

Moduli Spaces of Hyperkähler Manifolds and Mirror Symmetry

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Abstract

These lectures treat some of the basic features of moduli spaces of hyperkähler manifolds and in particular of K3 surfaces. The relation between the classical moduli spaces and the moduli spaces of conformal field theories is explained from a purely mathematical point of view. Recent results on hyperkähler manifolds are interpreted in this context. The second goal is to give a detailed account of mirror symmetry of K3 surfaces. The general principle, due to Aspinwall and Morrison, and various special cases (e.g. mirror symmetry for lattice polarized or elliptic K3 surfaces) are discussed.

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1 Introduction

Let (M, g) be a compact Riemannian manifold of dimension $2N$ with holonomy group $\mathrm{SU}(N)$. If $N > 2$ then there is a unique complex structure I on M such that g is a Kähler metric with respect to I . For given g and I a symplectic structure on M is naturally defined by the associated Kähler form $\omega = g(I(\), \cdot)$. This construction locally around g yields a decomposition of the moduli space of all Calabi-Yau metrics on M as $\mathcal{M}^{\text{met}}(M) \cong \mathcal{M}^{\text{cpl}}(M) \times \mathcal{M}^{\text{khl}}(M)$, where $\mathcal{M}^{\text{cpl}}(M)$ is the moduli space of complex structures on M and $\mathcal{M}^{\text{khl}}(M)$ is the moduli space of symplectic structures. Mirror symmetry in a first approximation predicts for any Calabi-Yau manifold (M, g) the existence of another Calabi-Yau manifold (M^v, g^v) together with an isomorphism $\mathcal{M}^{\text{met}}(M) \cong \mathcal{M}^{\text{met}}(M^v)$ which interchanges the two factors of the above decomposition, e.g. $\mathcal{M}^{\text{cpl}}(M) \cong \mathcal{M}^{\text{khl}}(M^v)$.

The picture has to be modified when we consider the second type of irreducible Ricci-flat manifolds. If the holonomy group of a $4n$ -dimensional manifold (M, g) is $\mathrm{Sp}(n)$, i.e. (M, g) is a hyperkähler manifold, then the moduli space of metrics near g does not split into the product of complex and kähler moduli as above, e.g. for a given hyperkähler metric there is a whole sphere S^2 of complex structures compatible with g . Hence mirror symmetry as formulated above for Calabi-Yau manifolds needs to be reformulated for hyperkähler manifolds. It still defines an isomorphism between the metric moduli spaces, but the relation between complex and symplectic structures are more subtle. Nevertheless, mirror symmetry is supposed to be much simpler for hyperkähler manifolds, as usually the mirror manifold M^v as a real manifold is M itself.

These notes intend to explain the analogue of the product decomposition of the moduli space of metrics on a Calabi-Yau manifold in the hyperkähler situation and to show how mirror symmetry for K3 surfaces, i.e. hyperkähler manifolds of dimension 4, is obtained by the action of a discrete group.

After recalling the main definitions and facts concerning the complex and metric structure of these manifolds in Section 2 we will soon turn to the global aspects of their moduli spaces. In Sections 3 and 4 we introduce these moduli spaces as well as the corresponding period domains. The geometric moduli spaces are studied via maps into the period domains. This will be explained in Section 5. Some of the main results about compact hyperkähler manifolds can be translated into global aspects of these maps.

Compared to other texts (e.g. [1]) on moduli spaces of K3 surfaces we will try to develop the theory as far as possible for compact hyperkähler manifolds of arbitrary dimension. The second main difference is that we also treat the less classical moduli spaces of certain CFTs. This will be done from a purely mathematical point of view by considering hyperkähler manifolds endowed with an additional B-field, i.e. a real cohomology class of degree two. This will lead to new features starting in Section 6, where we let act a certain discrete group on the various moduli spaces. This section follows papers by Aspinwall, Morrison, and others. Using this action mirror symmetry of K3 surfaces will be explained in Section 7. The advantage of this slightly technical approach is that various versions of mirror symmetry for (e.g. lattice polarized or elliptic) K3 surfaces can be explained by the same group action. Of course, explaining mirror symmetry in these terms is only possible for K3 surfaces or hyperkähler manifolds. Mirror symmetry for general Calabi–Yau manifolds will usually change the topology.

The text contains little or no original material. The main goal was to explain global phenomena of moduli spaces of K3 surfaces, or more generally of compact hyperkähler manifolds, and to give a concise introduction into the main constructions used in establishing mirror symmetry for K3 surfaces.

We encourage the reader to consult the survey [1] and the original articles [3, 4].

2 Basics

In this section we collect the basic definitions and facts concerning irreducible holomorphic symplectic manifolds and compact hyperkähler manifolds. Most of the material will be presented without proofs and we shall refer to other sources for more details (e.g. [6, 20]).

Definition 2.1 *An irreducible holomorphic symplectic manifold (IHS, for short) is a simply connected compact Kähler manifold X , such that $H^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic two-form σ .*

Since an IHS is in particular a compact Kähler manifold, Hodge decomposition holds. In degree two it yields

$$\begin{aligned} H^2(X, \mathbb{C}) &= H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X) \\ &= \mathbb{C}\sigma \oplus H^{1,1}(X) \oplus \mathbb{C}\bar{\sigma}. \end{aligned}$$

The existence of an everywhere non-degenerate two-form $\sigma \in H^0(X, \Omega_X^2)$ implies that the manifold has even complex dimension $\dim_{\mathbb{C}}(X) = 2n$. Moreover, σ induces an alternating homomorphism $\sigma : \mathcal{T}_X \rightarrow \Omega_X$. Since the two-form is everywhere non-degenerate, this homomorphism is bijective. Thus, the tangent bundle and the cotangent bundle of an IHS are isomorphic. Moreover, the canonical bundle $K_X = \Omega_X^{2n}$ is trivialized by the $(2n, 0)$ -form σ^n . Thus, an IHS has trivial canonical bundle and, therefore, vanishing first Chern class $c_1(X)$.

In dimension two IHS are also called K3 surfaces (K3=Kähler, Kodaira, Kummer). More precisely, by definition a K3 surface is a compact complex surface with trivial canonical bundle K_X and such that $H^1(X, \mathcal{O}_X) = 0$. It is a deep fact that any such surface is also Kähler [40]. Moreover, $H^1(X, \mathcal{O}_X) = 0$ does indeed imply that such a surface is simply-connected.

Here are the basic examples.

Examples 2.2 i) Any smooth quartic hypersurface $X \subset \mathbb{P}^3$ is a K3 surface, e.g. the Fermat quartic defined by $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$.

ii) Let $T = \mathbb{C}^2/\Gamma$ be a compact two-dimensional complex torus. The involution $x \mapsto -x$ has 16 fixed points and, thus, the quotient T/\pm is singular in precisely 16 points. Blowing-up those yields a *Kummer* surface $X \rightarrow T/\pm$, which is a K3 surface containing 16 smooth irreducible rational curves.

iii) An elliptic K3 surface is a K3 surface X together with a surjective morphism $\pi : X \rightarrow \mathbb{P}^1$. The general fibre of π is a smooth elliptic curve.

It is much harder to construct higher dimensional examples of IHS and all known examples are constructed by means of K3 surfaces or two-dimensional complex tori. The list of known examples has been discussed in length in the lectures of Lehn (see also [20]).

So far we have discussed IHS purely from the complex geometric point of view. However, the most important feature of this type of manifolds is the existence of a very special metric.

Definition 2.3 A compact oriented Riemannian manifold (M, g) of dimension $4n$ is called *hyperkähler* (HK, for short) if the holonomy group of g equals $\mathrm{Sp}(n)$. In this case g is called a *hyperkähler metric*.

Remark 2.4 If g is a hyperkähler metric, then there exist three complex structures I , J , and K on M , such that g is Kähler with respect to all three

of them and such that $K = I \circ J = -J \circ I$. Thus, I is orthogonal with respect to g and the Kähler form $\omega_I := g(I(\), \)$ is closed (similarly for J and K). Often, this is taken as a definition of a hyperkähler metric. Note that our condition is stronger, as we not only want the holonomy be contained in $\mathrm{Sp}(n)$, but be equal to it.

Proposition 2.5 *Let (M, g) be a HK. Then for any $(a, b, c) \in \mathbb{R}^3$ with $a^2 + b^2 + c^2 = 1$ the complex manifold $(M, aI + bJ + cK)$ is an IHS.*

Thus, for any HK (M, g) there exists a two-sphere $S^2 \subset \mathbb{R}^3$ of complex structures compatible with the Riemannian metric g .

Remark 2.6 Let (M, g) be a HK. The associated Kähler forms $\omega_I, \omega_J, \omega_K$ span a three-dimensional subspace $H_+^2(M, g) \subset H^2(M, \mathbb{R})$. In fact, this space will always be considered as a three-dimensional space endowed with the natural orientation. If $X = (M, I)$, then $H_+^2(M, g) = (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}} \oplus \mathbb{R}\omega_I$, where the orientation is given by the base $(\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), \omega_I)$. In order to see this, one verifies that the holomorphic two-form σ on $X = (M, I)$ can be given as $\sigma = \omega_J + i\omega_K$ (cf. [20]).

Definition 2.7 *Let X be an IHS. The Kähler cone $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ is the open convex cone of all Kähler classes on X , i.e. classes that can be represented by some Kähler form.*

The most important single result on IHS is the following consequence of the celebrated theorem of Calabi–Yau:

Theorem 2.8 *Let X be an IHS. Then for any $\alpha \in \mathcal{K}_X$ there exists a unique hyperkähler metric g on M , such that $\alpha = [\omega_I]$ for $\omega_I = g(I(\), \)$.*

Thus, on any IHS X the Kähler cone \mathcal{K}_X parametrizes all possible hyperkähler metrics g compatible with the given complex structure. Below we will explain how the Kähler cone \mathcal{K}_X can be described as a subset of $H^{1,1}(X)$.

Remark 2.9 Thus, an IHS X together with a Kähler class $\alpha \in \mathcal{K}_X$ is the same thing as a HK (M, g) together with a compatible complex structure I . As a short hand, we write $(X, \alpha) = (M, g, I)$ in this case.

Definition 2.10 *The BB(Beauville–Bogomolov)-form of an IHS X is the quadratic form on $H^2(X, \mathbb{R})$ given by*

$$q_X(\alpha) = \frac{n}{2} \int_X \alpha^2 (\sigma \bar{\sigma})^{n-1} + (1-n) \left(\int_X \alpha \sigma^{n-1} \bar{\sigma}^n \right) \left(\int_X \alpha \sigma^n \bar{\sigma}^{n-1} \right),$$

where $\sigma \in H^{2,0}(X)$ is chosen such that $\int_X (\sigma \bar{\sigma})^n = 1$

For any Kähler class $[\omega]$ we obtain a q_X -orthogonal decomposition $H^2(X, \mathbb{R}) = (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}} \oplus \mathbb{R}\omega \oplus H^{1,1}(X)_{\omega}$. Here, $H^{1,1}(X)_{\omega}$ is the space of ω -primitive real $(1,1)$ -classes. Note that we get a different decomposition for every Kähler class $[\omega] \in \mathcal{K}_X$, but that the quadratic form q_X does not depend on the chosen Kähler class.

The following proposition collects the main facts about the BB-form q_X .

Proposition 2.11 i) *For any Kähler class $[\omega] \in \mathcal{K}_X$ on an IHS X the BB-form q_X is positive definite on $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}} \oplus \mathbb{R}\omega$ and negative definite on $H^{1,1}(X)_{\omega}$.*

- ii) *There exists a positive real scalar λ_1 such that $q_X(\alpha)^n = \lambda_1 \cdot \int_X \alpha^{2n}$ for all $\alpha \in H^2(X)$.*
- iii) *There exists a positive real scalar λ_2 such that $\lambda_2 \cdot q_X$ is a primitive integral form on $H^2(X, \mathbb{Z})$.*
- iv) *There exists a positive real scalar λ_3 such that $q_X(\alpha) = \lambda_3 \cdot \int_X \alpha^2 \sqrt{\text{td}(X)}$ for all $\alpha \in H^2(X)$.*

After eliminating the denominator of $\sqrt{\text{td}(X)}$ by multiplying with a universal coefficient c_n that only depends on n we obtain an integral quadratic form $c_n \cdot \int_X \alpha^2 \sqrt{\text{td}(X)}$. In general this form need not be primitive, but this will be of no importance for us. Moreover, since any IHS has vanishing odd Chern classes, $\sqrt{\text{td}(X)} = \sqrt{\hat{A}(X)}$. (Everything that matters here is that $\sqrt{\text{td}(X)}$ is purely topological in this case.) Therefore, in these lectures we will use the following modified version of the BB-form.

Definition 2.12 *The BB-form q_X of an $2n$ -dimensional IHS X is given by*

$$q_X(\alpha) = c_n \cdot \int_X \alpha^2 \sqrt{\hat{A}(X)}.$$

With this definition we see that q_X only depends on the underlying manifold M , i.e. for two different hyperkähler metrics g and g' and two compatible complex structures I resp. I' the BB-forms with the above definition of $X = (M, I)$ and $X' = (M, I')$ coincide.

Note for $n = 1$ we have $c_1 = 1$ and thus q_X is nothing but the intersection pairing $\alpha \cup \alpha$ of the four-manifold underlying a K3 surface. The quadratic form in this case is even, unimodular and indefinite and can thus be explicitly determined:

Proposition 2.13 *The intersection form $(H^2(X, \mathbb{Z}), \cup)$ of a K3 surface X is isomorphic to the K3 lattice $2(-E_8) \oplus 3U$, where U is the standard hyperbolic plane $(\mathbb{Z}^2, (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))$.*

Definition 2.14 *The BB-volume of a HK (M, g) is*

$$q(M, g) := q_X([\omega_I]),$$

where $X = (M, I)$ is the IHS associated to one of the compatible complex structures I and ω_I is the induced Kähler form.

Note that the BB-volume does not depend on the chosen complex structure. Analogously one can define the volume of an IHS endowed with a Kähler class α as $q_X(\alpha)$. For a K3 surface one has $q(M, g) = \int \omega_I^2$, which is the usual volume up to the scalar factor $1/2$. In higher dimension the usual volume is of degree $2n$ and the BB-volume is quadratic. Of course, due to Proposition 2.11 one knows that up to a scalar factor $q(M, g)^n$ equals the standard volume, but this factor might a priori depend on the topology of M .

What makes the theory of K3 surfaces and higher-dimensional HK so pleasant is that they can be studied by means of their period.

Definition 2.15 *Let X be an IHS. The period of X is the lattice $(H^2(X, \mathbb{Z}), q_X)$ endowed with the weight-two Hodge structure $H^2(X, \mathbb{Z}) \otimes \mathbb{C} = H^2(X, \mathbb{C}) = \mathbb{C}\sigma \oplus H^{1,1}(X, \mathbb{C}) \oplus \mathbb{C}\bar{\sigma}$.*

Since $H^{1,1}(X, \mathbb{C})$ is orthogonal with respect to q_X and $\mathbb{C}\bar{\sigma}$ is the complex conjugate of $\mathbb{C}\sigma$, the period of the IHS X is in fact given by the lattice $(H^2(X, \mathbb{Z}), q_X)$ and the line $\mathbb{C}\sigma \subset H^2(X, \mathbb{C})$.

The theory of K3 surfaces is crowned by the so called Global Torelli Theorem (due to Pjateckii-Sapiro, Shafarevich, Burns, Rapoport, Looijenga, Peters, Friedman):

Theorem 2.16 *Let X and X' be two K3 surfaces and let $\varphi : H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z})$ be an isomorphism of their periods such that $\varphi(\mathcal{K}_X) \cap \mathcal{K}_{X'} \neq \emptyset$. Then there exists a unique isomorphism $f : X' \cong X$ such that $f^* = \varphi$.*

Moreover, an arbitrary isomorphism between the periods of two K3 surfaces is in general not induced by an isomorphism of the K3 surfaces, but the K3 surfaces are nevertheless isomorphic.

The uniqueness assertion in the Global Torelli Theorem is roughly proven as follows (cf. [29]): If f is a non-trivial automorphism of finite order with $f^* = \text{id}$ then the holomorphic two-form σ is invariant under f and the action at the fixed points is locally of the form $(u, v) \mapsto (\xi \cdot u, \xi^{-1} \cdot v)$. Using Lefschetz fixed point formula and again $f^* = \text{id}$ one finds that there are 24 fixed points. Thus, the minimal resolution \tilde{X} of the quotient $X/\langle f \rangle$ contains 24 pairwise disjoint curves. Moreover, one verifies that \tilde{X} is again a K3 surface. The last two statements together yield a contradiction.

The Global Torelli Theorem in the above version fails completely in higher dimensions. E.g. if $f : X \cong X$ is an automorphism of a K3 surface X such that $f^* = \text{id}$, then $f = \text{id}$. This does not hold in higher dimensions [7]. Even worse, due to a recent counterexample of Namikawa [36] one knows that higher dimensional IHS X and X' might have isomorphic periods without even being birational. A possible formulation of the Global Torelli Theorem in higher dimensions using derived categories was proposed in [36]. However, uniqueness is not expected. Compare the discussion in Section 5.4.

Often, a certain type of K3 surfaces is distinguished by the form of the period. We explain this in the three examples presented earlier. In fact, the proofs of these descriptions are all quite involved.

Example 2.17 i) Let X be a K3 surface such that $\text{Pic}(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ is generated by a class α with $\alpha^2 = 4$. Then X is isomorphic to a quartic hypersurface in \mathbb{P}^3 and α corresponds to $\mathcal{O}(1)$ (cf. [1, Exp. VI]).

ii) Let X be a K3 surface such that $\text{Pic}(X)$ contains 16 disjoint smooth irreducible rational curves $C_1, \dots, C_{16} \subset X$ such that $\sum [C_i] \in H^2(X, \mathbb{Z})$ is two-divisible. Then X is isomorphic to a Kummer surface.

This description of Kummer surfaces is not entirely in terms of the period. Later we will rather use the following description of an even more special type of K3 surfaces: Let X be a K3 surface such that the lattice $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{Z}}$ is of rank two and any vector x in this lattice satisfies $x^2 \equiv 0 \pmod{4}$. Then X is a Kummer surface. It turns out that K3 surfaces with this type of period are exactly the exceptional Kummer surfaces, i.e. Kummer surfaces with $\text{rk}(\text{Pic}(X)) = 20$ (cf. [1, Exp. VIII]).

iii) Let X be a K3 surface such that there exists a class $\alpha \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ with $\alpha^2 = 0$. Then X is an elliptic K3 surface. Clearly, if $X \rightarrow \mathbb{P}^1$ is an elliptic K3 surface then the class of the fibre defines such a class. But note that conversely not every class α with $\alpha^2 = 0$ is automatically a fibre class of some elliptic fibration, but by applying certain reflections it can be made into one (cf. [8]).

In order to get a better feeling for the set of all possible hyperkähler structures on an IHS X we shall discuss the Kähler cone in some more detail.

Definition 2.18 *The positive cone \mathcal{C}_X of an IHS X is the connected component of the open set $\{\alpha \mid q_X(\alpha) > 0\} \subset H^{1,1}(X, \mathbb{R})$ that contains the Kähler cone \mathcal{K}_X .*

(Here we use the fact that $q_X(\alpha) > 0$ for any Kähler class α .) Thus, $\mathcal{C}_X \cup (-\mathcal{C}_X)$ can be entirely read off the period of X . This is no longer possible for the Kähler cone, but one can at least try to find a minimal set of further geometric information that determines \mathcal{K}_X as an open subcone of \mathcal{C}_X .

Proposition 2.19 *The Kähler cone $\mathcal{K}_X \subset \mathcal{C}_X$ is the open subset of all $\alpha \in \mathcal{C}_X$ such that $\int_C \alpha > 0$ for all rational curves $C \subset X$. If X is a K3 surface it suffices to test smooth rational curves (cf. [1, 5, 20]).*

Since any smooth irreducible rational curve C in a K3 surface X defines a (-2) -class $[C] \in H^{1,1}(X, \mathbb{Z})$, one can use this result to show that for any class $\alpha \in \mathcal{C}_X$ there exists a finite number of smooth rational curves $C_1, \dots, C_k \subset X$ such that $s_{C_1} \dots s_{C_k}(\alpha) \in \overline{\mathcal{K}}_X$, where s_C is the reflection in the hyperplane $[C]^\perp$. Of course, these reflections s_C are contained in the discrete orthogonal group $O(\Gamma)$ of the lattice $\Gamma = (H^2(X, \mathbb{Z}), \cup)$.

3 Moduli spaces

Ultimately, we will be interested in moduli spaces of irreducible holomorphic symplectic manifolds (IHS), hyperkähler manifolds (HK), etc. In this section we will introduce moduli spaces of such manifolds endowed with an additional marking. A marking in general refers to an isomorphism of the second cohomology with a fixed lattice. The choice of such an isomorphism

gives rise to the action of a discrete group and the quotients by this group will eventually yield the true moduli spaces. For this section we fix a lattice Γ of signature $(3, b - 3)$ and an integer n .

