# Freudenthal duality and black holes: From groups of type $E_7$ to pre-homogeneous spaces

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Freudenthal duality can be defined as an anti-involutive, non-linear map acting on symplectic spaces. It was introduced in four-dimensional Maxwell-Einstein theories coupled to a non-linear sigma model of scalar fields.

In this short review, I will consider its relation to the $U$-duality Lie groups of type $E_7$ in extended supergravity theories, and comment on the relation between the Hessian of the black hole entropy and the pseudo-Euclidean, rigid special (pseudo)Kähler metric of the pre-homogeneous spaces associated to the $U$-orbits.

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## 1. Freudenthal Duality

We start and consider the following Lagrangian density in four dimensions (*cfr. e.g.*[1]):

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\varphi)\,\partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}(\varphi)\, F^{\Lambda}_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}(\varphi)\, \epsilon^{\mu\nu\rho\sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma}_{\rho\sigma}, \tag{1}$$

describing Einstein gravity coupled to Maxwell (Abelian) vector fields and to a non-linear sigma model of scalar fields (with no potential); note that $\mathcal{L}$ may -but does not necessarily need to - be conceived as the bosonic sector of $D = 4$ (*ungauged*) supergravity theory. Out of the Abelian two-form field strengths $F^{\Lambda}$'s, one can define their duals $G_{\Lambda}$, and construct a symplectic vector :

$$H := \left(F^{\Lambda}, G_{\Lambda}\right)^T, \quad {}^*G_{\Lambda|\mu\nu} := 2\frac{\delta \mathcal{L}}{\delta F^{\Lambda|\mu\nu}}. \tag{2}$$

We then consider the simplest solution of the equations of motion deriving from $\mathcal{L}$, namely a static, spherically symmetric, asymptotically flat, dyonic extremal black hole with metric[2]

$$ds^2 = -e^{2U(\tau)} dt^2 + e^{-2U(\tau)} \left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}\left(d\theta^2 + \sin\theta d\psi^2\right)\right], \tag{3}$$

where $\tau := -1/r$. Thus, the two-form field strengths and their duals can be fluxed on the two-sphere at infinity $S^2_\infty$ in such a background, respectively yielding the electric and magnetic charges of the black hole itself, which can be arranged in a symplectic vector $\mathcal{Q}$ :

$$p^{\Lambda} := \frac{1}{4\pi}\int_{S^2_\infty} F^{\Lambda}, \quad q_{\Lambda} := \frac{1}{4\pi}\int_{S^2_\infty} G_{\Lambda}, \tag{4}$$

$$\mathcal{Q} := \left(p^{\Lambda}, q_{\Lambda}\right)^T. \tag{5}$$

Then, by exploiting the symmetries of the background (3), the Lagrangian (1) can be dimensionally reduced from $D = 4$ to $D = 1$, obtaining a 1-dimensional effective Lagrangian ($' := d/d\tau$)[3]:

$$\mathcal{L}_{D=1} = (U')^2 + g_{ij}(\varphi)\varphi^{i\prime}\varphi^{j\prime} + e^{2U}V_{BH}(\varphi, \mathcal{Q}) \tag{6}$$

along with the Hamiltonian constraint[3]

$$(U')^2 + g_{ij}(\varphi)\varphi^{i\prime}\varphi^{j\prime} - e^{2U}V_{BH}(\varphi, \mathcal{Q}) = 0. \tag{7}$$

The so-called "effective black hole potential" $V_{BH}$ appearing in (6) and (7) is defined as[3]

$$V_{BH}(\varphi, \mathcal{Q}) := -\frac{1}{2}\mathcal{Q}^T\mathcal{M}(\varphi)\mathcal{Q}, \tag{8}$$

in terms of the symplectic and symmetric matrix[1]

$$\mathcal{M} := \begin{pmatrix} \mathbb{I} & -R \\ 0 & \mathbb{I} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & I^{-1} \end{pmatrix}\begin{pmatrix} \mathbb{I} & 0 \\ -R & \mathbb{I} \end{pmatrix} = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix}, \tag{9}$$

$$\mathcal{M}^T = \mathcal{M}; \quad \mathcal{M}\Omega\mathcal{M} = \Omega, \tag{10}$$

where $\mathbb{I}$ denotes the identity, and $R(\varphi)$ and $I(\varphi)$ are the scalar-dependent matrices occurring in (1); moreover, $\Omega$ stands for the symplectic metric ($\Omega^2 = -\mathbb{I}$). Note that, regardless of the invertibility of $R(\varphi)$ and as a consequence of the physical consistence of the kinetic vector matrix $I(\varphi)$, $\mathcal{M}$ is negative-definite; thus, the effective black hole potential (8) is positive-definite.

