Scattering Amplitudes from Generalized Recursion

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis[1]

Journal

1. "Recursion relations for scattering amplitudes with massive particles"

   Sourav Ballav and Arkajyoti Manna;

   JHEP (2021) 03: 295

2. "Recursion relations for scattering amplitudes with massive particles II: massive vector bosons"

   Sourav Ballav and Arkajyoti Manna;

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List of other Publications, Not included in the thesis

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   JHEP (2019) 10: 075

2. "Irrelevant deformations of chiral bosons "

   Subhroneel Chakrabarti, Divyanshu Gupta, Arkajyoti Manna, Madhusudhan Raman ;

   JHEP (2021) 02: 028

3. "Stokes phenomena in 3d $\mathcal{N} = 2$ $SQED_2$ and CP^1 models "

   Dharmesh Jain, Arkajyoti Manna;

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[1] As it is standard in the High Energy Physics Theory (hep-th) community the names of the authors on any paper appear in their alphabetical order.

4. "Renormalisation in $T\overline{T}$-Deformed (Non-)Integrable Theories "
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Arkajyoti Manna

Arkajyoti Manna

Presentations

1. *" Scattering amplitudes of massive vector bosons from generalised recursion"*, Indian Strings Meeting 2021, IIT Roorkee, December 12 - 17, 2021

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5. *"Scattering Amplitudes of Massive Vector Bosons from Generalized Recursion"*, Invited seminar at IISER, Pune, India on 24 September 2021.

6. *"New Recursion Relations for scattering amplitudes with massive particles"*, Invited seminar at Quantum Spacetime Seminar Series, TIFR-Mumbai on 26 April 2021.

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And

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Contents

Synopsis

Introduction

Quantum field theory is one of the basic foundations of modern theoretical physics which naturally unifies the principles of quantum mechanics and Poincare invariance. The object of significant physical interest in any quantum field theory is the "S-matrix" and its elements are known as scattering amplitudes. The conventional approach to obtain the amplitudes is to write down a Lagrangian for the theory underlying the scattering process, derive all the Feynman rules therein and then implement the Lehmann-Symanzik-Zimmermann (LSZ) theorem on external states. But this field theoretic description for massless particles with spin very quickly leads to huge off-shell redundancies from gauge symmetries and various field redefinitions, appearing in intermediate processes but that are absent in observables. This complexity in the computation of amplitudes grows rapidly with increasing number of particles involved. A famous example of this is the 6 page computation of the $2 \rightarrow 4$ gluon amplitude in [1]. Surprisingly, this huge result can be expressed into a single line and can be modified suitably to obtain the $n-$particle maximally helicity violating (MHV) gluon amplitude [2]. This enticing simplicity of scattering amplitudes fosters the development of a multitude of techniques, broadly alluded to as "On-shell methods".

The modern S-matrix program, powered by the on-shell methods, deals directly with the particles involved in scattering, without any allusion to quantum fields and their accompanying redundancies. In this approach, the scattering amplitudes are considered to be a function of external kinematic data and subjected to constraints imposed by physical principles like unitarity, locality, causality and various spacetime and internal symmetries like the additional Yang-Mills structures that appear in the case of self interacting massless spin 1 particles.

In last few decades, the S-matrix program of quantum field theory has witnessed a number of remarkable developments: a) the study of analytic structure of S-matrix has revealed strikingly new insights in our understanding of quantum field theory [3–6], b) on-shell techniques like Britto-Cachazo-Feng-Witten (BCFW) [7, 8] recursion relations and generalised unitarity [9] have enormously reduced the complexity of seemingly impossible computations which guided the next to leading order (NLO) revolution in Quantum Chromodynamics (QCD) and have even been used to calculate classical observables such as potential that models the merger of binary black-holes up to high order in post Newtonian and post Minkowskian expansion [10]. This thesis is devoted to the derivation of a new on-shell recursion scheme for computing scattering amplitudes in gauge theories in four spacetime dimensions, involving massive particles. The recursion relations are then used to calculate particular classes of amplitudes involving massive vector bosons in Higgsed Yang-Mills theories.

Background

This section includes a short review of necessary tools and techniques relevant to our thesis: the spinor helicity formalism for massive and massless particles in $(3+1)$ dimensions, classification of three particle amplitudes with specific configuration of external momenta and their high energy limit.

Spinor helicity formalism

Scattering amplitudes are Lorentz invariant objects and transform covariantly under little group which is the ISO(2) group for massless particles and SU(2) group for massive particles in four dimensions. Therefore, the massless and massive external states are labelled by the helicity (h) and SU(2) indices respectively. It is useful to label massive spin S state as a symmetric SU(2) tensor of rank $2S$ since the standard representation of SU(2) requires a preferred spin axis which breaks the rotational invariance of the S-matrix. Then the little group transformation of scattering amplitude involving massive as well as massless particles takes the following form [11]

$$\mathcal{A}^{h_j}_{I_1 I_2 \ldots I_{2S}}\left(p_i, p_j, \cdots\right) \to t^{-2h_j} W_{i,I_1}{}^{J_1} W_{i,I_2}{}^{J_2} \cdots W_{i,I_{2S}}{}^{J_{2S}} \mathcal{A}^{h}_{J_1 J_2 \ldots J_{2S}}\left(p_i, p_j, \cdots\right) . \tag{1}$$

Here one massless $j-$th particle with helicity h_j and one massive particle $i-$th particle are transformed under there respective little groups. The factor t^{-2h_j} is the ISO(2)~U(1) scaling and W_i's are SU(2) matrices in the fundamental representation. Since the scattering amplitudes are little group covariant, it is convenient to express them in terms of the so-called "spinor-helicity variables" that hardwires these little group transformation laws.

Particle with zero mass

To introduce these variables, we consider the SL(2,C) representation of momentum 4-vector given by the 2×2 hermitian matrix $p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = p_{\alpha\dot{\alpha}}$. In this representation, the norm of the 4-vector p^μ is given by the determinant of the matrix $p_{\alpha\dot{\alpha}}$- which is zero in the case of massless particles. Therefore $p_{\alpha\dot{\alpha}}$ is a rank-1 matrix for massless particles and can be expressed as

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} , \tag{2}$$

where λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ are two-component Weyl spinors, known as massless spinor-helicity variables. Since we can always rescale the spinor-helicity variables

$$\lambda_\alpha \longrightarrow t\lambda_\alpha \,, \qquad \tilde{\lambda}_{\dot{\alpha}} \longrightarrow t^{-1}\tilde{\lambda}_{\dot{\alpha}} \,, \tag{3}$$

it is impossible to assign unique spinor-helicity variables to express $p_{\alpha\dot{\alpha}}$. But this scaling is exactly the little group scaling for massless particle. Thus we identify λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ as objects having little group weight ± 1 respectively. Using spinor-helicity variables, we define Lorentz invariant but little group covariant angle and square brackets as

$$\langle ij\rangle := \lambda_i^\alpha \lambda_{j\alpha} \,, \qquad [ij] := \tilde{\lambda}_{i\dot{\alpha}}\tilde{\lambda}_j^{\dot{\alpha}} \,, \qquad 2p\cdot q = \langle pq\rangle[qp] \,. \tag{4}$$

These brackets are the basic building blocks of scattering amplitudes in spinor-helicity formalism. Massless spinor-helicity variables satisfy the Weyl equation

$$p_i|i\rangle = p_i|i] = 0 \,. \tag{5}$$

Particle with non-zero mass

The rank of the hermitian matrix $p_{\alpha\dot{\alpha}}$ is 2 for massive particles since the determinant is non vanishing. Therefore, $p_{\alpha\dot{\alpha}}$ is expressed as a linear combination of two rank-1 objects [11]

$$p_{\alpha\dot{\alpha}} = \sum_{I,J=1}^{2} \epsilon_{IJ}\lambda_\alpha^I \tilde{\lambda}_{\dot{\alpha}}^J \,, \tag{6}$$

where (I, J) are SU(2) little group indices for massive particle. The variables $\lambda_\alpha^I, \tilde{\lambda}_{\dot{\alpha}}^J$ are called massive spinor-helicity variables . Similar to the massless case, there is no unique way to fix these spinors, satisfying the above relation due to the following transformation

$$\lambda_\alpha^I \longrightarrow W^I{}_J \lambda_\alpha^J \qquad \tilde{\lambda}_{\dot{\alpha}}^J \longrightarrow (W^{-1})^J{}_K \tilde{\lambda}_{\dot{\alpha}}^K \,. \tag{7}$$

But for real momenta, it can be shown that W's are indeed SU(2) matrices with $\det(\lambda^I_\alpha) = \det(\tilde{\lambda}^J_{\dot{\alpha}}) = m$, identifying (7) as correct little group transformation for massive spinor helicity variables.

