Echoes of Infrared Universality:

Soft Theorems and Asymptotic Symmetries Beyond the Leading Order

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If there is one thing I have learned from writing this thesis, it is this: even infinity is not the limit.

Abstract

Over the past decade, asymptotic symmetries of gauge and gravity theories have been shown to underlie universal features of scattering amplitudes in the infrared regime. In particular, soft theorems can be reinterpreted as Ward identities of asymptotic symmetries. This thesis explores these connections from two complementary perspectives: the scattering amplitude approach, where soft theorems directly constrain the S-matrix, and the general relativistic approach, where they appear as diffeomorphisms preserving the asymptotic fall-offs of the metric.

In the first part, we extend the standard tree-level picture by incorporating loop corrections to soft theorems in quantum electrodynamics and gravity. We show that logarithmic terms in the photon and graviton soft expansions, absent at tree level, naturally arise from infrared effects and long-range interactions, and we interpret these corrections in terms of modified Ward identities for asymptotic symmetry charges. A detailed analysis reveals how the gravitational dressing of charged and massive states generates the additional loop-level structures observed in the subleading soft factors.

In the second part, we shift focus to the celestial representation and the full soft tower at tree level. Working in the Newman–Penrose formalism, we study the Einstein–Maxwell system and derive recursion relations for an infinite family of asymptotic charges directly from the equations of motion. We identify suitable quasi-conserved charges that, when smeared over the celestial sphere, close into the celestial $sw_{1+\infty}$ algebra—a symmetry structure unifying the gravitational $w_{1+\infty}$ and electromagnetic s-algebras. We further discuss the extension to Einstein–Yang–Mills theory and the interplay between gauge and gravitational couplings.

Taken together, these results provide new insights into the infrared structure of scattering amplitudes, the algebra of asymptotic symmetries, and their realization in theories with coupled gauge and gravitational interactions.

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Chapter 1

Introduction

Symmetries are a physicist's guiding light, illuminating regions where experiments cannot yet reach. Our universe is endowed with a wealth of symmetries, and these often provide the deepest insights into its underlying structure. Since the seminal work of Emmy Noether, which established a profound link between the mathematical symmetries of the laws of nature and the physical conservation laws they imply [1], the search for new such connections has been relentless. In this thesis, we ask: can symmetries teach us something fundamental about quantum gravity?

Over the decades, many researchers have sought to determine the ultimate symmetry group of our universe. In the latter half of the 20th century, general relativity [2] stood at the forefront of theoretical physics. One of its remarkable predictions was the existence of gravitational waves. In the early 1960s, Bondi, van der Burg, Metzner, and Sachs undertook a systematic analysis of gravitational waves in asymptotically flat spacetimes [3–5]. Their aim was to identify the symmetry group of such spacetimes, expecting to recover the familiar Poincaré group of flat spacetime. To their surprise, they found not merely the Poincaré group but an infinite-dimensional extension — now known as the *Bondi-van der Burg-Metzner-Sachs (BMS) group*. We will review this in detail in chapter 2. This discovery sparked a rich mathematical literature exploring the structure and implications of asymptotic symmetries [6, 7].

During the same period, quantum field theory (QFT) had matured into a well-established framework, with its predictions repeatedly confirmed by experiment. This maturity allowed researchers to probe deeper theoretical aspects, including the nature of *infrared divergences* and the universality of *soft theorems*. In QFT, the central object of study is the scattering matrix (or S-matrix), which encodes the transition from *in*-states defined in the far past to *out*-states in the far future. Scattering amplitudes are often plagued by ultraviolet divergences, i.e., they blow up at high energies, which can be tamed using renormalization techniques [8], yielding results that have been tested to extraordinary precision.

Massless particles, however, introduce a different challenge: infrared divergences. These cannot be handled by standard renormalization methods. A powerful way to study such divergences was introduced by Weinberg, who analyzed the *soft limit* — the regime where the energy of a massless particle tends to zero [9]. Remarkably, this limit imposes universal constraints on the S-matrix, known as *soft theorems* [9, 10]. We will review these and related developments in chapter 3.

For decades, the study of asymptotic symmetries in gravity and the analysis of soft theorems in QFT proceeded independently, largely as separate mathematical curiosities. In 2014, Strominger and collaborators [11, 12] demonstrated that these seemingly distinct topics are, in fact, two aspects of the same underlying structure in a theory of quantum gravity. In QFT, Ward identities associated with symmetries place constraints on the S-matrix; Strominger's work showed that the Ward identities for asymptotic symmetries reproduce precisely the known soft theorems. This unification has since become a cornerstone of the rapidly developing program known as *celestial holography* [13–16].

The main aim of this thesis is to investigate this relation between soft theorems and asymptotic symmetries in its most general setting, encompassing both massive and massless particles, as well as gravitational and electromagnetic interactions. There are several motivations for pursuing such a general treatment. Many of the most striking results in the literature have been derived in simplified contexts — typically at tree level, for massless particles, and with a single interaction — yet realistic physical processes often involve massive external states, multiple interactions, and quantum (loop) corrections. Extending the correspondence between asymptotic symmetries and soft theorems to include loop corrections tests its robustness and may reveal new structural features. Finally, a general framework is a necessary step toward applications ranging from precision scattering amplitudes to gravitational wave physics and, ultimately, to a flat space holographic correspondence.

1.1 Flat space holography

The quest for a quantum theory of gravity has a long and challenging history. Early attempts to unify quantum field theory (QFT) and general relativity (GR) encountered severe obstacles. In particular, the renormalization techniques that underpin the success of the Standard Model of particle physics [8] fail when applied to GR: perturbative quantum gravity is non-renormalizable [17], leaving many fundamental questions unanswered. This has motivated the search for a ultraviolet (UV) complete theory of gravity, with numerous approaches proposed over the years. Among the most influential of these is the *principle of holography* [18, 19].

Holography posits that a gravitational theory can be described by a dual non-gravitational theory living in one lower spacetime dimension. This idea is motivated by the observation that black hole entropy scales with the area of the event horizon [20, 21], suggesting that the fundamental degrees of freedom reside on the boundary, rather than in the bulk, of spacetime. While first articulated in heuristic form by 't Hooft [18] and Susskind [19], it was Maldacena [22] who provided the first concrete realization in the case of asymptotically anti–de Sitter (AdS) spacetimes. AdS spacetimes are solutions of the Einstein equations with a negative cormological constant. In this setting, the dual theory is a conformal field theory (CFT) living on the codimension-one boundary of AdS. The celebrated AdS/CFT correspondence has had remarkable success, particularly in using weakly coupled gravity in AdS to study strongly coupled CFTs [23, 24].

Our universe, however, has a small positive cosmological constant. While AdS remains a useful approximation in the near-horizon regions of certain black holes, it is not an accurate description at large distances. In many experimental contexts — at scales large compared to laboratory dimensions but small compared to cosmological scales — flat space is a good approximation. This raises the natural question: can the principle of holography be extended to asymptotically flat spacetimes?

In the development of AdS/CFT, a crucial role was played by the analysis of asymptotic symmetries. Prior to Maldacena's conjecture, Brown and Henneaux [25] had shown that the asymptotic symmetry group of three-dimensional AdS is precisely the conformal group in two dimensions, thereby laying the groundwork for holography. In the flat space context, the discovery of the BMS group [3, 5] and its modern connection to soft theorems [11] has provided a powerful motivation to study the asymptotic symmetries of flat spacetimes in greater generality. This program is rooted in the observation that a subalgebra of the BMS_4 group is isomorphic to a two-dimensional conformal algebra [26]. This insight has led to two complementary proposals: in one, the bms_4 algebra is identified with a celestial two-dimensional conformal algebra, giving rise to the framework of *celestial holography*; in the other, bms_4 is identified with a three-dimensional Carrollian conformal algebra, leading to the framework of *Carrollian holography* [27–29].

Understanding these symmetries and their associated charges is therefore an essential step toward formulating a holographic duality for flat space. Moreover, uncovering new aspects of soft theorems may in turn reveal previously unnoticed asymptotic symmetries, which could impose powerful constraints on putative dual theories [30–33].

1.2 Gravitational waves

The relation uncovered by Strominger and collaborators between soft theorems and asymptotic symmetries has a third vertex: an observational phenomenon known as the *gravitational wave memory effect* [34, 35]. The memory effect manifests as a permanent displacement between freely falling detectors following the passage of a gravitational wave. These three ingredients — asymptotic symmetries, soft theorems, and the memory effect — are now understood to be intimately connected, a relation often summarized in the so-called *infrared triangle* [13, 36].

In recent years, this connection has been clarified and extended. Soft theorems have emerged as a powerful tool for characterizing gravitational wave signals, making them directly relevant to gravitational wave astronomy [37, 38]. Improved understanding of the low-energy behaviour of scattering amplitudes can, in principle, inform predictions for waveform features accessible to detectors such as LIGO, Virgo, and LISA [39].

These relations have been explored in a variety of regimes, including the eikonal limit [40] and in analyses of the late time behaviour of gravitational radiation [41]. In particular, soft theorem considerations have shed new light on a previously known feature of gravitational waves — the *tail effect* [42]. Tails describe the characteristic way in which waveforms decay at late times due to backscattering off spacetime curvature, and their study benefits from the unified perspective provided by the infrared triangle.

The infrared triangle framework has also revealed entirely new types of memory effects associated with higher-order terms in the asymptotic expansion of the gravitational field [43–47]. There are genuine prospects of measuring these novel memory effects in the coming decades with experiments such as LISA and the Einstein Telescope [48, 49]. In turn, their detection would provide a unique observational window into determining the underlying symmetry structure of our universe [50].

1.3 Quantum gravity S-matrix

To define an S-matrix in quantum gravity, we must first specify the states to be scattered. In a quantum field theory in flat spacetime without gravity, asymptotic states are classified by the unitary irreducible representations (UIRs) of the Poincaré group, the symmetry group of Minkowski space [51]. In quantum gravity, however, the relevant asymptotic symmetries are no longer given by the Poincaré group but by the full symmetry group of asymptotically flat spacetimes — the BMS group. Accordingly, asymptotic states should be labeled by the UIRs of the BMS group (or its appropriate extensions) [52].

This mismatch may also underlie the infrared divergences of the gravitational S-matrix, as emphasized in [53, 54]. Earlier attempts to construct infrared-finite amplitudes introduced suitably *dressed* operators [55–57]. From the modern perspective, the reformulation of the S-matrix in terms of BMS representations may naturally clarify the role of such dressings and, by construction, lead to an infrared-finite amplitude [58].

A complete understanding of the gravitational S-matrix in this context therefore requires: (i) determining the asymptotic symmetry group in its full generality, and (ii) establishing whether these symmetries persist at the quantum level or are broken by quantum effects. As we shall see, soft theorems provide a powerful tool to address both of these questions.

1.4 Roadmap

The structure of this thesis is as follows. Chapters 2 and 3 review the existing literature on asymptotic symmetries and soft theorems, introducing the notation and conventions used throughout this work. This review is presented as a unified treatment up to one-loop and includes both massless and massive particles. It is not intended as an exhaustive account, but rather focuses on the material required for the main results.

Chapter 4 presents the first main result of this thesis, based on [59], extending the tree-level relations between soft theorems and asymptotic symmetries to one-loop. We unify results in the literature regarding loop corrections to soft theorems and show how they can be explained purely from symmetry arguments, thereby demonstrating the robustness of the underlying symmetries. Compared to [59], we also treat the cases of massless particles, charged particles, and soft photons. Chapter 5 turns to a more in-depth analysis of the symmetries of Einstein–Maxwell theory, deriving them directly from the asymptotic evolution equations, as presented in [60–62]. Building on the results of [59,62], we further present loop corrections to the soft photon theorem from graviton loops.

We conclude in Chapter 6 with a summary of our findings and a discussion of possible directions for future research.

The original results of this thesis are based on the following publications:

1. S. Agrawal, L. Donnay, K Nguyen and R. Ruzziconi, *Logarithmic soft graviton theorems from superrotation Ward identities*, *JHEP* **02** (2024) 120, arXiv:2309.11220.

2. S. Agrawal, P. Charalambous and L. Donnay, *Celestial $sw_{1+\infty}$ algebra in Einstein-Yang-Mills theory*, *JHEP* **03** (2025) 208, arXiv:2412.01647.

Additionally, the following works were also published during the course of this thesis but are not a part of the maintext (see appendices A and B respectively, where a part of the result is summarized):

1. S. Agrawal and K Nguyen, *Soft theorems and spontaneous symmetry breaking*, *Phys. Rev. D* **112** (2025) 2, arXiv:2504.10577.

2. S. Agrawal, P. Charalambous and L. Donnay, *Null infnity as an inverted extremal horizon: Matching an infinite set of conserved quantities for gravitational perturbations*, arXiv:2506.15526.

Chapter 2

Asymptotic Symmetries

We shall start this thesis by discussing asymptotic symmetries in gauge and gravity theories. In theories with dynamical gravity, defining global symmetries is notoriously difficult [63, 64]. In particular, black hole physics suggests that exact global symmetries cannot exist: any global charge carried by matter falling into a black hole is inaccessible to outside observers, and Hawking radiation [21] is expected to return a state with no memory of that charge, violating global conservation [65]. More generally, in quantum gravity frameworks such as string theory, all symmetries appear to be either gauged or explicitly broken, leading to the conjecture that exact global symmetries are incompatible with a consistent theory of gravity [66].

This is precisely why asymptotic symmetries become so interesting in gravity: while exact global symmetries are believed to be absent, gravitational theories do admit infinite-dimensional symmetry groups acting at the boundaries of spacetime. These asymptotic symmetries, such as the Bondi–van der Burg-Metzner-Sachs (BMS) group [3,4] in asymptotically flat spacetimes, are not global in the strict sense—they are associated with gauge transformations that have nontrivial action at infinity and thus correspond to surface charges measurable by distant observers. They provide a bridge between the gauge redundancies of gravity and physically observable quantities, encoding memory effects, soft theorems, and constraints on scattering [11]. Studying them offers a way to recover some of the organizing power of symmetries in a gravitational context, where true global invariances are forbidden.

We will therefore begin this chapter by introducing the notion of asymptotically flat spacetimes, which provide the natural stage for defining gravitational scattering and studying the associated symmetries at null infinity. We will determine their asymptotic symmetry group and construct the corresponding surface charges that encode the physically measurable quantities. Having established this framework for gravity, we will then turn to the analogous problem in electromagnetism, defining its asymptotic symmetries and associated charges, setting the stage for exploring their interplay with gravitational symmetries. For this chapter we will mostly be following the conventions in [14].

2.1 Minkowski spacetime

Let us first start by understanding the simplest case of flat space. The Minkowski space given by the metric,

$$ds^2 = -\,dt^2 + dr^2 \, + r^2\gamma_{AB}dx^A dx^B \,, \tag{2.1.1}$$

is the simplest solution to the Einstein equation [2] with zero cosmological constant. Here, γ_{AB} is the round sphere metric, A, B being the coordinates on the sphere. The Ricci curvature for the metric given above can be trivially seen to be zero. This is thus a solution which is flat everywhere. While the Minkowski spacetime has no boundaries, we can define its conformal completion by a change of coordinates defined by,

$$\tan U = t - r\,, \qquad \tan V = t + r\,. \tag{2.1.2}$$

The metric in this new coordinate system is given by,

$$ds^2 = \frac{1}{4\cos^2 U\ \cos^2 V}\left(-4dUdV + \sin^2(V-U)d\Omega\right)\,. \tag{2.1.3}$$

Rescaling the metric by a conformal factor given by $4\cos^2 U\ \cos^2 V$ defines an unphysical metric in which the physical metric is defined in the interior. The unphysical metric admits a completion, i.e., a boundary at infinity can be added where the unphysical metric is still well defined. The boundary is made of 5 pieces(see for example [67]),

- Future time-like infinity $i^+ : U = \frac{\pi}{2}, V = \frac{\pi}{2}$,
- Past time-like infinity $i^- : U = -\frac{\pi}{2}, V = -\frac{\pi}{2}$,
- space-like infinity $i^0 : U = -\frac{\pi}{2}, V = \frac{\pi}{2}$,
- Future null infinity $\mathscr{I}^+$: $U \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), V = \frac{\pi}{2}$,
- Past null infinity $\mathscr{I}^-$: $U = -\frac{\pi}{2}, V \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The null hypersurfaces, $\mathscr{I}$ are 3d manifolds with coordinates (U, x^A). The topology of this null hypersurface is thus $\mathbb{R} \times S^2$. The other three pieces of the boundary are simply points. While these points are singular in the coordinate system defined above, we can shift to a hyperbolic slicing where the structure of these points gets resolved.

This fixes the asymptotic structure of Minkowski spacetime at null infinity. The isometries of the Minkowski metric are given by the Poincaré group, consisting of spacetime translations and Lorentz transformations (rotations and boosts). The action of the Lorentz group on a vector V^μ can be written as

$$V'^\mu = \Lambda^\mu{}_\nu\, V^\nu, \tag{2.1.4}$$

where $\Lambda^\mu{}_\nu$ is a Lorentz transformation matrix satisfying

$$\eta_{\rho\sigma}\, \Lambda^\rho{}_\mu\, \Lambda^\sigma{}_\nu = \eta_{\mu\nu}\,. \tag{2.1.5}$$

This condition ensures that the Minkowski metric $\eta_{\mu\nu}$ is preserved under the transformation.

To resolve the structure of the other boundaries that appear as points in (2.1.3) it is instructive to look at the hyperbolic slicing of Minkowski spacetime. Looking first at the timelike infinity, the change of coordinates from spherical (t, r, x^A) to hyperbolic (τ, ρ, x^A) coordinates is given by (see e.g. [68]):

$$\tau := \sqrt{t^2 - r^2}, \quad \rho := \frac{r}{\sqrt{t^2 - r^2}}\,. \tag{2.1.6}$$

Note that this coordinate is only well defined on the patch $t > r$ of the full space. Minkowski metric in the hyperbolic coordinate system is seen to be given by

$$ds^2 = -d\tau^2 + \tau^2 h_{\alpha\beta} dy^\alpha dy^\beta\,, \tag{2.1.7}$$

with the unit hyperboloid metric

$$h_{\alpha\beta} dy^\alpha dy^\beta = \frac{d\rho^2}{1+\rho^2} + \rho^2 \gamma_{AB} dx^A dx^B\,. \tag{2.1.8}$$

From the above expression it is evident that in the limit of large (positive and negative) time τ, the metric approaches that of a hyperbolic surface, this surface is timelike infinity (future and past resp.).

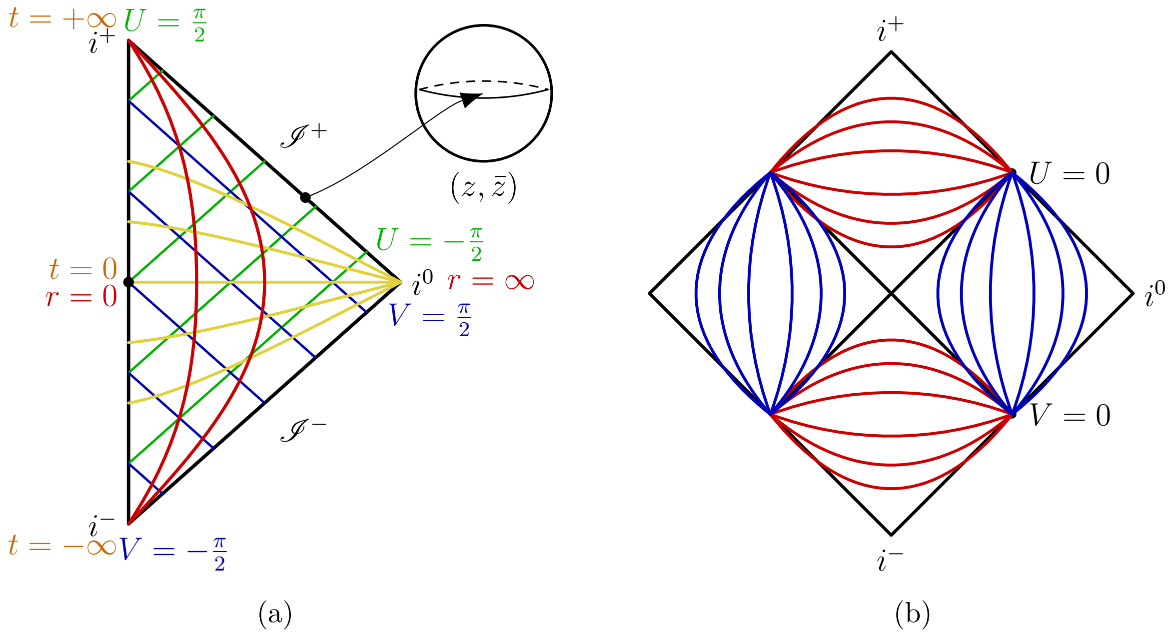


Figure 2.1: Compactified Minkowski spacetime in different coordinate systems. 2.1a is the conformal compactification with red line being constant r, yellow lines constant t, green line constant U and blue lines constant V. 2.1b shows the hyperbolic slicing with red lines showing constant τ surfaces in the region $|t|>r$ and blue lines showing constant ϱ slices in the complimentary region.

Similarly the coordinates that are well defined on the other patch are,

$$\tau := \sqrt{r^2-t^2}, \quad \varrho := \frac{r}{\sqrt{r^2-t^2}}\,. \tag{2.1.9}$$

Thus in the limit of large ρ (which is positive), space-like infinity is reached, which can be seen as a Lorentzian de-Sitter surface. As a final comment, we will be using complex stereographic coordinates on the 2-sphere, the round sphere metric can then be written as,

$$\gamma_{AB}dx^A dx^B = 2\,(1+z\bar{z})^2\,dzd\bar{z}\,. \tag{2.1.10}$$

2.2 Free scalar fields in Minkowski

When formulating a scattering problem in quantum field theory, it is natural to picture the process as asymptotic states in the far past evolving into asymptotic states in the far future via the action of the S-matrix (see for e.g. [13]). In the context of symmetries, their effect on scattering can be described by how they act on these asymptotic states, which in turn is captured by their action on the asymptotic phase space of the theory (see [69] for details). The goal of this section is to construct and understand this phase space explicitly for the simple case of a free scalar field in flat spacetime, providing a concrete setting in which to study the structure and action of asymptotic symmetries. We shall follow the conventions in [28].

In terms of the action of the Poincaré group, the free scalar fields transform under the spin-0 representation of the Lorentz group as

$$\phi'(x) = \phi(\Lambda^{-1}(x-a))\,, \tag{2.2.1}$$

where Λ is a Lorentz transformation and a is a spacetime translation vector. The scalar field obeys the Klein–Gordon equation,

$$(\Box - m^2)\,\phi = 0\,. \tag{2.2.2}$$

The asymptotics of the field are very different depending on whether it is massive or massless. This is because the geodesics of massive and massless particles are different. Referring back to the diagram of the Minkowski space, massless particles in the far past begin at $\mathscr{I}^-$ while in the far future end up at $\mathscr{I}^+$ while the worldline of massive particles originate at i^- and terminate at i^+. However as already noted, the nature of these two boundaries is very different, and require different coordinate systems to visualize. So we shall study these two cases separately.

2.2.1 Massless fields

The on-shell condition $p^2 = 0$ allows the massless momentum to be parametrized in terms of a positive energy ω and coordinates $(z, \bar{z})$ on the celestial sphere:

$$p^\mu(\omega, z, \bar{z}) = \frac{\omega}{2}\,(1 + z\bar{z},\, z + \bar{z},\, -i(z - \bar{z}),\, 1 - z\bar{z}) \equiv \frac{\omega}{2}\,\hat{p}^\mu(z, \bar{z})\,. \tag{2.2.3}$$

If the field also carries a spin index, the polarization co-vector can be expressed in a similar way,

$$\begin{aligned} \varepsilon^+_\mu &= \partial_z \hat{p}_\mu \\ \varepsilon^-_\mu &= \partial_{\bar{z}} \hat{p}_\mu \end{aligned} \tag{2.2.4}$$

Along with another vector defined as $n^\mu = \partial_z \partial_{\bar{z}} \hat{p}^\mu$, these satisfy the following identities,

$$\hat{p} \cdot n = -1, \qquad \varepsilon^+ \cdot \varepsilon^- = 1 \tag{2.2.5}$$

with all other contractions being zero. The Lorentz group acts on $(z, \bar{z})$ as the standard $SL(2, \mathbb{C})$ Möbius transformation

$$z \;\to\; \frac{az + b}{cz + d}. \tag{2.2.6}$$

The field admits the standard momentum-space expansion

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2p^0} \left[e^{-ip\cdot x}\, a(\vec{p}) + e^{ip\cdot x}\, a^\dagger(\vec{p})\right], \tag{2.2.7}$$

which, in the $(\omega, z, \bar{z})$ parametrization, becomes

$$\phi(x) = \int_0^\infty d\omega \int d^2z\, \frac{\omega\, \gamma_{z\bar{z}}}{16\pi^3} \left[e^{-ip\cdot x}\, a(\omega, z, \bar{z}) + e^{ip\cdot x}\, a^\dagger(\omega, z, \bar{z})\right], \tag{2.2.8}$$

In the far past, the massless field will asymptote to future null infinity. Near future null infinity $\mathscr{I}^+$, we use the retarded coordinates,

$$x^\mu = (u + r,\, r\,\hat{x}(z, \bar{z})), \qquad u = t - r, \tag{2.2.9}$$

and take $r \to \infty$ at fixed $(u, z, \bar{z})$. Using a saddle-point approximation, the field has the asymptotic expansion

$$\phi(u, r, z, \bar{z}) = \frac{1}{r}\,\phi_0(u, z, \bar{z}) + \frac{1}{r^2}\,\phi_1(u, z, \bar{z}) + \dots \tag{2.2.10}$$

with leading radiative data

$$\phi_0(u, z, \bar{z}) = \frac{1}{4\pi} \int_0^\infty d\omega \left[e^{-i\omega u}\, a(\omega, z, \bar{z}) + e^{i\omega u}\, a^\dagger(\omega, z, \bar{z})\right], \tag{2.2.11}$$

and all higher order terms fixed in terms of this leading data, as can be seen by expanding the Klein-Gordon equations. The operators $a(\omega, z, \bar{z})$ and $a^\dagger(\omega, z, \bar{z})$ create and annihilate quanta labeled by the direction on the celestial sphere at the point $(z, \bar{z})$, in the direction of the momentum. They satisfy the canonical commutation relations

$$\left[\, a(\omega, z, \bar{z}),\, a^\dagger(\omega', z', \bar{z}')\,\right] = \frac{16\pi^3}{\omega}\,\delta(\omega - \omega')\,\delta^2(z - z'), \tag{2.2.12}$$

with all others vanishing. These modes coordinatize the asymptotic phase space of the free massless scalar at $\mathscr{I}^+$, and can be interpreted as operator insertions on the celestial sphere:

$$\mathcal{O}_\omega(z, \bar{z}) := a^\dagger(\omega, z, \bar{z}). \tag{2.2.13}$$

2.2.2 Massive fields

For a massive scalar of mass m, the on-shell condition $p^2 = -m^2$ is naturally solved in hyperbolic coordinates $(\rho, z, \bar{z})$ on the future timelike hyperboloid:

$$p^\mu = \frac{m}{2\rho}\left(1 + \rho^2(1 + z\bar{z}),\, \rho^2(z + \bar{z}),\, -i\rho^2(z - \bar{z}),\, -1 + \rho^2(1 - z\bar{z})\right) \equiv m\,\hat{p}^\mu(\rho, z, \bar{z})\,. \tag{2.2.14}$$

In these coordinates, future timelike infinity i^+ corresponds to $\tau \to \infty$ at fixed $(\rho, z, \bar{z})$, where τ is the proper time. Taking this limit, the field behaves as

$$\phi(\tau, \rho, \hat{x}) = \frac{e^{-i\pi/4}\sqrt{m}}{2(2\pi\tau)^{3/2}}\left[a(m\rho, \hat{x})\,e^{-i\tau m} + a^\dagger(m\rho, \hat{x})\,e^{i\tau m}\right] + \mathcal{O}(\tau^{-5/2}), \tag{2.2.15}$$

localizing around the point $(\rho, z, \bar{z})$ on the hyperboloid determined by the momentum. Note however a difference in the two cases – while for massless particles the null time u was not fixed by their momentum and therefore particles thrown in at different times from past null infinity will also hit future null infinity at different times, all the massive particles with same final momentum will hit the same point at i^+. The creation and annihilation operators satisfy

$$\left[\, a(\vec{p}),\, a^\dagger(\vec{p}')\,\right] = (2\pi)^3\, 2E_{\vec{p}}\,\delta^{(3)}(\vec{p} - \vec{p}'), \tag{2.2.16}$$

with $E_{\vec{p}} = \sqrt{m^2 + \vec{p}^2}$.

Similar to the massless case these modes can be expressed as operators living on the hyperboloid,

$$\mathcal{R}_m(\rho, \hat{x}) := a^\dagger(m\rho, \hat{x})\,. \tag{2.2.17}$$

Massless limit and geometry

In the limit $\rho \to \infty$ of the massive parametrization (2.2.14), the momentum approaches that of a massless particle, and geometrically the boundary of the hyperboloid $\mathcal{H}^3$ is the celestial sphere CS^2. Matching the structure at null and timelike infinity yields

$$i^+\big|_\partial \;=\; \mathscr{I}^+_+, \tag{2.2.18}$$

where $\mathscr{I}^+_+$ denotes the future boundary of future null infinity.

Finally, when re-expressing scattering amplitudes as correlation functions on these boundaries, massless external particles are represented by operator insertions $\mathcal{O}_\omega(z, \bar{z})$ on the celestial sphere, while massive external particles of mass m correspond to operators $\mathcal{R}_m(\rho, z, \bar{z})$ inserted on the hyperboloid. An n-point scattering amplitude involving ℓ massless and $n - \ell$ massive particles can alternatively be expressed as,

$$\langle\, \mathcal{O}_{\omega_1,\eta_1}(z_1, \bar{z}_1)\, \ldots\, \mathcal{O}_{\omega_\ell,\eta_\ell}(z_\ell, \bar{z}_\ell)\, \mathcal{R}_{m_{\ell+1},\eta_{\ell+1}}(\rho_{\ell+1}, z_{\ell+1}, \bar{z}_{\ell+1})\, \ldots\, \mathcal{R}_{m_n,\eta_n}(\rho_n, z_n, \bar{z}_n)\,\rangle, \tag{2.2.19}$$

where η labels whether the particle is incoming or outgoing. This rewriting of momentum-space amplitudes as correlators will prove useful in later reformulations of the S-matrix.

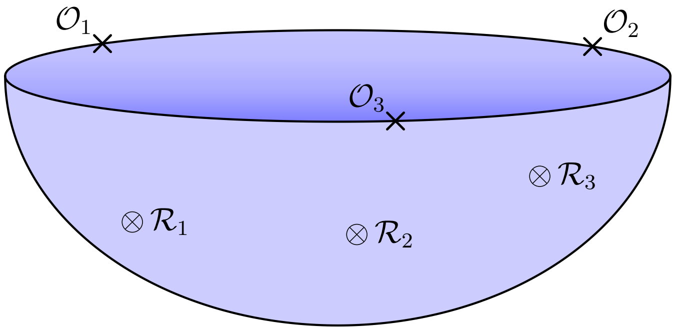


Figure 2.2: An example of the S-matrix visualized as operator insertions.

2.3 Asymptotically flat spacetimes

In the study of gravitational scattering, *asymptotic flatness* is the natural boundary condition ensuring that far from the scattering region, spacetime approaches Minkowski space. Informally, an asymptotically flat spacetime is one whose metric approaches the Minkowski metric at large distances, differing only in subleading terms [70–72]:

$$\lim_{r\to\infty} g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{r}\, h_{\mu\nu} + \mathcal{O}(r^{-2}), \tag{2.3.1}$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric and $h_{\mu\nu}$ encodes the leading deviation from flatness.

The condition of asymptotic flatness can be studied separately on the different boundaries of spacetimes as described in the previous section. In this section we focus on future null infinity $\mathscr{I}^+$, where gravitational radiation is naturally described in the *Bondi–Sachs* coordinate system $x^\mu = (u, r, z, \bar{z})$ [73].

The Bondi-Sachs coordinates were introduced by Bondi, van der Burg, Metzner, and Sachs for studying the behaviour of gravitational waves observed at large distances from point of emission [3, 4]. The idea is to foliate spacetime by null hypersurfaces $u =$ const. along which the gravitational waves travel (see [74, 75] for reviews). Each null hypersurface is generated by a congruence of null geodesics with tangent $l^\mu = \partial_r$. The complex coordinates[1] $x^A = (z, \bar{z})$ correspond to stereographic coordinates on the celestial sphere, which are constant along the null rays.

The *Bondi gauge* is defined by the conditions

$$g^{uu} = g^{uA} = 0, \qquad \partial_r \det\Big(r^{-2} g_{AB}\Big) = 0, \tag{2.3.2}$$

where A, B label sphere directions. These conditions preserve the interpretation of u as null time, and $(z, \bar{z})$ as constant along each null ray. The determinant condition $\partial_r \det(r^{-2} g_{AB}) = 0$ implies g_{AB} has unit sphere metric at leading order.

In this gauge the spacetime metric takes the general form [75],

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + g_{AB}(dx^A - U^A du)(dx^B - U^B du)\,, \tag{2.3.3}$$

where β, V, U^A, and g_{AB} are functions of $(u, r, z, \bar{z})$.

For example, the Minkowski metric in Bondi coordinates can be expressed as,

$$ds^2 = -du^2 - 2\,du\,dr + 2r^2\gamma_{z\bar{z}}\,dz\,d\bar{z}\,. \tag{2.3.4}$$

[1]Here and throughout this thesis we will use the convention where Greek letters are used for spacetime indices, small Roman letters for Euclidean space and capital Roman letters for transverse space.

In Bondi-Sachs gauge, an asymptotically flat spacetime can be defined to have the following large-r expansion [14]:

$$\begin{aligned} ds^2 = &\left(\frac{2M}{r} + \mathcal{O}(r^{-2})\right) du^2 - 2\left(1 + \mathcal{O}(r^{-2})\right) du\, dr \\ &+ \left(r^2 q_{AB} + rC_{AB} + \mathcal{O}(r^0)\right) dx^A dx^B \\ &+ \left[\frac{1}{2} D_B C_A{}^B + \frac{2}{3r}\left(N_A + \frac{1}{4} C_A{}^B D_C C_B{}^C\right) + \mathcal{O}(r^{-2})\right] du\, dx^A, \end{aligned} \tag{2.3.5}$$

where:

- q_{AB} is the unit S^2 metric,
- $C_{AB}(u, z, \bar{z})$ is the *shear tensor*, symmetric and traceless on the sphere,
- $N_{AB} = \partial_u C_{AB}$ is the *Bondi news*, encoding the gravitational radiation,
- $M(u, z, \bar{z})$ is the *Bondi mass aspect*,
- $N_A(u, z, \bar{z})$ is the *angular momentum aspect*,
- D_A is the covariant derivative with respect to q_{AB}.

The functions $M(u, z, \bar{z}), N_A(u, z, \bar{z}), C_{AB}(u, z, \bar{z})$ are defined to be living on the conformal boundary of spacetime, which is $\mathscr{I}^+$. In particular for Minkowski space, $M = 0$, $C_{AB} = 0$, and $N_A = 0$.

Imposing the vacuum Einstein equations $R_{\mu\nu} = 0$ yields evolution equations for M and N_A in terms of the free data C_{AB}:

$$\begin{aligned} \partial_u M &= -\frac{1}{8} N^{AB} N_{AB} + \frac{1}{8} \bar{\Delta} \bar{R} + \frac{1}{4} D_A D_B N^{AB}, \\ \partial_u N^A &= D^A M + \frac{1}{16} D^A \left(N_{BC} N^{BC}\right) - \frac{1}{4} N^{BC} D^A C_{BC} \\ &\quad - \frac{1}{4} D_B \left(C^{BC} N^A_C - N^{BC} C^A_C\right) - \frac{1}{4} D_B D^B D^C C^A_C \\ &\quad + \frac{1}{4} D_B D^A D^C C^B_C + \frac{1}{4} C^A_B D^B \bar{R}. \end{aligned} \tag{2.3.6}$$

Here $\bar{\Delta}$ and $\bar{R}$ are the Laplacian and scalar curvature on the unit sphere. The first equation is the *Bondi mass-loss formula* [3]: since $N^{AB} N_{AB} \geq 0$, gravitational radiation always reduces the total mass. The time evolution of the shear C_{AB} is *free data* at $\mathscr{I}^+$ and is directly related to the asymptotically free graviton in the scattering problem [6].

The above discussion applies to future null infinity $\mathscr{I}^+$. To describe past null infinity $\mathscr{I}^-$, one switches to *advanced coordinates* $(v, r, z, \bar{z})$ with $v = t + r$ and repeats the analysis, with v playing the role of advanced time. The resulting expansions and equations are analogous, but describe incoming radiation from $\mathscr{I}^-$.

The Bondi–Sachs form of the metric makes explicit the residual diffeomorphisms preserving asymptotic flatness [7]. These form the *Bondi–van der Burg–Metzner–Sachs* (BMS) group. The fields M, N_A, and C_{AB} transform in specific ways under these symmetries, and the associated conserved charges will play a central role in this thesis.

2.3.1 Asymptotic symmetries

The diffeomorphisms that preserve the Bondi–Sachs falloff conditions of an asymptotically flat spacetime form its *asymptotic symmetry group*. While in general the Noether charges associated with diffeomorphisms that act as gauge transformations vanish, large diffeomorphisms which act nontrivially at the boundary of spacetime can carry *nontrivial surface charges* [76, 77]. These asymptotic charges are not conserved in the usual sense: instead, they satisfy a *flux balance law* relating their change between two boundaries of $\mathscr{I}$ to the flux of radiation through the intervening portion of $\mathscr{I}$ [78, 79]. This reflects the fact that the boundary of an asymptotically flat spacetime is not a rigid box but a "leaky" surface through which energy can flow.

In what follows we restrict to the case where the metric on the two-dimensional transverse space is fixed to be the unit round sphere metric q_{AB}. In this setting, the most general infinitesimal diffeomorphism preserving the Bondi gauge and falloff conditions takes the schematic form

$$\xi = \mathcal{T}\,\partial_u + \mathcal{Y}^A\,\partial_A + \cdots \tag{2.3.7}$$

where:

- $\mathcal{T}(z, \bar{z})$ is an arbitrary smooth function on the sphere, generating *angle-dependent translations* of u along $\mathscr{I}$. These are the *supertranslations*, an infinite-dimensional generalization of the translation subgroup of the Poincaré group.

- $\mathcal{Y}^A(z, \bar{z})$ is a conformal Killing vector (CKV) of the sphere. In the *original* analysis of Bondi, van der Burg, Metzner, and Sachs, $\mathcal{Y}^A$ was taken to be globally well-defined on S^2, generating the Lorentz subgroup of the Poincaré group.

- In the *extended* analysis of Barnich and Troessaert [80], $\mathcal{Y}^A$ is allowed to be any *meromorphic* function on the sphere, generating the *superrotations*, which are two commuting copies of the centerless Virasoro (Witt) algebra.

In the extended case, which in this thesis we will refer to as BMS group, the asymptotic symmetry group is

$$\mathrm{bms}_4 \;\simeq\; \mathrm{Witt} \ltimes \mathbf{s}, \tag{2.3.8}$$

where $\mathbf{s}$ denotes the abelian algebra of supertranslations. Note that the Poincaré group is *not* a normal subgroup of the BMS group, a fact that underlies the phenomenon of *vacuum degeneracy* to be discussed later.

The vector fields generating these transformations have components

$$\begin{aligned}
\xi^u &= \mathcal{T} + u\,\alpha,\\
\xi^r &= -(r+u)\,\alpha + \partial_z\partial^z\mathcal{T} + \mathcal{O}(r^{-1}),\\
\xi^z &= \mathcal{Y}^z - \frac{1}{r}\,\partial^z(\mathcal{T} + u\,\alpha) + \mathcal{O}(r^{-2}),\\
\xi^{\bar{z}} &= \mathcal{Y}^{\bar{z}} - \frac{1}{r}\,\partial^{\bar{z}}(\mathcal{T} + u\,\alpha) + \mathcal{O}(r^{-2}),
\end{aligned} \tag{2.3.9}$$

where

$$\alpha \equiv \frac{1}{2}D_A\mathcal{Y}^A, \qquad \bar{\partial}\mathcal{Y} = 0, \qquad \partial\bar{\mathcal{Y}} = 0. \tag{2.3.10}$$

In the last expression we have defined $\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$, $\mathcal{Y} \equiv \mathcal{Y}^z$, $\bar{\mathcal{Y}} \equiv \mathcal{Y}^{\bar{z}}$, and D_A is the covariant derivative on the unit sphere. These vector fields are determined by requiring that they preserve both the Bondi gauge and the falloff conditions for the metric.

Applying $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$ to the Bondi–Sachs expansion, and introducing $f \equiv \mathcal{T} + u\alpha$ yields the transformation laws for the Bondi data:

$$\delta_\xi M = \left[f\,\partial_u + \mathcal{L}_\mathcal{Y} + \frac{3}{2}\partial_C\mathcal{Y}^C\right] M + \frac{1}{8}\partial_C\partial_B\partial_A\mathcal{Y}^A C^{BC} + \frac{1}{4}N^{AB}\partial_A\partial_B f + \frac{1}{2}\partial_A f\,\partial_B N^{AB},$$

$$\begin{aligned}\delta_\xi N_A = &\left[f\,\partial_u + \mathcal{L}_\mathcal{Y} + \partial_C\mathcal{Y}^C\right] N_A + 3M\,\partial_A f - \frac{3}{16}\partial_A f\,N^{BC}C_{BC} - \frac{1}{32}\partial_A\partial_B\mathcal{Y}^B\,C_{CD}C^{CD}\\ &+ \frac{1}{4}\left(2\partial_B f + \partial_B\partial_C\partial^C f\right)C^B{}_A - \frac{3}{4}\partial_B f\left(\partial^B\partial^C C_{AC} - \partial_A\partial^C C^B{}_C\right)\\ &+ \frac{3}{8}\partial_A\left(\partial_C\partial_B f\,C^{BC}\right) + \frac{1}{2}\left(\partial_A\partial_B f - \frac{1}{2}\bar\gamma_{AB}\partial_C\partial^C f\right)\partial_D C^{BD}\\ &+ \frac{1}{2}\partial_B f\,N^{BC}C_{AC},\end{aligned}$$

$$\delta_\xi C_{AB} = \left[f\,\partial_u + \mathcal{L}_\mathcal{Y} - \frac{1}{2}\partial_C\mathcal{Y}^C\right]C_{AB} - 2\partial_A\partial_B f + \bar\gamma_{AB}\,\partial_C\partial^C f,$$

$$\delta_\xi N_{AB} = \left[f\,\partial_u + \mathcal{L}_\mathcal{Y}\right]N_{AB} - \left(\partial_A\partial_B\partial_C\mathcal{Y}^C - \frac{1}{2}\bar\gamma_{AB}\,\partial_C\partial^C\partial_D\mathcal{Y}^D\right), \tag{2.3.11}$$

with the Bondi news given by $N_{AB} \equiv \partial_u C_{AB}$.

The BMS transformations close under a *modified* Lie bracket [26]:

$$[\xi_1, \xi_2]_\star \;=\; [\xi_1, \xi_2] - \delta_1\xi_2 + \delta_2\xi_1, \tag{2.3.12}$$

In terms of the symmetry parameters,

$$[\xi(\mathcal{T}_1, \mathcal{Y}_1, \bar{\mathcal{Y}}_1), \xi(\mathcal{T}_2, \mathcal{Y}_2, \bar{\mathcal{Y}}_2)]_\star = \xi(\mathcal{T}_{12}, \mathcal{Y}_{12}, \bar{\mathcal{Y}}_{12}), \tag{2.3.13}$$

with

$$\begin{aligned}\mathcal{T}_{12} &= \mathcal{Y}_1^z\,\partial_z\mathcal{T}_2 - \frac{1}{2}(\partial_z\mathcal{Y}_1^z)\mathcal{T}_2 + \text{c.c.} - (1\leftrightarrow 2),\\ \mathcal{Y}_{12}^z &= \mathcal{Y}_1^z\,\partial_z\mathcal{Y}_2^z - (1\leftrightarrow 2),\\ \mathcal{Y}_{12}^{\bar z} &= \mathcal{Y}_1^{\bar z}\,\partial_{\bar z}\mathcal{Y}_2^{\bar z} - (1\leftrightarrow 2).\end{aligned} \tag{2.3.14}$$

It is sometimes convenient to write $\mathcal{T} \equiv \xi(\mathcal{T}, 0, 0)$ for a pure supertranslation and $\mathcal{Y} \equiv \xi(0, \mathcal{Y}, 0)$ for a pure superrotation. The algebra then reads

$$\begin{aligned}&[\mathcal{T}_1, \mathcal{T}_2] = 0, \qquad\qquad [\mathcal{Y}_1, \bar{\mathcal{Y}}_2] = 0,\\ &[\mathcal{Y}_1, \mathcal{Y}_2] = \mathcal{Y}_1\,\partial\mathcal{Y}_2 - \mathcal{Y}_2\,\partial\mathcal{Y}_1,\\ &[\mathcal{Y}_1, \mathcal{T}_2] = \mathcal{Y}_1\,\partial\mathcal{T}_2 - \frac{1}{2}(\partial\mathcal{Y}_1)\,\mathcal{T}_2,\end{aligned} \tag{2.3.15}$$

together with the complex conjugate relations.[2]

We will see later that the existence of this infinite-dimensional symmetry group has deep implications for gravitational scattering, soft graviton theorems, and the vacuum structure of gravity.

[2]Strictly speaking, $\bar{\mathcal{Y}}$ is not the complex conjugate of $\mathcal{Y}$, but we will continue to use this notation.

2.3.2 Newman–Penrose formalism

In preparation for later chapters, it will be convenient to reformulate certain standard results in the language of the Newman–Penrose (NP) formalism [81]. This formalism expresses tensor equations in terms of a null tetrad basis, which is particularly well-suited for studying fields near null infinity.

A *null tetrad* is a set of four complexified vector fields

$$(l^\mu,\, n^\mu,\, m^\mu,\, \bar{m}^\mu) \tag{2.3.16}$$

satisfying the orthogonality and normalization conditions

$$l \cdot n = -1, \qquad m \cdot \bar{m} = 1, \qquad \text{all other inner products vanish.} \tag{2.3.17}$$

Here, l^μ and n^μ are real null vectors, while m^μ and $\bar{m}^\mu$ are null vectors complex conjugate to each other.

In this basis, various components of the Weyl tensor $C_{\mu\nu\rho\lambda}$ are encoded in the *Newman–Penrose scalars.* For example,

$$\Psi_0 \equiv -C_{\mu\nu\rho\lambda}\, l^\mu m^\nu l^\rho m^\lambda \;\equiv\; -C_{lmlm}, \tag{2.3.18}$$

is invariant under changes of the complex vector m^μ that preserve the null tetrad structure, provided the null congruence is generated by l^μ. If $\Psi_0 = 0$, the next scalar

$$\Psi_1 \equiv -C_{\mu\nu\rho\lambda}\, l^\mu m^\nu l^\rho n^\lambda \;\equiv\; -C_{lmln}, \tag{2.3.19}$$

becomes invariant instead. Proceeding in this way yields the hierarchy of Newman–Penrose scalars, defined by

$$\begin{aligned} &\Psi_0 := -C_{\ell m \ell m}\,, \quad \Psi_1 := -C_{\ell m \ell n}\,, \quad \Psi_2 := -C_{\ell m \bar{m} n}\,, \\ &\Psi_3 := -C_{\ell n \bar{m} n}\,, \quad \Psi_4 := -C_{\bar{m} n \bar{m} n}\,. \end{aligned} \tag{2.3.20}$$

If all the coefficients are zero, this implies that the spacetime is flat. These five complex scalar quantities encode the ten independent components of the Weyl tensor in a form adapted to the null directions l^μ and n^μ. They will play a central role in characterizing gravitational radiation and asymptotic properties of the spacetime.

