Geometric Phase of a Transported Oscillator

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An oscillator constrained to a plane that is transported along some surface will rotate by an angle dependent only on the path and the surface, not on the speed at which it is transported. This is thus an example of a geometric phase. We analyze this phase using the methods of parallel transport. This concept plays a key role in General Relativity, but it can also be applied in classical mechanics. The Foucault pendulum can be seen as an application of this analysis, where the surface is a sphere and the curve is a line of constant latitude. In view of some considerable confusion and erroneous treatments in the recent literature, we here present a rather simple way for visualizing the motion of the Foucault pendulum using concepts that are based on Frenet's formulae and the methods of parallel displacement.

I. INTRODUCTION

Consider a surface such as a mountainous terrain on the earth or moon. Suppose we drive a truck along this terrain and on the back of that truck is some sort of oscillator constrained to move in a plane. Suppose the truck drives gently so that the oscillator is not jerked around too much. Will there be a net effect on the oscillator as the truck drives from point A to B? The answer is "yes": the oscillator will rotate by some angle dependent on the surface and the path taken.

It is this angle which we derive in this paper. The Foucault pendulum can be seen as an example of this phenomenon. The bob of the pendulum is attached to a base that is anchored to the earth. As the earth spins around, the oscillating pendulum is transported along a circle of constant latitude while the normal vector to the plane rotates. The equivalent effect would occur if the earth were perfectly spherical, not rotating, and we were to drive the pendulum around the earth staying at constant latitude.

By considering the classical Foucault pendulum from this perspective of an oscillator being transported along a surface, we geometrize it. That is, we separate the geometrical properties from the dynamical properties. This allows for visualization and makes the connection to the concept of "parallel transport". The confusion surrounding visualization of the pendulum's motion is well expressed in Ref [1]. This reference also contains a geometric model for the pendulum and an argument using

differential geometry that the pendulum undergoes parallel transport. We want to follow a similar line, however avoiding the use of the full technology of differential geometry with Christoffel symbols. Instead, we use the more elementary and easily visualizable concepts of curvature and torsion of a curve by using the Frenet Formulae. The advantage of our approach is that the rotation of the Foucault pendulum, and the more general geometric phase of the transported oscillator can be related to simple properties of the surface and curve. Our result is that the rate of change of the rotation angle along the curve $d\beta/ds$ is given by $\kappa \sin \alpha$ where κ is the curvature of the curve and α is the angle between the normal vector to the invariant plane and the vector normal to the curve lying in the local "osculating plane".

The argument given in this paper is motivated by articles in the American Journal of Physics [2] by M. Kugler, and in Physical Review [3] by M. Kugler and S. Shtrikman. The authors derive an expression for the rotation angle of the oscillator. They use equations which they refer to as the "Frenet Formulae" involving torsion and curvature. Allthough it may appear so, they do not actually solve the problem we have set forth in this introduction. Their result is that $d\beta/ds = \tau$. However, this is really a dynamical result, not a geometrical one. τ can be related to the angular velocity of the plane in which the oscillator lies (though this is not stated in their paper). That it is not the torsion of a curve simply related to the transport is easily seen. If either the curve of constant latitude on the earth, or the curve traced out by the normal vector to the plane are chosen, these curves have zero torsion which incorrectly would imply no geometric phase. In this paper, we correctly apply the Frenet Formulae to relate the geometric phase to the properties of the curve and the surface. From this analysis, we give a simple way of visualizing the rotation of the Foucault pendulum and

the more general problem of the transported oscillator. [6]

II. FRENET'S FORMULAE AND PARALLEL TRANSPORT

To be more specific, let us consider a point particle constrained to move in a two-dimensional plane. In this plane, the particle moves under the influence of a cylindrically symmetric potential $V(r)$. Finally, let the plane (plate) move on a different manifold, e.g., on a sphere (see Fig. 1). This latter motion may be caused by an external force.
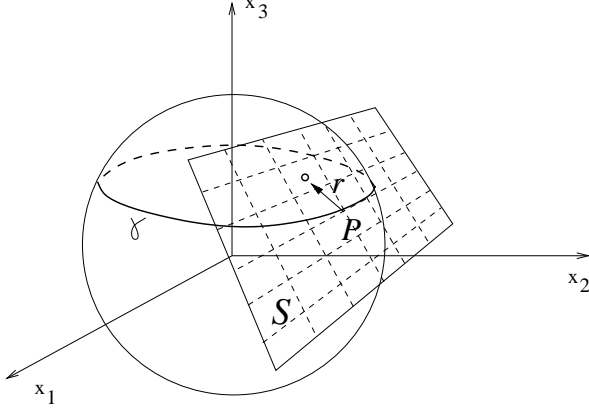

FIG. 1: Illustration of the plane S moving on a sphere along a curve γ touching at the point P. The dynamics of the Foucault pendulum can be thought of as being that of a particle constrained to this plane. r is the distance of the particle from P.

