, r the radial coordinate θ and ∅ the spherical angular coordinates.

This metric has the same shape as Kerr's black hole but with the difference in the definition of Δ, ρ and Σ, defined by Equation (2), Equation (3) and Equation (4) respectively,

$$\Delta = r^2 - 2Mr + a^2 + Q^2 \tag{2}$$

$$\rho^2 = r^2 + a^2 \cos^2\theta \tag{3}$$

$$\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta \tag{4}$$

2.1. Parameters of a black hole
There are only four exact solutions of Einstein's equations, describing black holes with or without load and angular momentum:

- The Schwarzschild solution, static and spherically symmetrical.
- Reissner-Nordstrom's solution, static, spherically symmetrical, depends on the mass M and the electrical charge Q.
- Kerr's solution, stationary, axisymmetric, depends on the mass M and the angular momentum J.
- The Kerr-Newman solution, stationary, axisymmetric, depends on the mass M, the angular momentum J and the electrical charge Q.

Of all these, the Kerr-Newman solution is the most general solution corresponding to the final equilibrium state of a black hole.

2.2. Singularities and horizons
Kerr-Newman space-time has coordinates singularities in the axis of symmetry (θ = 0) and in those values of r for which (Δ = 0). This last condition can be written by Equation (5) as,

$$\Delta = (r - r_+)(r - r_-) \tag{5}$$

Then, the event horizons of the Kerr-Newman black hole are found on the radios,

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2} \tag{6}$$

Equation (6) makes it possible to distinguish three cases: $M^2 < a^2 + Q^2$, $M^2 > a^2 + Q^2$ and $M^2 = a^2 + Q^2$. In addition, it has curvature singularities when, ρ = 0.

2.3. Event horizon area
To calculate the area of the event horizon, remember that the generalized volume element is obtained with Equation (7),

$$dV = \sqrt{|g|}d^n x, \tag{7}$$

where g is the determinant of the metric and n is the dimension of the variety.

If one considers a hypersurface of the Kerr-Newman variety, in Boyer-Lindquist coordinates as observed in Equation (1), with t = contand r = $r_{\pm}$, the generalized volume depends only on the θ and Ø coordinates, and corresponds to the area of the event horizon, according to Equation (8),

$$A = \int_0^\pi \int_0^{2\pi} \left[\sqrt{|g_{\theta\theta} g_{\emptyset\emptyset}|} \right] d\theta d\emptyset \tag{8}$$

Assessing the $r = r_+$ hypersurface, we have Equation (9),

$$A = \int_0^\pi \int_0^{2\pi} (r_+^2 + a^2) \sin\theta \, d\theta d\emptyset = 4\pi(r_+^2 + a^2) \tag{9}$$

From Equation (6) is obtained the area, Equation (10),

$$A = 4\pi \left[2M \left(M + \sqrt{M^2 - a^2 - Q^2} \right) - Q^2 \right] \tag{10}$$

The values of parameters M, a and Q may vary if the black hole absorbs particles with mass, angular momentum or charge, and consequently, the area A may vary. However, it can be shown that in no case can the area A decrease [6-9]. Thus, the area A of a black hole is similar to entropy in that it cannot decrease; as will be seen in section 3.3, this analogy is not only a formal coincidence, but an intrinsic property of black holes.

2.4. Surface gravity
To find surface gravity, we change the coordinates of Equation (1) to the coordinates of Eddington-Finkelstein [10], where singularities Δ= 0 do not exist. For this, the new temporal coordinates, ϑ, and angular coordinates, χ, are defined by means of Equation (11) and Equation (12).

$$d\vartheta = dt + \frac{r^2 + a^2}{\Delta} dr \tag{11}$$

$$d\chi = d\emptyset + \frac{a}{\Delta} dr \tag{12}$$

So, the Kerr-Newman metric takes the form of Equation (13),

$$ds^2 = \frac{\Delta - a^2 \sin^2\theta}{\rho^2} d\vartheta^2 + 2d\vartheta dr - \frac{2a(r^2 + a^2 - \Delta)\sin^2\theta}{\rho^2} d\vartheta d\chi - 2a\sin^2\theta dr d\chi - \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\rho^2} d\chi^2 \tag{13}$$