By virtue of the matrix $\mathcal{M}$, one can introduce a (scalar-dependent) *anti-involution* $\mathcal{S}$ in any Maxwell-Einstein-scalar theory described by (1) with a symplectic structure $\Omega$, as follows:

$$\mathcal{S}(\varphi) := \Omega\mathcal{M}(\varphi); \tag{11}$$

$$\mathcal{S}^2(\varphi) = \Omega\mathcal{M}(\varphi)\Omega\mathcal{M}(\varphi) = \Omega^2 = -\mathbb{I}; \tag{12}$$

in turn, this allows to define an anti-involution on the dyonic charge vector $\mathcal{Q}$, which has been called (scalar-dependent) *Freudenthal duality*[4–6]:

$$\mathfrak{F}(\mathcal{Q};\varphi) := -\mathcal{S}(\varphi)\mathcal{Q}; \tag{13}$$

$$\mathfrak{F}^2 = -\mathbb{I}, \quad (\forall\{\varphi\}). \tag{14}$$

By recalling (8) and (11), the action of $\mathfrak{F}$ on $\mathcal{Q}$, defining the so-called ($\varphi$-dependent) Freudenthal dual of $\mathcal{Q}$ itself, can be related to the symplectic gradient of the effective black hole potential $V_{BH}$:

$$\mathfrak{F}(\mathcal{Q};\varphi) = \Omega \frac{\partial V_{BH}(\varphi,\mathcal{Q})}{\partial \mathcal{Q}}. \tag{15}$$

Through the attractor mechanism[7], all this enjoys an interesting physical interpretation when evaluated at the (unique) event horizon of the extremal black hole (3) (denoted below by the subscript "$H$"); indeed

$$\partial_\varphi V_{BH} = 0 \Leftrightarrow \lim_{\tau\to-\infty} \varphi^i(\tau) = \varphi^i_H(\mathcal{Q}); \tag{16}$$

$$S_{BH}(\mathcal{Q}) = \frac{A_H}{4} = \pi \left. V_{BH}\right|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^T \mathcal{M}_H(\mathcal{Q})\mathcal{Q}, \tag{17}$$

where $S_{BH}$ and $A_H$ respectively denote the Bekenstein-Hawking entropy[8] and the area of the horizon of the extremal black hole, and the matrix horizon value $\mathcal{M}_H$ is defined as

$$\mathcal{M}_H(\mathcal{Q}) := \lim_{\tau\to-\infty} \mathcal{M}(\varphi(\tau)). \tag{18}$$

Correspondingly, one can define the (scalar-independent) horizon Freudenthal duality $\mathfrak{F}_H$ as the horizon limit of (13):

$$\widetilde{\mathcal{Q}} \equiv \mathfrak{F}_H(\mathcal{Q}) := \lim_{\tau\to-\infty} \mathfrak{F}(\mathcal{Q};\varphi(\tau)) = -\Omega\mathcal{M}_H(\mathcal{Q})\mathcal{Q} = \frac{1}{\pi}\Omega\frac{\partial S_{BH}(\mathcal{Q})}{\partial\mathcal{Q}}. \tag{19}$$

Remarkably, the (horizon) Freudenthal dual of $\mathcal{Q}$ is nothing but ($1/\pi$ times) the symplectic gradient of the Bekenstein-Hawking black hole entropy $S_{BH}$; this latter, from dimensional considerations, is only constrained to be an homogeneous function of degree two in $\mathcal{Q}$. As a result, $\widetilde{\mathcal{Q}} = \widetilde{\mathcal{Q}}(\mathcal{Q})$ is generally a complicated (non-linear) function, homogeneous of degree one in $\mathcal{Q}$.