The condition of asymptotic flatness can be translated to requiring the following fall-offs on the NP scalars,

$$\Psi_s = \mathcal{O}\left(\frac{1}{r^{5-s}}\right)\,. \tag{2.3.21}$$

These conditions are known as Sachs' peeling [5, 82]. This hierarchy reflects the fact that the leading radiative data is contained in Ψ_4 at $\mathscr{I}^+$ and in Ψ_0 at $\mathscr{I}^-$.

In the Newmann-Penrose formalism, the components of the Ricci tensor $R_{\mu\nu}$ are rearranged into 4 real and 3 complex scalars,

$$\begin{aligned} &\Phi_{00} := \frac{1}{2}R_{\ell\ell}\,, \quad \Phi_{11} := \frac{1}{4}\left(R_{\ell n} + R_{m\bar{m}}\right)\,, \quad \Phi_{22} := \frac{1}{2}R_{nn}\,, \quad \Lambda_{\mathrm{R}} := \frac{R}{24}\,, \\ &\Phi_{01} := \frac{1}{2}R_{\ell m} = \bar{\Phi}_{10}\,, \quad \Phi_{12} := \frac{1}{2}R_{nm} = \bar{\Phi}_{21}\,, \quad \Phi_{02} := \frac{1}{2}R_{mm} = \bar{\Phi}_{20}\,. \end{aligned} \tag{2.3.22}$$

Finally, the spacetime covariant derivatives ∇_μ are traded for the directional derivatives

$$\begin{pmatrix} D \\ \triangle \\ \delta \\ \bar{\delta} \end{pmatrix} := \begin{pmatrix} \ell^\mu \\ n^\mu \\ m^\mu \\ \bar{m}^\mu \end{pmatrix} \nabla_\mu \quad \Leftrightarrow \quad \nabla_\mu = -\ell_\mu \triangle - n_\mu D + m_\mu \bar{\delta} + \bar{m}_\mu \delta\,, \tag{2.3.23}$$

while the Christoffel symbols are rearranged into 12 complex spin coefficients

$$\begin{pmatrix}\kappa\\ \tau\\ \sigma\\ \rho\end{pmatrix} := -m^{\mu}\begin{pmatrix}D\\ \triangle\\ \delta\\ \bar{\delta}\end{pmatrix}\ell_{\mu}\,, \quad \begin{pmatrix}\pi\\ \nu\\ \mu\\ \lambda\end{pmatrix} := +\bar{m}^{\mu}\begin{pmatrix}D\\ \triangle\\ \delta\\ \bar{\delta}\end{pmatrix}n_{\mu}\,,$$
$$\begin{pmatrix}\epsilon\\ \gamma\\ \beta\\ \alpha\end{pmatrix} := +\frac{1}{2}\Big(\bar{m}^{\mu}\begin{pmatrix}D\\ \triangle\\ \delta\\ \bar{\delta}\end{pmatrix}m_{\mu} - n^{\mu}\begin{pmatrix}D\\ \triangle\\ \delta\\ \bar{\delta}\end{pmatrix}\ell_{\mu}\Big)\,. \tag{2.3.24}$$

In terms of the spin coefficients, an *eth* derivative operator can be defined as [83],

$$ð = \delta - (b-s)\,\bar{\alpha} - (b+s)\,\beta\,, \quad \bar{ð} = \bar{\delta} - (b+s)\,\alpha - (b-s)\,\bar{\beta}\,. \tag{2.3.25}$$

The *eth* operator acts on NP scalars with definite spin weight s as,

$$\left[\bar{ð}, ð\right]\varphi_s = s\frac{R}{2}\varphi_s\,, \tag{2.3.26}$$

where R is the Ricci scalar of the two-sphere. The boost weight b is defined similar to spin weight in terms of the thorn operator,

$$þ = D - (b+s)\,\epsilon - (b-s)\,\bar{\epsilon}\,, \quad þ' = \triangle - (b+s)\,\gamma - (b-s)\,\bar{\gamma}\,. \tag{2.3.27}$$

In particular, $ð$ raises the spin weight by one unit, while $\bar{ð}$ lowers the spin weight by one unit. These scalars of NP weight are defined in terms of the transverse frame vectors,

$$\varphi_s = \varphi_{\mu_1\mu_2\cdots\mu_s}m^{\mu_1}m^{\mu_2}\dots m^{\mu_s}\,. \tag{2.3.28}$$

A similar reformulation can be achieved for the electromagnetic fields. In the NP tetrad, the six independent components of the Maxwell field strength tensor can be encoded in three complex Maxwell NP scalars,

$$\Phi_0 := F_{\ell m}\,, \quad \Phi_1 := \frac{1}{2}\left(F_{\ell n} - F_{m\bar{m}}\right)\,, \quad \Phi_2 := F_{\bar{m}n}\,, \tag{2.3.29}$$

which similarly obey a peeling condition,

$$\Phi_s = \mathcal{O}\left(\frac{1}{r^{3-s}}\right)\,. \tag{2.3.30}$$

The spin weights and boost weights of each NP scalar with respect to the $ð$ and $þ$ operators is listed in the table 2.1,

For the case of Yang-Mills fields, the Maxwell NP scalars simply carry an additional colour index. We will be using this language extensively in chapter 5. For another example where the NP formalism is useful, see Appendix B.

2.3.3 Surface charges

As mentioned previously, asymptotic symmetries in gravity can have associated surface charges which, unlike global charges in field theory, are in general *not* strictly conserved but instead satisfy a flux-balance law. We now have the tools to find these surface charges explicitly.

Fundamental NP scalar	b	s
Φ	0	0
Φ_0	+1	+1
Φ_1	0	0
Φ_2	−1	−1
Ψ_0	+2	+2
Ψ_1	+1	+1
Ψ_2	0	0
Ψ_3	−1	−1
Ψ_4	−2	−2

Table 2.1: Boost-weights b and spin weights s of the fundamental NP scalars for the curvature tensors associated with scalar, electromagnetic and gravitational fields.

To make this precise within the Bondi–Sachs framework, it is useful to work with suitably *improved* versions of the Bondi mass aspect M and angular momentum aspect N_A that transform covariantly under the full BMS group [26, 79, 84]. These improved quantities are defined by,

$$\begin{aligned}\mathcal{M} &= M + \frac{1}{8}N^{AB}C_{AB},\\ \mathcal{N}_A &= N_A - u\,\partial_A M + \frac{1}{4}C^{BC}\partial_C C_{AB} + \frac{3}{32}\partial_A(C^{BC}C_{BC})\\ &\quad + \frac{u}{4}\partial_B\left[\left(\partial^B\partial^C - \frac{1}{2}N^{BC}\right)C_{AC}\right] - \frac{u}{4}\partial_B\left[\left(\partial_A\partial^C - \frac{1}{2}N_A{}^C\right)C^B_{\ C}\right].\end{aligned} \tag{2.3.31}$$

The extra terms are fixed so that the transformation rules of $\mathcal{M}$ and $\mathcal{N}_A$ under supertranslations and superrotations take a simple, covariant form, ensuring that the associated charges are well-defined at $\mathscr{I}$.

In the Newman–Penrose formalism, these improved quantities have a particularly compact form in terms of the leading-order Weyl scalars at null infinity [85]:

$$\begin{aligned}\mathcal{M} &= -\frac{1}{2}\left(\Psi_2^0 + \bar{\Psi}_2^0\right),\\ \mathcal{N}_A &= -\Psi_1^0 + u\,\partial_A\Psi_2^0,\end{aligned} \tag{2.3.32}$$

where Ψ_2^0 encodes the Coulombic (mass) part of the gravitational field, and Ψ_1^0 is related to the angular momentum aspect.

The surface charge[3] associated to a general BMS vector field ξ is then

$$Q_\xi = \frac{1}{16\pi G}\int d^2z\,\left(4\,\mathcal{T}\,\mathcal{M} + 2\,\mathcal{Y}^A\,\mathcal{N}_A\right), \tag{2.3.33}$$

where $\mathcal{T}(z,\bar{z})$ parametrizes a supertranslation and $\mathcal{Y}^A(z,\bar{z})$ parametrizes superrotation. Since $\mathcal{T}$ and $\mathcal{Y}^A$ are arbitrary functions or vector fields on the sphere, there is in fact an infinite tower of such charges, one for each mode of the symmetry parameter.

In the presence of gravitational radiation, these charges satisfy the flux-balance law

$$\partial_u Q_\xi = \int d^2z\,F_\xi \neq 0, \tag{2.3.34}$$

which expresses the non-conservation of asymptotic charges due to radiation crossing $\mathscr{I}$. In vacuum, where the Bondi news tensor N_{AB} vanishes, the flux F_ξ is zero and the charges are conserved.

[3]This charge is defined at the past and future boundaries of $\mathscr{I}^+$ and $\mathscr{I}^-$ respectively.

The explicit form of the fluxes can be obtained from the evolution equations for the Bondi mass and angular momentum aspects (2.3.6). For supertranslations and superrotations one finds

$$\begin{aligned} F_{\mathcal{T}} &= \frac{1}{16\pi G}\,\mathcal{T}\left(\partial_z^2 N^{\bar{z}\bar{z}} + \frac{1}{2}C^{\bar{z}\bar{z}}\,\partial_u N^{zz} + \text{c.c.}\right), \\ F_{\mathcal{Y}} &= \frac{1}{16\pi G}\,\mathcal{Y}^z\Big(- u\,\partial_z^3 N^{\bar{z}\bar{z}} + C^{zz}\,\partial_z N^{\bar{z}\bar{z}} - \frac{u}{2}\,\partial_z C^{zz}\,\partial_u N^{\bar{z}\bar{z}} - \frac{u}{2}\,C^{zz}\,\partial_z\partial_u N^{\bar{z}\bar{z}}\Big) + \text{c.c.} \end{aligned} \tag{2.3.35}$$

Here $F_{\mathcal{T}}$ measures the energy flux through null infinity weighted by the supertranslation parameter $\mathcal{T}$, and $F_{\mathcal{Y}}$ measures the angular momentum and superrotation flux.

2.3.4 Phase space

As noted earlier, the Poincaré group is not a normal subgroup of the full BMS group. Consequently, the choice of a particular Poincaré subgroup is tied to the choice of a vacuum configuration at null infinity [11, 36, 43]. This leads to an *infinite degeneracy of vacua*, each labelled by its associated gravitational memory. We now illustrate this structure in a simple example.

Consider acting on the Minkowski metric, for which $C_{zz} = 0$ with a finite supertranslation. The resulting metric is still a solution of the Einstein equations and satisfies the same asymptotic fall-off conditions, yet it differs from Minkowski by a shift in the asymptotic shear:

$$C_{zz} = -2\,\partial^2\mathcal{T}\,. \tag{2.3.36}$$

If we parameterize the shear in terms of a scalar potential as

$$C_{zz} = \partial^2 C^{(0)}, \tag{2.3.37}$$

then under a supertranslation the potential transforms as

$$\delta_{\mathcal{T}} C^{(0)} = -2\,\mathcal{T}. \tag{2.3.38}$$

This transformation law is characteristic of a *Goldstone mode* for spontaneously broken supertranslation symmetry (For a review of the subject, see for example [86]). In this sense, the vacuum degeneracy of asymptotically flat gravity can be interpreted as arising from spontaneous symmetry breaking of the BMS group down to the Poincaré group[4].

In more general spacetimes, such vacuum transitions are physical and measurable. A permanent shift in the shear C_{zz} between early and late times at $\mathscr{I}^+$ corresponds to a *gravitational memory effect* [35, 36], which can arise from the passage of gravitational radiation through null infinity. Therefore, the phase space of asymptotically flat gravity is not comprised solely of radiative modes; it also includes *zero modes* associated with vacuum transitions [69, 87].

Concretely, the radiative and soft data on $\mathscr{I}^+$ are encoded in the asymptotic shear $C_{zz}(u, z, \bar{z})$ and the Bondi news tensor $N_{zz} \equiv \partial_u C_{zz}$.

We impose the following large-$|u|$ fall off conditions [59, 88, 89], which are compatible with the BMS action on the phase space [42, 90, 91]:

$$C_{zz} = (u + C_{\pm})N_{zz}^{\text{vac}} - 2\,\partial^2 C_{\pm} + \frac{1}{u}\,C_{zz}^{L,\pm} + o(u^{-1})\,, \quad N_{zz} = N_{zz}^{\text{vac}} - \frac{C_{zz}^{L,\pm}}{u^2} + o(u^{-2})\,. \tag{2.3.39}$$

Here, $C_{\pm}(z, \bar{z})$ denote the boundary values of the *supertranslation field* at the corners $\mathscr{I}^+_{\pm}$ of null infinity, and they encode the displacement memory effect [36]. The subleading $1/u$ terms in (2.3.39), $C_{zz}^{L,\pm}(z, \bar{z})$, correspond to the presence of *gravitational tails* [92–95]. Such fall offs are consistent with the asymptotic structure of Christodoulou–Klainerman (CK) spacetimes [96],

[4] For a more field theoretic interpretation, see Appendix A.

which describe the global stability of Minkowski space. In particular, they have been shown to be present in physically relevant spacetimes describing scattering processes [93]. Under a generic BMS transformation $(\mathcal{T}, \mathcal{Y})$, these fields transform as

$$\delta_{\xi(\mathcal{T},\mathcal{Y})} C^{\pm} = \left(\mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} - \frac{1}{2}\partial\mathcal{Y} - \frac{1}{2}\bar{\partial}\bar{\mathcal{Y}}\right) C^{\pm} + \mathcal{T} . \tag{2.3.40}$$

It is convenient to introduce the sum and difference of the boundary supertranslation fields,

$$C^{(0)} \equiv \frac{1}{2}\left(C^{+} + C^{-}\right), \quad \mathcal{N} \equiv \frac{1}{2}\left(C^{+} - C^{-}\right), \tag{2.3.41}$$

where $C^{(0)}$ represents *Goldstone mode* of supertranslation, while $\mathcal{N}$ captures the net memory between past and future boundaries of $\mathscr{I}^{+}$.

The vacuum news tensor $N_{zz}^{vac}(z)$ [88, 97], identified with the tracefree part of the Geroch tensor [71, 98, 99], is given in terms of a Liouville field $\varphi(z)$,

$$N_{zz}^{vac} = \frac{1}{2}(\partial\varphi)^2 - \partial^2\varphi . \tag{2.3.42}$$

The latter encodes the refraction/velocity kick memory effects [88]. To make the covariance of the expressions under the superrotations manifest, it is useful to introduce the derivative operators [60, 98, 100, 101]

$$\mathscr{D}\phi_{h,\bar{h}} = [\partial - h\partial\varphi]\phi_{h,\bar{h}} , \qquad \bar{\mathscr{D}}\phi_{h,\bar{h}} = [\bar{\partial} - \bar{h}\bar{\partial}\bar{\varphi}]\phi_{h,\bar{h}} , \tag{2.3.43}$$

which, when acting on conformal fields $\phi_{h,\bar{h}}$, produce conformal fields of weights $(h+1, \bar{h})$ and $(h, \bar{h}+1)$, respectively. From these, we construct the leading soft (zero-mode) fields,

$$C_{zz}^{(0)} = -2\mathscr{D}_z^2 C^{(0)}, \quad \mathcal{N}_{zz}^{(0)} = -4\mathscr{D}_z^2\mathcal{N} . \tag{2.3.44}$$

The soft modes can then be checked to transform as conformal fields of weight $\left(\frac{3}{2}, -\frac{1}{2}\right)$ under a BMS vector field $\xi(\mathcal{T}, \mathcal{Y}, \bar{\mathcal{Y}})$,

$$\begin{aligned} \delta_{\xi(\mathcal{T},\mathcal{Y})}\, C_{zz}^{(0)} &= (\mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \frac{3}{2}\partial\mathcal{Y} - \frac{1}{2}\bar{\partial}\bar{\mathcal{Y}})C_{zz}^{(0)} - 2\,\partial^2\mathcal{T} , \\ \delta_{\xi(\mathcal{T},\mathcal{Y})}\, \mathcal{N}_{zz}^{(0)} &= (\mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \frac{3}{2}\partial\mathcal{Y} - \frac{1}{2}\bar{\partial}\bar{\mathcal{Y}})\mathcal{N}_{zz}^{(0)} . \end{aligned} \tag{2.3.45}$$

To isolate the purely radiative degrees of freedom from these soft modes, we define the *shifted fields*:

$$\tilde{C}_{zz} \equiv C_{zz} - C_{zz}^{(0)} - u\, N_{zz}^{\text{vac}} , \qquad \tilde{N}_{zz} \equiv N_{zz} - N_{zz}^{\text{vac}} , \tag{2.3.46}$$

which transform homogeneously under BMS transformations:

$$\begin{aligned} \delta_\xi(\mathcal{T}, \mathcal{Y})\, \tilde{C}_{zz} &= (\mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \tfrac{3}{2}\partial\mathcal{Y} - \tfrac{1}{2}\bar{\partial}\bar{\mathcal{Y}})\tilde{C}_{zz} + (\mathcal{T} + u\,\alpha)\,\tilde{N}_{zz} , \\ \delta_\xi(\mathcal{T}, \mathcal{Y})\, \tilde{N}_{zz} &= (\mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + 2\,\partial\mathcal{Y})\tilde{N}_{zz} + (\mathcal{T} + u\,\alpha)\,\partial_u\tilde{N}_{zz} , \end{aligned} \tag{2.3.47}$$

with $\alpha \equiv \frac{1}{2}(\partial\mathcal{Y} + \bar{\partial}\bar{\mathcal{Y}})$.

The zero-mode field $\mathcal{N}_{zz}^{(0)}$ in (2.3.44) can be re-expressed as the u-integral of the shifted news tensor. In addition, a subleading mode can be defined as

$$\mathcal{N}_{zz}^{(0)} = \int_{-\infty}^{+\infty} du\, \tilde{N}_{zz}(u, z, \bar{z}) , \qquad \mathcal{N}_{zz}^{(1)} = \int_{-\infty}^{+\infty} du\, u\, \tilde{N}_{zz}(u, z, \bar{z}) . \tag{2.3.48}$$

With the variables defined above, the gravitational phase space at null infinity naturally decomposes into a direct sum of a *soft* sector and a *hard* sector. The soft sector consists of the zero modes, which are associated with soft or low-energy gravitons and encode memory effects, while the hard sector contains the finite-energy, radiative modes. Explicitly,

$$\begin{aligned}\Gamma^{\text{soft}} &= \Big\{C^{(0)}_{zz}, C^{(0)}_{\bar z\bar z}, \mathcal{N}^{(0)}_{zz}, \mathcal{N}^{(0)}_{\bar z\bar z}, \Pi_{zz}, \Pi_{\bar z\bar z}, N^{\text{vac}}_{zz}, N^{\text{vac}}_{\bar z\bar z}\Big\}\,,\\ \Gamma^{\text{hard}} &= \Big\{\tilde C_{zz}(u,z,\bar z),\ \tilde C_{\bar z\bar z}(u,z,\bar z),\ \tilde N_{zz}(u,z,\bar z),\ \tilde N_{\bar z\bar z}(u,z,\bar z)\Big\}\,.\end{aligned} \tag{2.3.49}$$

The additional soft variable Π_{zz} is defined as a combination of the remaining soft modes as,

$$\Pi_{zz} \equiv 2\,\mathcal{N}^{(1)}_{zz} + C^{(0)}_{zz}\,\mathcal{N}^{(0)}_{zz} + (\varphi+\bar\varphi)\,\Delta C^{L}_{zz}\,, \tag{2.3.50}$$

where ΔC^L_{zz} denotes the difference between the value of logarithmic term appearing in the large-u expansion (2.3.39) at the two boundaries of $\mathscr{I}^+$. Its transformation law follows directly from the BMS variations of the individual fields on the right-hand side of (2.3.50), yielding

$$\delta_\xi\,\Pi_{zz} = \Big(\mathcal{Y}\,\partial + \bar{\mathcal{Y}}\,\bar\partial + \partial\mathcal{Y} - \bar\partial\bar{\mathcal{Y}}\Big)\,\Pi_{zz} - \mathcal{T}\,\mathcal{N}^{(0)}_{zz} \tag{2.3.51}$$

The phase space described in (2.3.49) is equipped with Poisson structure, which encodes the canonical commutation relations. The non-vanishing brackets are

$$\begin{aligned}\{\tilde N_{zz}(u), \tilde C_{\bar w\bar w}(u')\} &= -16\pi G\,\delta^{(2)}(z-w)\,\delta(u-u')\,,\\ \{\mathcal{N}^{(0)}_{zz}, C^{(0)}_{\bar w\bar w}\} &= -16\pi G\,\delta^{(2)}(z-w)\,,\\ \{\Pi_{zz}, N^{\text{vac}}_{\bar w\bar w}\} &= -16\pi G\,\delta^{(2)}(z-w)\,.\end{aligned} \tag{2.3.52}$$

The structure (2.3.52) makes manifest an important property: the *hard* and *soft* sectors form two mutually commuting subalgebras. In other words, radiative degrees of freedom Poisson-commute with the zero modes. This separation persists at the level of symmetry generators, as we now discuss.

We recall from Sec. 2.3.3 that the BMS charges are not conserved in the strict sense but obey a flux-balance law. Integrating the flux density (2.3.35) over null infinity, we define the *BMS fluxes*

$$\mathcal{F}_\xi \equiv \int du\,d^2z\;F_\xi\,, \tag{2.3.53}$$

where $\xi \equiv (\mathcal{T}, \mathcal{Y}, \bar{\mathcal{Y}})$ parametrizes the BMS vector field.

A direct computation using the brackets (2.3.52) shows that the flux functionals form a representation of the same algebra as the BMS charges themselves:

$$\{\mathcal{F}_{(\mathcal{T}_1,\mathcal{Y}_1,\bar{\mathcal{Y}}_1)}, \mathcal{F}_{(\mathcal{T}_2,\mathcal{Y}_2,\bar{\mathcal{Y}}_2)}\} = -\mathcal{F}_{[(\mathcal{T}_1,\mathcal{Y}_1,\bar{\mathcal{Y}}_1),\,(\mathcal{T}_2,\mathcal{Y}_2,\bar{\mathcal{Y}}_2)]}\,, \tag{2.3.54}$$

where the bracket on the right-hand side is the standard BMS Lie bracket (2.3.15). This algebraic closure ensures that the action of the fluxes on phase-space variables reproduces the infinitesimal BMS transformations.

The fluxes $\mathcal{F}_\xi$ naturally decompose into a *hard* contribution, depending solely on the radiative fields $(\tilde C_{zz}, \tilde N_{zz})$, and a *soft* contribution, depending only on the zero modes $(C^{(0)}_{zz}, \mathcal{N}^{(0)}_{zz}, \Pi_{zz}, N^{\text{vac}}_{zz})$. For supertranslations, this split reads:

$$\begin{aligned}\mathcal{F}^{\text{hard}}_{\mathcal{T}} &= -\frac{1}{16\pi G}\int du\,d^2z\;\mathcal{T}\;\tilde N_{zz}\tilde N_{\bar z\bar z}\,,\\ \mathcal{F}^{\text{soft}}_{\mathcal{T}} &= \frac{1}{8\pi G}\int d^2z\;\mathcal{T}\;\mathscr{D}^2\mathcal{N}^{(0)}_{\bar z\bar z}\,.\end{aligned} \tag{2.3.55}$$

Similarly, for superrotations one finds

$$\begin{aligned}\mathcal{F}^{\text{hard}}_{\mathcal{Y}} &= \frac{1}{16\pi G}\int du\,d^2z\;\mathcal{Y}\Big[\frac{3}{2}\tilde C_{zz}\,\partial\tilde N_{\bar z\bar z} + \frac{1}{2}\tilde N_{\bar z\bar z}\,\partial\tilde C_{zz} + \frac{u}{2}\,\partial\Big(\tilde N_{zz}\tilde N_{\bar z\bar z}\Big)\Big],\\ \mathcal{F}^{\text{soft}}_{\mathcal{Y}} &= \frac{1}{16\pi G}\int d^2z\;\mathcal{Y}\Big[-\mathscr{D}^3\Big(\mathcal{N}^{(1)}_{\bar z\bar z} + \frac{1}{2}(\varphi+\bar\varphi)\,\Delta C^L_{\bar z\bar z}\Big) + \frac{3}{2}C^{(0)}_{zz}\,\mathscr{D}\mathcal{N}^{(0)}_{\bar z\bar z} + \frac{1}{2}\mathcal{N}^{(0)}_{\bar z\bar z}\,\mathscr{D}C^{(0)}_{zz}\Big].\end{aligned} \tag{2.3.56}$$

By construction, each flux piece acts non-trivially only on its own sector of the phase space:

$$\begin{aligned}
\{\mathcal{F}^{\text{hard}}_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}, \tilde{C}_{zz}\} &= \delta_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}\tilde{C}_{zz}, & \{\mathcal{F}^{\text{soft}}_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}, \tilde{C}_{zz}\} &= 0,\\
\{\mathcal{F}^{\text{hard}}_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}, \tilde{N}_{zz}\} &= \delta_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}\tilde{N}_{zz}, & \{\mathcal{F}^{\text{soft}}_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}, \tilde{N}_{zz}\} &= 0,\\
\{\mathcal{F}^{\text{soft}}_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}, C^{(0)}_{zz}\} &= \delta_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}C^{(0)}_{zz}, & \{\mathcal{F}^{\text{hard}}_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}, C^{(0)}_{zz}\} &= 0,\\
\{\mathcal{F}^{\text{soft}}_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}, \mathcal{N}^{(0)}_{zz}\} &= \delta_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}\mathcal{N}^{(0)}_{zz}, & \{\mathcal{F}^{\text{hard}}_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}, \mathcal{N}^{(0)}_{zz}\} &= 0,\\
\{\mathcal{F}^{\text{soft}}_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}, \Pi_{zz}\} &= \delta_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}\Pi_{zz}, & \{\mathcal{F}^{\text{hard}}_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}, \Pi_{zz}\} &= 0.
\end{aligned} \tag{2.3.57}$$

Eq. (2.3.57) makes explicit that the soft and hard fluxes generate BMS transformations on their respective subspaces and act trivially on the complementary sector. This decoupling will play a central role in the discussion of soft graviton theorems and their relation to the Ward identities of the BMS group.

In fact, the decoupling of the two sectors extends to the algebra of the fluxes themselves. The soft and hard components of $\mathcal{F}_\xi$ each form an independent representation of the BMS algebra, with no cross-interaction:

$$\begin{aligned}
\{\mathcal{F}^{\text{soft}}_{\mathcal{T}_1,\mathcal{Y}_1,\bar{\mathcal{Y}}_1}, \mathcal{F}^{\text{hard}}_{\mathcal{T}_2,\mathcal{Y}_2,\bar{\mathcal{Y}}_2}\} &= 0,\\
\{\mathcal{F}^{\text{soft/hard}}_{\mathcal{T}_1,\mathcal{Y}_1,\bar{\mathcal{Y}}_1}, \mathcal{F}^{\text{soft/hard}}_{\mathcal{T}_2,\mathcal{Y}_2,\bar{\mathcal{Y}}_2}\} &= -\mathcal{F}^{\text{soft/hard}}_{[(\mathcal{T}_1,\mathcal{Y}_1,\bar{\mathcal{Y}}_1),\,(\mathcal{T}_2,\mathcal{Y}_2,\bar{\mathcal{Y}}_2)]}\,.
\end{aligned} \tag{2.3.58}$$

Equation (2.3.58) shows that each sector realises the BMS symmetry independently: the soft fluxes generate supertranslations and superrotations on the zero-mode subspace, while the hard fluxes generate the same transformations on the radiative subspace.

This property motivates treating the two sectors separately. From now on, we will analyse the soft and hard contributions in turn. The *soft sector* contains only the gravitational zero modes discussed here. The *hard sector* includes both the radiative modes of the gravitational field and, when present, contributions from matter fields coupled to gravity. We will review the structure of the matter field interaction in the next section.

Finally, for simplicity of presentation in the remainder of this thesis, we will work in a conformal frame where the Liouville fields are set to zero,

$$\varphi = 0 = \bar{\varphi}\,, \tag{2.3.59}$$

so that the covariant sphere derivatives $\mathscr{D}$ reduce to the standard derivatives $(\partial, \bar{\partial})$ on the unit two-sphere.

2.4 Action on matter fields

So far, the transformations generated by the BMS fluxes have been considered only in their action on the gravitational variables defined at null infinity. However, in any realistic setting where gravity interacts with matter, these symmetries also act non-trivially on the matter fields. The effect is encoded through Einstein's equations with a non-vanishing matter stress-energy tensor $T_{\mu\nu}$.

In the presence of matter, the evolution equations (2.3.6) acquire additional contributions from the matter stress tensor. At the level of the fluxes, this amounts to modifying the expressions for the hard sector by adding appropriate stress-tensor terms. Explicitly, the hard supertranslation and superrotation fluxes become

$$
\begin{aligned}
\mathcal{F}_{\mathcal{T}}^{\text{hard}} &= -\frac{1}{16\pi G}\int du\, d^2z\, \mathcal{T}\left(\tilde{N}_{zz}\tilde{N}_{\bar{z}\bar{z}} + 16\pi G\, T_{uu}^{(2)}\right), \\
\mathcal{F}_{\mathcal{Y}}^{\text{hard}} &= \frac{1}{16\pi G}\int du\, d^2z\, \mathcal{Y}\bigg[\frac{3}{2}\tilde{C}_{zz}\,\partial\tilde{N}_{\bar{z}\bar{z}} + \frac{1}{2}\tilde{N}_{\bar{z}\bar{z}}\,\partial\tilde{C}_{zz} + \frac{u}{2}\,\partial\left(\tilde{N}_{zz}\tilde{N}_{\bar{z}\bar{z}}\right) \\
&\qquad\qquad + 16\pi G\left(\frac{u}{2}\,\partial T_{uu}^{(2)} - T_{uz}^{(2)}\right)\bigg],
\end{aligned}
\tag{2.4.1}
$$

where the superscript $^{(2)}$ indicates that this term is the coefficient of r^{-2} in the large-r expansion of the corresponding stress-tensor component. We shall follow this convention also for other fields and their asymptotic expansions.

As a first step, we will work in the simplest situation where the matter stress tensor corresponds to a free field. For concreteness, let us consider a free massless scalar field ϕ with stress tensor

$$
T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \eta_{\mu\nu}\,(\partial\phi)^2. \tag{2.4.2}
$$

Near null infinity, using (2.2.11), the leading contribution to T_{uu} takes the form

$$
T_{uu}^{(2)} = (\partial_u\varphi)^2 = \frac{1}{16\pi^2}\int d\omega\, d\omega'\, \omega\omega'\left[a(\omega,z,\bar{z})\, a^\dagger(\omega',z,\bar{z})\, e^{-i(\omega-\omega')u} + \cdots\right], \tag{2.4.3}
$$

where φ denotes the appropriately rescaled field on $\mathscr{I}^+$, and $a(\omega,z,\bar{z})$ and $a^\dagger(\omega,z,\bar{z})$ are the annihilation and creation operators introduced in section 2.2.

Using the mode expansion above, along with the canonical commutation relations, one can compute the action of the hard fluxes on the matter creation and annihilation operators. For a creation operator $a_h^\dagger(\omega,z,\bar{z})$ of helicity h, the result is [102]:

$$
\begin{aligned}
\{\mathcal{F}_{\mathcal{T},\mathcal{Y},\bar{\mathcal{Y}}}^{\text{hard}},\, a_h^\dagger(\omega,z,\bar{z})\} = \bigg[&\mathcal{T}(z,\bar{z})\,\omega + \left(\mathcal{Y}(z)\,\partial + \bar{\mathcal{Y}}(\bar{z})\,\bar{\partial}\right) \\
&- \frac{1}{2}\left(\partial\mathcal{Y} + \bar{\partial}\bar{\mathcal{Y}}\right)\omega\,\partial_\omega + \frac{h}{2}\left(\partial\mathcal{Y} - \bar{\partial}\bar{\mathcal{Y}}\right)\bigg]a_h^\dagger(\omega,z,\bar{z}).
\end{aligned}
\tag{2.4.4}
$$

For the Poincaré subgroup of the BMS group, this reduces to the familiar canonical action of translations and Lorentz transformations on the momentum-space modes of a scalar field.

2.4.1 Charges at time-like boundary

Up to this point, our discussion of BMS fluxes has focused entirely on massless matter fields, whose dynamics and asymptotic behaviour are naturally captured at null infinity. However, in order to account for the action of asymptotic symmetries on *massive* fields, one must resolve the geometry of future and past time-like infinities $i^\pm$. This was accomplished by Campiglia and Laddha in [103] using the hyperbolic slicing of Minkowski space described in (2.1.6). Here we briefly review the essential results.

The key observation is that the symmetry parameters $(\mathcal{T},\mathcal{Y}^A)$ defined on the celestial sphere at null infinity can be extended into the interior of the hyperboloid that represents time-like infinity. Given the scalar function $\mathcal{T}$ and the vector field $\mathcal{Y}^A$ on S^2, one can define the corresponding bulk functions on the unit hyperboloid $\mathcal{H}$ as

$$
\begin{aligned}
\mathcal{T}_{\mathcal{H}}(\rho,w,\bar{w}) &= \int d^2z\, \mathcal{G}^{(3)}(\rho,w,\bar{w};z,\bar{z})\,\mathcal{T}(z,\bar{z}), \\
\mathcal{Y}_{\mathcal{H}}^\alpha(\rho,w,\bar{w}) &= \int d^2z\, \mathcal{G}_A^\alpha(\rho,w,\bar{w};z,\bar{z})\,\mathcal{Y}^A(z,\bar{z}).
\end{aligned}
\tag{2.4.5}
$$

Here, the intertwining kernels $\mathcal{G}^{(3)}$ and $\mathcal{G}^{\alpha}_{A}$ are bulk-to-boundary propagators on the hyperboloid, explicitly given by [68]

$$\begin{aligned}
\mathcal{G}^{(3)}(\rho, w, \bar{w}; z, \bar{z}) &= \frac{1}{\pi}\left(\frac{\rho}{1+\rho^2|z-w|^2}\right)^3 , \\
\mathcal{G}^{\alpha}_{w}(\rho, w, \bar{w}; z, \bar{z})\,\partial_\alpha &= \frac{i}{4\pi}\,\partial_w^3\left[\frac{\hat{p}^\mu\,\varepsilon^-_{\mu\nu}(\hat{q})\,\hat{q}^\lambda}{\hat{p}\cdot\hat{q}}\,J_\lambda{}^\nu(\hat{p})\right] ,
\end{aligned} \tag{2.4.6}$$

where $J_{\mu\nu}(\hat{p})$ is the angular momentum operator acting on a particle of normalized momentum $\hat{p}^\mu(\rho, z, \bar{z})$ as parametrized in (2.2.14). Similarly $\hat{q}(z, \bar{z})$ is a massless momentum as parametrized in (2.2.3), and the polarization co-vector $\varepsilon^-_{\mu\nu} = \varepsilon^-_\mu \varepsilon^-_\nu$ was defined in (2.2.4).

These extensions are chosen so that the vector field generating the asymptotic symmetry on $\mathcal{H}$ is

$$\xi^\tau(\rho, \hat{x}) = \mathcal{T}_{\mathcal{H}}(\rho, \hat{x}), \qquad \xi^\alpha(\rho, \hat{x}) = \mathcal{Y}^\alpha_{\mathcal{H}}(\rho, \hat{x}). \tag{2.4.7}$$

In the large-ρ limit, the properties of the bulk-to-boundary propagators ensure that ξ^μ matches onto the large-u behaviour of the BMS vector fields in Eq. (2.3.9), after an appropriate change of coordinates. This guarantees that the symmetry transformations agree on the intersection of $\mathscr{I}^+$ with i^+, as they must. A completely analogous construction applies at past time-like infinity i^-.

The charges associated with these transformations at $i^\pm$ can be obtained directly from the matter stress-energy tensor via the standard expression [69, 104]

$$Q[\xi] = \int_\Sigma d\Sigma_\mu\, \xi_\nu\, T^{\mu\nu}, \tag{2.4.8}$$

where Σ is the relevant hypersurface at infinity. When Σ is taken to be $\mathscr{I}^+$, one recovers the matter part of the hard BMS fluxes in (2.4.1). When Σ is taken to be a $\tau \to \infty$ slice of the hyperboloid at i^+, one instead finds the massive-field contributions:

$$\begin{aligned}
Q^{i^+}_{\mathcal{T}} &= \lim_{\tau\to\infty} \tau^3 \int d^3Y\, \mathcal{T}_{\mathcal{H}}(Y)\, T_{\tau\tau}(Y), \\
Q^{i^+}_{\mathcal{Y}} &= \lim_{\tau\to\infty} \tau^3 \int d^3Y\, \mathcal{Y}^\alpha_{\mathcal{H}}(Y)\, T_{\tau\alpha}(Y),
\end{aligned} \tag{2.4.9}$$

where the hyperboloid coordinates are collectively denoted $Y^\alpha = (\rho, z, \bar{z})$.

Evaluating these charges on one-particle states amounts to computing their commutators with the creation and annihilation operators for massive modes. If we denote these modes by $b(p)$, one finds using eq (2.2.16)

$$\begin{aligned}
\left[Q^{i^+}_{\mathcal{T}}, b(p)\right] &= -i\, m\, \mathcal{T}_{\mathcal{H}}(\hat{p})\, b(p), \\
\left[Q^{i^+}_{\mathcal{Y}}, b(p)\right] &= i\, \mathcal{Y}^\alpha_{\mathcal{H}}(\hat{p})\, \partial_\alpha b(p),
\end{aligned} \tag{2.4.10}$$

where $\hat{p}$ denotes the direction of the massive momentum on the hyperboloid.

Finally, with both null and time-like infinity contributions included, the total flux of a BMS generator ξ takes the form

$$\mathcal{F}_\xi = \mathcal{F}^{\text{soft}}_\xi + \mathcal{F}^{\text{hard}}_\xi + Q^{i^+}_\xi . \tag{2.4.11}$$

The expression above refers to the future boundary. An analogous formula holds at the past boundary.

2.5 Antipodal matching at space-like boundary

The associated charges of BMS symmetry can be defined both at past and at future null infinity. However, in order to formulate conservation laws, one must relate these charges across the entire spacetime boundary. Since past and future null infinity are disconnected, this requires passing through spatial infinity, i^0. The procedure of *antipodal matching* provides this link: it identifies the charges at the "early" edge of $\mathscr{I}^+$ with those at the "late" edge of $\mathscr{I}^-$, up to an antipodal map on the celestial sphere [105–110]. This matching is essential for connecting scattering data between the in- and out-states, for deriving Ward identities of the $\mathcal{S}$-matrix from asymptotic symmetries, and for relating soft theorems to memory effects.

To make this matching precise, we need a coordinate system that resolves the neighborhood of spatial infinity, which is inaccessible in the standard Bondi coordinates. A natural choice is given by the Beig–Schmidt coordinates [111], a generalization of the hyperbolic slicing of Minkowski spacetime (2.1.9), in which the relevant asymptotic expansions and residual symmetries can be analyzed explicitly. We now review this construction and show how it leads to the antipodal identification of BMS charges.

According to the ansatz of Beig and Schmidt, a spacetime is said to be asymptotically flat near space-like infinity if there exists a coordinate system in which, as $\varrho \to \infty$, the metric takes the form

$$\begin{aligned} ds^2 \xrightarrow{\varrho\to\infty} &\left(1+\frac{2\sigma}{\varrho}+\frac{\sigma^2}{\varrho^2}+o(\varrho^{-2})\right)d\varrho^2+o(\varrho^{-1})\,d\varrho\,d\varphi^\alpha \\ &+\Big(\varrho^2\,\hat{h}_{\alpha\beta}+\varrho(\hat{k}_{\alpha\beta}-2\sigma\,\hat{h}_{\alpha\beta})+\log\varrho\,\hat{i}_{\alpha\beta}+\hat{j}_{\alpha\beta}+o(\varrho^0)\Big)\,d\varphi^\alpha d\varphi^\beta, \end{aligned} \tag{2.5.1}$$

where the fields $\sigma, \hat{h}_{\alpha\beta}, \hat{k}_{\alpha\beta}, \hat{i}_{\alpha\beta}, \hat{j}_{\alpha\beta}$ are functions of the Lorentzian de Sitter coordinates φ^α. In the present discussion we will focus only on the structures relevant for supertranslations.

Infinitesimal diffeomorphisms preserving these fall-off conditions are parameterized as

$$\begin{aligned} \zeta^\rho &= H\log\varrho+\omega+o(\varrho^0), \\ \zeta^\alpha &= \chi^\alpha+\frac{\log\varrho}{\varrho}D^\alpha H+\frac{1}{\varrho}D^\alpha(H+\omega)+o(\varrho^{-1}), \end{aligned} \tag{2.5.2}$$

where χ^α are the six Killing vectors of the unit hyperboloid, H contains four constants corresponding to *log supertranslations*, and ω is a free function generating *Spi supertranslations* (supertranslations at spatial infinity).

To reduce this enlarged symmetry group to the BMS group, further conditions are imposed. First, the *log supertranslations* generated by H are removed by requiring σ to be parity-even under a combination of time reversal $\tau \to -\tau$ and anti-podal map $x^A \to -x^A$, which we define by the operator $\Upsilon^*_{i^0}$. Next, to restrict the Spi supertranslations, the Compère–Dehouck condition is imposed:

$$D^\alpha \hat{k}_{\alpha\beta}=0, \qquad \hat{k}^\alpha_{\ \alpha}=0, \tag{2.5.3}$$

where D_α is the covariant derivative on the unit hyperboloid. This constraint forces ω to satisfy

$$(D^2+3)\,\omega=0. \tag{2.5.4}$$

The solutions to this equation decompose into parts of definite parity under $\Upsilon^*_{i^0}$. To further reduce to a single copy of Spi supertranslations, one imposes an additional parity condition on $\hat{k}_{\alpha\beta}$:

$$\Upsilon^*_{i^0}\,\hat{k}_{\alpha\beta}=-\hat{k}_{\alpha\beta}. \tag{2.5.5}$$

With these restrictions, the free function ω behaves on the two boundaries of spatial infinity as

$$\begin{aligned}\omega(\tau, x^A) &\xrightarrow{\tau\to+\infty} -\frac{1}{2}e^{\tau}\,\mathcal{T}(x^A),\\ \omega(\tau, x^A) &\xrightarrow{\tau\to-\infty} +\frac{1}{2}e^{-\tau}\,\mathcal{T}(-x^A),\end{aligned} \tag{2.5.6}$$

where $\mathcal{T}$ is an arbitrary function on the boundary S^2. Upon matching with the supertranslation charges defined at $\mathscr{I}^+$ and $\mathscr{I}^-$, $\mathcal{T}$ is recognized as the supertranslation parameter itself. The two limits above therefore correspond to the supertranslation parameters on $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$, related by an *antipodal map* on the sphere.

Consequently, the supertranslation charges satisfy the matching condition

$$Q(\mathscr{I}^+_-) = Q(\mathscr{I}^-_+) \equiv Q(i^0). \tag{2.5.7}$$

While this equality has been explicitly demonstrated for supertranslations, we will assume that it extends to the higher BMS generators as well, in particular to superrotations [102, 112]. This antipodal matching will serve as a key ingredient in deriving constraints from BMS symmetry in subsequent chapters.

It is worth emphasizing that, although the charges are not strictly conserved along a given null infinity, they do match when propagated from past to future null infinity through spatial infinity. The physical implications of this matching will be discussed in the next chapter.

2.6 QED asymptotic symmetries

In ordinary gauge theory, gauge transformations are considered redundancies of the description and therefore have vanishing Noether charges [113, 114]. However, when one studies gauge transformations that *do not* vanish at infinity, the situation changes dramatically. Such transformations, known as *asymptotic symmetries*, act non-trivially on the physical phase space and are associated with physically meaningful surface charges [13, 115, 116]. These are the analogs, in gauge theory, of the BMS symmetries encountered in gravity, and they play an equally important role in relating symmetries to memory effects and to the soft theorems of scattering amplitudes [117].

The asymptotic symmetry group (ASG) of a gauge theory can be defined as

$$\text{ASG} = \frac{\text{Residual Gauge Transformations}}{\text{Trivial Gauge Transformations}}, \tag{2.6.1}$$

where "trivial" means those transformations whose action on the phase space vanishes [77]. In QED, requiring a finite energy flux through null infinity constrains the asymptotic behavior of the gauge field in retarded Bondi coordinates to

$$A_z \sim \mathcal{O}(1), \quad A_r \sim \mathcal{O}\left(\frac{1}{r^2}\right), \quad A_u \sim \mathcal{O}\left(\frac{1}{r}\right). \tag{2.6.2}$$

The residual gauge transformations preserving these falloffs are

$$\varepsilon = \varepsilon(z, \bar{z}) + \mathcal{O}\left(\frac{1}{r}\right), \tag{2.6.3}$$

where the leading term is an arbitrary function on the celestial sphere. Since this parameter does not vanish at infinity, it labels a genuine physical symmetry — a *Large Gauge Transformation* (LGT) — rather than a redundancy [115].

Applying Noether's theorem to these LGTs yields surface charges defined on the boundary of spacetime. At future null infinity, the charge is

$$Q_\varepsilon^+ = \frac{1}{e^2}\int_{\mathscr{I}^+_-} \varepsilon\, {}^*F = \frac{1}{e^2}\int_{\mathscr{I}^+_-} d^2z\, \gamma_{z\bar{z}}\, \varepsilon\, F_{ru}^{(2)}, \tag{2.6.4}$$

with a similar expression on $\mathscr{I}^-$. The superscript $^{(2)}$ denotes the coefficient of r^{-2} in the large-r expansion.

This charge generically contains both "soft" and "hard" parts. To separate them, we use Maxwell's equations near $\mathscr{I}^+$,

$$\partial_u F_{ru}^{(2)} + \partial_z F_{u\bar{z}}^{(0)} + \partial_{\bar{z}} F_{uz}^{(0)} + e^2 j_u^{(2)} = 0, \tag{2.6.5}$$

to rewrite Q_ε as

$$Q_\varepsilon = -\frac{1}{e^2}\int du\, d^2z \left(\partial_z \varepsilon\, F_{u\bar{z}}^{(0)} + \partial_{\bar{z}} \varepsilon\, F_{uz}^{(0)}\right) + \int du\, d^2z\, \varepsilon\, j_u^{(2)}. \tag{2.6.6}$$

The first term is the *soft* photon contribution, determined entirely by the zero-frequency modes of the gauge field. The second term is the *hard* contribution, which depends on the matter current $j_u^{(2)}$ and encodes the charged particle content of the theory.

Just as with supertranslations in gravity, these large gauge charges satisfy an *antipodal matching condition* across spatial infinity:

$$Q_\varepsilon^+ = Q_\varepsilon^-. \tag{2.6.7}$$

This ensures that the LGT parameter ε is the same (up to the antipodal map) on the past and future boundaries, a property essential for connecting scattering amplitudes to conserved charges.

2.6.1 QED phase space

The QED phase space at null infinity can be organised in complete analogy with the gravitational case. The starting point is the large-u behaviour of the gauge potential[5]:

$$A_z^{(0)}\Big|_{\mathscr{I}^+_\pm} = -2\,\partial \mathcal{A}_\pm + \frac{1}{u}\, A_z^{L,\pm} + \mathcal{O}(u^{-2}), \tag{2.6.8}$$

where $\mathcal{A}^\pm$ are the constant modes at early/late retarded times and $A_z^{L,\pm}$ encode subleading behaviour. We define

$$\mathscr{A} = \frac{1}{2}\left(\mathcal{A}^+ + \mathcal{A}^-\right), \qquad \mathscr{F} = \frac{1}{2}\left(\mathcal{A}^+ - \mathcal{A}^-\right). \tag{2.6.9}$$

The combination $\mathscr{A}_z := \partial \mathscr{A}$ transforms under a large gauge transformation as

$$\delta_\varepsilon \mathscr{A}_z = \partial \varepsilon, \tag{2.6.10}$$

identifying $\mathscr{A}$ as the Goldstone boson associated with spontaneously broken large gauge symmetry.

The radiative degrees of freedom are conveniently defined by subtracting off this Goldstone mode:

$$\tilde{A}_z^{(0)} = A_z^{(0)} - \mathscr{A}_z. \tag{2.6.11}$$

[5] Here again we have a $1/u$ term in analogy to eq (2.3.39) which accounts for the *electromagnetic tails* [94].