3.1 Moduli spaces of marked IHS

Definition 3.1 *A marked IHS is a pair (X, φ) consisting of an IHS of complex dimension $2n$ and a lattice isomorphism $\varphi : (H^2(X, \mathbb{Z}), q_X) \cong \Gamma$. We say that two marked IHS (X, φ) and (X', φ') are equivalent, $(X, \varphi) \sim (X', \varphi')$, if there exists an isomorphism $f : X \cong X'$ of complex manifolds such that $\varphi' = \varphi \circ f^*$.*

Definition 3.2 *The moduli space of marked IHS is the space*

$$\mathcal{T}_\Gamma^{\text{cpl}} := \{(X, \varphi) = \text{marked IHS}\}/\sim .$$

A priori, $\mathcal{T}_\Gamma^{\text{cpl}}$ is just a set, but, as we will see later, it can be endowed with the structure of a topological space locally isomorphic to a complex manifold of dimension $b - 2$.

Let X be an IHS and φ a marking of X . If $\mathcal{X} \rightarrow \text{Def}(X)$ is the universal deformation of $X = \mathcal{X}_0$, then $\text{Def}(X)$ is a smooth germ of dimension $h^1(X, \mathcal{T}_X)$. We may represent $\text{Def}(X)$ by a small disc in $\mathbb{C}^{h^1(X, \mathcal{T}_X)}$. The marking φ induces in a canonical way a marking φ_t of the fibre \mathcal{X}_t for any $t \in \text{Def}(X)$. Using the Local Torelli Theorem (cf. Section 5) we see that the induced map $\text{Def}(X) \rightarrow \mathcal{T}_\Gamma^{\text{cpl}}$ is injective, i.e. any two fibres of the family $\mathcal{X} \rightarrow \text{Def}(X)$ define non-equivalent marked IHS. The various $\text{Def}(X) \subset \mathcal{T}_\Gamma^{\text{cpl}}$ for all possible choices of X and markings φ cover the moduli space $\mathcal{T}_\Gamma^{\text{cpl}}$. Since the universal deformation $\mathcal{X} \rightarrow \text{Def}(X)$ of $X = \mathcal{X}_0$ is, at the same time, also the universal deformation of all its fibres \mathcal{X}_t , one can define a natural topology on $\mathcal{T}_\Gamma^{\text{cpl}}$ by gluing the complex manifolds $\text{Def}(X)$. Thus, locally $\mathcal{T}_\Gamma^{\text{cpl}}$ is a smooth complex manifold of dimension $h^1(X, \mathcal{T}_X) = b - 2$. However, $\mathcal{T}_\Gamma^{\text{cpl}}$ is not a complex manifold, as it does not need to be Hausdorff. In fact, no example is known, where $\mathcal{T}_\Gamma^{\text{cpl}}$ would be Hausdorff and conjecturally this never happens.

A family $(\mathcal{X}, \varphi) \rightarrow S$ of marked IHS is a family $\mathcal{X} \rightarrow S$ of IHS of dimension $2n$ and a family of markings φ_t of the fibres \mathcal{X}_t locally constant with respect to t .

The universality of $\mathcal{X} \rightarrow \text{Def}(X)$ immediately implies the following

Lemma 3.3 *If $(\mathcal{X}, \varphi) \rightarrow S$ is a family of marked IHS, then there exists a canonical holomorphic map $\eta : S \rightarrow \mathcal{T}_\Gamma^{\text{cpl}}$, such that $\eta(t) = [(\mathcal{X}_t, \varphi_t)]$. \square*

Remark 3.4 In order to construct a universal family over $\mathcal{T}_\Gamma^{\text{cpl}}$ one would need to glue universal families $\mathcal{X} \rightarrow \text{Def}(X)$, $\mathcal{Y} \rightarrow \text{Def}(Y)$, where (X, φ) and (Y, ψ) are marked IHS, over the intersection $\text{Def}(X) \cap \text{Def}(Y) \subset \mathcal{T}_\Gamma^{\text{cpl}}$. This is only possible if for $t \in \text{Def}(X) \cap \text{Def}(Y)$ there exists a unique isomorphism $f : \mathcal{X}_t \cong \mathcal{Y}_t$ with $\psi_t = \varphi_t \circ f^*$. For K3 surfaces the uniqueness can be ensured due to the strong version of the Global Torelli Theorem (see Thm. 2.16), but in higher dimensions this fails. Thus, $\mathcal{T}_\Gamma^{\text{cpl}}$ is, in general, only a coarse moduli space.

3.2 Moduli spaces of marked HK

Definition 3.5 *A marked HK is a triple (M, g, φ) , where (M, g) is a compact HK of dimension $4n$ in the sense of Proposition 2.3 and φ is an isomorphism $(H^2(M, \mathbb{Z}), q) \cong \Gamma$. Two triples (M, g, φ) , (M', g', φ') are equivalent, $(M, g, \varphi) \sim (M', g', \varphi')$, if there exists an isometry $f : (M, g) \cong (M', g')$ with $\varphi' = \varphi \circ f^*$.*

Definition 3.6 *The moduli space of marked HK is the space*

$$\mathcal{T}_\Gamma^{\text{met}} := \{(M, g, \varphi) = \text{marked HK}\}/\sim.$$

A slightly different approach towards $\mathcal{T}_\Gamma^{\text{met}}$ will be explained in Section 3.5. There, the manifold M is fixed and only the metric g is allowed to vary.

3.3 Moduli spaces of marked complex HK or Kähler IHS

Recall (cf. Remark 2.9) that there is a bijection between HK with a compatible complex structure and IHS with a chosen Kähler class. Thus, the two moduli spaces are naturally equivalent.

Definition 3.7 *A marked complex HK is a tuple (M, g, I, φ) , where (M, g, φ) is a marked HK and I is a compatible complex structure on (M, g) . A marked Kähler IHS is a triple (X, α, φ) , where (X, φ) is a marked IHS and $\alpha \in \mathcal{K}_X$ is a Kähler class. Two marked complex HK (M, g, I, φ) , (M', g', I', φ') are equivalent if there exists an isometry $f : (M, g) \cong (M', g')$ with $I = f^* I'$ and $\varphi' = \varphi \circ f^*$. Analogously, one defines the equivalence of marked Kähler IHS.*

Note that the equivalence relation is compatible with the natural bijection $\{(M, g, I, \varphi)\} \leftrightarrow \{(X, \alpha, \varphi)\}$.

Definition 3.8 *The moduli space of complex HK or, equivalently, of Kähler IHS is the space*

$$\begin{aligned} \mathcal{T}_\Gamma &:= \{(M, g, I, \varphi) = \text{marked complex HK}\}/\sim \\ &= \{(X, \alpha, \varphi) = \text{marked Kähler IHS}\}/\sim. \end{aligned}$$

Obviously, there are two forgetful maps $m : (M, g, I, \varphi) \mapsto (M, g, \varphi)$ and $c : (X, \alpha, \varphi) \mapsto (X, \varphi)$. The following diagram is the hyperkähler version of the product decomposition of the metric moduli space for Calabi–Yau manifolds.

$$\begin{array}{ccc} \mathcal{T}_\Gamma & \xrightarrow{m} & \mathcal{T}_\Gamma^{\text{met}} \\ \downarrow c & & \\ \mathcal{T}_\Gamma^{\text{cpl}} & & \end{array}$$

Proposition 3.9 *The set \mathcal{T}_Γ has the structure of a real manifold of dimension $3(b-2)$. The fibre $c^{-1}(X, \varphi) = \mathcal{K}_X$ is a real manifold of dimension $b-2$. The fibre $m^{-1}(M, g, \varphi)$ is naturally isomorphic to the complex manifold \mathbb{P}^1 . The induced map $c : \mathbb{P}^1 = m^{-1}(M, g, \varphi) \rightarrow \mathcal{T}_\Gamma^{\text{cpl}}$ is a holomorphic embedding. The map $m : c^{-1}(X, \varphi) \rightarrow \mathcal{T}_\Gamma^{\text{met}}$ is a real embedding. \square*

The line $\mathbb{P}^1 \subset \mathcal{T}_\Gamma^{\text{cpl}}$ is also called ‘twistor line’. Disposing of a global deformation like this, is one of the key tools in studying moduli spaces of IHS.

3.4 CFT moduli spaces of HK

From a geometric point of view the following moduli space is an almost trivial extension of \mathcal{T}_Γ . However, it will become of central interest in later sections, when we will let act the full modular group on it. This group action will relate very different HK and thus gives rise to mirror symmetry phenomena.

Definition 3.10 *A marked complex HK with a B-field is a tuple (M, g, I, B, φ) , where (M, g, I, φ) is a marked complex HK and $B \in H^2(M, \mathbb{R})$. Two such tuples (M, g, I, B, φ) , $(M', g', I', B', \varphi')$ are equivalent if there exists an isometry $f : (M, g) \cong (M', g')$ with $I = f^*I'$, $\varphi' = \varphi \circ f^*$, and $f^*(B') = B$.*

Definition 3.11 *The $(2, 2)$ -CFT moduli space of HK is the space*

$$\mathcal{T}_\Gamma^{(2,2)} := \{(M, g, I, B, \varphi) = \text{marked complex HK with B-field}\}/\sim.$$

Clearly, the moduli space $\mathcal{T}_\Gamma^{(2,2)}$ is naturally isomorphic to $\mathcal{T}_\Gamma \times \Gamma \otimes \mathbb{R}$ by mapping (M, g, I, B, φ) to $((M, g, I, \varphi), \varphi_{\mathbb{R}}(B))$. In particular, $\mathcal{T}_\Gamma^{(2,2)}$ is a real manifold of dimension $4b - 6$.

Analogously, one defines the $(4, 4)$ -CFT moduli space

$$\mathcal{T}_\Gamma^{(4,4)} := \{(M, g, B, \varphi) = \text{marked HK with B-field}\}/\sim.$$

In particular, there is a natural forgetful map $\mathcal{T}_\Gamma^{(2,2)} \rightarrow \mathcal{T}_\Gamma^{(4,4)}$ which is surjective with fibre S^2 .

3.5 Moduli spaces without markings

All previous moduli spaces parametrize various geometric objects with an additional marking of the second cohomology. Of course, what we are really interested in are the true moduli spaces $\mathcal{M}_\Gamma^{\text{cpl}}$, $\mathcal{M}_\Gamma^{\text{met}}$, \mathcal{M}_Γ , $\mathcal{M}_\Gamma^{(2,2)}$, and $\mathcal{M}^{(4,4)}$. E.g. $\mathcal{M}_\Gamma^{\text{cpl}}$ is the moduli space of IHS X of dimension $2n$ such that $(H^2(X, \mathbb{Z}), q_X)$ is isomorphic to Γ , but without actually fixing the isomorphism. Analogously for the other spaces. In other words one has:

$$\begin{aligned} \mathcal{M}_\Gamma^{\text{cpl}} &= \text{O}(\Gamma) \setminus \mathcal{T}_\Gamma^{\text{cpl}}, & \mathcal{M}_\Gamma^{\text{met}} &= \text{O}(\Gamma) \setminus \mathcal{T}_\Gamma^{\text{met}}, & \mathcal{M}_\Gamma &= \text{O}(\Gamma) \setminus \mathcal{T}_\Gamma, \\ \mathcal{M}_\Gamma^{(2,2)} &= \text{O}(\Gamma) \setminus \mathcal{T}_\Gamma^{(2,2)}, & \mathcal{M}_\Gamma^{(4,4)} &= \text{O}(\Gamma) \setminus \mathcal{T}_\Gamma^{(4,4)}. \end{aligned}$$

The Teichmüller spaces \mathcal{T}_Γ^* are in general better behaved. E.g. the moduli spaces are usually singular at points that correspond to manifolds with a bigger automorphism group than expected. This usually leads to orbifold singularities. However, sometimes the passage from the Teichmüller space to the moduli space is really ill-behaved. E.g. the action of $\text{O}(\Gamma)$ on $\mathcal{T}_\Gamma^{\text{cpl}}$ is not properly discontinuous. Thus, $\mathcal{T}_\Gamma^{\text{cpl}}$ which already is not Hausdorff, becomes even worse when divided out by $\text{O}(\Gamma)$ (cf. the discussion in Section 6).

There is yet another approach to these moduli spaces where one actually fixes the underlying manifold and constructs the moduli space as a quotient of the space of hyperkähler metrics by the diffeomorphism group. We will briefly discuss this.

Let M be a compact oriented differentiable manifold of real dimension $4n$ and let q_M be the quadratic form on $H^2(M, \mathbb{Z})$ given by $q_M(\alpha) = c_n \cdot \int_M \alpha^2 \sqrt{\hat{A}(M)}$. We write $\Gamma = (H^2(M, \mathbb{Z}), q_M)$ and call this identification φ_0 .

By $\text{Diff}(M)$ we denote the group of orientation-preserving diffeomorphisms of M . In fact, at least for $b_2 \neq 6$, the group $\text{Diff}(M)$ is the full diffeomorphism group of M , as any orientation-reversing diffeomorphism f would induce an isomorphism $(H^2(M, \mathbb{Z}), q_M) \cong (H^2(M, \mathbb{Z}), -q_M)$ which is impossible for $b_2(M) \neq 6$. The set of all hyperkähler metrics g on M is denoted by $\text{Met}^{\text{HK}}(M)$. Clearly, $\text{Diff}(M)$ acts naturally on $\text{Met}^{\text{HK}}(M)$ by $(f, g) \mapsto f^*g$.

Definition 3.12 *The group $\text{Diff}_o(M) \subset \text{Diff}(M)$ is the connected component of $\text{Diff}(M)$ containing the identity $\text{id}_M \in \text{Diff}(M)$. The group $\text{Diff}_*(M) \subset \text{Diff}(M)$ is the kernel of the natural representation $\text{Diff}(M) \rightarrow \text{O}(H^2(M, \mathbb{Z}), q_M)$.*

Mapping $g \in \text{Met}^{\text{HK}}(M)$ to $(M, g, \varphi_0) \in \mathcal{T}_\Gamma^{\text{met}}$ induces a commutative diagram

$$\begin{array}{ccc} \text{Met}^{\text{HK}}(M)/\text{Diff}_*(M) & \xrightarrow{\eta} & \mathcal{T}_\Gamma^{\text{met}} \\ \downarrow & & \downarrow \\ \text{Met}^{\text{HK}}(M)/\text{Diff}(M) & \xrightarrow{\bar{\eta}} & \mathcal{M}_\Gamma^{\text{met}} \end{array}$$

Note that η is well-defined. Indeed, if $f \in \text{Diff}_*(M)$, then $(M, g, \varphi_0) \sim (M, f^*g, \varphi_0 \circ f^*) = (M, f^*g, \varphi_0)$.

Remark 3.13 It seems essentially nothing is known about the quotient of the natural inclusion $\text{Diff}_o(M) \subset \text{Diff}_*(M)$, not even for K3 surfaces, i.e. $n = 1$.

Clearly, the image of η (and $\bar{\eta}$) can contain only those HK $(M', g', \varphi) \in \mathcal{T}_\Gamma^{\text{met}}$ whose underlying real manifold M' is diffeomorphic to M . Let $\mathcal{T}_\Gamma^{\text{met}}(M)$ and $\mathcal{M}_\Gamma^{\text{met}}(M)$ denote the union of all those connected components.

i) In general, $\eta : \text{Met}^{\text{HK}}(M)/\text{Diff}_*(M) \rightarrow \mathcal{T}_\Gamma^{\text{met}}(M)$ is injective, but not surjective.

The injectivity is clear. Let us explain why surjectivity fails in general. If $(M, g, \varphi) \in \text{Im}(\eta)$ and $\psi \in \text{O}(H^2(M, \mathbb{Z}), q_M)$, then $(M, g, \varphi_0 \circ \psi) \in \text{Im}(\eta)$ if and only if there exists $f \in \text{Diff}(M)$ with $f^* = \psi$ but $\text{Diff}(M) \rightarrow \text{O}(H^2(M, \mathbb{Z}), q_M)$ is not necessarily surjective. E.g. for K3 surfaces the image does not contain $-\text{id}$ and, more precisely, $\text{O}(H^2(M, \mathbb{Z}), \cup)/\text{Diff}(M) \cong \mathbb{Z}/2\mathbb{Z}$

(cf. 5.3). However in this case the situation is rather simple, as $\mathcal{T}_\Gamma^{\text{met}}$ consists of two components, interchanged by id_{H^2} , and $\text{Met}^{\text{HK}}(M)/\text{Diff}_*(M)$ is one of them. For higher dimensional HK nothing is known about the image of $\text{Diff}(M) \rightarrow \text{O}(H^2(M, \mathbb{Z}), q_M)$.

ii) The map $\bar{\eta} : \text{Met}^{\text{HK}}(M)/\text{Diff}(M) \rightarrow \mathcal{M}_\Gamma^{\text{met}}(M)$ is bijective.

Indeed, if $(M, g, \varphi) \in \mathcal{T}_\Gamma^{\text{met}}(M)$, then $[(M, g, \varphi_0)] = [(M, g, (\varphi_0 \varphi^{-1})\varphi)] = (\varphi_0 \varphi^{-1})[(M, g, \varphi)] = [(M, g, \varphi)] \in \mathcal{M}_\Gamma^{\text{met}}(M)$. Thus, $\bar{\eta}$ is surjective. If $\bar{\eta}(M, g) = \bar{\eta}(M, g')$, then there exists $\psi \in \text{O}(\Gamma)$ such that $(M, g, \varphi_0) \sim (M, g', \psi \circ \varphi_0)$ and hence there exists $f \in \text{Diff}(M)$ with $f^*g = g'$ (note that for $b_2(M) = 6$ one would have to argue that f can be chosen orientation-preserving) and $\varphi_0 = \psi \circ \varphi_0 \circ f^*$. Thus, $[(M, g)] = [(M, g')] \in \text{Met}^{\text{HK}}(M)/\text{Diff}(M)$ and hence $\bar{\eta}$ is injective.

One last word concerning the stabilizer of the action of $\text{Diff}(M)$. Clearly, the stabilizer of a hyperkähler metric g is the isometry group $\text{Isom}(M, g)$ of (M, g) . This group is compact (cf. [9]). Hence the stabilizer of $g \in \text{Met}^{\text{HK}}(M)$ is a compact group. Moreover, $\text{Isom}(M, g) \cap \text{Diff}_*(M)$ is finite. Indeed, if $f \in \text{Isom}(M, g)$ then f maps any g -compatible complex structure I to another g -compatible complex structure f^*I . If in addition $f^* = \text{id}$ on $H^*(M, \mathbb{R})$, then the map $I \mapsto f^*I$ must also be the identity. Hence $f \in \text{Aut}(M, I)$ for any g -compatible complex structure I . Since $H^0((M, I), \mathcal{T}) = 0$, the latter group is discrete and, therefore, $\text{Aut}(M, I) \cap \text{Isom}(M, g)$ is finite. Hence, the action of $\text{Diff}_*(M)$ on $\text{Met}^{\text{HK}}(M)$ has finite stabilizer.

4 Period domains

The moduli spaces that have been introduced in the last section will be studied by means of various period maps. In this section we define and discuss the spaces in which these maps take their values, the period domains.

Let Γ be a lattice of signature (m, n) . The standard example for Γ is the K3 lattice $2(-E_8) \oplus 3U$, where U denotes the hyperbolic plane $(\mathbb{Z}^2, (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))$. However, Γ might in general be non-unimodular. This will be of no importance in this section, as only the real vector space $\Gamma_{\mathbb{R}} := \Gamma \otimes \mathbb{R}$ is going to be used. In fact, usually we will work with an arbitrary vector space V , but Γ will nevertheless occur in the notation. I hope this will not lead to any confusion.

4.1 Positive (oriented) subspaces

Let V be a real vector space that is endowed with a bilinear form $\langle \cdot, \cdot \rangle$ of signature (m, n) , e.g. $V = \Gamma_{\mathbb{R}}$. We will also write x^2 for $\langle x, x \rangle$. Fix $k \leq m$ and consider the space of all k -dimensional subspaces $W \subset V$ such that $\langle \cdot, \cdot \rangle$ restricted to W is positive definite. We will denote this space by $\text{Gr}_k^P(V)$. Clearly, $\text{Gr}_k^P(V)$ is an open non-empty subset of the Grassmannian $\text{Gr}_k(V)$.