Unlike the massless spinor-helicity variables $\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}$, the dotted and undotted massive spinor-helicity variables are related to each other via Dirac equation

$$p_{\alpha\dot{\alpha}}\lambda^\alpha_I = -m\tilde{\lambda}_{I\dot{\alpha}}\ ; \quad p_{\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_I = m\lambda_{I\alpha}\,. \tag{8}$$

Therefore the scattering amplitude involving massive particles can be expressed in terms of only either λ^α_I or $\tilde{\lambda}_{I\dot{\alpha}}$ as opposed to amplitude with only massless particles. This feature of the amplitude proves extremely useful to classify all possible three-particle amplitudes [11] involving both massive and massless particles.

Three particle amplitudes

The basic goal of on-shell recursion relation (like BCFW) is to construct higher point amplitudes from three particle amplitudes. We briefly review all the required three-particle amplitudes which will be used as basic building blocks to construct four- and higher-point amplitudes in this section.

Massless amplitude

The three particle kinematics of massless particles strongly constrains the structure of the amplitude: it can be a function of either λ_α or $\tilde{\lambda}_{\dot{\alpha}}$. Apart from an overall coupling, the rest of the structure of an amplitude involving particles with helicities (h_1, h_2, h_3) gets constrained by the little group scaling

$$\mathcal{A}_3^{h_1h_2h_3}[1,2,3] = g[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2}\ ; \quad h_1+h_2+h_3 > 0$$

$$= g^{'} \langle 12 \rangle^{h_3 - h_1 - h_2} \langle 23 \rangle^{h_1 - h_2 - h_3} \langle 31 \rangle^{h_1 - h_2 - h_3} \, ; \quad h_1 + h_2 + h_3 < 0 \, . \tag{9}$$

The conditions on sum of helicities ensure that the amplitude has a smooth vanishing limit in Minkowski signature as individual brackets vanish in this signature for real momenta.

Massive amplitude

There are two classes of three-point amplitudes involving both massive and massless particles: two massless-one massive and two massive-one massless. The latter further involves two sub classes: i)with different mass and ii) with same mass particles. In this thesis, we mostly consider the "minimally coupled"[2] three-particle amplitudes involving a massless particle of helicity $|h|$ and a pair of massive particles of mass m and spin S [11]:

$$\mathcal{A}^{+h}_{3,\text{min}}(\mathbf{1}, \mathbf{2}, 3^h) = g x_{12}^h \frac{\langle \mathbf{12} \rangle^{2S}}{m^{2S-1}} \; ; \quad \mathcal{A}^{-h}_{3,\text{min}}(\mathbf{1}, \mathbf{2}, 3^{-h}) = g x_{12}^{-h} \frac{[\mathbf{12}]^{2S}}{m^{2S-1}} \, , \tag{10}$$

where the x_{12} factor arises due to the degeneracy of masses and is defined as follows

$$x_{12} = \frac{\langle \zeta | p_1 | 3]}{m \langle \zeta 3 \rangle} \quad \text{or} \quad x_{12}^{-1} = \frac{\langle 3 | p_1 | \zeta]}{m [3 \zeta]} \tag{11}$$

with ζ being a reference spinor. We have omitted the SU(2) little group indices of massive spinor helicity variables for convenience. Instead, we will be using products of bold face spinor helicity variables which is defined as symmetric product of normal spinor helicity brackets. For example,

$$\langle \mathbf{12} \rangle^2 = \langle 1^{I_1} 2^{J_1} \rangle \langle 1^{I_2} 2^{J_2} \rangle + \langle 1^{I_2} 2^{J_1} \rangle \langle 1^{I_1} 2^{J_2} \rangle \, , \tag{12}$$

$$\langle 3\mathbf{2} \rangle^2 = \langle 3 2^{J_1} \rangle \langle 3 2^{J_2} \rangle \, . \tag{13}$$

[2]In this thesis, we follow the definition of [11] for "minimal coupling". Here, the"minimal coupled" amplitude involving massive particles are defined to be the three-particle amplitudes whose leading contribution to the high energy limit is dominated by opposite helicity massless particles. This is **not** the conventional terminology for "minimal coupling" exists in the literature. In this approach, the interactions are introduced by covariantizing the kinetic term. I thank the referee of this thesis for pointing out this issue. See section 2.4 for more details.

Generalized Recursion

The BCFW recursion relations are a huge step in the direction of formulating a strikingly simple on-shell method to compute massless scattering amplitudes. In this section, we will generalize these well known recursion relations by combining complexification of massless as well as massive external states. Let us consider the massive p_i and massless momenta p_j are analytically continued to complex plane while staying on-shell

$$p_i^\mu \to \widehat{p}_i^\mu = p_i^\mu - zr^\mu \, ; \qquad p_j^\mu \to \widehat{p}_j^\mu = p_j^\mu + zr^\mu . \tag{14}$$

Here z is the deformation parameter and r^μ is lightlike shift vector orthogonal to both of the momenta. The scattering amplitude$\mathcal{A}_n(z = 0)$ with undeformed momenta can be obtained from the deformed one using Cauchy's theorem

$$\mathcal{A}_n = \widehat{\mathcal{A}}_n(0) = \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{\widehat{\mathcal{A}}_n(z)}{z} dz = - \sum_{z_I} Res \left(\frac{\widehat{\mathcal{A}}_n(z)}{z} \right)_{z=z_I} + \mathcal{R}_n . \tag{15}$$

The contour Γ_0 encloses the pole at origin and $\mathcal{R}_n$ is the boundary term at infinity. All other simple pole locations of the amplitude are denoted by z_I.

The tree-level amplitude, $\widehat{\mathcal{A}}_n(z)$ has extremly simple analytic structure- it can have only simple poles in the form of propagator $\frac{1}{P^2 - m^2}$. When the propagator goes onshell the residue of $\widehat{\mathcal{A}}_n(z)$ factorizes into two lower point on-shell subamplitudes. Hence we can write

$$\widehat{\mathcal{A}}_n(z) = \sum_I \widehat{\mathcal{A}}_{l+1} \frac{1}{\widehat{P}_I^2 - m^2} \widehat{\mathcal{A}}_{r+1} + \sum_J \widehat{\tilde{\mathcal{A}}}_{l+1}(z_J) \frac{1}{\widehat{P}_J^2} \widehat{\tilde{\mathcal{A}}}_{r+1}(z_J) \, , \tag{16}$$

where I, J corresponds to all possible internal states. In the complex plane, we re-write the deformed propagator as follows

$$\widehat{P}_I^2|_{z_I} = m^2 \Rightarrow (P_I + z_I\, p_j)^2 = m^2 \longrightarrow \frac{1}{\widehat{P}_I^2 - m^2} = -\frac{z_I}{z - z_I}\frac{1}{{P_I}^2 - m^2}\,. \tag{17}$$

Assuming that the boundary term $\mathcal{R}$ vanishes, the physical amplitude can be constructed recursively by using [12, 13].

$$\mathcal{A}_n = \sum_I \widehat{\mathcal{A}}_{l+1}(z_I)\frac{1}{P_I^2 - m^2}\widehat{\mathcal{A}}_{r+1}(z_I) + \sum_J \widehat{\tilde{\mathcal{A}}}_{l+1}(z_J)\frac{1}{P_J^2}\widehat{\tilde{\mathcal{A}}}_{r+1}(z_J)\,. \tag{18}$$

It is important to emphasize that the recursion technique works only for the amplitudes that can be constructed using a set of three particle amplitudes. This excludes theories with contact terms like $\lambda\phi^4$ theory.