The plate S touches the manifold at a point P and traces out a curve γ. Associated with γ is an orthonormal frame of unit vectors $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$, where $\mathbf{t}$ is tangent to the curve, $\mathbf{n}$ is perpendicular to the curve, lying in the plane that locally contains the curve, and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. This frame satisfies the Frenet formulae of differential geometry [4]:

$$\begin{aligned}
\dot{\mathbf{t}} &= v\kappa\,\mathbf{n} \\
\dot{\mathbf{n}} &= -v\kappa\,\mathbf{t} + v\tau\,\mathbf{b}, \\
\dot{\mathbf{b}} &= -v\tau\,\mathbf{n}.
\end{aligned} \tag{1}$$

Here, $\kappa(t)$ corresponds to the curvature and $\tau(t)$ to the torsion of the curve γ. v is the speed of the parametrization of the curve γ. The equations for a particle moving on the plate, where the plate continuously changes its orientation, are generally very complicated. Therefore it will be our main goal, when dealing with the motion of a particle, to introduce a local inertial frame; this will greatly simplify our equations of motion. Incidentally, in differential geometry one usually uses the arc length s to parametrize the curve γ and writes $\frac{d}{ds}$ instead of $\frac{d}{dt}$. The relation between the two derivatives for our Frenet-vectors is, e.g., $\mathbf{t}' = \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dt}\frac{dt}{ds} = \dot{\mathbf{t}}\frac{1}{v}$.

If we want to describe motion in the plane S, the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is not particularly convenient, because no two unit vectors provide a basis for S. $\mathbf{t}$ lies in S, but $\mathbf{n}$ and $\mathbf{b}$ do not. We thus define $\mathbf{s}$ to be a unit vector perpendicular to S at P, and $\mathbf{m} = \mathbf{s} \times \mathbf{t}$. Then $\{\mathbf{t}, \mathbf{m}\}$ provides a basis for S. To better visualize our various unit vectors we refer to Figures 2 and 3.
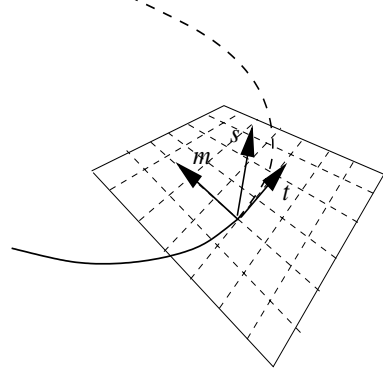

FIG. 2: Orthonormal frame $\{\mathbf{t}, \mathbf{m}, \mathbf{s}\}$, $\mathbf{t} = \mathbf{m} \times \mathbf{s}$.
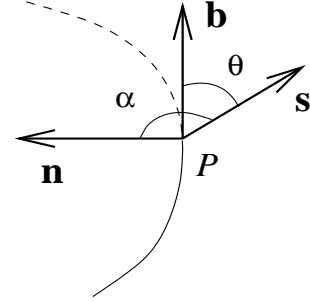

FIG. 3: Definition of angles α and $\theta = \alpha - \frac{\pi}{2}$.

A simple rotation about the $\mathbf{t}$ axis takes us from the $\{\mathbf{t}, \mathbf{m}, \mathbf{s}\}$ system to the $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ system; see Fig. 4.
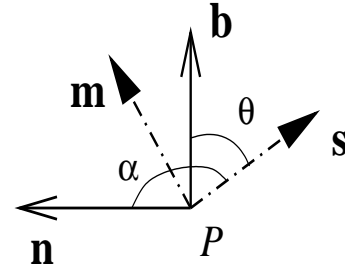

FIG. 4: Transformation of $\{\mathbf{m}, \mathbf{s}\}$ system into $\{\mathbf{n}, \mathbf{b}\}$ system.