The corresponding zero mode,

$$\mathscr{F}_z^{(0)} \equiv \int du\, \partial_u \tilde{A}_z^{(0)} = 2\partial \mathscr{F}, \tag{2.6.12}$$

is the operator that creates and annihilates soft photons, analogous to the role played by the soft graviton modes in the gravitational phase space.

The hard sector is determined by charged matter. For massless fields, the relevant part of the large gauge charge is seen from 2.6.6 to be,

$$Q_\varepsilon = \int du\, d^2z\, \varepsilon\, j_u^{(2)}, \tag{2.6.13}$$

with a particle with charge e contributing to the conserved current

$$j_u = -ie\left(\phi\, \partial_u \phi^* - \phi^*\, \partial_u \phi\right). \tag{2.6.14}$$

Introducing a unit vector on the hyperboloid $Y^\alpha(\rho, z, \bar{z})$, massive charged fields contribute through time-like infinity:

$$Q_\varepsilon^{i^+} = \int d^3Y\, \varepsilon_{\mathcal{H}}(Y)\, j_\tau^{(3)}(Y), \tag{2.6.15}$$

where the large gauge parameter is extended to the hyperboloid by [116]

$$\varepsilon_{\mathcal{H}}(Y) = \int d^2z\, \mathcal{G}^{(2)}(Y; z, \bar{z})\, \varepsilon(z, \bar{z}), \tag{2.6.16}$$

and the bulk-to-boundary propagator is [68]

$$\mathcal{G}^{(2)}(\rho, w, \bar{w}; z, \bar{z}) = \frac{1}{\pi}\left(\frac{\rho}{1 + \rho^2|z - w|^2}\right)^2. \tag{2.6.17}$$

The action of these charges on matter is simply that of a gauge transformation with parameter ε for the massless modes at $\mathscr{I}$, and parameter $\varepsilon_{\mathcal{H}}$ for the massive modes at $i^\pm$.

This completes our brief but self-contained review of asymptotic symmetries in gravity and QED. While many important subtleties have been omitted here (for which the reader is referred to dedicated reviews such as [13, 14]), the material above will be sufficient for the purposes of this thesis. In particular, we now have the parallel gravitational and electromagnetic cases in hand, both exhibiting a universal structure of soft modes, hard matter contributions, and antipodal matching conditions across spatial infinity.

Chapter 3

Soft Theorems

The low-energy behavior of scattering amplitudes involving external photons and gravitons is governed by a set of universal relations known as soft theorems. These theorems describe how an amplitude factorizes in the limit where the energy of one of the external gauge bosons or gravitons becomes small (soft). Remarkably, the structure of this factorization is independent of the details of the high-energy process—it depends only on the nature of the soft particle and the momenta and charges of the hard external states.

The modern study of soft theorems originates with the seminal work of Weinberg in 1965 [9], who derived the universal leading behavior of scattering amplitudes in the soft photon and soft graviton limits. His result showed that the insertion of a soft photon or graviton into an arbitrary hard process yields a factorized expression, with a universal "soft factor" multiplying the amplitude without the soft particle. Over the years, further work extended this to subleading orders, notably in gravity, where new universal terms were discovered [118–120]. Initially, these results were viewed as isolated technical facts of perturbative quantum field theory, obtained through explicit diagrammatic analysis, without an apparent symmetry principle behind them.

In recent years, however, a powerful new perspective has emerged: soft theorems can be understood as Ward identities of asymptotic symmetries acting at null infinity ($\mathscr{I}$). This insight—pioneered by Strominger and collaborators [11,12,112]—has revealed a deep equivalence between the infrared structure of gauge and gravitational theories, the geometry of spacetime boundaries, and the asymptotic symmetry group.

In what follows, we will primarily present results for gravitons, noting along the way the analogous statements for photons in QED. The photon case will be discussed explicitly where the details differ from gravity.

3.1 Soft factorization

3.1.1 Leading soft theorem

Consider a scattering process with n incoming particles and m outgoing particles. If, in addition to these hard particles, the process emits a photon or graviton whose energy ω tends to zero (the *soft limit*), the amplitude exhibits a universal factorization property: it splits into a part depending solely on the momentum and polarization of the emitted soft particle, and another part depending only on the momenta of the hard external states. Explicitly, for a soft graviton,

$$\lim_{\omega\to 0}\mathcal{M}_{n\to m+1}(p_i;\omega\hat{q}) = \frac{\kappa}{2\omega}\sum_{i=1}^{n+m}\frac{p_i\cdot\varepsilon(\hat{q})\cdot p_i}{p_i\cdot\hat{q}}\,\mathcal{M}_{n\to m}(p_i) + \mathcal{O}(\omega^0)\,. \tag{3.1.1}$$

Here:

- $\mathcal{M}_{n\to m}$ is the scattering amplitude without the soft particle, with the hard momenta denoted p_i.
- The momentum of the soft particle is $\omega\hat{q}$, with ω its energy and $\hat{q}$ a null vector specifying its direction.
- $\varepsilon(\hat{q})$ is the polarization tensor of the soft graviton, symmetric, traceless, and transverse to $\hat{q}$.
- $\kappa = \sqrt{32\pi G}$ is the gravitational coupling constant.

In much of what follows, we will use the *all-incoming* convention, where all momenta are taken to be incoming. In this convention, an additional factor η_i is introduced:

$$\eta_i = \begin{cases} +1, & \text{for incoming hard particles,} \\ -1, & \text{for outgoing hard particles.} \end{cases}$$

We will then write the amplitudes simply as $\mathcal{M}_n$ when the number of external hard legs is n.

The factorization in eq. (3.1.1) defines the *leading soft factor* $S^{(0)}_{\text{gr}}$, which captures the most divergent behavior in the $\omega \to 0$ limit. For a photon, the polarization tensor is replaced by a polarization vector, and the leading soft factor becomes

$$S^{(0)}_{\text{em}} = \sum_i e_i \eta_i \, \frac{\varepsilon(\hat{q}) \cdot p_i}{\hat{q} \cdot p_i} \,, \tag{3.1.2}$$

where e_i is the charge of the i^{th} particle, and the sum run over all hard particles.

In what follows we will use subscripts *gr* and *em* to distinguish gravitational and electromagnetic cases respectively. When the subscript is omitted, the statement is intended to hold for both cases.

A striking feature of $S^{(0)}$ is its *universality*: it is completely independent of the detailed dynamics of the hard process. It depends only on the external hard particle momenta (and charges, in QED), but not on the specific interactions or the number of loops in the hard amplitude. This universality strongly suggests that the soft theorem is the consequence of an underlying symmetry principle, rather than an accident of perturbation theory. In fact, as we will see later, it corresponds to a Ward identity of an asymptotic symmetry acting at null infinity ($\mathscr{I}$).

It is also useful to write eq. (3.1.1) in a form where only the leading divergence is singled out:

$$\lim_{\omega \to 0} \omega \, \mathcal{M}_{n+1}(p_i; \omega\hat{q}) = S^{(0)} \, \mathcal{M}_n(p_i) \,. \tag{3.1.3}$$

This "projected" form of the soft theorem will play an important role in later parts of the thesis, especially when we discuss the connection to charges and Ward identities.

3.1.2 Subleading soft theorem

The universality of the leading soft factor naturally raises the question of whether similar factorization properties persist at subleading orders in the soft expansion. In other words, we ask whether in the $\omega \to 0$ limit the $(n+1)$-point amplitude can still be written as

$$\lim_{\omega \to 0} \mathcal{M}_{n+1}(p_i; \omega\hat{q}) = S(\omega) \, \mathcal{M}_n(p_i) + \cdots \,, \tag{3.1.4}$$

where $S(\omega)$ contains all the factorizing ω-dependence, while the ellipsis denotes terms that do not necessarily factorize in this manner. An example of such non-factorizing term will be shown in section 5.7.

At tree level in gravity, Cachazo and Strominger showed that the *subleading* term,i.e. the term of order ω^0 in the expansion of the soft graviton factor, is also universal and admits a factorized form [120]. Explicitly, one finds

$$\begin{aligned}\lim_{\omega\to 0}\,(1+\omega\partial_\omega)\,\mathcal{M}_{n+1} &= S^{(1)}_{\rm gr}\,\mathcal{M}_n \\ &= -\frac{i\kappa}{2}\sum_{i=1}^{n}\frac{p_i^\mu\,\varepsilon_{\mu\nu}(\hat q)\,q_\lambda}{p_i\cdot q}\left(J_i^{\lambda\nu}+S_i^{\lambda\nu}\right) \\ &\equiv S^{(1)J}_{\rm gr}+S^{(1)S}_{\rm gr}\,,\end{aligned} \tag{3.1.5}$$

where:

- $J_i^{\mu\nu}=-i\left(p_i^\mu\frac{\partial}{\partial p_{i\nu}}-p_i^\nu\frac{\partial}{\partial p_{i\mu}}\right)$ is the orbital angular momentum operator for the i-th hard particle,
- $S_i^{\mu\nu}$ is the spin angular momentum operator of the i-th particle.

The differential operator $(1+\omega\partial_\omega)$ projects out the $\mathcal{O}(\omega^{-1})$ term, isolating the subleading contribution in the expansion. While $S^{(1)}_{\rm gr}$ depends on the spins of the hard external particles, this dependence appears in a universal operatorial form, making the result valid for particles of arbitrary spin.

A closely related result was obtained much earlier by Low for QED [10], giving the subleading soft photon theorem. At tree level, the universal factor is

$$S^{(1)}_{\rm em}=-\frac{i}{2}\sum_{i=1}^{n}e_i\frac{\varepsilon_\nu(\hat q)\,q_\lambda}{p_i\cdot q}\left(J_i^{\lambda\nu}+S_i^{\lambda\nu}\right)\equiv S^{(1)J}_{\rm em}+S^{(1)S}_{\rm em}\,, \tag{3.1.6}$$

where e is the gauge coupling and $\varepsilon_\nu(\hat q)$ is the photon polarization vector.

In both gravity and QED, the subleading factors decompose naturally into an "orbital" piece $S^{(1)J}$ involving $J_i^{\mu\nu}$ and a "spin" piece $S^{(1)S}$ involving $S_i^{\mu\nu}$.

It is important to stress that, unlike the leading soft theorem which holds to all orders in perturbation theory (and even non-perturbatively), the above subleading results are derived explicitly at tree level. This immediately raises the question of whether these universal structures survive once loop corrections are included. As we will discuss later, the answer turns out to be more subtle than in the leading case, and infrared effects play a crucial role in determining the fate of the subleading soft theorem beyond tree level.

3.1.3 The soft tower

The factorization properties discussed above for the leading and subleading soft theorems naturally suggest a possible *hierarchy* of soft factors at higher orders in the soft expansion. At tree level, one may consider the following ansatz for the soft factor (3.1.4):

$$S_n(\omega)=\frac{1}{\omega}S^{(0)}+S^{(1)}+\omega\,S^{(2)}+\mathcal{O}(\omega^2)\,, \tag{3.1.7}$$

where $S^{(0)}$ is the leading soft factor, $S^{(1)}$ is the subleading one, and $S^{(2)}$ is the subsubleading term, and in general we call $S^{(\ell)}$, the (sub)$^\ell$leading soft factor.

As before, the $(n+1)$-point amplitude with one graviton or photon of momentum $\omega\hat q$ going soft can be projected to extract the ℓ-th term in this expansion:

$$\lim_{\omega\to 0}\partial_\omega^\ell\,(1+\omega\partial_\omega)\,\mathcal{M}_{n+1}=S^{(\ell)}\,\mathcal{M}_n+\cdots\,, \tag{3.1.8}$$

where the ellipsis denotes non-factorizing terms.

It is then natural to ask whether higher-order terms in this expansion also admit a *universal* form. This question was addressed by Hamada and Shiu [121] and later by Li, Lin, and Zhang [122], who showed that *projected soft theorems* exist for these higher orders at tree level. The general tree-level expression takes the form:

$$\begin{aligned}\lim_{\omega\to 0}\partial_\omega^\ell\left(1+\omega\partial_\omega\right)\mathcal{M}_{m+1}^{\mu\nu}=&\sum_{i=1}^{m}\frac{1}{(\ell+1)!}\frac{q^\alpha q^\beta}{p_i\cdot q}\left(J_i+S_i\right)^{\mu\alpha}\left(J_i+S_i\right)^{\nu\beta}\left(q\cdot\partial_i\right)^{\ell-1}\mathcal{M}_m\\&+q^{\alpha_1}\cdots q^{\alpha_\ell}\sum_{i,j}L^\mu{}_{\alpha_i}L^\nu{}_{\alpha_j}D_\ell^{\alpha_1\cdots\hat{\alpha}_i\cdots\hat{\alpha}_j\cdots\alpha_\ell}\,.\end{aligned}\tag{3.1.9}$$

The first term above is factorizing, while the second term contains the *non-factorizing* contributions at ℓ^{th} order, involving auxiliary functions L and D. These latter terms are in general non-zero and not fixed by symmetry arguments.

However, by projecting with a totally symmetric, traceless tensor $\Omega_{\mu(\nu\alpha_1\cdots\alpha_\ell)}$ (symmetric in $(\nu\alpha_1\cdots\alpha_\ell)$ and traceless in all indices), one can eliminate the non-factorizing terms and obtain a *universal* result:

$$\begin{aligned}&\lim_{\omega\to 0}\left[\left(\Omega^{\mu(\nu\alpha_1\cdots\alpha_\ell)}+\Omega^{\nu(\mu\alpha_1\cdots\alpha_\ell)}\right)\partial_{\alpha_1}\cdots\partial_{\alpha_\ell}\right]\partial_\omega^\ell\left(1+\omega\partial_\omega\right)\mathcal{M}_{m+1}^{\mu\nu}\\&\quad=\left(\Omega^{\mu(\nu\alpha_1\cdots\alpha_\ell)}+\Omega^{\nu(\mu\alpha_1\cdots\alpha_\ell)}\right)\partial_{\alpha_1}\cdots\partial_{\alpha_\ell}\\&\qquad\qquad\left[\sum_{i=1}^{m}\frac{1}{(\ell+1)!}\frac{q^\alpha q^\beta}{p_i\cdot q}\left(J_i+S_i\right)^{\mu\alpha}\left(J_i+S_i\right)^{\nu\beta}\left(q\cdot\partial_i\right)^{\ell-1}\mathcal{M}_m\right].\end{aligned}\tag{3.1.10}$$

From this general structure, the *subsubleading* soft factor for gravity can be read off as [123]

$$S_{\text{gr}}^{(2)}=\sum_{i=1}^{m}\frac{\varepsilon_{\mu\nu}\,q^\alpha q^\beta}{2\,p_i\cdot q}\,\left(J_i+S_i\right)^{\mu\alpha}\left(J_i+S_i\right)^{\nu\beta}\,.\tag{3.1.11}$$

Interestingly, at tree level this term does not contain any non-factorizing contribution, and is therefore completely fixed by the spin and momenta of the hard particles.

A similar tower of projected soft theorems exists for QED. In this case, the projected relation reads:

$$\begin{aligned}&\lim_{\omega\to 0}\left[\Omega^{\mu\alpha_1\cdots\alpha_\ell}\,\partial_{\alpha_1}\cdots\partial_{\alpha_\ell}\right]\partial_\omega^\ell\left(1+\omega\partial_\omega\right)\mathcal{M}_{n+1}^{\mu}\\&\quad=\Omega^{\mu\alpha_1\cdots\alpha_\ell}\,\partial_{\alpha_1}\cdots\partial_{\alpha_\ell}\left[\sum_{i=1}^{n}\frac{1}{(\ell+1)!}\frac{e_i}{p_i\cdot q}\,q^\nu\left(J_i+S_i\right)^{\mu\nu}\left(q\cdot\partial_i\right)^\ell\mathcal{M}_n\right].\end{aligned}\tag{3.1.12}$$

The existence of this "soft tower" indicates that there is a deeper universal structure in the infrared behavior of scattering amplitudes, at least at tree level. Whether this universality persists once loop effects are included remains an open and nontrivial question, to which we now turn.

3.2 Infrared divergences

Theories with long-range interactions or massless particles inevitably encounter the problem of *infrared (IR) divergences*. These arise because of a virtual graviton or photon in the loops accessing the soft energy region. In such situations, the loop integrals develop divergences in the small momentum region [124].

Weinberg showed that these divergences can be *exponentiated*, allowing the amplitude to be factorized into a divergent prefactor and a finite remainder. For the case of IR divergences originating from *virtual graviton* loops, the factorization takes the form [9]

$$\mathcal{M}_n=\exp\left[\frac{1}{\epsilon}\frac{\kappa^2}{2(8\pi)^2}\sum_{ij}\eta_i\eta_j\,m_im_j\,\frac{1+\beta_{ij}^2}{\beta_{ij}\sqrt{1-\beta_{ij}^2}}\left(i\pi\,\delta_{\eta_i,\eta_j}-\frac{1}{2}\ln\frac{1+\beta_{ij}}{1-\beta_{ij}}\right)\right]\mathcal{M}_n^{\text{finite}}\,.\tag{3.2.1}$$

Here ϵ is the dimensional regularization parameter, with loop integrals evaluated in $d = 4 - 2\epsilon$ dimensions, and $\kappa = \sqrt{32\pi G}$ is the gravitational coupling constant.

In the above expression, the relative velocity between particles i and j is defined as

$$\beta_{ij} \equiv \sqrt{1 - (\hat{p}_i \cdot \hat{p}_j)^{-2}}\,, \tag{3.2.2}$$

where $\hat{p}_i$ denotes the normalized momentum satisfying $\hat{p}_i^2 = -1$. For timelike momenta $\hat{p}_i, \hat{p}_j$ we note that $0 \leq \beta_{ij} < 1$. If eiter momentum is null, then $\beta_{ij} = 1$ and the massless limit of the above expression must be carefully taken. It is convenient to introduce the shorthand quantities[1]

$$\sigma_n = \frac{1}{2(8\pi)^2} \sum_{ij} \eta_i \eta_j\, m_i m_j\, \frac{1+\beta_{ij}^2}{\beta_{ij}\sqrt{1-\beta_{ij}^2}} \left(i\pi\, \delta_{\eta_i,\eta_j} - \frac{1}{2} \ln \frac{1+\beta_{ij}}{1-\beta_{ij}} \right)\,, \tag{3.2.3}$$

$$\hat{\sigma}'_{n+1}(\hat{q}) = \frac{1}{2(4\pi)^2} \sum_{i=1}^{n} (p_i \cdot \hat{q})\, \ln(\hat{p}_i \cdot \hat{q})\,. \tag{3.2.4}$$

With this notation, the factorization of the amplitude simplifies to

$$\mathcal{M}_n = \exp\left(-\frac{\kappa^2}{\epsilon}\, \sigma_n \right) \mathcal{M}_n^{\text{finite}}\,. \tag{3.2.5}$$

The function σ'_{n+1} will prove useful later because when factorizing an amplitude including a graviton, the exponential factor can be expressed as $\sigma_{n+1} = \sigma_n + \sigma'_{n+1}$.

The exponential factor contains both a real and an imaginary part. The imaginary part is often referred to as the *Coulomb phase*, which in the case of electromagnetism arises from the long-range $1/r$ behavior of the Coulomb potential.

Unlike ultraviolet divergences, which can often be absorbed into a redefinition of parameters,[2] infrared divergences cannot be removed in this way. However, as shown by Bloch and Nordsieck [125] for QED (see also [55]), Kinoshita, Lee, and Nauenberg [126, 127] for the Standard Model, and later by Weinberg for gravity [9], the divergences *cancel* in inclusive cross sections once one sums over final states that differ by the emission of arbitrarily soft bosons. Since the measurable quantity in experiments such as those at the LHC is the *cross section* [128] rather than the S-matrix element itself, these IR divergences were historically viewed as harmless. In the modern context, however, with the renewed interest in understanding the structure of the S-matrix itself, these divergences become physically significant.

For *virtual soft photon* loops, the factorization takes a slightly different form:

$$\mathcal{M}_n = e^{-\frac{1}{\epsilon}\lambda_n}\, \mathcal{M}_n^{\text{finite}}\,, \tag{3.2.6}$$

where the phase factor λ_n is defined for compactness as

$$\lambda_n = \frac{1}{16\pi^2} \sum_{ij} \eta_i \eta_j\, e_i e_j\, \beta_{ij}^{-1} \left(i\pi\, \delta_{\eta_i,\eta_j} - \frac{1}{2} \ln \frac{1+\beta_{ij}}{1-\beta_{ij}} \right)\,. \tag{3.2.7}$$

Finally, consider the case where one of the particles in β_{ij} becomes massless. The above expressions simplify to:

$$\begin{aligned} \sigma_{ij} &= \frac{1}{(8\pi)^2}\, \eta_i \eta_j\, |p_i \cdot p_j| \left(i\pi\, \delta_{\eta_i,\eta_j} - \ln|\hat{p}_i \cdot \hat{p}_j| \right) = \frac{1}{(8\pi)^2}\, (p_i \cdot p_j)\, \ln(\hat{p}_i \cdot \hat{p}_j)\,, \\ \lambda_{ij} &= \frac{1}{8\pi^2} \eta_i \eta_j e_i e_j \ln(\hat{p}_i \cdot \hat{p}_j)\,. \end{aligned} \tag{3.2.8}$$

This massless limit will be particularly relevant when discussing loop corrections to the soft theorems in the next subsection.

[1] $\delta_{\eta_i,\eta_j} = 1$ only for $\eta_i = \eta_j$ and $i \neq j$, otherwise it is 0.
[2] See for example [8].

3.2.1 Corrections to the subleading soft factor

In Section 3.1.1, we saw that the leading soft factor is exact to all orders in perturbation theory, both for gravitons and photons. This robustness follows from its derivation via general arguments that do not depend on the details of the dynamics beyond gauge invariance and Lorentz symmetry. By contrast, the subleading soft factor was originally derived from explicit tree-level computations and is therefore natural to question whether loop corrections can modify it.

Bern, Davies, and Nohle [129] addressed this question by analyzing the infrared (IR) divergent structure of scattering amplitudes. Their key observation is that when loop effects are included, the subleading soft factor acquires a universal IR-divergent correction. We now briefly review their result for the case of soft graviton emission in a scattering process involving massive external particles.

Gravity case. From Eq. (3.2.5), the n-point amplitude factorizes in the presence of IR divergences as

$$\mathcal{M}_n = e^{-\frac{\kappa^2}{\epsilon}\sigma_n}\,\mathcal{M}_n^{\text{finite}}, \tag{3.2.9}$$

where σ_n is given in Eq. (3.2.3). If we now consider an $(n+1)$-point amplitude in which the additional particle is a graviton with momentum $q^\mu = \omega\hat{q}^\mu$ that will be taken soft, the corresponding factorization reads

$$\mathcal{M}_{n+1} = e^{-\frac{\kappa^2}{\epsilon}\sigma_n}\, e^{-\frac{\kappa^2}{\epsilon}\sigma'_{n+1}(\hat{q})}\,\mathcal{M}_{n+1}^{\text{finite}}, \tag{3.2.10}$$

with $\sigma'_{n+1}(\hat{q})$ defined in Eq. (3.2.4). Here, the two exponentials separate the contribution from the hard n-particle configuration (σ_n) and the additional to-be-soft graviton (σ'_{n+1}).

Starting from the soft expansion ansatz

$$S(\omega) = \frac{1}{\omega}S_{\text{gr}}^{(0)} + S_{\text{gr}}^{(1)} + \omega\, S_{\text{gr}}^{(2)} + \mathcal{O}(\omega^2), \tag{3.2.11}$$

and taking the soft limit of the graviton emitted in the $(n+1)$-point amplitude we have,

$$\mathcal{M}_{n+1} = e^{-\frac{\kappa^2}{\epsilon}\sigma_n}\, e^{-\frac{\kappa^2}{\epsilon}\sigma'_{n+1}(\hat{q})}\left[\frac{1}{\omega}S_{\text{gr}}^{(0)} + S_{\text{gr}}^{(1)} + \omega\, S_{\text{gr}}^{(2)} + \mathcal{O}(\omega^2)\right]\mathcal{M}_n^{\text{finite}}. \tag{3.2.12}$$

On the left hand side of this equation, we shall instead take the soft limit first and then factorize,

$$\begin{aligned}
&\left[\frac{1}{\omega}S_{\text{gr}}^{(0)} + S_{\text{gr}}^{(1)} + \omega\, S_{\text{gr}}^{(2)} + \mathcal{O}(\omega^2)\right]\mathcal{M}_n \\
&= e^{-\kappa^2\frac{1}{\epsilon}\sigma_n} e^{-\frac{\kappa^2}{\epsilon}\sigma'_{n+1}(\hat{q})}\left[\frac{1}{\omega}S_{\text{gr}}^{(0)} + S_{\text{gr}}^{(1)} + \omega\, S_{\text{gr}}^{(2)} + \mathcal{O}(\omega^2)\right]\mathcal{M}_n^{\text{finite}} \\
&\left[\frac{1}{\omega}S_{\text{gr}}^{(0)} + S_{\text{gr}}^{(1)} + \omega\, S_{\text{gr}}^{(2)} + \mathcal{O}(\omega^2)\right] e^{-\kappa^2\frac{1}{\epsilon}\sigma_n}\mathcal{M}_n^{\text{finite}} \\
&= e^{-\kappa^2\frac{1}{\epsilon}\sigma_n} e^{-\frac{\kappa^2}{\epsilon}\sigma'_{n+1}(\hat{q})}\left[\frac{1}{\omega}S_{\text{gr}}^{(0)} + S_{\text{gr}}^{(1)} + \omega\, S_{\text{gr}}^{(2)} + \mathcal{O}(\omega^2)\right]\mathcal{M}_n^{\text{finite}}
\end{aligned} \tag{3.2.13}$$

Matching the infrared divergences on the two sides, shows that the subleading soft factor must receive an additional IR-divergent term:

$$\lim_{\omega\to 0}\,(1+\omega\,\partial_\omega)\,\mathcal{M}_{n+1} = \left[S_{\text{gr}}^{(1)} + \frac{1}{\epsilon}\sigma'_{n+1}\,S_{\text{gr}}^{(0)} - \frac{1}{\epsilon}S_{\text{gr},i}^{(1)J}\,\sigma_n^i\right]\mathcal{M}_n, \tag{3.2.14}$$

where $S^{(1)J}_{\mathrm{gr},i}$ denotes the *orbital* part of the subleading soft operator acting only on the i^{th} hard particle, and σ_n^i denotes the contribution of that particle to σ_n. Explicitly,

$$S^{(1)J}_{\mathrm{gr},i}\,\sigma_n^i = -\frac{i\kappa}{8(8\pi)^2}\sum_{i=1}^{n}\frac{\varepsilon_{\mu\nu}\,q_\rho\,p_i^\nu}{p_i\cdot q}\,J_i^{\mu\rho}\sum_j \eta_i\eta_j\,m_i m_j\,\frac{1+\beta_{ij}^2}{\beta_{ij}\sqrt{1-\beta_{ij}^2}}\left(i\pi\,\delta_{\eta_i,\eta_j}-\frac{1}{2}\ln\frac{1+\beta_{ij}}{1-\beta_{ij}}\right). \tag{3.2.15}$$

The infrared divergent piece of Eq. (3.2.14) is also seen to be universal in its dependence on the details of the external hard particles, it only depends on their momenta. Note that this derivation only points to the loop corrections that are infrared divergent, the loop corrections that are infrared finite may continue to add up to all loop orders.

Subsubleading order. A similar analysis can be carried out for the subsubleading soft factor. Matching the IR-divergent terms up to two loops yields, for the most divergent piece $(1/\epsilon^2)$,

$$\begin{aligned}&\lim_{\omega\to 0}\partial_\omega\,(1+\omega\,\partial_\omega)\,\mathcal{M}_{n+1}\Big|_{1/\epsilon^2}\\&\quad=\left[(\sigma'_{n+1})^2 S^{(0)}_{\mathrm{gr}}+\sigma'_{n+1}S^{(1)}_{\mathrm{gr}}\,\sigma_n+\sum_i\frac{\varepsilon^{\mu\nu}\hat{q}^\rho\hat{q}^\sigma}{p_i\cdot\hat{q}}\left(J^i_{\mu\rho}\,\sigma_n\right)\left(J^i_{\nu\sigma}\,\sigma_n\right)\right]\mathcal{M}_n.\end{aligned} \tag{3.2.16}$$

The less singular $1/\epsilon$ part receives contributions at both one and two loops and is not universal. In general, the $(\text{sub})^k$-leading soft factor receives IR-divergent loop corrections up to k loops.

QED case. For photons, the situation is simpler because there is no photon self-interaction. The IR factorization for the n-point amplitude is

$$\mathcal{M}_n = e^{-\frac{1}{\epsilon}\lambda_n}\,\mathcal{M}_n^{\text{finite}}, \tag{3.2.17}$$

with λ_n defined in Eq. (3.2.7).

Repeating the same mismatch analysis between taking the soft limit before and after IR factorization, we find

$$\lim_{\omega\to 0}(1+\omega\,\partial_\omega)\,\mathcal{M}_{n+1} = \left[S^{(1)}_{\mathrm{em}}-\frac{1}{\epsilon}S^{(1)J}_{\mathrm{em},i}\,\lambda_n^i\right]\mathcal{M}_n. \tag{3.2.18}$$

Mixed gravity + QED case. When the hard particles carry both mass and charge, IR divergences arise from both virtual graviton and photon exchanges. The corrections to the soft graviton theorem in this case are

$$\lim_{\omega\to 0}(1+\omega\,\partial_\omega)\,\mathcal{M}_{n+1} = \left[S^{(1)}_{\mathrm{gr}}+\frac{1}{\epsilon}\sigma'_{n+1}S^{(0)}_{\mathrm{gr}}-\frac{1}{\epsilon}S^{(1)J}_{\mathrm{gr},i}\,\sigma_n^i+\frac{1}{\epsilon}S^{(1)J}_{\mathrm{gr},i}\,\lambda_n^i\right]\mathcal{M}_n, \tag{3.2.19}$$

and the corrections to the soft photon theorem read

$$\lim_{\omega\to 0}(1+\omega\,\partial_\omega)\,\mathcal{M}_{n+1} = \left[S^{(1)}_{\mathrm{em}}+\frac{1}{\epsilon}\sigma'_{n+1}S^{(0)}_{\mathrm{em}}-\frac{1}{\epsilon}S^{(1)J}_{\mathrm{em},i}\,\sigma_n^i+\frac{1}{\epsilon}S^{(1)J}_{\mathrm{em},i}\,\lambda_n^i\right]\mathcal{M}_n, \tag{3.2.20}$$

The loop-induced IR-divergent corrections to the subleading soft factors have a clear origin: they reflect the fact that soft emission and IR factorization do not commute beyond leading order. In the leading soft theorem, the emission of a very low-energy boson can be factorized cleanly, because its effect on the rest of the process is purely eikonal and independent of the details of the hard scattering. At subleading order, however, the soft emission operator involves angular momentum generators acting on the hard legs, which do not commute with the IR-divergent exponential factors arising from virtual soft exchanges. This mismatch produces new universal $1/\epsilon$ (and at higher orders $1/\epsilon^2$) terms whose structure is fixed entirely by the kinematics and charges/masses of the external states. In this way, the corrections are not a breakdown of universality but rather a shift of the universal structure to include the interplay between real and virtual soft modes. The fact that these corrections can be expressed in a universal form for the most divergent pieces strengthens the interpretation of soft theorems as symmetry statements valid even in the presence of loop effects, provided the IR structure is properly accounted for.

3.3 Soft theorems as Ward identities

The universal structure of soft factorization, long regarded as an intriguing coincidence, eventually found a deeper explanation in terms of hidden symmetries of the S-matrix. In a landmark series of works [11, 12, 102, 112, 116, 130], it was demonstrated that the soft theorems can be reinterpreted as Ward identities associated with asymptotic symmetries—precisely the large gauge transformations and BMS transformations we encountered in Chapter 2.

This realization unites two previously separate lines of investigation: on the one hand, the study of infrared behavior in scattering amplitudes and their universal soft limits; on the other, the classification of symmetries at null and time-like infinity. From this perspective, the universal terms in the soft expansion are not accidental but are instead fixed by symmetry principles acting on the gravitational or gauge fields at asymptotic boundaries.

In the remainder of this section, we will make this correspondence precise, showing explicitly how the insertion of a soft particle in the S-matrix is determined a Ward identity for an asymptotic symmetry. This will also clarify how the charges defined in the asymptotic phase space formalism translate directly into constraints on scattering amplitudes.

3.3.1 Leading soft graviton theorem

The leading soft graviton theorem can be reinterpreted as a Ward identity for the S-matrix, arising from the conservation of supertranslation charge across spatial infinity [12, 103]. To make this connection precise, we begin with the expression for the total supertranslation flux, including both massless and massive field contributions, as derived in Chapter 2:

$$
\begin{aligned}
\mathcal{F}_{\mathcal{T}}^{\text{soft}} &= \frac{1}{8\pi G}\int d^2z\,\mathcal{T}(z,\bar z)\,\bar\partial^2\mathcal{N}_{zz}^{(0)}\,,\\
\mathcal{F}_{\mathcal{T}}^{\text{hard}} &= -\frac{1}{16\pi G}\int du\,d^2z\,\mathcal{T}(z,\bar z)\left(\tilde N_{zz}\tilde N_{\bar z\bar z}+16\pi G\,T_{uu}^{(2)}\right)\\
Q_{\mathcal{T}}^{i^+} &= \frac{1}{4}\int d^3Y\,\mathcal{T}_{\mathcal{H}}(Y)\,T_{\tau\tau}^{(3)}(Y)\,.
\end{aligned}
\tag{3.3.1}
$$

The total flux is thus a sum of a *soft* term, acting on the soft phase space, and a *hard* term, acting on the matter and radiative phase space. $\mathcal{T}$ is the function parameterizing supertranslations and $\mathcal{T}_{\mathcal{H}}$ is its extension to timelike infinity. From Chapter 2, we also recall the action of the hard supertranslation charge on the leading phase space variables:

$$
\begin{aligned}
\left\{F_{\mathcal{T}}^{\text{hard}},\tilde C_{zz}(z,\bar z)\right\} &= \mathcal{T}(z,\bar z)\,\partial_u\tilde C_{zz}\,,\\
\left[F_{\mathcal{T}}^{\text{hard}},\,a^\dagger(\omega,z,\bar z)\right] &= \mathcal{T}(z,\bar z)\,\omega\,a^\dagger(\omega,z,\bar z)\,,\\
\left[Q_{\mathcal{T}}^{i^+},b(p)\right] &= -i\,m\,\mathcal{T}_{\mathcal{H}}(\hat p)\,b(p)\,.
\end{aligned}
\tag{3.3.2}
$$

The first line follows directly from the Poisson bracket structure of the gravitational radiative phase space (2.3.52) upon using eq. (2.3.47), while the remaining lines describe the action on creation operators for massless particles and on annihilation operators for massive particles of arbitrary spin.

To relate the gravitational part of the hard charge to an S-matrix statement, we must express $\tilde{C}_{zz}$ in terms of the ladder operators. Recalling the large-r expansion from Eq. (2.3.5), the leading Bondi shear is proportional to the graviton field at null infinity. On $\mathscr{I}^+$, this takes the form

$$\tilde{C}_{zz}(u,z,\bar{z}) = \kappa \lim_{r\to\infty} \frac{1}{r}\, h^{\text{out}}_{zz}(r,u,z,\bar{z})\,. \tag{3.3.3}$$

With this identification, the *soft* part of the supertranslation charge can be written explicitly in terms of the creation and annihilation operators for gravitons. From Eq. (2.3.48) in Chapter 2, we have

$$\mathcal{N}^{(0)}_{zz}(z,\bar{z}) = -\frac{\kappa}{8\pi^2} \lim_{\omega\to 0^+} \omega \left[a_+(\omega,z,\bar{z}) + a_-^\dagger(\omega,z,\bar{z})\right], \tag{3.3.4}$$

where the ladder operators $a_\pm(\omega,z,\bar{z})$ create or annihilate gravitons of definite helicity. The helicity index is carried implicitly, with the polarization tensor for positive helicity given by

$$\varepsilon^{+\mu\nu}(\hat{q}) = \partial_z \hat{q}^\mu\, \partial_z \hat{q}^\nu\,,$$

where $\hat{q}^\mu(z,\bar{z})$, parametrizing the null direction of the graviton momentum, is given by Eq. (2.2.3).

From Eq. (3.3.4), it is evident that $\mathcal{N}^{(0)}_{zz}$ precisely inserts or removes a zero-energy graviton, thereby shifting the vacuum state. This observation underpins the connection between soft graviton insertions and vacuum degeneracy under supertranslation action.

The conservation of supertranslation charge across spatial infinity follows from imposing *antipodal matching* of the fluxes at $\mathscr{I}^+$ and $\mathscr{I}^-$, as shown in section 2.5. In operator language, this is equivalent to the statement that the full supertranslation charge commutes with the S-matrix:

$$\left[Q_{\mathcal{T}}(i^0),\, \mathcal{S}\right] = 0\,. \tag{3.3.5}$$

Equation (3.3.5) is the supertranslation Ward identity. In terms of the fluxes and timelike charges, the antipodal map can be expressed as,

$$Q_{\mathcal{T}}(i^0) = Q_{\mathcal{T}}(\mathscr{I}^+_+) - \mathcal{F}_{\mathcal{T}}(\mathscr{I}^+) = Q_{\mathcal{T}}(\mathscr{I}^-_-) + \mathcal{F}_{\mathcal{T}}(\mathscr{I}^-) \tag{3.3.6}$$

Here $\mathcal{F}_{\mathcal{T}}$ denotes the supertranslation flux evaluated at the future ($\mathscr{I}^+$) or past ($\mathscr{I}^-$) null boundary, while $Q_{\mathcal{T}}(\mathscr{I}^+_+) = Q^{i^+}_{\mathcal{T}_\mathcal{H}}$ and $Q_{\mathcal{T}}(\mathscr{I}^-_-) = Q^{i^-}_{\mathcal{T}_\mathcal{H}}$ are the charges defined at future and past timelike infinity, acting on massive states.

Using the Eqs. (3.3.5) and (3.3.6), the conservation of supertranslation charge across spatial infinity can be imposed on the S-matrix as

$$\langle \text{out}|\left(\mathcal{F}_{\mathcal{T}}(\mathscr{I}^+) - Q_{\mathcal{T}}(\mathscr{I}^+_+)\right)\mathcal{S} + \mathcal{S}\left(Q_{\mathcal{T}}(\mathscr{I}^-_-) + \mathcal{F}_{\mathcal{T}}(\mathscr{I}^-)\right)|\text{in}\rangle = 0\,. \tag{3.3.7}$$

Splitting both sides of Eq. (3.3.7) into their soft and hard components, we obtain

$$\begin{aligned}&\langle \text{out}|\left(\mathcal{F}^{\text{hard}}_{\mathcal{T}}(\mathscr{I}^+) - Q_{\mathcal{T}}(\mathscr{I}^+_+)\right)\mathcal{S} + \mathcal{S}\left(Q_{\mathcal{T}}(\mathscr{I}^-_-) + \mathcal{F}^{\text{hard}}_{\mathcal{T}}(\mathscr{I}^-)\right)|\text{in}\rangle \\ &\quad = -\langle \text{out}|\,\mathcal{F}^{\text{soft}}_{\mathcal{T}}(\mathscr{I}^+)\,\mathcal{S} + \mathcal{S}\,\mathcal{F}^{\text{soft}}_{\mathcal{T}}(\mathscr{I}^-)\,|\text{in}\rangle.\end{aligned} \tag{3.3.8}$$

Using Eq. (3.3.4), the soft flux on $\mathscr{I}^+$ can be expressed in terms of graviton creation and annihilation operators. Crossing symmetry allows us to rewrite an incoming graviton insertion as an outgoing graviton of opposite helicity [8]. Thus, the right-hand side of Eq. (3.3.8) becomes

$$\lim_{\omega\to 0} \omega \int d^2z\, \mathcal{T}(z,\bar{z})\, \bar{\partial}^2 \langle \text{out}|\, a_+(\omega,z,\bar{z})\,\mathcal{S}\,|\text{in}\rangle. \tag{3.3.9}$$

From the hard charge action given in Eq. (3.3.2), the left-hand side of Eq. (3.3.8) evaluates to

$$\left(\sum_{i \text{ massless}} \mathcal{T}(z_i, \bar{z}_i)\, \omega_i + \sum_{i \text{ massive}} m_i\, \mathcal{T}_{\mathcal{H}}(\rho_i, z_i, \bar{z}_i) \right) \langle \text{out}|\, \mathcal{S}\, |\text{in}\rangle . \tag{3.3.10}$$

The Ward identity takes its simplest form for the meromorphic choice

$$\mathcal{T}(z, \bar{z}) = \frac{\bar{z} - \bar{w}}{z - w}\,. \tag{3.3.11}$$

Inserting Eq. (3.3.11) into Eqs. (3.3.9)–(3.3.10), using the momentum parametrization of section 2.2, and using the identities [68]

$$\begin{aligned} \partial_{\bar{z}}^2 \frac{\bar{z} - \bar{w}}{z - w} &= 2\pi\, \delta^2(z - w), \\ \mathcal{G}^{(3)}(\rho_i, z_i, \bar{z}_i; z, \bar{z}) &= -\frac{1}{2\pi\, m_i} \partial_{\bar{z}}^2\, S_{\text{gr},i}^{(0)+}(m_i, \rho_i, z_i, \bar{z}_i; z, \bar{z}), \\ S_i^{(0)+}(\omega_i, z_i, \bar{z}_i; z, \bar{z}) &= \omega_i \frac{\bar{z}_i - \bar{z}}{z_i - z}, \end{aligned} \tag{3.3.12}$$

we find that Eq. (3.3.8) reduces to

$$\lim_{\omega \to 0} \omega\, \langle \text{out}|\, a_+(\omega, w, \bar{w})\, \mathcal{S}\, |\text{in}\rangle = S_{\text{gr}}^{(0)+}\, \langle \text{out}|\, \mathcal{S}\, |\text{in}\rangle, \tag{3.3.13}$$

which is precisely the leading soft graviton theorem.

From the soft theorem back to the Ward identity. Conversely, one can *derive* the supertranslation Ward identity starting from the leading soft theorem. Apply the projector

$$\int d^2 z\, \mathcal{T}(z, \bar{z})\, \bar{\partial}^2 \tag{3.3.14}$$

to both sides of the leading soft relation (3.3.13). Using the identities in (3.3.12), integrating by parts on the celestial sphere, and substituting the hard action rules (3.3.2), one immediately recovers the Ward identity (3.3.8). In this way, the entire *family* of supertranslation Ward identities follows from the single soft theorem, with different choices of the meromorphic test function $\mathcal{T}(z, \bar{z})$ selecting the corresponding charge.

Since the leading soft factor is non-perturbatively exact, this equivalence shows that supertranslations are exact symmetries of the full quantum-gravity S-matrix. Finally, note that in the discussion above we projected onto the + helicity by choosing $\mathcal{T}(z, \bar{z}) = \frac{\bar{z} - \bar{w}}{z - w}$. The opposite helicity follows by taking the complex conjugate flux (which is the same because it follows from Eq. (2.3.44) that $\partial^2 \mathcal{N}_{\bar{z}\bar{z}}^{(0)} = \bar{\partial}^2 \mathcal{N}_{zz}^{(0)}$) with the meromorphic choice,

$$\mathcal{T}(z, \bar{z}) = \frac{z - w}{\bar{z} - \bar{w}}\,, \tag{3.3.15}$$

which reproduces the negative-helicity soft graviton theorem and the same Ward identity.

3.3.2 Subleading soft graviton theorem

Given the success at leading order, it is natural to ask whether the *subleading* soft factorization also admits a symmetry interpretation. The answer turns out to be more subtle: while a tree-level correspondence with asymptotic symmetries was found in [112], loop effects modify the structure, as discussed in Section 3.2.1. In this subsection we follow the steps of [102, 103, 112] to outline the tree-level derivation in close analogy to the leading case, while highlighting the differences that arise beyond tree level.

Analogous to (3.3.4), the subleading soft mode in Eq. (2.3.48) can be expressed in terms of creation and annihilation operators as

$$\mathcal{N}^{(1)}_{zz}(z,\bar z) = -\frac{\kappa}{8\pi^2}\lim_{\omega\to 0^+}(1+\omega\partial_\omega)\left[a_+(\omega,z,\bar z)+a^\dagger_-(\omega,z,\bar z)\right]. \tag{3.3.16}$$

This expression follows from expanding the shear in terms of ladder operators and replacing u inside the du integral with a ω derivative acting on the exponential $e^{\pm i\omega u}$.

At tree level, the subleading soft theorem for an outgoing + helicity graviton reads

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)\langle \text{out}|a_+(\omega,z,\bar z)\mathcal{S}|\text{in}\rangle = S^{(1)}_{\text{gr}}\langle\text{out}|\mathcal{S}|\text{in}\rangle\,, \tag{3.3.17}$$

where $S^{(1)}$ acts as a differential operator on the external hard states. For a *massless* hard particle with momentum $\omega_i\hat q_i(z_i,\bar z_i)$ and helicity h, we can express the subleading soft factor using Eq. (2.2.3) as [102],

$$S^{(1)+}_{\text{gr},i} = \frac{(\bar z_i-\bar z)^2}{z_i-z}\,\partial_{\bar z} - \frac{1}{2}\left[\partial_{\bar z}\frac{(\bar z_i-\bar z)^2}{z_i-z}\right]\omega_i\,\partial_{\omega_i} + \frac{h}{2}\left[\partial_{\bar z}\frac{(\bar z_i-\bar z)^2}{z_i-z}\right]. \tag{3.3.18}$$

Interestingly, from Eq. (2.4.4), this term can be seen as the action of superrotation flux on a massless particle with the supperrotation parameter given by $\bar{\mathcal{Y}} = \frac{(\bar z_i-\bar z)^2}{z_i-z}$.

For *massive* hard particles the corresponding identity takes the form [68]

$$\mathcal{G}^\alpha_{\bar z}(\rho_i,z_i,\bar z_i;z,\bar z)\,\partial_\alpha = -\frac{1}{4\pi}\partial^3_{\bar z}S^{(1)+}_{\text{gr},i}(\rho_i,z_i,\bar z_i;z,\bar z)\,, \tag{3.3.19}$$

as can be verified from eqs. (2.4.6) and (3.1.5).

Using the above expressions, the subleading soft theorem may be written as

$$\begin{aligned}&\frac{8\pi}{\kappa}\langle\text{out}|\mathcal{N}^{(1)}_{zz}(z,\bar z)\,\mathcal{S}\,|\text{in}\rangle\\ &\quad=\sum_{i\,\text{massless}}\langle\text{out}|\big[\mathcal{F}^{\text{hard}}_{\bar{\mathcal{Y}}=\frac{(\bar z_i-\bar z)^2}{z_i-z}},\mathcal{S}\big]|\text{in}\rangle + \sum_{i\,\text{massive}}S^{(1)+}_{\text{gr},i}(\rho_i,z_i,\bar z_i;z,\bar z)\,\langle\text{out}|\mathcal{S}|\text{in}\rangle.\end{aligned} \tag{3.3.20}$$

Here we have expressed the massless part at the action of superrotation $\bar{\mathcal{Y}}(z,\bar z)$, while left the massive part without any change for now.

