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Beam Heating by Wall Resistivity

the circulating current in the Doubler for $N \times 10^{13}$ circulating protons may be written:

$$I(t) = N \times 10^{13} \times 1.6 \times 10^{-19} \times \frac{3 \times 10^{8}}{6283} \left[1 + \sum_{n=1}^{\infty} A_n \cos(n\omega t) \right] \text{ Amps}$$

$$= 7.64 \times 10^{-2} N \left[1 + \sum_{n=1}^{\infty} A_n \cos(n\omega t) \right] \text{ Amps}$$

where A_n are the normalized Fourier cosine coeficients for the particular bucket shape and width, and ω represents the fundamental RF (and rotational) frequency ($2\pi \times 53.1$ MHz and $2\pi \times 47.7$ kHz respectively).

We consider 3 specific pulse shapes (T_p is the full width at half maximum)

Pulse shape	$F(t)$	A_n
Gaussian	$e^{-\frac{(4 \ln 2) t^2}{T_p^2}}$	$2 e^{-(n\omega T_p)^2 / 16 \ln 2}$
Parabolic	$1 - 2\left(\frac{t}{T_p}\right)^2 ; \ \lvert t \rvert < \frac{T_p}{\sqrt{2}}$	$\frac{12}{(n\omega T_p)^2} \left[\frac{\sqrt{2}}{n\omega T_p} \sin\left(\frac{n\omega T_p}{\sqrt{2}}\right) - \cos\left(\frac{n\omega T_p}{\sqrt{2}}\right) \right]$
Triangular	$1 - \frac{\lvert t \rvert}{T_p} ; \ \lvert t \rvert < T_p$	$\frac{4}{(n\omega T_p)^2} \left[1 - \cos(n\omega T_p) \right]$

The image currents flow in the walls of the vacuum chamber in a layer δ thick (δ = skin depth) $\times$ πd wide (d = diameter of beam tube). We assume a wall resistivity of $\rho = 0.53 \times 10^{-6}$ ohm-m at 4°K and a beam tube diameter of 2.60" (0.066 m)

Neglecting for the moment the subharmonics of the RF frequency

We use $\omega = 2\pi \times 53.1\ \mathrm{MHz} = 3.34 \times 10^{8}\ \mathrm{sec}^{-1}$

$$\therefore \frac{\text{Resistance}}{\text{unit length}} = \frac{\rho}{\pi d \delta} = \frac{\rho}{\pi d}\sqrt{\frac{4\pi \times 10^{-7} n\omega}{2\rho}} = 5.09\sqrt{n} \times 10^{-2}\ \frac{\text{ohms}}{m}$$

Hence the total heating in the cold region of the doubler due to wall resistivity (cold region = 5860 m) is

$$W = \sum_{n=1}^{\infty} \langle I^2 \rangle_n\ 5.09 \times 10^{-2}\sqrt{n} \times 5860 \quad \text{watts}$$

$$\text{where } \langle I^2 \rangle_n = \frac{\int_0^{2\pi} I^2(t)\, d(\omega t)}{\int_0^{2\pi} d(\omega t)}$$

$$= \frac{(7.64 \times 10^{-2} N)^2}{2\pi} \int_0^{2\pi} \left[1 + \sum_{n=1}^{\infty} A_n \cos(n\omega t)\right]\left[1 + \sum_{m=1}^{\infty} A_m \cos(m\omega t)\right] d(\omega t)$$

$$= \frac{(7.64 \times 10^{-2} N)^2}{2\pi} A_n^2 \int_0^{2\pi} \cos^2(n\omega t)\, d(\omega t) \quad \text{(neglecting dc component)}$$

$$= \frac{(7.64 \times 10^{-2} N)^2}{2} = 2.92 \times 10^{-3} N^2 \quad \text{amps}^2$$

$$\therefore W = 2.92 \times 10^{-3} N^2 \times 5.09 \times 10^{-2} \times 5860 \sum_{n=1}^{\infty} A_n^2 \sqrt{n} \quad \text{watts}$$

$$W = 0.871\, N^2 \sum_{n=1}^{\infty} A_n^2 \sqrt{n} \quad \text{watts}$$

This is shown on the following page for a variety of pulse widths. These values agree very well with Sho Ohnuma's values (UPC# 33) where he has used the parameter "total pulse width" ($= 2\sqrt{6}\,\sigma \cong 2.08\, T_p$). (recall that $T_p = \sqrt{8 \ln 2}\,\sigma = 2.35\,\sigma$).