In order to describe $\text{Gr}_k^P(V)$ as a homogeneous space we consider the natural action of $O(V)$ on $\text{Gr}_k^P(V)$ given by $(\varphi, W) \mapsto \varphi(W)$. The stabilizer of a point $W_0 \in \text{Gr}_k^P(V)$ is $O(W_0) \times O(W_0^\perp)$. Since the action is transitive, one obtains the following description

$$\boxed{\text{Gr}_k^P(V) \cong O(V)/O(W_0) \times O(W_0^\perp) \cong O(m, n)/O(k) \times O(m - k, n)}$$

The second isomorphism depends on the choice of a basis of the spaces W_0 and W_0^\perp .

Next consider the space $\text{Gr}_k^{\text{po}}(V)$ of all oriented positive subspaces $W \subset V$ of dimension k . Clearly, the natural map $\text{Gr}_k^{\text{po}}(V) \rightarrow \text{Gr}_k^P(V)$ is a $2 : 1$ cover. Again, $O(V)$ acts transitively on $\text{Gr}_k^{\text{po}}(V)$ and the stabilizer of an oriented positive subspace W_0 is $SO(W_0) \times O(W_0^\perp)$. Thus,

$$\boxed{\text{Gr}_2^{\text{po}}(V) \cong O(V)/SO(W_0) \times O(W_0^\perp) \cong O(m, n)/SO(k) \times O(m - k, n)}$$

4.2 Planes and complex lines

For $k = 2$ the space $\text{Gr}_2^{\text{po}}(V)$ allows an alternative description. It turns out that there is a natural bijection between this space and the space

$$Q_\Gamma := \{x \mid x^2 = 0, (x + \bar{x})^2 > 0\} \subset \mathbb{P}(\Gamma_{\mathbb{C}}),$$

where we use the \mathbb{C} -linear extension of $\langle \cdot, \cdot \rangle$. Note that the second condition in the definition of Q_Γ is well posed, i.e. independent of the representative $x \in \Gamma_{\mathbb{C}}$ of the line $x \in \mathbb{P}(\Gamma_{\mathbb{C}})$, as long as the first condition $x^2 = 0$ is satisfied. Clearly, Q_Γ is an open subset of a non-singular quadric hypersurface in $\mathbb{P}(\Gamma_{\mathbb{C}})$.

To any $x \in Q_\Gamma$ one associates the plane $W_x := \Gamma_{\mathbb{R}} \cap (x\mathbb{C} \oplus \bar{x}\mathbb{C}) \subset \Gamma_{\mathbb{R}}$ endowed with the orientation given by $(\text{Re}(x), \text{Im}(x))$. Since $x\mathbb{C} \oplus \bar{x}\mathbb{C}$ is invariant under conjugation, this space is indeed a real plane. Moreover, $W_{\lambda x} =$

$\Gamma_{\mathbb{R}} \cap (\lambda x \mathbb{C} \oplus \bar{\lambda} \bar{x} \mathbb{C}) = W_x$ and $(\operatorname{Re}(\lambda x), \operatorname{Im}(\lambda x)) = (\operatorname{Re}(x), \operatorname{Im}(x)) \begin{pmatrix} \operatorname{Re}(\Lambda) & \operatorname{Im}(\lambda) \\ -\operatorname{Im}(\lambda) & \operatorname{Re}(\Lambda) \end{pmatrix}$, where the matrix has positive determinant. Hence, the oriented plane W_x is well-defined, i.e. it only depends on $x \in \mathbb{P}(\Gamma_{\mathbb{C}})$. It is positive, since $(\lambda x + \bar{\lambda} \bar{x})^2 = \lambda \bar{\lambda} (x + \bar{x})^2 > 0$ for $\lambda \neq 0$.

Conversely, if $W \in \operatorname{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}})$, then choose a positively oriented orthonormal basis $w_1, w_2 \in W$ and set $x := w_1 + iw_2$. Then $W = W_x$ and $x^2 = 0$, $(x + \bar{x})^2 = (2w_1)^2 > 0$. Moreover, $x \in \mathbb{P}(\Gamma_{\mathbb{C}})$ does not depend on the choice of the basis and any $x \in Q_{\Gamma}$ can be written in this form.

Thus, one has a bijection

$$Q_{\Gamma} \cong \operatorname{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}})$$

4.3 Planes and three-spaces

For our purpose the spaces $\operatorname{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}})$, $\operatorname{Gr}_3^{\text{po}}(\Gamma_{\mathbb{R}})$, and $\operatorname{Gr}_4^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ are the most interesting ones. In the next two sections we will study how they are related to each other. To this end let us first introduce the space

$$\operatorname{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) := \{(P, \omega) \mid P \in \operatorname{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}}), \omega \in P^{\perp} \subset \Gamma_{\mathbb{R}}, \omega^2 > 0\}.$$

Clearly, this space projects naturally to $\operatorname{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}})$ by $(P, \omega) \mapsto P$. The fibre over the point P is the quadratic cone $\{\omega \mid \omega^2 > 0\} \subset P^{\perp} \subset \Gamma_{\mathbb{R}}$. If Γ has signature $(3, b-3)$, this cone consists of exactly two connected components, which can be identified with each other by $\omega \mapsto -\omega$. Thus, the fibre of $\operatorname{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \rightarrow \operatorname{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}})$ over P in this case is the disjoint union of two copies of a connected cone, which will be called \mathcal{C}_P .

In fact, $\operatorname{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \rightarrow \operatorname{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}})$ is a trivial cover, i.e. $\operatorname{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}})$ splits into two components. This can either be deduced from the fact that $\operatorname{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}}) \cong \operatorname{O}(3, b-3)/(\operatorname{SO}(2) \times \operatorname{O}(1, b-3))$ is simply connected (cf. Section 4.7) or from the following argument: If we fix an oriented positive three-space $F \in \operatorname{Gr}_3^{\text{po}}(\Gamma_{\mathbb{R}})$, then the orthogonal projection $P \oplus \mathbb{R}\omega \rightarrow F$ for any $\omega \in \pm \mathcal{C}_X$ must be an isomorphism, since F^{\perp} is negative. Thus, we can single out one of the two connected components of $\pm \mathcal{C}_P$ by requiring that $P \oplus \mathbb{R}\omega \cong F$ is compatible with the orientations on both spaces.

Mapping (P, ω) to the oriented positive three-space $F(P, \omega) := P \oplus \omega \mathbb{R}$ and the scalar $\omega^2 \in \mathbb{R}_{>0}$ defines a map $\operatorname{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \rightarrow \operatorname{Gr}_3^{\text{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0}$. The map is surjective and the fibre over a point (F, λ) can be identified with the set of all $\omega \in F$ with $\omega^2 = \lambda$ which is a two-dimensional sphere.

Thus, one has the following diagram

$$\begin{array}{ccc}
 \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) & \xrightarrow{s^2} & \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \cong \left(\mathrm{O}(m, n)/\mathrm{SO}(3) \times \mathrm{O}(m-3, n) \right) \times \mathbb{R}_{>0} \\
 \downarrow \pm c_P & & \\
 \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}}) & \xrightarrow{\cong} & \mathrm{O}(m, n)/\mathrm{SO}(2) \times \mathrm{O}(m-2, n)
 \end{array}$$

Note that the two natural compositions $S^2 \subset \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \rightarrow \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})$ and $c_P \subset \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \rightarrow \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0}$ are both injective.

4.4 Three- and four-spaces

From now on we will assume that Γ has signature $(3, b-3)$. Furthermore, let us fix a standard basis (w, w^*) of U , i.e. $w^2 = w^{*2} = 0$ and $\langle w, w^* \rangle = 1$. We will see that the space of four-spaces in $\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}$ relates naturally to the space of three-spaces in $\Gamma_{\mathbb{R}}$. Explicitly, we will show

$$\begin{aligned}
 \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \times \Gamma_{\mathbb{R}} &\cong \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \\
 &\cong \mathrm{O}(4, b-2)/\mathrm{SO}(4) \times \mathrm{O}(b-2)
 \end{aligned}$$

The second isomorphism follows from Section 4.1. The first one is given as follows.

$$\phi : (F, \alpha, B) \mapsto \Pi := B' \mathbb{R} \oplus F',$$

where $F' := \{f - \langle f, B \rangle w \mid f \in F\}$ and $B' := B + \frac{1}{2}(\alpha - B^2)w + w^*$. Clearly, $\langle f - \langle f, B \rangle w, B' \rangle = \langle f - \langle f, B \rangle w, B + w^* \rangle = 0$ and thus the decomposition is orthogonal. Furthermore, $(f - \langle f, B \rangle w)^2 = f^2 > 0$ for $0 \neq f \in F$ and $B'^2 = B^2 + \alpha - B^2 = \alpha > 0$. Hence, Π is a positive four-space. Its orientation is induced by the orientation of $F \cong F'$ and the decomposition $\Pi = B' \mathbb{R} \oplus F'$.

In order to see that ϕ is bijective we study the inverse map $\psi : \Pi \mapsto (F, B'^2, B)$, where F , B' , and B are defined as follows: One first introduces $F' := \Pi \cap w^\perp$. This space is of dimension three, since otherwise $\Pi \subset w^\perp = \Gamma_{\mathbb{R}} \oplus w \mathbb{R}$ and the latter space does not contain any positive four-space. Again by the positivity of Π one finds $w \notin F' \subset \Pi$. Hence, $F := \pi(F') \subset \Gamma_{\mathbb{R}}$ is a positive three-space, where $\pi : \Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}} \rightarrow \Gamma_{\mathbb{R}}$ is the natural projection. Furthermore, there exists a $B' \in \Pi$ such that $\Pi = B' \mathbb{R} \oplus F'$ is an orthogonal

splitting. As before B' cannot be contained in w^\perp . Thus, one can rescale B' such that $\langle B', w \rangle = 1$. This determines B' uniquely. Since $B' \in \Pi$, one has $B'^2 > 0$. The B-field is by definition $B := \pi(B')$. One easily verifies that ψ and ϕ are indeed inverse to each other.

4.5 Pairs of planes

The last space we will discuss in this series of period domains is the space of orthogonal oriented positive planes in $\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}$, i.e.

$$\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) = \{(H_1, H_2) \mid H_i \in \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}), H_1 \perp H_2\}.$$

Using the same techniques as before this space can also be described as an homogeneous space as follows

$$\begin{aligned} \mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) &\cong \mathrm{O}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})/\mathrm{SO}(H_1) \times \mathrm{SO}(H_2) \times \mathrm{O}((H_1 \oplus H_2)^\perp) \\ &\cong \mathrm{O}(4, b-2)/\mathrm{SO}(2) \times \mathrm{SO}(2) \times \mathrm{O}(b-2), \end{aligned}$$

for some chosen point $(H_1, H_2) \in \mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$, respectively basis of the spaces H_1 , H_2 , and $(H_1 \oplus H_2)^\perp$.

We will be interested in the natural projection

$$\pi : \mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \rightarrow \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}), (H_1, H_2) \mapsto \Pi := H_1 \oplus H_2$$

and in the injection

$$\gamma : \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} \hookrightarrow \mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$$

which is compatible with $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \rightarrow \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}})$.

Let us first study the projection. Using the above description of both spaces as homogeneous spaces this map corresponds to dividing by $\mathrm{SO}(4)/(\mathrm{SO}(2) \times \mathrm{SO}(2))$. The fibre of π over $\Pi \in \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ is canonically isomorphic to $\mathrm{Gr}_2^{\mathrm{po}}(\Pi)$ via $(H_1, H_2) \mapsto H_1$. The inverse image of $H \in \mathrm{Gr}_2^{\mathrm{po}}(\Pi)$ is (H, H^\perp) , where H^\perp gets its orientation from Π and the decomposition $\Pi = H \oplus H^\perp$.

Thus, one obtains the following description of the fibre

$$\pi^{-1}(\Pi) \cong \mathrm{Gr}_2^{\mathrm{po}}(\Pi) \cong S^2 \times S^2.$$

The second isomorphism is derived as in Section 4.2 from

$$\mathrm{Gr}_2^{\mathrm{po}}(\Pi) = \{x \in \mathbb{P}(\Pi_{\mathbb{C}}) \mid x^2 = 0\} \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Note that $(x + \bar{x})^2 > 0$ is automatically satisfied, for $\langle \cdot, \cdot \rangle$ on Π is positive by assumption.

Let us now turn to the injection γ , which is defined as follows. We set $\gamma((P, \omega), B) = (H_1, H_2)$ with

$$H_1 := \{x - \langle x, B \rangle w \mid x \in P\}$$

and

$$H_2 := \left(\frac{1}{2}(\alpha - B^2)w + w^* + B \right) \mathbb{R} \oplus (\omega - \langle \omega, B \rangle w) \mathbb{R},$$

where as before (w, w^*) is the standard basis of U and $\alpha = \omega^2$.

The isomorphism $P \cong H_1$, $x \mapsto x - \langle x, B \rangle w$ endows H_1 with an orientation. A natural orientation of H_2 is given by definition. Observe that H_1 only depends on P and B , whereas H_2 depends on ω and B . One easily verifies that the map γ is injective and that it commutes with the projections to

$$\mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \times \mathbb{R}_{>0} \times \Gamma_{\mathbb{R}} \cong \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}).$$

Recall that the fibre of $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} \rightarrow \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ is S^2 , whereas the fibre of $\pi : \mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \rightarrow \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ is $S^2 \times S^2$. It can be checked that the embedding γ does not identify the fibre S^2 neither with the diagonal nor with one of the two factors. In algebro-geometric terms $S^2 \subset S^2 \times S^2$ is a hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to the Segre embedding in \mathbb{P}^3 .

Remark 4.1 Note that the projection $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} \rightarrow \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \rightarrow \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})$ does not extend, at least not canonically, to a map $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \rightarrow \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})$. Geometrically this will be interpreted by the fact that not any point in the $(2, 2)$ -CFT moduli space of K3 surfaces canonically defines a complex structure. More recently, it has become clear that generalized K3 surfaces, a notion that relies on Hitchin's generalized Calabi-Yau structures [22], might be useful to give a geometric interpretation to every $N = (2, 2)$ -SCFT (see [27])

We summarize the discussion of this paragraph in the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) & \xrightarrow{s^2 \times s^2} & \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \\
\uparrow & & \parallel \\
\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} & \xrightarrow{s^2} & \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \times \Gamma_{\mathbb{R}} \\
\downarrow \Gamma_{\mathbb{R}} & & \downarrow \Gamma_{\mathbb{R}} \\
\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) & \xrightarrow{s^2} & \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \\
\downarrow \pm c_P & & \\
\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}}) & &
\end{array}$$

4.6 Calculations in the Mukai lattice

We shall indicate how the formulae change if we pass to the Mukai bilinear form. This will enable us to make the description of the various period spaces and period maps compatible with conventions used elsewhere. We include this discussion for completeness, but it is not necessary for the understanding of the later sections.

Clearly the hyperbolic lattice U with the standard basis w, w^* is isomorphic to $-U$ via $w \mapsto -w, w^* \mapsto w^*$. This extends to a lattice isomorphism

$$\eta : \Gamma \oplus U \cong \Gamma \oplus (-U) =: \tilde{\Gamma}.$$

For any oriented four-manifold M underlying a K3 surfaces we can identify $H^*(M, \mathbb{Z})$ endowed with the standard intersection pairing with $\Gamma \oplus U$ such that $w^* = 1 \in H^0(M, \mathbb{Z})$, $w = [\mathrm{pt}] \in H^4(M, \mathbb{Z})$, and $\Gamma \cong H^2(M, \mathbb{Z})$. Then $\tilde{\Gamma}$ is naturally isomorphic to $H^*(M, \mathbb{Z})$ with the Mukai pairing $(\alpha_0 + \alpha_2 + \alpha_4, \beta_0 + \beta_2 + \beta_4)_{\tilde{\Gamma}} = -\alpha_0\beta_4 - \alpha_4\beta_0 + \alpha_2\beta_2$, where $\alpha_i, \beta_i \in H^i(M, \mathbb{Z})$.

The identification of $\Gamma \oplus U$ and $\tilde{\Gamma}$ with the cohomology of a K3 surface induces a ring structure on both lattices, i.e. in both cases we define $(\lambda w^* + x + \mu w)^2 := \lambda^2 w^* + 2\lambda x + (2\lambda\mu + x^2)w$. Note that η does not respect these ring structures.

Using the ring structure on $\tilde{\Gamma}_{\mathbb{R}}$ we can let act any element $B_0 \in \Gamma_{\mathbb{R}}$ on $\tilde{\Gamma}_{\mathbb{R}}$ via its exponential $\exp(B_0) = w^* + B_0 + (B_0^2/2)w$.

Lemma 4.2 *For any $B_0 \in \Gamma_{\mathbb{R}}$ one has $\exp(B_0) \in \mathrm{O}(\tilde{\Gamma}_{\mathbb{R}})$.*

Proof. This results from the following straightforward calculation

$$\begin{aligned}
 & (\exp(B_0) \cdot (\lambda w^* + x + \mu w))^2_{\tilde{\Gamma}} \\
 &= \left(\lambda w^* + (\lambda B_0 + x) + (\mu + \langle B_0, x \rangle + \lambda \frac{B_0^2}{2}) w \right)^2_{\tilde{\Gamma}} \\
 &= x^2 - 2\lambda\mu = (\lambda w^* + x + \mu w)^2_{\tilde{\Gamma}}.
 \end{aligned}$$

□

Later we shall study the map φ_{B_0} associated to any $B_0 \in \Gamma_{\mathbb{R}}$ (see Section 6.2. By definition $\varphi_{B_0} \in O(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ acts on $\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}$ by

$$\begin{aligned}
 w &\mapsto w, \quad w^* \mapsto B_0 + w^* - \frac{B_0^2}{2}w, \\
 x &\mapsto x - \langle B_0, x \rangle w, \quad \text{for } x \in \Gamma_{\mathbb{R}}.
 \end{aligned}$$

Let us compare $\exp(B_0)$ with φ_{B_0} ,

Proposition 4.3 *Under the isomorphism $\eta : \Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}} \cong \tilde{\Gamma}_{\mathbb{R}}$ the automorphism φ_{B_0} corresponds to the action of $\exp(B_0)$, i.e. $\eta \circ \varphi_{B_0} = \exp(B_0) \circ \eta$.*

Proof. By definition, $\exp(B_0)$ acts by

$$\begin{aligned}
 w &\mapsto w, \quad w^* \mapsto (w^* + B_0 + \frac{B_0^2}{2}w) \cdot w^* = B_0 + w^* + \frac{B_0^2}{2}w, \\
 x &\mapsto (w^* + B_0 + \frac{B_0^2}{2}w) \cdot x = x + \langle B_0, x \rangle w, \quad \text{for } x \in \Gamma_{\mathbb{R}},
 \end{aligned}$$

which yields the assertion. □

The isomorphism η induces a natural isomorphism $Gr_{2,2}^{po}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \cong Gr_{2,2}^{po}(\tilde{\Gamma}_{\mathbb{R}})$. In order, to describe the image $\eta(H_1, H_2)$ we will use the identification $Q_{\tilde{\Gamma}} \cong Gr_2^{po}(\tilde{\Gamma}_{\mathbb{R}})$ established in Section 4.2. The positive plane associated to an element $x \in \tilde{\Gamma}_{\mathbb{C}}$ with $[x] \in Q_{\tilde{\Gamma}}$ will be denoted by P_x , i.e. P_x is spanned by $\text{Re}(x)$ and $\text{Im}(x)$. Clearly, $\exp(B) \cdot P_x = P_{\exp(B)x}$.

Let $(P, \omega) \in Gr_{2,1}^{po}(\Gamma_{\mathbb{R}})$ and $B \in \Gamma_{\mathbb{R}}$. We denote $(H_1, H_2) := \gamma((P, \omega), 0)$ and $(H_1^B, H_2^B) := \gamma((P, \omega), B)$. Then a direct calculation shows $\varphi_B(H_1, H_2) = (H_1^B, H_2^B)$ and therefore

Corollary 1 $\eta(H_1^B, H_2^B) = \exp(B) \cdot \eta(H_1, H_2) = \exp(B) \cdot (P_{\sigma}, P_{\exp(i\omega)}) = (P_{\exp(B)\sigma}, P_{\exp(B+i\omega)})$.

Proof. The only thing that needs a proof is $\eta(H_2) = P_{\exp(i\omega)}$. But this follow immediately from the definition of H_2 . \square

In particular, via η the image of $\gamma : \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} \rightarrow \mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ can be identified with $\exp(\Gamma_{\mathbb{R}}) \cdot \left(\eta(\gamma(\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}))) \right)$.

4.7 Topology of period domains

Let us study some basic aspects of the topology of the period domains that are of interest for us. Let Γ be a lattice of signature $(3, b - 3)$. We will consider the spaces:

$$\begin{aligned} \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}}) &\cong \mathrm{O}(3, b - 3)/\mathrm{SO}(2) \times \mathrm{O}(1, b - 3) \\ \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) &\cong \mathrm{O}(4, b - 2)/\mathrm{SO}(4) \times \mathrm{O}(b - 2) \\ \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) &\cong \mathrm{O}(3, b - 3)/\mathrm{SO}(3) \times \mathrm{O}(b - 3) \\ \mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) &\cong \mathrm{O}(4, b - 2)/\mathrm{SO}(2) \times \mathrm{SO}(2) \times \mathrm{O}(b - 2) \end{aligned}$$

For simplicity we will suppose that $b > 3$.