Acting on both sides with

$$\int d^2z\,\bar{\mathcal{Y}}\,\bar\partial^3 \tag{3.3.21}$$

and using the identity

$$\bar\partial^3\frac{(\bar z_i-\bar z)^2}{z_i-z} = 2\pi\delta^2(z-z_i)\,,$$

one obtains

$$\frac{8\pi}{\kappa}\int d^2z\,\bar{\mathcal{Y}}\,\langle\text{out}|\bar\partial^3\mathcal{N}^{(1)}_{\bar z\bar z}(z,\bar z)\,\mathcal{S}\,|\text{in}\rangle = \langle\text{out}|\big[\mathcal{F}^{\text{hard}}_{\bar{\mathcal{Y}}}+Q^{i^+}_{\bar{\mathcal{Y}}},\mathcal{S}\big]|\text{in}\rangle. \tag{3.3.22}$$

n expressing the massive part as charge on timelike infinity, we have made use of Eqs. (3.3.19) and (2.4.10). Adding the contribution from the opposite helicity mode yields

$$\frac{8\pi}{\kappa}\int d^2z\,\langle\text{out}|(\bar{\mathcal{Y}}\,\bar\partial^3\mathcal{N}^{(1)}_{zz}+\mathcal{Y}\,\partial^3\mathcal{N}^{(1)}_{\bar z\bar z})\mathcal{S}|\text{in}\rangle = \langle\text{out}|\big[\mathcal{F}^{\text{hard}}_{\mathcal{Y},\bar{\mathcal{Y}}}+Q^{i^+}_{\mathcal{Y},\bar{\mathcal{Y}}},\mathcal{S}\big]|\text{in}\rangle. \tag{3.3.23}$$

This is the Ward identity for the charge

$$Q_{\mathcal{Y},\bar{\mathcal{Y}}} = \frac{2}{\kappa^2}\int d^2z\,(\mathcal{Y}\,\partial^3_z\mathcal{N}^{(1)}_{zz}+\bar{\mathcal{Y}}\,\partial^3_{\bar z}\mathcal{N}^{(1)}_{\bar z\bar z}) + \mathcal{F}^{\text{hard}}_{\mathcal{Y},\bar{\mathcal{Y}}} + Q^{i^+}_{\mathcal{Y},\bar{\mathcal{Y}}}\,, \tag{3.3.24}$$

which matches the superrotation charge defined in Eq. (2.3.56) up to terms in the soft flux.

At tree level this Ward identity is equivalent to the subleading soft theorem. However, as reviewed in Section 3.2.1, the subleading soft factor receives infrared-divergent loop corrections, already at one-loop. Understanding the precise nature of these corrections is a central question that we will return to in the next chapter. Moreover, to derive the full Ward identity of the superrotation flux in Eq. (2.3.56) requires understanding how to quantize the Goldstone mode. We will be dealing with this in the next sections.

3.3.3 Leading soft photon theorem

For the case of the leading soft photon theorem, most of the steps from the graviton analysis carry over directly, with the only difference being that the relevant asymptotic symmetry is the *large gauge transformation* of electromagnetism, presented in section 2.6.

The large gauge charge on future null infinity $\mathscr{I}^+$, including both massless and massive charged matter, can be written as

$$Q^+_\varepsilon = -\frac{1}{e^2}\int_{\mathscr{I}^+} d^2z\,\varepsilon\,\bar\partial\,\mathscr{F}^{(0)}_z + \int_{\mathscr{I}^+} du\,d^2z\,\varepsilon\,j^{(2)}_u + \int_{\mathcal{H}^+} d^3Y\,\varepsilon_{\mathcal{H}}\,j^{(3)}_\tau\,, \tag{3.3.25}$$

where $\mathscr{F}^{(0)}_z$ was defined in Eq. (2.6.12), $\varepsilon(z,\bar z)$ is the large gauge parameter on $\mathscr{I}^+$ and $\varepsilon_{\mathcal{H}}$ its extension onto time-like infinity $\mathcal{H}^+$, given by Eq. (2.6.16), and j is the $U(1)$ matter current.

Imposing antipodal matching across spatial infinity i^0 amounts to demanding the conservation of the large gauge charge:

$$\langle\text{out}|\left(Q^{\text{hard}}_\varepsilon(\mathscr{I}^+)\,\mathcal{S} - \mathcal{S}\,Q^{\text{hard}}_\varepsilon(\mathscr{I}^-)\right)|\text{in}\rangle = -\langle\text{out}|\left(Q^{\text{soft}}_\varepsilon(\mathscr{I}^+)\,\mathcal{S} - \mathcal{S}\,Q^{\text{soft}}_\varepsilon(\mathscr{I}^-)\right)|\text{in}\rangle\,. \tag{3.3.26}$$

The hard part of the charge acts on creation operators for charged particles as

$$\begin{aligned}\left[Q^{\text{hard}}_\varepsilon,\, b^\dagger(\omega,z,\bar z)\right] &= iq\,\varepsilon(z,\bar z)\,b^\dagger(\omega,z,\bar z)\,,\\ \left[Q^{\text{hard}}_\varepsilon,\, d^\dagger(m,\rho,z,\bar z)\right] &= iq\,\varepsilon_{\mathcal{H}}(\rho,z,\bar z)\,d^\dagger(m,\rho,z,\bar z)\,,\end{aligned} \tag{3.3.27}$$

where $b^\dagger$ and $d^\dagger$ create massless and massive charged particles of electric charge q, respectively.

The soft part of the charge is expressed in terms of the photon field modes by identifying $\tilde A^{(0)}_z$ with the free photon field on $\mathscr{I}^+$:

$$\mathscr{F}^{(0)}_z(z,\bar z) = \frac{1}{4\pi}\lim_{\omega\to 0^+}\omega\left[a_+(\omega,z,\bar z) + a^\dagger_-(\omega,z,\bar z)\right], \tag{3.3.28}$$

where $a_\pm(\omega,z,\bar z)$ annihilate outgoing photons of definite helicity.

Substituting (3.3.27) and (3.3.28) into (3.3.26), and using crossing symmetry to replace an incoming photon with an outgoing photon of opposite helicity, the Ward identity takes the form

$$\begin{aligned}&\lim_{\omega\to 0}\frac{\omega}{4\pi}\int d^2z\,\varepsilon(z,\bar z)\,\bar\partial\,\langle\text{out}|a_+(\omega,z,\bar z)\mathcal{S}|\text{in}\rangle\\ &\qquad = \left(\sum_{\text{massless}} iq_i\,\varepsilon(z_i,\bar z_i) + \sum_{\text{massive}} iq_i\,\varepsilon_{\mathcal{H}}(\rho_i,z_i,\bar z_i)\right)\langle\text{out}|\mathcal{S}|\text{in}\rangle.\end{aligned} \tag{3.3.29}$$

To recover the soft theorem, we make the choice of gauge parameter

$$\varepsilon(z,\bar z) = \frac{1}{z-w}\,, \tag{3.3.30}$$

which matches the + helicity leading soft photon factor in the stereographic parametrization of section 2.2. This choice, together with the identities

$$\begin{aligned} \bar{\partial}\frac{1}{z-w} &= 2\pi\,\delta^{(2)}(z-w)\,, \\ \mathcal{G}^{(2)}(\rho_i, z_i, \bar{z}_i; z, \bar{z}) &= \frac{1}{e_i}\bar{\partial}\, S^{(0)+}_{\text{em},i}(\rho_i, z_i, \bar{z}_i; z, \bar{z})\,, \end{aligned} \tag{3.3.31}$$

reduces (3.3.29) to

$$\lim_{\omega\to 0}\omega\,\langle \text{out}|a_+(\omega, w, \bar{w})\mathcal{S}|\text{in}\rangle = S^{(0)+}_{\text{em}}\,\langle \text{out}|\mathcal{S}|\text{in}\rangle\,, \tag{3.3.32}$$

which is precisely the leading soft photon theorem.

The derivation can also be reversed: starting from the soft theorem, acting with

$$\int d^2z\,\varepsilon(z,\bar{z})\,\bar{\partial} \tag{3.3.33}$$

on both sides, and using the identities in (3.3.31), one recovers the Ward identity (3.3.26) for an arbitrary large gauge parameter $\varepsilon(z,\bar{z})$. Since the leading soft factor is exact to all orders in perturbation theory, this shows that large gauge transformations are exact symmetries of the full QED S-matrix [115, 116].

3.3.4 Subleading soft photon theorem

In close analogy with the gravitational case, the universal form of the subleading soft photon theorem (3.1.6) suggests it might be possible to formulate it as a Ward identity. Unlike gravity, however, QED possesses only a single known family of asymptotic charges — those associated with large gauge transformations — and these are already fully captured by the leading soft factor. Nevertheless, this does not preclude us from "reverse engineering" a putative Ward identity directly from the subleading soft theorem. In this subsection, we outline such a construction following [131].

We begin by introducing the *subleading soft photon operator*, in direct analogy with the gravitational case,

$$\mathscr{F}^{(1)}_z \equiv \int du\, u\,\partial_u \tilde{A}^{(0)}_z, \tag{3.3.34}$$

which can be expressed in terms of photon creation and annihilation operators as

$$\mathscr{F}^{(1)}_z(z,\bar{z}) = \frac{1}{4\pi}\lim_{\omega\to 0^+}(1+\omega\partial_\omega)\left[a_+(\omega,z,\bar{z}) + a^\dagger_-(\omega,z,\bar{z})\right]. \tag{3.3.35}$$

Starting from the tree-level subleading soft photon theorem (3.1.6) and acting on both sides with the differential operator

$$\int d^2z\,\Upsilon(z,\bar{z})\,\partial^2, \tag{3.3.36}$$

we obtain the following relation:

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)\int d^2z\,\Upsilon(z,\bar{z})\,\partial^2\,\langle \text{out}|a_-(\omega,z,\bar{z})\mathcal{S}|\text{in}\rangle = \int d^2z\,\Upsilon(z,\bar{z})\,S^{(1)-}_{\text{em}}\,\langle \text{out}|\mathcal{S}|\text{in}\rangle. \tag{3.3.37}$$

Here Υ will be the parameter related to the subleadin transformation.

In analogy with the superrotation case in gravity, we demand that the action of the subleading soft photon factor on massless particles can be recast as a transformation generated by a vector field $V(z,\bar{z};w,\bar{w})$ satisfying

$$\partial^2 V(z,\bar{z};w,\bar{w}) = 2\pi\,\delta^2(z-w). \tag{3.3.38}$$

This motivates the definition of *hard charges* for QED,

$$Q^{\rm hard}_{\Upsilon} = \int du\, d^2z\, \Upsilon(z,\bar{z}) \left(j^{(2)}_z - u\, \partial_u j^{(2)}_u \right), \tag{3.3.39}$$

for which one can verify that

$$S^{(1)-}_{{\rm em},i}(z,\bar{z}) = \left[Q^{\rm hard}_{\Upsilon = \frac{z-z_i}{\bar{z}-\bar{z}_i}}, \mathcal{S} \right]. \tag{3.3.40}$$

For massive external particles, the analogous statement can be formulated by introducing a bulk-to-boundary propagator $\mathcal{G}^{\alpha}(Y_i;\hat{q})$ such that

$$e_i\, \mathcal{G}^{\alpha}(Y_i;\hat{q})\, \partial_\alpha = \partial^2 S^{(1)-}_{{\rm em},i}(Y_i;\hat{q}). \tag{3.3.41}$$

Integrating by parts in (3.3.37) and substituting the above definitions, the subleading soft photon theorem can then be rewritten as the Ward identity associated with the following *putative charge*:

$$Q_{\varepsilon'} = \int d^2z\, \Upsilon(z,\bar{z})\, \partial^2_z \mathscr{F}^{(1)}_z + \int du\, d^2z\, \Upsilon(z,\bar{z}) \left(j^{(2)}_z - u\, \partial_u j^{(2)}_u \right) + \int d^3Y\, \Upsilon^{\alpha}_{\mathcal{H}}\, j^{(3)}_{\alpha}, \tag{3.3.42}$$

where the extension of the transformation parameter Υ to timelike infinity is given by,

$$\Upsilon^{\alpha}_{\mathcal{H}}(Y) = \int d^2z\, \mathcal{G}^{\alpha}(Y;z,\bar{z})\, \Upsilon(z,\bar{z}) \tag{3.3.43}$$

It was shown in [132] that this charge can be interpreted as an *overleading large gauge transformation.* The terminology 'overleading' refers to gauge transformations which are linearly divergent in r near null infinity, hence lying outside the usual asymptotic symmetry algebra of QED. As in the gravitational case, the subleading soft photon factor receives infrared-divergent loop corrections: at one-loop, such corrections were found to modify the structure of the Ward identity above. We will revisit these loop effects in detail in the next chapter.

3.4 Dressings and invariant states

Infrared (IR) divergences in gauge and gravitational scattering amplitudes are not merely a perturbative inconvenience, but rather a symptom of a deeper structural mismatch between the states used in the LSZ reduction formula and the true asymptotic states of the theory. In theories with long-range interactions, such as QED or gravity, the gauge fields sourced by hard particles decay only as a power law in time and space, and therefore cannot be neglected even in the far asymptotic region. Physically, this means that an isolated charged or gravitating particle is never truly "bare" — it is always accompanied by an infinite cloud of soft gauge quanta which encode its long-range behavior.

If one naively uses bare Fock states of free particles as asymptotic states in the S-matrix, the calculation inevitably includes virtual soft exchanges that give rise to IR divergences. The resolution proposed by Faddeev and Kulish [56] was to modify the asymptotic states themselves so that they already contain the correct soft photon (or graviton) profile. This leads to a well-defined, IR-finite scattering operator between physically dressed states.

In their construction, a physical one-particle asymptotic state of momentum p is written as

$$|p\rangle_D = e^{R_f} |p\rangle_I, \tag{3.4.1}$$

where the subscript I denotes the usual interaction picture state, the subscript D denotes a *dressed state*, and R_f is a *dressing operator* which creates a coherent state of soft photons. Schematically, R_f takes the form

$$R_f \sim \int_{\rm soft} \frac{d^3k}{(2\pi)^3 2\omega_k} \left[f(k,p)\, a^{\dagger}_{\pm}(k) - f^*(k,p)\, a_{\pm}(k) \right], \tag{3.4.2}$$

Evaluating the S-matrix between such dressed states produces a result that is infrared finite up to an overall divergent *phase*, which has no observable consequence. This construction also forces a redefinition of the asymptotic Hilbert space: it must be enlarged to include coherent states of soft photon in addition to the usual particle excitations. In this enlarged space, the action of the large gauge symmetry on physical states is realized trivially.

For example, the leading soft photon theorem becomes

$$
{}_D\langle \text{out}| \left[Q_\varepsilon^{\text{hard}}, \mathcal{S}\right] |\text{in}\rangle_D = 0, \tag{3.4.3}
$$

which is simply the Ward identity for large gauge transformations in a sector where all states are gauge-invariant.

The gravitational analogue of this construction was later developed in [57], where the role of the photon field is replaced by the linearized gravitational field generated by a massive or massless particle. The resulting *gravitational dressing* ensures that the dressed state is a BMS supertranslation-invariant eigenstate of the asymptotic charges [58]. The coherent state structure of this dressing mirrors that of the Faddeev–Kulish photons.

In the remaining part of this thesis, we will continue to work with bare operators, even though they possess infrared divergences. Nevertheless, there are two important lessons to retain from this section: firstly, in the presence of long-range interactions, bulk fields do *not* asymptote to free fields at null infinity; secondly, from the perspective of asymptotic symmetries, correlators of supertranslation- (or large-gauge-) invariant operators are automatically free of infrared divergences.

In the next section, we will exploit this viewpoint to reinterpret the universal infrared factorization of scattering amplitudes.

3.5 Soft S-matrix

In Section 2.2 we showed that the scattering matrix can be rewritten in terms of operator insertions on the timelike-infinity hyperboloid and its boundary. This operator-based reformulation provides a natural stage on which to express both the infrared (IR) factorization properties of scattering amplitudes and the soft theorems. In the present section, we focus on rephrasing these results in this language following the works in [133, 134].

The leading soft graviton theorem can be expressed, in hyperboloid language, as the Ward identity of a *soft current* $\mathcal{N}^{(0)}$ defined on the boundary of the hyperboloid:

$$
\begin{aligned}
&\Big\langle \mathcal{N}^{(0)}_{zz}(z,\bar z) \prod_{\text{massless}} \mathcal{O}_{\omega_i,\eta_i}(z_i,\bar z_i) \prod_{\text{massive}} \mathcal{R}_{m_i,\eta_i}(\rho_i,z_i,\bar z_i)\Big\rangle \\
&\quad = \left(\sum_{\text{massless}} \frac{z_i - z}{\bar z_i - \bar z} + \sum_{\text{massive}} \int d^2w\, \mathcal{G}^{(3)}(Y_i; w,\bar w)\, \frac{w-z}{\bar w - \bar z}\right) \times \\
&\qquad\qquad\qquad \Big\langle \prod_{\text{massless}} \mathcal{O}_{\omega_i,\eta_i}(z_i,\bar z_i) \prod_{\text{massive}} \mathcal{R}_{m_i,\eta_i}(\rho_i,z_i,\bar z_i)\Big\rangle.
\end{aligned} \tag{3.5.1}
$$

In this way, the soft theorem becomes a local current algebra statement at the boundary of $\mathcal{H}^+$.

From the Ward identity above, the action of the soft current on the hard operators can be rephrased as the operator product expansions (OPEs):

$$
\begin{aligned}
\mathcal{N}^{(0)}_{zz}(z,\bar z)\, \mathcal{O}_{\omega_i,\eta_i}(z_i,\bar z_i) &\sim \frac{z_i - z}{\bar z_i - \bar z}\, \mathcal{O}_{\omega_i,\eta_i}(z_i,\bar z_i), \\
\mathcal{N}^{(0)}_{zz}(z,\bar z)\, \mathcal{R}_{m_i,\eta_i}(\rho_i,z_i,\bar z_i) &\sim \int d^2w\, \mathcal{G}^{(3)}(\rho_i,z_i,\bar z_i; w,\bar w)\, \frac{w-z}{\bar w - \bar z}\, \mathcal{R}_{m_i,\eta_i}(\rho_i,z_i,\bar z_i).
\end{aligned} \tag{3.5.2}
$$

Since the Ward identity of the corresponding charge does *not* vanish when summed over all insertions, the soft current must also have a nontrivial action on the vacuum. This corresponds to the creation of soft gravitons, which does not change the total energy of the state, leading to an infinite degeneracy of vacua. As emphasized earlier, this is the hallmark of spontaneous symmetry breaking (SSB) of the asymptotic symmetry in question.

In quantum field theory, spontaneous breaking of a symmetry implies that charged operators can be factorized into a part that carries the symmetry transformation, and another operator that is invariant under the broken symmetry:

$$\mathcal{O}_{\omega,\eta}(z,\bar{z}) \;=\; \mathcal{W}_{\omega,\eta}(z,\bar{z})\,\tilde{\mathcal{O}}(z,\bar{z}). \tag{3.5.3}$$

Here $\tilde{\mathcal{O}}$ is invariant under the soft current, while $\mathcal{W}_{\omega,\eta}$ accounts for the full transformation. Being invariant under the action of the soft current, $\tilde{\mathcal{O}}$ corresponds to the dressed operator of section 3.4. In general, the operaotor $\mathcal{W}_{\omega,\eta}$ can be written as the exponential of the Goldstone mode for the underlying broken current. It was identified in section 2.3.4 that $C^{(0)}(z,\bar{z})$ is the Goldstone mode associated with the spontaneously broken supertranslation symmetry. Thus we write

$$\mathcal{W}_{\omega,\eta} \;=\; \exp[\,i\,\eta\,\omega\,C^{(0)}(z,\bar{z})\,]. \tag{3.5.4}$$

Demanding that this factorization reproduces the OPE (3.5.2) fixes the action of $\mathcal{N}^{(0)}$ on the Goldstone mode:

$$\mathcal{N}^{(0)}_{zz}\,C^{(0)} \;=\; \frac{z_i - z}{\bar{z}_i - \bar{z}}. \tag{3.5.5}$$

This expression agrees precisely with the supertranslation action on the Goldstone mode derived earlier in Eq. (2.3.45).

A completely analogous decomposition can be carried out for operators creating *massive* external states. In this case, the decoupling of the Goldstone mode requires reintroducing the bulk-to-boundary propagator $\mathcal{G}^{(3)}$ that was defined in eq 2.4.6. The factorization then takes the form

$$\mathcal{R}_{m,\eta}(\rho,z,\bar{z}) \;=\; \exp\!\left[im\eta\int d^2w\,\mathcal{G}^{(3)}(\rho,z,\bar{z};w,\bar{w})\;C^{(0)}(w,\bar{w})\right]\,\tilde{\mathcal{R}}_{m,\eta}(\rho,z,\bar{z}). \tag{3.5.6}$$

Here $\tilde{\mathcal{R}}_{m,\eta}$ is invariant under the soft current, while the exponential factor encodes the transformation of the full operator entirely through the Goldstone mode $C^{(0)}$.

The action of $\mathcal{N}^{(0)}$ on $C^{(0)}$ is sufficient to reproduce its action on $\mathcal{R}_{m,\eta}$, exactly as in the massless case. Moreover, taking the large-ρ limit of the massive factorization recovers the massless result. In what follows, we will focus on *massive* external states for concreteness, with the understanding that the massless case arises smoothly in the appropriate limit.

Following the factorization of the individual operators, the full S-matrix correlator also admits a corresponding factorization. For a general scattering process involving both massless and massive external states, we may write

$$\begin{aligned}
&\Big\langle \prod_{\text{massless}} \mathcal{O}_{\omega_i,\eta_i}(z_i,\bar{z}_i) \prod_{\text{massive}} \mathcal{R}_{m_i,\eta_i}(\rho_i,z_i,\bar{z}_i)\Big\rangle \\
&\quad= \Big\langle \prod_{\text{massless}} \mathcal{W}_{\omega_i,\eta_i}(z_i,\bar{z}_i) \prod_{\text{massive}} \mathcal{X}_{m_i,\eta_i}(\rho_i,z_i,\bar{z}_i)\Big\rangle \times \\
&\qquad\quad \Big\langle \prod_{\text{massless}} \widetilde{\mathcal{O}}_{\omega_i,\eta_i}(z_i,\bar{z}_i) \prod_{\text{massive}} \widetilde{\mathcal{R}}_{m_i,\eta_i}(\rho_i,z_i,\bar{z}_i)\Big\rangle.
\end{aligned} \tag{3.5.7}$$

Here: - $\mathcal{W}$ and $\mathcal{X}$ contain the exponential soft factors built from the Goldstone mode $C^{(0)}$, - $\widetilde{\mathcal{O}}$ and $\widetilde{\mathcal{R}}$ are the "dressed" hard operators invariant under the soft current $\mathcal{N}^{(0)}$.

As demonstrated in Section 3.4, correlators of operators that are *invariant* under the soft current are free from infrared divergences upto a divergent phase. The factorization above is therefore nothing but a reformulation of Weinberg's universal infrared factorization: the IR-divergent part is entirely captured by the correlation functions of the exponential dressings, while the remaining correlator is infrared finite.

Identifying the two factors explicitly, we write

$$\begin{aligned} \langle \prod_{\text{massless}} \widetilde{\mathcal{W}}_{\omega_i,\eta_i}(z_i,\bar{z}_i) \prod_{\text{massive}} \widetilde{\mathcal{X}}_{m_i,\eta_i}(\rho_i,z_i,\bar{z}_i)\rangle &= \exp\left(-\frac{\kappa^2}{\epsilon}\,\sigma_n\right), \\ \langle \prod_{\text{massless}} \widetilde{\mathcal{O}}_{\omega_i,\eta_i}(z_i,\bar{z}_i) \prod_{\text{massive}} \widetilde{\mathcal{R}}_{m_i,\eta_i}(\rho_i,z_i,\bar{z}_i)\rangle &= \mathcal{M}_n^{\text{finite}}\,. \end{aligned} \tag{3.5.8}$$

Demanding consistency with the universal IR factorization, fixes the $C^{(0)}$ two point function in terms of the infrared regulator ϵ as

$$\langle C^{(0)}(z_i,\bar{z}_i)C^{(0)}(z_j,\bar{z}_j)\rangle = -\frac{i\kappa^2}{\epsilon}\,|z_i-z_j|^2\left(\ln|z_i-z_j|^2 - i\pi\delta_{\eta_i,\eta_j}\right), \tag{3.5.9}$$

Thus, the entire IR divergence of the S-matrix is encoded in the $C^{(0)}$ two-point function, making the Goldstone mode the natural carrier of the long-range behavior.

The two-point function of the Goldstone mode also enables a direct evaluation of its insertions into S-matrix correlators. These insertions will prove particularly useful in the next chapter when discussing the role of soft modes in loop corrections. For a single insertion, we can express the operator as a derivative of an exponential,

$$\langle \text{out}|C^{(0)}(z,\bar{z})\,\mathcal{S}\,|\text{in}\rangle = \frac{\partial}{\partial\omega}\,\langle \text{out}|e^{i\omega\,C^{(0)}(z,\bar{z})}\,\mathcal{S}\,|\text{in}\rangle\Big|_{\omega=0}. \tag{3.5.10}$$

Using the two-point function this expression yields

$$\langle \text{out}|C^{(0)}(z,\bar{z})\,\mathcal{S}_n\,|\text{in}\rangle = -\frac{i}{\epsilon}\,\sigma'_{n+1}\,\langle \text{out}|\mathcal{S}_n\,|\text{in}\rangle. \tag{3.5.11}$$

Here, σ'_{n+1} denotes the kinematic factor obtained from σ_n by including an additional graviton, as defined in Eq. (3.2.4).

A completely analogous rewriting can be performed for the case of soft photons. We begin by expressing the leading soft photon theorem as the insertion of a soft current on the boundary of the hyperboloid,

$$\begin{aligned} &\langle \mathscr{F}^{(0)}(z,\bar{z}) \prod_{\text{massless}} \mathcal{O}_{\omega_i,\eta_i}(z_i,\bar{z}_i) \prod_{\text{massive}} \mathcal{R}_{m_i,\eta_i}(\rho_i,z_i,\bar{z}_i)\rangle \\ &= \left(\sum_{\text{massless}} \frac{1}{\bar{z}_i-\bar{z}} + \sum_{\text{massive}} \int d^2w\,\mathcal{G}^{(2)}\,\frac{1}{\bar{z}_i-\bar{z}}\right)\times \\ &\qquad\qquad \langle \prod_{\text{massless}} \mathcal{O}_{\omega_i,\eta_i}(z_i,\bar{z}_i) \prod_{\text{massive}} \mathcal{R}_{m_i,\eta_i}(\rho_i,z_i,\bar{z}_i)\rangle. \end{aligned} \tag{3.5.12}$$

Here, the soft current $\mathscr{F}^{(0)}$ generates large gauge transformations, whose action on single operators is

$$\begin{aligned} \mathscr{F}^{(0)}(z,\bar{z})\,\mathcal{O}_{\omega_i,\eta_i}(z_i,\bar{z}_i) &\sim \frac{1}{\bar{z}_i-\bar{z}}\,\mathcal{O}_{\omega_i,\eta_i}(z_i,\bar{z}_i)\,, \\ \mathscr{F}^{(0)}(z,\bar{z})\,\mathcal{R}_{m_i,\eta_i}(\rho_i,z_i,\bar{z}_i) &\sim \int d^2w\,\mathcal{G}^{(3)}(\rho_i,z_i,\bar{z}_i)\,\frac{1}{\bar{z}_i-\bar{z}}\,\mathcal{R}_{m_i,\eta_i}(\rho_i,z_i,\bar{z}_i)\,. \end{aligned} \tag{3.5.13}$$

This current is also spontaneously broken, with the corresponding Goldstone mode identified as $\mathscr{A}$. Charged operators can thus be factorized in analogy with the gravitational case:

$$\mathcal{R} \;=\; \exp\Big[iq\int d^2w\,\mathcal{G}^{(2)}\,\mathscr{A}(w)\Big]\,\widetilde{\mathcal{R}}\,, \tag{3.5.14}$$

where $\widetilde{\mathcal{R}}$ is invariant under the action of the soft current. This factorization carries over to the full S-matrix correlator, and matching to the standard infrared factorization identifies the two-point function of the Goldstone mode as

$$\langle \mathscr{A}(z,\bar{z})\,\mathscr{A}(w,\bar{w})\rangle = \frac{ie^2}{\epsilon}\left(\ln|z-w| - i\pi\delta_{\eta_i,\eta_j}\right)\,. \tag{3.5.15}$$

The insertion of the Goldstone mode into S-matrix correlators is then computed exactly as in the graviton case:

$$\langle \text{out}|\mathscr{A}(z,\bar{z})\,\mathcal{S}_n\,|\text{in}\rangle = \frac{i}{\epsilon}\lambda'_{n+1}\,\langle \text{out}|\mathcal{S}_n\,|\text{in}\rangle\,. \tag{3.5.16}$$

where $\lambda'_{n+1}(\hat{q})$ can be defined analogously to $\sigma'_{n+1}(\hat{q})$ as

$$\lambda'_{n+1}(\hat{q}) = \frac{1}{4\pi^2}\sum_{i=1}^{n} e_i\,\ln(\hat{p}_i\cdot\hat{q})\,. \tag{3.5.17}$$

In summary, we have shown that the universal infrared behaviour of both gauge and gravitational scattering amplitudes at tree level can be encapsulated entirely in terms of the insertion of soft currents and their associated Goldstone modes on the hyperboloid and its boundary. This reformulation sets the stage for the next chapter, where these relations will be extended to incorporate one-loop corrections.

Chapter 4

Logarithmic Soft Theorems and Ward Identities

In the previous two chapters, we established the deep connection between universal soft theorems and the Ward identities of asymptotic symmetries. Starting from the factorization properties of tree-level scattering amplitudes, we showed how the leading and subleading soft limits in gravity and gauge theory can be reinterpreted as consequences of spontaneously broken symmetries acting at null infinity and timelike infinity. We also reformulated these results in a boundary-field language, from where we were able to quantize the respective Goldstone modes, and where Weinberg's infrared factorization emerges naturally from operator factorization.

While this framework is exact at tree level, it does not capture the full story. Loop corrections (both classical and quantum) modify the naive factorization and, in certain cases, introduce qualitatively new features such as infrared logarithms and loop-induced mixing between soft and hard sectors. These effects arise in both classical and quantum regimes, and their proper understanding is essential for a consistent picture of the interplay between soft theorems and asymptotic symmetries.

The aim of this chapter is to extend the tree-level correspondence to incorporate the leading loop corrections — in particular, the logarithmic terms (see eqs (4.1.3)–(4.1.6)) that appear in the soft expansion of amplitudes [94, 95, 135]. We will see how these logarithmic soft theorems can be reformulated as Ward identities, providing a symmetry-based interpretation for their structure. This analysis represents the first main set of original results in this thesis.

Our primary focus will be on loop corrections to the subleading soft graviton theorem and their relation to superrotation symmetry. Superrotations are of central importance to the program of flat-space holography, as they generate the two-dimensional conformal group on the celestial sphere [112]. For superrotations to be genuine symmetries of the full quantum gravity S-matrix, their Ward identities must reproduce the complete, loop-corrected subleading soft graviton theorem. Achieving this requires a detailed understanding of the loop corrections themselves, which were shown by Sahoo and Sen to include a term divergent as $\ln\omega$ [94]. In this chapter, we will match this result with the infrared-divergent loop corrections derived in section 3.2.1, and then demonstrate how they can emerge from superrotation Ward identities in section 4.2. For related previous results see [136–139].

4.1 Logarithmic soft theorem

In the previous chapter, we saw that incorporating infrared divergences and adopting a specific ansatz for the soft factor leads to an infrared-divergent correction to the subleading soft factor. However, that approach did not provide an independent justification for the assumed expansion. A more direct analysis was carried out by Sahoo and Sen, who demonstrated that, if no such analytic expansion in the soft energy ω is imposed from the outset, the soft factor naturally acquires terms that diverge logarithmically as $\omega \to 0$.

The analytic structure of soft theorems had been implicitly assumed in most earlier derivations, including those based on asymptotic symmetry arguments. The appearance of a $\ln \omega$ term in the loop-corrected expansion therefore came as a surprise, revealing that the soft limit at loop level is more subtle than previously anticipated. In particular, it indicates that infrared divergences can qualitatively modify the hierarchy of terms in the soft expansion, and it motivates a re-examination of the symmetry interpretation of subleading orders.

Schematically, we define the soft factor as

$$S(p_i, \omega\hat{q}) \;=\; \lim_{\omega\to 0} \frac{\mathcal{M}^{\text{tree}}_{n+1} + \mathcal{M}^{\text{1-loop}}_{n+1}}{\mathcal{M}^{\text{tree}}_{n} + \mathcal{M}^{\text{1-loop}}_{n}}, \tag{4.1.1}$$

without assuming that it admits a regular power series expansion in ω. Under these conditions, the result of [94] shows that the soft factor takes the form

$$S \;=\; \frac{1}{\omega} S^{(0)} \;+\; \ln\omega\, S^{(\ln)} \;+\; S^{(1)} \;+\; \mathcal{O}(\omega \ln \omega). \tag{4.1.2}$$

We thus encounter a $\ln \omega$ term which is more singular than the subleading $\mathcal{O}(\omega^0)$ term traditionally called the subleading soft factor. This represents an important modification of the soft expansion. These loop corrections necessarily introduce an infrared length scale R in order to make the argument of the logarithm dimensionless, i.e. $\ln(\omega R)$. In what follows we will focus on the $\ln \omega$ dependence and absorb the $\ln R$ pieces into the $\mathcal{O}(\omega^0)$ terms of the expansion.

The explicit expressions for these logarithmic corrections, as derived in [44, 94, 95, 135, 140, 141], are remarkably universal: they depend only on the momenta of the hard external particles and not on other details of the scattering process. We now list the results of [94] in detail, beginning with the purely gravitational case and then including additional gauge and mixed interactions.

(i) Pure gravity

For an $(n+1)$-point amplitude with a single soft graviton, the logarithmic soft factor in pure gravity takes the form

$$\begin{aligned}
-\frac{4}{\kappa^2} S^{(\ln)}_{\text{gr}} = {}& \frac{i\kappa}{8\pi} \sum_i \frac{\varepsilon_{\mu\nu} p_i^\mu p_i^\nu}{p_i\cdot q} \sum_j \delta_{\eta,\eta_j}\, q\cdot p_j \\
&+ \frac{i\kappa}{16\pi} \sum_i \frac{\varepsilon_{\mu\nu} p_i^\nu q_\rho}{p_i\cdot q} \sum_{j\neq i} \delta_{\eta_i,\eta_j}\, (p_i\cdot p_j)\, (p_i^\mu p_j^\rho - p_j^\mu p_i^\rho) \frac{2(p_i\cdot p_j)^2 - 3p_i^2 p_j^2}{\left[(p_i\cdot p_j)^2 - p_i^2 p_j^2\right]^{3/2}} \\
&- \frac{\kappa}{8\pi^2} \sum_i \frac{\varepsilon_{\mu\nu} p_i^\nu p_i^\nu}{p_i\cdot q} \sum_j q\cdot p_j\, \ln|\hat{q}\cdot\hat{p}_j| \\
&- \frac{\kappa}{32\pi^2} \sum_i \frac{p_i^\mu \varepsilon_{\mu\nu} q_\lambda}{p_i\cdot q} \left(p_i^\lambda \frac{\partial}{\partial p_{i\nu}} - p_i^\nu \frac{\partial}{\partial p_{i\lambda}} \right) \times \\
&\qquad \sum_{j\neq i} \frac{2(p_i\cdot p_j)^2 - p_i^2 p_j^2}{\sqrt{(p_i\cdot p_j)^2 - p_i^2 p_j^2}} \ln\left(\frac{p_i\cdot p_j + \sqrt{(p_i\cdot p_j)^2 - p_i^2 p_j^2}}{p_i\cdot p_j - \sqrt{(p_i\cdot p_j)^2 - p_i^2 p_j^2}} \right).
\end{aligned} \tag{4.1.3}$$

Recall that p_i are the momenta of the external hard particles directed along $\hat{p}_i$, $q = \omega\hat{q}$ is the momentum of the emitted soft graviton with energy ω and polarization tensor given by $\varepsilon_{\mu\nu}$, and κ is the gravitational coupling constant defined in eq (3.1.1).

(ii) Electromagnetic correction to soft graviton

If the hard particles also carry an electric charge q_i, there is an additional electromagnetic contribution to the soft graviton logarithmic term:

$$\begin{aligned}\Delta_{\rm em}S^{(\ln)}_{\rm gr} = &-\frac{i\kappa}{2}\sum_i \frac{\varepsilon_{\mu\nu}p_i^\nu q_\rho}{p_i\cdot q}\sum_{\substack{j\neq i\\ \eta_i\eta_j=1}}\frac{e_ie_j}{4\pi}\frac{m_i^2m_j^2\left(p_j^\rho p_i^\mu - p_j^\mu p_i^\rho\right)}{\left[(p_i\cdot p_j)^2 - m_i^2m_j^2\right]^{3/2}} \\ &+\frac{\kappa}{32\pi^2}\sum_i \frac{p_i^\mu\varepsilon_{\mu\nu}q_\lambda}{p_i\cdot q}\left(p_i^\lambda\frac{\partial}{\partial p_{i\nu}} - p_i^\nu\frac{\partial}{\partial p_{i\lambda}}\right)\times \\ &\sum_{j\neq i}\frac{2e_ie_j(p_i\cdot p_j)}{\sqrt{(p_i\cdot p_j)^2 - p_i^2p_j^2}}\ln\left(\frac{p_i\cdot p_j + \sqrt{(p_i\cdot p_j)^2 - p_i^2p_j^2}}{p_i\cdot p_j - \sqrt{(p_i\cdot p_j)^2 - p_i^2p_j^2}}\right),\end{aligned} \tag{4.1.4}$$

where the charges e_i enter only through the multiplicative factors q_iq_j. This term vanishes in the neutral-particle limit.

(iii) Purely electromagnetic: soft photon

In the case of a soft photon, the purely electromagnetic logarithmic factor reads

$$\begin{aligned}S^{(\ln)}_{\rm em} = &-i\sum_i e_i\frac{\varepsilon_\mu q_\rho}{p_i\cdot q}\sum_{\substack{j\neq i\\ \eta_i\eta_j=1}}\frac{e_ie_j}{4\pi}\frac{m_i^2m_j^2\left(p_j^\rho p_i^\mu - p_j^\mu p_i^\rho\right)}{\left[(p_i\cdot p_j)^2 - m_i^2m_j^2\right]^{3/2}} \\ &-\frac{1}{16\pi^2}\sum_i e_i\frac{\varepsilon_\nu q_\lambda}{p_i\cdot q}\left(p_i^\lambda\frac{\partial}{\partial p_{i\nu}} - p_i^\nu\frac{\partial}{\partial p_{i\lambda}}\right)\times \\ &\sum_{j\neq i}\frac{2e_ie_j(p_i\cdot p_j)}{\sqrt{(p_i\cdot p_j)^2 - p_i^2p_j^2}}\ln\left(\frac{p_i\cdot p_j + \sqrt{(p_i\cdot p_j)^2 - p_i^2p_j^2}}{p_i\cdot p_j - \sqrt{(p_i\cdot p_j)^2 - p_i^2p_j^2}}\right).\end{aligned} \tag{4.1.5}$$

Here, only the electromagnetic coupling e_ie_j appears, and no gravitational coupling κ is present.

(iv) Gravitational correction to soft photon

Finally, gravitational interactions induce an additional correction to the logarithmic soft photon factor:

$$\begin{aligned}\Delta_{\rm gr}S^{(\ln)}_{\rm em} = &\frac{i\kappa^2}{8\pi}\sum_i e_i\frac{\varepsilon_\mu p_i^\mu}{p_i\cdot q}\sum_j \delta_{\eta,\eta_j}\, q\cdot p_j \\ &+\frac{i\kappa^2}{16\pi}\sum_i e_i\frac{\varepsilon_\mu q_\rho}{p_i\cdot q}\sum_{j\neq i}\delta_{\eta_i,\eta_j}\,(p_i\cdot p_j)\,(p_i^\mu p_j^\rho - p_j^\mu p_i^\rho)\frac{2(p_i\cdot p_j)^2 - 3p_i^2p_j^2}{\left[(p_i\cdot p_j)^2 - p_i^2p_j^2\right]^{3/2}} \\ &-\frac{\kappa^2}{8\pi^2}\sum_i e_i\frac{\varepsilon_\mu p_i^\mu}{p_i\cdot q}\sum_j q\cdot p_j\,\ln|\hat{q}\cdot\hat{p}_j| \end{aligned} \tag{4.1.6}$$

$$\begin{aligned}&-\frac{\kappa^2}{32\pi^2}\sum_i e_i\frac{\varepsilon_\nu q_\lambda}{p_i\cdot q}\left(p_i^\lambda\frac{\partial}{\partial p_{i\nu}} - p_i^\nu\frac{\partial}{\partial p_{i\lambda}}\right)\times \\ &\sum_{j\neq i}\frac{2(p_i\cdot p_j)^2 - p_i^2p_j^2}{\sqrt{(p_i\cdot p_j)^2 - p_i^2p_j^2}}\ln\left(\frac{p_i\cdot p_j + \sqrt{(p_i\cdot p_j)^2 - p_i^2p_j^2}}{p_i\cdot p_j - \sqrt{(p_i\cdot p_j)^2 - p_i^2p_j^2}}\right).\end{aligned} \tag{4.1.7}$$

The structure closely mirrors that of (4.1.3), reflecting the universality of gravitational effects.

A striking feature of all the expressions above is that they are *one-loop exact*: the logarithmic terms do not receive any further corrections at higher loop orders in the coupling constants.

At first sight, the explicit expressions in Eqs. (4.1.3)–(4.1.6) may appear rather unwieldy. However, by borrowing the compact operator definitions introduced in the previous chapter, they can be rewritten in a much more transparent form.

Using the definition of the kinematic factors σ_n and $\hat{\sigma}'_{n+1}(\hat{q})$ from Sec. 3.2.1, together with the tree-level leading soft operator $S^{(0)}_{\rm gr}(\hat{q})$ and the angular-momentum part of the subleading soft operator $S^{(1)J}_{\rm gr}(\hat{q})$, the logarithmic soft graviton factor in eq. (4.1.3) can be expressed compactly as

$$S^{(\ln)}_{\rm gr}(\hat{q}) \;=\; \Big[\hat{\sigma}'_{n+1}(\hat{q})\, S^{(0)}_{\rm gr}(\hat{q}) \;-\; S^{(1)J}_{\rm gr}(\hat{q})\,\sigma_n\Big]\,, \tag{4.1.8}$$

where all quantities are defined purely in terms of the external hard momenta, and we have further compactified the notation $S^{(1)J}_{\rm gr}(\hat{q})\,\sigma_n \equiv S^{(1)J}_{{\rm gr},i}(\hat{q})\,\sigma^i_n$. In this form, it becomes manifest that the coefficient of the $\ln\omega$ term is *identical* to the coefficient of the infrared-divergent factor appearing in the subleading soft graviton theorem (3.2.14).

A similar rewriting applies to the soft photon case. The logarithmic soft photon factor of eq. (4.1.5) can be written as

$$S^{(\ln)}_{\rm em}(\hat{q}) \;=\; S^{(1)J}_{\rm em}(\hat{q})\,\lambda_n, \tag{4.1.9}$$

again matching precisely the coefficient of the infrared-divergent term in the subleading soft photon factor (3.2.18). Similarly the mixed terms (4.1.4) and (4.1.6) can be seen to match with those given in eqs (3.2.19) and (3.2.20) respectively.

The fact that in both gravity and gauge theory the logarithmic term is governed by the same kinematic coefficient that controls the IR-divergent part of the subleading soft theorem is highly nontrivial. As we shall see in the next section, this correspondence admits a more mathematical explanation.

4.1.1 Logarithmic soft photon theorem

In this subsection, we take a brief detour to rederive the logarithmic soft photon theorem through an explicit one-loop computation in scalar QED. While this is less general than the approach of [94], it will shed light on the direct relation between the $\ln\omega$ term and the coefficient of the infrared (IR) divergent contribution discussed earlier. The key technical twist here will be to reverse the order of limits compared to [94]: we will first take the $\omega\to 0$ (soft) limit at fixed ϵ in dimensional regularisation, and only afterwards send $\epsilon\to 0$.

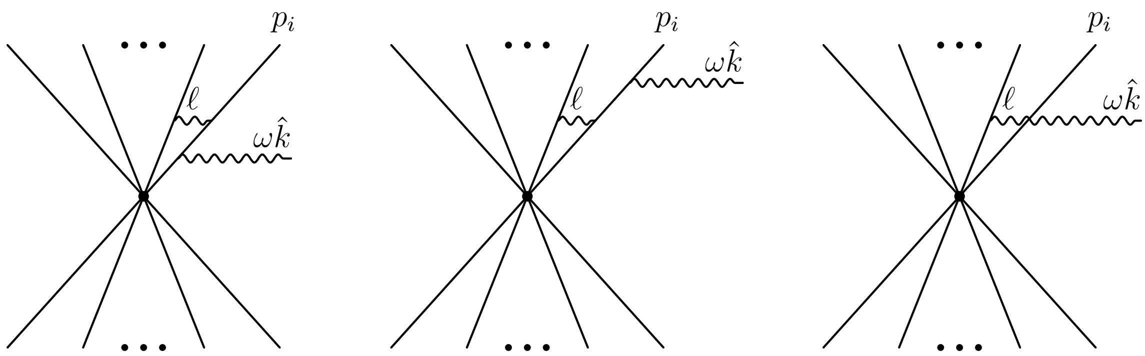
We regulate the loop integral by working in $D = 4-2\epsilon$ dimensions. In [94], the integrals were computed directly in $D = 4$ and the $\omega\to 0$ limit was taken last, producing the characteristic $\ln\omega$ behaviour. Here, by taking the soft limit first, we will be able to explicitly match the coefficient of $\ln\omega$ to the IR-divergent $1/\epsilon$ factor.

Consider first the one-loop diagrams where the soft photon attaches to the scalar leg inside the loop (type A in Fig. 4.1). The corresponding contribution is given by the integral

$$\sum_{i,j}\int d^D\ell\,\frac{(2p_i+k+2\ell)^\mu\,(4p_i{\cdot}p_j-2p_i{\cdot}\ell-2p_j{\cdot}\ell-\ell^2)}{(2p_j{\cdot}\ell+\ell^2)^2\,(2p_i{\cdot}\ell+\ell^2)\,\ell^2\,(2p_i{\cdot}(k+\ell)+(k+\ell)^2)}\,. \tag{4.1.10}$$

Since we are taking the soft limit first, we expand in powers of ω and keep only terms up to $\mathcal{O}(\omega^0)$, setting all higher orders to zero. Moreover, because we are interested in IR divergences, we retain only terms in the integrand scaling as ℓ^{-4} in the loop momentum. From the denominator scaling $\mathcal{O}(\ell^5)$, only one term contributes with the desired behaviour:

$$\int d^D\ell\,\frac{\ell^\mu\,p_i\cdot p_j}{(p_j\cdot\ell)(p_i\cdot\ell)(\ell^2)^2}\,. \tag{4.1.11}$$



(a) Type A (b) Type B (c) Type C

Figure 4.1: One-loop diagrams contributing to infrared corrections to subleading soft photon factor. Hard lines are scalars and wavy lines are photons. ℓ labels the loop momentum, $k = \omega\hat{k}$ is the momentum of the emitted soft photon with energy ω and p_i's label the hard momenta. There will be other one-loop diagrams as well but they will not have any effect on the subleading soft photon factorization.

It is convenient to define the symmetric loop integral

$$K_{ij} \;=\; \int d^D\ell \, \frac{1}{(p_i\cdot\ell)(p_j\cdot\ell)(\ell^2)^2}\,, \tag{4.1.12}$$

so that the above expression becomes

$$\int d^D\ell \, \frac{\ell^\mu \, p_i\cdot p_j}{(p_j\cdot\ell)(p_i\cdot\ell)(\ell^2)^2} \;=\; p_i\cdot p_j \, \frac{\partial}{\partial p_i^\mu} K_{ij}\,. \tag{4.1.13}$$

Next, consider diagrams where the soft photon is emitted from an external scalar leg with an internal photon exchange (type B in Fig. 4.1):

$$\int d^D\ell \, \frac{(2p_i+k)^\mu \, (4p_i{\cdot}p_j - 2p_i{\cdot}\ell + 2p_j{\cdot}\ell + 4p_j{\cdot}k - 2k\cdot\ell)}{(2p_j{\cdot}\ell+\ell^2)\,(2p_i\cdot k+\ell^2)\,\ell^2\,(2p_i(k+\ell)+(k+\ell)^2)}\,. \tag{4.1.14}$$

Here there are three relevant contributions:

1. The first term is of order ω^{-1}:

$$\sum_i \frac{p_i^\mu}{p_i\cdot k} \sum_{n,m} \int d^D\ell \, \frac{p_n\cdot p_m}{(p_n\cdot\ell)(p_m\cdot\ell)(\ell^2)^2} = \frac{1}{\omega} S^{(0)} \mathcal{M}_n^{\text{1-loop}}\,, \tag{4.1.15}$$

which simply corresponds to the leading soft factor multiplying the one-loop amplitude.