Little group covariant massive-massless shift

These on-shell recursions have been proved to be most efficient when the scattering amplitudes are expressed in spinor helicity basis. Therefore, the momentum shifts (14) should be realized in terms of spinor helicity variables. Moreover, it is paramount that the complex shift of spinor helicity variables obey little group covariance as the amplitude is covariant under little group. By choosing the shift vector to be $r_{\alpha\dot{\alpha}} = \frac{p_{i\alpha\dot{\beta}}}{m}\tilde{\lambda}_j^{\dot{\beta}}\tilde{\lambda}_{j\dot{\alpha}}$ in (14), we propose [14] the combined complex deformation of spinor helicity variables as follows

$$\widehat{\lambda}_{j\alpha} = \lambda_{j\alpha} + \frac{z}{m}p_{i\alpha\dot{\beta}}\tilde{\lambda}_j^{\dot{\beta}}\,; \qquad \widehat{\tilde{\lambda}}_{j\dot{\alpha}} = \tilde{\lambda}_{j\dot{\alpha}}, \tag{19}$$

$$\widehat{\lambda}^I_{i\alpha} = \lambda^I_{i\alpha}\,; \qquad \widehat{\tilde{\lambda}}^I_{i\dot{\alpha}} = \tilde{\lambda}^I_{i\dot{\alpha}} - \frac{z}{m}\tilde{\lambda}_{j\dot{\alpha}}[i^I j]\,. \tag{20}$$

We characterize these complex shifts as $[\mathbf{i}\, j^h\rangle$-shift where we have bold faced the massive spinor helicity variable ($|\mathbf{i}] \equiv \tilde{\lambda}_i^I$) instead of keeping the SU(2) index and h is the helicity of the $j-$th particle. It is crucial to note that these complex shifts are manifestly little group covariant and therefore can be implemented directly into the spinor helicity representation

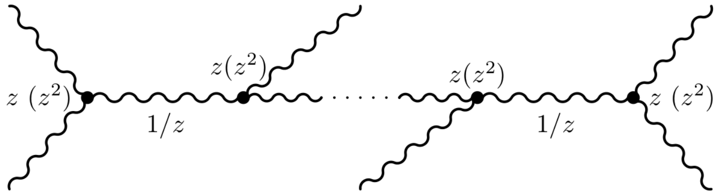


Figure 1: Individual Feynman diagrams contributing to $\widehat{A}(z)$ diverge at large z for gauge and Einstein gravity. The vertices grows as z, z^2 for gauge theory and Einstein gravity and overcompensate for the $\frac{1}{z}$ dependence of the propagators

of scattering amplitudes.

Large z behaviour of scattering amplitudes

On-shell recursion techniques are one of the most preferred tools in modern approaches to compute scattering amplitudes since it requires only on-shell three particle amplitude as input data. However, the contour arguments in earlier section reveals that the amplitude (as a function of complex momenta) must decay in the limit $z \to \infty$ for the on-shell recursion to work. This behaviour of deformed amplitude narrows down the space the allowed class of theories in which on-shell techniques can be implemented.

Remarkably, at least for some helicity combination of external particles, the amplitude does vanish at large z. For Yang-Mills theories, Britto-Cachazo-Feng-Witten (BCFW) showed that the deformed amplitude scales as follows at large-z [7, 8]

$$\mathcal{A}^{[-+\rangle}, \mathcal{A}^{[++\rangle}, \mathcal{A}^{[--\rangle} \propto \frac{1}{z} \qquad \mathcal{A}^{[+-\rangle} \propto z^3 \,.$$

Here the superscripts indicate the kind of spinor helicity variables to be analytically continued along with helicity assignment. Their analysis showed that $[-+\rangle, [++\rangle, [--\rangle$ are valid shifts to construct gluon amplitudes.

In our work [14], we classify all possible massive-massless shifts required for the generalized recursion method to execute in massive scalar QCD and Higgsed Yang-Mills

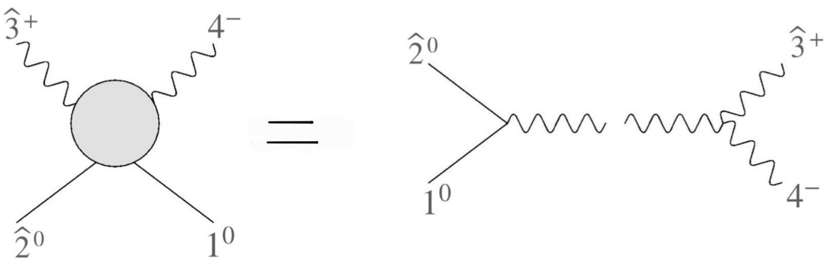


Figure 2: Colour ordered diagram for four particle amplitude with $[\mathbf{2}3^+\rangle$ shift

theories.

Scalar QCD

We start with Yang-Mills theory coupled with massive scalars. We consider an n-particles amplitude $\widehat{A_n}(z)$ including analytically continued one scalar and one gluon momenta. The scalar QCD Lagrangian is

$$\mathcal{L} = -\frac{1}{4}\text{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \mathcal{L}_{GF}\left(\partial_\mu A^\mu\right) - \frac{1}{2}|D_\mu\phi|^2 - \frac{1}{2}m^2|\phi|^2 - \frac{\lambda}{8}(|\phi|^2)^2 . \tag{21}$$

Let us consider the most simple amplitude that can be constructed from generalized recursion: the $2 \to 2$ scattering amplitude of a pair of massive scalars and gluons. Also, we need to consider only colour ordered amplitude since the full colour dressed amplitude can be constructed by using well-known colour decomposition rules [15–18]. There exists only a single diagram due to the adjacent $[\mathbf{2}3^+\rangle$ shift with massless exchange.

This amplitude can be constructed by gluing the three point scalar-gluon and pure gluon vertices. The relevant terms in the Lagrangian contributing to these vertices are

$$\mathcal{L}_3 = ig\left[\partial_\mu\phi A^\mu\phi^* - A_\mu\phi\partial^\mu\phi^* - \frac{1}{2}\text{Tr}(\partial^\mu A^\nu)[A_\mu, A_\nu]\right] . \tag{22}$$

In [14], we have shown that the vertex factors are independent of z when a single scalar and gluon line are analytically continued to complex plane. However, the (complex) propagator scales as $O(1/z)$, leading to an overall scaling of the amplitude as $O(1/z)$.

All higher point amplitudes with adjacent scalar-gluon shift will be suppressed by more $\frac{1}{z}$ factors due to more number of propagators. So we conclude that the scalar-gluon shift is a valid shift for generalized recursion[3].

Higgssed Yang-Mills theory

Since the classifcation of the generalized on-shell recursion is technical and lengthy, in this section, we briefly summarize the main concepts.

Our analysis is inspired by an analogous proof of the BCFW shifts for gluon amplitudes by Arkani-Hamed and Kaplan [19]. However conceptually there is a key difference that we highlight below. The proof regarding the validity of the BCFW shift for massless particles considered a set up where a highly boosted gluon was scattered off a background of low energy massless fields. For real momenta, this corresponds to the familiar Eikonal scattering. The background was referred to as a soft background. In the Eikonal approximation, the helicity of the highly boosted particle was conserved. This conservation law was shown to be a consequence the so-called spin-Lorentz symmetry which was then used to constrain the amplitude at large z.

In our case [14], the soft background is replaced by a static background which is a collection of massive and soft massless particles. Our set up is hence closer to the scattering of a boosted gluon off a heavy scattering center surrounded by a cloud of soft gluons. It was shown that, the resulting outgoing states are a highly boosted massive spin-1 boson and a highly boosted gluon. At infinite boost, the dominant contribution to the amplitude is again coming from helicity preserving terms in the Lagrangian. Thus, as in the case of massless theories, this contribution is constrained by the spin-Lorentz symmetry. We then use the Ward identity for massless gluons to constrain the sub-leading behaviour of the

[3] It is worthwhile to note that this conclusion continues to hold for massless scalars.

amplitude and show that for a particular class of shifts, which we refer to as valid shifts, the amplitude vanishes as $\frac{1}{z}$ for large z.