Lemma 4.4 *The group $\mathrm{O}(k, \ell)$ with $k, \ell > 0$ has exactly four connected components.*

Proof. Write $\mathrm{O} := \mathrm{O}(k, \ell)$. Then there are the following disjoint unions $\mathrm{O} = \mathrm{O}^+ \cup \mathrm{O}^-$, $\mathrm{O} = \mathrm{O}_+ \cup \mathrm{O}_-$, and $\mathrm{O} = \mathrm{O}_+^+ \cup \mathrm{O}_-^+ \cup \mathrm{O}_+^- \cup \mathrm{O}_-^-$. Here, O_{\pm}^{\pm} are defined as follows: Write $\mathbb{R}^{k+\ell} = W_0 \oplus W_0^{\perp}$ with $W_0 \subset \mathbb{R}^{k+\ell}$ a maximal positive subspace, which is endowed with an orientation. Then let O^+ and O^- (respectively, O_+ and O_-) be the subsets of all linear maps $A \in \mathrm{O}$ such that the orthogonal projection $AW_0 \rightarrow W_0$ (respectively, $AW_0^{\perp} \rightarrow W_0^{\perp}$) is orientation preserving resp. orientation reversing. By definition $\mathrm{O}_+^+ = \mathrm{O}^+ \cap \mathrm{O}_+$, etc. For any $A_0 \in \mathrm{O}_{\pm}^{\pm}$ the map $\mathrm{O}_+^{\pm} \rightarrow \mathrm{O}_{\pm}^{\pm}$, $A \mapsto AA_0$ defines a homeomorphism. Thus, it suffices to show that O_+^+ is connected. \square

Note that $\mathrm{O}_+^+(k, l)$ is the connected component of the identity. It will thus also be denoted $\mathrm{O}_o(m, n)$.

Corollary 2 *The space $\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})$ is connected, whereas the spaces $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}})$, $\mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}})$, $\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$, and $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ consist of two connected components.*

Proof. Use the obvious fact that the inclusion $\mathrm{SO}(2) \times \mathrm{O}(1, b-3) \subset \mathrm{O}(3, b-3)$ respects the decomposition into connected components, i.e. $\mathrm{O}_{\pm}^{\pm}(1, b-3) \subset \mathrm{O}_{\pm}^{\pm}(3, b-3)$. Thus, $\pi_0(\mathrm{SO}(2) \times \mathrm{O}(1, b-3)) \cong \pi_0(\mathrm{O}(3, b-3))$. Similarly for $\mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}})$. Here $\mathrm{O}(3, b-3)$ has four connected components, but $\pi_0(\mathrm{SO}(3) \times \mathrm{O}(b-3)) = \mathbb{Z}/2\mathbb{Z}$, i.e. the components O_{\pm}^{\pm} do not intersect the image of the inclusion. Hence, $\pi_0(\mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}})) \cong \mathbb{Z}/2\mathbb{Z}$. The remaining assertions are proved analogously. \square

We are also interested in the fundamental groups of these spaces. In order to compute those, we recall the following classical facts.

Proposition 4.5 *One has $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$, $\pi_1(\mathrm{SO}(k)) = \mathbb{Z}/2\mathbb{Z}$ for $k > 2$, and $\pi_1(\mathrm{O}_o(k, \ell)) \cong \pi_1(\mathrm{SO}(k)) \times \pi_1(\mathrm{SO}(\ell))$.*

Proof. The first assertion follows from $\mathrm{SO}(2) \cong S^1$. The universal cover of $\mathrm{SO}(k)$ for $k \geq 3$ is the two-to-one cover $\mathrm{Spin}(k) \rightarrow \mathrm{SO}(k)$. The isomorphism in the last assertion is induced by the natural inclusion $\mathrm{SO}(k) \times \mathrm{SO}(\ell) \hookrightarrow \mathrm{O}_o(k, \ell)$. \square

Corollary 3 *All the Grassmannians $\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})$, $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}})$, $\mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}})$, $\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$, and $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ are simply-connected, i.e. every connected component is simply connected.*

Proof. Since

$$\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}}) = \mathrm{O}(3, b-3)/\mathrm{SO}(2) \times \mathrm{O}(1, b-3) \cong \mathrm{O}_o(3, b-3)/\mathrm{SO}(2) \times \mathrm{O}_o(1, b-3),$$

we may use the exact sequence

$$\pi_1(\mathrm{SO}(2) \times \mathrm{O}_o(1, b-3)) \xrightarrow{a} \pi_1(\mathrm{O}_o(3, b-3)) \rightarrow \pi_1(\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})) \rightarrow \pi_0(\) \cong \pi_0(\).$$

The map a is compatible with the natural isomorphisms $\pi_1(\mathrm{SO}(2) \times \mathrm{O}_o(1, b-3)) \cong \pi_1(\mathrm{SO}(2)) \times \pi_1(\mathrm{O}_o(1, b-3)) \cong \pi_1(\mathrm{SO}(2)) \times \pi_1(\mathrm{SO}(1)) \times \pi_1(\mathrm{SO}(b-3))$, $\pi_1(\mathrm{O}_o(3, b-3)) \cong \pi_1(\mathrm{SO}(3)) \times \pi_1(\mathrm{SO}(b-3))$ and the natural maps $\mathbb{Z} \cong \pi_1(\mathrm{SO}(2)) \times \pi_1(\mathrm{SO}(1)) \rightarrow \pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$. Thus, a is surjective and hence $\pi_1(\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})) = 0$. The other assertions are proved analogously. \square

Remark 4.6 Eventually, we list the real dimensions of our period spaces, which can easily be computed starting from $\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \cong Q_{\Gamma}$. We have $\dim \mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}}) = 2(b-2)$, $\dim \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) = 3(b-2)$, $\dim \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) = 3(b-3)$, $\dim \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) = 4(b-2)$, and $\dim \mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) = 4(b-1)$.

4.8 Density results

Here we shall be interested in those points $P \in Q_\Gamma$ whose orthogonal complement $P^\perp \subset \Gamma_{\mathbb{R}}$ contains integral elements $\alpha \in \Gamma$ of given length. For simplicity we shall assume that Γ is the K3 lattice $2(-E_8) \oplus 3U$, but all we will use is that Γ is even of index $(3, b - 3)$ and that any primitive isotropic element of Γ can be complemented to a sublattice of Γ which is isomorphic to the hyperbolic plane. First note the following easy fact.

Lemma 4.7 *If $0 \neq \alpha \in \Gamma_{\mathbb{R}}$ then $\alpha^\perp \cap Q_\Gamma$ is not empty.*

Proof. Indeed, $\alpha^\perp \subset \Gamma_{\mathbb{R}}$ is a hyperplane containing at least two linearly independent orthogonal positive vectors x, y . Thus, $P := \langle x, y \rangle \in \alpha^\perp \cap Q_\Gamma$. \square

The quadric in $\mathbb{P}(\Gamma_{\mathbb{C}})$ defined by the quadratic form $\langle \cdot, \cdot \rangle$ on Γ will be denoted Z , its real points form the set $Z_{\mathbb{R}} = \mathbb{P}(\Gamma_{\mathbb{R}}) \cap Z$.

Proposition 4.8 *Let $0 \neq \alpha \in \Gamma$. Then the set*

$$\bigcup_{g \in O(\Gamma)} g(\alpha^\perp \cap Q_\Gamma) = \bigcup_{g \in O(\Gamma)} g(\alpha)^\perp \cap Q_\Gamma$$

is dense in Q_Γ .

Proof. We start out with the following observation: Let $\Gamma = \Gamma' \oplus U$ be an orthogonal decomposition and let (v, v^*) be a standard basis of U . For $B \in \Gamma'$ with $B^2 \neq 0$ we define $\varphi_B \in O(\Gamma)$ by $\varphi_B(v) = v$, $\varphi_B(v^*) = B + v^* - B^2/2 \cdot v$, and $\varphi_B(x) = x - \langle B, x \rangle v$ for $x \in \Gamma'$. It is easy to see that indeed with this definition $\varphi_B \in O(\Gamma)$. (We shall study a similarly defined automorphism $\varphi_B \in O(\Gamma \oplus U)$ in Section 6).

This automorphism has the remarkable property that for any $y \in \Gamma_{\mathbb{R}}$ one has

$$\lim_{k \rightarrow \infty} \varphi_N^k[y] = [v] \in \mathbb{P}(\Gamma_{\mathbb{R}}).$$

In particular, we find that in the closure of the orbit $O := O(\Gamma) \cdot [\alpha] \subset \mathbb{P}(\Gamma_{\mathbb{R}})$ there exists an isotropic vector, i.e. $\overline{O} \cap Z_{\mathbb{R}} \neq \emptyset$.

In order to prove the assertion of the proposition we have to show that for any $P \in Q_\Gamma$ there exists an automorphism $g \in O(\Gamma)$ such that $g(\alpha)$ is arbitrarily close to P^\perp . Indeed, in this case we find a codimension two subspace $W \subset \Gamma_{\mathbb{R}}$ close to P^\perp containing $g(\alpha)$ and, therefore, $W^\perp \in Q_\Gamma$ is close to P and orthogonal to $g(\alpha)$.

Since P^\perp contains some isotropic vector, it suffices to show that any vector $[y] \in Z_{\mathbb{R}} \subset \mathbb{P}(\Gamma_{\mathbb{R}})$ is contained in \overline{O} . As explained before, $\overline{O} \cap Z_{\mathbb{R}} \neq \emptyset$. On the other hand, $\overline{O} \cap Z_{\mathbb{R}}$ is closed and $O(\Gamma)$ -invariant. Thus, it suffices to show that any $O(\Gamma)$ -orbit $O_y := O(\Gamma) \cdot [y] \subset Z_{\mathbb{R}}$ is dense. This is proved in two steps.

i) The closure \overline{O}_y contains the subset $\{[x] \in Z \mid x \in \Gamma\}$. Indeed, for any $x \in \Gamma$ primitive with $x^2 = 0$ one finds an orthogonal decomposition $\Gamma = \Gamma' \oplus U$ with $x = v$, where (v, v^*) is a standard basis of the hyperbolic plane U . If we choose $B \in \Gamma'$ with $B^2 \neq 0$, then $\lim_{k \rightarrow \infty} \varphi_B^k[y] = [v] = [x]$, as we have seen before. Hence, $[x] \in \overline{O}_y$.

ii) The set $\{[x] \in Z \mid x \in \Gamma\}$ is dense in Z . Indeed, if we write $\Gamma = \Gamma' \oplus U$ as before, then the dense open subset $V \subset Z_{\mathbb{R}}$ of points of the form $[x' + \lambda v + v^*]$ with $\lambda \in \mathbb{R}$, $x' \in \Gamma'_{\mathbb{R}}$ is the affine quadric $\{(x', \lambda) \mid 2\lambda + x'^2 = 0\} \subset \Gamma_{\mathbb{R}} \times \mathbb{R}$ and thus is given as the graph of the rational polynomial $\Gamma'_{\mathbb{R}} \rightarrow \mathbb{R}$, $x' \mapsto -x'^2/2$. Therefore, the rational points are dense in V .

Combining both steps yields the assertion. \square

Corollary 4 *For any $m \in \mathbb{Z}$ the subset*

$$\{P \in Q_{\Gamma} \mid \text{there exists a primitive } \alpha \in \Gamma \cap P^\perp \text{ with } \alpha^2 = 2m\}$$

is dense in Q_{Γ} .

Proof. In order to apply the proposition we only have to ensure that there is a primitive element $0 \neq \alpha \in \Gamma$ with $\alpha^2 = 2m$. If (w, w^*) is the standard base of a copy of the hyperbolic plane U contained in Γ , we can choose $\alpha = w + mw^*$. \square

In fact, if $\alpha_1, \alpha_2 \in \Gamma$ are primitive elements with $\alpha_1^2 = \alpha_2^2$ then there exists an automorphism $\varphi \in O(\Gamma)$ with $\varphi(\alpha_1) = \alpha_2$ (cf. [29, Thm.2.4] or Remark 7.4). Thus, the assertion of the corollary is essentially equivalent to the proposition (see [1] page 111). Note that for general HKs we don't know which values of $2m$ can be realized.

As a further trivial consequence, one finds that the set of those $P \in Q_{\Gamma}$ such that $P^\perp \cap \Gamma \neq 0$ is dense in Q_{Γ} . One can now go on and ask for those $P \in Q_{\Gamma}$ such that $P^\perp \cap \Gamma$ has higher rank. Those with maximal rank, i.e. $\text{rk}(P^\perp \cap \Gamma) = \text{rk}(\Gamma) - 2$, are called exceptional. An equivalent definition is

Definition 4.9 *A period point $P \in Q_{\Gamma}$ is exceptional if $P \subset \Gamma_{\mathbb{R}}$ is defined over \mathbb{Q} , i.e. $P \in Q_{\Gamma} \cap \mathbb{P}(\Gamma_{\mathbb{Q}(i)})$.*

Clearly, P is exceptional if there exist linearly independent elements $\alpha_1, \dots, \alpha_{\text{rk}(\Gamma)-2} \in \Gamma$ such that $P \subset \alpha_i^\perp$ for all i . Note that if $P \in Q_\Gamma$ is exceptional, the orthogonal complement P^\perp always contains a lattice vector $x \in \Gamma$ with $x^2 > 0$ (use that Γ has signature $(3, b-3)$).

Next we will prove that also the exceptional points are dense in Q_Γ . For K3 surfaces one can add further restrictions.

Definition 4.10 *Let Γ be the K3 lattice. A period point $P \in Q_\Gamma$ is called exceptional Kummer if $P \subset \Gamma_{\mathbb{R}}$ is defined over \mathbb{Q} and for all $x \in P \cap \Gamma$ one has $x^2 \equiv 0 \pmod{4}$.*

Proposition 4.11 *Let Γ be the K3 lattice. Then the set of exceptional Kummer points $P \in Q_\Gamma$ is a dense subset of Q_Γ .*

Proof. We first prove the following statement. Let L be an arbitrary lattice. Then the set

$$\{[x] \mid x \in L \text{ is primitive and } x^2 \equiv 0 \pmod{4}\} \subset \mathbb{P}(L_{\mathbb{R}})$$

is empty or dense. Indeed, if $[x]$ is contained in this set and $y \in L$ is arbitrary, then $[x + N \cdot y] \in \mathbb{P}(L_{\mathbb{R}})$ converges towards $[y]$ for $N \rightarrow \infty$. Moreover, $(x + N \cdot y)^2 \equiv x^2 \equiv 0 \pmod{4}$ if N is even. If $y \in L$ is primitive and $y \neq x$ then there exist arbitrarily large even N such that $x + N \cdot y$ is again primitive. Since the set of all $[y]$ with $y \in L$ primitive is dense in $\mathbb{P}(L_{\mathbb{R}})$, this proves the assertion.

Now let $P \in Q_\Gamma$ be spanned by orthogonal vectors $y_1, y_2 \in \Gamma_{\mathbb{R}}$. Then by what was explained before we can find $x_1 \in \Gamma$ primitive with $x_1^2 \equiv 0 \pmod{4}$ such that $[x_1]$ is arbitrarily close to $[y_1] \in \mathbb{P}(\Gamma_{\mathbb{R}})$. Furthermore, choose $x_2 \in x_1^\perp \subset \Gamma$ primitive and arbitrarily close to $y_2 \in y_1^\perp$ with $x_2^2 \equiv 0 \pmod{4}$ and set $P' := (\mathbb{Z}x_1 \oplus \mathbb{Z}x_2)_{\mathbb{R}}$. Such an element x_2 can be found, as $x_1^\perp \subset \Gamma$ contains a copy of the hyperbolic plane U and thus an element whose square is divisible by four, e.g. $2v + v^*$, where (v, v^*) is a standard basis of U . Then P' is close to P and $(ax_1 + bx_2)^2 = a^2x_1^2 + b^2x_2^2 \equiv 0 \pmod{4}$. \square

We leave it to the reader to modify the above proof to obtain

Corollary 5 *Let Γ be an arbitrary lattice of signature $(3, b-3)$. Then the set of exceptional period points is dense in Q_Γ .* \square

5 Period maps

The aim of this section is to compare the various moduli spaces introduced in Section 3 with the period domains of Section 4 via period maps \mathcal{P}^{cpl} , \mathcal{P} , \mathcal{P}^{met} , $\mathcal{P}^{(2,2)}$, and $\mathcal{P}^{(4,4)}$.

5.1 Definition of the period maps

The period maps we are about to define will fit into the following two commutative diagrams:

$$\begin{array}{ccc}
 \mathcal{P}^{\text{cpl}} : & \mathcal{T}_{\Gamma}^{\text{cpl}} \longrightarrow \text{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}}) \cong Q_{\Gamma} \\
 & \uparrow & \uparrow \\
 \mathcal{P} : & \mathcal{T}_{\Gamma} \longrightarrow \text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \\
 & \downarrow s^2 & \downarrow s^2 \\
 \mathcal{P}^{\text{met}} : & \mathcal{T}_{\Gamma}^{\text{met}} \longrightarrow \text{Gr}_3^{\text{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0}
 \end{array}$$

and

$$\begin{array}{ccc}
 \mathcal{P}^{(2,2)} : & \mathcal{T}_{\Gamma}^{(2,2)} \longrightarrow \text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} \longrightarrow \text{Gr}_{2,2}^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \\
 & \downarrow s^2 & \downarrow s^2 \quad \searrow s^2 \times s^2 \\
 \mathcal{P}^{(4,4)} : & \mathcal{T}_{\Gamma}^{(4,4)} \longrightarrow \text{Gr}_4^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})
 \end{array}$$

The latter should be compatible with the two diagrams

$$\begin{array}{ccc}
 \mathcal{T}_{\Gamma}^{(2,2)} \longrightarrow \mathcal{T}_{\Gamma} & \text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} \longrightarrow \text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \\
 \downarrow & \downarrow & \downarrow \\
 \mathcal{T}_{\Gamma}^{(4,4)} \longrightarrow \mathcal{T}_{\Gamma}^{\text{met}} & \text{Gr}_4^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \longrightarrow \text{Gr}_3^{\text{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0}
 \end{array}$$

and the period maps \mathcal{P} and \mathcal{P}^{met} .

The definition of the maps \mathcal{P} , \mathcal{P}^{met} , \mathcal{P}^{cpl} , $\mathcal{P}^{(2,2)}$, and $\mathcal{P}^{(4,4)}$ is straightforward. Let $(X, \alpha, \varphi) = (M, g, I, \varphi) \in \mathcal{T}_{\Gamma}$ and $B \in H^2(X, \mathbb{R}) = H^2(M, \mathbb{R})$ a B-field. By σ we denote a generator of $H^{2,0}(X)$.

Then we set:

$$\begin{aligned}
\mathcal{P}^{\text{cpl}}(X, \varphi) &= [\varphi(\sigma)] \in Q_\Gamma \subset \mathbb{P}(\Gamma_{\mathbb{C}}) \\
&= \varphi \langle \text{Re}(\sigma), \text{Im}(\sigma) \rangle \in \text{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}}) \\
\mathcal{P}(X, \alpha, \varphi) &= (\mathcal{P}^{\text{cpl}}(X, \varphi), \varphi(\alpha)) \in \text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \\
\mathcal{P}^{\text{met}}(M, g, \varphi) &= (\varphi(H_+^2(M, g)), q(M, g)) \in \text{Gr}_3^{\text{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \\
\mathcal{P}^{(2,2)}(M, g, I, B, \varphi) &= (\mathcal{P}(M, g, I, \varphi), \varphi(B)) \in \text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} \\
\mathcal{P}^{(4,4)}(M, g, B, \varphi) &= (\mathcal{P}^{\text{met}}(M, g, \varphi), \varphi(B)) \in \text{Gr}_3^{\text{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \times \Gamma_{\mathbb{R}} \\
&\cong \text{Gr}_4^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})
\end{aligned}$$

We leave it to the reader to verify that all period maps are $O(\Gamma)$ -equivariant and that one indeed obtains the above commutative diagrams.

Also note that there is a natural $O(\Gamma \oplus U)$ -action on the two period domains $\text{Gr}_{2,2}^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ and $\text{Gr}_4^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$, but the image of $\mathcal{P}^{(2,2)}$ (or its closure) is not left invariant under this action.

5.2 Geometry and period maps

Without going too much into the details we collect in the following some important results about period maps. In particular, we will translate geometric results, like the Global Torelli Theorem into global properties of the period maps.