2. For the next terms, we use the expansion

$$\frac{1}{2p_i\cdot(k+\ell)} \left[1 + \frac{k\cdot\ell}{p_i\cdot\ell}\right]^{-1} = \frac{1}{2p_i\cdot\ell}\left(1 - \frac{k\cdot\ell}{p_i\cdot\ell}\right) + \mathcal{O}(\omega)\,. \tag{4.1.16}$$

This yields two IR-divergent $\mathcal{O}(\omega^0)$ contributions:

$$\frac{p_i^\mu}{p_i\cdot k} \int d^D\ell \, \frac{(p_i\cdot p_j)(k\cdot\ell) + (p_j\cdot k)(p_i\cdot\ell)}{(p_j\cdot\ell)(p_i\cdot\ell)^2(\ell^2)^2} = \frac{p_i\cdot p_j}{p_i\cdot k} p_i^\mu \, k^\nu \frac{\partial}{\partial p_i^\nu} K_{ij} + \frac{p_j\cdot k}{p_i\cdot k} p_i^\mu \, K_{ij}\,. \tag{4.1.17}$$

Finally, the contribution from the diagrams of type C in Fig. 4.1 can be evaluated by using

$$\int d^D\ell \, \frac{2p_j^\mu}{(p_i\cdot\ell)(p_j\cdot\ell)(\ell^2)^2} = p_j^\mu \, K_{ij} \,. \tag{4.1.18}$$

Using the identity $\sum_{i,j} p_i \cdot p_j \, K_{ij} = \frac{1}{\epsilon}\lambda_n$, the soft factorisation of the amplitude becomes

$$\mathcal{M}_{n+1}^{\text{tree}} + \mathcal{M}_{n+1}^{\text{1-loop}} = S^{(0)}\left(\mathcal{M}_n^{\text{tree}} + \mathcal{M}_n^{\text{1-loop}}\right) + \frac{1}{\epsilon}\left(S^{(1)}_{\text{em},i}\lambda_n^i\right)\mathcal{M}_n^{\text{tree}} + \mathcal{O}(\omega^0)\,. \tag{4.1.19}$$

If instead of dimensional regularisation we used an energy cutoff as in Weinberg's original analysis [9], the IR pole would be replaced by

$$\frac{1}{\epsilon} \quad \longleftrightarrow \quad \ln\frac{\Lambda}{\lambda}\,, \tag{4.1.20}$$

where λ is the IR regulator and Λ is a dividing energy scale separating "soft" from "hard" virtual photons. Identifying Λ with the soft photon energy ω and λ with the inverse detector size R^{-1}, we see that $\frac{1}{\epsilon}$ in dimensional regularisation directly maps to $\ln(\omega R)$. In this way, the coefficient of the $\ln\omega$ term in the Sahoo–Sen result is precisely the same as that of the IR-divergent $1/\epsilon$ term obtained here.

4.1.2 Logarithmic soft graviton theorem

We now turn to the gravitational case, where the structure of the logarithmic term in the soft expansion (cf. eq. (4.1.2)) can be dissected in a way analogous to the soft photon computation of the previous subsection. However, the gravitational case is richer: the $\ln\omega$ term originates from long-range interactions among the external hard particles as well as from self-interaction effects of the radiated field. The structure that is common to both photon and graviton is that the full logarithmic soft factor contains both a *real* and an *imaginary* part, each with a distinct physical origin.

The imaginary part of the logarithmic soft graviton factor (and likewise for the photon case) can already be obtained in the classical theory, without invoking quantum loop corrections. Physically, this contribution arises from the iterative backreaction of the scattering process on the background metric: as the hard particles scatter, they source a long-range metric perturbation that subsequently influences the emission of the soft graviton. A detailed derivation of this iterative procedure can be found in [95] (see [142] for understanding these terms as a classical limit of the quantum correlators), where it is shown that the backreaction produces a logarithmic modification emitted radiation in the classical limit. For later reference, we will refer to this as the *classical logarithmic soft factor*. This modification was also the reason for considering $1/u$ tail terms in eq. (2.3.39), as a logarithmically divergent term in the small frequency domain signals a $1/u$ fall-off at late times [93].

In contrast, the real part of the logarithmic soft factor is purely quantum mechanical in origin. It arises from loop diagrams in which an additional soft graviton is emitted.

The full (complex) logarithmic soft graviton factor can be written schematically as

$$S_{\text{gr}}^{(\ln)} \;=\; \sigma'_{n+1}\, S_{\text{gr}}^{(0)} \;-\; S_{\text{gr}}^{(1)J}\,\sigma_n\,, \tag{4.1.21}$$

where σ_n denotes the infrared-divergent factor defined in eq. (3.2.3), σ'_{n+1} defined in eq. (3.2.4), and $S_{\text{gr}}^{(0)}$ and $S_{\text{gr}}^{(1)}$ are the leading and subleading tree-level soft graviton factors, respectively. This decomposition is not just a formal rewriting: each term has a clear and distinct physical origin.

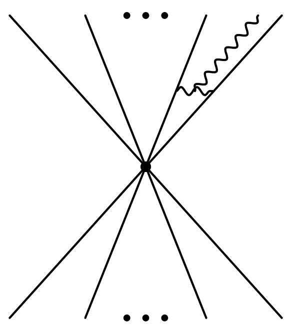


Figure 4.2: Additional contribution to the logarithmic soft graviton factor which is missing for the photon. The wavy line denotes a graviton here. This contribution is a result of graviton self-interaction.

- The term $S_{\rm gr}^{(1)}\,\sigma_n$ is *common* to both the soft photon (where it appears as $S_{\rm em}^{(1)}\,\lambda_n$) and soft graviton cases. As seen in the explicit QED computation of the previous subsection (see Fig. 4.1), this term originates from the long-range interactions between external hard particles.
- The term $\sigma'_{n+1}\,S_{\rm gr}^{(0)}$ is *unique* to gravity. It originates from the *self-interaction* of the gravitational field itself (see Fig. 4.2). It accounts for the gravitational drag of the background metric on the graviton. Interestingly, a similar phenomenon can occur in electromagnetism when gravitational interactions are also included: in that case, electromagnetic radiation experiences a gravitational drag, as reflected in eq. (4.1.6).

We summarise the above discussion in Table 4.1, where we separate the contributions into their *classical* and *quantum* origins, and indicate whether they arise from long-range interactions or from gravitational self-interactions. The notation $\mathrm{Im}[\cdots]$ indicates the imaginary part of the complete factor, while the absence of Im denotes the full contribution.

	Classical	**Quantum**
Long-range interaction	$\mathrm{Im}[S_{\rm gr}^{(1)}\sigma_n]$	$S_{\rm gr}^{(1)}\sigma_n$
Gravitational drag	$\mathrm{Im}[\sigma'_{n+1}S_{\rm gr}^{(0)}]$	$\sigma'_{n+1}S_{\rm gr}^{(0)}$

Table 4.1: Physical origin of the different terms in the logarithmic soft graviton factor. Long-range interaction terms appear in both gravity and gauge theory, while gravitational drag terms are unique to gravity. The "Classical" column corresponds to contributions obtained from iterative backreaction in the classical theory, while the "Quantum" column contains contributions from loop effects.

This decomposition will be particularly important in the next section, where we reinterpret these contributions as arising from Ward identities of asymptotic symmetries at one-loop.

4.1.3 The soft pyramid

We have now seen that at subleading order in the soft expansion, loop corrections generate a term proportional to $\ln\omega$, which is *more leading* than the ω^0 term. This is a sharp departure from the naive tree-level hierarchy of the soft expansion, where each subsequent order was suppressed by an additional power of ω.

As discussed in section 3.1.3, at tree level there exists an *entire tower* of universal soft theorems: leading, subleading, sub-subleading, and so on. However, once loop corrections are included, this tower is deformed into a richer structure: at each order in ω, additional logarithmic enhancements appear. At higher loops, one encounters higher powers of $\ln\omega$ multiplying the corresponding tree-level power of ω [44, 141–143]. The resulting structure can be visualised as a "soft pyramid," schematically summarised in Table 4.2.

Soft order	**Tree level**	**1-loop**	**2-loop**	**n-loop**
Leading	ω^{-1}			
Subleading	ω^0	$\ln\omega$		
Sub-subleading	ω	$\omega\ln\omega$	$\omega(\ln\omega)^2$	
(Sub)^n-leading	ω^{n-1}	$\omega^{n-1}\ln\omega$	$\omega^{n-1}(\ln\omega)^2$	$\omega^{n-1}(\ln\omega)^n$

Table 4.2: Structure of the "soft pyramid." At tree level the soft expansion forms a tower in powers of ω, while loop corrections generate a hierarchy of logarithmic enhancements. The $\ln\omega$ term at subleading order is one-loop exact, but higher powers of $\ln\omega$ appear at higher loops.

Such logarithmically enhanced behaviour is present in both gauge theory and gravity. While the universality of the coefficients in this extended expansion is still an open question, certain partial results are known. For example, in the case of soft photons, it was shown in [144] that the *classical* coefficients of the terms of order $\omega^n(\ln\omega)^{n+1}$ are universal. For gravity, the universality properties of higher-loop logarithmic coefficients remain less well understood.

Although our main interest in this work lies in the one-loop logarithmic soft factors, it is worth noting a striking example at two loops. The $\omega(\ln\omega)^2$ soft graviton factor was computed in [140] and found to be[1]

$$\begin{aligned}\frac{1}{\kappa^3}S^{\text{2-loop}}_{\text{gr}} = \frac{1}{2}\sum_{i=1}^{N}(\sigma'_{n+1})^2\,\frac{\varepsilon_{\mu\nu}\,p_i^\mu p_i^\nu}{p_i\cdot k} \\ +\sum_{i=1}^{N}\sigma'_{n+1}\,\frac{\varepsilon_{\mu\nu}\,p_i^\mu k^\rho}{p_i\cdot k}\left(p_i^\nu\,\frac{\partial\sigma_n}{\partial p_{i\,\rho}} - p_i^\rho\,\frac{\partial\sigma_n}{\partial p_{i\,\nu}}\right) \\ +\frac{1}{2}\sum_{i=1}^{N}\frac{\varepsilon_{\mu\nu}\,k^\rho k^\sigma}{p_i\cdot k}\left(p_i^\mu\,\frac{\partial\sigma_n}{\partial p_{i\,\rho}} - p_i^\rho\,\frac{\partial\sigma_n}{\partial p_{i\,\mu}}\right)\left(p_i^\nu\,\frac{\partial\sigma_n}{\partial p_{i\,\sigma}} - p_i^\sigma\,\frac{\partial\sigma_n}{\partial p_{i\,\nu}}\right)\,.\end{aligned} \tag{4.1.22}$$

Here σ_n and σ'_{n+1} are the same infrared-divergent coefficients of eqs. (3.2.3) and (3.2.4), and $\varepsilon_{\mu\nu}$ is the graviton polarisation tensor. Remarkably, the result (4.1.22) matches *exactly* with the infrared divergent factor in eq. (3.2.16), upon the usual identification $\ln\omega \leftrightarrow 1/\epsilon$.

This observation reinforces the theme running through this chapter: logarithmically enhanced soft factors and infrared divergences are two sides of the same coin. With this understanding in place, we now move on to investigate how the Ward identities of subleading asymptotic symmetries encode these loop-corrected soft theorems.

4.2 Subleading Ward identities

Recall that at tree level, the subleading soft graviton theorem can be expressed as a constraint on the $(n+1)$-point amplitude of the form

$$\lim_{\omega\to 0}\,(1+\omega\,\partial_\omega)\,\mathcal{M}_{n+1} \;=\; S^{(1)}\,\mathcal{M}_n\,. \tag{4.2.1}$$

[1]See Appendix C for a derivation from subsubleading Ward identities.

Here the projector $(1+\omega\partial_\omega)$ acts on the analytic ω-expansion of the soft factorisation, eliminating all terms except the constant $\mathcal{O}(\omega^0)$ term in the expansion.

When logarithmic terms in ω are present, however, this simple story changes. The operator in (4.2.1) no longer annihilates all non-constant terms: it now acts nontrivially on the $\ln\omega$ piece, producing an infrared-divergent contribution to the subleading soft limit. Inserting the general expansion (4.1.2) into (4.2.1) gives

$$\lim_{\omega\to 0}(1+\omega\,\partial_\omega)\,\mathcal{M}_{n+1} = \lim_{\omega\to 0}\Big[\ln\omega\; S^{(\ln)} + S^{(1)}\Big]\,\mathcal{M}_n\,. \tag{4.2.2}$$

The appearance of the $\ln\omega$ term here is exactly what one expects from the loop-corrected subleading soft theorem (cf. eq. (3.2.14)), once the relation between $S^{(\ln)}$ and the infrared-divergent coefficient is used. This agrees with the discussion in the previous section: to study the loop corrections to the subleading soft theorem, it is sufficient to work in dimensional regularisation and analyse the infrared-divergent structure of the amplitude.

The same observation applies in gauge theory. In the soft photon case, the modification in (4.2.2) matches precisely with the infrared-divergent terms obtained from explicit loop calculations, confirming that the $\ln\omega$ enhancement is a universal feature of both gravitational and electromagnetic scattering.

From the perspective of asymptotic symmetries, the tree-level subleading soft graviton theorem is equivalent to the Ward identity for superrotations. If superrotations are to remain a genuine symmetry of the *full* quantum gravity $\mathcal{S}$-matrix, their Ward identity must reproduce the complete, loop-corrected subleading soft theorem — including the $\ln\omega$ (or equivalently $1/\epsilon$) infrared-divergent term. In other words, the presence of the logarithmic factor does not merely alter the coefficient of the tree-level soft factor; it imposes an additional requirement on the symmetry algebra itself.

Finally, we note that the $\ln\omega$ term can also be isolated into an *independent* soft theorem, obtained by using the projection operator

$$\lim_{\omega\to 0}\Big(\partial_\omega\,\omega^2\,\partial_\omega\Big)\,\mathcal{M}_{n+1} \;=\; S^{(\ln)}\,\mathcal{M}_n\,. \tag{4.2.3}$$

This operator annihilates all analytic terms and singles out the $\ln\omega$ piece of the soft expansion. Although we will not explore eq. (4.2.3) in detail here, we note that the classical part of this relation was recently connected to asymptotic symmetries in [145, 146].

4.2.1 Superrotation

We are now in a position to ask whether the *logarithmic soft factor*, when viewed as a correction to the subleading soft graviton theorem, can be obtained directly from the Ward identities of superrotation. As discussed in section 3.3.2, the subleading soft graviton theorem can be expressed as the Ward identity of certain charges which differ from the standard superrotation charges by an additional "soft" contribution, given by

$$\Delta\mathcal{F}^{\text{soft}}_{\mathcal{Y}} = \frac{1}{16\pi G}\int d^2z\,\mathcal{Y}\left(\frac{3}{2}C^{(0)}_{zz}\,\mathscr{D}\mathcal{N}^{(0)}_{\bar z\bar z} + \frac{1}{2}\mathcal{N}^{(0)}_{\bar z\bar z}\,\mathscr{D}C^{(0)}_{zz}\right). \tag{4.2.4}$$

Here $C^{(0)}_{zz}$ denotes the supertranslation Goldstone mode, $\mathcal{N}^{(0)}_{\bar z\bar z}$ is the corresponding zero mode of the News tensor, $\mathscr{D}$ denotes the sphere covariant derivative, and $\mathcal{Y}$ is the superrotation parameter.

Following [139], and using integration by parts on the sphere, the expression above can be recast in the equivalent form

$$\Delta F^{\text{soft}}_{\mathcal{Y}} = \frac{2}{\kappa^2}\int d^2z\,\mathcal{Y}\Big[-\partial^3(C^{(0)}\mathcal{N}^{(0)}_{\bar z\bar z}) + 3\bar\partial^2\mathcal{N}^{(0)}_{zz}\partial C^{(0)} + C^{(0)}\partial\bar\partial^2\mathcal{N}^{(0)}_{zz}\Big]\,, \tag{4.2.5}$$

where $\kappa^2 = 32\pi G$ and $(\partial, \bar{\partial})$ are derivatives with respect to $(z, \bar{z})$ on the celestial sphere.

We now insert the modified soft flux (4.2.5) into the $\mathcal{S}$-matrix Ward identity, using the insertion formulae for the Goldstone mode $C^{(0)}$ and the News zero mode $\mathcal{N}_{zz}^{(0)}$ from section 3.5:

$$\begin{aligned} C^{(0)} &\longrightarrow \text{Goldstone insertion, eq. (3.5.11)}, \\ \mathcal{N}_{zz}^{(0)} &\longrightarrow \text{zero-mode insertion, eq. (3.5.1)}. \end{aligned}$$

This yields

$$\begin{aligned} &\langle \text{out}|\Delta F_{\mathcal{Y}}^{\text{soft}}(\mathscr{I}^+)\,\mathcal{S} + \mathcal{S}\,\Delta F_{\mathcal{Y}}^{\text{soft}}(\mathscr{I}^-)|\text{in}\rangle \\ &= -\frac{i\kappa}{8\pi\epsilon}\int d^2z\,\mathcal{Y}\left[\partial^3(\hat{\sigma}'_{n+1}\,\hat{S}_n^{(0)-}) - \hat{\sigma}'_{n+1}\,\partial\bar{\partial}^2\hat{S}_n^{(0)+} - 3\partial\hat{\sigma}'_{n+1}\,\bar{\partial}^2\hat{S}_n^{(0)+}\right]\langle \text{out}|\mathcal{S}|\text{in}\rangle \\ &= -\frac{i\kappa}{16\pi\epsilon}\int d^2z\,\mathcal{Y}\,\partial^3(\hat{\sigma}'_{n+1}\,\hat{S}_n^{(0)-})\,\langle \text{out}|\mathcal{S}|\text{in}\rangle, \end{aligned} \tag{4.2.6}$$

where, in the second equality, we have used the identity (5.18) of [139] to combine the three terms in the first line.

When the result (4.2.6) is included along with the remaining soft and hard contributions, the Ward identity takes the form

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)\mathcal{M}_{n+1} = \left[\frac{1}{\epsilon}\sigma'_{n+1}S^{(0)} + S^{(1)\,\text{tree}}\right]\mathcal{M}_n. \tag{4.2.7}$$

The first (divergent) term here, proportional to $\sigma'_{n+1}S^{(0)}$, corresponds to the *gravitational drag* on the soft graviton, as discussed earlier. Its appearance from a modification of the soft flux relative to the tree-level expression is thus physically natural.

We are left with the second contribution to the logarithmic soft factor, namely $S_n^{(1)J}\,\sigma_n$, which arises from the *long-range interaction between the hard particles* themselves. Based on the analogy with the previous step, we expect this term to originate from a modification of the *hard flux* of the superrotation charge.

Recall from section 2.4 that, when computing the action of hard charges on external states, we implicitly assumed the bulk fields to become free at null and timelike infinity. In reality, fields coupled to gravity (or gauge fields) remain *interacting* at these boundaries due to long-range forces. This is exactly the situation encountered in section 3.4, where we accounted for such interactions by dressing the bare field operators.

The asymptotic limit of a scalar filed interacting with gravity can equivalently be understood as replacing the free field mode operators with *dressed*[2] mode operators. For massive fields, this amounts to performing the replacement [33]

$$b(\hat{p})^\dagger \quad \mapsto \quad \tilde{b}(\hat{p})^\dagger = \exp\left[-\frac{im}{2}\int d^2\hat{q}\,\mathcal{G}(\hat{p};\hat{q})\,C^{(0)}(\hat{q})\right]b(\hat{p})^\dagger\,, \tag{4.2.8}$$

as follows from eq. (3.5.6). Recall that $\mathcal{G}^{(3)}$ is the bulk-to-boundary propagator given by eq (2.4.6) and $C^{(0)}$ is the supertranslation Goldstone mode found in section 2.3.4.

Similarly, for a massless creation operator we perform the replacement,

$$d(\hat{q})^\dagger \quad \mapsto \quad \tilde{d}(\hat{q})^\dagger = \exp\left[-i\omega\,C^{(0)}(\hat{q})\right]d(\hat{q})^\dagger. \tag{4.2.9}$$

Before proceeding with superrotation, we check that this dressing does *not* modify the supertranslation charge. For the massive case, this is immediate: the exponential factor is

[2]We are using the term dressing here in a different context than the Fadeev-Kulish dressings of section 3.4. The Fadeev-Kulish dressed operators asymptote to free field operators, thus effectively *undressing* the modifications due to the dressings considered here.

independent of the time coordinate τ, and being a pure phase, cancels between the field and its conjugate in $T^{(3)}_{\tau\tau}$. The charge is then

$$Q^{i^+}_{\mathcal{T}} = \frac{im^2}{32(2\pi)^3}\int_{\mathcal{H}} d^3\hat{p}\,\mathcal{T}_{\mathcal{H}}(\hat{p})\,b^\dagger b. \tag{4.2.10}$$

For massless particles, starting from eq. (2.2.11), the matter contribution to the supertranslation charge is

$$F^{\text{matter}}_{\mathcal{T}} = \int du\,d^2z\,\mathcal{T}(z,\bar{z})\int d\omega_1\,d\omega_2\;\omega_1\omega_2\,e^{-i(\omega_1-\omega_2)u}d^\dagger d\,e^{iC^{(0)}(\omega_1-\omega_2)}. \tag{4.2.11}$$

The u-integral gives $\delta(\omega_1-\omega_2)$, setting the exponential to unity and leaving

$$F^{\text{matter}}_{\mathcal{T}} = \int d\omega\,d^2z\,\mathcal{T}(z,\bar{z})\,\omega^2 d^\dagger d\,, \tag{4.2.12}$$

which is exactly the free-field result. For hard gravitons, a similar check can be performed as in the case of massless particles. This confirms that the leading soft graviton theorem, associated with supertranslations, is exact to all orders.

The free-field hard superrotation charge for massive particles is

$$Q^{i^+(free)}_{\mathcal{Y}} = \frac{im^2}{32(2\pi)^3}\int_{\mathcal{H}} d^3\hat{p}\,\mathcal{Y}^a_{\mathcal{H}}(\hat{p})\left(b^\dagger\partial_a b - \partial_a b^\dagger\,b\right). \tag{4.2.13}$$

Replacing $b^\dagger$ by its dressed form adds a new term:

$$Q^{i^+}_{\mathcal{Y}} = Q^{i^+(free)}_{\mathcal{Y}} - \Delta Q^{i^+}_{\mathcal{Y}}\,, \tag{4.2.14}$$

with

$$\Delta Q^{i^+}_{\mathcal{Y}} = \frac{m^3}{32(2\pi)^3}\int d^2\hat{q}\int d^3\hat{p}\,\mathcal{Y}^a_{\mathcal{H}}(\hat{p})\,\partial_a\mathcal{G}(\hat{p};\hat{q})\,C^{(0)}(\hat{q})\,b^\dagger(\hat{p})b(\hat{p}). \tag{4.2.15}$$

The analogous new term for massless particles, after performing the u-integral, is

$$\Delta\mathcal{F}^{\text{matter}}_{\mathcal{Y}} = \frac{1}{16\pi^2}\int d\omega\,d^2z\,\left(\mathcal{Y}\,\partial C^{(0)} - \frac{1}{2}\partial\mathcal{Y}\,C^{(0)}\right)\omega^2 d^\dagger(\omega,z,\bar{z})d(\omega,z,\bar{z}). \tag{4.2.16}$$

In arriving to this expression we have made use of the fact that $uT^{(2)}_{uu}$ can be traded for an ω derivative, that is,

$$\begin{aligned}&\int dud^2z\,\mathcal{Y}\frac{u}{2}\partial_z T^{(2)}_{uu}\\ &\quad= -\int dud^2z\,\partial_z\mathcal{Y}\frac{1}{2}\int d\omega_1 d\omega_2\,(\omega_1\omega_2)(e^{i(\omega_1-\omega_2)C^{(0)}}a^\dagger_{s2}a_{s1}\partial_{\omega_1}e^{-i(\omega_1-\omega_2)u}+\cdots)\,,\end{aligned} \tag{4.2.17}$$

and used integration by parts on the ω derivative.

The gravitational hard flux receives an analogous correction from self-interactions:

$$\Delta\mathcal{F}^{\text{hard}}_{\mathcal{Y}} = \int du\,d^2z\,\tilde{N}_{zz}\tilde{N}_{\bar{z}\bar{z}}\left(\mathcal{Y}\,\partial C^{(0)} - \frac{1}{2}\partial\mathcal{Y}\,C^{(0)}\right). \tag{4.2.18}$$

The result of these new terms on the Ward identity of superrotation can now be obtained by inserting them in the S-matrix correlator and using the following identitities,

$$\sigma_n = -\frac{\epsilon}{8\kappa^2}\sum_{ij=1}^{n} m_i m_j\int d^2\hat{q}\,d^2\hat{q}'\,\mathcal{G}(\hat{p}_i;\hat{q})\mathcal{G}(\hat{p}_j;\hat{q}')\langle C^{(0)}(\hat{q})C^{(0)}(\hat{q}')\rangle\,, \tag{4.2.19}$$

$$\begin{aligned}\hat{\sigma}'_{n+1}(\hat{q}') &= -\frac{\epsilon}{2\kappa^2}\sum_{i=1}^{n} m_i \int d^2\hat{q}\,\mathcal{G}(\hat{p}_i;\hat{q})\,\langle C^{(0)}(\hat{q}')C^{(0)}(\hat{q})\rangle \\ &= \frac{1}{(8\pi)^2}\sum_{i=1}^{n} m_i \int d^2\hat{q}\,\mathcal{G}(\hat{p}_i;\hat{q})\,(\hat{q}\cdot\hat{q}')\ln(\hat{q}\cdot\hat{q}')\,,\end{aligned} \tag{4.2.20}$$

such that

$$\partial^3 S_n^{(1)J-}(\hat{q})\,\sigma_n = \frac{1}{2}\sum_i^n m_i \int d^2\hat{q}'\,\mathcal{G}(\hat{p}_i;\hat{q}')\,\partial^3 S_n^{(1)J-}(\hat{q})\,\hat{\sigma}'_{n+1}(\hat{q}')\,. \tag{4.2.21}$$

For the case of massive external particles we have,

$$\begin{aligned}&\langle \text{out}|\Delta Q_{\mathcal{Y}}^{i+}\mathcal{S} - \mathcal{S}\Delta Q_{\mathcal{Y}}^{i-}|\text{in}\rangle \\ &\quad = \frac{1}{16}\sum_i m_i \int d^2\hat{q}\,\mathcal{Y}^a_{\mathcal{H}}(\hat{p}_i)\partial_a\mathcal{G}(\hat{p}_i;\hat{q})\,\langle \text{out}|C^{(0)}(\hat{q})\mathcal{S}|\text{in}\rangle \\ &\quad = \frac{-i\kappa^2}{16\epsilon}\sum_i m_i \int d^2\hat{q}\,\mathcal{Y}^a_{\mathcal{H}}(\hat{p}_i)\partial_a\mathcal{G}(\hat{p}_i;\hat{q})\,\hat{\sigma}'_{n+1}(\hat{q})\,\langle \text{out}|\mathcal{S}|\text{in}\rangle \\ &\quad = \frac{i}{32}\sum_{ij} m_i m_j \int d^2\hat{q}'\,\mathcal{G}(\hat{p}_j;\hat{q}') \int d^2\hat{q}\,\mathcal{Y}^a_{\mathcal{H}}(\hat{p}_i)\partial_a\mathcal{G}(\hat{p}_i;\hat{q})\,\langle C^{(0)}(\hat{q})C^{(0)}(\hat{q}')\rangle\,\langle \text{out}|\mathcal{S}|\text{in}\rangle \\ &\quad = \frac{-i\kappa}{32\pi\epsilon}\sum_j m_j \int d^2\hat{q}'\,\mathcal{G}(\hat{p}_j;\hat{q}') \int d^2\hat{q}\,\mathcal{Y}(\hat{q})\,\partial^3 S_n^{(1)J-}(\hat{q})\,\hat{\sigma}'_{n+1}(\hat{q}')\,\langle \text{out}|\mathcal{S}|\text{in}\rangle \\ &\quad = \frac{-i\kappa}{16\pi\epsilon}\int d^2\hat{q}\,\mathcal{Y}(\hat{q})\,\partial^3 S_n^{(1)J-}(\hat{q})\,\sigma_n\,\langle \text{out}|\mathcal{S}|\text{in}\rangle\,.\end{aligned} \tag{4.2.22}$$

In the first equality we acted on the massive external states with the particle number operator

$$b(\hat{p})^\dagger\, b(\hat{p})\,|p'\rangle = m^{-2}(2\pi)^3\,(2E_{\hat{p}})\,\delta^3(\hat{p}-\hat{p}')\,|p'\rangle\,. \tag{4.2.23}$$

The second equality is obtained from (3.5.11), while the third equality relies on (4.2.20). The fourth equality holds thanks to (3.3.19) together with (4.2.20). The last equality follows directly from (4.2.21).

For massless particles, a similar computation gives

$$\begin{aligned}&\langle \text{out}|\Delta\mathcal{F}_{\mathcal{Y}}^{\text{matter}}\mathcal{S} - \mathcal{S}\Delta\mathcal{F}_{\mathcal{Y}}^{\text{matter}}|\text{in}\rangle \\ &\quad = \frac{1}{8}\sum_i \omega_i \left[\mathcal{Y}(z_i,\bar{z}_i)\partial_{z_i} - \frac{1}{2}\partial_{z_i}\mathcal{Y}(z_i,\bar{z}_i)\right]\langle \text{out}|C^{(0)}(\hat{q}_i)\mathcal{S}|\text{in}\rangle \\ &\quad = \frac{-i\kappa^2}{8\epsilon}\sum_i \omega_i \left[\mathcal{Y}(z_i,\bar{z}_i)\partial_{z_i} - \frac{1}{2}\partial_{z_i}\mathcal{Y}(z_i,\bar{z}_i)\right]\sigma_n\,\langle \text{out}|\mathcal{S}|\text{in}\rangle.\end{aligned} \tag{4.2.24}$$

To arrive to the first line we have made use of the fact the $\Delta F_{\mathcal{Y}}^{\text{matter}}$ is same as $\mathcal{F}_{\mathcal{T}}^{\text{matter}}$ with $\mathcal{T} = \mathcal{Y}\partial - \frac{1}{2}\partial\mathcal{Y}$. A similar computation follows for the case of $\Delta\mathcal{F}_{\mathcal{Y}}^{\text{hard}}$.

Now choosing the superrotation parameter of section 3.3.2, $\mathcal{Y} = \frac{(z-w)^2}{\bar{z}-\bar{w}}$ and using eq. (3.3.18) we arrive at the following Ward identity,

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)\langle \text{out}|a^\dagger(\omega,z,\bar{z})\mathcal{S}|\text{in}\rangle = \left[\frac{1}{\epsilon}\,S^{(1)J}\sigma_n - \frac{1}{\epsilon}\sigma'_n S^{(0)} + S^{(1)}\right]\langle \text{out}|\mathcal{S}|\text{in}\rangle\,, \tag{4.2.25}$$

Where we have also inserted the full soft flux and collected all the terms without the soft graviton creation operator on the right hand side. As explained in section 4.1.1, since we are evaluating the operator insertions in dimensional regularization we get a $\frac{1}{\epsilon}$. Translating[3] $1/\epsilon \to \ln\omega$ the result above can be expressed in terms of $\log\omega$.

[3]See appendix D for how this relation can be understood at the level of operator insertion.

The first divergent term is from gravitational drag (4.2.6), and the second from long-range hard particle interactions (4.2.22), (4.2.24), precisely matching the two parts of the logarithmic soft factor in eq. (4.1.3).

4.2.2 Asymptotic QED

We now turn to the case of *asymptotic QED*, with the goal of recovering the *full* subleading soft photon theorem, including the *logarithmic corrections*, by closely following the steps that were successful in the gravitational case.

Unlike gravity, QED has no self-interaction terms in the photon field. This has an important consequence, the soft part of the charge receives *no* additional contributions from loop effects involving only photons.

This can be checked explicitly: starting from the subleading photon charge (3.3.42), one may apply the renormalization substitution (2.6.11). The form of the charge remains unchanged, so the tree-level and loop-corrected expressions for the soft charge are identical.

The only remaining source of a logarithmic correction, just as in the gravitational case, is the *long-range interaction between the hard charged particles*. This is the electromagnetic analogue of the gravitational tail effect. The physical mechanism is the same: asymptotic charged states are not truly free at null and timelike infinity because they remain coupled to the electromagnetic field.

Following the same logic as in section 3.4 and in the superrotation analysis, this effect is incorporated by dressing the bare creation and annihilation operators with the LGT Goldstone mode.

For a massive charged scalar creation operator $b(\hat{p})^\dagger$, the electromagnetic dressing is

$$b(\hat{p})^\dagger \quad \mapsto \quad \tilde{b}(\hat{p})^\dagger = \exp\left[-\frac{im}{2}\int d^2\hat{q}\,\mathcal{G}^{(2)}(\hat{p};\hat{q})\,\mathscr{A}(\hat{q})\right] b(\hat{p})^\dagger, \tag{4.2.26}$$

where $\mathscr{A}(\hat{q})$ is the Goldstone mode for large $U(1)$ gauge transformations on the celestial sphere (2.6.9), and $\mathcal{G}^{(2)}$ is the electromagnetic analogue of the gravitational Green's function given in eq (2.6.17).

For a massless charged particle creation operator $d(\hat{q})^\dagger$, the dressing is

$$d(\hat{q})^\dagger \quad \mapsto \quad \tilde{d}(\hat{q})^\dagger = \exp\left[-i\omega\,\mathscr{A}(\hat{q})\right] d(\hat{q})^\dagger. \tag{4.2.27}$$

As in the gravitational supertranslation case, the leading $U(1)$ large gauge charge is *unchanged* by this replacement. The reason is again that the dressing factor is a pure phase with no u or τ dependence, so it cancels between the field and its complex conjugate when inserted into the expression for the leading charge. This ensures that the leading soft photon theorem holds exactly to all orders.

The dressing *does* modify the hard part of the *subleading* large gauge charge. Expressing the hard charges in terms of the dressed operators gives the new contributions:

$$\begin{aligned}\Delta Q^{i^+}_{\Upsilon} &= \int d^2\hat{q}\int d^3\hat{p}\;\Upsilon^\alpha_{\mathcal{H}}(\hat{p})\,\partial_\alpha\mathcal{G}^{(2)}(\hat{p};\hat{q})\,\mathscr{A}(\hat{q})\,b^\dagger(\hat{p})b(\hat{p}),\\ \Delta Q^{\text{hard}}_{\Upsilon} &= \int d\omega\,d^2z\,\left(\Upsilon\,\partial\mathscr{A} - \frac{1}{2}\partial\Upsilon\,\mathscr{A}\right)\omega^2\,d^\dagger(\omega,z,\bar{z})\,d(\omega,z,\bar{z}),\end{aligned} \tag{4.2.28}$$

where Υ is the subleading large gauge parameter on the sphere. The first line is the massive-particle contribution; the second is from massless charged particles.

There is also an analogous contribution from the antiparticle (opposite charge) sector, which acts on the other half of the complex field.

The electromagnetic analogue of the gravitational σ_n is the quantity λ_n, which collects the effect of long-range electromagnetic interactions between pairs of charged particles. It can be written as

$$\begin{aligned} \lambda_n &= -\frac{\epsilon}{4} \sum_{i,j=1}^{n} m_i m_j \int d^2\hat{q}\, d^2\hat{q}'\, \mathcal{G}^{(2)}(\hat{p}_i;\hat{q})\, \mathcal{G}^{(2)}(\hat{p}_j;\hat{q}')\, \langle \mathscr{A}(\hat{q})\, \mathscr{A}(\hat{q}')\rangle, \\ \partial^2 S^{(1)J-}_{\text{em}}(\hat{q})\, \lambda_n &= \frac{1}{2} \sum_{i=1}^{n} m_i \int d^2\hat{q}'\, \mathcal{G}^{(2)}(\hat{p}_i;\hat{q}')\, \partial^2 S^{(1)J-}_{\text{em}}(\hat{q})\, \hat{\lambda}'_{n+1}(\hat{q}'). \end{aligned} \tag{4.2.29}$$

These expressions are the QED counterparts of (4.2.20) and (4.2.21) in the gravitational case, and can be checked using eqs (3.2.7), (2.6.17), and (3.1.6).

Following exactly the same steps as in the gravitational computation, the insertions of the new hard terms into the $\mathcal{S}$-matrix yield

$$\begin{aligned} \langle \text{out}|\Delta Q^{i+}_{\Upsilon}\,\mathcal{S} - \mathcal{S}\Delta Q^{i-}_{\Upsilon}\,|\text{in}\rangle &= \frac{-i}{2\pi\epsilon} \sum_{m_i \neq 0} \int d^2\hat{q}\; \Upsilon(\hat{q})\, \partial^2 S^{(1)J-}_{\text{em},i}(\hat{q})\, \lambda^i_n\, \langle \text{out}|\mathcal{S}|\text{in}\rangle, \\ \langle \text{out}|\Delta Q^{\text{hard}}_{\Upsilon}\mathcal{S} - \mathcal{S}\Delta Q^{\text{hard}}_{\Upsilon}|\text{in}\rangle &= \frac{-i}{\epsilon} \sum_{m_i=0} \omega_i \left[\Upsilon(z_i,\bar{z}_i)\partial_{z_i} - \frac{1}{2}\partial_{z_i}\Upsilon(z_i,\bar{z}_i)\right] \lambda^i_n\, \langle \text{out}|\mathcal{S}|\text{in}\rangle. \end{aligned} \tag{4.2.30}$$

These terms are the electromagnetic analogues of (4.2.22) and (4.2.24) in the gravitational case, and together they reconstruct the logarithmic part of the subleading soft photon theorem.

Finally, we choose the large gauge parameter to be $\Upsilon = \frac{z-w}{\bar{z}-\bar{w}}$ and use the expressions for $S^{(1)}$ from eq. (3.3.40). Inserting these into the Ward identity and recalling that $1/\epsilon$ maps to $\ln \omega$ in dimensional regularization, we find

$$\lim_{\omega\to 0} \langle \text{out}|a^\dagger(\omega,z,\bar{z})\,\mathcal{S}\,|\text{in}\rangle = \left[\frac{1}{\epsilon}\, S^{(1)J}_{\text{em,i}}\, \lambda^i_n + S^{(1)}_{\text{em}}\right] \langle \text{out}|\mathcal{S}|\text{in}\rangle. \tag{4.2.31}$$

This is the full *subleading soft photon theorem*, now including the *logarithmic correction* from the long-range electromagnetic interaction between charged particles.

Just as in gravity, the logarithmic enhancement arises entirely from the hard part of the asymptotic charge, due to the persistent coupling of charged particles to the soft sector. The absence of photon self-interactions makes the analysis technically simpler, but the structure of the correction and its origin are directly parallel to the gravitational case.

4.2.3 Gauge–Gravity Interactions

Up to this point we have restricted attention to theories with *either* purely gravitational interactions or purely electromagnetic interactions. When the two long-range forces are both present, the logarithmic soft factors receive *additional* corrections. For the soft graviton factor, the relevant modification is given in eq. (4.1.4), while for the soft photon factor it is given in eq. (4.1.6).

In this section we focus on the gravitational soft theorem and account for the correction arising from electromagnetic interactions. The analogous electromagnetic correction to the *soft photon factor* requires an understanding of photon–graviton interactions, which we will return to in the next chapter (see secion 5.7) when studying Einstein–Maxwell theory in detail.

If a particle moving in a gravitational background carries also an electric charge (coupling to the photon), then *both* long-range interactions must be included in its asymptotic description. The correct asymptotic creation operators for such a particle are obtained by combining the gravitational dressing from eq. (3.5.6)and (3.5.4) with the electromagnetic dressing from eq. (4.2.26) and (4.2.27).

For massive particles:

$$b(\hat{p})^\dagger \quad \mapsto \quad \tilde{b}(\hat{p})^\dagger = \exp\left[-\frac{im}{2}\int d^2\hat{q}\,\mathcal{G}^{(2)}(\hat{p};\hat{q})\,\mathscr{A}(\hat{q})\right] b(\hat{p})^\dagger, \tag{4.2.32}$$

and for massless charged particles:

$$d(\hat{q})^\dagger \quad \mapsto \quad \tilde{d}(\hat{q})^\dagger = \exp\left[-i\omega\,\mathscr{A}(\hat{q})\right] d(\hat{q})^\dagger, \tag{4.2.33}$$

where $\mathscr{A}(\hat{q})$ is the Goldstone mode associated with large $U(1)$ gauge transformations on the celestial sphere, and $\mathcal{G}^{(2)}$ is the corresponding electromagnetic Green's function. These dressing factors multiply the *already present* gravitational dressing (not shown explicitly here), so the full asymptotic operator is a product of gravitational and electromagnetic exponentials.

Electromagnetic interactions produce additional contributions to the *hard* part of the superrotation charge. These come from the phase factors in the dressed operators above and are structurally identical to the QED case discussed earlier, except that here they appear in the *gravitational* Ward identity.

The new terms are:

$$\begin{aligned}
\Delta_{\rm em}Q^{i+}_{\mathcal{Y}} &= \int d^2\hat{q}\int d^3\hat{p}\;\mathcal{Y}^\alpha_{\mathcal{H}}(\hat{p})\,\partial_\alpha\mathcal{G}^{(2)}(\hat{p};\hat{q})\,\mathscr{A}(\hat{q})\;b^\dagger(\hat{p})b(\hat{p}),\\
\Delta_{\rm em}\mathcal{F}^{\rm hard}_{\mathcal{Y}} &= \int d\omega\,d^2z\,\left(\mathcal{Y}\,\partial\mathscr{A}-\frac{1}{2}\partial\mathcal{Y}\,\mathscr{A}\right)\omega^2\,d^\dagger(\omega,z,\bar{z})\,d(\omega,z,\bar{z}),
\end{aligned} \tag{4.2.34}$$

where Υ is the relevant mode of the superrotation vector field on the sphere.

The effect of the $\mathscr{A}$-dependent terms on the Ward identity can be computed using the same correlator techniques as before. This yields:

$$\begin{aligned}
\langle\text{out}|\Delta Q^{i+}_{\mathcal{Y}}\mathcal{S}-\mathcal{S}\Delta Q^{i-}_{\mathcal{Y}}|\text{in}\rangle &= \frac{-i}{16\pi\epsilon}\sum_{m_i\neq 0}\int d^2\hat{q}\;\mathcal{Y}(\hat{q})\,\partial^3 S^{(1)J-}_{\text{gr},i}(\hat{q})\,\lambda^i_n\,\langle\text{out}|\mathcal{S}|\text{in}\rangle,\\
\langle\text{out}|\Delta Q^{\rm hard}_{\mathcal{Y}}\mathcal{S}-\mathcal{S}\Delta Q^{\rm hard}_{\mathcal{Y}}|\text{in}\rangle &= \frac{-i}{8\epsilon}\sum_{m_i=0}\omega_i\left[\mathcal{Y}(z_i,\bar{z}_i)\,\partial_{z_i}-\frac{1}{2}\partial_{z_i}\mathcal{Y}(z_i,\bar{z}_i)\right]\lambda_n\,\langle\text{out}|\mathcal{S}|\text{in}\rangle.
\end{aligned} \tag{4.2.35}$$

Here, λ_n is the same function appearing in the soft photon logarithmic factor.

Combining these electromagnetic hard-flux corrections with the purely gravitational ones reproduces the full logarithmic correction to the subleading soft graviton theorem *in the presence of electromagnetism*. Physically, this reflects the fact that in gauge–gravity systems, the asymptotic dynamics of a charged particle is influenced by *both* the long-range gravitational field and the long-range electromagnetic field, and both must be incorporated in the asymptotic charges if the Ward identities are to capture the complete soft theorem.

Concluding remarks

In this chapter we have taken the first step beyond the tree-level structure of soft theorems by systematically incorporating loop effects. Focusing on the subleading soft graviton and photon theorems, we have identified the logarithmic corrections that arise at one-loop, traced their origin to long-range interactions and self-interactions, and shown how they fit naturally within the Ward identities of the corresponding asymptotic symmetries. Along the way, we extended the analysis to include theories with simultaneous gauge and gravitational interactions, thereby completing the picture of loop-induced logarithmic factors in this setting.

From the perspective of the "soft pyramid" shown in table 4.2, our exploration in this chapter corresponds to moving *horizontally* along the loop-order axis while staying fixed at the first two steps in the ω-expansion (leading and subleading orders). In the next chapter, we will instead move *down* along the ω-expansion axis: keeping ourselves at tree level, but going beyond the leading ω^{-1} order to explore the structure of higher-order terms. We will do this in the concrete context of Einstein–Maxwell theory, where gravitational and electromagnetic radiation are present simultaneously, allowing us to investigate the interplay of their higher-order soft limits and the corresponding asymptotic symmetry structure.

Chapter 5

Celestial $sw_{1+\infty}$ Algebra

Our study of asymptotic symmetries so far has been centered on their connection to soft theorems. Up to this point, these soft theorems have been expressed in the *momentum basis*, where they impose constraints on scattering amplitudes. In the framework of *celestial holography*, however, momentum-space soft theorems are mapped to *conformally soft theorems* in the celestial basis [147–150]. In this picture, the soft theorem translates into a Ward identity for conformal currents, in complete analogy with the leading soft graviton current relation in eq. (3.5.1) (see, e.g., [31] and references therein for a review).

In section 3.1.3 we saw that, at tree level, gauge and gravity amplitudes obey an *infinite tower* of soft theorems. Within celestial holography, this soft tower was related to an infinite tower of charges in [151,152], which were shown to close the so-called $w_{1+\infty}$ algebra. The action of these charges on the S-matrix reproduces the entire tree-level soft tower. This result drew renewed attention to the long-known structures of self-dual gravity, which also possesses an infinite-dimensional symmetry algebra [153–157]. Labeling the generators by w^p_m, the $w_{1+\infty}$ Poisson bracket reads

$$\{w^p_m, w^q_n\} = \left[m(q-1) - n(p-1)\right] w^{p+q-2}_{m+n} \,. \tag{5.0.1}$$

Here p, q run over positive half-integers,

$$p \in \left\{1, \frac{3}{2}, 2, \frac{5}{2}, \dots\right\},$$

with the value inherited from the conformal dimension Δ of the soft graviton current (recall from eq (2.3.45) that soft modes transform as conformal fields of weight $(h, \bar{h})$, therefore their conformal dimension is given by $\Delta = h + \bar{h}$) via $\Delta_{\text{gr}} = 4 - 2p$. The mode indices m, n are restricted to the range $1 - p \leq m \leq p - 1$, making (5.0.1) the *wedge algebra* of $w_{1+\infty}$ (see [158] for a review of W-algebras). The action of these symmetries on general-spin massless celestial primaries and on massive scalars was studied in [159,160].

For non-abelian gauge theories, an analogous infinite tower of symmetries is organized into the so-called s-algebra [151,152],

$$\{s^{p,b}_{n'},\, s^{q,a}_{n}\} = i f^{ab}{}_{c}\, s^{q+p-1,\,c}_{n+n'} \,, \tag{5.0.2}$$

where a, b are color indices and p is related to the conformal dimension of the celestial gluon operator as $\Delta_{\text{gl}} = 3 - 2p$. In the abelian case, the above bracket vanishes. Nonetheless, both photon and gluon currents couple to gravitons, and their mixed commutators with the w-generators take the form

$$\{w^p_m,\, s^{q,a}_n\} = \left[m(q-1) - n(p-1)\right] s^{p+q-2,\,a}_{m+n} \,. \tag{5.0.3}$$

The set of commutation relations (5.0.1)–(5.0.3) defines the $sw_{1+\infty}$ *algebra*. This algebra was originally derived from collinear limits of celestial operator product expansions (OPEs) involving positive-helicity gravitons and gluons [151, 152, 161]. In this chapter, we will set aside the OPE derivation and instead attempt to reconstruct these charges directly from an analysis of the Einstein–Maxwell equations of motion. Our focus will be on understanding how the $sw_{1+\infty}$ structure is encoded into the (*truncated*) asymptotic phase space of Einstein-Maxwell theory.