It can be proved that the entropy $S_{BH}$ itself is invariant along the flow in the charge space $\mathcal{Q}$ defined by the symplectic gradient (or, equivalently, by the horizon Freudenthal dual) of $\mathcal{Q}$ itself :

$$S_{BH}(\mathcal{Q}) = S_{BH}(\mathfrak{F}_H(\mathcal{Q})) = S_{BH}\left(\frac{1}{\pi}\Omega\frac{\partial S_{BH}(\mathcal{Q})}{\partial\mathcal{Q}}\right) = S_{BH}\left(\widetilde{\mathcal{Q}}\right). \tag{20}$$

It is here worth pointing out that this invariance is pretty remarkable: the (semi-classical) Bekenstein-Hawking entropy of an extremal black hole turns out to be invariant under a generally non-linear map acting on the black hole charges themselves, and corresponding to a symplectic gradient flow in their corresponding vector space.

For other applications and instances of Freudenthal duality, see[9–11].

## 2. Groups of Type $E_7$

The concept of Lie groups *of type* $E_7$ as introduced in the 60s by Brown[12], and then later developed *e.g.* by[13–17].

Starting from a pair $(G, \mathbf{R})$ made of a Lie group $G$ and its faithful representation $\mathbf{R}$, the three axioms defining $(G, \mathbf{R})$ as a group of type $E_7$ read as follows:

(1) Existence of a (unique) symplectic invariant structure $\Omega$ in $\mathbf{R}$:

$$\exists!\Omega \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R}, \tag{21}$$

which then allows to define a symplectic product $\langle \cdot, \cdot \rangle$ among two vectors in the representation space $\mathbf{R}$ itself:

$$\langle Q_1, Q_2 \rangle := Q_1^M Q_2^N \Omega_{MN} = - \langle Q_2, Q_1 \rangle . \tag{22}$$

(2) Existence of (unique) rank-4 completely symmetric invariant tensor ($K$-tensor) in $\mathbf{R}$:

$$\exists! K \equiv \mathbf{1} \in (\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R})_s , \tag{23}$$

which then allows to define a degree-4 invariant polynomial $I_4$ in $\mathbf{R}$ itself:

$$I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q. \tag{24}$$

(3) Defining a triple map $T$ in $\mathbf{R}$ as

$$T : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}; \tag{25}$$

$$\langle T(Q_1, Q_2, Q_3), Q_4 \rangle : = K_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q, \tag{26}$$

it holds that

$$\langle T(Q_1, Q_1, Q_2), T(Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q. \tag{27}$$

This property makes a group of type $E_7$ amenable to a description as an automorphism group of a *Freudenthal triple system* (or, equivalently, as the conformal groups of the underlying Jordan triple system - whose a Jordan algebra is a particular case - ).

All electric-magnetic duality ($U$-duality[a]) groups of $\mathcal{N} \geqslant 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are of type $E_7$. Among these, degenerate groups of type $E_7$ are those in which the $K$-tensor is actually reducible, and thus $I_4$ is the square of a quadratic invariant polynomial $I_2$. In fact, in general, in theories with electric-magnetic duality groups of type $E_7$ holds that

$$S_{BH} = \pi \sqrt{|I_4(\mathcal{Q})|} = \pi \sqrt{|K_{MNPQ} \mathcal{Q}^M \mathcal{Q}^N \mathcal{Q}^P \mathcal{Q}^Q|}, \tag{28}$$

whereas in the case of degenerate groups of type $E_7$ it holds that $I_4(\mathcal{Q}) = (I_2(\mathcal{Q}))^2$, and therefore the latter formula simplifies to

$$S_{BH} = \pi \sqrt{|I_4(\mathcal{Q})|} = \pi |I_2(\mathcal{Q})| . \tag{29}$$

Simple, non-degenerate groups of type $E_7$ relevant to $\mathcal{N} \geqslant 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are reported *e.g.* in Table 1 of[20]. Semi-simple, non-degenerate groups of type $E_7$ of the same kind are given by

[a]Here $U$-duality is referred to as the "continuous" symmetries of[18]. Their discrete versions are the $U$-duality non-perturbative string theory symmetries introduced by Hull and Townsend[19].