Local Torelli. *The map $\mathcal{P}^{\text{cpl}} : \mathcal{T}_\Gamma^{\text{cpl}} \rightarrow Q_\Gamma$ is holomorphic and locally (in $\mathcal{T}_\Gamma^{\text{cpl}}$) an isomorphism (cf. [6]).*

Recall that $\mathcal{T}_\Gamma^{\text{cpl}}$ has a natural complex structure, but that the underlying topological space is not Hausdorff. On the other hand, Q_Γ is an open subset of a non-singular quadric in $\mathbb{P}(\Gamma_{\mathbb{C}})$ and, therefore, a nice complex manifold.

Of course, the Local Torelli Theorem in the above version immediately carries over to the other period maps. Thus, \mathcal{P} , \mathcal{P}^{met} , $\mathcal{P}^{(2,2)}$, and $\mathcal{P}^{(4,4)}$ are all locally injective. Since the Teichmüller spaces \mathcal{T}_Γ , $\mathcal{T}_\Gamma^{\text{met}}$, $\mathcal{T}_\Gamma^{(2,2)}$, and $\mathcal{T}^{(4,4)}$ are all Hausdorff, this shows that except \mathcal{P}^{cpl} all period maps define covering maps on their open images.

Twistor lines. *Under the period map \mathcal{P}^{cpl} the twistor line $\mathbb{P}^1 = c(m^{-1}(M, g, \varphi)) \subset \mathcal{T}_\Gamma^{\text{cpl}}$ (cf. Proposition 3.9) is identified with a quadric in some linear subspace $\mathbb{P}^2 \subset \mathbb{P}(\Gamma_{\mathbb{C}})$.*

Indeed, the \mathbb{P}^2 is given as $\mathbb{P}(\varphi(H_+^2(M, g)_\mathbb{C})) \subset \mathbb{P}(\Gamma_\mathbb{C})$.

Surjectivity of the period map. *The map $\mathcal{P}^{\text{cpl}} : \mathcal{T}_\Gamma^{\text{cpl}} \rightarrow Q_\Gamma$ maps every connected component of $\mathcal{T}_\Gamma^{\text{cpl}}$ onto Q_Γ (cf. [24]).*

Analogous statements for the other period maps do not hold. In these cases the assertion has to be modified. To see this let us look at the fibres of $\mathcal{T}_\Gamma \rightarrow \mathcal{T}_\Gamma^{\text{cpl}}$ over (X, φ) . By definition of \mathcal{T}_Γ this is the Kähler cone \mathcal{K}_X which, via the period map \mathcal{P} , is identified with an open subcone of the positive cone $\mathcal{C}_{\mathcal{P}^{\text{cpl}}(X, \varphi)}$ which is just one of the two connected components of the fibre of $\text{Gr}_{2,1}^{\text{po}}(\Gamma_\mathbb{R}) \rightarrow Q_\Gamma$ over $\mathcal{P}^{\text{cpl}}(X, \varphi)$. For a very general marked IHS $(X, \varphi) \in \mathcal{T}_\Gamma^{\text{cpl}}$ the Kähler cone \mathcal{K}_X is maximal, i.e. $\mathcal{K}_X = \mathcal{C}_X$. Thus, for those points \mathcal{P} maps the fibre of $\mathcal{T}_\Gamma \rightarrow \mathcal{T}_\Gamma^{\text{cpl}}$ bijectively onto one of the connected components \mathcal{C}_P or $-\mathcal{C}_P$ of the fibre of $\text{Gr}_{2,1}^{\text{po}}(\Gamma_\mathbb{R}) \rightarrow Q_\Gamma$ over $P = \mathcal{P}^{\text{cpl}}(X, \varphi)$. For special marked IHS (X, φ) , which usually (e.g. for K3 surfaces) nevertheless form a dense subset of $\mathcal{T}_\Gamma^{\text{cpl}}$, the Kähler cone is strictly smaller.

Density of the image. *The image of every connected component of \mathcal{T}_Γ under the period map \mathcal{P} is dense in the connected component of the period domain $\text{Gr}_{2,1}^{\text{po}}(\Gamma_\mathbb{R})$ containing it. Analogous statements hold true for \mathcal{P}^{met} , $\mathcal{P}^{(2,2)}$, and $\mathcal{P}^{(4,4)}$.*

Let us say a few words about how the density is proved and how the boundary $\text{Gr}_{2,1}^{\text{po}}(\Gamma_\mathbb{R}) \setminus \mathcal{P}(\mathcal{T}_\Gamma)$ can be interpreted.

Since \mathcal{P}^{cpl} is surjective, we may consider $(X, \varphi) \in \mathcal{T}_\Gamma^{\text{cpl}}$ and study the fibre of $\text{Gr}_{2,1}^{\text{po}}(\Gamma_\mathbb{R}) \rightarrow \text{Gr}_2^{\text{po}}(\Gamma_\mathbb{R}) \cong Q_\Gamma$ over $\mathcal{P}^{\text{cpl}}(X, \varphi)$, which is $\pm\varphi(\mathcal{C}_X)$. The \pm -sign distinguishes the two connected components of $\text{Gr}_{2,1}^{\text{po}}(\Gamma_\mathbb{R})$. The image of the fibre $\mathcal{T}_\Gamma \rightarrow \mathcal{T}_\Gamma^{\text{cpl}}$ over (X, φ) is the open subcone $\varphi(\mathcal{K}_X) \subset \varphi(\mathcal{C}_X)$. We will discuss its boundary and its complement: If $\alpha \in \mathcal{C}_X$ is general, then there exists $(X', \varphi') \in \mathcal{T}_\Gamma^{\text{cpl}}$ which cannot be separated from (X, φ) such that $\mathcal{P}(X, \varphi) = \mathcal{P}(X', \varphi')$ and $\varphi(\alpha) \in \varphi'(\mathcal{K}_{X'})$ (see [20]). (Moreover, X and X' are birational.) Thus, the disjoint union $\bigcup \varphi(\mathcal{K}_X)$ over all (X, φ) in the same connected component and with the same period $\mathcal{P}(X, \varphi) \in Q_\Gamma$ is dense in $\varphi(\mathcal{C}_X)$.

For a point $\alpha \in \partial\varphi(\mathcal{K}_X)$ there always exists a rational curve $C \subset X$ with $\int_C \alpha = 0$ (see [10]), i.e. under the degenerate Kähler structure α the volume of the rational curve C shrinks to zero. Thus, points in the boundary

of $\mathcal{P}(\mathcal{K}_X)$ should be thought of as singular IHS/HK which are obtained by contracting certain rational curves. Unfortunately, neither are we able to make this statement more precise nor do we know that any point $\alpha \in \varphi(\mathcal{C}_X)$ is actually contained in the closure of some $\varphi'(\mathcal{K}_{X'})$, where (X', φ') is as above. However, for K3 surfaces the situation is much better understood (cf. [28]).

Projective IHS. *The set of projective marked IHS forms a countable dense union of hyperplane section of Q_Γ . If $\mathcal{M}_\Gamma^{\text{proj}} \subset \mathcal{M}_\Gamma$ denotes the set of all Kähler IHS for which the underlying IHS is projective, then $\mathcal{M}_\Gamma^{\text{proj}} \rightarrow \mathcal{M}_\Gamma^{\text{met}}$ is surjective.*

In fact, due to a general projectivity criterion for surfaces and an analogous result for IHS (cf. [20]) one knows that an IHS X is projective if and only if there exists an integral $(1, 1)$ -class α with $q(\alpha) > 0$. Thus, $(X, \varphi) \in \mathcal{T}_\Gamma^{\text{cpl}}$ is projective if and only if $\mathcal{P}^{\text{cpl}}(X, \varphi)$ is contained in a hyperplane orthogonal to some $\alpha \in \Gamma$ with $\alpha^2 > 0$. As we have seen before, the set of such periods is dense in the period domain Q_Γ . Since the fibre of $\mathcal{T}_\Gamma \rightarrow \mathcal{T}_\Gamma^{\text{met}}$ is identified with a quadric curve $\mathbb{P}^1 \subset \mathbb{P}(\Gamma_{\mathbb{C}})$ under the projection $\mathcal{T}_\Gamma \rightarrow \mathcal{T}_\Gamma^{\text{cpl}}$ and as such is intersected non-trivially by every such hyperplane, the fibre contains at least one Kähler (X, α, φ) with X projective. In other words, for any hyperkähler metric g on a manifold M at least one of the complex structures $\lambda = aI + bJ + cK$ defines a projective IHS. In fact, the set of projective IHS is also dense among the (M, λ) .

Finiteness. *The induced period maps*

$$\begin{aligned} \overline{\mathcal{P}} : \quad \mathcal{M}_\Gamma &\rightarrow \text{O}(\Gamma) \setminus \text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \\ \overline{\mathcal{P}}^{\text{met}} : \quad \mathcal{M}_\Gamma^{\text{met}} &\rightarrow \text{O}(\Gamma) \setminus \text{Gr}_3^{\text{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \\ &\cong \text{O}(\Gamma) \setminus \text{O}(3, b-3)/\text{SO}(3) \times \text{O}(b-3) \times \mathbb{R}_{>0} \end{aligned}$$

are finite trivial covers of their images, i.e. every moduli space has only finitely many connected components and each connected component is mapped bijectively onto its image.

The same holds for the period map

$$\overline{\mathcal{P}}^{\text{cpl}} : \mathcal{M}_\Gamma^{\text{cpl}} \rightarrow \text{O}(\Gamma) \setminus Q_\Gamma \cong \text{O}(\Gamma) \setminus \text{O}(3, b-3)/\text{SO}(2) \times \text{O}(1, b-3)$$

except for non-Hausdorff points in the fibers.

Note that e.g. $\mathcal{T}_\Gamma^{\text{cpl}}$ might *a priori* have infinitely many components. That this is no longer possible for the quotient $\mathcal{M}_\Gamma^{\text{cpl}} = \text{O}(\Gamma) \backslash \mathcal{T}_\Gamma^{\text{cpl}}$ is a consequence of the finiteness result in [26, Thm. 4.3] which says that there are only finitely many different deformation types of IHS with the same BB-form q_X . Since Q_Γ is simply connected and \mathcal{P}^{cpl} is surjective, the cover \mathcal{P}^{cpl} has to be trivial. In fact, in order to make this precise one first should construct the ‘Hausdorff reduction’ of $\mathcal{T}_\Gamma^{\text{cpl}}$ by identifying all points that cannot be separated from each other. This Hausdorff space then is an honest étale cover of the simply connected space Q_Γ and, therefore, consists of several copies of Q_Γ .

We leave it to the reader to deduce similar statements for the maps $\mathcal{P}^{(2,2)}$ and $\mathcal{P}^{(4,4)}$.

Remark 5.1 This is essentially all that is known in the general case. For **K3 surfaces** however the above results can be strengthened considerably as follows. The Global Torelli for K3 surfaces shows that $\mathcal{T}_\Gamma^{\text{cpl}}$ consists of two connected components which are identified with each other by $(X, \varphi) \mapsto (X, -\varphi)$ and which are not distinguished by \mathcal{P}^{cpl} . The two components are separated by the map $\mathcal{P} : \mathcal{T}_\Gamma \rightarrow \text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}})$, which is injective in the case of K3 surfaces. Analogously, \mathcal{P}^{met} , $\mathcal{P}^{(2,2)}$, and $\mathcal{P}^{(4,4)}$ are all injective.

The density results of Section 4.8 together with the description of the periods of our list of examples of K3 surfaces in Section 2 and the above information about the period maps (i.e. the Global Torelli Theorem) yield:

Proposition 5.2 *The following three sets are dense in the moduli space of marked K3 surfaces:* i) $\{(X, \varphi) \mid X \subset \mathbb{P}^3 \text{ is a quartic hypersurface}\}$,
ii) $\{(X, \varphi) \mid X \text{ is an elliptic K3 surface}\}$, and
iii) $\{(X, \varphi) \mid X \text{ is a(n exceptional) Kummer surface}\}$. \square

5.3 The diffeomorphism group of a K3 surface

Proposition 5.3 *Let X be a K3 surface. The image of the natural map $\rho : \text{Diff}(X) \rightarrow \text{O}(H^2(X, \mathbb{Z}), \cup)$ is the subgroup $\text{O}^+(H^2(X, \mathbb{Z}), \cup)$, which is of index two.*

Recall (cf. Section 4.7) that O^+ is the group of all $A \in \text{O}$ that preserve the orientation of positive three-space (but not necessarily of a negative 19-space). The proposition is due to Borcea [11], who showed the inclusion

$O^+ \subset \text{Im}(\rho)$, and Donaldson [17], who showed equality. We only reproduce Borcea's argument here.

Proof. First note the following. If (X_t, φ_t) is a connected path in $\mathcal{T}_\Gamma^{\text{cpl}}$, then there exists a sequence of diffeomorphisms $f_t : X_0 \cong X_t$ such that $\varphi_0 \circ f^* = \varphi_t$.

Let now φ be any marking of X and consider $(X, \varphi) \in \mathcal{T}_\Gamma^{\text{cpl}}$. By \mathcal{T}_0 we denote the connected component of $\mathcal{T}_\Gamma^{\text{cpl}}$ that contains this point. Pick $A \in O^+(H^2(X, \mathbb{Z}), \cup)$. Then A acts on $\mathcal{T}_\Gamma^{\text{cpl}}$ and Q_Γ by $\varphi A \varphi^{-1}$ and the period map $\mathcal{P}^{\text{cpl}} : \mathcal{T}_\Gamma^{\text{cpl}} \rightarrow Q_\Gamma$ is equivariant. Since the restriction of the period map \mathcal{P}^{cpl} yields a surjective map $\mathcal{T}_0 \rightarrow Q_\Gamma$, there exists a marked K3 surface (X', φ') with $\mathcal{P}^{\text{cpl}}(X', \varphi') = A\mathcal{P}^{\text{cpl}}(X, \varphi) = \mathcal{P}^{\text{cpl}}(X, A\varphi)$. If X is a general K3 surface such that $\mathcal{K}_X \cong \mathcal{C}_X$, then $\pm\varphi'^{-1} \circ (\varphi A) : H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z})$ is an isomorphism of periods mapping \mathcal{K}_X to $\mathcal{K}_{X'}$. By the Global Torelli Theorem there exists a (unique) isomorphism $g : X' \cong X$ such that $g^* = \pm\varphi'^{-1} \circ (\varphi A)$. By the remark above we also find a diffeomorphism $f : X \cong X'$ such that $\varphi \circ f^* = \varphi'$. Hence, $\varphi \circ f^* g^* = \pm(\varphi A)$ and thus $(g \circ f)^* = \pm A$ is realized by a diffeomorphism of X . In fact, the sign must be "+", as g^* , f^* , and A preserve the orientation of a positive three-space.

It remains to show that $-\text{id}$ is not contained in the image and this was done by Donaldson using zero-dimensional moduli spaces of stable bundles on a double cover of the projective plane. \square

Remark 5.4 In the proof above we used the assumption that $n = 1$ twice: When we applied the Global Torelli Theorem and, of course, when using Donaldson invariants. The surjectivity which is also crucial holds true also for $n > 1$. Somehow, the use of the Global Torelli Theorem seems a little strong, as we have no need to know that g^* is induced by a biholomorphic map, a diffeomorphism would be enough.

In [36] Namikawa constructs an example of two four-dimensional IHS X and X' together with an isomorphism of their periods which preserves the Kähler cone, but such that X and X' are not even birational. To be more precise, he considers generalized Kummer varieties $X = K_2(T)$ and $X' = K_2(T^*)$ associated to a complex torus T and its dual T^* . As the moduli space of complex tori is connected, one can endow X and X' with markings φ respectively φ' such that (X, φ) and (X', φ') are contained in the same connected component \mathcal{T}_0 of $\mathcal{T}_\Gamma^{\text{cpl}}$. His example shows that $O^+(\Gamma)$ does not preserve \mathcal{T}_0 , i.e. there exists $A \in O^+$ such that $(X', A\varphi') \notin \mathcal{T}_0$ (with

$\mathcal{P}(X', A\varphi') = \mathcal{P}(X, \varphi)$). Indeed, after identifying non-separated points in $\mathcal{T}_\Gamma^{\text{cpl}}$ the period map $\mathcal{P}^{\text{cpl}} : \mathcal{T}_\Gamma^{\text{cpl}} \rightarrow Q_\Gamma$ is a covering and thus, since Q_Γ is simply connected, every connected component \mathcal{T}_0 of $\mathcal{T}_\Gamma^{\text{cpl}}$ is generically mapped one-to-one onto Q_Γ .

5.4 (Derived) Global Torelli Theorem

Before discussing the action of $O(\Gamma \oplus U)$ from the mirror symmetry point of view we shall explain that a derived version of the Global Torelli Theorem can be formulated by means of this action.

First, we reformulate the classical Global Torelli Theorem for K3 surfaces as follows:

Theorem 5.5 *Let X and X' be two K3 surfaces. Then X and X' are isomorphic if and only if their images $\mathcal{P}^{\text{cpl}}(X, \varphi)$ and $\mathcal{P}^{\text{cpl}}(X', \varphi')$ are contained in the same $O(\Gamma)$ -orbit in Q_Γ .*

(Of course, the choice of φ and φ' does not matter.)

In order to formulate the derived version of this, which consists in weakening the isomorphism of X and X' to an equivalence of their derived categories, we need to complete the picture of the various period maps as follows.

The diagram in Section 5.1 can be enriched by adding a moduli space that naturally contains the complex moduli space $\mathcal{T}_\Gamma^{\text{cpl}}$ and the complex period domain $\text{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}})$ such that the group $O(\Gamma \oplus U)$ acts naturally on the latter. We introduce the commutative diagram:

$$\begin{array}{ccc} \text{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}^C & \xrightarrow{\delta} & \text{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \\ \uparrow & & \uparrow \pi \\ \text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}^C & \xrightarrow{\gamma} & \text{Gr}_{2,2}^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \end{array}$$

Here, π is the projection $(H_1, H_2) \mapsto H_1$ and δ is given by $\delta : (P, B) \mapsto \{x - \langle x, B \rangle w \mid x \in P\}$. This obviously yields the above commutative diagram. Moreover, π is equivariant with respect to the natural $O(\Gamma \oplus U)$ -action on both spaces. But note that ι and $\tilde{\xi} = \iota \circ \xi$ do not descend to $\text{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$.

Choosing a vanishing B-field for any marked K3 surface (X, φ) yields a map $\mathcal{T}_\Gamma^{\text{cpl}} \rightarrow \text{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}}) \xrightarrow{\delta} \text{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$.

Analogously to the discussion of the embedding γ in Section 4.5, one finds that the image of δ is not invariant under the $O(\Gamma \oplus U)$ -action, but

it might of course happen that the image of a marked K3 surface (X, φ) under some $\psi \in O(\Gamma \oplus U) \setminus O(\Gamma)$ is mapped to the period of another K3 surface (X', φ') . At least for algebraic K3 surfaces, when this happens can be explained in terms of derived categories. This is due to a result of Orlov [38] which is based on [31].

Theorem 5.6 *Two algebraic K3 surfaces X and X' have equivalent derived categories*

$$D^b(\mathrm{Coh}(X)) \text{ and } D^b(\mathrm{Coh}(X'))$$

if and only if their images $\delta\mathcal{P}(X, \varphi)$ and $\delta\mathcal{P}(X', \varphi')$ are contained in the same $O(\Gamma \oplus U)$ -orbit in $\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$. \square

(Again, the choice of the markings φ and φ' is inessential.)

Remark 5.7 There is a conjecture that generalizes the above results to K3 surfaces with rational B-field $B \in H^2(X, \mathbb{Q})$. The derived categories in this case have to be replaced by twisted derived categories, where one derives the abelian category of coherent sheaves over an Azumaya algebra (cf. [13]).

The following result due to Hosono, Lian, Oguiso, Yau [23] and independently to Ploog [39] should be regarded as an analogue of the fact that the image of $\mathrm{Aut}(X) \rightarrow O^+(H^2(X, \mathbb{Z}))$ is the set of Hodge isometries. At the same time it is ‘mirror’ to the result of Borcea discussed above.

Theorem 5.8 *Let X be a projective K3 surface. Then the image of*

$$\mathrm{Autequ}(D^b(\mathrm{Coh}(X)) \rightarrow O(H^*(X, \mathbb{Z})))$$

contains the set of Hodge isometries contained in O^+ .

Here the Hodge structure on $H^*(X, \mathbb{Z})$ is a weight-two Hodge structure given by $H^{2,0}(X) \subset H^*(X, \mathbb{C})$. As had been pointed out by Szendrői in [42], mirror symmetry suggests that the image should be exactly O^+ . This would be the analogue of Donaldson’s result.