We will build upon the results and methods developed for pure gravity and Yang–Mills theory in [60, 61, 162]. In section 5.1 we review the Einstein–Maxwell system in the Newman–Penrose formalism introduced earlier in section 2.3.2. Following the analysis of the above works, in section 5.3 we study the asymptotic equations of motion and identify a recursion relation for a family of putative charges. These charges turn out to be divergent and non-conserved even in non-radiative vacuum configurations. We therefore construct, in section 5.4, a set of quasi-conserved charges derived from them. In section 5.5, we smear these quasi-conserved charges with test functions over the celestial sphere and compute their algebra, showing that it matches precisely the $sw_{1+\infty}$ structure (5.0.1)–(5.0.3). Section 5.6 briefly reviews how this framework extends to the Einstein–Yang–Mills system. Our results are explicitly perturbative and valid at tree level only (for a recent non-linear treatment, see [163–166]). Finally, in section 5.7 we return to the logarithmic soft theorem analysis of the previous chapter, combining it with the results obtained here to show how gravitational loops contribute to the logarithmic soft photon theorem.

5.1 Einstein-Maxwell NP scalars

In Section 2.3.2 the Newmann-Penrose formalism was introduced for a generic spacetime. In this Section, we shall specialize this to asymptotically flat spacetimes useful for our analysis. To do this, let us first express a generic metric in the Bondi coordiantes (u, r, x^A),

$$ds^2 = \frac{V}{r}e^{2B}du^2 - 2e^{2B}dudr + g_{AB}(dx^A - U^A du)(dx^B - U^B du)\,, \tag{5.1.1}$$

V,U^A, B, and g_{AB} are all functions of the coordinates. To specialize this metric to asymptotically flat spacetimes, the following boundary conditions need to be imposed that are consistent with eq (2.3.5),

$$\begin{gathered} g_{AB}(u, r, x^C) = r^2 q_{AB}(x^C) + r\, C_{AB}(u, x^C) + D_{AB} + \mathcal{O}\left(r^{-1}\right)\,, \\ g_{uu} = \mathcal{O}\left(1\right)\,, \quad g_{ur} = -1 + \mathcal{O}\left(r^{-2}\right) \quad \text{and} \quad g_{uA} = \mathcal{O}\left(1\right)\,. \end{gathered} \tag{5.1.2}$$

With this choice we see the remaining functions can be expressed as,

$$V = 2M + \mathcal{O}(r^{-1}), \quad B = \mathcal{O}(r^{-2}), \quad U^A = -\frac{1}{2r^2}D_B C^{AB} + \mathcal{O}(r^{-3}) \tag{5.1.3}$$

For the generic class of spacetime metric (5.1.1), we make the following choice of the null tetrad,

$$\ell = \partial_r\,, \quad n = e^{-2B}\left(\partial_u + \frac{V}{2r}\partial_r + U^A\partial_A\right)\,, \quad m = \frac{1}{r}e^A\partial_A\,, \quad \bar{m} = \frac{1}{r}\bar{e}^A\partial_A\,, \tag{5.1.4}$$

It can be checked that all the inner products among the tetrad vectors are zero except $\ell \cdot n = -1$ and $m \cdot \bar{m} = +1$. The dyad vector $e^A = e^A(u, r, x^B)$ is a complex dyad for the $2d$ inverse spatial metric $r^2 g^{AB}$, i.e. $r^2 g^{AB} = e^A\bar{e}^B + \bar{e}^A e^B$, such that and $g_{AB}e^A e^B = 0$ and $r^{-2}g_{AB}e^A\bar{e}^B = +1$. This complex dyad itself admits an asymptotic expansion of the form

$$e^A(u, r, x^B) = \varepsilon^A(x^B) + \frac{1}{r}\sum_{n=0}^{\infty}\frac{e^{(n)A}(u, x^B)}{r^n}\,, \tag{5.1.5}$$

with the zeroth order part satisfying $q^{AB} = 2\varepsilon^{(A}\bar{\varepsilon}^{B)}$.

Our aim now is to study the Einsteni-Maxwell system in asymptotically flat spacetimes and identify the asymptotic symmetries. So we will be imposing these boundary conditions, along with the boundary conditions for gauge field in eq (2.6.2), for the action of Einstein-Maxwell theory[1],

$$S = \frac{1}{2\kappa^2}\int d^4x \left[R - \frac{1}{2}F^2\right] \tag{5.1.6}$$

with $\kappa^2 = 8\pi G$. The approach we will be taking is not to solve the complete Einstein-Maxwell system for the fields but only to study the asymptotic forms of these equations near null infinity ($\mathscr{I}$). We will study these equations in the Newman-Penrose (NP) formalism to extract useful information efficiently. with the zeroth order part satisfying $q^{AB} = 2\varepsilon^{(A}\bar{\varepsilon}^{B)}$. Once the choice of tetrad is made, the NP scalars can be defined as in section 2.3.2. The 'peeling' conditions for the NP scalars [81, 167–169] in presence of the inteactions can then be expressed as,

$$\begin{aligned}
\Phi_0 &= \sum_{s=0}^{\infty} \frac{1}{r^{3+s}}\Phi_0^s(u, x^A)\,,\\
\Phi_1 &= r^{-2}\Phi_1^0 - r^{-3}\bar{\eth}\Phi_0^0 + \mathcal{O}\left(r^{-4}\right)\,,\\
\Phi_2 &= r^{-1}\Phi_2^0 - r^{-2}\bar{\eth}\Phi_1^0 + \mathcal{O}\left(r^{-3}\right)\,,\\
\Psi_0 &= \sum_{s=0}^{\infty} \frac{1}{r^{5+s}}\Psi_0^s(u, x^A)\,,\\
\Psi_1 &= r^{-4}\Psi_1^0 - r^{-5}\left(\bar{\eth}\Psi_0^0 - 3\bar{\Phi}_1^0\Phi_0^0\right) + \mathcal{O}\left(r^{-6}\right)\,,\\
\Psi_2 &= r^{-3}\Psi_2^0 - r^{-4}\left(\bar{\eth}\Psi_1^0 - 2\bar{\Phi}_1^0\Phi_1^0\right) + \mathcal{O}\left(r^{-5}\right)\,,\\
\Psi_3 &= r^{-2}\Psi_3^0 - r^{-3}\left(\bar{\eth}\Psi_2^0 - \bar{\Phi}_1^0\Phi_2^0\right) + \mathcal{O}\left(r^{-4}\right)\,,\\
\Psi_4 &= r^{-1}\Psi_4^0 - r^{-2}\bar{\eth}\Psi_3^0 + \mathcal{O}\left(r^{-3}\right)\,.
\end{aligned} \tag{5.1.7}$$

In the above expressions, all near-$\mathscr{I}$ modes are functions of u and x^A. The subleading near-$\mathscr{I}$ modes of the NP scalars of lower spin weights are fixed by the hypersurface equations of motion (5.2.3) and (5.2.10), while $\eth$ and $\bar{\eth}$ are the GHP "edth" operators with respect to the 2-dimensional boundary transverse space $q_{AB}dx^A dx^B$[2].

The leading near-$\mathscr{I}$ modes of the lowest-spin weight NP scalars are related to the radiative fields according to

$$\begin{aligned}
\Phi_2^0 &= -\bar{F} = -\partial_u \bar{A}\,,\\
\Psi_4^0 &= \partial_u \bar{N} = \partial_u^2 \bar{C}\,,
\end{aligned} \tag{5.1.8}$$

where we have defined the negative helicity photon field $\bar{A}$ (and field strength $\bar{F} = \partial_u \bar{A}$) and the negative helicity gravitational shear $\bar{C}$ (of news $\bar{N} = \partial_u \bar{C}$) as

$$\begin{aligned}
\bar{A} &:= \bar{\varepsilon}^A A_A^{(0)}\,, \quad \bar{F} := \bar{\varepsilon}^A F_{uA}^{(0)} = \partial_u \bar{A}\,,\\
\bar{C} &:= \frac{1}{2}\bar{\varepsilon}^A\bar{\varepsilon}^B C_{AB}\,, \quad \bar{N} := \frac{1}{2}\bar{\varepsilon}^A\bar{\varepsilon}^B N_{AB} = \partial_u \bar{C}\,.
\end{aligned} \tag{5.1.9}$$

Similarly, the positive helicity counterparts, A, F, C and N, are obtained by replacing $\bar{\varepsilon}^A$ with ε^A in the above expressions[3].

[1]Compared to the Maxwell action using canonical field variables, $S_{\text{Maxwell}} = \int d^4x\sqrt{-g}\left[-\frac{1}{4}F_{\text{can}}^2\right]$, here we have performed the field redefinition $F_{\text{can}} = \frac{1}{\kappa}F$.

[2]Note that, in our conventions, $\eth^{\text{here}} = \eth^{\,[162]} = -\bar{\eth}^{\,[169]}$.

[3]We remark here that, compared to [60, 162],

$$C^{\text{here}} = \frac{1}{2}C^{\,[60]} = -C^{\,[162]} \quad \text{and} \quad N^{\text{here}} = -\frac{1}{2}\bar{N}^{\,[60]} = -N^{\,[162]}\,.$$

5.2 Equations of motion in NP formalism

To achieve our aim of studying the Einstein–Maxwell equations in the Newman–Penrose (NP) formalism, we must first recall the form of these equations in their standard covariant form. We begin with the equations of motion for the electromagnetic field derived from the action[4]

$$S_{\text{Maxwell}} = \int d^4x\sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\right], \tag{5.2.1}$$

are the Maxwell field equations $\nabla^\nu F_{\nu\mu} = J_\mu$ and the Maxwell Bianchi identities $\nabla_{[\rho}F_{\mu\nu]} = 0$. In the NP formalism, these are rearranged into the following 4 complex equations of motion

$$(\triangle + 1\mu - 2\gamma)\,\Phi_0 - (\delta - 2\tau + 0\beta)\,\Phi_1 - \sigma\Phi_2 = -\frac{1}{2}J_m\,, \tag{5.2.2a}$$

$$(\triangle + 2\mu + 0\gamma)\,\Phi_1 - (\delta - 1\tau + 2\beta)\,\Phi_2 - \nu\Phi_0 = -\frac{1}{2}J_n\,, \tag{5.2.2b}$$

$$\left(\bar{\delta} + 1\pi - 2\alpha\right)\Phi_0 - (D - 2\rho + 0\epsilon)\,\Phi_1 - \kappa\Phi_2 = -\frac{1}{2}J_\ell\,, \tag{5.2.3a}$$

$$\left(\bar{\delta} + 2\pi + 0\alpha\right)\Phi_1 - (D - 1\rho + 2\epsilon)\,\Phi_2 - \sigma\Phi_0 = -\frac{1}{2}J_{\bar{m}}\,. \tag{5.2.3b}$$

These can be written more compactly as

$$\begin{aligned}&[\triangle + (2-s)\,\mu - 2s\gamma]\,\Phi_{1-s} - [\delta - (1+s)\,\tau + 2\,(1-s)\,\beta]\,\Phi_{2-s}\\ &\quad - s\sigma\Phi_{3-s} - (1-s)\,\nu\Phi_{-s} = J^{(1)}_{s-1,s}\,,\end{aligned} \tag{5.2.4a}$$

$$\begin{aligned}&\left[\bar{\delta} + (2-s)\,\pi - 2s\alpha\right]\Phi_{1-s} - [D - (1+s)\,\rho + 2\,(1-s)\,\epsilon]\,\Phi_{2-s}\\ &\quad - s\kappa\Phi_{3-s} - (1-s)\,\lambda\Phi_{-s} = J^{(1)}_{s,s-1}\,,\end{aligned} \tag{5.2.4b}$$

where it is understood that only the range $0 \leq s \leq +1$ gives non-trivial equations. The terms "$J^{(1)}_{b,s}$" refer to terms of boost-weight b and spin weight s that are switched on in the presence of sources, and can be read directly from (5.2.2)-(5.2.3), e.g. $J^{(1)}_{0,1} = -\frac{1}{2}J_m$. For us this will be 0 however as we will be working in elecrovaccuum.

Furthermore, the stress-energy momentum tensor can be decomposed in NP scalars in a fashion similar to the Ricci tensor. In particular,

$$\begin{aligned}&\mathrm{T}_{00} := \frac{1}{2}T_{\ell\ell}\,,\quad \mathrm{T}_{11} := \frac{1}{4}\left(T_{\ell n} + T_{m\bar{m}}\right),\quad \mathrm{T}_{22} := \frac{1}{2}T_{nn}\,,\quad \Lambda_{\mathrm{T}} := \frac{T}{24}\,,\\ &\mathrm{T}_{01} := \frac{1}{2}T_{\ell m} = \bar{\mathrm{T}}_{10}\,,\quad \mathrm{T}_{12} := \frac{1}{2}T_{nm} = \bar{\mathrm{T}}_{21}\,,\quad \mathrm{T}_{02} := \frac{1}{2}T_{mm} = \bar{\mathrm{T}}_{20}\,.\end{aligned} \tag{5.2.5}$$

For electromagnetism, in the absence of sources,

$$T_{\mu\nu} = F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{5.2.6}$$

and the stress-energy-momentum-NP scalars acquire the remarkably simple expression in terms of the Maxwell-NP scalars

$$\Lambda_{\mathrm{T}} = 0\,,\quad \mathrm{T}_{\mathrm{ab}} = \Phi_{\mathrm{a}}\bar{\Phi}_{\mathrm{b}}\,,\quad \mathrm{a},\mathrm{b} \in \{0,1,2\}\,. \tag{5.2.7}$$

[4] Compared to section 5.1, $A^{5.1}_\mu = \frac{1}{\sqrt{8\pi G}}A^{\text{here}}_\mu$.

Then, for general-relativistic gravity with a cosmological constant Λ, the Einstein-Hilbert equations of motion, $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$, for a purely electromagnetic stress-energy momentum tensor in the absence of electromagnetic sources reduce to the following equations in the NP formalism

$$\Lambda_{\rm R} = \frac{\Lambda}{6}\,, \quad \Phi_{\rm ab} = 8\pi G \mathrm{T}_{\rm ab} = 8\pi G \Phi_{\rm a} \bar{\Phi}_{\rm b}\,. \tag{5.2.8}$$

We also remark that if one redefines the Maxwell fields according to $A_\mu \to \frac{1}{\sqrt{8\pi G}} A_\mu$, such that the Einstein-Hilbert-Maxwell action in the absence of electromagnetic sources reads $S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[R - 2\Lambda - \frac{1}{2} F_{\mu\nu} F^{\mu\nu}\right]$, then the Einstein-Hilbert equations of motion in the NP formalism reduce to $\Phi_{\rm ab} = \Phi_{\rm a} \bar{\Phi}_{\rm b}$.

For the gravitational case, the equations of motion of interest are purely mathematical identities that do not rely on some particular theory gravity, as long as gravity is geometric. These are the differential Bianchi identities $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$. They are repackaged into the following 8 independent Binachi identities within the NP formalism[5]:

$$\begin{aligned}&(\triangle + 1\mu - 4\gamma)\,\Psi_0 - (\delta - 4\tau - 2\beta)\,\Psi_1 - 3\sigma\Psi_2 = \\ &\quad + (D - \bar{\rho} - 2\epsilon + 2\bar{\epsilon})\,\Phi_{02} - (\delta + 2\bar{\pi} - 2\beta)\,\Phi_{01} + \bar{\lambda}\Phi_{00} - 2\sigma\Phi_{11} + 2\kappa\Phi_{12}\,,\end{aligned} \tag{5.2.9a}$$

$$\begin{aligned}&(\triangle + 2\mu - 2\gamma)\,\Psi_1 - (\delta - 3\tau + 0\beta)\,\Psi_2 - 2\sigma\Psi_3 + 1\nu\Psi_0 = -2\delta\Lambda_{\rm R} \\ &\quad + \left(\bar{\delta} - \bar{\tau} - 2\alpha + 2\bar{\beta}\right)\Phi_{02} - (\triangle + 2\bar{\mu} - 2\gamma)\,\Phi_{01} - 2\tau\Phi_{11} + \bar{\nu}\Phi_{00} + 2\rho\Phi_{12}\,,\end{aligned} \tag{5.2.9b}$$

$$\begin{aligned}&(\triangle + 3\mu + 0\gamma)\,\Psi_2 - (\delta - 2\tau + 2\beta)\,\Psi_3 - 1\sigma\Psi_4 - 2\nu\Psi_1 = +2\triangle\Lambda_{\rm R} \\ &\quad + (D - \bar{\rho} + 2\epsilon + 2\bar{\epsilon})\,\Phi_{22} - (\delta + 2\bar{\pi} + 2\beta)\,\Phi_{21} - 2\pi\Phi_{12} + 2\mu\Phi_{11} + \bar{\lambda}\Phi_{20}\,,\end{aligned} \tag{5.2.9c}$$

$$\begin{aligned}&(\triangle + 4\mu + 2\gamma)\,\Psi_3 - (\delta - 1\tau + 4\beta)\,\Psi_4 - 3\nu\Psi_2 = \\ &\quad + \left(\bar{\delta} - \bar{\tau} + 2\alpha + 2\bar{\beta}\right)\Phi_{22} - (\triangle + 2\bar{\mu} + 2\gamma)\,\Phi_{21} + 2\nu\Phi_{11} - 2\lambda\Phi_{12} + \bar{\nu}\Phi_{20}\,,\end{aligned} \tag{5.2.9d}$$

$$\begin{aligned}&\left(\bar{\delta} + 1\pi - 4\alpha\right)\Psi_0 - (D - 4\rho - 2\epsilon)\,\Psi_1 - 3\kappa\Psi_2 = \\ &\quad + (D - 2\bar{\rho} - 2\epsilon)\,\Phi_{01} - (\delta + \bar{\pi} - 2\bar{\alpha} - 2\beta)\,\Phi_{00} + 2\kappa\Phi_{11} - 2\sigma\Phi_{10} + \bar{\kappa}\Phi_{02}\,,\end{aligned} \tag{5.2.10a}$$

$$\begin{aligned}&\left(\bar{\delta} + 2\pi - 2\alpha\right)\Psi_1 - (D - 3\rho + 0\epsilon)\,\Psi_2 - 2\kappa\Psi_3 - 1\lambda\Psi_0 = -2D\Lambda_{\rm R} \\ &\quad + \left(\bar{\delta} - 2\bar{\tau} - 2\alpha\right)\Phi_{01} - (\triangle + \bar{\mu} - 2\gamma - 2\bar{\gamma})\,\Phi_{00} + 2\rho\Phi_{11} - 2\tau\Phi_{10} + \bar{\sigma}\Phi_{02}\,,\end{aligned} \tag{5.2.10b}$$

$$\begin{aligned}&\left(\bar{\delta} + 3\pi - 0\alpha\right)\Psi_2 - (D - 2\rho + 2\epsilon)\,\Psi_3 - 1\kappa\Psi_4 - 2\lambda\Psi_1 = +2\bar{\delta}\Lambda_{\rm R} \\ &\quad + (D - 2\bar{\rho} + 2\epsilon)\,\Phi_{21} - (\delta + \bar{\pi} - 2\bar{\alpha} + 2\beta)\,\Phi_{20} + \bar{\kappa}\Phi_{22} - 2\pi\Phi_{11} + 2\mu\Phi_{10}\,,\end{aligned} \tag{5.2.10c}$$

$$\begin{aligned}&\left(\bar{\delta} + 4\pi + 2\alpha\right)\Psi_3 - (D - 1\rho + 4\epsilon)\,\Psi_4 - 3\lambda\Psi_2 = \\ &\quad + \left(\bar{\delta} - 2\bar{\tau} + 2\alpha\right)\Phi_{21} - (\triangle + \bar{\mu} + 2\gamma - 2\bar{\gamma})\,\Phi_{20} + 2\nu\Phi_{10} + \bar{\sigma}\Phi_{22} - 2\lambda\Phi_{11}\,,\end{aligned} \tag{5.2.10d}$$

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[5]Some of these expressions can also be re-expressed by using the twice-contracted Bianchi identities, $\nabla^\nu R_{\mu\nu} - \frac{1}{2}\nabla_\mu R = 0$ which in the NP formalism read:

$$\begin{aligned}&(\delta + \bar{\pi} - 2\tau - 2\bar{\alpha})\,\Phi_{10} - (D - 2\rho - 2\bar{\rho})\,\Phi_{11} + \left(\bar{\delta} + \pi - 2\bar{\tau} - 2\alpha\right)\Phi_{01} - (\triangle + \mu + \bar{\mu} - 2\gamma - 2\bar{\gamma})\,\Phi_{00} \\ &\qquad - 3D\Lambda_{\rm R} - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21} + \bar{\sigma}\Phi_{02} + \sigma\Phi_{20} = 0\,, \\ &(\delta + 2\bar{\pi} - \tau + 2\beta)\,\Phi_{21} - (D - \rho - \bar{\rho} + 2\epsilon + 2\bar{\epsilon})\,\Phi_{22} + \left(\bar{\delta} + 2\pi - \bar{\tau} + 2\bar{\beta}\right)\Phi_{12} - (\triangle + 2\mu + 2\bar{\mu})\,\Phi_{11} \\ &\qquad - 3\triangle\Lambda_{\rm R} + \nu\Phi_{01} + \bar{\nu}\Phi_{10} - \lambda\Phi_{02} - \bar{\lambda}\Phi_{20} = 0\,, \\ &(\delta + 2\bar{\pi} - 2\tau)\,\Phi_{11} - (D - 2\rho - \bar{\rho} + 2\bar{\epsilon})\,\Phi_{12} + \left(\bar{\delta} + \pi - \bar{\tau} - 2\alpha + 2\bar{\beta}\right)\Phi_{02} - (\triangle + \mu + 2\bar{\mu} - 2\gamma)\,\Phi_{01} \\ &\qquad - 3\delta\Lambda_{\rm R} - \kappa\Phi_{22} + \bar{\nu}\Phi_{00} - \bar{\lambda}\Phi_{10} + \sigma\Phi_{21} = 0\,.\end{aligned}$$

The above equations can also be written more compactly as

$$\begin{aligned}&[\triangle + (3-s)\,\mu - 2s\gamma]\,\Psi_{2-s} - [\delta - (2+s)\,\tau + 2\,(1-s)\,\beta]\,\Psi_{3-s}\\&\quad - (1+s)\,\sigma\Psi_{4-s} - (2-s)\,\nu\Psi_{1-s} = J^{(2)}_{s-1,s}\,,\end{aligned} \tag{5.2.11a}$$

$$\begin{aligned}&\left[\bar{\delta} + (3-s)\,\pi - 2s\alpha\right]\Psi_{2-s} - [D - (2+s)\,\rho + 2\,(1-s)\,\epsilon]\,\Psi_{3-s}\\&\quad - (1+s)\,\kappa\psi_{4-s} - (2-s)\,\lambda\Psi_{1-s} = J^{(2)}_{s,s-1}\,,\end{aligned} \tag{5.2.11b}$$

where it is understood that only the range $-1 \le s \le +2$ gives non-trivial equations and "$J^{(2)}_{b,s}$" refers to terms of boost-weight b and spin weight s that are switched on in the presence of gravitational sources, i.e. for non-Ricci-flat geometries. These sources can also be written in a collective form, after splitting the above equations into two overlapping branches, one for $0 \le s \le +2$ and one for $-1 \le s \le +1$ (with overlap for $0 \le s \le +1$). For $0 \le s \le +2$,

$$\begin{aligned}J^{(2)}_{s-1,s} &= s\,[\kappa\Phi_{3-s,2} - \sigma\Phi_{3-s,1}] - (s-2)\,[\nabla_{s-1s,s}\Lambda_{\mathrm{R}} - \pi\Phi_{1-s,2} + \mu\Phi_{1-s,1}]\\&\quad + [D - \bar{\rho} + 2\,(1-s)\,\epsilon + 2\bar{\epsilon}]\,\Phi_{2-s,2} - [\delta + 2\bar{\pi} + 2\,(1-s)\,\beta]\,\Phi_{2-s,1} + \bar{\lambda}\Phi_{2-s,0}\,,\end{aligned} \tag{5.2.12a}$$

$$\begin{aligned}J^{(2)}_{s,s-1} &= s\,[\kappa\Phi_{3-s,1} - \sigma\Phi_{3-s,0}] - (s-2)\,[\nabla_{s,s-1}\Lambda_{\mathrm{R}} - \pi\Phi_{1-s,1} + \mu\Phi_{1-s,0}]\\&\quad + \bar{\kappa}\Phi_{2-s,2} + [D - 2\bar{\rho} + 2\,(1-s)\,\epsilon]\,\Phi_{2-s,1} - [\delta + \bar{\pi} - 2\bar{\alpha} + 2\,(1-s)\,\beta]\,\Phi_{2-s,0}\,,\end{aligned} \tag{5.2.12b}$$

while, for $-1 \le s \le +1$,

$$\begin{aligned}J^{(2)}_{s-1,s} &= (s-1)\,[\lambda\Phi_{-s,2} - \nu\Phi_{-s,1}] - (s+1)\,[\nabla_{s-1,s}\Lambda_{\mathrm{R}} - \rho\Phi_{2-s,2} + \tau\Phi_{2-s,1}]\\&\quad + \left[\bar{\delta} - \bar{\tau} - 2s\alpha + 2\bar{\beta}\right]\Phi_{1-s,2} - [\triangle + 2\bar{\mu} - 2s\gamma]\,\Phi_{1-s,1} + \bar{\nu}\Phi_{1-s,0}\,,\end{aligned} \tag{5.2.13a}$$

$$\begin{aligned}J^{(2)}_{s,s-1} &= (s-1)\,[\lambda\Phi_{-s,1} - \nu\Phi_{-s,0}] - (s+1)\,[\nabla_{s,s-1}\Lambda_{\mathrm{R}} - \rho\Phi_{2-s,1} + \tau\Phi_{2-s,0}]\\&\quad + \bar{\sigma}\Phi_{1-s,2} + \left[\bar{\delta} - 2\bar{\tau} - 2s\alpha\right]\Phi_{1-s,1} - [\triangle + \bar{\mu} - 2s\gamma - 2\bar{\gamma}]\,\Phi_{1-s,0}\,.\end{aligned} \tag{5.2.13b}$$

In the above equations, $\nabla_{b,s}$, with $-1 \le s, b \le +1$ and $|s| \ne |b|$, is the directional derivative of boost-weight b and spin weight s, namely, $\nabla_{+1,0} = D$, $\nabla_{-1,0} = \triangle$, $\nabla_{0,+1} = \delta$ and $\nabla_{0,-1} = \bar{\delta}$.

5.3 Recursion equations

We are at a stage where from the Einstein-Maxwell system, we can extract relation for certain putative charges. Let us start with the pure Maxwell theory. The Maxwell-NP equations (5.2.2)-(5.2.3) can be combined to get the following separated second order equation for Φ_0

$$\frac{1}{r}\partial_u\partial_r\left(r^3\Phi_0\right) = \eth\bar{\eth}\Phi_0 + \frac{R\,[q]}{4r}\partial_r\left(r^2\partial_r\left(r\Phi_0\right)\right)\,, \tag{5.3.1}$$

where $R \coloneqq R\,[q]$ is the Ricci scalar of the boundary metric, which we take to be constant, and we are working in the absence of Maxwell sources. In terms of the near-$\mathscr{I}$ modes Φ_0^s introduced in the previous section, these read [170],

$$\dot{\Phi}_0^{s+1} = -\frac{1}{s+1}\left(\bar{\eth}\eth + \frac{R}{4}s\,(s+3)\right)\Phi_0^s\,, \tag{5.3.2}$$

where the dot stands for the u-derivative and we have made use of the property

$$\left[\bar{\eth}, \eth\right]\varphi_s = s\frac{R}{2}\varphi_s \tag{5.3.3}$$

for an object object φ_s of spin weight s. To proceed, we rewrite the expansion (5.1.7) of Φ_0 as

$$\Phi_0 = \frac{\mathcal{Q}_1^{em}}{r^3} + \sum_{s=2}^{\infty} \frac{1}{r^{s+2}} \frac{(-1)^{s+1}}{(s-1)!} \left(\bar{\eth}^{s-1} \mathcal{Q}_s^{em} + \tilde{\Phi}_0^{s-1} \right), \tag{5.3.4}$$

where in the first term $\mathcal{Q}_1^{em} = \Phi_0^0$. The second term defines (in an implicit way) what will be referred to as the electromagnetic higher spin charges $\mathcal{Q}_s^{em}$, whose gravitational analogs were first introduced in [60, 162]. These charges are of higher spin weight, namely, $\mathcal{Q}_s^{em}$ has spin weight s. The last term in (5.3.4) contains the remaining pieces of Φ_0^s; for pure Maxwell theory, all $\tilde{\Phi}_0$'s are linear in the fields. Recall that the operator $\eth$ (resp. $\bar{\eth}$) raises (resp. lowers) the spin weight by one unit, hence all the terms in this expansion have the correct spin weight.
The decomposition of Φ_0^s in terms of spin weighted spherical harmonics starts from $\ell = 1$. Then at linear order, (5.3.4) can be viewed as splitting this decomposition into a part that starts from $\ell = s$ (the higher spin charges $\mathcal{Q}_s^{em}$) and a remaining part with $1 \leq \ell \leq s-1$ (the $\tilde{\Phi}_0^{s-1}$ part). The above allows us to write a recursion relation for the higher spin charges separately[6],

$$\begin{aligned} \bar{\eth}^{n+1} \dot{\mathcal{Q}}_{n+2}^{em} &= \left(\bar{\eth}\eth + \frac{1}{2} n(n+3) \right) \bar{\eth}^n \mathcal{Q}_{n+1}^{em} \\ &= \bar{\eth}\eth\bar{\eth}^n \mathcal{Q}_{n+1}^{em} + \frac{1}{2} n(n+3) \bar{\eth}^n \mathcal{Q}_{n+1}^{em} \\ &= -\Big(\sum_{\ell=2}^{n+1} \ell \bar{\eth}^n - \bar{\eth}^{n+1}\eth\Big) \mathcal{Q}_{n+1}^{em} + \frac{1}{2} n(n+3) \bar{\eth}^n \mathcal{Q}_{n+1}^{em} \\ &= -\Big(\frac{1}{2} n(n+3) \bar{\eth}^n - \bar{\eth}^{n+1}\eth\Big) \mathcal{Q}_{n+1}^{em} + \frac{1}{2} n(n+3) \bar{\eth}^n \mathcal{Q}_{n+1}^{em}, \end{aligned} \tag{5.3.5}$$

where we have repeatedly used (5.3.3). This then implies the linear recursion relation for the pure Maxwell sector,

$$\dot{\mathcal{Q}}_s^{em} = \eth \mathcal{Q}_{s-1}^{em}, \quad s \geq +2. \tag{5.3.6}$$

This relation can in fact be extended to also include the cases $0 \leq s \leq +1$ after defining the corresponding electromagnetic charges as

$$\mathcal{Q}_{-1}^{em} := \Phi_2^0, \quad \mathcal{Q}_0^{em} := \Phi_1^0, \tag{5.3.7}$$

$$\dot{\mathcal{Q}}_s^{gr} = \eth \mathcal{Q}_{s-1}^{gr} + (s+1)\, C \mathcal{Q}_{s-2}^{gr}. \tag{5.3.8}$$

The above equation is modified once electromagnetic interactions are included, as we shall now discuss.

We now go beyond the (linear) pure Maxwell case and consider the (non-linear) Einstein-Maxwell system. The relevant equations of motion for the Maxwell field coupled to gravity are (5.2.2) in the absence of electromagnetic sources, while, for the gravitational field, one looks at the Bianchi identities (5.2.9) with the gravitational sources coming from the Maxwell stress-energy-momentum tensor according to the Einstein field equations (5.2.8). Expanding all of these equations in terms of the near-$\mathscr{I}$ modes of the NP scalars, the first few orders read [169],

$$\begin{aligned} \dot{\Psi}_0^0 - \eth\Psi_1^0 - 3C\Psi_2^0 + 3\Phi_0^0\bar{\Phi}_2^0 &= 0, \\ \dot{\Psi}_0^1 + 4\bar{\eth}\left(C\Psi_1^0\right) + \bar{\eth}\eth\Psi_0^0 + 4\bar{\Phi}_1^0\eth\Phi_0^0 + 8C\Phi_1^0\bar{\Phi}_1^0 + 4\Phi_0^1\bar{\Phi}_2^0 &= 0, \\ \dot{\Psi}_1^0 - \eth\Psi_2^0 - 2C\Psi_3^0 + 2\Phi_1^0\bar{\Phi}_2^0 &= 0, \\ \dot{\Psi}_2^0 - \eth\Psi_3^0 - C\Psi_4^0 + \Phi_2^0\bar{\Phi}_2^0 &= 0, \\ \dot{\Phi}_0^0 - \eth\Phi_1^0 - C\Phi_2^0 &= 0, \\ \dot{\Phi}_0^1 + \bar{\eth}\eth\Phi_0^0 + 2\bar{\eth}\left(C\Phi_1^0\right) &= 0, \\ \dot{\Phi}_1^0 - \eth\Phi_2^0 &= 0. \end{aligned} \tag{5.3.9}$$

[6]Note also that decomposing the equations for $\tilde{\Phi}_0^s$ we get $\dot{\tilde{\Phi}}_{0(s,m)}^s = 0$, where $\tilde{\Phi}_{0(\ell,m)}^s$ are the spherical harmonic modes of $\tilde{\Phi}_0^s$; these are the linearized NP constants for electrodynamics [170].

The data needed to determine a solution of the Einstein-Maxwell system are Ψ_1^0, Ψ_2^0, Ψ_0^s and Φ_1^0, Φ_2^0, Φ_0^s, plus the free data of the gravitational shear C and the asymptotic photon field F [168]. Now, by inspecting the form of subleading terms in (5.1.7) and following the treatment of [60, 162], we write the expansion of Ψ_0 in terms of higher spin charges $\mathcal{Q}_s$ as,

$$\Psi_0 = \frac{\mathcal{Q}_2^{gr}}{r^5} + \sum_{s=3}^{\infty} \frac{1}{r^{s+3}} \frac{(-1)^s}{(s-2)!} \left(\bar{\eth}^{s-2} \mathcal{Q}_s^{gr} - (s+1)\, \bar{\mathcal{Q}}_0^{em} \bar{\eth}^{s-3} \mathcal{Q}_{s-1}^{em} + \tilde{\Psi}_0^{s-2} \right) , \tag{5.3.10}$$

which reduces to the expression in [162] in the absence of electromagnetic charges. In the full Einstein-Maxwell theory, the $\tilde{\Psi}_0^s$ and $\tilde{\Phi}_0^s$ in (5.3.4) are some generic functions, including possibly non-linear and non-local quantities. We will however see that these do not contribute to the first few charge aspects but might appear at further order. For the remaining NP near-$\mathscr{I}$ modes, we also write

$$\begin{aligned} \Phi_{1-s}^0 &= \mathcal{Q}_s^{em} , \qquad -1 \le s \le 1 , \\ \Psi_{2-s}^0 &= \mathcal{Q}_s^{gr} , \qquad -2 \le s \le 2 . \end{aligned} \tag{5.3.11}$$

With this, one can study the evolution equations (5.3.9) in terms of the higher spin charges. We see that, for $s = 0, 1, 2$, the following equations hold

$$\dot{\mathcal{Q}}_s^{em} = \eth \mathcal{Q}_{s-1}^{em} + sC\mathcal{Q}_{s-2}^{em} , \tag{5.3.12}$$

where the extra term compared to (5.3.6) encodes a non-linear contribution due to the gravitational interactions. We also find that, for $s = -1, 0, 1, 2, 3$,

$$\dot{\mathcal{Q}}_s^{gr} = \eth \mathcal{Q}_{s-1}^{gr} + (s+1)\, C\mathcal{Q}_{s-2}^{gr} + (s+1)\, F\mathcal{Q}_{s-1}^{em} , \tag{5.3.13}$$

where the last term generalizes (5.3.8) to include the presence of Maxwell fields.

The insight of [60] was the realization that the $w_{1+\infty}$ algebra appears when extending the validity of the higher spin charges recursion relations for *all* values of s. In the Maxwell-Einstein case, the role of the $\tilde{\Psi}_0^s$ and $\tilde{\Phi}_0^s$ quantities[7] is thus precisely to absorb all relevant terms in such a way that the electromagnetic and gravitational charges obey, by definition, the recursions relations (5.3.12) for all $s \ge 0$ and (5.3.13) for all $s \ge -1$, respectively. While a systematic understanding of this mechanism is still lacking, the relation between the gravitational recursion relations and equations for self-dual gravity in twistor space was shown in [171]. In the next sections, we will show that the definition of recursion higher spin charges in Einstein-Maxwell satisfying (5.3.12), (5.3.13) (and, in Section 5.6, their generalization for Einstein-Yang-Mills) allows to derive the $sw_{1+\infty}$ algebra.

5.4 Quadratic quasi-conserved charges

The radiative degrees of freedom are encoded in the negative spin weight charges,

$$\mathcal{Q}_{-1}^{em} = -\bar{F} , \qquad \mathcal{Q}_{-1}^{gr} = \eth \bar{N} , \qquad \mathcal{Q}_{-2}^{gr} = \partial_u \bar{N} . \tag{5.4.1}$$

In the absence of radiation, namely, when the above quantities vanish, only the spin weight-zero charges $\mathcal{Q}_0^{em}$, $\mathcal{Q}_0^{gr}$ are conserved, according to the evolution equations (5.3.12), (5.3.13),[8]

$$\partial_u \mathcal{Q}_0 \overset{\text{non-rad}}{=} 0 . \tag{5.4.2}$$

[7]As noted in [162], a clever choice of tetrad might result in simplification of the expression for these auxiliary functions.

[8]When dropping the ^{em} and ^{gr} superscripts we mean that the relation applies for both the electromagnetic and gravitational charges.

Nevertheless, one can perturbatively construct combinations of the recursion charges and the boundary gauge/metric fields that satisfy quasi-conservation equations, i.e., that are conserved in the absence of radiation, for all $s \geq 0$. Building on the works of [60, 162], let us then consider the following 'renormalized charge aspects'

$$\begin{aligned}
\tilde{q}^{em}_s &= \sum_{n=0}^{s} \frac{(-u)^n}{n!} \eth^n \mathcal{Q}^{em}_{s-n} - \sum_{\ell=2}^{s}\sum_{n=0}^{\ell-2} \frac{(-1)^n \ell}{(s-\ell)!} \eth^{s-\ell}(\partial_u^{-(n+1)}((-u)^{s-\ell}C)\eth^n \mathcal{Q}^{em}_{\ell-2-n}) + \mathcal{O}(\mathbb{F}^3), \\
\tilde{q}^{gr}_s &= \sum_{n=0}^{s} \frac{(-u)^n}{n!} \eth^n \mathcal{Q}^{gr}_{s-n} - \sum_{\ell=2}^{s}\sum_{n=0}^{\ell-2} \frac{(-1)^n (\ell+1)}{(s-\ell)!} \eth^{s-\ell}(\partial_u^{-(n+1)}((-u)^{s-\ell}C)\eth^n \mathcal{Q}^{gr}_{\ell-2-n}) \\
&\quad - \sum_{\ell=1}^{s}\sum_{n=0}^{\ell-1} \frac{(-1)^n (\ell+1)}{(s-\ell)!} \eth^{s-\ell}(\partial_u^{-(n+1)}((-u)^{s-\ell}F)\eth^n \mathcal{Q}^{em}_{\ell-1-n}) + \mathcal{O}(\mathbb{F}^3),
\end{aligned} \tag{5.4.3}$$

where $s \geq 0$ and the notation $\mathcal{O}(\mathbb{F}^3)$ emphasizes that we are working up to quadratic order, suppressing potential terms that are of cubic or higher order in the fields. These charges are conserved in non-radiative vacuum defined by $\mathcal{Q}^{\text{em}}_{-1} = \mathcal{Q}^{\text{gr}}_{-1} = \mathcal{Q}^{\text{gr}}_{-2} = 0$ up to quadratic order. We can see this by taking the u-derivative of the renormalized charges. For example, for the electromagnetic bare charge,

$$\begin{aligned}
\partial_u \tilde{q}^{em}_s &= -\sum_{n=0}^{s} \frac{(-u)^{n-1}}{(n-1)!} \eth^n \mathcal{Q}^{em}_{s-n} + \sum_{n=0}^{s} \frac{(-u)^n}{n!} \eth^n \dot{\mathcal{Q}}^{em}_{s-n} \\
&\quad + \sum_{\ell=2}^{s}\sum_{n=0}^{\ell-2} \frac{(-1)^n \ell}{(s-\ell)!} \eth^{s-\ell}(\partial_u^{-n}((-u)^{s-\ell}C)\eth^n \mathcal{Q}^{em}_{\ell-2-n}) \\
&\quad + \sum_{\ell=2}^{s}\sum_{n=0}^{\ell-2} \frac{(-1)^n \ell}{(s-\ell)!} \eth^{s-\ell}(\partial_u^{-(n+1)}((-u)^{s-\ell}C)\eth^n \dot{\mathcal{Q}}^{em}_{\ell-2-n})
\end{aligned}$$

Then using the eqs. (5.3.12) and (5.3.13), and keeping terms at most quadratic in fields,

$$\begin{aligned}
\partial_u \tilde{q}^{em}_s &= -\sum_{n=1}^{s} \frac{(-u)^{n-1}}{(n-1)!} \eth^n \mathcal{Q}^{em}_{s-n} + \sum_{n=0}^{s} \frac{(-u)^n}{n!} \eth^n (\eth \mathcal{Q}^{em}_{s-n-1} - (s-n) C \mathcal{Q}^{em}_{s-n-2}) \\
&\quad + \sum_{\ell=2}^{s}\sum_{n=0}^{\ell-2} \frac{(-1)^n \ell}{(s-\ell)!} \eth^{s-\ell}(\partial_u^{-n}((-u)^{s-\ell}C)\eth^n \mathcal{Q}^{em}_{\ell-2-n}) \\
&\quad + \sum_{\ell=2}^{s}\sum_{n=0}^{\ell-2} \frac{(-1)^n \ell}{(s-\ell)!} \eth^{s-\ell}(\partial_u^{-(n+1)}((-u)^{s-\ell}C)\eth^{n+1} \mathcal{Q}^{em}_{\ell-2-n-1}) + \mathcal{O}(\mathbb{F}^3)
\end{aligned}$$

This expression can then be simplified to give,

$$\partial_u \tilde{q}^{em}_s = \frac{(-u)^s}{s!} \eth^{s+1} \mathcal{Q}^{em}_{-1} + \sum_{\ell=1}^{s} \frac{(-1)^\ell \ell}{(s-\ell)1!} \eth^{s-\ell}(\partial_u^{1-\ell}((-u)^{s-\ell}C)\eth^{\ell-1} \mathcal{Q}^{em}_{-1}) + \mathcal{O}(\mathbb{F}^3) \tag{5.4.4}$$

The evolution of the renormalized gravitational charges can be similarly obtained which upto quadratic order in fields give,

$$\begin{aligned}
\partial_u \tilde{q}^{gr}_s &= \frac{(-u)^s}{s!} \eth^s (\eth \mathcal{Q}^{gr}_{-1} - C \mathcal{Q}^{gr}_{-2}) + \sum_{\ell=1}^{s} \frac{(-1)^\ell (\ell+1)}{(s-\ell)!} \eth^{s-\ell}(\partial_u^{1-\ell}((-u)^{s-\ell}C)\eth^{\ell-1} \mathcal{Q}^{gr}_{-1}) \\
&\quad + \frac{(-u)^s}{s!} \eth^s (F \mathcal{Q}^{em}_{-1}) + \sum_{\ell=1}^{s} \frac{(-1)^\ell (\ell+1)}{(s-\ell)!} \eth^{s-\ell}(\partial_u^{-\ell}((-u)^{s-\ell}F)\eth^{\ell} \mathcal{Q}^{em}_{-1}) + \mathcal{O}(\mathbb{F}^3)
\end{aligned} \tag{5.4.5}$$

Therefore the time evolution of the renormalized charges vanishes upto quadratic order in non-radiative vacuum.

We can now iteratively integrate the recursion relations (5.3.12), (5.3.13) at each order, starting from the linear order. This is achieved employing the anti-derivative the iterated anti-derivative operator ∂_u^{-n}, $n \geq 0$, defined as [162, 172]

$$\partial_u^{-n} \mathcal{F}(u) := \int_{+\infty}^{u} du_1 \int_{+\infty}^{u_1} du_2 \cdots \int_{+\infty}^{u_{n-1}} du_n \, \mathcal{F}(u_n) \,, \tag{5.4.6}$$

for any function $\mathcal{F}(u)$ (see also Appendix E.1). This is a well-defined operation for integrating the evolution equations for $\mathcal{Q}_s$ as long as

$$\bar{N} = \mathcal{O}\left(|u|^{-(1+s+\varepsilon)}\right) = \bar{F} \quad \text{as } u \to \pm\infty \quad \text{and} \quad \lim_{u \to +\infty} \mathcal{Q}_s = 0 \,, \tag{5.4.7}$$

with $0 < \varepsilon < 1$. The former fall-off conditions on the news tensor and the radiative part of the field strength tensor are the minimum conditions needed to access the sub^s-leading soft theorem [98, 173], while the condition that $\varepsilon \notin \mathbb{N}$ ensures the absence of logarithmic corrections [92, 174].

Soft/hard decomposition[9]

At each order and for arbitrary spin s, the recursion charges can be perturbatively expanded as[10]

$$\mathcal{Q}_s^{em}(u, x^A) = \sum_{k=1}^{\lfloor \frac{s+1}{2} \rfloor + 1} \mathcal{Q}_s^{k,em} \,, \qquad \mathcal{Q}_s^{gr} = \sum_{k=1}^{\lfloor \frac{s+2}{2} \rfloor + 1} \mathcal{Q}_s^{k,gr}(u, x^A) \,, \tag{5.4.8}$$

where each term $\mathcal{Q}_s^{k,em}$ and $\mathcal{Q}_s^{k,gr}$ contains one insertion of the radiation fields $\bar{F}$, $\bar{N}$ and $k-1$ insertions of the radiative data A, C. For our purposes, it will be sufficient to work out the soft ($k = 1$) and leading hard ($k = 2$) contributions. Starting from (5.4.1), the equations (5.3.12), (5.3.13) can then be solved recursively. Up to the quadratic order, we get

$$\mathcal{Q}_{s \geq -1}^{1,em} = -(\partial_u^{-1} \eth)^{s+1} \bar{F} \,, \tag{5.4.9a}$$

$$\mathcal{Q}_{s \geq 0}^{2,em} = -\sum_{\ell=0}^{s} \ell \, (\partial_u^{-1} \eth)^{s-\ell} \partial_u^{-1} (C (\partial_u^{-1} \eth)^{\ell-1} \bar{F}) \,, \tag{5.4.9b}$$

$$\mathcal{Q}_{s \geq -2}^{1,gr} = (\partial_u^{-1} \eth)^{s+2} \partial_u \bar{N} \,, \tag{5.4.9c}$$

$$\mathcal{Q}_{s \geq -1}^{2,gr} = \sum_{\ell=0}^{s} (\ell + 1) \, (\partial_u^{-1} \eth)^{s-\ell} \partial_u^{-1} \left(C (\partial_u^{-1} \eth)^{\ell} \partial_u \bar{N} - F (\partial_u^{-1} \eth)^{\ell} \bar{F} \right) \,. \tag{5.4.9d}$$

Similarly, the quasi-conserved charge aspects (5.4.3) can be split into soft, leading-order hard and higher order terms,

$$\tilde{q}_s^{em}(u, x^A) = \sum_{k=1}^{\lfloor \frac{s+1}{2} \rfloor + 1} \tilde{q}_s^{k,em}(u, x^A) \,, \qquad \tilde{q}_s^{gr}(u, x^A) = \sum_{k=1}^{\lfloor \frac{s+2}{2} \rfloor + 1} \tilde{q}_s^{k,gr}(u, x^A) \,. \tag{5.4.10}$$

[9] Note in this chapter what we call soft and hard is different from the phase space separation defined in chapter 2. Here we are simply calling the linear piece as soft and quadratic piece as hard, while the quadratic piece will contain both radiative and zero modes.

[10] The upper limits of the sums follow from the observation that the terms in the recursion relations that increase the number of insertions of the radiative data come from charge aspects whose spin-weight is offset by 2 units, e.g. $\partial_u \mathcal{Q}_s^{em} \supset s \, C \mathcal{Q}_{s-2}^{em}$.