Through this change of variables, it is found that in the r = $r_{\pm}$ hypersurfaces we have the Killing vectors for the coordinates, represented by Equation (14),

$$\psi_{\pm} = \frac{\partial}{\partial\vartheta} + \frac{a}{(r_+^2 + a^2)} \frac{\partial}{\partial\chi} \tag{14}$$

The Killing $\psi_{\pm}$ vector can be written to the Boyer-Lindquist coordinates as,

$$\psi_{\pm} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial t} \tag{15}$$

The quantity Ω in Equation (15) is the angular velocity of the black hole, which is constant at the horizon and is written by Equation (16),

$$\Omega = \frac{a}{r_{\pm}^2 + a^2} \tag{16}$$

Using Equation (6) you have,

$$\Omega = \frac{a}{2M\left(M + \sqrt{M^2 - a^2 - Q^2}\right) - Q^2} \tag{17}$$

Equation (17) is used below to calculate the Hawking temperature of a Kerr-Newman black hole.

2.4.1. Killing horizons. Be a horizon of Killing N and be ξ your normal Killing vector. The surface gravity k is defined as the proportionality constant between the $\xi^{\vartheta} \nabla_{\vartheta} \xi^{\mu}$ vectors and over ξ^{μ} [11]. The value of k is defined by Equation (18),

$$k_{\pm} = \frac{r_{\pm} - r_{\mp}}{2(r_{\pm}^2 + a^2)} \tag{18}$$

For $r = r_+$ and according to Equation (6), surface gravity can be expressed by Equation (19),

$$k = \frac{r_{\pm} - r_{\mp}}{2(r_{\pm}^2 + a^2)} \tag{19}$$

In the case of $M^2 = a^2 + Q^2$, $k = 0$ would be equivalent to an extreme black hole, which has lost its horizon of events.

3. Black hole thermodynamics
In 1973 Bardeen, Carter and Hawking formulated the four laws of classical black hole mechanics [12], which are analogous to the four laws of thermodynamics. This was also supported by the works of Chistodoulou [9] and Bekenstein [13,14].

3.1. Zero law: constancy of surface gravity on the horizon
Since the event horizon of a black hole is a Killing horizon, it can be assured that Equation (20) is satisfied:

$$\xi_{\mu} \nabla_{\vartheta} k = -\xi_{\mu} \nabla_{\vartheta} \rho \xi^{\rho} = 0 \tag{20}$$

As ξ is perpendicular to any vector tangent t to the horizon, it is concluded that $t^{\mu} \nabla_u k = 0$, i.e. k is constant over the horizon, as can also be corroborated in Equation (20) where the expression for k is independent of r, θ and $\emptyset$.

3.2. First law: Differential relationship between mass, area, load and angular momentum
Given an asymptotically flat and stationary space-time, the area of the horizon can only depend on the parameters M; J and Q, therefore $A = A(M, J, Q)$. When a stationary black hole is disturbed, a new stationary black hole state is reached with increased mass, load and angular momentum parameters in dM; dQ and dJ, respectively, then the area of the event horizon undergoes an infinitesimal change,

$$dA = \left(\frac{\partial A}{\partial M}\right) dM + \left(\frac{\partial A}{\partial J}\right) dJ + \left(\frac{\partial A}{\partial Q}\right) dQ \tag{21}$$

From Equation (10), Equation (17) and Equation (19) we find the differentials: $\frac{\partial A}{\partial M}$, $\frac{\partial A}{\partial J}$ and $\frac{\partial A}{\partial Q}$, replacing in Equation (21), clearing dM, we obtain Equation (22),

$$dM = \frac{k}{8\pi} dM + \Omega dJ + \varphi dQ, \tag{22}$$

where $\varphi = \frac{Q}{r}$ has been defined, represented in Equation (23),

$$\varphi = \frac{Q\left(M + \sqrt{M^2 - a^2 - Q^2}\right)}{2M\left(M + \sqrt{M^2 - a^2 - Q^2}\right) - a^2 - Q^2} \tag{23}$$

In essence, the first law of the mechanics of black holes is limited to establishing that the energy of the black hole is conserved.