6 Discrete group actions

All spaces considered in Section 4 are quotients either of $O(\Gamma_{\mathbb{R}})$ or $O(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$. So from a mathematical point of view it seems very natural to study the action of the discrete groups $O(\Gamma)$ respectively $O(\Gamma \oplus U)$ on these spaces. In fact, in order to obtain moduli spaces of unmarked (complex) HK or (kähler) IHS with or without B-fields. one has to divide out by a smaller group. But in [4] it is argued that dividing out $T_{\Gamma}^{(4,4)}$ or $T_{\Gamma}^{(2,2)}$ by $O(\Gamma \oplus U)$ yields the true moduli space of CFTs on K3 surfaces. In order to recover the full symmetry of the situation they proceed as follows:

- i) **Maximal discrete subgroups.** Find a discrete group G that acts on a certain moduli space of relevant theories and show that it is maximal in the sense that any bigger group would no longer act properly discontinuously. (Recall that the quotient of a properly discontinuous group action is Hausdorff.)
- ii) **Geometric symmetries.** Describe the part of G (the geometric symmetries) that identifies geometrically identical theories and the part that is responsible for trivial identifications (e.g. integral shifts of the B-field).
- iii) **Mirror symmetries.** Show that G is generated by the symmetries in ii) and a few others that are responsible for mirror symmetry phenomena.

6.1 Maximal discrete subgroups

We first recall the following facts:

- Let G be a topological group which is Hausdorff and locally compact. If $K \subset G$ is a compact subgroup then any other subgroup H acts properly discontinuously from the left on the quotient space G/K if and only if $H \subset G$ is a discrete subgroup. (For the elementary proof see e.g. [46, Lemma 3.1.1].)
- Let L be a non-trivial definite even unimodular lattice and let $q \geq 3$. Then $O(L \oplus U^{\oplus q}) \subset O(L_{\mathbb{R}} \oplus U_{\mathbb{R}}^{\oplus q})$ is a maximal discrete subgroup (cf. [2]).

The second result in particular applies to the K3 surface lattice $\Gamma = 2(-E_8) \oplus 3U$ and yields that $O(\Gamma) \subset O(\Gamma_{\mathbb{R}})$ and $O(\Gamma \oplus U) \subset O(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ are both maximal discrete subgroups.

The group $O(\Gamma)$ acts on $\text{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}})$ and $\text{Gr}_3^{\text{po}}(\Gamma_{\mathbb{R}})$. As we have seen

$$\text{Gr}_2^{\text{po}}(\Gamma_{\mathbb{R}}) \cong O(3, 19)/SO(2) \times O(1, 19) \text{ and } \text{Gr}_3^{\text{po}}(\Gamma_{\mathbb{R}}) \cong O(3, 19)/SO(3) \times O(19).$$

In the second case we are in the above situation, i.e. the quotient is taken with respect to the compact subgroup $\mathrm{SO}(3) \times \mathrm{O}(19)$. Hence, $\mathrm{O}(\Gamma)$ acts properly discontinuously on $\mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}})$ and there is no bigger subgroup of $\mathrm{O}(\Gamma_{\mathbb{R}})$ than $\mathrm{O}(\Gamma)$ with the same property. However, the action of $\mathrm{O}(\Gamma)$ on $\mathrm{Gr}_2^{\mathrm{po}}(\Gamma_{\mathbb{R}})$ is badly behaved, as the subgroup $\mathrm{SO}(2) \times \mathrm{O}(1, 19)$ is not compact. In fact, in the proof of Proposition 4.8 we have already seen that the action of $\mathrm{O}(\Gamma)$ is not properly discontinuous.

We are more interested in the action of $\mathrm{O}(\Gamma \oplus U)$ on $\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) \cong \mathrm{O}(4, 20)/(\mathrm{SO}(4) \times \mathrm{O}(20))$. Again $\mathrm{O}(\Gamma \oplus U)$ is maximal discrete and $\mathrm{SO}(4) \times \mathrm{O}(20)$ is compact. Hence, there is no bigger properly discontinuous subgroup action on $\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$. Analogously, one finds that $\mathrm{O}(\Gamma \oplus U)$ is a maximal discrete subgroup of $\mathrm{O}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ acting properly discontinuously on $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$.

Presumably, all these arguments also apply to any HK manifold, but details need to be checked. (Recall that $(H^2(X, \mathbb{Z}), q_X)$ is not necessarily unimodular in higher dimensions.)

6.2 Geometric symmetries

We will try to identify “geometric” symmetries and integral shifts of the B-field inside $\mathrm{O}(\Gamma \oplus U)$. To this end we use the identification

$$\phi : \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \times \Gamma_{\mathbb{R}} \cong \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$$

described in Section 4.4.

The natural inclusion $\mathrm{O}(\Gamma) \subset \mathrm{O}(\Gamma \oplus U)$ is compatible with this isomorphism, i.e. if $\Pi = \phi(F, \alpha, B)$ and $\varphi \in \mathrm{O}(\Gamma) \subset \mathrm{O}(\Gamma \oplus U)$, then $\varphi(\Pi) = \phi(\varphi(F), \alpha, \varphi(B))$. This is a straightforward calculation which we leave to the reader. Clearly, $\mathrm{O}(\Gamma)$ acts naturally on all spaces \mathcal{T} , $\mathcal{T}^{\mathrm{met}}$, $\mathcal{T}^{\mathrm{cpl}}$, $\mathcal{T}^{(2,2)}$, and $\mathcal{T}^{(4,4)}$ and the period maps are equivariant. Thus, $\mathrm{O}(\Gamma)$ is the subgroup that identifies geometrically equivalent theories.

Next let $B_0 \in \Gamma$ and let $\varphi_{B_0} \in \mathrm{O}(\Gamma \oplus U)$ be the automorphism $w \mapsto w$, $w^* \mapsto B_0 + w^* - (B_0^2/2)w$, and $x \in \Gamma \mapsto x - \langle B_0, x \rangle w$. One easily verifies that this really defines an isometry. We claim that if $\Pi = \phi(F, \alpha, B)$, then $\varphi_{B_0}(\Pi) = \phi(F, \alpha, B + B_0)$.

In order to do this let us more generally consider an element $\varphi \in \mathrm{O}(\Gamma \oplus U)$ such that $\varphi(w) = w$. For $\Pi \in \mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$, let $\tilde{\Pi} := \varphi(\Pi)$. Then $\tilde{F}' = \tilde{\Pi} \cap w^{\perp} = \varphi(\Pi) \cap \varphi(w)^{\perp} = \varphi(\Pi \cap w^{\perp}) = \varphi(F')$. Moreover, one has the two orthogonal splittings $\tilde{\Pi} = \tilde{B}' \mathbb{R} \oplus \tilde{F}'$ and $\tilde{\Pi} = \varphi(B') \mathbb{R} \oplus \varphi(F')$, where \tilde{B}' is

determined by $\langle \tilde{B}', w \rangle = 1$. Since $\langle \varphi(B'), w \rangle = \langle \varphi(B'), \varphi(w) \rangle = \langle B', w \rangle = 1$, one concludes $\tilde{B}' = \varphi(B')$. In particular, $\tilde{B}'^2 = B'^2$. The B-field B is given by $B' = \alpha w + w^* + B$. Hence, $\tilde{B}' = \alpha w + \varphi(w^*) + \varphi(B)$ and thus the B-field determined by \tilde{B}' is nothing but $\varphi(B)$.

All this applied to $\varphi = \varphi_{B_0}$ one finds that under the isomorphism $\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}}) = \mathrm{Gr}_3^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \mathbb{R}_{>0} \times \Gamma_{\mathbb{R}}$ the integral B-shift by B_0 that maps (F, α, B) to $(F, \alpha, B + B_0)$ corresponds to φ_{B_0} .

We leave it to the reader to verify that also the $\mathrm{O}(\Gamma \oplus U)$ -action on $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ is well-behaved in the sense that $\mathrm{O}(\Gamma) \subset \mathrm{O}(\Gamma \oplus U)$ and the maps φ_{B_0} for $B_0 \in \Gamma$ act on the subspace $\gamma(\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}) \subset \mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ in the natural way.

6.3 Mirror symmetries

The next result (due to C. T. C. Wall, [45]) explains which additional group elements have to be added in order to pass from $\mathrm{O}(\Gamma)$ to $\mathrm{O}(\Gamma \oplus U)$.

Proposition 6.1 *Let Γ be a unimodular lattice of index (k, ℓ) with $k, \ell \geq 2$. Then $\mathrm{O}(\Gamma \oplus U)$ is generated by the following three subgroups:*

$$\mathrm{O}(\Gamma), \quad \mathrm{O}(U), \quad \text{and} \quad \{\varphi_{B_0} \mid B_0 \in \Gamma\}.$$

□

Thus, the result applies to the K3 surface lattice $2(-E_8) \oplus 3U$, but presumably something similar can be said for the case of the lattice $2(-E_8) \oplus 3U \oplus 2(n-1)\mathbb{Z}$, which is realized by the Hilbert scheme of a K3 surface.

In [4] passing from $\mathrm{O}(\Gamma)$ to $\mathrm{O}(\Gamma \oplus U)$ is motivated on the base of physical insight. As usual in mathematical papers on mirror symmetry we will take this for granted and rather study the effects of these additional symmetries in geometrical terms. Thus, the rest of this paragraph is devoted to the study a few special elements of $\mathrm{O}(\Gamma \oplus U)$ that are not contained in the subgroup generated by $\mathrm{O}(\Gamma)$ and $\{\varphi_{B_0} \mid B_0 \in \Gamma\}$. In particular, we will be interested in their induced action on $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$.

So far we have argued that $\mathrm{O}(\Gamma \oplus U)$ is a maximal discrete subgroup of $\mathrm{O}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ that acts on the two period spaces that interest us: $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ and $\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$. However, there seems to be a bigger group which naturally and properly discontinuously acts on the space $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ (which thus cannot be realized as a subgroup of $\mathrm{O}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$).

Definition 6.2 *The group $\tilde{O}(\Gamma \oplus U)$ is the group acting on $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ which is generated by $O(\Gamma \oplus U)$ and the involution $\iota : (H_1, H_2) \mapsto (H_2, H_1)$.*

Here \bar{H} is the space H with the opposite orientation. Note that one could actually go further and consider the maps $(H_1, H_2) \mapsto (H_1, \bar{H}_2)$ or $(H_1, H_2) \mapsto (\bar{H}_1, H_2)$. However, for the versions of mirror symmetry that will be discussed in these lectures ι will do.

Before turning to the mirror map that interests us most in Section 6.4 let us discuss a few more elementary cases:

$-\mathbf{id}_{\mathbf{U}}$

Consider the automorphism $\psi_0 \in O(\Gamma \oplus U)$ that acts trivially on Γ and as $-\mathbf{id}$ on U .

Lemma 6.3 *The automorphism ψ_0 preserves the subspace $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ and acts on it by*

$$((P, \omega), B) \mapsto ((P, -\omega), -B).$$

Proof. If $(H_1, H_2) = \gamma((P, \omega), B)$, then by definition of ψ_0 :

$$\psi_0(H_1) = \{x + \langle x, B \rangle w \mid x \in P\} = \{x - \langle x, (-B) \rangle w \mid x \in P\}$$

and

$$\begin{aligned} \psi_0(H_2) &= \left(\frac{1}{2}(\alpha - B^2)(-w) - w^* + B\right)\mathbb{R} \oplus (\omega + \langle \omega, B \rangle w)\mathbb{R} \\ &= -\left(\frac{1}{2}(\alpha - (-B)^2)w + w^* - B\right)\mathbb{R} \oplus (\omega - \langle \omega, (-B) \rangle w)\mathbb{R} \end{aligned}$$

Thus, the sign of ω has to be changed in order to get the correct orientation $\psi_0(H_2)$. \square

$\mathbf{w} \leftrightarrow \mathbf{w}^*$

Consider the automorphism $\psi_1 \in O(\Gamma \oplus U)$ that acts trivially on Γ and by $\psi_1(w) = w^*$, $\psi_1(w^*) = w$ on U .

Lemma 6.4 *The automorphism ψ_1 preserves the subspace $\{((P, \omega), B) \mid B \in (P, \omega)^\perp, \alpha \neq B^2\}$ of $\text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ and acts on it by*

$$((P, \omega), B) \mapsto \frac{2}{\alpha - B^2}((P, \omega), B).$$

Proof. Indeed, by definition of ψ_1 one has $\psi_1(H_1) = H_1$ and

$$\begin{aligned} \psi_1(H_2) &= \left(\frac{1}{2}(\alpha - B^2)w^* + w + B \right) \mathbb{R} \oplus (\omega - \langle \omega, B \rangle w^*) \mathbb{R} \\ &= \left(w^* + \frac{2}{\alpha - B^2}w + \frac{2}{\alpha - B^2}B \right) \mathbb{R} \oplus \left(\frac{2}{\alpha - B^2}\omega \right) \mathbb{R} \end{aligned}$$

Then check that for $\tilde{\omega} := \frac{2}{\alpha - B^2}\omega$ and $\tilde{B} := \frac{2}{\alpha - B^2}B$ one indeed has $\frac{2}{\alpha - B^2} = \frac{1}{2}(\tilde{\omega}^2 - \tilde{B}^2)$. \square

It is interesting to observe that on the yet smaller subspace $\{((P, \omega), 0)\}$ the automorphism ψ_1 acts by $(P, \omega) \mapsto \frac{2}{\omega^2}(P, \omega)$. In the geometric context this will be interpreted as inversion of the volume or, in physical language, T-duality.

Remark 6.5 Nahm and Wendland argue that $w \leftrightarrow w^*$ occurs as an automorphism of the orbifold $(2, 2)$ -SCFT associated to a Kummer surface. Thus, it has to be added as a global symmetry to the subgroup $\langle O(\Gamma), \{\varphi_B \mid B \in \Gamma\} \rangle$. Due to the result of Wall, one thus obtains the full $O(\Gamma \oplus U)$ -action on $\text{Gr}_4^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$.

Note that in the original argument Aspinwall and Morrison had used another additional symmetry. Writing $\Gamma \oplus U = (-E_8 \oplus 2U) \oplus (-E_8 \oplus 2U)$ allows one to consider the interchange of the two summands $(-E_8 \oplus 2U) \leftrightarrow (-E_8 \oplus 2U)$ as an element in $O(\Gamma \oplus U)$. This additional automorphism, which together with $\langle O(\Gamma), \{\varphi_B \mid B \in \Gamma\} \rangle$ also generates the whole $O(\Gamma \oplus U)$ -action on $\text{Gr}_4^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$, is realized as an automorphism of a certain Gepner model. For the details of both approaches we have to refer to the original articles.

6.4 The mirror map $U \leftrightarrow U'$

If the lattice can be written as $\Gamma = \Gamma' \oplus U'$, where U' is a copy of the hyperbolic plane U , then by Wall's result Proposition 6.1 the group $\tilde{O}(\Gamma \oplus U)$ is generated by $O(\Gamma)$, $\{\varphi_{B_0} \mid B_0 \in \Gamma\}$, the involution ι , and $\xi \in O(\Gamma \oplus$

U) which is the identity on Γ' and switches U and U' . Here we use an isomorphism $U \cong U'$ which we fix once and for all. We consider $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ as a subspace of $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ via the injection γ .

Neither ι nor ξ leave the subspace $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ invariant. Indeed, if $((P, \omega), B)$ then $H_1 \subset \Gamma_{\mathbb{R}} \oplus \mathbb{R}w$ and $H_2 \not\subset \Gamma_{\mathbb{R}} \oplus \mathbb{R}w$ and therefore $(H_2, H_1) = \iota(H_1, H_2)$ cannot be contained in the image of γ . Similarly, for a general (H_1, H_2) the pair of planes $(\xi(H_1), \xi(H_2))$ will not satisfy $\xi(H_1) \subset \Gamma_{\mathbb{R}} \oplus \mathbb{R}w$.

Definition 6.6 $\tilde{\xi} := \iota \circ \xi \in \tilde{O}(\Gamma \oplus U)$.

By definition, $\tilde{\xi}$ acts naturally on $\mathrm{Gr}_{2,2}^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ and $\mathrm{Gr}_4^{\mathrm{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$. The action on the latter coincides with the action of ξ . We will show that $\tilde{\xi}$ can be used to identify certain subspaces of $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$, but the whole $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ will again not be invariant. Maybe it is worth emphasizing that $\tilde{\xi}$ is an involution. Indeed, ι commutes with the action of $O(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ and both transformations ι and ξ are of order two.

Note that different decompositions $\Gamma = \Gamma' \oplus U'$ yield different ξ , which then relate different pairs of subspaces of $\mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$. The following easy lemma shows that we dispose of such a decomposition whenever we find a hyperbolic plane contained in Γ .

Lemma 6.7 *If U' is a hyperbolic plane contained in a lattice Γ , then $\Gamma = U'^{\perp} \oplus U'$.*

Proof. Choose a basis (v, v^*) of U' that corresponds to the basis (w, w^*) of U under the identification $U' \cong U$. Furthermore, let $\Gamma' := U'^{\perp}$ and let V be the subspace of the \mathbb{Q} -vector space $\Gamma_{\mathbb{Q}}$ that is orthogonal to $U'_{\mathbb{Q}}$. Thus, $\Gamma_{\mathbb{Q}} = V \oplus U'_{\mathbb{Q}}$. Clearly, $\Gamma' \subset V$ and, conversely, for any $v \in V$ there exists $\lambda \in \mathbb{Q}^*$ with $\lambda v \in V \cap \Gamma \subset \Gamma'$. Hence, $V = \Gamma'_{\mathbb{Q}}$. Let $x \in \Gamma$ and write $x = y + (\lambda v + \mu v^*)$ with $y \in V$ and $\lambda, \mu \in \mathbb{Q}$. Then $\langle x, v \rangle, \langle x, v^* \rangle \in \mathbb{Z}$ implies $\lambda, \mu \in \mathbb{Z}$ and, therefore, $y = x - (\lambda v + \mu v^*) \in \Gamma \cap V = \Gamma'$. Thus, $\Gamma = \Gamma' \oplus U'$. \square

For the rest of this section we fix the orthogonal splitting $\Gamma = \Gamma' \oplus U'$ together with an identification $U' = U$. By $\mathrm{pr} : \Gamma_{\mathbb{R}} \rightarrow \Gamma'_{\mathbb{R}}$ we denote the orthogonal projection.

Proposition 6.8 *Let $((P, \omega), B) \in \mathrm{Gr}_{2,1}^{\mathrm{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ such that $\omega, B \in \Gamma'_{\mathbb{R}} \oplus \mathbb{R}v$. Then the $\tilde{\xi}$ -mirror image $((P^v, \omega^v), B^v) := \tilde{\xi}((P, \omega), B)$ is again contained*

in $\text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$. It is explicitly given as

$$\begin{aligned}\sigma^v &:= \frac{1}{\langle \text{Re}(\sigma), v \rangle} \left(\text{pr}(B + i\omega) - \frac{1}{2}(B + i\omega)^2 v + v^* \right) \\ B^v + i\omega^v &:= \frac{1}{\langle \text{Re}(\sigma), v \rangle} (\text{pr}(\sigma) - \langle \sigma, B \rangle v)\end{aligned}$$

Here, we have replaced P by the corresponding line $[\sigma] \in Q_{\Gamma} \subset \mathbb{P}(\Gamma_{\mathbb{C}})$. Furthermore, we have chosen σ such that $\text{Im}(\sigma)$ is orthogonal to v .

Proof. By definition the positive plane P is contained in ω^{\perp} . Since the intersection of ω^{\perp} with $\Gamma'_{\mathbb{R}} \oplus \mathbb{R}v$ and $\Gamma'_{\mathbb{R}} \oplus \mathbb{R}v^*$ have both only one positive direction, P cannot be contained in either of them. Thus, we may choose σ such that $v^{\perp} \cap P = \text{Im}(\sigma)\mathbb{R}$ and $\langle \text{Re}(\sigma), v \rangle \neq 0$. This justifies the above choices. Also note that ω^v and B^v do not change when σ is changed by a real scalar. The defining equations for $B^v + i\omega^v$ and σ^v are spelled out as follows

$$\begin{aligned}\sigma^v &:= \frac{1}{\langle \text{Re}(\sigma), v \rangle} \left(-\frac{1}{2}(B + i\omega + v^*)^2 v + B + i\omega + v^* \right) \\ \omega^v &:= \frac{1}{\langle \text{Re}(\sigma), v \rangle} (\text{Im}(\sigma) - \langle \text{Im}(\sigma), v^* \rangle v - \langle \text{Im}(\sigma), B \rangle v) \\ B^v &:= \frac{1}{\langle \text{Re}(\sigma), v \rangle} (\text{Re}(\sigma) - \langle \text{Re}(\sigma), v \rangle v^* - \langle \text{Re}(\sigma), v^* \rangle v - \langle \text{Re}(\sigma), B \rangle v)\end{aligned}$$

Let us now compute $\tilde{\xi}(H_1, H_2)$. We denote $\gamma((\sigma^v, \omega^v), B^v)$ by (H_1^v, H_2^v) , where σ^v, ω^v , and B^v are as above.