Unlike massless amplitudes, scattering amplitudes with massive particles contains an additional longitudinal mode at large z. However, this mode of amplitude is related to the amplitude involving massless scalars and gluons via the Ward identity for the spontaneously broken gauge theory [20]. We have determined that the scalar-gluon amplitudes decay as $O\left(\frac{1}{z}\right)$ in the limit $z \to \infty$ in previous section (see footnote 1).

To summarize, on one hand we have classified in [14] that the little group covariant shifts $[\mathbf{m}+\rangle$ and $[-\mathbf{m}\rangle$ are legitimate for generalized recursion to construct higher point amplitudes. On the other hand, we established that $[+\mathbf{m}\rangle$ and $[\mathbf{m}-\rangle$ are invalid shifts for the generalized recursion to work since the amplitude does not decay at large z. In the following table, we collect the large z behaviour of generalized recursion due to various types of shifts.

Massive-massless shift	**Large z behaviour**
$[\mathbf{m}+\rangle$	$1/z$
$[\mathbf{m}-\rangle$	z^2
$[+\mathbf{m}\rangle$	z^2
$[-\mathbf{m}\rangle$	$1/z$

Vector boson amplitude with arbitrary number of gluons

The purpose of this section is to explore the potential of being able to derive new classes of $n-$particle amplitudes by using generalized recursion relations within the Yang-Mills theory in Higgssed phase. We take the first step in this direction by considering scattering amplitudes involving two massive vector bosons and an arbitrary number of gluons in [21]. In particular, we study the following colour ordered massless configurations: a) identical helicity gluons and b) one flipped helicity gluon that is colour adjacent to massive bosons. These two classes of amplitudes reduce to maximally helicity violating (MHV)

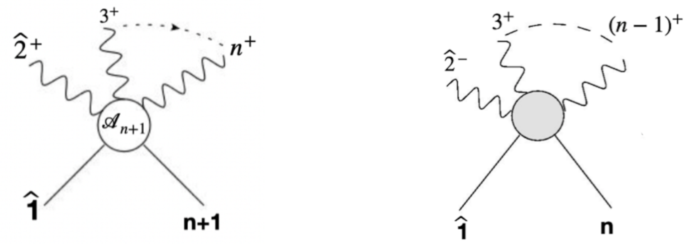


Figure 3: Massive analogues of MHV (left) and NMHV (right) amplitude.

and the next-to-maximally helicity violating (NMHV) gluon amplitudes respectively in the high energy limit. We are considering the "massive" analogue of MHV and NMHV amplitudes since they are the stepping stones for constructing more complicated gauge theory amplitudes.

Scattering of massive vector bosons with identical gluons

We obtain the vector boson amplitude with all positive helicity gluons in two different ways: firstly we relate this amplitude to one involving a pair of massive scalars and $(n-2)$ positive helicity gluons via:

$$\mathcal{A}_n[\mathbf{1}, 2^+, \ldots, (n-1)^+, \mathbf{n}] = \frac{\langle \mathbf{1n}\rangle^2}{m^2} \mathcal{A}_n[\mathbf{1}^0, 2^+, \ldots, (n-1)^+, \mathbf{n}^0] \,. \tag{23}$$

This proposal is a covariant expression of a result that has appeared previously in the literature [22]. Secondly, we shall prove this relation inductively by making use of the generalized recursion. The n-point massive-scalar gluon amplitude is already known [23]:

$$\mathcal{A}_n[\mathbf{1}^0, 2^+, \cdots, (n-1)^+, \mathbf{n}^0] = g^{n-2} \frac{m^2 [2| \prod_{k=3}^{n-2}\big((s_{1\ldots k}-m^2) - \not{p}_k \not{p}_{1,k-1}\big)|n-1]}{(s_{12}-m^2)(s_{123}-m^2)\cdots(s_{12\ldots(n-2)}-m^2)\langle 23\rangle\langle 34\rangle\cdots\langle (n-2)(n-1)\rangle} \,, \tag{24}$$

where the Mandelstam variables and $p_{1,l}$ are defined as follows

$$s_{1\ldots l} := (p_1 + \cdots + p_l)^2 \,, \qquad p_{1,l} := p_1 + \cdots + p_l \,. \tag{25}$$

We denote the spinor brackets appearing inside the product in the numerator as follows

$$[a|\not{p}_i\not{p}_j|b] = \tilde{\lambda}_{a\dot{\alpha}} p_i^{\dot{\alpha}\alpha} p_{j\alpha\dot{\beta}} \tilde{\lambda}_b^{\dot{\beta}} \,. \tag{26}$$

Substituting the scalar amplitude in (4.1), we find the following simple expression for the n-point amplitude with a pair massive vector bosons and $(n-2)$ positive helicity gluons (for $n > 3$):

$$\mathcal{A}_n[\mathbf{1}, 2^+, \cdots, (n-1)^+, \mathbf{n}] = g^{n-2} \frac{\langle \mathbf{1n}\rangle^2 [2| \prod_{k=3}^{n-2}\big((s_{1\ldots k}-m^2)-\not{p}_k\not{p}_{1,k-1}\big)|n-1]}{(s_{12}-m^2)(s_{123}-m^2)\cdots(s_{12\ldots(n-2)}-m^2)\langle 23\rangle\langle 34\rangle\cdots\langle (n-2)(n-1)\rangle} \,. \tag{27}$$

Since this expression is already in agreement with the four and five particle amplitudes derived in [14], we derive the $(n+1)$- particle amplitude using the generalized recursion with $[\mathbf{1}2^+\rangle$ shift. With this particular shift, all possible channels that contribute to the $\mathcal{A}_{n+1}$ amplitude are shown in Figure 4.

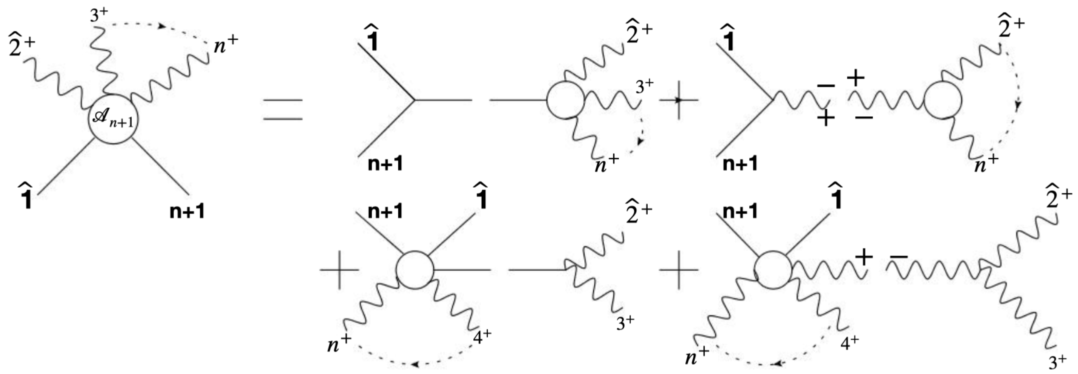


Figure 4: Generalized recursion with $[\mathbf{1}2^+\rangle$ shift

The first three diagrams do not contribute to the amplitude due to following reasons: a) the first diagram vanishes due to the vanishing of the right subamplitude involving a single massive vector boson, b) the second diagram vanishes since the pure gluon amplitude with either all positive helicity gluons or a single negative helicity gluon is zero and c) the third diagram vanishes because a massive vector boson cannot decay into two positive helicity gluons. Thus we only have to consider a single diagram. This demonstrates the simplicity in calculation in using the new recursion relations.