The rotation of the basis vectors is simply given by

$$\begin{aligned}
\mathbf{s} &= \cos\alpha\,\mathbf{n} + \sin\alpha\,\mathbf{b} \\
\mathbf{m} &= \sin\alpha\,\mathbf{n} - \cos\alpha\,\mathbf{b}.
\end{aligned} \tag{2}$$

The inverse of this transformation reads

$$\mathbf{n} = \cos\alpha\ \mathbf{s} + \sin\alpha\ \mathbf{m} \qquad (3)$$
$$\mathbf{b} = \sin\alpha\ \mathbf{s} - \cos\alpha\ \mathbf{m}.$$

These equations, together with (1), enable us to write for the time derivatives:

$$\dot{\mathbf{t}} = (v\kappa)\sin\alpha\ \mathbf{m} + (v\kappa)\cos\alpha\ \mathbf{s}\ , \qquad (4)$$
$$\dot{\mathbf{m}} = -(v\kappa)\sin\alpha\ \mathbf{t} + (\dot\alpha + v\tau)\ \mathbf{s}\ ,$$
$$\dot{\mathbf{s}} = -(v\kappa)\cos\alpha\ \mathbf{t} - (\dot\alpha + v\tau)\ \mathbf{m}\ .$$

At this stage it is very convenient to introduce still another transformation, namely from the $\{\mathbf{t}, \mathbf{m}; \mathbf{s}\}$ system to a local inertial frame $\{\mathbf{u}_1, \mathbf{u}_2; \mathbf{s}\}$. Local inertial frames are defined as frames where the basis vectors undergo parallel transport. Parallel transport applied to an orthonormal set of vectors $\{\mathbf{u}_1, \mathbf{u}_2\}$ means that the change in the vectors has no components along the direction of the original vectors,

$$\mathbf{u}_i \cdot \frac{d\mathbf{u}_j}{dt} = 0\ ,\quad i,j = 1,2. \qquad (5)$$

It should be noted that the $\{\mathbf{t}, \mathbf{m}\}$ vectors do not undergo parallel transport. The basis vectors that do undergo parallel transport differ from the $\{\mathbf{t}, \mathbf{m}\}$ system by a rotation around the $\mathbf{s}$ axis; see Fig. 5.
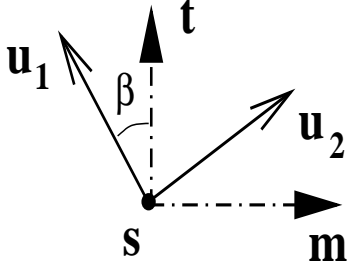

FIG. 5: Transformation of $\{\mathbf{t}, \mathbf{m}\}$ to local inertial frame $\{\mathbf{u}_1, \mathbf{u}_2\}$. Definition of angle β.

$$\mathbf{u}_1 = \cos\beta\ \mathbf{t} - \sin\beta\ \mathbf{m}\ , \qquad (6)$$
$$\mathbf{u}_2 = \sin\beta\ \mathbf{t} + \cos\beta\ \mathbf{m}\ .$$

Let us determine the angle β. For this we need $\dot{\mathbf{u}}_i$. These can be easily calculated and we obtain:

$$\dot{\mathbf{u}}_1 = (v\kappa\sin\alpha - \dot\beta)\mathbf{u}_2 + [v\kappa\cos\alpha\cos\beta - \sin\beta(\dot\alpha + v\tau)]\mathbf{s},$$
$$\dot{\mathbf{u}}_2 = (\dot\beta - v\kappa\sin\alpha)\mathbf{u}_1 + [v\kappa\cos\alpha\sin\beta + \cos\beta(\dot\alpha + v\tau)]\mathbf{s}.$$
$$(7)$$

For $\mathbf{u}_1\cdot\dot{\mathbf{u}}_1 = \mathbf{u}_1\cdot\dot{\mathbf{u}}_2 = \mathbf{u}_2\cdot\dot{\mathbf{u}}_1 = \mathbf{u}_2\cdot\dot{\mathbf{u}}_2 = 0$ to be satisfied, we have to choose

$$\dot\beta = v\kappa\ \sin\alpha. \qquad (8)$$

Under this condition, equations (7) reduce to

$$\dot{\mathbf{u}}_1 = [v\kappa\cos\alpha\cos\beta - \sin\beta(\dot\alpha + v\tau)]\mathbf{s} \equiv a\mathbf{s}\ ,$$
$$\dot{\mathbf{u}}_2 = [v\kappa\cos\alpha\sin\beta + \cos\beta(\dot\alpha + v\tau)]\mathbf{s} \equiv b\mathbf{s}.$$
$$(9)$$