Let us start with the soft ($k = 1$) contributions. Using (5.4.9) and the integral Leibniz rule (E.1.5), we can write the soft charges as

$$\tilde{q}_s^{1,em} = -\partial_u^{-1}\left(\frac{(-u)^s}{s!}\eth^{s+1}\bar{F}\right), \qquad \tilde{q}_s^{1,gr} = \partial_u^{-1}\left(\frac{(-u)^s}{s!}\eth^{s+2}\bar{N}\right). \tag{5.4.11}$$

For the leading hard pieces ($k = 2$), following the strategy of [162], it turns out sufficient to find their evolution equations, since this will be the only part of the quadratic charges that contribute in the derivation of the $sw_{1+\infty}$ algebra, as we will see in the next section. Using the quasi-conservation equations (5.4.4), we arrive at

$$\partial_u \tilde{q}_s^{2,em} = \sum_{\ell=0}^{s} \frac{(-1)^\ell \ell}{(s-\ell)!}\eth^{s-\ell}(\partial_u^{1-\ell}((-u)^{s-\ell}C)\eth^{\ell-1}\bar{F}), \tag{5.4.12a}$$

$$\begin{aligned}\partial_u \tilde{q}_s^{2,gr} = {}& \frac{(-u)^s}{s!}\eth^s(C\partial_u\bar{N}) - \sum_{\ell=1}^{s}\frac{(-1)^\ell(\ell+1)}{(s-\ell)!}\eth^{s-\ell}(\partial_u^{1-\ell}((-u)^{s-\ell}C)\eth^\ell\bar{N}) \\ & - \sum_{\ell=0}^{s}\frac{(-1)^\ell(\ell+1)}{(s-\ell)!}\eth^{s-\ell}(\partial_u^{-\ell}((-u)^{s-\ell}F)\eth^\ell\bar{F}).\end{aligned} \tag{5.4.12b}$$

5.5 Celestial $sw_{1+\infty}$ algebra

In this section, we derive the linearized charge algebra of electromagnetic and gravitational higher spin charges. This will be achieved by computing the Poisson bracket among the celestial charges, defined from the quasi-conserved quantities (5.4.3) as

$$q_s(x^A) := \lim_{u\to-\infty}\tilde{q}_s(u,x^A), \tag{5.5.1}$$

at linear order in the fields [60, 162]. Performing again a perturbative expansion over the asymptotic fields as in (5.4.10), we see that the linearized bracket only involves the soft and the leading order hard parts of these charges,

$$\left\{q_s(x^A), q_{s'}(x'^A)\right\} = \left\{q_s^1(x^A), q_{s'}^2(x'^A)\right\} + \left\{q_s^2(x^A), q_{s'}^1(x'^A)\right\} + \mathcal{O}\left(\mathbb{F}^2\right). \tag{5.5.2}$$

The goal is thus to obtain the expressions for the two above bracket contributions, and the final algebra will simply be a smeared version of (5.5.2).

The electromagnetic soft celestial charge is easily seen to be given by

$$q_s^{1,em} = \eth^{s+1}\bar{F}_s, \quad \bar{F}_s := \int_{-\infty}^{+\infty} du\,\frac{(-u)^s}{s!}\bar{F}, \tag{5.5.3}$$

namely it descends from the negative helicity (sub)^s-leading soft photon operator. Similarly, the gravitational soft celestial charge,

$$q_s^{1,gr} = -\eth^{s+2}\bar{N}_s, \quad \bar{N}_s := \int_{-\infty}^{+\infty} du\,\frac{(-u)^s}{s!}\bar{N}, \tag{5.5.4}$$

descends from the negative helicity (sub)^s-leading soft graviton operator [60].

Strictly speaking, the bracket $\left\{q_s^2(x^A), \mathcal{F}(u', x'^A)\right\}$, for some $\mathcal{F}(u, x^A)$, in general diverges, because of the divergent behavior of $q_s^2(x^A) = \lim_{u\to-\infty}\tilde{q}_s^2(u,x^A)$. It will be regularized according to the prescription

$$\left\{q_s^2(x^A), \mathcal{F}(u', x'^A)\right\} = \lim_{u\to-\infty}\partial_u^{-1}\left\{\partial_u\tilde{q}_s^2(u,x^A), \mathcal{F}(u', x'^A)\right\}. \tag{5.5.5}$$

We are now ready to derive the current algebra, starting from the canonical Poisson brackets on the radiative phase space at $\mathscr{I}^+$ (see e.g. [175])[11],

$$\left\{A(u,z), \bar{F}\left(u',z'\right)\right\} = \kappa^2\,\delta(u-u')\delta(z,z')\,, \tag{5.5.6a}$$
$$\left\{C(u,z), \bar{N}(u',z')\right\} = \kappa^2\,\delta(u-u')\delta(z,z')\,. \tag{5.5.6b}$$

In the above relations, we charted as usual the 2-dimensional transverse metric q_{AB} with complex coordinates $(z,\bar{z})$ such that $q_{AB}\left(x^C\right)dx^A dx^B = 2q_{z\bar{z}}\left(z,\bar{z}\right)\,dzd\bar{z}$ [12].

From the canonical relations (5.5.6), the Poisson brackets for the soft charges can be readily obtained using (5.5.3), (5.5.4),

$$\left\{q_s^{1,em}(z), F(u',z')\right\} = \kappa^2\,\frac{(-u')^{s-1}}{(s-1)!}\eth_z^{s+1}\delta(z,z')\,, \tag{5.5.7a}$$
$$\left\{q_s^{1,gr}(z), N(u',z')\right\} = -\kappa^2\,\frac{(-u')^{s-1}}{(s-1)!}\eth_z^{s+2}\delta(z,z')\,. \tag{5.5.7b}$$

The case of the hard part is slightly more involved. Let us detail the computation for one of the brackets involving the action of the renormalized gravitational charges on the electromagnetic potential. Using (5.5.5), the $u\to-\infty$ limit is well-defined and by means of (5.4.12) gives the following hard action,

$$\begin{aligned}\left\{q_s^{2,gr}(z), \bar{A}(u',z')\right\} &= \lim_{u\to-\infty}\partial_u^{-1}\left\{\partial_u\tilde{q}_s^{2,gr}(u,z), \bar{A}(u',z')\right\}\\ &= -\kappa^2\sum_{\ell=0}^{s}\frac{(-1)^\ell\,(\ell+1)}{(s-\ell)!}\eth_z^{s-\ell}\lim_{u\to-\infty}\partial_u^{-1}\left(\left\{\partial_u^{-\ell}((-u)^{s-\ell}F(u,z)), \bar{A}(u',z')\right\}\eth_z^\ell\bar{F}(u,z)\right)\end{aligned} \tag{5.5.8}$$

To massage this, let us rewrite it by defining

$$\mathcal{A}_{s\ell}\left(z;u',z'\right) := \lim_{u\to-\infty}\partial_u^{-1}\left(\mathcal{B}_{s\ell}\left(u,z;u',z'\right)\eth_z^\ell\bar{F}(u,z)\right)\,, \tag{5.5.9}$$

with

$$\mathcal{B}_{s\ell}\left(u,z;u',z'\right) := \left\{\partial_u^{-\ell}\left(\frac{(-u)^{s-\ell}}{(s-\ell)!}F(u,z)\right), \bar{A}(u',z')\right\}\,, \tag{5.5.10}$$

such that

$$\left\{q_s^{2,gr}(z), \bar{A}(u',z')\right\} = -\sum_{\ell=0}^{s}\frac{(-1)^\ell\,(\ell+1)}{(s-\ell)!}\eth_z^{s-\ell}\mathcal{A}_{s\ell}\left(z;u',z'\right)\,. \tag{5.5.11}$$

The repeated integral $\partial_u^{-\ell}$ in $\mathcal{B}_{s\ell}$ can be distributed using (E.1.6) to get

$$\mathcal{B}_{s\ell}\left(u,z;u',z'\right) = \sum_{n=0}^{s-\ell}\binom{s-n-1}{\ell-1}\frac{(-u)^n}{n!}\left\{\partial_u^{-(s-n)}F(u,z), \bar{A}(u',z')\right\}\,. \tag{5.5.12}$$

To proceed, we borrow from Appendix E.2 the following bracket

$$\begin{aligned}\left\{\partial_u^{-n}F(u,z), \bar{A}(u',z')\right\} &= -\frac{\kappa^2}{2}\left(\partial_u^{-n} + (-1)^n\,\partial_{u'}^{-n}\right)\delta\left(u-u'\right)\delta(z,z')\\ &= -\frac{\kappa^2}{2}\frac{\left(u-u'\right)^{n-1}}{(n-1)!}\Theta\left(u-u'\right)\delta(z,z')\,,\end{aligned} \tag{5.5.13}$$

[11]Note again here the unconventional normalization we are using for the field strength, namely $F = \kappa F_{\text{can}}$.

[12]The 2-dimensional δ-function density is defined as $\delta(z,z') := q_{z\bar{z}}^{-1}\delta\left(z-z'\right)\delta\left(\bar{z}-\bar{z}'\right)$.

where $\Theta(t) = \theta(t) - \theta(-t)$ is the antisymmetrized Heaviside function, satisfying $\frac{d}{dt}\Theta(t) = 2\delta(t)$. Using this in $\mathcal{B}_{s\ell}$, the sum over n turns out to be a binomial expansion sum with the end result

$$\mathcal{B}_{s\ell}(u,z;u',z') = -\frac{\kappa^2}{2}\frac{(-u')^{s-\ell}}{(s-\ell)!}\frac{(u-u')^{\ell-1}}{(\ell-1)!}\Theta(u-u')\,\delta(z,z')\,. \tag{5.5.14}$$

Next, using the definition of the repeated integral from Appendix E.1, one can show that

$$\begin{aligned}\partial_u^{-1}\left(\frac{(u-u')^{\ell-1}}{(\ell-1)!}f(u)\,\Theta(u-u')\right) = (-1)^{\ell-1}\Big\{&2\partial_{u'}^{-\ell}f(u')\,\theta(u'-u)\\ &+\Theta(u-u')\int_{+\infty}^{u}du''\frac{(u'-u'')^{\ell-1}}{(\ell-1)!}f(u'')\Big\}\,.\end{aligned} \tag{5.5.15}$$

Plugging the expression of $\mathcal{B}_{s\ell}$ into $\mathcal{A}_{s\ell}$ and taking the $u\to-\infty$ limit of the above equation, we then find

$$\begin{aligned}&\mathcal{A}_{s\ell}(z;u',z')\\ &= \kappa^2\delta(z,z')(-1)^{\ell}\frac{(-u')^{s-\ell}}{(s-\ell)!}\left\{\partial_{u'}^{1-\ell}\eth_{z'}^{\ell}\bar{A}(u',z') + \frac{1}{2}\eth_{z'}^{\ell}\int_{-\infty}^{+\infty}du''\frac{(u'-u'')^{\ell-1}}{(\ell-1)!}\bar{F}(u'',z')\right\}\\ &= \kappa^2\delta(z,z')\frac{(-1)^s}{(s-\ell)!}(\Delta_{u'}-1)_{s-\ell}\,\partial_{u'}^{1-s}\eth_{z'}^{\ell}\bar{A}(u',z')\,.\end{aligned} \tag{5.5.16}$$

In the last line we have used the identity $u^n = (\Delta_u - 1)_n\,\partial_u^{-n}$, where $\Delta_u := u\partial_u + 1$ and $(x)_n$ is the falling factorial operation, that follows from (E.1.10). We also realized there that the integral over the entire real line that appears turns out to vanish. This is because we can write the integrand as a total derivative since it is the same as the integrand of an anti-derivative,

$$\frac{(u'-u)^{\ell-1}}{(\ell-1)!}\bar{F}(u,z') = \partial_u\left(\partial_u^{-\ell}\bar{F}(u,z')\right)\,. \tag{5.5.17}$$

Then, the fact that $0\le\ell\le s$ and the decaying conditions for the radiative fields are sufficient to conclude that

$$\int_{-\infty}^{+\infty}du''\frac{(u'-u'')^{\ell-1}}{(\ell-1)!}\bar{F}(u'',z') = -\lim_{u\to-\infty}\partial_u^{-\ell}\bar{F}(u,z') = 0\,. \tag{5.5.18}$$

Finally, the bracket $\left\{q_s^{2,gr}(z)\,,\bar{A}(u',z')\right\}$ can be extracted from the above to be

$$\left\{q_s^{2,gr}(z),\bar{A}(u',z')\right\} = -\kappa^2\sum_{\ell=0}^{s}\frac{(-1)^{s-\ell}(\ell+1)}{(s-\ell)!}(\Delta_{u'}-1)_{s-\ell}\partial_{u'}^{1-s}\eth_{z'}^{\ell}\bar{A}(u',z')\eth_z^{s-\ell}\delta(z,z')\,. \tag{5.5.19}$$

A similar computation can be performed to obtain the action of the hard electromagnetic charges on the gravitational shear,

$$\left\{q_s^{2,em}(z),\bar{C}(u',z')\right\} = \kappa^2\sum_{n=0}^{s}\frac{(-1)^{s-n}n}{(s-n)!}(\Delta_{u'}-2)_{s-n}\partial_{u'}^{1-s}\eth_{z'}^{n-1}\bar{A}(u',z')\eth_z^{s-n}\delta(z,z')\,. \tag{5.5.20}$$

The remaining brackets are found to be given by

$$
\begin{aligned}
\left\{q_s^{2,gr}(z), A(u',z')\right\} &= -\kappa^2 \sum_{n=0}^{s} \frac{(-1)^{s-n}(n+1)}{(s-n)!}(\Delta_{u'}+1)_{s-n}\partial_{u'}^{1-s}\eth_{z'}^{n} A(u',z')\eth_z^{s-n}\delta(z,z'),\\
\left\{q_s^{2,em}(z), C(u',z')\right\} &= 0,\\
\left\{q_s^{2,em}(z), A(u',z')\right\} &= -\kappa^2 \sum_{n=0}^{s} \frac{(-1)^{s-n}n}{(s-n)!}(\Delta_{u'}+1)_{s-n}\partial_{u'}^{1-s}\eth_{z'}^{n-1} C(u',z')\eth_z^{s-n}\delta(z,z'),\\
\left\{q_s^{2,em}(z), \bar{A}(u',z')\right\} &= 0,\\
\left\{q_s^{2,gr}(z), C(u',z')\right\} &= -\kappa^2 \sum_{n=0}^{s} \frac{(-1)^{s-n}(n+1)}{(s-n)!}(\Delta_{u'}+2)_{s-n}\partial_{u'}^{1-s}\eth_{z'}^{n} C(u',z')\eth_z^{s-n}\delta(z,z'),\\
\left\{q_s^{2,gr}(z), \bar{C}(u',z')\right\} &= -\kappa^2 \sum_{n=0}^{s} \frac{(-1)^{s-n}(n+1)}{(s-n)!}(\Delta_{u'}-2)_{s-n}\partial_{u'}^{1-s}\eth_{z'}^{n} \bar{C}(u',z')\eth_z^{s-n}\delta(z,z').
\end{aligned}
\tag{5.5.21}
$$

The action of the charges on the negative helicity photon field or gravitational follows using the same methods as above. For their action on the positive helicity fields, the corresponding brackets are derived using the steps outlined in [162][13]

From the above action of the charges on the radiative data, one can deduce their action on the sub^s-leading soft operators $\bar{F}_s$, $\bar{N}_s$ defined in (5.5.3) by means of pseudo-differential calculus identities (see Appendix E.1), as shown in the gravity case in [60]. Starting from (5.5.19), we find, using identities (E.1.10),

$$
\begin{aligned}
&\left\{q_s^{2,gr}(z), \bar{F}_{s'}(z')\right\}\\
&= -\kappa^2 \int_{-\infty}^{+\infty} du\, \frac{(-u)^{s'}}{s'!}\partial_u \sum_{n=0}^{s} \frac{(-1)^{s-n}(n+1)}{(s-n)!}(\Delta_u-1)_{s-n}\partial_u^{1-s}\eth_{z'}^{n}\bar{A}(u,z')\eth_z^{s-n}\delta(z,z')\\
&= -\kappa^2 \sum_{n=0}^{s}\int_{-\infty}^{+\infty} du\, \frac{(-1)^{n+1}(n+1)}{s'!(s-n)!}\frac{(\Delta_u-s')_{s-n}}{(\Delta_u-s'-1)_{s-1}}(-u)^{s'+s-1}\eth_{z'}^{n}\bar{F}(u,z')\eth_z^{s-n}\delta(z,z')\\
&= -\kappa^2 \sum_{n=0}^{s}\binom{s'+s-n-1}{s'-1}(n+1)\,\eth_{z'}^{n}\bar{F}_{s+s'-1}(z')\eth_z^{s-n}\delta(z,z')\,.
\end{aligned}
\tag{5.5.22}
$$

To arrive at the last line, one uses some combinatorial identities and assumes that the field strength $\bar{F}$ falls off as $\mathcal{O}(u^{-1-s-\epsilon})$ at large u for every s, such that one can set $\Delta_u \approx 0$ inside the integral (see e.g. Appendix B in [60]).

Similarly, we compute the action of the electromagnetic charge on soft graviton operators,

$$
\left\{q_s^{2,em}(z), \bar{N}_{s'}(z')\right\} = \kappa^2 \sum_{n=0}^{s}\binom{s'+s-n}{s'}\,n\,\eth_{z'}^{n-1}\bar{F}_{s+s'-1}(z')\eth_z^{s-n}\delta(z,z')\,. \tag{5.5.23}
$$

It is then straightforward, from (5.5.3), (5.5.4), to obtain the brackets between the linear and

[13]See, in particular, Section 6.4.2 there, baring in mind that the canonical brackets of [162] come with an opposite sign compared to what we use here.

quadratic charges,

$$
\begin{aligned}
&\{q_s^{2,gr}(z), q_{s'}^{1,em}(z')\} = -\kappa^2 \sum_{n=0}^{s} \begin{pmatrix} s'+s-n-1 \\ s'-1 \end{pmatrix} (n+1)\eth_{z'}^{s'+1}(\eth_{z'}^n \bar{F}_{s+s'-1}(z')\eth_z^{s-n}\delta(z,z'))\,,\\
&\{q_s^{1,gr}(z), q_{s'}^{2,em}(z')\} = \kappa^2 \sum_{n=0}^{s'} \begin{pmatrix} s'+s-n \\ s \end{pmatrix} n\eth_z^{s+2}(\eth_z^{n-1}\bar{F}_{s+s'-1}(z)\eth_{z'}^{s'-n}\delta(z',z)),\\
&\{q_s^{1,em}(z), q_{s'}^{2,em}(z')\} = 0\,, \qquad\qquad (5.5.24)\\
&\{q_s^{2,gr}(z), q_{s'}^{1,gr}(z')\} = \kappa^2 \sum_{n=0}^{s} \begin{pmatrix} s'+s-n \\ s' \end{pmatrix} (n+1)\eth_{z'}^{s'+2}(\eth_{z'}^n \bar{N}_{s+s'-1}(z')\eth_z^{s-n}\delta(z,z'))\,.
\end{aligned}
$$

Finally, we derive the algebra at the level of smeared charges [162], defined as[14]

$$Q_s^{em}(\sigma_s) = \frac{1}{\kappa^2}\oint \sigma_s(z,\bar{z})\, q_s^{em}(z,\bar{z})\,, \tag{5.5.25a}$$

$$Q_s^{gr}(\tau_s) = \frac{1}{\kappa^2}\oint \tau_s(z,\bar{z})\, q_s^{gr}(z,\bar{z})\,. \tag{5.5.25b}$$

They involve the pairing between the higher spin charges q_s and higher spin symmetry parameters σ_s, τ_s of helicity $-s$. The linearized algebra involves the quadratic brackets between the smeared charges. For the mixed gravity-gauge field bracket,

$$\{Q_s^{gr}(\tau_s), Q_{s'}^{em}(\sigma_{s'})\}^{(1)} = \left\{\mathcal{Q}_s^{1,gr}(\tau_s), \mathcal{Q}_{s'}^{2,em}(\sigma_{s'})\right\} + \left\{Q_s^{2,gr}(\tau_s), Q_{s'}^{1,em}(\sigma_{s'})\right\}\,, \tag{5.5.26}$$

the two above brackets can then be calculated using the bare charge brackets (5.5.24) and integrating by parts, as demonstrated, for instance, in Appendix I of [162] for the case of pure gravity. We find,

$$
\begin{aligned}
&\left\{Q_s^{1,gr}(\tau_s), Q_{s'}^{2,em}(\sigma_{s'})\right\} = -s' Q_{s+s'-1}^{1,em}(\sigma_{s'}\eth\tau_s)\\
&-\frac{1}{\kappa^2}\oint \sum_{k=0}^{s'-1}\sum_{p=1}^{s+k+1} (-1)^k \frac{(s+s'+1)_2}{(k+s+2)_2} \begin{pmatrix} s+s'-1 \\ s \end{pmatrix} \begin{pmatrix} s'-1 \\ k \end{pmatrix} \begin{pmatrix} s+k+1 \\ p \end{pmatrix} \eth\tau_s \eth^p \sigma_{s'} \eth^{s+s'-p}\bar{F}_{s+s'-1},\\
&\left\{Q_s^{2,gr}(\tau_s), Q_{s'}^{1,em}(\sigma_{s'})\right\} = (s+1) Q_{s+s'-1}^{1,em}(\tau_s \eth\sigma_{s'}) \qquad\qquad (5.5.27)\\
&+\frac{1}{\kappa^2}\oint \sum_{k=0}^{s}\sum_{p=1}^{s'+k} (-1)^k \frac{(s+s'+1)_2}{(k+s'+1)_2} \begin{pmatrix} s+s'-1 \\ s'-1 \end{pmatrix} \begin{pmatrix} s \\ k \end{pmatrix} \begin{pmatrix} s'+k \\ p \end{pmatrix} \eth\sigma_{s'} \eth^p \tau_s \eth^{s+s'-p}\bar{F}_{s+s'-1}\,.
\end{aligned}
$$

Substituting the expressions back in (5.5.26), the second lines of each term remarkably combine to give an integral over a total derivative. Together with the pure gravity bracket which can be derived in a similar way [60, 162] and the vanishing pure Maxwell bracket, we thus arrive at the final algebra

$$
\begin{aligned}
&\{Q_s^{gr}(\tau_s), Q_{s'}^{gr}(\tau'_{s'})\}^{(1)} = Q_{s+s'-1}^{1,gr}((s+1)\tau_s\eth\tau'_{s'} - (s'+1)\tau'_{s'}\eth\tau_s)\,,\\
&\{Q_s^{gr}(\tau_s), Q_{s'}^{em}(\sigma_{s'})\}^{(1)} = Q_{s+s'-1}^{1,em}((s+1)\tau_s\eth\sigma_{s'} - s'\sigma_{s'}\eth\tau_s)\,, \qquad (5.5.28)\\
&\{Q_s^{em}(\sigma_s), Q_{s'}^{em}(\sigma'_{s'})\}^{(1)} = 0\,.
\end{aligned}
$$

To make contact with the notation of [152], let us introduce the modes

$$
\begin{aligned}
&Q_{k,l}^{s,gr} := Q_s^{gr}\left(\tau_{k,l}^s\right)\,, \quad \tau_{k,l}^s = z^{1+s-k}\bar{z}^{1-s-l}\,,\\
&Q_{k,l}^{s,em} := Q_s^{em}\left(\sigma_{k,l}^s\right)\,, \quad \sigma_{k,l}^s = z^{\frac{1}{2}+s-k}\bar{z}^{\frac{1}{2}-s-l}\,,
\end{aligned} \tag{5.5.29}
$$

[14] The trasnverse 2-dimensional integral here is abbreviated as $\oint := \int d^2z\, q_{z\bar{z}}\,(z,\bar{z})$.

in terms of which (5.5.28) reads

$$\begin{aligned}
&\left\{Q^{s,gr}_{k,l}, Q^{s',em}_{k',l'}\right\} = \left(k\, s' + \left(k' - \tfrac{1}{2}\right)(s+1)\right) Q^{s+s'-1,em}_{k+k'-1,l+l'}\,, \\
&\left\{Q^{s,em}_{k,l}, Q^{s',em}_{k',l'}\right\} = 0\,, \\
&\left\{Q^{s,gr}_{k,l}, Q^{s',gr}_{k',l'}\right\} = \left(k\,(s'+1) - k'\,(s+1)\right) Q^{s+s'-1,gr}_{k+k'-1,l+l'}\,.
\end{aligned} \tag{5.5.30}$$

With the mode re-labeling

$$q^{s,gr}_k(\bar z) = -2w^{\frac{s+3}{2}}_{\frac{s+1}{2}-k}(\bar z)\,, \qquad q^{s,em}_k(\bar z) = -2is^{\frac{s+2}{2}}_{\frac{s+1}{2}-k}(\bar z)\,, \tag{5.5.31a}$$

we finally get,

$$\begin{aligned}
&\{w^p_m, w^q_n\} = (m(q-1) - n(p-1))w^{p+q-2}_{m+n}\,, \\
&\{w^p_m, s^q_n\} = (m(q-1) - n(p-1))s^{p+q-2}_{m+n}\,, \\
&\{s^p_m, s^q_n\} = 0\,,
\end{aligned} \tag{5.5.32}$$

which is the Einstein-Maxwell $sw_{1+\infty}$ algebra we wished to derive. Notice that the indices-independent numerical factor $(-2i)$ relating $q^{s,em}_k$ and s^p_m can in principle be any number in the current Einstein-Maxwell setup. However, as we will see in the next section, the Einstein–Yang-Mills $sw_{1+\infty}$ algebra precisely fixes this normalization constant to be as above.

5.6 Generalization to Einstein-Yang-Mills

The analysis of the previous section be genearlized once color indices are also included in the gauge fields. A brief summary of this generalization is presented in this section, for a detailed review the reader is referred to [61, 62]. Starting with the recursion relations, in presence of non-abelian interactions these are modified as,

$$\partial_u \mathcal{Q}^{(j)}_s = \eth \mathcal{Q}^{(j)}_{s-1} - ig_{\rm YM}\left[A, \mathcal{Q}^{(j)}_{s-1}\right] + (j+s-1)\sum_{i=0}^{j-1} \sigma^{(2-i)} \bullet \mathcal{Q}^{(j-i)}_{s-2+i}\,, \tag{5.6.1}$$

with $-j+1 \le s \le +j$, where

$$\sigma^{(2)} := C \quad \text{and} \quad \sigma^{(1)} := F^a T_a\,, \tag{5.6.2}$$

and we have furthermore introduced a "bullet" operation which contracts indices in the color space, i.e.

$$\begin{aligned}
\sigma^{(2)} \bullet \mathcal{Q}^{(2)}_s &= \sigma^{(2)} \mathcal{Q}^{(2)}_s\,, \quad \sigma^{(2)} \bullet \mathcal{Q}^{(1)}_s = \sigma^{(2)} \mathcal{Q}^{(1)a}_s T_a\,, \\
\sigma^{(1)} \bullet \mathcal{Q}^{(2)}_s &= \sigma^{(1)a} \mathcal{Q}^{(2)}_s T_a\,, \quad \sigma^{(1)} \bullet \mathcal{Q}^{(1)}_s = \delta_{ab}\sigma^{(1)a} \mathcal{Q}^{(1)b}_s\,.
\end{aligned} \tag{5.6.3}$$

The above recursion relations are exact for $-j \le s \le +j$, but can be extended to all $s \ge -j$ by redefining the subleading modes of the spin weight $+j$ NP scalar ψ_0. Defining from these the quasi-conserved charges,

$$\begin{aligned}
\tilde q^{(j)}_s &= \sum_{n=0}^{s} \alpha_n \eth^n \mathcal{Q}^{(j)}_{s-n} - ig_{\rm YM}\sum_{\ell=1}^{s}\sum_{n=0}^{\ell-1} \eth^{s-\ell}[(-\partial_u)^{-(n+1)}(\alpha_{s-\ell}A), \eth^n \mathcal{Q}^{(j)}_{\ell-n-1}] \\
&+ \sum_{i=0}^{j-1}\sum_{\ell=2-i}^{s}\sum_{n=0}^{\ell-2+i} (\ell+j-1)\,\eth^{s-\ell}((-\partial_u)^{-(n+1)}(\alpha_{s-\ell}\sigma^{(2-i)}) \bullet \eth^n \mathcal{Q}^{(j-i)}_{\ell-n-2+i}) + \mathcal{O}\left(\mathbb{F}^3\right)\,,
\end{aligned} \tag{5.6.4}$$

where $\tilde{q}_s^{(1)} = \tilde{q}_s^{gl}$, $\tilde{q}_s^{(2)} = \tilde{q}_s^{gr}$. As before, these charges are conserved in non-radiative configurations up to quadratic order in the fields. More explicitly, they satisfy the following evolution equation

$$\begin{aligned}\partial_u \tilde{q}_s^{(j)} &= \alpha_s \eth^s \partial_u \mathcal{Q}_0^{(j)} - i g_{\rm YM} \sum_{\ell=1}^{s} \eth^{s-\ell}[(-\partial_u)^{-\ell}(\alpha_{s-\ell}A), \eth^\ell \mathcal{Q}_{-1}^{(j)}] \\ &+ \sum_{i=0}^{j-1}\sum_{\ell=1}^{s}(\ell+j-1)\,\eth^{s-\ell}((-\partial_u)^{1-\ell-i}(\alpha_{s-\ell}\sigma^{(2-i)}) \bullet \eth^{\ell-1+i}\mathcal{Q}_{-1}^{(j-i)}) + \mathcal{O}\left(\mathbb{F}^3\right) .\end{aligned} \tag{5.6.5}$$

We can now start iteratively integrating the evolution equations (5.6.5) for all $s \geq -j+1$ and sepearting the 'soft' and 'hard' parts. A similar decomposition can be then obtained for the quasi-conserved charges (5.6.4). We shall skip the intermediate steps that follow closely the ones demonstrated for the caes of Einstein-Maxwell. We want now the algebra of the smeared charges, which are defined as,

$$Q_s^{(j)}\left(Z_s\right) = \frac{1}{\kappa^2}\oint Z_s(z) q_s^{(j)}(z)\,, \tag{5.6.6}$$

with $Z_s(z)$ the spin weight $-s$ smearing function and we are reminding here that we are using the abbreviation of the spatial arguments, e.g. $Z_s(z) := Z_s\left(z,\bar{z}\right)$. Following the steps from the previous sections, we find the following $sw_{1+\infty}$ algebra

$$\begin{aligned}\left\{Q_s^{(1)a}\left(Z_s\right), Q_{s'}^{(1)b}\left(Z'_{s'}\right)\right\}^{(1)} &= 2g_{\rm YM} f_{cd}{}^a \delta^{bd} Q_{s+s'}^{1(1)c}\left(Z_s Z'_{s'}\right)\,, \\ \left\{Q_s^{(2)}\left(Z_s\right), Q_{s'}^{(1)a}\left(Z'_{s'}\right)\right\}^{(1)} &= Q_{s+s'-1}^{1(1)a}\left(\left(s+1\right)Z_s\eth Z'_{s'} - s' Z'_{s'}\eth Z_s\right)\,, \\ \left\{Q_s^{(2)}\left(Z_s\right), Q_{s'}^{(2)}\left(Z'_{s'}\right)\right\}^{(1)} &= Q_{s+s'-1}^{1(2)}\left(\left(s+1\right)Z_s\eth Z'_{s'} - \left(s'+1\right)Z'_{s'}\eth Z_s\right).\end{aligned} \tag{5.6.7}$$

Last, let us define the modes $Q_{s;k,\ell}^{(j)}$ in the holomorphic basis $Z_{s;k,\ell}^{(j)}$ for each spacetime spin j according to

$$Q_{s;k,\ell}^{(j)} := Q_s^{(j)}\left(Z_{s;k,\ell}^{(j)}\left(z,\bar{z}\right)\right)\,, \quad Z_{s;k,\ell}^{(j)}\left(z,\bar{z}\right) = z^{\frac{j}{2}+s-k}\bar{z}^{\frac{j}{2}-s-\ell}\,. \tag{5.6.8}$$

Then, the $sw_{1+\infty}$ algebra reads

$$\begin{aligned}&\left\{Q_{s;k,\ell}^{(j)}, Q_{s';k',\ell'}^{(j')}\right\}^{(1)} = \delta_{j1}\delta_{j'1}\mathcal{O}\left(g_{\rm YM}\right) + \left(1-\delta_{j1}\delta_{j'1}\right)\mathbf{1}_c^{-1}\times \\ &\left[\left(k+\tfrac{j}{2}-1\right)\left(s'+j'-1\right) - \left(k'+\tfrac{j'}{2}-1\right)\left(s+j-1\right)\right] Q_{s+s'-1;k+k'-1,\ell+\ell'}^{1(j+j'-2)}\,,\end{aligned} \tag{5.6.9}$$

with

$$\left\{Q_{s;k,\ell}^{(1)a}, Q_{s';k',\ell'}^{(1)b}\right\}^{(1)} = 2g_{\rm YM} f_{cd}{}^a\delta^{bd} Q_{s+s';k+k'-\frac{1}{2},\ell+\ell'-\frac{1}{2}}^{1(1)c}\,. \tag{5.6.10}$$

In summary,

$$\begin{aligned}\left\{Q_{s;k,\ell}^{(1)a}, Q_{s';k',\ell'}^{(1)b}\right\}^{(1)} &= 2g_{\rm YM} f_{cd}{}^a\delta^{bd} Q_{s+s';k+k'-\frac{1}{2},\ell+\ell'-\frac{1}{2}}^{1(1)c}\,, \\ \left\{Q_{s;k,\ell}^{(2)}, Q_{s';k',\ell'}^{(1)a}\right\}^{(1)} &= \left[ks' - \left(k'-\tfrac{1}{2}\right)\left(s+1\right)\right] Q_{s+s'-1;k+k'-1,\ell+\ell'}^{1(1)a}\,, \\ \left\{Q_{s;k,\ell}^{(2)}, Q_{s';k',\ell'}^{(2)}\right\}^{(1)} &= \left[k\left(s'+1\right) - k'\left(s+1\right)\right] Q_{s+s'-1;k+k'-1,\ell+\ell'}^{1(2)}\,.\end{aligned} \tag{5.6.11}$$

These can be brought in the $sw_{1+\infty}$ algebra as originally written in [152] according to the matching

$$Q_{s;k}^{(1)a}\left(\bar{z}\right) = -2i\, s_{\frac{s+1}{2}-k}^{\frac{s+2}{2},a}\left(\bar{z}\right)\,, \quad Q_{s;k}^{(2)} = -2w_{\frac{s+1}{2}-k}^{\frac{s+3}{2}}\left(\bar{z}\right)\,, \tag{5.6.12}$$

where $Q^{(j)}_{s;k}(\bar{z})$ refers to the partial modes

$$Q^{(j)}_{s;k}(\bar{z}) := Q^{(j)}_{s}\left(Z^{(j)}_{s;k}(z,\bar{z})\right), \quad Z^{(j)}_{s;k}(z,\bar{z}) = z^{\frac{j}{2}+s-k} f(\bar{z}) \tag{5.6.13}$$

for arbitrary anti-holomorphic functions $f(\bar{z})$. Then, one recovers

$$\begin{aligned} \left\{s^{p,a}_{m}, s^{q,b}_{n}\right\} &= i g_{\mathrm{YM}} f_{cd}{}^{a} \delta^{bd} s^{p+q-1,c}_{m+n}, \\ \{w^{p}_{m}, s^{q,a}_{n}\} &= (m(q-1) - n(p-1))\, s^{p+q-2,a}_{m+n}, \\ \{w^{p}_{m}, w^{q}_{n}\} &= (m(q-1) - n(p-1))\, w^{p+q-2}_{m+n}. \end{aligned} \tag{5.6.14}$$

We have thus shown that the $sw_{1+\infty}$ algebra (5.6.14), which organizes the current algebra soft sector of celestial CFT [151, 176], emerges from the structure of subleading equations of motion in Einstein–Yang-Mills theory. The underlying analysis, which was first reported in [60, 61, 162], is based on the identification of a truncation of the phase space which allows to write recursion relations for an infinite tower of higher spin weight charges. We have established the closure of this renormalized higher spin charge algebra at linear order.

One of the striking lessons one can learn from this work is that it would have seemed a priori impossible to predict the emergence of the $sw_{1+\infty}$ algebra solely from an asymptotic spacetime analysis without the insights from the celestial current algebra (or twistor theory). This is one of many examples highlighting the value of bridging different approaches to flat space holography: On one hand, the lowest-spin generators of this algebra (supertranslations and superrotations), identified long ago as asymptotic symmetries of flat spacetimes, were at the foundation of celestial holography. In return, insights from celestial OPEs subsequently revealed the presence of an infinite extension of these symmetries.

5.7 Gravitational correction to the logarithmic soft photon theorem

In the previous chapter we demonstrated that the logarithmic soft photon theorem arises naturally from the Ward identity of a subleading large gauge charge. In particular, in section 3.3.4 we constructed this charge directly from the soft theorem, and in section 4.2.2 we verified that its Ward identity reproduces the logarithmic corrections to the subleading soft photon factor when only electromagnetic interactions are considered.

We now revisit the problem from the complementary point of view of the equations of motion, and study how the inclusion of gravitational interactions modifies the soft charge. This provides a dynamical derivation of the logarithmic corrections and clarifies their relation to the mixed gauge–gravitational sector.

We begin with Maxwell's equations coupled to a charged scalar field, written in the Newman–Penrose (NP) formalism. The relevant evolution equations for the NP scalars Φ_s are

$$\begin{aligned} \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\Phi_1 - \frac{\gamma^{z\bar{z}}}{r}\left(\sqrt{\gamma_{z\bar{z}}}\bar{\partial} + \bar{\partial}\sqrt{\gamma_{z\bar{z}}}\right)\Phi_0 &= \tfrac{1}{2}J_l, \\ \left(\frac{\partial}{\partial r} + \tfrac{1}{r}\right)\Phi_2 - \frac{\sqrt{\gamma^{z\bar{z}}}}{r}\bar{\partial}\Phi_1 &= \tfrac{1}{2}J_{\bar{m}}, \\ \left(2\frac{\partial}{\partial u} - \frac{\partial}{\partial r} - \tfrac{1}{r}\right)\Phi_0 - \frac{2\sqrt{\gamma^{z\bar{z}}}}{r}\partial\Phi_1 &= -J_m, \\ \left(2\frac{\partial}{\partial u} - \frac{\partial}{\partial r} - \tfrac{2}{r}\right)\Phi_1 - \frac{2\gamma^{z\bar{z}}}{r}\left(\sqrt{\gamma_{z\bar{z}}}\partial + \partial\sqrt{\gamma_{z\bar{z}}}\right)\Phi_2 &= -J_n. \end{aligned} \tag{5.7.1}$$

Together with charge conservation,

$$\frac{1}{r^2}\left(\partial_r(r^2 j_u) - \gamma^{z\bar{z}}(\partial j_{\bar{z}} + \bar{\partial} j_z)\right) = A_r j_u - \gamma^{z\bar{z}}(A_z j_{\bar{z}} + A_{\bar{z}} j_z)\,, \tag{5.7.2}$$

these equations encode the asymptotic dynamics of the photon field.

Expanding near future null infinity using eq (5.1.7), and specializing to the flat sphere coordinates (2.1.10), the leading equation of motion reads

$$\partial_u \Phi_1^0 = \partial \Phi_2^0 - j_u^{(2)}\,, \tag{5.7.3}$$

whose integrated form yields the usual large gauge transformation charge

$$Q_0^{\text{em}} = \int d^2z\, \lambda(z,\bar{z}) \partial \mathscr{F}^{(0)} - \int du\, d^2z\, \lambda(z,\bar{z}) j_u^{(2)}\,. \tag{5.7.4}$$

As shown in section 3.3.3, the Ward identity of Q_0 reproduces the leading soft photon theorem.

At the next order one finds

$$\partial_u \Phi_0^0 = \partial \Phi_1^0 - j_z^{(2)}\,, \tag{5.7.5}$$

which defines the subleading charge moment

$$\mathcal{Q}_1^{\text{em}} = \lim_{u\to-\infty} (\Phi_0^0 - u\partial\Phi_1^0)\,. \tag{5.7.6}$$

From this, the smeared charge

$$Q_1^{\text{em}} = \int d^2z\, \Upsilon(z,\bar{z}) \partial^2 \mathscr{F}^{(1)} - \int du\, d^2z\, \Upsilon(z,\bar{z}) (j_z^{(2)} - u\partial j_u^{(2)}) \tag{5.7.7}$$

can be constructed. This is precisely the charge obtained in section 3.3.4, whose Ward identity reproduces the subleading soft photon theorem, including the logarithmic corrections.

The inclusion of gravitational interactions modifies the evolution equation to eq (5.3.9). Then the subleading term is seen to modify according to

$$\partial_u \Phi_0^0 = \partial \Phi_1^0 - j_z^{(2)} + C_{zz}\Phi_2^0\,, \tag{5.7.8}$$

where the last term encodes the coupling to the asymptotic shear C_{zz}. Consequently, the subleading charge receives an additional contribution:

$$\begin{aligned} Q_1^{\text{em}} = &\int d^2z\, \Upsilon(z,\bar{z}) \Big(\partial^2 \mathscr{F}^{(1)} - \partial^2 C^{(0)} \mathscr{F}^{(0)}\Big) \\ &- \int du\, d^2z\, \Upsilon(z,\bar{z}) \Big(j_z^{(2)} - u\partial j_u^{(2)} - \tilde{C}_{zz} F_{u\bar{z}}^{(2)}\Big)\,. \end{aligned} \tag{5.7.9}$$

The first new term in the soft sector, proportional to $\partial^2 C^{(0)} \mathscr{F}^{(0)}$ arises from using (2.3.44) and (2.3.46). This term reproduces the gravitational correction to the logarithmic soft photon theorem derived in eq. (4.1.6).

Acting on the S-matrix, this contribution yields

$$\begin{aligned} &\langle \text{out}|\Delta Q_1^{\text{em,soft}} \mathcal{S} + \mathcal{S} \Delta Q_1^{\text{em,soft}}|\text{in}\rangle \\ &= \int d^2z\, \Upsilon \left[\partial^2\left(\sigma'_{n+1} S_{\text{em}}^{(0)}\right) - \partial\left(\sigma'_{n+1}\partial S_{\text{em}}^{(0)}\right)\right] \langle \text{out}|\mathcal{S}|\text{in}\rangle \\ &= \int d^2z\, \Upsilon \left[\partial^2\left(\sigma'_{n+1} S_{\text{em}}^{(0)}\right) - \partial\left(\sigma'_{n+1}\left(\sum_{m=0}\delta^2(z-z_i) + \sum_{m\neq 0}\mathcal{G}^{(2)}(\hat{p}_i;z,\bar{z})\right)\right)\right] \langle \text{out}|\mathcal{S}|\text{in}\rangle\,, \end{aligned} \tag{5.7.10}$$

where the intermediate steps use the soft-mode insertions (3.5.12), (3.5.16), and the identities (3.3.31). Now fixing the gauge parameter as before to $\Upsilon = \frac{z-w}{\bar{z}-\bar{w}}$, and integrating by parts, we find,

$$\langle \text{out}|\Delta Q_1^{\text{em,soft}}\mathcal{S} + \mathcal{S}\Delta Q_1^{\text{em,soft}}|\text{in}\rangle = \sigma'_{n+1} S_{\text{em}}^{(0)}\langle \text{out}|\mathcal{S}|\text{in}\rangle\,. \tag{5.7.11}$$

In reaching the final expression we also had to make use of the following relations,

$$\partial_z \frac{z-w}{\bar{z}-\bar{w}} = S_{\text{em}}^{(0)}(z;w)\,, \qquad \int d^2z\, S_{\text{em}}^{(0)}(z;w)\mathcal{G}^{(2)}(\hat{p};z) = S_{\text{em}}^{(0)}(\hat{p};w) \tag{5.7.12}$$

Thus together with replacing the matter operators to account for gravitational interactions, we reproduce the full logarithmic correction to subleading soft photon theorem.

The final additional term in eq. (5.7.9) contributes to the hard charge rather than the soft one. Importantly, this term scales as ω^0 in the soft expansion and does *not* factorize in the standard sense: it cannot be written as a universal multiplicative factor multiplying the n-point amplitude. Instead, it leads to mixing between amplitudes with different external content. Concretely, when a hard graviton (respectively, hard photon) is present in the $n+1$-point amplitude, the soft limit produces a term equivalent to the amplitude without the soft photon but with the hard graviton (resp. photon) replaced by a hard photon (resp. graviton) of opposite helicity.

This non-factorizing behavior is a genuine one-loop effect. It is nevertheless interesting to note that such a non-factorizing term can also be obtained from Ward identities of asymptotic charges, demonstrating once again the reach of the symmetry perspective beyond the naive factorization paradigm. In this sense, the gravitational correction to the logarithmic soft photon theorem both confirms the universality of the infrared structure and illustrates how loop effects can generate qualitatively new features.

Chapter 6

Conclusion

The central theme of this thesis has been the deep interplay between soft theorems and asymptotic symmetries. We have shown how this relationship extends beyond the well-studied tree-level and leading-order regime, encompassing both loop corrections in gravity and gauge theory as well as higher-order terms in the soft expansion. Together, these results illustrate that the universal structures governing infrared physics are far richer than previously appreciated.

Soft pyramid. The results presented here should be regarded as only the tip of the iceberg, as already hinted at by the hierarchical structure of the "soft pyramid" summarized in table 4.2. A natural direction for future work is to uncover the underlying organizing principles that account for the universal features of each entry in this expansion. For electromagnetism, such universality has recently been established in [144], while in gravity analogous conjectures for $2 \to 2$ scattering with a soft graviton were put forward in [143]. These developments strongly suggest that a symmetry-based explanation should exist for the full tower of corrections. Such a structure may also shed light on the assumed Schwarzian fall-offs encountered in chapter 5 and on proposed relaxations of the peeling condition [177]. A preliminary step in this direction, extending the subleading soft graviton theorem to the subsubleading order, has been included in appendix C.

Finally, let us comment on renormalization of subleading charges. The charges constructed in Chapter 4 formally diverge when the fall-offs (2.3.39) are imposed, which is consistent with the fact that their Ward identities are themselves divergent. Thus, the construction should be viewed as a *regularization* of the divergence. A natural next step is to *renormalize* the charges by combining our analysis with the Ward identities of [145]. Concretely, one may organize the superrotation charge as

$$\mathcal{F}_{\mathcal{Y}} \;=\; \underbrace{\mathcal{F}_{\mathcal{Y}}^{\text{hard}} + \Delta\mathcal{F}_{\mathcal{Y}}^{\text{hard}}}_{\text{interacting hard sector}} \;+\; \underbrace{\mathcal{F}_{\mathcal{Y}}^{\text{soft}} + \Delta\mathcal{F}_{\mathcal{Y}}^{\text{soft}}}_{\text{soft sector}} \;+\; \frac{1}{\epsilon}\Big(Q_{\ln}^{\text{hard}} + Q_{\ln}^{\text{soft}}\Big), \tag{6.0.1}$$

where $Q_{\ln}^{\text{hard}}$ and $Q_{\ln}^{\text{soft}}$ are the logarithmic counterterms defined in [145], and ϵ is the dimensional regulator. The corresponding Ward identity would then be extracted from the *finite* part of the loop-corrected subleading soft graviton theorem. Equivalently, one may implement this by acting on the $(n{+}1)$-point amplitude with a projector that removes the $1/\epsilon$ (or, after a change of scheme, $\ln\omega$) divergence:

$$\lim_{\omega\to 0}\Big(1 + \omega\partial_\omega - \ln\omega\;\partial_\omega\,\omega^2\partial_\omega\Big)\,\langle a^\dagger(\omega,z,\bar{z})\,\mathcal{S}\rangle \;=\; S^{(1)}\,\langle\mathcal{S}\rangle, \tag{6.0.2}$$

so that only the finite, universal subleading soft factor $S^{(1)}$ survives on the right-hand side. A complete understanding of the renormalized charge algebra and its potential central extensions [178] in this scheme is left for future work.