Scattering of massive vector bosons with flipped helicity gluon

In this section we discuss, the massive analog of the NMHV amplitude in which the external particle configuration consists of a pair of massive vector bosons, one minus helicity gluon adjacent to one of the massive bosons and arbitrary number of plus helicity gluons. At the onset of our discussion, it is important to note that this particular amplitude can be easily computed by using a single covariant recursion formula.

If one had used the usual BCFW shift to compute this amplitude, one would end up with subamplitudes involving the same configuration as the one we set out to compute (i.e. involving two massive vector bosons and helicity flipped gluons). See the second diagram in Figure 5. In the absence of an ansatz one would need to use the recursion relation iteratively to compute those subamplitudes that appear in a given recursion. This would make the computation technically involved

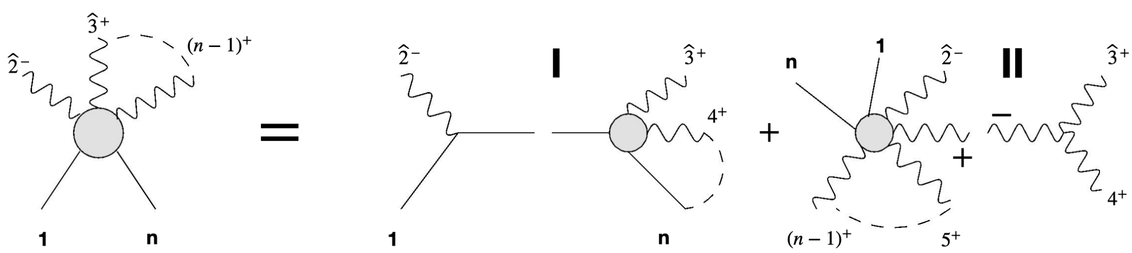


Figure 5: BCFW recursion with $[2^-3^+\rangle$ shift

Instead, we derive the n-point amplitude by complexifying massive momentum p_1 and massless momentum p_2 and taking the $[2^-\mathbf{1}\rangle$ shift in spinor helicity basis:

$$\widehat{|2]} = |2] + \frac{z}{m} p_1 |2\rangle \,, \qquad \widehat{|1^I\rangle} = |1^I\rangle - \frac{z}{m}\langle 2 1^I\rangle |2\rangle \,. \tag{28}$$

Due to this particular shift, all possible different scattering channels that contribute to

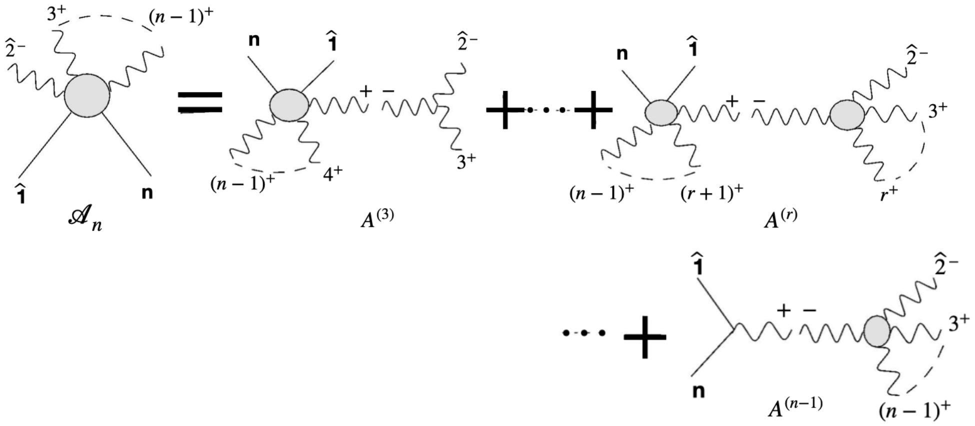


Figure 6: Generalized recursion with $[\mathbf{1}2^{+}\rangle$ shift

the amplitude in the generalized recursion are shown in Figure 4.3. All the lower point amplitudes, either vector boson amplitude with all positive helicity gluons or pure gluon amplitudes has already been computed. Therefore this example set forth one of the many utilities of generalized recursion relation.

The computation of the n-particle amplitude is technical and involved, but nonetheless we find a compact expression for this in [21]

$$\mathcal{A}_n[\mathbf{1},2^-,3^+,\ldots,\mathbf{n}] = g^{n-2}\left[\frac{(\langle 2|p_1|\mathbf{n}]\langle 2\mathbf{1}\rangle+\langle 2|p_n|\mathbf{1}]\langle 2\mathbf{n}\rangle+2m\langle \mathbf{12}\rangle\langle 2\mathbf{n}\rangle)^2}{s_{1n}\langle 23\rangle\langle 34\rangle\cdots\langle (n-2)(n-1)\rangle\big(\langle 2|p_1\cdot p_n|n-1\rangle+m^2\langle 2(n-1)\rangle\big)}\right.$$
$$\left. + \sum_{r=3}^{n-2}\frac{\langle 2|p_{3,r}\cdot\prod_{k=r+1}^{n-2}\big\{(s_{1\cdots k}-m^2)-\not{p}_k\not{p}_{1,k-1}\big\}|n-1]\big(\langle 2|\not{p}_1\not{p}_{3,r}|2\rangle\langle \mathbf{1n}\rangle+p_{2,r}^2\langle \mathbf{12}\rangle\langle 2\mathbf{n}\rangle\big)^2\langle r(r+1)\rangle}{s_{23\ldots r}(s_{12\ldots r}-m^2)\ldots(s_{12\ldots(n-2)}-m^2)\langle 23\rangle\langle 34\rangle\ldots\langle (n-2)(n-1)\rangle\langle 2|\not{p}_1\not{p}_{2,r-1}|r\rangle\langle 2|\not{p}_1\not{p}_{2,r}|r+1\rangle}\right]. \tag{29}$$

This is a completely new gauge theory amplitude result and as a simple check, a few lower-point amplitudes are obtained by independent methods and shown to match the expected result [21]. Additionally, we have considered the high energy limit of this amplitude and can reproduce the NMHV amplitude [24] for the specific ordering of external particles considered here.

Plan of the thesis

The subject of our investigations in this thesis is to study a new class of on-shell recursion relations for gauge theory amplitudes involving massive as well as massless particles in (3 + 1) dimensions. It will consist of the following chapters:

1. Chapter 1 provides a general introduction to the modern on-shell approach to scattering amplitudes.

2. Chapter 2 reviews all the background materials needed for this thesis. This includes spinor helicity formalism for both massive and massless particles in (3 + 1) dimensions and classification of the three particle amplitudes.

3. In chapter 3, we introduce the generalized recursion relation and provide practical examples for computing a few types of amplitudes in gauge theory.

4. Chapter 3.3 is devoted to the classification of all possible little group covariant massive-massless shifts for massive scalar QCD and Higgssed Yang-Mills theory.

5. We conclude in chapter 4 with one of the many advantages of the new recursion relation by deriving new gauge theory amplitudes along with a short outlook and future directions in chapter 5.

List of Figures

Chapter 1

Introduction

The S-matrix program of quantum field theory has witnessed a number of remarkable developments over past few decades. On the one hand, the study of the analytic structure of S-matrix reveals significantly new insights with the potential to revolutionise the entire edifice of quantum field theory [3–6]. On the other hand, on-shell techniques like Britto-Cachazo-Feng-Witten (BCFW) [7,8] recursion relations and generalised unitarity [9] have made seemingly impossible computations possible within a few pages. The latter developments are directly responsible for the NLO revolution in QCD. More recently, these on-shell methods have even been used in computing classical observables such as the potential in the binary black-hole problems up to high order in the Post Newtonian and the Post Minkowskian expansion [10].