Now, we calculate β for the Foucault pendulum. Since γ is obviously a circle, the curvature κ at each point is simply the inverse of the radius of the circle. Using R as the Earth radius and θ as the angle between $\mathbf{s}$ and the polar axis x_3, we have

$$r = R\sin\theta\ ,\ \kappa = \frac{1}{r}\ =\ \frac{1}{R\sin\theta}\ ,\ \alpha = \frac{\pi}{2} + \theta\ ,$$
$$\dot\beta = v\kappa\sin(\frac{\pi}{2} + \theta)\ =\ v\kappa\cos\theta\ . \qquad (10)$$

Since $\dot\beta$ gets transformed into β' when switching from t to s, we find

$$\beta'(s) = \kappa\cos\theta = \frac{1}{R\sin\theta}\cos\theta = \frac{1}{R}\cot\theta\ . \qquad (11)$$

Letting $\beta(0) = 0$, we obtain for the value of β at the end of one revolution of the Earth:

$$\beta_{\text{final}} = \int_0^{2\pi R\sin\theta}\beta'(s)ds = 2\pi R\sin\theta\frac{1}{R}\cot\theta$$

or

$$\beta_{\text{final}} = 2\pi\cos\theta, \qquad (12)$$

which is the expected formula for the Foucault pendulum.

We thus see that the general result (8) gives the correct answer for the special case of the Foucault pendulum. By changing from time to arclength for the independent variable, we see that the general expression for β is given by

$$\beta(s) = \int_0^s \kappa(s')\sin\alpha(s')\ ds', \qquad (13)$$

from which it is clear that the result is independent of the speed of parametrization. Thus, β is a true example of a geometric phase or Hannay angle, the classical analogue of the Berry phase [5].

III.   EQUATIONS OF MOTION IN THE PLANE

In this section we want to find the equations of motion in the plane S. These can be expressed either in terms of the $\{\mathbf{t}, \mathbf{m}\}$ basis vectors or, much more conveniently, in the $\{\mathbf{u}_1, \mathbf{u}_2\}$ frame. If the particle is constrained to the $\{\mathbf{t}, \mathbf{m}\}$ plane, we have that $r_3 = \dot{r}_3 = \ddot{r}_3 = 0$, $\vec{r}\cdot\mathbf{s} = 0$, with $\vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$ and $\vec{r} = r_1\mathbf{t} + r_2\mathbf{m}$. To identify the equations of motion $m\ddot{r}_i = \frac{-\partial V}{\partial r_i}$ with $|\vec{r}| = \sqrt{r_1^2 + r_2^2}$, we then would have to compute $\ddot{\mathbf{t}}$ and $\ddot{\mathbf{m}}$. We have done the calculation and found

$$m\ddot{r}_1 = -\frac{\partial V}{\partial r_1} + m\{\ \},$$
$$m\ddot{r}_2 = -\frac{\partial V}{\partial r_2} + m\{\ \},$$

where the contents of the curly brackets are quite complicated and besides teach us little.

The situation changes substantially when we now formulate the equation of motion relative to the $\{\mathbf{u}_1, \mathbf{u}_2\}$ system. In the two-dimensional S plane we have

$$\vec{r} = \rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_2 \ , \quad |\vec{r}| = \sqrt{\rho_1^2 + \rho_2^2} .$$

A first time derivative yields

$$\dot{\vec{r}} = \dot{\rho}_1 \mathbf{u}_1 + \rho_1 \dot{\mathbf{u}}_1 + \dot{\rho}_2 \mathbf{u}_2 + \rho_2 \dot{\mathbf{u}}_2 .$$

Here we make use of our result as stated in (9):

$$\begin{aligned} \dot{\mathbf{u}}_1 &= a\mathbf{s} \ , \\ \dot{\mathbf{u}}_2 &= b\mathbf{s} \ . \end{aligned}$$