3.3. Second law: Hawking areas theorem

Accepting the conjecture of cosmic censorship, the cross-sectional area of the future event horizon cannot diminish,

$$\partial A \geq 0 \tag{24}$$

In Equation (24), the area of the horizon can be expressed in terms of the irreducible mass M_{ir},

$$A = 16\pi M_{ir}^2, \tag{25}$$

where the irreducible mass M_{ir} of Equation (25), in turn, has the form of Equation (26),

$$M_{ir} = \frac{1}{2}\sqrt{2M\left(M + \sqrt{M^2 - a^2 - Q^2}\right) - a^2 - Q^2} \tag{26}$$

Therefore, the second law can be written by Equation (27), alternatively as,

$$\partial M_{ir} \geq 0 \tag{27}$$

This equation establishes an upper limit for the extraction of energy from a rotating black hole.

3.4. Third law: Superficial gravity is never annulled

Given following the analysis of Equation (19), we see that $k = 0$ when $M^2 = a^2 + Q^2$, in turn we see that from Equation (6) that for $r_+ = r_- = M$ we arrive at the same condition. If we continue increasing the angular momentum or the charge we will have a black hole without horizon; this is in contradiction with the so-called hypothesis of cosmic censorship, which tells us that, to safeguard the laws of physics, there can be no naked singularities in nature where these laws are violated. The fact that superficial gravity is annulled for the extreme case with a maximum value of $a^2 + Q^2$, where a horizon still exists, is associated with the so-called third law of black hole mechanics. This law establishes that it is impossible to reach $k = 0$, with a finite succession of physical processes.

Once again, we can see that there is a clear relationship with thermodynamics: The above theorem is surprisingly similar to the third law of thermodynamics which tells us that $T = 0$, cannot be reached with a finite number of thermodynamic transformations.

4. Hawking temperature

From the classical point of view black holes have a perfect absorption, because they absorb all the radiation that reaches them, but do not emit anything and therefore their physical temperature must be absolute zero. But if the black hole is at 0K, its entropy must also be zero. And if its entropy is zero, it is possible to diminish the entropy of the universe by throwing inside a certain amount of matter that

has a high entropy. For this reason, from the classical point of view, the laws of black hole mechanics only resemble the laws of thermodynamics, but there is no direct connection between them. However, Hawking found that black holes actually emit radiation with a thermal spectrum [1,2], while Bekenstein suggested that there is an entropy associated with the black hole proportional to the area of its event horizon [15]. More precisely, Bekenstein proposed that the entropy of a black hole is represented by Equation (28),

$$S = \frac{\gamma k_B c^3}{\hbar G} A, \tag{28}$$

where γ is a constant dimensionless. Hawking found that γ corresponds to $^1/_4$, based on the application of quantum field theory over curved spaces to black holes. Then, the entropy of a black hole can be written according to Equation (29) as,

$$S = \frac{1}{4} \frac{k_B c^3}{\hbar G} A \tag{29}$$

In the system of natural units ($\hbar = G = c = 1$), we have Equation (30),

$$S = \frac{A}{4} \tag{30}$$

Using Equation (11), entropy is expressed by Equation (31) as,

$$S = \pi(r^2 + a^2) = \frac{A}{4} \tag{31}$$

Differentiating Equation (31) with respect to r and assuming that parameter a is much smaller than r and M we have Equation (32),

$$dS = 2\pi r dr \tag{32}$$

According to Equation (6), for a and Q much less than M, Equation (33) is deduced,

$$dS = 2\pi \frac{r^2}{\sqrt{M^2 - a^2 - Q^2}} dM \tag{33}$$

The black hole can be considered as a system in a state of thermodynamic equilibrium that obeys the first law, Equation (22), in the following form,

$$dM = TdS + \Omega dS + \varphi dQ \tag{34}$$

For the case of neutral black holes ($dQ = 0$), replacing Equation (17) and Equation (33) in Equation (34), Equation (35) is obtained,

$$dM = 2\pi T \frac{r^2}{\sqrt{M^2 - a^2 - Q^2}} dM + \frac{a}{r^2 + a^2} dJ \tag{35}$$