The subject of our exploration in this thesis is a new class of on-shell recursion techniques, which we use to compute tree level scattering amplitudes in gauge theories in four space-time dimensions, involving massless as well as massive particles. The utility of recursion technique is the construction of higher-point amplitudes by gluing pairs of lower point amplitudes in a specific manner. Within a large class of recursion relations that already exists for computing amplitudes, a subset of them are called "On-shell" recursions, in which the three particle vertices serve as the only input to these recursion relations. Since the three-point amplitude vanishes for real momenta in Lorentzian signature, the on-shell recursion schemes are based on a analytic continuation of amplitudes into the complex domain by complexifying the external momenta. Simple contour arguments show that the desired amplitude is a residue of a meromorphic function with a simple pole at infinity. For the class of quantum field theories in which the amplitude decays at infinite momentum, the on-shell recursion relations completely determine the tree level scattering matrix from the three point amplitudes. By far the most widely used and efficient on-shell recursion is the BCFW recursion scheme in which precisely two of the external massless momenta are complexified while both the conservation of momentum and on mass shell conditions are maintained. For a succinct discussion of these ideas see [25].

In general, scattering amplitudes are functions of the kinematic space of (generalised) Mandelstam invariants built out of the external momenta. Since the amplitude is subject to several non-linear constraints like the Gram determinant conditions, this space is rather complicated. Therefore, the choice of appropriate variables to span the kinematic space becomes crucial in efficient computations of the scattering amplitudes. In four dimensions, it turns out that the most optimal variables, in which the constraints can be made trivial are the so-called spinor helicity variables. These variables are manifestly on-shell and therefore can directly describe the "on-shell physics", without any reference to the quantum fields and their huge gauge redundancies and field redefinitions. In fact, the motivation to pursue an alternative approach to the S-matrix program owes a lot to the aspiration of getting rid of off-shell redundancies. Implementing the BCFW recursion relations in deriving massless scattering amplitudes in spinor helicity formalism renders the computation of tree level amplitudes in gauge theories and gravity strikingly simple as opposed to the usual approach to obtain scattering amplitudes by summing over Feynman diagrams.

The traditional field-theoretic method for describing the massless particles involves introduction of gauge redundancies- leaving enough room for improvement. On-shell methods make no reference to the quantum fields and their accompanying redundancies. At first glance, the advantages of using spinor helicity variables seem to be absent for massive particles since they do not involve the off-shell redundancies that exist for massless particles. This is indeed true but, as was shown in [11], apart from a rather straightforward resolution of the technical issues [1], the spinor helicity formalism for massive particles allows us to understand the low energy behaviour of massless amplitudes via Higgs mechanism from an on-shell perspective. Although the spinor helicity formalism for massive particles has been known for decades, these variables were not the most suitable candidates for computation of the S-matrix as they were not little group covariant [26–30]. However, the spinor helicity formalism developed by Arkani-Hamed et al. [11] introduced on-shell variables that transform covariantly under the little group SU(2). In terms of these variables, the scattering amplitudes remain Lorentz invariant but transform covariantly under the little group also in the massive case. This formalism also allows one to take appropriate massless limits of an amplitude involving massive particles in a systematic manner. We use this formalism throughout this paper in computing massive scattering amplitudes.

However, even in the absence of a suitable on-shell formulation like spinor-helicity variables for massive particles, tree amplitudes involving a set of massive and at least two massless particles have been analysed using the BCFW recursion [12, 29, 31]. For example, in [12] several lower point tree level amplitudes with massive scalars and gluons were computed using the conventional BCFW relations. In [29] this method was extended to computations of scattering amplitude involving massive vector bosons, fermions and gluons. In a notable work [32], the authors derived recursion relations for all possible Born amplitudes in QCD and gave a closed form expression for amplitude involving two massive quarks scattering with an arbitrary number of gluons. Recently in a beautiful paper, Ochirov [31] generalised this computation by using the newly developed massive spinor helicity formalism of [11] and in particular proposed formulae [2] for two specific $n-$ par-

[1] Since the little group for massive particles is SU(2) instead of ISO(2) for massless particles, the massive spinor helicity variables carry an extra SU(2) index as opposed to their massless counterpart.

[2] which was proved by using the principle of induction and the BCFW recursion by the same author.

ticle amplitudes consistent with results previously derived [23,32]. Finally, in [13,27,33] scattering amplitudes involving massive particles have been first computed using multi-line complex shifts. However in these works, the external momenta of massive particles are decomposed in terms of certain massless vectors in terms of which the computations are brought closer to that of amplitude involving only massless particles. Recently, in [34,35] recursion relation for massive supersymmetric amplitudes have been developed using a massive super-BCFW shift.

Since the work of [11] has unified the kinematic space of both massive and massless external momenta into the little group covariant spinor helicity formalism, several questions arise naturally as a consequence of this discovery. For instance, (i) is it possible to derive “BCFW type" recursion relations by complexifying massive instead of massless external states.? (ii) Can these new recursion relations be used to derive new scattering amplitudes that are difficult to construct using the BCFW recursion?

In this thesis, we answer these questions by proposing a generalization of the BCFW recursion to the case where one massive and one massless external momentum get complexified [14]. There has been earlier work in this direction in [36] in which a particular massive-massless shift was proposed and it was used to compute four particle amplitude involving two massive vector bosons and two photons. We extend this by giving a classification of all possible massive-massless shifts and show that not all possible shifts lead to a valid recursion relation. The little group covariant realization of complex momentum shift in terms the spinor helicity formalism allows us to work with the formalism of [11] seamlessly. We then derive several lower point amplitudes in massive scalar QCD and Higgsed Yang-Mills theory using the new recursion relations and find perfect agreement with the results in [13,29]. In the process, we obtain the five point amplitude involving a pair of massive vector bosons and arbitrary helicity gluons. This serves as the first new scattering amplitude result involving gluons.

As mentioned previously, we provide a proof for the validity of the massive-massless shift to recursively construct amplitudes in massive scalar QCD and Higgsed Yang-Mills theory. A comprehensive study of the validity of BCFW shifts for different theories has previously appeared in the seminal work by Arkani-Hamed et. al. [19]. Our proof for massive-massless shift follows similar line of work but is more involved due to presence of an additional longitudinal mode corresponding to the massive particle. For completeness, we also extend the proof of [19] to the case of massive scattering amplitudes and prove the validity of the massless-massless shift for these theories.

We then explore the usefulness of the covariant recursion relations by computing two distinct classes of amplitudes involving two massive vector bosons and an arbitrary number of gluons with specific helicity [21]. The two classes are such that in the high energy limit, these amplitudes reduce to "maximally helicity violating" (MHV) and the "next-to-maximally helicity violating" (NMHV) gluon amplitudes respectively. We use the new recursion relations to derive the scattering amplitudes for both these classes. We provide an inductive proof for the first class of amplitudes and we show that for the second class of amplitudes, the massive-massless shift can be used to recursively construct the amplitude. This shows the practical utility of the new recursion relations derived in [14] .

This thesis is organised as follows. We begin with a short introduction on the spinor helicity formalism in four spacetime dimensions in chapter 2 along with a description of the three particle amplitudes that serve as the basic building blocks of the on-shell recursion technique. In the next chapter 3, we introduce the covariant recursion relations and use it to derive several four and five particle amplitudes in massive scalar QCD and Higgsed Yang-Mills theory. Then we move onto the central theme of this thesis in section 3.3: classification of all the valid massive-massless shifts in these theories. We also give an alternative derivation of the new five particle amplitude involving two massive vector bosons and gluons with arbitrary helicity using BCFW shift (see section 3.5.1). We discuss the massive analogues of the "maximally helicity violating" (MHV) and the "next-to-maximally helicity violating" (NMHV) gluon amplitudes in chapter 4. We conclude in chapter 5 with a short summary of the thesis and outline some immediate open questions.

Chapter 2

Background

In this chapter, we discuss the spinor helicity formalism in four spacetime dimensions. Spinor helicity formalism is an especially powerful notational tool to represent scattering amplitudes in (3 + 1) dimensional gauge and gravity theories by providing suitable variables, describing the external kinematics, that hardwire the correct little group transformations and are concurrently associated with appropriate representation of the on-shell momentum. We begin by reviewing the concept of the little group and the associated transformation of scattering amplitudes. This will pave our way to introduce the spinor helicity variables for on-shell momenta of massive and massless particles in four dimensions.