$G = SL(2,\mathbb{R}) \times SO(2,n)$ and $G = SL(2,\mathbb{R}) \times SO(6,n)$, with $\mathbf{R} = (\mathbf{2}, \mathbf{2}+\mathbf{n})$ and $\mathbf{R} = (\mathbf{2}, \mathbf{6}+\mathbf{n})$, respectively relevant for $\mathcal{N}=2$ and $\mathcal{N}=4$ supergravity. Moreover, degenerate (simple) groups of type $E_7$ relevant to the same class of theories are $G = U(1,n)$ and $G = U(3,n)$, with complex fundamental representations $\mathbf{R} = \mathbf{n}+\mathbf{1}$ and $\mathbf{R} = \mathbf{3}+\mathbf{n}$, respectively relevant for $\mathcal{N}=2$ and $\mathcal{N}=3$ supergravity[16].

The classification of groups of type $E_7$ is still an open problem, even if some progress have been recently made *e.g.* in[21] (in particular, *cfr.* Table D therein).

In all the aforementioned cases, the scalar manifold is a *symmetric* cosets $\frac{G}{H}$, where $H$ is the maximal compact subgroup (with symmetric embedding) of $G$. Moreover, the $K$-tensor can generally be expressed as[17]

$$K_{MNPQ} = -\frac{n(2n+1)}{6d}\left[t^{\alpha}_{MN} t_{\alpha|PQ} - \frac{d}{n\,(2n+1)}\Omega_{M(P}\Omega_{Q)N}\right], \tag{30}$$

where $\dim \mathbf{R} = 2n$ and $\dim G = d$, and $t^{\alpha}_{MN}$ denotes the symplectic representation of the generators of $G$ itself. Thus, the horizon Freudenthal duality can be expressed in terms of the $K$-tensor as follows[4]:

$$\mathfrak{F}_H\left(\mathcal{Q}\right)_M \equiv \widetilde{\mathcal{Q}}_M = \frac{\partial\sqrt{\left|I_4\left(\mathcal{Q}\right)\right|}}{\partial \mathcal{Q}^M} = \epsilon\frac{2}{\sqrt{\left|I_4\left(\mathcal{Q}\right)\right|}} K_{MNPQ}\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q, \tag{31}$$

where $\epsilon := I_4/\left|I_4\right|$; note that the horizon Freudenthal dual of a given symplectic dyonic charge vector $\mathcal{Q}$ is well defined only when $\mathcal{Q}$ is such that $I_4\left(\mathcal{Q}\right) \neq 0$. Consequently, the invariance (20) of the black hole entropy under the the horizon Freudenthal duality can be recast as the invariance of $I_4$ itself:

$$I_4\left(\mathcal{Q}\right) = I_4\left(\widetilde{\mathcal{Q}}\right) = I_4\left(\Omega\frac{\partial\sqrt{\left|I_4\left(\mathcal{Q}\right)\right|}}{\partial \mathcal{Q}}\right). \tag{32}$$

In absence of "flat directions" at the attractor points (namely, of unstabilized scalar fields at the horizon of the black hole), and for $I_4 > 0$, the expression of the matrix $\mathcal{M}_H\left(\mathcal{Q}\right)$ at the horizon can be computed to read

$$\mathcal{M}_{H|MN}(\mathcal{Q}) = -\frac{1}{\sqrt{I_4}}\left(2\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - 6K_{MNPQ}\mathcal{Q}^P\mathcal{Q}^Q + \mathcal{Q}_M\mathcal{Q}_N\right), \tag{33}$$

and it is invariant under horizon Freudenthal duality:

$$\mathfrak{F}_H\left(\mathcal{M}_H\right)_{MN} := \mathcal{M}_{H|MN}(\widetilde{\mathcal{Q}}) = \mathcal{M}_{H|MN}(\mathcal{Q}). \tag{34}$$