Beam Heating (watts per 10^{13} protons) for 5860 m cold region

Tp(nsec) FWHM	Gaussian	Parabolic	Triangular
0.10	1158.08	1493.09	1175.21
0.20	446.27	627.70	442.53
0.30	243.08	346.90	242.47
0.40	157.98	225.99	157.76
0.50	113.12	161.98	113.08
0.60	86.12	123.40	86.10
0.70	68.40	98.03	68.40
0.80	56.03	80.30	56.04
0.90	47.00	67.35	47.01
1.00	40.17	57.55	40.18
1.10	34.86	49.93	34.87
1.20	30.63	43.85	30.64
1.30	27.19	38.92	27.20
1.40	24.36	34.86	24.37
1.50	21.99	31.46	22.00
1.60	19.99	28.58	20.00
1.70	18.27	26.12	18.28
1.80	16.80	24.00	16.80
1.90	15.51	22.15	15.51
2.00	14.38	20.53	14.39
2.10	13.38	19.10	13.39
2.20	12.50	17.83	12.51
2.30	11.71	16.70	11.72
2.40	11.00	15.68	11.01
2.50	10.37	14.77	10.37
2.60	9.79	13.94	9.79
2.70	9.27	13.19	9.27
2.80	8.79	12.50	8.79
2.90	8.35	11.88	8.35
3.00	7.95	11.30	7.95
3.10	7.58	10.77	7.58
3.20	7.24	10.28	7.24
3.30	6.93	9.83	6.93
3.40	6.63	9.41	6.64
3.50	6.36	9.02	6.37
3.60	6.11	8.66	6.11
3.70	5.87	8.32	5.88
3.80	5.65	8.01	5.66
3.90	5.45	7.71	5.45
4.00	5.25	7.43	5.26
4.10	5.07	7.17	5.08
4.20	4.90	6.93	4.90
4.30	4.74	6.70	4.74
4.40	4.59	6.48	4.59
4.50	4.44	6.27	4.45
4.60	4.31	6.08	4.31
4.70	4.18	5.89	4.18
4.80	4.06	5.72	4.06
4.90	3.94	5.55	3.94

There are 3 main corrections to these beam heating calculations:

1. As pointed out by Jim Griffin (UPC 69), the subharmonics of 53 MHz due to gaps between booster batches and non-uniform filling of the ring adds additional structure to the beam resulting in additional heating.

2. Displacement of the beam from the center of the beam pipe produces a non-uniform azimuthal distribution of image currents and therefore increases the value of $\langle I^2 \rangle$.

3. Use of a non-circular beam tube also creates a non-uniform azimuthal distribution of image currents, even for centered beams, and therefore also increases beam heating.

The additional heating due to the subharmonics of 53.1 MHz seem to be adequately estimated by multiplying the above calculated heating values by $\frac{1113}{\#\text{filled buckets}}$. this adequately accounts for the increased peak currents in each filled bucket and the reduced number of filled buckets. It does not include the specific subharmonic structure, however, which is likely to be at frequencies $\lesssim$ 700 kHz or so and therefore a relatively small contribution due to the $\sqrt{n}$ factor.

We now examine the effect of an off-center beam in a round beam tube. Consider a line charge $+q$ at a position r_0, θ_0 relative to the origin. The potential for such a charge may be expressed as a harmonic expansion in r, θ (see Smythe sec 4.02) for $r > r_0$:

$$V_0(r,\theta) = \frac{q}{2\pi\epsilon_0}\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r_0}{r}\right)^n (\cos n\theta_0 \cos n\theta + \sin n\theta_0 \sin n\theta) - \ln r\right]$$

Similarly a charge $-q$ at a position r_1, θ_0 for $r < r_1$ is:

$$V_1(r,\theta) = \frac{-q}{2\pi\epsilon_0}\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{r_1}\right)^n (\cos n\theta_0 \cos n\theta + \sin n\theta_0 \cos n\theta - \ln r_1)\right]$$

Each of these satisfies Laplace's equation, as does a linear sum + a constant: (We here choose $\theta_0 = 0$ for convenience)

$$V_s(r,\theta) = \frac{q}{2\pi\epsilon_0}\left[\sum_{n=1}^{\infty} \frac{1}{n}\left[\left(\frac{r_0}{r}\right)^n - \left(\frac{r}{r_1}\right)^n\right]\cos n\theta - \ln(r/r_1) + \text{constant}\right]$$

We can make this potential zero at $r = a$ if $\frac{r_0}{a} = \frac{a}{r_1}$ and const $= \ln\left(\frac{a}{r_1}\right)$