By doing $a \ll M$ and using the $J = aM$ definition, one finds that temperature, Equation (36),

$$T = \frac{1}{2\pi} \frac{\sqrt{M^2 - a^2 - Q^2}}{r^2 + a^2} \tag{36}$$

According to Equation (6), the Hawking temperature for a Kerr-Newman black hole can finally be expressed by Equation (37), in the form,

$$T = \frac{\sqrt{M^2 - a^2 - Q^2}}{2\pi\left(2M^2 + 2M\sqrt{M^2 - a^2 - Q^2} - Q^2\right)} \tag{37}$$

It is possible to graph (Figure 1), the Hawking Temperature of a Kerr Newman black hole as a function of the parameters M, a and Q, taking $a = 1\text{x}10^4 m_p$, being m_p the Planck mass.
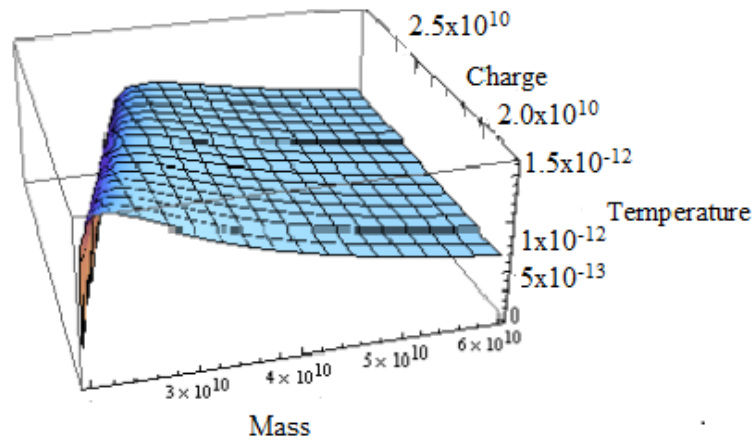

Figure 1. Hawking temperature of a Kerr Newman black hole as a function of M, a and Q, with, $a = 1\text{x}10^4 m_p$.

In Figure 1, a value was set for the angular momentum a, specific to the black hole, its mass M and load Q are varied, and the Hawking temperature variation is found. It is observed that for values of Q and a, other than zero, the temperature is maximum and then quickly tends to absolute zero even if one has a certain amount of mass without evaporating from the black hole.

In addition, from the Equation (37), it is inferred that when $Q = a = 0$, for M other than zero has to $T = \frac{1}{8\pi M}$. The temperature of the black hole decreases as the mass increases, so it has a specific negative heat and cools down as it absorbs mass. There is a unique behavior along the extreme limit $J = M\sqrt{M^2 - Q^2}$, which is the same $M^2 = a^2 + Q^2$.

Mass, load and specific angular momentum are given in Planck mass units ($m_p = \sqrt{\hbar c / G} = 2.2\text{x}10^{-8}$Kg); temperature is given in Planck temperature units ($T_p = \frac{m_p c^2}{k_B} = 1.4\text{x}10^{32}$K).

5. Conclusions

Surface gravity was found, which establishes a limit for the black hole consistent with the third law of thermodynamics. From this, an expression is established for the Hawking temperature of a Kerr-Newman black hole as a function of its mass M, angular moment J and load Q. As the black hole loses mass, its temperature increases inversely proportional. The temperature of the black hole decreases when the mass increases, therefore, it has a specific negative heat.

When the black hole is energized, its temperature decreases rather than increases. The effect of the charge and the angular momentum (Q and a different from zero) is that the black hole reaches the maximum temperature value and it suddenly decays to absolute zero before losing its mass altogether, unlike the Schwarzschild case.

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