The space H_1^v is spanned by the real and imaginary part of $\sigma^v - \langle \sigma^v, B^v \rangle w$. A simple calculation yields

$$\langle \sigma^v, B^v \rangle = -\langle \text{Re}(\sigma), v \rangle^{-1} (\langle B, v^* \rangle + i\langle \omega, v^* \rangle).$$

Thus, H_1^v is spanned by

$$\begin{aligned}& B + v^* + \frac{1}{2}(\omega^2 - B^2 - 2\langle B, v^* \rangle)v + \langle B, v^* \rangle w \\ &= \frac{1}{2}(\omega^2 - B^2)v + v^* + (B - \langle B, v^* \rangle v) + \langle B, v^* \rangle w \\ &= \xi \left(\frac{1}{2}(\omega^2 - B^2)w + w^* + (B - \langle B, v^* \rangle v) + \langle B, v^* \rangle v \right) \\ &= \xi \left(\frac{1}{2}(\omega^2 - B^2)w + w^* + B \right)\end{aligned}$$

and

$$\begin{aligned}
& \omega - \langle \omega, B + v^* \rangle v + \langle \omega, v^* \rangle w \\
&= (\omega - \langle \omega, v^* \rangle v) + \langle \omega, v^* \rangle w - \langle \omega, B \rangle v \\
&= \xi (\omega - \langle \omega, v^* \rangle v + \langle \omega, v^* \rangle v - \langle \omega, B \rangle w) \\
&= \xi (\omega - \langle \omega, B \rangle w).
\end{aligned}$$

Hence, $H_1^v = \xi(H_2)$. Similarly, one proves $H_2^v = \xi(H_1)$. First one computes

$$\begin{aligned}
\omega^{v2} &= \langle \text{Re}(\sigma), v \rangle^{-2} \text{Im}(\sigma)^2, \\
B^{v2} &= \langle \text{Re}(\sigma), v \rangle^{-2} (\text{Re}(\sigma)^2 - 2\langle \text{Re}(\sigma), v \rangle \langle \text{Re}(\sigma), v^* \rangle) \\
\langle \omega^v, B^v \rangle &= -\langle \text{Re}(\sigma), v \rangle^{-1} \langle \text{Im}(\sigma), v^* \rangle,
\end{aligned}$$

where one uses $\langle \text{Im}(\sigma), v \rangle = 0$. Since $\text{Im}(\sigma)^2 = \text{Re}(\sigma)^2$, this yields

$$\omega^{v2} - B^{v2} = 2\langle \text{Re}(\sigma), v \rangle^{-1} \langle \text{Re}(\sigma), v^* \rangle.$$

Hence, H_2^v is spanned by

$$\begin{aligned}
& \frac{1}{2}(\omega^{v2} - B^{v2})w + w^* + B^v \\
&= \langle \text{Re}(\sigma), v \rangle^{-1} \langle \text{Re}(\sigma), v^* \rangle w + w^* \\
&\quad + \langle \text{Re}(\sigma), v \rangle^{-1} (\text{Re}(\sigma) - \langle \text{Re}(\sigma), v \rangle v^* \langle \text{Re}(\sigma), v^* \rangle v - \langle \text{Re}(\sigma), B \rangle v)
\end{aligned}$$

and

$$\begin{aligned}
& \omega^v - \langle \omega^v, B^v \rangle w \\
&= \langle \text{Re}(\sigma), v \rangle^{-1} ((\text{Im}(\sigma) - \langle \text{Im}(\sigma), v^* \rangle v) - \langle \text{Im}(\sigma), B \rangle v + \langle \text{Im}(\sigma), v^* \rangle w)
\end{aligned}$$

Thus, $\xi(H_2^v)$ is generated by $\text{Re}(\sigma) - \langle \text{Re}(\sigma), B \rangle w$ and $\text{Im}(\sigma) - \langle \text{Im}(\sigma), B \rangle w$. Hence, $H_2^v = \xi(H_1)$. \square

Examples 6.9 The proposition can be used to identify certain subspaces of $\text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ via the mirror map $\tilde{\xi}$. We will present a few examples, which will be interpreted geometrically later on. As the B-field from a geometric point of view is not well-understood, we will be especially interested in those points with vanishing B-field.

i) Fix an orthogonal decomposition $\Gamma'_{\mathbb{R}} = V \oplus V^v$, such that both subspaces V and V^v contain a positive line. The automorphism $\tilde{\xi} \in \tilde{O}(\Gamma \oplus U)$ induces a bijection between the two subspaces:

$$\{((P, \omega), B) \mid B, \omega \in V, P \subset V^v \oplus U'_{\mathbb{R}}\} \text{ and } \{((P, \omega), B) \mid B, \omega \in V^v, P \subset V \oplus U'_{\mathbb{R}}\}.$$

Note that in this case the formulae for $(\sigma^v, \omega^v, B^v)$ simplify slightly to:

$$\sigma^v = \frac{1}{\langle \text{Re}(\sigma), v \rangle} \left(-\frac{1}{2} (B + i\omega)^2 v + v^* + B + i\omega \right), \quad \omega^v = \frac{1}{\langle \text{Re}(\sigma), v \rangle} (\text{Im}(\sigma) - \langle \text{Im}(\sigma), v^* \rangle v),$$

and

$$B^v = \frac{1}{\langle \text{Re}(\sigma), v \rangle} (\text{Re}(\sigma) - \langle \text{Re}(\sigma), v \rangle v^* - \langle \text{Re}(\sigma), v^* \rangle v).$$

From here it is easy to verify that $\tilde{\xi}$ maps these two subspaces into each other. Note that $B^v + i\omega^v$ is up to the scalar factor $\langle \text{Re}(\sigma), v \rangle^{-1}$ nothing but the projection of $\sigma \in V_{\mathbb{C}}^v \oplus U'_{\mathbb{C}}$ to $V_{\mathbb{C}}^v$.

ii) It might be interesting to see what happens in the previous example if we set the B-field zero. Under the assumption of **i)** the symmetry $\tilde{\xi}$ induces a bijection between the following two subspaces

$$\begin{aligned} & \{((P, \omega), B = 0) \mid \omega \in V, \text{Re}(\sigma) \in U'_{\mathbb{R}}, \text{Im}(\sigma) \in V^v\} \\ \text{and } & \{((P, \omega), B = 0) \mid \omega \in V^v, \text{Re}(\sigma) \in U'_{\mathbb{R}}, \text{Im}(\sigma) \in V\} \end{aligned}$$

Indeed, $\text{Im}(\sigma^v) = \langle \text{Re}(\sigma), v \rangle^{-1} (-\langle B, \omega \rangle v + \omega)$ and $\text{Re}(\sigma^v) = \langle \text{Re}(\sigma), v \rangle^{-1} (\frac{1}{2}(\omega^2 - B^2)v + v^* + B)$. Thus, if $B = 0$ one has $\text{Im}(\sigma^v) = \langle \text{Re}(\sigma), v \rangle^{-1} \omega$ and $\text{Re}(\sigma^v) = \langle \text{Re}(\sigma), v \rangle^{-1} (\frac{\omega^2}{2}v + v^*)$, and $B^v = 0$. Conversely, if $\text{Re}(\sigma^v) \in U'_{\mathbb{R}}$ then $B = 0$. Moreover, $\text{Im}(\sigma) \in V$ implies $\omega \in V$ and $\omega^v \in V^v$ implies $\text{Im}(\sigma) \in V^v$. Eventually, $B^v = 0$ yields $\text{Re}(\sigma) \in U'_{\mathbb{R}}$.

iii) In this example we will not need any further decomposition of $\Gamma'_{\mathbb{R}}$. The automorphism $\tilde{\xi} \in \tilde{O}(\Gamma \oplus U)$ induces an involution on the subspace

$$\{((P, \omega), B) \mid \omega, B \in \Gamma'_{\mathbb{R}} \oplus \mathbb{R}v\} \subset \text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}).$$

This follows again easily from the explicit description of $(\sigma^v, \omega^v, B^v)$.

iv) Also in **iii)** one finds a smaller subset parametrizing only objects with trivial B-field that is left invariant by $\tilde{\xi}$. Indeed, the subspace

$$\{((P, \omega), 0) \mid \omega \in \Gamma'_{\mathbb{R}}, P \cap U'_{\mathbb{R}} \neq 0\}$$

is mapped onto itself under $\tilde{\xi}$. \square

Remark 6.10 If $((P, \omega), B)$ such that $B^\vee = 0$ and $\varphi \in O(\Gamma')$, then also $\tilde{\xi}((\varphi(P), \varphi(\omega)), \varphi(B))$ has vanishing B-field. Geometrically this is used to argue that if the mirror X^\vee of X has vanishing B-field then the same holds for the mirror of f^*X under any diffeomorphism f of X with $f^*|_{U'} = \text{id}$. The assertion is an immediate consequence of the explicit description of B^\vee given above (cf. [42]).

Note that $\tilde{\xi}$ is by far the most interesting automorphism considered so far, as it really mixes the ‘complex direction’ σ with the ‘metric direction’ (ω, B) . However, at least for the case of the K3 lattice $\Gamma = 2(-E_8) \oplus 3U$ the automorphisms ξ respectively $\{-\text{id}_U, w \leftrightarrow w^*\}$ together with $\{O(\Gamma), \varphi_{B_0 \in \Gamma}\}$ generate both the same group, namely $O(\Gamma \oplus U)$. This is a consequence of Proposition 6.1, where one uses $\xi O(U)\xi = O(U')$ and thus $O(U) \subset \langle \xi, O(\Gamma) \rangle$. So in this sense, $\xi \in O(\Gamma \oplus U)$ as an automorphism of $\text{Gr}_{2,2}^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$ is not more or less interesting than those in 6.3 and 6.3, but for the latter ones the interesting things happen outside the ‘geometric world’ of $\text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$.

7 Geometric interpretation of mirror symmetry

7.1 Lattice polarized mirror symmetry

Let Γ as before be the K3 lattice $2(-E_8) \oplus 3U$ and fix a sublattice $N \subset \Gamma$ of signature $(1, r)$.

Definition 7.1 An N -polarized marked K3 surface is a marked K3 surface (X, φ) such that $N \subset \varphi(\text{Pic}(X))$.

Note that any N -polarized K3 surface is projective. If $\mathcal{T}_\Gamma^{\text{cpl}}$ is the moduli space of marked K3 surfaces we denote by $\mathcal{T}_{N \subset \Gamma}^{\text{cpl}}$ the subspace that consists of N -polarized marked K3 surfaces. Analogously, one defines

$$\mathcal{T}_{N \subset \Gamma}^{(2,2)} \subset \mathcal{T}_\Gamma^{(2,2)}$$

as the subset of all marked Kähler K3 surfaces with B-field (X, ω, B, φ) such that $N \subset \text{Pic}(X)$ and $\omega, B \in N_{\mathbb{R}}$. Here and in the following, we omit the marking in the notation, i.e. the identification $H^2(X, \mathbb{Z}) \cong \Gamma$ via φ will be understood.

The condition $N \subset \text{Pic}(X)$ is in fact equivalent to $V := N_{\mathbb{R}} \subset \text{Pic}(X)_{\mathbb{R}}$. The latter can furthermore be rephrased as $V \subset (H^{2,0}(X) \oplus H^{0,2}(X))^{\perp}$, i.e. $\sigma \in V_{\mathbb{C}}^{\perp}$.

By construction there exists a natural map

$$\mathcal{T}_{N \subset \Gamma}^{(2,2)} \rightarrow \mathcal{T}_{N \subset \Gamma}^{\text{cpl}}.$$

The fibre over $(X, \varphi) \in \mathcal{T}_{N \subset \Gamma}^{\text{cpl}}$ is isomorphic to $V_{\mathbb{R}} + i(\mathcal{K}_X \cap V_{\mathbb{R}})$ via $(\omega, B) \mapsto B + i\omega$.

Using the period map, the space $\mathcal{T}_{N \subset \Gamma}^{(2,2)}$ can be realized as a subspace of $\text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}} \subset \text{Gr}_{2,2}^{\text{po}}(\Gamma_{\mathbb{R}} \oplus U_{\mathbb{R}})$. Its closure $\overline{\mathcal{T}}_{N \subset \Gamma}^{(2,2)}$ consists of all points $((P, \omega), B) \in \text{Gr}_{2,1}^{\text{po}}(\Gamma_{\mathbb{R}}) \times \Gamma_{\mathbb{R}}$ such that $P \subset V^{\perp}$ and $\omega, B \in V$. Indeed, via the period map $\mathcal{T}_{N \subset \Gamma}^{(2,2)}$ is identified with an open subset of $\{((P, \omega), B) \mid B, \omega \in V, P \subset V^{\perp}\}$ and the latter is irreducible.

Let us now assume that the orthogonal complement $N^{\perp} \subset \Gamma$ contains a hyperbolic plane $U' \subset N^{\perp}$. Then $N^{\perp} = N^{\vee} \oplus U'$ by Lemma 6.7 for some sublattice $N^{\vee} \subset \Gamma$ of signature $(1, 18 - r)$. The real vector space $N_{\mathbb{R}}^{\vee}$ is denoted by V^{\vee} . As above one introduces $\mathcal{T}_{N^{\vee} \subset \Gamma}^{(2,2)}$ and $\mathcal{T}_{N^{\vee} \subset \Gamma}^{\text{cpl}}$.

Proposition 7.2 *The mirror symmetry map $\tilde{\xi}$ associated to the splitting $\Gamma = \Gamma' \oplus U'$ induces a bijection*

$$\overline{\mathcal{T}}_{N \subset \Gamma}^{(2,2)} \cong \overline{\mathcal{T}}_{N^{\vee} \subset \Gamma}^{(2,2)}.$$

Proof. By the description of $\overline{\mathcal{T}}_{N \subset \Gamma}^{(2,2)}$ as the set $\{((P, \omega), B) \mid B, \omega \in V, P \subset V^{\perp}\}$, it suffices to show that the mirror map identifies the two sets $\{((P, \omega), B) \mid B, \omega \in V, P \subset V^{\perp}\}$ and $\{((P, \omega), B) \mid B, \omega \in V^{\vee}, P \subset (V^{\vee})^{\perp}\}$, which has been observed already in the Examples in Section 6.4. \square

Remark 7.3 i) In general, we cannot expect to have a bijection $\mathcal{T}_{N \subset \Gamma}^{(2,2)} \cong \mathcal{T}_{N^{\vee} \subset \Gamma}^{(2,2)}$. Indeed, for a point in $\mathcal{T}_{N \subset \Gamma}^{(2,2)}$ that corresponds to a triple $((P, \omega), B)$ the image $((P^{\vee}, \omega^{\vee}), B^{\vee}) = \tilde{\xi}((P, \omega), B)$ might admit a (-2) -class $c \in (P^{\vee})^{\perp} \cap \Gamma$ with $\langle c, \omega^{\vee} \rangle = 0$. In fact, these two conditions on the (-2) -class c translate into the equations $\langle c + \langle c, v \rangle B, \omega \rangle = 0$ and $\langle c - \langle c, v \rangle v^*, \text{Im}(\sigma) \rangle = 0$. To exclude this possibility one would need to derive from this fact that there exists a (-2) -class c' with $\langle c', \sigma \rangle = 0$ and $\langle c', \omega \rangle = 0$ and this doesn't seem possible in general.

One should regard this phenomenon as a very fortunate fact. As points in the boundary are interpreted as singular K3 surfaces, it enables us to compare smooth K3 surfaces with singular ones. One should try to construct examples of (singular) Kummer surfaces in this context.

ii) Also note that if $\omega \in \Gamma$, i.e. ω corresponds to a line bundle, then ω^v does not necessarily have the same property.

iii) We also remark that the lattices N and N^v are rather unimportant in all this. Indeed, what really matters are the two decompositions $\Gamma = \Gamma \oplus U'$ and $\Gamma'_{\mathbb{R}} = V \oplus V^v$.

To conclude this section, we shall compare the above discussion with [16]. Let $N \subset \Gamma$ and $N^\perp = N^v \oplus U'$ be as before. Following [16] one defines

$$\Omega := N_{\mathbb{R}}^v \oplus i(N_{\mathbb{R}}^v \cap \mathcal{C}) \text{ and } D_N := Q_{\Gamma} \cap \mathbb{P}(N_{\mathbb{C}}^\perp).$$

Then by [16, Thm.4.2, Rem.4.5] the map

$$\alpha : \Omega \rightarrow D_N, z \mapsto [z - \frac{1}{2}z^2 \cdot v + v^*]$$

is an isomorphism. This map obviously coincides with $(B + i\omega) \mapsto [\sigma^v]$ as described in Proposition 6.8, since for $B, \omega \in N_{\mathbb{R}}^v \subset \Gamma'_{\mathbb{R}}$ one has $\text{pr}(B + i\omega) = B + i\omega$. Thus, the map α coincides with the map given by the isomorphism $\overline{\mathcal{T}}_{N^v \subset \Gamma}^{(2,2)} \cong \overline{\mathcal{T}}_{N \subset \Gamma}^{(2,2)}$. To make this precise note that $\mathcal{T}_{N \subset \Gamma}^{\text{cpl}} \cong D_N$ via the period map and that $((P, \omega), B) \mapsto B + i\omega$ defines a surjection $\overline{\mathcal{T}}_{N^v \subset \Gamma}^{(2,2)} \twoheadrightarrow \Omega$. This yields a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{T}}_{N^v \subset \Gamma}^{(2,2)} & \xrightarrow{\cong} & \overline{\mathcal{T}}_{N \subset \Gamma}^{(2,2)} \\ \downarrow & & \downarrow \\ \Omega & \xrightarrow{\cong} & D_N \cong \mathcal{T}_{N \subset \Gamma}^{\text{cpl}} \end{array}$$

which emphasizes the fact that the mirror isomorphism identifies Kähler deformations with complex deformations.

Remark 7.4 i) We also mention the following result of Looijenga and Peters [29], which shows that lattices of small rank can always be realized. Let Γ be the K3 lattice and N any even lattice of rank at most three. Then there exists a primitive embedding $N \subset \Gamma$. If the rank is smaller than three then this primitive embedding is unique up to automorphisms of Γ , i.e. elements of $O(\Gamma)$. An even more general version of this result can be found in [37].

ii) Moduli spaces of polarized K3 surfaces and their compactifications have been treated in detail by Looijenga, Friedman, Scatonne and many others (see [41]).

7.2 Mirror symmetry by hyperkähler rotation

Let X be a K3 surface with Kähler class ω_I . Assume that a decomposition $H^2(X, \mathbb{Z}) = \Gamma = \Gamma' \oplus U'$ together with an isomorphism $\xi : U' \cong U$ has been fixed. As before, we denote by (v, v^*) the basis of U' that corresponds to (w, w^*) .

Proposition 7.5 *Let $\omega_I \in \Gamma'_{\mathbb{R}}$ be a Kähler class on X and assume that $\sigma_I \in H^{2,0}(X)$ with $\sigma_I \bar{\sigma}_I = 2\omega_I^2$ can be chosen such that $\text{Re}(\sigma_I) \in U'_{\mathbb{R}}$, $\text{Im}(\sigma_I) \in \Gamma'_{\mathbb{R}}$, and $\langle \text{Re}(\sigma_I), v \rangle = 1$. Then the $\tilde{\xi}$ -mirror of $(X, \omega_I, B = 0)$ is given by the formula*

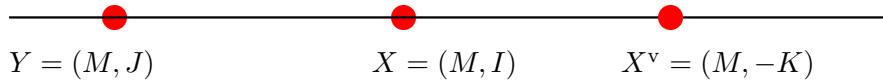
$$\begin{aligned}\sigma^v &:= \frac{1}{\langle \text{Re}(\sigma_I), v \rangle} (\text{Re}(\sigma_I) + i\omega_I) \\ \omega^v &:= \frac{1}{\langle \text{Re}(\sigma_I), v \rangle} \text{Im}(\sigma_I) \text{ and } B^v = 0\end{aligned}$$

Proof. Using $\text{Im}(\sigma_I) \in \Gamma'_{\mathbb{R}}$ and the general formula given in the proof of Proposition 6.8, we find that it is enough to prove $\text{Re}(\sigma_I) = (\omega_I^2/2)v + v^*$, but this follows immediately from the assumption $\text{Re}(\sigma_I) \in U'_{\mathbb{R}}$ and $\text{Re}(\sigma_I)^2 = \omega_I^2$. \square

Note that, if we only know that $\text{Re}(\sigma_I) \in U_{\mathbb{R}}$, the metric, and hence σ_I and ω_I , can be rescaled such that $\langle (\sigma_I), v \rangle = 1$. But scaling the metric changes the complex structure of the mirror. This will be important when we discuss the large Kähler and complex structure limits.