$\therefore\ r_1 = \frac{a^2}{r_0}$ and constant $= \ln(r_0/a)$

$$\therefore\ V_s(r,\theta) = \frac{q}{2\pi\epsilon_0}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r_0}{r}\right)^n - \left(\frac{r_0 r}{a^2}\right)^n\right]\cos n\theta - \ln(r/a)\right]$$

this potential is zero on the surface of a cylinder of radius $r=a$, and can be considered as the result of the charge q at $r=r_0$ and a surface charge density σ on the walls of the cylinder. Specifically

$$\sigma(\theta) = |\epsilon_0\vec{E}| = -\epsilon_0\vec{\nabla}V_s(r,\theta)\Big|_{r=a} = -\epsilon_0\frac{\partial V_s(r,\theta)}{\partial r}\Big|_{r=a}$$

$$= -\frac{q}{2\pi}\left[\sum\left[\left(\frac{r_0}{r}\right)^n + \left(\frac{r_0 r}{a^2}\right)^n\right]\frac{\cos n\theta}{r} + \frac{1}{r}\right]_{r=a}$$

$$= -\frac{q}{2\pi a}\left[1 + \sum_{n=1}^{\infty}2\left(\frac{r_0}{a}\right)^n\cos n\theta\right]$$

It is seen that $\int_0^{2\pi}\sigma(\theta)\,a\,d\theta = -q$ as expected

We now examine $$\frac{\langle\sigma^2(\theta)\rangle_{r=r_0}}{\langle\sigma^2(\theta)\rangle_{r=0}} = \frac{1}{2\pi}\int_0^{2\pi}\left(1 + \sum_{n=1}^{\infty}4\left(\frac{r_0}{a}\right)^{2n}\cos^2 n\theta\right)d\theta$$

$$= 1 + 2\sum_{n=1}^{\infty}\left(\frac{r_0}{a}\right)^{2n}$$

$$\frac{\langle\sigma^2(\theta)\rangle_{r=r_0}}{\langle\sigma^2(\theta)\rangle_{r=0}} = \left[\frac{2}{1-\left(\frac{r_0}{a}\right)^2}\right] - 1$$

and $\int_0^{2\pi}\cos n\theta\,d\theta = 0$

since $\int_0^{2\pi}\cos n\theta\cos m\theta\,d\theta = \pi\,\delta(m-n)$ and $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ $(x<1)$

As the heating is proportional to $\langle\sigma^2(\theta)\rangle$, the above factor represents the additional multiplicative factor in the heating calculation for off center beams.

Table of Geometrical factor $\dfrac{\langle \sigma^2(\Theta) \rangle_{r=R}}{\langle \sigma^2(\Theta) \rangle_{r=0}}$ for various radius beam pipes

Beam position R (INCHES)	— Beam pipe radius — A=1.2"	A=1.3"	A=1.4"
0.00	1.0000	1.0000	1.0000
0.10	1.0140	1.0119	1.0103
0.20	1.0571	1.0485	1.0417
0.30	1.1333	1.1125	1.0963
0.40	1.2500	1.2092	1.1778
0.50	1.4202	1.3472	1.2924
0.60	1.6667	1.5414	1.4500
0.70	2.0316	1.8167	1.6667
0.80	2.6000	2.2190	1.9697
0.90	3.5714	2.8409	2.4087
1.00	5.5455	3.8986	3.0833

For a beampipe of approximate radius $r(\Theta) = 1.3''\left(1 \pm \frac{1}{17}\cos 4\Theta\right)$ which adequately represents the actual case, the potential seems difficult to calculate in closed form. However, the real geometrical factor is probably within the range of values shown above. Specifically, the $A = 1.4''$ column probably represents the present "diamond" shape beam pipe ◇, while the $A = 1.2''$ represents the "square" shape ▢ proposed to reduce radiation induced quenching.

In summarizing, the resistive wall beam heating for the entire 5860 m cryogenic length of the machine may be expressed as a series of factors:

$$W = (14.4 \text{ Watts}) \times N^2 \times \left(\frac{1113}{\#\text{ filled buckets}}\right) \times \left(\frac{2\text{ nsec}}{T_p}\right)^{1.5} \times \left(\begin{array}{c}\text{Geometric}\\ \text{factor}\end{array}\right)$$

N: # $\times 10^{13}$ protons

$\left(\frac{2\text{ nsec}}{T_p}\right)^{1.5}$: Assumes Gaussian with full width at half max = T_p

Geometric factor: beam position and bore tube geometry (see above table)

For 5×10^{13}, 1079 filled buckets, $T_p = 1.5$ nsec, beam 0.5" off center in "square" beam pipe

$$W \sim 14.4 \times 5^2 \times \frac{1113}{1079} \times \left(\frac{2}{1.5}\right)^{1.5} \times 1.42 = 812 \text{ watts}$$