The second time derivative is then given by

$$\ddot{\vec{r}} = (\ddot{\rho}_1 \mathbf{u}_1 + 2\dot{\rho}_1 \dot{\mathbf{u}}_1 + \rho_1 \ddot{\mathbf{u}}_1) + (\ddot{\rho}_2 \mathbf{u}_2 + 2\dot{\rho}_2 \dot{\mathbf{u}}_2 + \rho_2 \ddot{\mathbf{u}}_2)$$

with

$$\begin{aligned} \ddot{\mathbf{u}}_1 &= \dot{a}\mathbf{s} + a\dot{\mathbf{s}} \ , \\ \ddot{\mathbf{u}}_2 &= \dot{b}\mathbf{s} + b\dot{\mathbf{s}} \ , \end{aligned}$$

so that $\ddot{\vec{r}}$ is given by

$$\begin{aligned} \ddot{\vec{r}} = {}& \ddot{\rho}_1 \mathbf{u}_1 + 2\dot{\rho}_1 a\mathbf{s} + \rho_1 (\dot{a}\mathbf{s} + a\dot{\mathbf{s}}) \\ &+ \ddot{\rho}_2 \mathbf{u}_2 + 2\dot{\rho}_2 b\mathbf{s} + \rho_2 (\dot{b}\mathbf{s} + b\dot{\mathbf{s}}) . \end{aligned}$$

If we here drop the $\mathbf{s}$ dependence we are left with

$$\ddot{\vec{r}} \rightarrow \ddot{\rho}_1 \ \mathbf{u}_1 + \rho_1 a \ \dot{\mathbf{s}} + \ddot{\rho}_2 \ \mathbf{u}_2 + \rho_2 b \ \dot{\mathbf{s}} \ .$$

The time derivative $\dot{\mathbf{s}}$ can be easily calculated with the aid of (4) and the inverse of (6),

$$\begin{aligned} \mathbf{t} &= \cos\beta \mathbf{u}_1 + \sin\beta \mathbf{u}_2 \ , \\ \mathbf{m} &= -\sin\beta \mathbf{u}_1 + \cos\beta \mathbf{u}_2 . \end{aligned}$$

The desired expression for $\dot{\mathbf{s}}$ is rather simple:

$$\dot{\mathbf{s}} = -a\mathbf{u}_1 - b\mathbf{u}_2 \ .$$

From here we obtain in the two-dimensional $\{\mathbf{u}_1, \mathbf{u}_2\}$ plane:

$$\ddot{\vec{r}} = (\ddot{\rho}_1 - a^2 \rho_1 - ab\rho_2)\mathbf{u}_1 + (\ddot{\rho}_2 - b^2 \rho_2 - ab\rho_1)\mathbf{u}_2$$

Finally, we find in our local inertial frame the equations of motion:

$$\begin{aligned} m\ddot{\rho}_1 &= -\frac{\partial V}{\partial \rho_1} + ma^2 \rho_1 + mab\rho_2 \ , \qquad (14) \\ m\ddot{\rho}_2 &= -\frac{\partial V}{\partial \rho_2} + mb^2 \rho_2 + mab\rho_1 \ . \end{aligned}$$

Note that velocity-dependent forces (Coriolis forces) no longer appear relative to the inertial $\{\mathbf{u}_1, \mathbf{u}_2\}$ frame. But this was the goal of our exercise in going from the $\{\mathbf{t}, \mathbf{m}\}$

system to the $\{\mathbf{u}_1, \mathbf{u}_2\}$ system. One also can say that the Coriolis forces are automatically included in the rotation of the inertial frame vector $\mathbf{u}_1, \mathbf{u}_2$.

Finally we want to study the equations for the special case $\alpha = \text{const}$, i.e., $\dot{\alpha} = 0$ and $\dot{\tau} = 0$, which is realized for our simple Foucault pendulum. In this case we find

$$\begin{aligned} a^2 &= (v\kappa)^2 \cos^2 \alpha \cos^2 \beta \ , \\ b^2 &= (v\kappa)^2 \cos^2 \alpha \sin^2 \beta \ , \\ ab &= (v\kappa)^2 \cos^2 \alpha \sin\beta \cos\beta \ , \end{aligned}$$