2.1 Little group

The main purpose of spinor helicity formalism is to trivialise a part of the intricate physics of scattering that traces back to the basic but fundamental question− "what is a particle?" and the accompanying ideas of Wigner's "Little group" , which govern the kinematics involved in the scattering of particles.

To elucidate this point, let us review the standard text book theory on Wigner's classification of one-particle states [37, 38] in accordance with their transformation under the Poincare group. We follow the discussion in Weinberg [39]. In order to define one-particle states, it should be noted that the momentum generators (P^μ) of the inhomogeneous Lorentz group (or Poincare group) commute with each other. Therefore, it is natural to express the one-particle states in terms of the eigenvectors of the momentum operator:

$$P^\mu|\psi_{p,\sigma}\rangle = p^\mu|\psi_{p,\sigma}\rangle\,, \tag{2.1}$$

where σ denotes all other possible degrees of freedom (that commute with P^μ) such as helicity for massless particles. Since, these one-particle states transform with a phase factor ($e^{ip\cdot x}$) due to translation group element, we now consider their transformation properties under homogeneous Lorentz transformations. If we denote the quantum operator responsible for these transformation as $U(\Lambda)$ then by definition,

$$U^{-1}(\Lambda)P^\mu U(\Lambda) = \left(\Lambda^{-1}\right)_\nu{}^\mu P^\nu = \Lambda^\mu{}_\nu P^\nu\,. \tag{2.2}$$

We can then find the action of the Lorentz group on an one-particle momentum state as

$$P^\mu U(\Lambda)|\psi_{p,\sigma}\rangle = \Lambda^\mu{}_\nu p^\nu U(\Lambda)|\psi_{p,\sigma}\rangle\,. \tag{2.3}$$

Therefore the one-particle state, after acted on by $U(\Lambda)$ remains an eigenstate of momentum operator, as expected, but with eigenvalue $(\Lambda p)^\mu$. This suggests that we can represent the state $U(\Lambda)|\psi_{p,\sigma}\rangle$ as a linear combination of $|\psi_{\Lambda p,\sigma}\rangle$ states

$$U(\Lambda)|\psi_{p,\sigma}\rangle = \sum_{\sigma'} C_{\sigma,\sigma'}(\Lambda, p)|\psi_{\Lambda p,\sigma'}\rangle\,. \tag{2.4}$$

Here the matrices $C_{\sigma,\sigma'}(\Lambda, p)$ furnish a representation of the quantum Lorentz transformations. Without any loss of generality, one can always choose linear combination of $|\psi_{p,\sigma}\rangle$ for some σ in such a way that the matrices $C_{\sigma,\sigma'}(\Lambda, p)$ are block diagonal. In this case, these matrices furnish irreducible unitary representation of the Poincare group that can naturally be identified with a specific type of "particles". This is the precise sense in which we define particles in quantum field theory. In order to find the structure of such an irreducible representation $C_{\sigma,\sigma'}(\Lambda, p)$, we now introduce the notion of the Little group. Note that we can write any momentum 4-vector as

$$p^\mu = L^\mu{}_\nu(p)\, k^\nu\,, \tag{2.5}$$

where k^ν is a standard reference momentum which we take to be $k^\mu \equiv (E; 0, 0, E)$ for massless particles and $k^\mu \equiv (m; 0, 0, 0)$ for massive particles. Now the one-particle states with arbitrary momentum p^μ are related to the one-particle states with standard momentum k^μ as

$$|\psi_{p,\sigma}\rangle = N(p)U(L(p))|\psi_{k,\sigma}\rangle\,, \tag{2.6}$$

where $U(L(p))$ is the unitary quantum operator associated to Lorentz transformation $L^\mu{}_\nu(p)$ and the normalization $N(p)$ is fixed by demanding orthogonality of one-particle states. With this definition, the action of $U(\Lambda)$ on $|\psi_{p,\sigma}\rangle$ (from equation (2.6)) is

$$U(\Lambda)|\psi_{p,\sigma}\rangle = N(p)U(L(\Lambda p))U[L^{-1}(\Lambda p)\Lambda L(p)]|\psi_{k,\sigma}\rangle \tag{2.7}$$

Here we have used

$$U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2)\,. \tag{2.8}$$

Let us consider the action of the combined Lorentz transformations in the quantum operator that directly acts on the state and find that it gives back k^μ

$$(L^{-1}(\Lambda p))^\mu{}_\nu(\Lambda p)^\nu = k^\nu\,. \tag{2.9}$$

Thus we have found a subset of Lorentz transformations that leaves the standard momentum 4-vector k^μ invariant. This subset is called -"Little group" and denoted as W

$$W := L^{-1}(\Lambda p)\Lambda L(p)\,, \quad \Rightarrow W^\mu{}_\nu k^\nu = k^\mu\,. \tag{2.10}$$

In general, this group is not trivial. For example, it is the ISO(2) group for massless particles and SU(2) group for massive particles. Therefore, the problem of finding the matrices $C_{\sigma,\sigma'}(\Lambda, p)$ has been reduced to finding the little group representation $D_{\sigma,\sigma'}[W(\Lambda, p)]$

$$U(\Lambda)|\psi_{p,\sigma}\rangle = \frac{N(p)}{N(\Lambda p)} \sum_{\sigma'} D_{\sigma,\sigma'}[W(\Lambda, p)]|\psi_{p,\sigma'}\rangle\,, \tag{2.11}$$

Representations of a larger group (for instance the Poincare group) derived from the representations of its subgroup are known as induced representations. Finally, by using the orthogonality condition of one-particle states

$$\langle\psi_{k',\sigma'}|\psi_{k,\sigma}\rangle = 2k^0\delta^3(\vec{k} - \vec{k}')\delta_{\sigma,\sigma'}\,, \tag{2.12}$$

we determine the normalization that turns out to be unity. Including the additional U(1) factor due to translation, we find the desired transformation property for one-particle state as

$$U(\Lambda)|\psi_{p,\sigma}\rangle = e^{-i(\Lambda p\cdot a)} \sum_{\sigma'} D_{\sigma,\sigma'}[W(\Lambda, p)]|\psi_{p,\sigma'}\rangle\,. \tag{2.13}$$

As a consequence, we conclude that the one-particle states are labelled by the momentum of the particles and transform under some representation of the little group. Since the S-matrix is defined to be the inner product of asymptotically free "in" and "out" states constituted with product of one-particle momentum states, the scattering amplitude, which is related to the S-matrix as $S = 1 + iA$, is naturally labelled by momentum and little group indices of external on-shell particles. The Poincare invariance of the amplitude then tells us that

$$A_{\sigma_1,\cdots,\sigma_n}(p_1,p_2,\cdots,p_n) = \delta^{(4)}(p_1+p_2+\cdots+p_n)\,\mathcal{A}_{\sigma_1,\cdots,\sigma_n}(p_1,p_2,\cdots,p_n)\,, \tag{2.14}$$

$$\mathcal{A}^{\Lambda}_{\sigma_1,\cdots,\sigma_n}(p_1,p_2,\cdots,p_n) = \prod_{i=1}^{n} D_{\sigma_i,\sigma_i'}[W(\Lambda,p)]\mathcal{A}_{\sigma_1',\cdots,\sigma_n'}(\Lambda p_1,\Lambda p_2,\cdots,\Lambda p_n)\,. \tag{2.15}$$

For completeness, it is worthwhile to mention that the so called scattering amplitudes that we compute using the Feynman diagrams in usual quantum field theory are typically Lorentz covariant tensors and do not have the desired transformation properties (2.15). Therefore, polarization tensors (for particles with spin) are introduced in order to convert the "Feynman amplitude" into "Natural amplitude" which is manifestly Little group covariant. The polarization tensors are bi-fundamentals under both Lorentz group and the Little group and transforms in the following way

$$e_{\mu,\sigma\delta}(\vec{\Lambda p}) = \Lambda_{\mu}{}^{\nu} D_{\sigma\sigma'}(W) D_{\delta\delta'}(W) e_{\nu,\sigma'\delta'}(\vec{p})\,, \tag{2.16}$$

where $D_{\sigma\sigma'}$ is in the spin-$\frac{1}{2}$ representation of the Little group. The natural amplitude is then related to the Feynman amplitude as

$$\mathcal{A}_{\sigma_1\ldots\sigma_{2S}}(\vec{p}) = \left(\prod_{i=1}^{2S} e^{\mu_i}{}_{\sigma_i\sigma_i'}(\vec{p})\right)\mathcal{A}_{\mu_1,\ldots,\mu_{2S}}\,. \tag{2.17}$$

The amplitude, carrying only little group indices, is called Natural amplitude (in LHS) and manifestly transforms as (2.15). Here we have shown the Little group indices for a single particle with spin S. In general, we have to introduce polarization tensors for all spinning particles that are involved in the scattering.