## 3. Duality Orbits, Rigid Special Kähler Geometry and Pre-Homogeneous Vector Spaces

For $I_4 > 0$, $\mathcal{M}_H\left(\mathcal{Q}\right)$ given by (33) is one of the two possible solutions to the set of equations[22]

$$\begin{cases} M^T\left(\mathcal{Q}\right)\Omega M\left(\mathcal{Q}\right) = \epsilon\Omega; \\ M^T\left(\mathcal{Q}\right) = M\left(\mathcal{Q}\right); \\ \mathcal{Q}^T M\left(\mathcal{Q}\right)\mathcal{Q} = -2\sqrt{\left|I_4\left(\mathcal{Q}\right)\right|}, \end{cases} \tag{35}$$

which describes symmetric, purely $\mathcal{Q}$-dependent structures at the horizon; they are symplectic or anti-symplectic, depending on whether $I_4 > 0$ or $I_4 < 0$, respectively. Since in the class of (super)gravity $D = 4$ theories discussed the sign of $I_4$ actually determines a stratification of the representation space $\mathbf{R}$ of charges into distinct orbits of the action of $G$ into $\mathbf{R}$ itself (usually named duality orbits), the symplectic or anti-symplectic nature of the solutions to the system (35) is $G$-invariant, and supported by the various duality orbits of $G$ (in particular, by the so-called "large" orbits, for which $I_4$ is non-vanishing).

One of the two possible solutions to the system (35) reads[22]

$$M_+(\mathcal{Q}) = -\frac{1}{\sqrt{|I_4|}}\left(2\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - 6\epsilon K_{MNPQ}\mathcal{Q}^P\mathcal{Q}^Q + \epsilon\mathcal{Q}_M\mathcal{Q}_N\right);$$

$$\mathfrak{F}_H\left(M_+\right)_{MN} \ : \ = M_{+|MN}(\widetilde{\mathcal{Q}}) = \epsilon M_{+|MN}(\mathcal{Q}).$$

For $\epsilon = +1 \Leftrightarrow I_4 > 0$, it thus follows that

$$M_+(\mathcal{Q}) = \mathcal{M}_H\left(\mathcal{Q}\right), \tag{36}$$

as anticipated.

On the other hand, the other solution to system (35) reads[22]

$$M_-\left(\mathcal{Q}\right) = \frac{1}{\sqrt{|I_4|}}\left(\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - 6\epsilon K_{MNPQ}\mathcal{Q}^P\mathcal{Q}^Q\right); \tag{37}$$

$$\mathfrak{F}_H\left(M_-\right)_{MN} \ : \ = M_{-|MN}(\widetilde{\mathcal{Q}}) = \epsilon M_{-|MN}(\mathcal{Q}). \tag{38}$$

By recalling the definition of $I_4$ (24), it is then immediate to realize that $M_-\left(\mathcal{Q}\right)$ is the (opposite of the) Hessian matrix of ($1/\pi$ times) the black hole entropy $S_{BH}$:

$$M_{-|MN}\left(\mathcal{Q}\right) = -\partial_M\partial_N\sqrt{|I_4|} = -\frac{1}{\pi}\partial_M\partial_N S_{BH}. \tag{39}$$

The matrix $M_-\left(\mathcal{Q}\right)$ is the (opposite of the) pseudo-Euclidean metric of a non-compact, non-Riemannian rigid special Kähler manifold related to the duality orbit of the black hole electromagnetic charges (to which $\mathcal{Q}$ belongs), which is an example of pre-homogeneous vector space (PVS)[23]. In turn, the nature of the rigid special manifold may be Kähler or pseudo-Kähler, depending on the existence of a $U(1)$ or $SO(1,1)$ connection[b].

From its definition, a PVS is a finite-dimensional vector space V together with a subgroup $G$ of $GL(V)$, such that $G$ has an open dense orbit in $V$. PVS are subdivided into two types (type 1 and type 2), according to whether there exists an homogeneous polynomial on $V$ which is invariant under the semi-simple (reductive) part of $G$ itself. For more details, see *e.g.*[23,25–27].

Amazingly, simple, non-degenerate groups of type $E_7$ (relevant to $D = 4$ Einstein (super)gravities with symmetric scalar manifolds) *almost* saturate the list of irreducible PVS with unique $G$-invariant polynomial of degree 4 ([25]; also *cfr.* Table

[b]For a thorough introduction to special Kähler geometry, see *e.g.*[24].

2 of Ref. 20); in particular, the parameter $n$ characterizing each PVS can be interpreted as the number of centers of the regular solution in the (super)gravity theory with electric-magnetic duality ($U$-duality) group given by $G$.

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