Celestial holography. Beyond their intrinsic interest, these relations have far-reaching implications for flat space holography. One of the most compelling motivations for studying asymptotic symmetries is that they provide the candidate symmetry algebras of putative holographic duals. For instance, starting from the BMS algebra in eq. (2.3.15), one may expand the supertranslation and superrotation parameters into Laurent modes [26],

$$\mathcal{T}(z,\bar{z}) = \sum_{k,\ell\in\frac{\mathbb{N}}{2}} \mathcal{T}_{k,\ell}\, z^{\frac{1}{2}-k}\bar{z}^{\frac{1}{2}-\ell}\,, \qquad \mathcal{Y}(z) = \sum_{m\in\mathbb{N}} \mathcal{Y}_m z^{1-m}\,. \tag{6.0.3}$$

In this basis, the bms_4 commutators take a form that makes explicit the two commuting copies of the Witt (centerless Virasoro) algebra in semi-direct sum with the abelian ideal of supertranslations. This observation lies at the heart of the connection between asymptotic symmetries and celestial conformal field theory (CCFT).

In celestial holography, tree-level soft theorems map under *Mellin transform* to Ward identities of conformally soft currents on the celestial sphere [151, 179, 180]. The logarithmic soft graviton theorem, in particular, can be understood as a loop-corrected Ward identity of the superrotation symmetry (or celestial stress tensor) [59, 139]. Moreover, collinear limits of amplitudes correspond to operator product expansions (OPEs) on the celestial sphere [181–184], which have been shown to acquire loop-level corrections [185–187]. Relatedly, possible loop corrections to the structure of the celestial algebra have begun to be explored in [185–195]. These developments suggest that an important future direction is to understand how loop effects deform the full $sw_{1+\infty}$ algebra and its realization on the celestial sphere.

Carrollian holography. There is a complementary approach to flat space holography – Carrollian holography [27–29]. On the Carrollian side, the S-matrix is translated to a Carrollian correlator and similar to the celestial case, the collinear limits in the momentum basis can be understood as OPEs in the Carrollian basis [196–199]. Soft theorems in Carrollian holography have been related to Ward identities of spontaneously broken global symmetries [200](see appendix A), and recent works suggest that the celestial $w_{1+\infty}$ algebra may also have a Carrollian realization [201]. However, the implications of loop corrections and gravitational tails in this framework remain unexplored, presenting a fertile direction for future work.

For clarity, the comparison between the celestial and Carrollian approaches to flat space holography has been summarized in table 6.1.

Momentum basis	**Celestial basis**	**Carrollian basis**
S-matrix	Celestial correlators	Carrollian correlators
Collinear limit	Celestial OPE	Carrollian OPE
Tree-level soft theorems	Conformal current Ward identities	Ward identities of spontaneously broken global symmetries
Soft tower	Conformally soft currents	Carrollian ascendants of hard gravitons
Logarithmic soft theorem	Complementary tower of conformally soft currents	??

Table 6.1: Comparison of celestial and Carrollian holographic correspondences for soft theorems.

Twistors. The celestial $w_{1+\infty}$ algebra admits a remarkably natural geometric realization in *twistor space* (see [202] for a review), where the generators act as Poisson diffeomorphisms [155, 156]. The connection between this twistor realization and the celestial OPE of conformally soft gravitons was elucidated in [203]. From a Carrollian perspective, the representation of $w_{1+\infty}$ symmetries on gravitational data at $\mathscr{I}$ was derived explicitly from twistor space through the Penrose transform in [204]. In particular, this construction illustrates how the correspondence between twistor space and $\mathscr{I}$ gives rise to a non-local spacetime representation: for a Carrollian zero-rest-mass field ϕ_j of arbitrary spin $j \in \mathbb{Z}$, eq. (3.8) of [204] reads[1]

$$\delta_{\tau_s}\phi_j = \sum_{\ell=0}^{s} \frac{(\ell+1)}{(s-\ell)!} (\bar{\eth}_z^{s-\ell}\tau_s)\, \partial_u^{1+j} \Big(u^{s-\ell}\, \partial_u^{-\ell-j} \bar{\eth}_z^{\ell} \phi_j(u,z) \Big) \,. \tag{6.0.4}$$

Interestingly, this same expression can be reproduced by generalizing to arbitrary spin j the canonical bracket of the smeared charges obtained in section 5.6, namely

$$\delta_{\tau_s}\phi_j = \{Q_s^{2,gr}(\tau_s), \phi_j(u,z)\} \,. \tag{6.0.5}$$

Further connections between twistor theory and celestial charges were established in [171], where the gravitational canonical celestial charges of [60, 162] were related to twistor space via a BF twistor action for self-dual gravity. An exciting open question is whether the full Einstein-Yang-Mills set of charges and canonical brackets in (5.0.3) can also be derived directly from a twistor construction.

Multipoles and memories. Beyond twistor space, an intriguing connection between celestial $w_{1+\infty}$ charges and multipole expansions of the gravitational field near null and spatial infinity was pointed out in [205]. In the linearized theory, the infinite tower of celestial charges was shown to correspond to canonical multipole moments; see also [91, 206–209] for related developments. These works suggest that the higher levels of the celestial tower might encode novel gravitational memory effects, subleading in strength but universal in character. Uncovering a precise relationship between higher memory effects, $w_{1+\infty}$ symmetries, and scattering amplitudes could open a new infrared "web" of connections linking symmetry, dynamics, and radiation in gauge and gravity theories in asymptotically flat spacetimes.

Black holes. Finally, while the discussion in this thesis has focused primarily on the asymptotic structure at null infinity, asymptotically flat spacetimes can also contain black holes, which introduce additional boundaries associated with event horizons. Symmetries near horizons have been studied in recent years, revealing analogs of supertranslations and superrotations on these null surfaces [210]. To fully understand how such horizon symmetries constrain black holes within asymptotically flat spacetimes, it is crucial to develop a unified framework treating null infinity and horizons on equal footing [211]. One step in this direction is illustrated in appendix B. Incorporating black holes into the analysis of asymptotic symmetries, and clarifying their relationship to horizon multipole moments [205, 212], represents a rich and challenging avenue for future exploration.

In summary, the results of this thesis highlight how soft theorems and asymptotic symmetries continue to reveal new infrared structures in gauge and gravity theories. By going beyond the leading order, both in the soft expansion and at loop level, we have seen how universal patterns persist but also acquire richer deformations, pointing toward deeper organizing principles yet

[1] In the notation of [204], one has $s = n-1$ and the generators are $\tau_{n-1} = g_{\alpha(n)}\bar{\lambda}^{\dot{\alpha}(n)}$.

to be uncovered. Whether approached through celestial holography, Carrollian field theories, twistor space, or horizon symmetries, the common thread is clear: infrared physics encodes a universal language that bridges scattering amplitudes, geometry, and symmetry. Unraveling this language further promises not only to sharpen our understanding of flat space holography, but also to illuminate the fundamental role of universality in the quantum structure of spacetime.

Appendix A

Soft theorems in Carrollian holography

In the introduction, we briefly talked about flat space holography. There are two roads to flat space hologarphy – Celestial and Carrollian. In Celestial Holography, the soft theorems have been shown to correspond to Ward identities of conformal current operators, a fact we already touched on in chapter 5. What about Carrollian Holography? Here we briefly review the results of [200] where the Carrollian analogue of soft theorems was shown to imply spontaneous breaking of global symmetries. Our focus will be for the case of massless scattering. In this context we shall also briefly review some basics of Carrollian field theory.

A Carrollian manifold is endowed with a conformal equivalence class of degenerate metrics, with standard representative

$$ds^2_{\mathscr{I}} = 0\,\mathrm{d}u^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,. \tag{A.0.1}$$

Here x^i are cartesian stereographic coordinates on the celestial sphere S^2, and we will denote the full set of coordinates by $\mathbf{x} = (u, \vec{x})$. Of particular interest is the realisation of $\mathscr{I}$ as the future or past component of the null conformal boundary of Minkowski spacetime. The intertwining relation is given through the modified Fourier transform [28, 196, 213, 214]

$$O_{\Delta,J}(\mathbf{x}) = O_{\Delta,J}(u,\vec{x}) = \int_0^\infty d\omega\,\omega^{\Delta-1}e^{i\omega u}\,a_J^\dagger(p(\omega,\vec{x}))\,. \tag{A.0.2}$$

This transform can be applied to momentum $\mathcal{S}$-matrix elements S_n, thereby defining the *Carrollian amplitudes*

$$\langle O^{\eta_1}_{\Delta_1,J_1}(\mathbf{x}_1)\ldots O^{\eta_n}_{\Delta_n,J_n}(\mathbf{x}_n)\rangle \equiv \prod_{k=1}^n\int_0^\infty d\omega_k\,\omega^{\Delta_k-1}e^{i\eta_k\omega_k u_k}S_n(1^{J_1}\ldots n^{J_n})\,, \tag{A.0.3}$$

where $\eta_k = \pm 1$ depending whether the particle is ingoing (+) or outgoing (-).

As usual, symmetries yield Ward identities satsfied by correlation functions. In particular, computing the expectation value of $[Q_\lambda\,, O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)]$ in the vacuum state $|0\rangle$ yields the (integrated) Ward identity

$$\begin{aligned}&\langle 0|Q_\lambda\,O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n) - O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)\,Q_\lambda|0\rangle\\ &\qquad = -i\sum_{a=1}^n e_a\,\lambda(\vec{x}_a)\langle 0|O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)|0\rangle\,.\end{aligned} \tag{A.0.4}$$

Similarly, the Ward identity resulting from BMS symmetries reads

$$\begin{aligned}&\langle 0|Q_T\,O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n) - O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)\,Q_T|0\rangle\\ &\qquad = i\sum_{a=1}^n T(\vec{x}_a)\partial_{u_a}\langle 0|O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)|0\rangle\,.\end{aligned} \tag{A.0.5}$$

In this basis we can then rewrite the soft theorems. The leading soft photon theorem is then given by,

$$\begin{aligned}\lim_{\omega\to 0}\omega\Big[\langle O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)\,a_i^\dagger(\omega q(\vec{y}))\rangle - \langle\, a_i(\omega q(\vec{y}))\,O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)\rangle\Big]&\\ = \sum_{a=1}^{n} e_a\,\frac{(y^i - x_a^i)}{|\vec{y}-\vec{x}_a|^2}\,\langle O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)\rangle\,.&\end{aligned} \tag{A.0.6}$$

At this point we are able to make contact with the Ward identity (A.0.4). Indeed we see that the right-hand side of (A.0.6) reproduces the right-hand of (A.0.4) if we choose the particular symmetry parameter

$$\lambda(\vec{x};\vec{y},\varepsilon_i) = \frac{(y^i - x^i)}{|\vec{y}-\vec{x}|^2}\,, \tag{A.0.7}$$

which is determined in terms of the momentum coordinate $\vec{y}$ and the polarization label i of the soft photon. Since the right-hand is nonzero, *it necessarily implies spontaneous breaking* of the $\mathrm{U}(1)_{\text{large}}$ symmetry within the Carrollian field theory framework.

Spontaneous breaking of BMS symmetries can be diagnosed in the same way. Starting from the soft graviton theorem and going to the Carrollian basis using (A.0.2), we obtain

$$\begin{aligned}\lim_{\omega\to 0}\omega\Big[\langle O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)\,a_i^\dagger(\omega q(\vec{y}))\rangle - \langle a_i(\omega q(\vec{y}))\,O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)\rangle\Big]&\\ = -i\kappa\sum_{a=1}^{n}\frac{2(y^i - x_a^i)^2}{|\vec{y}-\vec{x}_a|^2}\,\partial_{u_a}\langle O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)\rangle\,.&\end{aligned} \tag{A.0.8}$$

The right-hand side of (A.0.8) reproduces the right-hand of (A.0.5) if we choose the particular symmetry parameter

$$T(\vec{x};\vec{y},\varepsilon_i) = \frac{2(y^i - x^i)^2}{|\vec{y}-\vec{x}|^2}\,, \tag{A.0.9}$$

which is determined in terms of the momentum coordinate $\vec{y}$ and the polarization label i of the soft graviton. Since the right-hand is nonzero, it implies spontaneous breaking of BMS supertranslations.

This spontaneous symmetry breaking can also be connected to an appropriately defined Goldstone's theorem. The details of this can be found in [200].

Appendix B

Extremal Reissner-Nordström black holes

Let us see the Newman-Penrose formalism in action using the example of an extremal Reissner-Nordström (ERN) black hole geometry. This is particularly interesting because the geometry has a special property of being 'self-dual' under the spatial inversions – the discrete Couch-Torrence conformal isometry identified in [215]. Utilizing this property, it is possible to study the NP scalars both at the black hole horizon and at null infinity and extract new pairings between near-horizon and near–null infinity data which dictate the one-to-one matching between infinite towers of conserved quantities. This will just be a very quick review of the results presented in [216], interested readers can refer to the paper and also check [217–223]. The conformal isometry can be seen at the level of the ERN metric which in Schwarzschild-like $\left(t, r, x^A\right)$ coordinates, is described by the line element

$$ds^2_{\text{ERN}} = -\frac{\Delta(r)}{r^2}dt^2 + \frac{r^2}{\Delta(r)}dr^2 + r^2 d\Omega_2^2 \tag{B.0.1}$$

with $d\Omega_2^2 = \gamma_{AB}\left(x^C\right)dx^A dx^B = d\theta^2 + \sin^2\theta\, d\phi^2$ the line element on $\mathbb{S}^2$. The discriminant function is a perfect square, $\Delta(r) = (r-M)^2$, whose double root at $r = M$ determines the radial location of the degenerate horizon, with M the ADM mass of the black hole. This geometry describes an isolated, asymptotically flat, non-rotating and electrically charged black hole solution of the general-relativistic electrovacuum field equations, whose electric charge Q attains its critical (extremal) value, $Q^2 = M^2$ (in CGS units). The Couch-Torrence (CT) spatial inversion symmetry [215] is then given by,

$$r \xrightarrow{\text{CT}} \tilde{r} = \frac{Mr}{r-M} \Rightarrow ds^2_{\text{ERN}} = \Omega^{-2} d\tilde{s}^2_{\text{ERN}}\,, \quad \Omega = \frac{\tilde{r}-M}{M} = \frac{M}{r-M}\,, \tag{B.0.2}$$

where $d\tilde{s}^2_{\text{ERN}} = -\frac{\Delta(\tilde{r})}{\tilde{r}^2}dt^2 + \frac{\tilde{r}^2}{\Delta(\tilde{r})}d\tilde{r}^2 + \tilde{r}^2 d\Omega_2^2$ is the same ERN black hole geometry, but with $\tilde{r}$ replacing r.

Keeping this in mind, we study the geometry of this spacetime near null infinity and near the horizon. To study the near-$\mathscr{I}^+$ or near-$\mathscr{I}^-$ modes, we will use retarded or advanced Eddington-Finkelstein coordinates, $\left(u, r, x^A\right)$ or $\left(v, r, x^A\right)$, respectively,

$$\begin{aligned} ds^2_{\text{ERN}} &= -\left(1-\frac{M}{r}\right)^2 du^2 - 2\,du dr + r^2 d\Omega_2^2 \\ &= -\left(1-\frac{M}{r}\right)^2 dv^2 + 2\,dv dr + r^2 d\Omega_2^2\,. \end{aligned} \tag{B.0.3}$$

A set of null tetrad vectors[1] $\{\ell, n, m, \bar{m}\}$, adapted to $\mathscr{I}^+$ would then be

$$\ell = \partial_r \,, \quad n = \partial_u - \frac{1}{2}\left(1 - \frac{M}{r}\right)^2 \partial_r \,, \quad m = \frac{1}{r}\varepsilon^A_{\mathbb{S}^2}\partial_A \,, \quad \bar{m} = \frac{1}{r}\bar{\varepsilon}^A_{\mathbb{S}^2}\partial_A \,, \tag{B.0.4}$$

with $\varepsilon^A_{\mathbb{S}^2}$ a complex dyad for the round 2-sphere. Charting the 2-sphere by spherical coordinates (θ, ϕ), a convenient choice of this complex dyad is

$$\varepsilon^A_{\mathbb{S}^2}\partial_A = \frac{1}{\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) . \tag{B.0.5}$$

To study the near-$\mathscr{H}^+$ or near-$\mathscr{H}^-$ modes, we will instead use advanced or retarded Eddington-Finkelstein coordinates, $\left(v, \rho, x^A\right)$ and $\left(u, \rho, x^A\right)$, respectively, $\rho = r - M$ being the affine radial coordinate centered at the horizon,

$$\begin{aligned} ds^2_{\text{ERN}} &= -\frac{\rho^2}{(M+\rho)^2}dv^2 + 2\,dvd\rho + (M+\rho)^2\,d\Omega^2_2 \\ &= -\frac{\rho^2}{(M+\rho)^2}du^2 - 2\,dud\rho + (M+\rho)^2\,d\Omega^2_2 \,. \end{aligned} \tag{B.0.6}$$

Then, a set of null tetrad vectors adapted to $\mathscr{H}^+$ would be

$$\ell = -\partial_\rho \,, \quad n = \partial_v + \frac{1}{2}\frac{\rho^2}{(M+\rho)^2}\partial_\rho \,, \quad m = \frac{1}{M+\rho}\varepsilon^A_{\mathbb{S}^2}\partial_A \,, \quad \bar{m} = \frac{1}{M+\rho}\bar{\varepsilon}^A_{\mathbb{S}^2}\partial_A \,, \tag{B.0.7}$$

with $\varepsilon^A_{\mathbb{S}^2}$ the same complex dyad for the 2-sphere as in Eq. (B.0.5).

With this choice of tetrad and using the scalars and spin coefficients defines in section 2.3.2, the equation of motion for a spin-s perturbation can be expressed as,

$$\begin{aligned} &{}_{\mathscr{I}^+}\mathbb{T}_s\psi_s = 0 \,, \\ &{}_{\mathscr{I}^+}\mathbb{T}_s \coloneqq (r-M)^{-2s}\,\partial_r\,(r-M)^{2(s+1)}\,\partial_r + 2\eth'_{\mathbb{S}^2}\eth_{\mathbb{S}^2} - 2\left(r^2\partial_r + (2s+1)\,r\right)\partial_u \,, \end{aligned} \tag{B.0.8}$$

when using the near-$\mathscr{I}$-adapted tetrad and coordinates, see Eq. (B.0.4), and after multiplying by $-2r^2$, or to

$$\begin{aligned} &{}_{\mathscr{H}^+}\mathbb{T}_s\psi_s = 0 \,, \\ &{}_{\mathscr{H}^+}\mathbb{T}_s \coloneqq \rho^{-2s}\partial_\rho\,\rho^{2(s+1)}\partial_\rho + 2\eth'_{\mathbb{S}^2}\eth_{\mathbb{S}^2} + 2\left((M+\rho)^2\,\partial_\rho + (2s+1)\,(M+\rho)\right)\partial_v \,, \end{aligned} \tag{B.0.9}$$

when using the near-$\mathscr{H}$-adapted tetrad and coordinates, see Eq. (B.0.7), and after multiplying by $-2\,(M+\rho)^2$. The spin-weight s master variable ψ_s is directly related to the fundamental NP scalars according to

$$\psi_s = (\Phi_2)^{\frac{|s|-s}{2}} \times \begin{cases} \Phi & \text{for scalar perturbations } (s=0)\,; \\ \phi_{1-s} & \text{for electromagnetic perturbations } (s=\pm 1)\,; \\ \Psi_{2-s} & \text{for gravitational perturbations } (s=\pm 2)\,, \end{cases} \tag{B.0.10}$$

Using the peeling properties, we can now expand the NP scalars ψ_s into near-$\mathscr{I}$ modes,

$$\psi_s \sim \frac{1}{(r-M)^{2s+1}}\sum_{n=0}^{\infty}\frac{\psi^{(n)}_s\left(u, x^A\right)}{(r-M)^n} \coloneqq \psi_s\left(u, r, x^A\right) , \tag{B.0.11}$$

[1] We are using the sign convention $m \cdot \bar{m} = -\ell \cdot n = +1$, such that $g_{ab} = -2\ell_{(a}n_{b)} + 2m_{(a}\bar{m}_{b)}$.

After expanding the near-$\mathscr{I}$ modes into spin-weight s spherical harmonics,

$$\psi_s^{(n)}\left(u,x^A\right) = \sum_{\ell=|s|}^{\infty}\sum_{m=-\ell}^{\ell}\psi_{s\ell m}^{(n)}(u)\ {}_sY_{\ell m}\left(x^A\right)\,, \tag{B.0.12}$$

It can be checked using B.0.8 that the following quantity,

$${}_sN_{\ell m} = \psi_{s\ell m}^{(\ell-s+1)}(u) + \frac{2\ell+1}{\ell-s+1}M\psi_{s\ell m}^{(\ell-s)}(u) + \frac{\ell+s}{\ell-s+1}M^2\psi_{s\ell m}^{(\ell-s-1)}(u)\ ,\ell\geq|s|\,, \tag{B.0.13}$$

is conserved, i.e., $\partial_u\,{}_sN_{\ell m} = 0$. This is the tower of linearly conserved NP quantities identified in [81] for generic asymptotically flat spacetimes.

We can obtain an analogous result for near the horizon. We first expand the NP scalar ψ_s in near-$\mathscr{H}$ modes,

$$\psi_s \sim \sum_{n=0}^{\infty}\hat{\psi}_s^{(n)}\left(v,x^A\right)\left(\frac{\rho}{M}\right)^n \coloneqq \hat{\psi}_s\left(v,\rho,x^A\right)\,, \tag{B.0.14}$$

and then expand into spin-weight s spherical harmonics,

$$\hat{\psi}_s^{(n)}\left(v,x^A\right) = \sum_{\ell=|s|}^{\infty}\sum_{m=-\ell}^{\ell}\hat{\psi}_{s\ell m}^{(n)}(v)\ {}_sY_{\ell m}\left(x^A\right)\,. \tag{B.0.15}$$

Then using (B.0.9) we once again identify a tower of conserved quantities – the so called Aretakis charges, given by,

$$\begin{gathered}{}_sA_{\ell m} \coloneqq \hat{\psi}_{s\ell m}^{(\ell-s+1)}(v) + \frac{2\ell+1}{\ell-s+1}\hat{\psi}_{s\ell m}^{(\ell-s)}(v) + \frac{\ell+s}{\ell-s+1}\hat{\psi}_{s\ell m}^{(\ell-s-1)}(v)\,,\\ \Rightarrow\quad \partial_v\,{}_sA_{\ell m} = 0\,,\quad \ell\geq|s|\,.\end{gathered} \tag{B.0.16}$$

These two towers of conserved quantities can then be shown to map onto each other under CT inversion with

$$\psi_s^{(n)}\left(u,x^A\right) = M^{n+2s+1}\hat{\psi}_s^{(n)}\left(v\mapsto u,x^A\right)\,. \tag{B.0.17}$$

It is then straightforward to see that the Newman-Penrose charges exactly match the Aretakis charges,

$$\begin{aligned}{}_sN_{\ell m} &= \psi_{s\ell m}^{(\ell-s+1)(u)} + \frac{2\ell+1}{\ell-s+1}M\psi_{s\ell m}^{(\ell-s)}(u) + \frac{\ell+s}{\ell-s+1}M^2\psi_{s\ell m}^{(\ell-s-1)}(u)\\ &= M^{\ell+s+2}\left[\hat{\psi}_{s\ell m}^{(\ell-s+1)}(v\mapsto u) + \frac{2\ell+1}{\ell-s+1}\hat{\psi}_{s\ell m}^{(\ell-s)}(v\mapsto u) + \frac{\ell+s}{\ell-s+1}\hat{\psi}_{s\ell m}^{(\ell-s-1)}(v\mapsto u)\right]\,,\end{aligned} \tag{B.0.18}$$

$$\therefore\ {}_sN_{\ell m} = M^{\ell+s+2}{}_sA_{\ell m}\,,\quad \ell\geq|s|\,. \tag{B.0.19}$$

Once again we skipped over many minute details which can be found in [216].

Appendix C

Logarithmic correction to the subsubleading soft graviton factor

In the main text we discussed in detail how loop-corrected subleading soft factors match with corresponding asymptotic symmetries. At tree level, the universal expression for the subsubleading soft graviton factor was given in eq. (3.1.11), while in eq. (4.1.22) we saw that it acquires a universal two-loop correction proportional to $\omega(\ln\omega)^2$. In this appendix, we sketch how this correction can be understood from the perspective of subsubleading charges originally proposed in [104, 172].

Let us begin with the subsubleading charge for massless matter fields defined in [104]. At future timelike infinity i^+ it takes the form[1]

$$Q_2^{i^+} = \int_{i^+} d^3V\, T^{(3)}_{\alpha\beta}\, \mathcal{Z}^{\alpha\beta}\,, \tag{C.0.1}$$

with

$$\mathcal{Z}^{\alpha\beta}(y^\alpha) = \int d^2z\, \mathcal{Z}^{AB}(x^A)\, \mathcal{G}^{\alpha\beta}_{AB}(x^A; y^\alpha)\,. \tag{C.0.2}$$

Here, $\mathcal{G}$ is the intertwining operator defined by

$$\mathcal{G}^{\alpha\beta}_{AB}(x^A; y^\alpha)\, \partial_\alpha\partial_\beta \;=\; \partial_A^2\partial_B^2\, S^{(2)}\,. \tag{C.0.3}$$

The action of this charge on an incoming state is then

$$Q_2^{i^+}|\text{in}\rangle = \int_{i^+} d^3y \int d^2z\, \mathcal{Z}^{AB}(x^A)\, \mathcal{G}^{\alpha\beta}_{AB}(x^A; y^\alpha)\, \partial_\alpha\partial_\beta\, b^\dagger b\, |\text{in}\rangle\,. \tag{C.0.4}$$

Choosing the symmetry parameter such that

$$\partial_A^2\partial_B^2\mathcal{Z}^{AB} \;=\; \delta^2(z-w)\,, \tag{C.0.5}$$

and using eqs. (C.0.3)–(C.0.4), one recovers the standard tree-level Ward identity,

$$\langle\text{out}|[Q_2^{i^+}, \mathcal{S}]|\text{in}\rangle = S^{(2)}\, \langle\text{out}|\mathcal{S}|\text{in}\rangle\,. \tag{C.0.6}$$

To incorporate loop corrections, we now dress the asymptotic fields at i^+ with gravitational interactions. For example, an interacting massive scalar behaves at late times as

$$\varphi_{FK}(\tau,\rho,\hat{x}) \overset{\tau\to\infty}{=} \frac{\sqrt{m}}{2(2\pi\tau)^{3/2}}\Big(e^{-im\int \mathcal{G}^{(3)}C^{(0)}}b(\rho,\hat{x})\, e^{-i\tau m} + \text{h.c.}\Big) + e^{i\alpha_1}\mathcal{O}(\tau^{-5/2})\,. \tag{C.0.7}$$

[1] We drop the $^{\text{gr}}$ label throughout this appendix, as all charges discussed here are purely gravitational.

This dressing modifies the charge at i^+ by introducing terms linear and quadratic in $C^{(0)}$. The linear piece diverges only as $1/\epsilon$, whereas we are after contributions that behave as $\omega(\ln\omega)^2$, corresponding to ϵ^{-2}. Thus, the relevant term is the quadratic $C^{(0)}$ contribution:

$$\begin{aligned}\int_{i^+} d^3y \int d^2z\, \mathcal{Z}^{AB}(x^A)\, \mathcal{G}_{AB}^{\alpha\beta}(x^A;y^\alpha)\, \partial_\alpha C^{(0)}\, \partial_\beta C^{(0)}\, T^{(0)}_{\tau\tau}|\text{in}\rangle \\ = \frac{1}{\epsilon^2}\, \frac{\varepsilon^{\mu\nu} q^\rho q^\sigma}{p_i\cdot q}\, J^i_{\mu\rho}(\sigma_n)\, J^i_{\nu\sigma}(\sigma_n)\, |\text{in}\rangle\,.\end{aligned} \tag{C.0.8}$$

We note that at subsubleading order one may also expect contributions from the subleading dressing proposed in [224]; however, since these effects are IR-finite, they affect only the non-universal ϵ^{-1} terms and not the universal ϵ^{-2} piece.

The soft part of the charge can be treated similarly. A useful shortcut is to note that if one had dressed the shear and news tensors instead of the matter fields, the additional soft flux would also have been generated. This motivates us to replace

$$\int d^2z\, \mathcal{Z}^{AB}\, \partial_A\partial^3\mathcal{N}_B^{(2)} \;\longrightarrow\; \int d^2z\, \mathcal{Z}^{AB}\, \partial_A\partial^3\mathcal{N}_B^{(2)} + \int d^2z\, \mathcal{Z}^{AB}\, \partial_A\partial^3\big(C^{(0)}\mathcal{N}_B^{(1)}\big)\,. \tag{C.0.9}$$

Adding this soft contribution to the matter piece yields

$$\begin{aligned}&\lim_{\omega\to 0} \partial_\omega(1+\omega\partial_\omega)\, \langle\text{out}|\, a^\dagger(\omega,z,\bar z)\, \mathcal{S}\, |\text{in}\rangle\Big|_{1/\epsilon^2} \\ &= \Big((\sigma'_{n+1})^2\, S^{(0)} + \sigma'_{n+1}\, S^{(1)}\sigma_n + \sum_i \frac{\varepsilon^{\mu\nu}\hat q^\rho \hat q^\sigma}{p_i\cdot\hat q}\, J^i_{\mu\rho}(\sigma_n)\, J^i_{\nu\sigma}(\sigma_n)\Big)\langle\text{out}|\mathcal{S}|\text{in}\rangle\,,\end{aligned} \tag{C.0.10}$$

where we used

$$\partial^4\sigma'_{n+1} \;=\; 2\pi\sum_i \delta^2(z_i - z)\,. \tag{C.0.11}$$

Combining the hard and soft contributions, we indeed reproduce the full $\omega(\ln\omega)^2$ correction to the subsubleading soft graviton factor.

Appendix D

$C^{(0)}$ insertion in Grammer–Yennie decomposition

In the main text it was shown that the $C^{(0)}$ insertion in the S-matrix produces a divergence of order $\frac{1}{\epsilon}$, with ϵ the infrared regulator. In this appendix we revisit the same effect directly from the operator insertion perspective, tracking in particular the origin of the $\log\omega$ divergence in the S-matrix.

In the Grammer–Yennie regularization scheme [94, 225], all infrared divergent contributions to scattering amplitudes can be factorized into an exponential,

$$\mathcal{A}_n \;=\; e^{K_{gr}}\,\mathcal{A}_n^{\text{IR}-\text{finite}}\,. \tag{D.0.1}$$

We focus on the case where one of the external legs corresponds to a graviton, which will eventually be taken soft to mimic the effect of the $C^{(0)}$ insertion. For amplitudes with at least one massless external state, the exponent reads

$$K_{gr} \;=\; \frac{i}{2}\sum_{a\neq b}(p_a\!\cdot\! p_b)^2\int\frac{d^4l}{(2\pi)^4}\,\frac{1}{l^2-i\epsilon}\,\frac{1}{(p_a\cdot l-l^2/2-i\epsilon)(p_b\cdot l-l^2/2+i\epsilon)}\,. \tag{D.0.2}$$

While in dimensional regularization this reproduces the familiar divergent structure, here we keep the explicit integral form in order to expose the logarithmic behavior.

From section 3.5 we know that the S-matrix can equivalently be written as an expectation value of operator insertions, with separate factors responsible for the finite and divergent pieces,

$$\langle\mathcal{W}_1\cdots\mathcal{W}_n\rangle \;=\; e^{K_{gr}}\,, \tag{D.0.3}$$

where each insertion is of the form

$$\mathcal{W}_i \;=\; \exp\big[i\omega\,C^{(0)}(z_i,\bar z_i)\big]\,. \tag{D.0.4}$$

To compute the $C^{(0)}$ insertion, we differentiate with respect to the soft graviton energy:

$$\begin{aligned}\langle C^{(0)}(z,\bar z)\,\mathcal{S}\rangle &= \lim_{\omega\to 0}\,(-i\partial_\omega)\,\langle\mathcal{W}_\omega\mathcal{W}_1\cdots\mathcal{W}_n\rangle\,\mathcal{A}_n^{\text{IR}-\text{finite}}\\ &= \lim_{\omega\to 0}\,(-i\partial_\omega)\,\exp\!\left[\frac{i}{2}\sum_{a=1}^{n}(p_a\cdot k)^2\int\frac{d^4l}{(2\pi)^4}\frac{1}{l^2-i\epsilon}\,\frac{1}{(p_a\cdot l-i\epsilon)(2k\cdot l-l^2+i\epsilon)}\right]\langle\mathcal{S}\rangle,\end{aligned} \tag{D.0.5}$$

where $k=\omega\hat q(z,\bar z)$ is the momentum of the to-be-soft graviton. Following the prescription of [94], the soft limit is taken only at the very end of the computation.

The integral is IR-finite when the loop momentum l is larger than k, but for the region $l < \omega$ one picks up a logarithmic divergence. Expanding, one finds

$$\langle C^{(0)}(z,\bar{z})\,\mathcal{S}\rangle = \lim_{\omega\to 0}\ \frac{1}{2}\sum_{a=1}^{n}\frac{1}{4\pi}\big(\ln\tfrac{1}{\omega}+\ln\tfrac{1}{R}\big)\,|\hat{q}\cdot p_a|\Big(\delta_{\eta_a,-1}-\tfrac{i}{2\pi}\ln(\hat{q}\cdot\hat{p}_a)\Big)\,\langle\mathcal{S}\rangle, \tag{D.0.6}$$

where we kept only the logarithmically divergent contributions. This makes transparent the mapping

$$\tfrac{1}{\epsilon}\ \ \longleftrightarrow\ \ \ln\frac{\Lambda}{\lambda}\,, \tag{D.0.7}$$

between dimensional and cut-off regularizations.

We can also attempt to reconstruct the two-point function of the Goldstone operator in this scheme. With an upper cut-off at ω, the regulated Grammer–Yennie factor is

$$K^{\rm reg}_{gr}\ =\ \frac{i}{2}\sum_{a\neq b}\frac{1}{4\pi}\ \ln\tfrac{1}{\omega}\,\omega_a\omega_b\,|\hat{q}_a\!\cdot\hat{q}_b|\Big(\delta_{\eta_a\eta_b,1}-\tfrac{i}{2\pi}\ln(\hat{q}_a\!\cdot\hat{q}_b)\Big)\,, \tag{D.0.8}$$

from which we read off

$$\langle C^{(0)}_a\,C^{(0)}_b\rangle\ =\ \frac{1}{4\pi}\ \ln\tfrac{1}{\omega}\,|\hat{q}_a\!\cdot\hat{q}_b|\Big(\delta_{\eta_a\eta_b,1}-\tfrac{i}{2\pi}\ln(\hat{q}_a\!\cdot\hat{q}_b)\Big)\,. \tag{D.0.9}$$

Note that in this construction the lower cut-off remains the same across the two integrals, and hence the correlation function captures only the universal divergent piece.

Appendix E

Supplementary material

E.1 Various identities in pseudo-differential calculus

In this appendix, we collect a number of mathematical identities that have proved to be useful in deriving the $w_{1+\infty}$ charge algebra. First of all, we are using the iterated anti-derivative operator ∂_u^{-n}, $n \geq 0$, defined as the n'th repeated integral with base point $+\infty$ [162, 172]

$$\left(\partial_u^{-n}\mathcal{F}\right)(u) := \int_{+\infty}^{u} du_1 \int_{+\infty}^{u_1} du_2 \cdots \int_{+\infty}^{u_{n-1}} du_n\, \mathcal{F}(u_n) \tag{E.1.1}$$

for any scalar function $\mathcal{F}(u)$. This is a well-defined operation as long as

$$\mathcal{F}(u) = o\left(u^{-n}\right) \quad \text{as } u \to +\infty\,. \tag{E.1.2}$$

The action of the iterated anti-derivative operator can be equivalently written as the single integral

$$\left(\partial_u^{-n}\mathcal{F}\right)(u) = \int_{+\infty}^{u} du' \frac{(u-u')^{n-1}}{(n-1)!}\mathcal{F}(u') \tag{E.1.3}$$

by virtue of the Cauchy formula for repeated integration. In the following, and also in the main text, we loosen our notation and write $\partial_u^{-n}\mathcal{F}(u)$ in place of $\left(\partial_u^{-n}\mathcal{F}\right)(u)$. The anti-derivative operator obeys the generalized integral Leibniz rule,

$$\partial_u^{-1}\left(f(u)\,g(u)\right) = \sum_{n=0}^{\infty}(-1)^n\left(\partial_u^n f(u)\right)\left(\partial_u^{-(n+1)}g(u)\right)\,, \tag{E.1.4}$$

which, in particular, implies that

$$\partial_u^{-1}\left(\frac{(-u)^s}{s!}f(u)\right) = \sum_{n=0}^{s}\frac{(-u)^n}{n!}\partial_u^{-(s-n+1)}f(u)\,. \tag{E.1.5}$$

This can be generalized to

$$\partial_u^{-\ell}\left(\frac{(-u)^s}{s!}f(u)\right) = \sum_{n=0}^{s}\binom{s-n+\ell-1}{\ell-1}\frac{(-u)^n}{n!}\partial_u^{-(s-n+\ell)}f(u)\,, \tag{E.1.6}$$

for all $\ell \in \mathbb{N}$.

We will also make frequent use of the following distributional identities involving the δ-functional [60],

$$\partial_u^n \delta(u-u') = (-1)^n \partial_{u'}^n \delta(u-u')\,, \tag{E.1.7a}$$

$$f(u)\,\partial_u^n \delta(u-u') = (-1)^n \partial_{u'}^n f(u')\,\delta(u-u')\,, \tag{E.1.7b}$$

$$\begin{aligned}\partial_u^{-n} \delta(u-u') &= \frac{(u-u')^{n-1}}{(n-1)!}\partial_u^{-1}\delta(u-u') \\ &= -\frac{(u-u')^{n-1}}{(n-1)!}\,\theta(u'-u)\,,\end{aligned} \tag{E.1.7c}$$

with $n \in \mathbb{N}$ and where $\theta(x)$ is the Heaviside step function. Using (E.1.7a)-(E.1.7b) and the distributional property of the derivative operator, one can then show that

$$\begin{aligned} f(u)\,\partial_u^n \delta(u-u') &= (-1)^n f(u)\,\partial_{u'}^n \delta(u-u') \\ &= (-1)^n \partial_{u'}^n \left(f(u)\,\delta(u-u')\right) \\ &= (-1)^n \partial_{u'}^n \left(f(u')\,\delta(u-u')\right) \\ &= (-1)^n \sum_{m=0}^{n} \binom{n}{m} \partial_{u'}^m f(u')\,\partial_{u'}^{n-m}\delta(u-u') \\ &= \sum_{m=0}^{n} (-1)^m \binom{n}{m} \partial_{u'}^m f(u')\,\partial_{u}^{n-m}\delta(u-u')\,. \end{aligned} \tag{E.1.8}$$

This will be applied in the main text in the form

$$\eth_z^{s-\ell}\left(f(z)\eth_z^{\ell-m}\delta\left(z-z'\right)\right) = \sum_{n=m}^{\ell} (-1)^{n-m} \binom{\ell-m}{n-m} \eth_{z'}^{n-m} f(z')\eth_z^{s-n}\delta\left(z-z'\right) \tag{E.1.9}$$

to find the action of the celestial charges on the gravitational shear and the gluon field.

Last, the following manipulations have proved to be particularly useful [60]

$$u^n \partial_u^n = (\Delta_u - 1)_n\,, \tag{E.1.10a}$$

$$\partial_u^n u^n = (\Delta_u + n - 1)_n\,, \tag{E.1.10b}$$

$$u^{-n}\partial_u^{-n} = (\Delta_u + n - 1)_n^{-1}\,, \tag{E.1.10c}$$

$$\partial_u^k (\Delta_u + \alpha)_n = (\Delta_u + \alpha + k)_n \partial_u^k\,, \tag{E.1.10d}$$

$$u^k (\Delta_u + \alpha)_n^{\pm 1} = (\Delta_u + \alpha - k)_n^{\pm 1} u^k\,, \tag{E.1.10e}$$

with $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$, and where $(x)_n := x(x-1)\ldots(x-n+1)$, $(x)_0 = 1$, is the falling factorial, while $\Delta_u := u\partial_u + 1$. A nice property of the operator Δ_u is that it, as well as any analytic function of it, integrate to zero, when integrated over the entire real line,

$$\int_{-\infty}^{+\infty} du\,\Delta_u g(u) = 0\,, \tag{E.1.11}$$

assuming the function $g(u)$ falls off sufficiently fast at large u, namely, $g(u) = o\left(u^{-1}\right)$ above. For the cases encountered in this work, namely, for the fall-conditions of the radiative fields $\bar{N}$ and $\bar{F}$, this is indeed the case for all the integrals involved in deriving the action of the celestial charges on the sub^s-leading soft operators, and allows to simply set $\Delta_u = 0$ under such expressions,

$$\int_{-\infty}^{+\infty} du\, f(\Delta_u)\, g(u) = f(0)\int_{-\infty}^{+\infty} du\, g(u)\,. \tag{E.1.12}$$

E.2 Brackets between anti-derivatives of fundamental fields

In this appendix, we collect a list of the Poisson brackets between the fundamental fields and their anti-derivatives that are needed for finding the action of the celestial charges onto the photon/gluon field and the gravitational shear.

From the Einstein-YM action functional,

$$S\left[g,A\right]=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R-\frac{1}{2}\operatorname{tr}\left(\mathrm{F}^2\right)\right]\,, \tag{E.2.1}$$

the symplectic structure on $\mathscr{I}^+$ can be worked out to be

$$\begin{aligned}\Omega_{\mathscr{I}^+}&=\frac{1}{\kappa^2}\oint\int_{-\infty}^{+\infty}du\left[\frac{1}{4}\delta C_{AB}\wedge\delta N^{AB}+\operatorname{tr}\left(\delta A_A^{(0)}\wedge\delta\partial_u A^{(0)A}\right)\right]\\&=\frac{1}{\kappa^2}\oint\int_{-\infty}^{+\infty}du\left[\delta C\wedge\delta\bar{N}+\delta_{ab}\delta A^a\wedge\delta\bar{F}^b+(\text{c.c.})\right]\\&=\frac{1}{\kappa^2}\oint\int_{-\infty}^{+\infty}du\sum_{j=1}^{2}\left(\delta C^{(j)}\wedge\delta\bar{N}^{(j)}+(\text{c.c.})\right)\,,\end{aligned} \tag{E.2.2}$$

from which we extract the following fundamental Poisson brackets

$$\begin{aligned}\left\{C(u,z),\bar{N}(u',z')\right\}&=\kappa^2\,\delta(u-u')\delta(z,z')\,,\\\left\{A^a(u,z),\bar{F}^b(u',z')\right\}&=\kappa^2\,\delta(u-u')\delta(z,z')\delta^{ab}\,,\end{aligned} \tag{E.2.3}$$

or, more compactly, using the spacetime-spin-j notation developed in Section 5.6,

$$\left\{C^{(j)}(u,z),\bar{N}^{(j')}(u',z')\right\}=\kappa^2\,\delta(u-u')\delta(z,z')\delta^{j,j'}\,\mathbf{1}_{\mathrm{c}}^{-1}\,, \tag{E.2.4}$$

where $\mathbf{1}_{\mathrm{c}}^{-1}$ refers to the color space inverse metric structure as dictated by the color of the objects involved in the bracket. The above brackets also come with their complex conjugate pair for the opposite helicity quantities,

$$\left\{\bar{C}^{(j)}(u,z),N^{(j')}(u',z')\right\}=\kappa^2\,\delta(u-u')\delta(z,z')\delta^{j,j'}\,\mathbf{1}_{\mathrm{c}}^{-1}\,. \tag{E.2.5}$$

It will also be useful to have at hand the Poisson bracket among $C^{(j)}$'s. Integrating the above, we get

$$\left\{C^{(j)}(u,z),\bar{C}^{(j')}(u',z')\right\}=-\frac{\kappa^2}{2}\,\Theta(u-u')\delta(z,z')\delta^{j,j'}\,\mathbf{1}_{\mathrm{c}}^{-1}\,, \tag{E.2.6}$$

where $\Theta\left(t\right)=\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{d\omega}{\omega}e^{i\omega t}$, with $\Theta\left(t\neq 0\right)=\operatorname{sign}\left\{t\right\}$[1]. More generally,

$$\begin{aligned}\left\{C^{(j)}(u,z),\partial_{u'}^{-n}\bar{N}^{(j')}(u',z')\right\}&=\frac{\kappa^2}{2}\left(\partial_{u'}^{-n}+(-1)^n\,\partial_u^{-n}\right)\delta(u-u')\delta(z,z')\delta^{j,j'}\,\mathbf{1}_{\mathrm{c}}^{-1}\\&=-\frac{\kappa^2}{2}\frac{(u'-u)^{n-1}}{(n-1)!}\Theta(u-u')\delta(z,z')\delta^{j,j'}\,\mathbf{1}_{\mathrm{c}}^{-1}\,,\end{aligned} \tag{E.2.7}$$

where in the last line we have used (E.1.7c).

To derive this bracket, let us focus without loss of generality to the $j=j'=2$ case. Our first step is then to notice that, for generic $n\geq 0$,

$$\begin{aligned}\left\{C(u,z),\bar{N}(u',z')\right\}&=\left\{C(u,z),\partial_{u'}^{n}\partial_{u'}^{-n}\bar{N}(u',z')\right\}\\&=\partial_{u'}^{n}\left\{C(u,z),\partial_{u'}^{-n}\bar{N}(u',z')\right\}\,,\end{aligned} \tag{E.2.8}$$

[1] Notice it satisfies $\frac{d}{dt}\Theta\left(t\right)=2\delta\left(t\right)$.

which can be integrated to get

$$\begin{aligned}\left\{C(u,z),\partial_{u'}^{-n}\bar{N}(u',z')\right\} &= \partial_{u'}^{-n}\left\{C(u,z),\bar{N}(u',z')\right\}+\kappa^2 c_n\left(u-u'\right)\delta(z,z')\\ &= \kappa^2\delta(z,z')\left(\partial_{u'}^{-n}\delta\left(u-u'\right)+c_n\left(u-u'\right)\right)\\ &= -\kappa^2\delta(z,z')\left(\frac{(u'-u)^{n-1}}{(n-1)!}\theta\left(u-u'\right)-c_n\left(u-u'\right)\right),\end{aligned} \tag{E.2.9}$$

for some integration functions $c_n\left(u-u'\right)$ such that $\partial_{u'}^n c_n\left(u-u'\right)=0$. Acting with $\partial_{u'}$ on the above equation, one then sees that the integration functions satisfy the recursion relation

$$\partial_{u'}c_n\left(u-u'\right)=c_{n-1}\left(u-u'\right), \tag{E.2.10}$$

with $c_0\left(u-u'\right)=0$ set by the already known canonical bracket. Furthermore, we know from the antisymmetry of the $\left\{C(u,z),\bar{C}(u',z')\right\}$ bracket that

$$c_1\left(u-u'\right)=\frac{1}{2}=\frac{1}{2}\left(\theta\left(u-u'\right)+\theta\left(u'-u\right)\right). \tag{E.2.11}$$

The general solution to the above recursion relation is then

$$c_n\left(u-u'\right)=\frac{1}{2}\frac{(u'-u)^{n-1}}{(n-1)!}+\sum_{m=0}^{n-2}a_m\frac{(u'-u)^m}{m!}, \tag{E.2.12}$$

where a_m is the integration constant that enters when integrating the recursion relation for $n=m$. We choose to set all of these integration constants to be zero, e.g. by imposing the parity condition $c_n\left(u'-u\right)=(-1)^{n-1}c_n\left(u-u'\right)$,

$$c_n\left(u-u'\right)=\frac{1}{2}\frac{(u'-u)^{n-1}}{(n-1)!}=\frac{1}{2}\left(\partial_{u'}^{-n}-(-1)^n\partial_u^{-n}\right)\delta\left(u-u'\right), \tag{E.2.13}$$

which results into

$$\begin{aligned}\left\{C(u,z),\partial_{u'}^{-n}\bar{N}(u',z')\right\} &= \frac{\kappa^2}{2}\left(\partial_{u'}^{-n}+(-1)^n\partial_u^{-n}\right)\delta(u-u')\delta(z,z')\\ &= -\frac{\kappa^2}{2}\frac{(u'-u)^{n-1}}{(n-1)!}\Theta(u-u')\delta(z,z')\end{aligned} \tag{E.2.14}$$

as promised. Fixing these integration constants as above is consistent with the resulting action of the celestial charges on the negative helicity gravitational shear that were reported in [60,162].

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