Corollary 6 *If (X, ω_I) is a K3 surface as in the proposition then the mirror K3 surface X^v is obtained by hyperkähler rotation to $-K$.*

Proof. Observe that $\text{Re}(\sigma_I) = \omega_J$. Thus, $\sigma^v = \langle \text{Re}(\sigma_I), v \rangle^{-1}(\omega_J + i\omega_I)$. On the other hand, $\sigma_{-K} = \omega_J + i\omega_I$. Hence, X^v is given by the complex structure $-K$. \square



There is one tiny subtlety. If we compute also the mirror Kähler form ω^v , we obtain $-\omega_{-K}$. But this is of no importance, as we can always apply the harmless global transformation $-\text{id} \in O(\Gamma)$.

Remark 7.6 Thus, in the very special case that $H^2(X, \mathbb{Z}) = \Gamma' \oplus U'$ such that $\text{Re}(\sigma_I) \in U'_{\mathbb{R}}$ and $\omega_I \in \Gamma'_{\mathbb{R}}$, mirror symmetry is given by hyperkähler rotation. However, the phenomenon seems rather accidental and one should maybe not expect that there is a deeper interplay between mirror symmetry and hyperkähler rotation. E.g. one can check that the solution of the Maurer-Cartan equation given by the Tian-Todorov coordinates is not the one obtained from hyperkähler rotation and deforming the hyperkähler structure.

7.3 Mirror symmetry for elliptic K3 surfaces

Here we will discuss one geometric instance where special K3 surfaces as treated in the last section naturally occur.

Let $\pi : Y \rightarrow \mathbb{P}^1$ be an elliptic K3 surface with a section $\sigma_0 \subset Y$. The cohomology class f of the fibre and $[\sigma_0]$ generate a sublattice $U' \subset H^2(Y, \mathbb{Z})$. It can be identified with the standard hyperbolic plane by choosing as a basis $v = f$ and $v^* = f + \sigma_0$. Thus, we obtain a decomposition $\Gamma := H^2(Y, \mathbb{Z}) = \Gamma' \oplus U'$ together with an isomorphism $\xi : U' \cong U$.

Let us now study the action of $\tilde{\xi}$ on K3 surfaces that are related to Y . If we fix a HK-metric g on Y , then we may write $Y = (M, J)$, where J is one of the compatible complex structures $\{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}$ associated with g . A holomorphic two-form on Y can be given as $\sigma_J = \omega_K + i\omega_I$. The reason why the complex structure that defines Y is denoted J is that we will actually not describe the mirror of Y , but rather of $X := (M, I)$.

Clearly X inherits the torus fibration from Y which gives rise to a differentiable map $\pi : X \rightarrow \mathbb{P}^1$.

Lemma 7.7 *The torus fibration $\pi : X \rightarrow \mathbb{P}^1$ is a SLAG fibration.*

Proof. Indeed, since the holomorphic two-form σ_J vanishes on any holomorphic curve in Y , the form $\omega_I = \text{Im}(\sigma_J)$ vanishes in particular on every fibre of $X \rightarrow \mathbb{P}^1$, i.e. all fibres are Lagrangian. Moreover, since $\sigma_I = \omega_J + i\omega_K$ and $\omega_K = \frac{1}{2}(\sigma_J + \bar{\sigma}_J)$, we see that $\text{Im}(\sigma_I)|_{\pi^{-1}(t)} = 0$ and $\text{Re}(\sigma_I)|_{\pi^{-1}(t)} = \omega_J|_{\pi^{-1}(t)}$. Hence, the (smooth) fibres are special Lagrangians of phase 0. \square

Again as a consequence of the general formula in Proposition 6.8 one computes the mirror of (X, ω_I) explicitly.

Proposition 7.8 *The $\tilde{\xi}$ -mirror of (X, ω_I) is the K3 surface X^v given by the period*

$$\sigma^v = \frac{1}{\text{vol}(f)} \left(\frac{\omega_I^2}{2} f + \sigma_0 + f + i\omega_I \right),$$

which is endowed with the Kähler class

$$\omega^v = \frac{1}{\text{vol}(f)} \text{Im}(\sigma_I),$$

where $\text{vol}(f) = \langle \omega_J, f \rangle$ is the volume of the fibre of the elliptic fibration $Y \rightarrow \mathbb{P}^1$. \square

Of course, an explicit formula could also be given for the mirror of X endowed with the Kähler form ω_I and an auxiliary B-field. We leave this to the reader.

Remark 7.9 A priori, the mirror K3 surface could be singular, i.e. there could be a (-2) -class $c \in \Gamma$ such that $\langle c, \omega^v \rangle = \langle c, \sigma^v \rangle = 0$. For such a class we would have $\langle c, \omega_I \rangle = \langle c, \text{Im}(\sigma_I) \rangle = 0$. Of course, if we also had $\langle c, \omega_I \rangle = 0$, then already (X, ω_I) would be singular.

If in addition we choose $\omega_J := (\alpha/2 + 1)f + \sigma_0$ is a Kähler class for some $\alpha > 0$, e.g. when Y has Picard number two, then $\text{Re}(\sigma_I)$ satisfies the condition of Proposition 7.5, i.e. $\text{Re}(\sigma) \in U'_\mathbb{R}$ $\text{Im}(\sigma_I) \in \Gamma'_\mathbb{R}$, and $\langle \text{Re}(\sigma), v \rangle = 1$. Hence, in this case the mirror of X is given by the complex structure $-K$ and the fibration is still a SLAG fibration.

Remark 7.10 If we go back to the more general case, where ω_J on Y might be arbitrary, then we still see that $\omega^v, \text{Im}(\sigma^v) \in \langle v, v^* \rangle^\perp$, i.e. at least cohomologically the classes f and σ_0 are still Lagrangian on the mirror, as in the more special case above where X^v was given by $-K$.

7.4 FM transformation and mirror symmetry

Here we shall explain how the mirror symmetry map $\tilde{\xi}$ can be viewed as the action of the FM-transformation on cohomology. Let us first recall the setting of the previous section. Let $\pi : Y \rightarrow \mathbb{P}^1$ be an elliptic K3 surface with

a section σ_0 . The cohomology class of the fibre will be denoted by f and the complex structure by J , i.e. $Y = (M, J)$. Assume that $[\omega_J] = (\alpha/2+1)f + \sigma_0$ is a Kähler class on Y . Then let $X = (M, I)$, where I and $K = IJ$ are the other two compatible complex structures associated to the hyperkähler metric underlying $[\omega_J]$. Then $\pi : X \rightarrow \mathbb{P}^1$ is a SLAG fibration. Furthermore, consider the dual elliptic fibration $\pi : Y^\vee \rightarrow \mathbb{P}^1$. Since $\pi : Y \rightarrow \mathbb{P}^1$ has a section, there is a canonical isomorphism $Y \cong Y^\vee$ compatible with the projection.

Let $\mathcal{P} \rightarrow Y \times_{\mathbb{P}^1} Y^\vee = Y \times_{\mathbb{P}^1} Y$ be the relative Poincaré sheaf and let $\tilde{FM} : H^*(Y, \mathbb{Z}) \cong H^*(Y, \mathbb{Z})$ denote the cohomological Fourier-Mukai transformation $\beta \mapsto q_*(p^*\beta \cdot \text{ch}(\mathcal{P}))$. A standard calculation shows (cf. [12]):

Lemma 7.11 $\tilde{FM} = \xi$

Proof. For the Fourier-Mukai transform on the level of derived categories one has $FM(\mathcal{O}_f) = \mathcal{O}_{\sigma_0 \cap f}$ and $FM(\mathcal{O}_{\sigma_0}(-1)) = \mathcal{O}_Y$. Since for the Mukai vectors one has $v(\mathcal{O}_f) = f$, $v(\mathcal{O}_{\sigma_0}(-1)) = \sigma_0$, $v(\mathcal{O}_{\sigma_0 \cap f}) = [\text{pt}] \in H^4(Y)$, and $v(\mathcal{O}_Y) = [\text{pt}] + [Y] \in H^4(Y) \oplus H^0(Y)$, passing to cohomology yields the result. \square

Thus, on a purely cohomological level, this fits nicely with the expectation that SLAGs on X should correspond to holomorphic objects on the mirror X^\vee which happens to be $(M, -K)$ as was explained before. Indeed, the two SLAGs f and σ_0 on X are first hyperkähler rotated to holomorphic objects in Y , namely the holomorphic fibre respectively section of the elliptic fibration $Y \rightarrow \mathbb{P}^1$. The FM-transforms of those are $k(\sigma_0 \cap \pi^{-1}(t))$ respectively \mathcal{O}_Y . Although we still have to hyperkähler rotate from $Y^\vee = Y$ to $X^\vee = (M, -K)$ the cycles $k(\sigma_0 \cap \pi^{-1}(t))$ and \mathcal{O}_Y stay holomorphic. Thus, the mirror symmetry map $\tilde{\xi}$, after interpreting it as Fourier-Mukai transform on Y , maps SLAGs on X to holomorphic cycles on X^\vee . So the picture is roughly the following

$$\begin{array}{ccc}
 X = (M, I) & & \\
 \downarrow \text{HK-rotation} & & \\
 Y = (M, J) & \xrightarrow{\text{FM}=\xi} & Y = (M, J) \\
 & & \downarrow \text{HK-rotation} \\
 & & X^\vee = (M, -K)
 \end{array}$$

7.5 Large complex structure limit

Mirror symmetry for Calabi-Yau manifolds suggests that the moduli space of complex structures on one Calabi-Yau manifold should be canonically isomorphic to the moduli space of Kähler structures on its dual. In fact, this should literally only be true near certain limit points. The limit point for the complex structure is called large complex structure limit and should correspond via mirror symmetry to the large Kähler limit. Moreover, according to the SYZ version of mirror symmetry, near the large complex structure limit the Calabi-Yau manifold is a Lagrangian fibration and while approaching the limit the Lagrangian fibres shrink to zero.

Let us consider a K3 surface X together with a sequence of Kähler classes ω_t such that the volume of X with respect to ω_t goes to infinity for $t \rightarrow \infty$. We will discuss in particular cases, what happens to the mirror X_t^v of (X, ω_t) . In order to incorporate the SYZ picture we will later concentrate on mirror symmetry for elliptic K3 surfaces.

There is a technical definition of the large complex structure limit due to Morrison [30]. For a one-parameter family of K3 surfaces it goes as follows:

Definition 7.12 *A family of K3 surfaces*

$$\mathcal{X}^v \rightarrow D^*$$

over the punctured disc $D^* := \{z \mid 0 < |z| < 1\}$ is a large complex structure limit if the monodromy operator T on $H^2(X^v, \mathbb{Z})$, where $X^v := \mathcal{X}_t^v$ for some $t \neq 0$, is maximally unipotent, i.e. $(T - 1)^2 \neq 0$ but $(T - 1)^3 = 0$, and $N := \log(T)$ induces a filtration

$W_0 := \text{Im}(N^2)$, $W_1 := \text{Im}(N|_{\ker(N^2)})$, $W_2 := \text{Im}(N)$, and $W_3 := \ker(N^2)$,
such that $\dim(W_0) = \dim(W_1) = 1$ and $\dim(W_2) = 2$.

Geometrically such families arise as type III degenerations of K3 surfaces, as studied by Kulikov (cf. [19]).

Remark 7.13 The weight filtration W_* always satisfies $W_0 \perp W_3$ (To see this one use $\langle (T - 1)(a), b \rangle = -\langle T(a), (T - 1)(b) \rangle$). In particular, $W_0 = W_1$ is spanned by an isotropic vector $v \in \Gamma$.

Conversely, any isotropic vector $v \in \Gamma$ together with an additional B-field $B_0 \in \Gamma \setminus \mathbb{Z}v$ defines a weight filtration $W_0 = W_1 = \langle v \rangle$, $W_2 = \langle v, B_0 \rangle$, $W_3 = v^\perp$ as above.

Let us check that the mirror of a large Kähler limit is a large complex structure limit in the above sense. We fix a decomposition $\Gamma = \Gamma' \oplus U'$ as before. Now consider a K3 surface X with a family of Kähler structures $\omega_t \in \Gamma'_{\mathbb{R}}$.

Recall that by Proposition 6.8 the period of the mirror X_t^v of (X, ω_t) is given by

$$[\sigma_t^v] = [\text{pr}(i\omega_t) + (\omega_t^2/2)v + v^*] \in \mathbb{P}(\Gamma_{\mathbb{C}}).$$

(For simplicity we assume $B = 0$.) For $\omega_t^2 \rightarrow \infty$ the period point of X_t^v converges to $[v]$. The loop around the large Kähler limit is given by $sB_0 + i\omega_t$ with $s \in [0, 1]$. For $B_0 \in \Gamma'$ the periods of the mirror for $s = 0$ and $s = 1$ are thus given by

$$i\omega_t + (\omega_t^2/2)v + v^* \text{ resp. } i\omega_t - \langle \omega_t, B_0 \rangle v + (\omega_t^2/2)v + v^* + B_0 - (B_0^2/2)v.$$

Thus, the induced monodromy is given by $T \in \text{O}(\Gamma)$ with

$$T(v) = v, \quad T(v^*) = v^* + B_0 - (B_0^2/2)v, \quad \text{and } T(x) = x - \langle B_0, x \rangle v \text{ for } x \in \Gamma'.$$

Note that this is an ‘internal B-shift’ by B_0 with respect to the decomposition $\Gamma = \Gamma' \oplus U'$ of the type we have encountered already in the proof of Proposition 4.8. Hence, $T - 1$ maps $v \mapsto 0$, $v^* \mapsto B_0 - (B_0^2/2)v$, $x \mapsto -\langle x, B_0 \rangle v$ for $x \in \Gamma'$ and, therefore, $W_0 = W_1$ is spanned by v and W_2 is spanned by v and B_0 .

This yields

Proposition 7.14 *Let us consider the mirror map induced by the decomposition $\Gamma = \Gamma' \oplus U'$. For any choice of $0 \neq B_0 \in \Gamma'$ the mirror of the large Kähler limit $(X, \omega_t \in \Gamma'_{\mathbb{R}}, B = 0)$ is a large complex structure limit. The choice of B_0 corresponds to choosing a component of the boundary divisor around which the monodromy is considered.* \square

In [16] the relation between the choice of the decomposition $\Gamma = \Gamma' \oplus U'$ and the large complex structure limit is expressed by: ‘the choice of the isotropic vector is the analog of MS1’ (MS1: the choice of a boundary point with maximally unipotent monodromy).

The SYZ conjecture can be incorporated into this picture without too much trouble: Let $\mathcal{X}^v \rightarrow D^*$ be the large complex structure limit obtained as above endowed with the mirror Kähler structures ω_t^v . The space W_0 is spanned by v .

Proposition 7.15 *The mirror $(\mathcal{X}_t^v, \omega_t^v)$ of $(X, \omega_t) \in \Gamma'_{\mathbb{R}}$ admits a SLAG fibration $\mathcal{X}_t^v \rightarrow \mathbb{P}^1$ with fibre class v . The volume of the fibre converges to zero for $t \rightarrow 0$.*

Proof. Recall that the holomorphic volume form σ is always chosen such that $\langle \text{Im}(\sigma), v \rangle = 0$. See the discussion at the beginning of the proof of Proposition 6.8. Moreover, if an additional Kähler form ω is chosen one requires $\sigma \wedge \bar{\sigma} = 2\omega^2$.

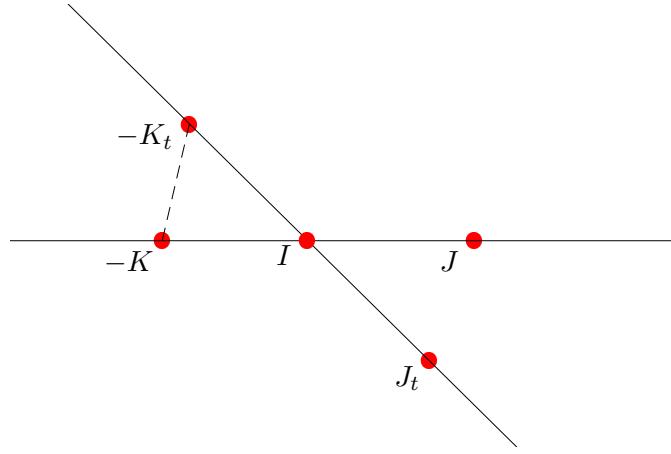
The class v is isotropic, as was remarked before. It thus satisfies a necessary condition for a fibre class. Furthermore, using the formulae for the mirror one finds that $\langle \omega_t^v, v \rangle = \langle \text{Im}(\sigma_t^v), v \rangle = 0$, where we use $\omega_t \in \Gamma'_{\mathbb{R}}$. Thus, on the level of cohomology the class v is a SLAG.

Also note that $\langle \text{Re}(\sigma_t^v), v \rangle = \langle \text{Re}(\sigma_t), v \rangle^{-1}$. Although the complex structure of X does not change, we have to rescale σ in order to ensure that $\sigma \bar{\sigma} = 2\omega_t^2$. The conditions on the choice of the real and imaginary part of σ_t imply that $\langle \text{Re}(\sigma_t), v \rangle \rightarrow \infty$. Hence, $\langle \text{Re}(\sigma_t^v), v \rangle \rightarrow 0$. Therefore, the volume of the SLAG fibre class v on \mathcal{X}_t^v approaches zero.

It thus remains to realize v geometrically. Here one uses hyperkähler rotation as before. Indeed, rotating with respect to the Kähler form ω_t^v one finds a complex structure with respect to which v is of type $(1, 1)$ and can thus be realized as the fibre class of an elliptic fibration (modulo the action of the Weyl group). \square

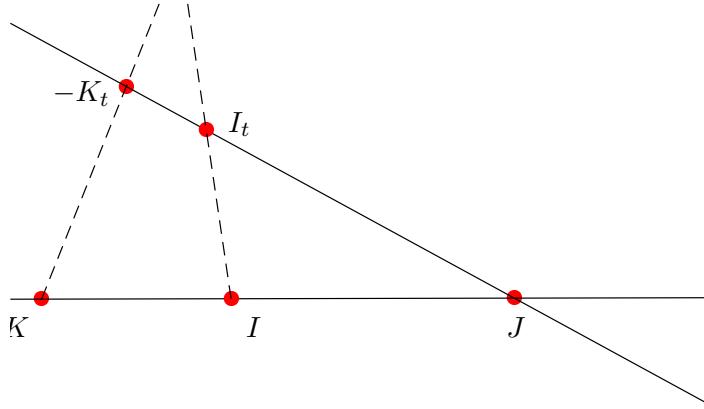
It is tempting to apply this discussion to the case of elliptic K3 surfaces or to the case where mirror symmetry is described by hyperkähler rotation. However, it seems impossible to follow the mirror to the large complex structure limit and at the same time to be able to obtain the mirror by a hyperkähler rotation. This also supports the point of view expressed in 7.6, that mirror symmetry and hyperkähler rotation are only related to each other in a very restricted sense.

Let us recall the setting. We fix an elliptic K3 surface $Y = (M, J) \rightarrow \mathbb{P}^1$ with a section σ_0 and assume that the Kähler form ω_J is of the form $(\alpha/2 + 1)f + \sigma_0$. The holomorphic two-form σ_J on Y is chosen such that $\sigma_J \bar{\sigma}_J = 2\omega_J^2$. In this case the mirror of $X = (M, I)$ endowed with $\omega_I = \text{Im}(\sigma_J)$ is $(M, -K)$. In principle there are two ways one could try to combine the passage of the mirror to the complex structure limit and the interpretation of the mirror by hyperkähler rotation. First, one could fix the complex structure I , i.e. the K3 surface $X = (M, I)$, and change the Kähler form ω_I .



However, in this case we keep σ_I . Rescaling is not permitted if the Kähler form on (M, J_t) is supposed to be of the form $(\alpha/2 + 1)f + \sigma_0$. Thus, $\langle \text{Re}(\sigma_I), f \rangle = \langle \omega_{J_t}, f \rangle \equiv 1$. Hence, the volume of the SLAG fibres in the mirror does not converge to zero. This yields a contradiction.

The second possibility is to freeze $Y = (M, J)$ and to change $\omega_J = (\alpha/2 + 1)f + \sigma_0$ by considering $(t^2\alpha/2 + 1)f + \sigma_0$ with $t \rightarrow \infty$. In this case, we compute the mirror X_t^v of $X_t = (M, I_t)$ endowed with $t\omega_I$, which stays of type $(1, 1)$ with respect to the changing complex structure I_t .



In this case, one finds that the complex structures $-K_t$ and I_t converge. But as before, the fibre volume $\langle \text{Re}(\sigma_{I_t}), f \rangle = \langle (t^2\alpha/2 + 1)f + \sigma_0, f \rangle$ does not converge to zero.

The large complex structure limit was studied by Gross and Wilson on a much deeper level in [21]. They studied the corresponding sequence of Kähler metrics (or at least a very good approximation of those) and could indeed show that the fibres of the elliptic fibration are shrunk to zero. Moreover, they managed to describe the limit metric on the base sphere $S^2 = \mathbb{P}^1$.

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