and the equations of motion (14) turn into

$$\begin{aligned} m\ddot{\rho}_1 &= -\frac{\partial V}{\partial \rho_1} + m(v\kappa)^2 \cos^2 \alpha (\cos^2 \beta \ \rho_1 + \sin\beta\cos\beta \rho_2), \\ m\ddot{\rho}_2 &= -\frac{\partial V}{\partial \rho_2} + m(v\kappa)^2 \cos^2 \alpha (\sin^2 \beta \ \rho_2 + \sin\beta\cos\beta \rho_1). \end{aligned}$$

$$(15)$$

These equations can be simplified further by introducing a modified potential $U = V + W$, where W is given by

$$\begin{aligned} W &= -\frac{m}{2}(v\kappa)^2 \cos^2 \alpha (\rho_1 \cos\beta + \rho_2 \sin\beta) \ , v\kappa = \omega \\ &= -\frac{m}{2}\omega^2 r^2 \cos^2 \alpha \ . \end{aligned} \qquad (16)$$

In this way, equations (15) can be written in the simple form

$$m\ddot{\rho}_1 = -\frac{\partial U}{\partial \rho_1} \ , \qquad (17)$$

$$m\ddot{\rho}_2 = -\frac{\partial U}{\partial \rho_2} \ .$$

For the special case $\alpha = \frac{\pi}{2}$ (tip of the North-Pole), we obtain the relations

$$\begin{aligned} m\ddot{\rho}_1 &= -\frac{\partial V}{\partial \rho_1} \ , \\ m\ddot{\rho}_2 &= -\frac{\partial V}{\partial \rho_2} \ . \end{aligned}$$

Related to this special situation ($\tau = 0$, $\alpha = \frac{\pi}{2}$) is the final angle β ($\dot{\beta} = v\kappa = \omega$):

$$\beta_{\text{final}} = \int_0^T dt\dot{\beta} = \omega T = \frac{2\pi}{T}T = 2\pi \ , \qquad (18)$$

as is to be expected for the Foucault pendulum at the North Pole.

By the way, this obvious result can directly be obtained from equations (1) with $\alpha = \frac{\pi}{2}$ and $\tau = 0$. This simple exercise is left to the reader. He/She will find that the transformation of the $\{\mathbf{n}, \mathbf{t}\}$ system to the inertial frame $\{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = \cos\varphi \ \mathbf{t} - \sin\varphi \ \mathbf{n}$, $\mathbf{v}_2 = \sin\varphi \ \mathbf{t} + \cos\varphi \ \mathbf{n}$, still results in $\dot{\varphi} = v\kappa = \omega$, but now the change in the unit vector is $\dot{\mathbf{v}}_1 = \dot{\mathbf{v}}_2 = 0$. φ_{final} is of course given by $\varphi_{\text{final}} = 2\pi$.

IV.   CONCLUSION

In this article we derived an expression for the geometric phase of an oscillator constrained to a plane that is transported along some surface. As an application of this analysis, we considered the Foucault pendulum. We described in great detail the motion of a particle in a two-dimensional plane, which in turn is transported along some curve on a manifold. As such we have chosen a sphere for demonstrative purposes. If the path taken by the plate follows a certain latitude, then as the pendulum swings back and forth, its plane of oscillation rotates clockwise in the Northern hemisphere. After the Earth has completed one revolution, the plane of oscillation of the pendulum will not have made a rotation of 2π until the Earth has advanced by an additional angle of $2\pi(1 - \cos\theta)$. This is exactly the solid angle of the cap bounded by the curve,which the tip of $\mathbf{s}$ traces on the unit sphere. We obtained this well-known result by analyzing the particle's motion on the plate with respect to a local inertial frame. It was then fairly easy to identify the above-mentioned deficit angle. We also presented the rather simple equations of motion with respect to the basis that undergoes parallel displacement. In the special case of the plate located at the tip of the North Pole, we verified that the period of the plane of oscillation is identical to the period of the Earth's rotation, the deficit angle being zero in this case. In summary, using the language of differential geometry, we "geometrized" the Foucault pendulum and proved quite generally how an observable, well-known dynamical result can also be obtained from a purely geometrical point of view. Finally, it should be reemphasized that although we have focussed on the special case of the Foucault pendulum because of its general interest, our result for the geometric phase of a transported oscillator constrained to the plane ($d\beta/ds = \kappa\sin\alpha$) applies to an arbitrary surface and curve.

V.   ACKNOWEDGEMENTS

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[6] Some of these derivations can also be found in the Reed College undergraduate thesis (1998) by B. Nash, along with more discussion of issues surrounding transporting oscillators.