2.2 Massless particles

The Pauli Lubanski vector, which is defined in terms of inhomogeneous Lorentz symmetry generators $(J^{\mu\nu}; P^{\sigma})$ as

$$W_\mu := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^{\sigma}\,, \tag{2.18}$$

is related to the momentum 4-vector for massless particles via the following relation

$$W^\mu = \widehat{h}P^\mu\,, \tag{2.19}$$

where the operator $\widehat{h}$ is called the helicity of massless particle. Since $p^0 = |\vec{p}|$ for massless particle, the helicity can be expressed in terms of the spin (of massless particle) projection onto the direction of momentum 3-vector

$$\widehat{h} = \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|}\,. \tag{2.20}$$

This can be considered as the defining relation for helicity. Moreover, for particles travelling at the speed of light, there exists no Lorentz boost which can change the direction of its momentum and this is why $\widehat{h}$ is a Lorentz invariant object. Therefore, all the massless 1-particle momentum states in the scattering matrix can be labelled by the helicity of the particle.

Traditionally, scattering amplitudes are expressed in terms of the Lorentz invariant products of external momentum 4-vector (p^μ), known as: the Mandelstam variables, defined as $s_{ij} := (p_i + p_j)^2$. However, all the fundamental massless particles in the Standard model have non zero helicity, which in turn can be utilized to label the momentum states and therefore the scattering amplitude. Thus, it is useful to find kinematic variables that transform under a smaller representation of the Lorentz group. The lowest possible representations of Lorentz group are the spinor representations denoted by $\left(0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 0\right)$. They are the spin-$\frac{1}{2}$ representations of two different SU(2) groups needed to describe the Lorentz group SO(3,1) [1]. In terms of the spinor representation, a momentum 4-vector p^μ can be expressed as a 2×2 hermitian matrix $p_{\alpha\dot{\alpha}} = p_\mu \sigma^\mu_{\alpha\dot{\alpha}}$, where $\sigma^0_{\alpha\dot{\alpha}}$ is the identity matrix and $\vec{\sigma}_{\alpha\dot{\alpha}}$ are the Pauli matrices. Here $(\alpha, \dot{\alpha})$ are labelling the two different SU(2) spinor indices describing the Lorentz group. In this representation, the matrix $p_{\alpha\dot{\alpha}}$ is related to the 4-vector p^μ as

$$p_{\alpha\dot{\alpha}} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}. \tag{2.21}$$

The determinant of the matrix $p_{\alpha\dot{\alpha}}$ gives the square of the norm of the 4-vector: $p_\mu p^\mu = p^2$. For massless particle, the determinant of the matrix $p_{\alpha\dot{\alpha}}$ is vanishing. Therefore, the momentum vector of a massless particle in spinor representation can be written in terms of a pair of 2-component Weyl spinors $(\lambda_{j\alpha}, \tilde{\lambda}_{j\dot{\alpha}})$ as

$$p_{j,\alpha\dot{\alpha}} := \lambda_{j\alpha}\tilde{\lambda}_{j\dot{\alpha}} \equiv |j\rangle[j|\,. \tag{2.22}$$

Here $(\lambda_{j\alpha}, \tilde{\lambda}_{j\dot{\alpha}})$ are the "spinor helicity variables" and j is the particle index. Since one can always rescale the spinor-helicity variables as

$$\lambda_\alpha \longrightarrow t\lambda_\alpha\,, \qquad \tilde{\lambda}_{\dot{\alpha}} \longrightarrow t^{-1}\tilde{\lambda}_{\dot{\alpha}}\,, \tag{2.23}$$

it is impossible to assign unique spinor-helicity variables to a given on-shell momentum p^μ, which remains invariant under this scaling. Now we show that this scaling is actually associated with the little group transformations for massless particles. First, we note that,

[1] The topology of inhomogeneous Lorentz group is SL(2,$\mathbb{C}$)/ $\mathbb{Z}_2 \ltimes \mathbb{R}^4$.

in SU(2) basis, an arbitrary momentum is related to the standard momentum as

$$p_{\alpha\dot{\alpha}} = S_{\alpha}{}^{\beta} R_{\dot{\alpha}}{}^{\dot{\beta}} k_{\beta\dot{\beta}} \,, \tag{2.24}$$

where recall that the standard momentum for massless particle is $k^{\mu} = (E, 0, 0, E)$. If we express $k_{\alpha\dot{\alpha}}$ in the spinor helicity formalism as

$$k_{\alpha\dot{\alpha}} = \lambda_{\alpha}^{(k)} \tilde{\lambda}_{\dot{\alpha}}^{(k)} \,, \tag{2.25}$$

then the spinor helicity variables for $k_{\alpha\dot{\alpha}}$ and $p_{\alpha\dot{\alpha}}$ are related as

$$\lambda_{\alpha} = S_{\alpha}{}^{\beta} \lambda_{\beta}^{(k)} \,, \quad \tilde{\lambda}_{\dot{\alpha}} = R_{\dot{\alpha}}{}^{\dot{\beta}} \tilde{\lambda}_{\dot{\beta}}^{(k)} \,, \tag{2.26}$$

where $S_{\alpha}{}^{\beta}$ and $R_{\dot{\alpha}}{}^{\dot{\beta}}$ are the standard left and right handed generators of the Lorentz group written in SU(2) basis. Since the little group does not change $k_{\alpha\dot{\alpha}}$, the spinor helicity variables can transform under the little group as

$$\lambda_{\alpha}^{(k)} \longrightarrow t \lambda_{\alpha}^{(k)} \,, \qquad \tilde{\lambda}_{\dot{\alpha}}^{(k)} \longrightarrow t^{-1} \tilde{\lambda}_{\dot{\alpha}}^{(k)} \,, \tag{2.27}$$

where t is an arbitrary complex number. Now using (2.26), we can find the transformations of the spinor helicity variables $\lambda_{\alpha}, \tilde{\lambda}_{\dot{\alpha}}$ and it turns out to be exactly (2.23). Therefore, we can identify (2.23) as the little group scaling for massless particles.

Requiring the 4-momentum to be real in Minkowski signature, the spinor helicity variables are related to each other as

$$(\lambda_{\alpha})^{\star} = \pm \tilde{\lambda}_{\dot{\alpha}} \,, \tag{2.28}$$

where the choice of the sign is associated to the sign convention of the energy of the particle. We always consider the energy to be positive and therefore consider only the +

sign in the above reality condition. For real Lorentzian momentum, the complex number t becomes an $U(1)$ phase factor [2].

Since the spinor indices can be raised and lowered with the anti symmetric Levi-Civita tensor as

$$\lambda_\alpha = \epsilon_{\alpha\beta}\lambda^\beta , \qquad \tilde{\lambda}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\beta}} , \tag{2.29}$$

it can be easily seen that the spinor helicity variables are on-shell variables in the following sense

$$p_{\alpha\dot{\alpha}}\lambda^\alpha = 0 = p_{\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}} , \tag{2.30}$$

making them a suitable candidate for expressing the scattering amplitudes. In spinor-helicity formalism, we now introduce the Lorentz invariant and little group covariant angle and square brackets which will repeatedly appear in the rest of the thesis as

$$\langle ij\rangle := \lambda_i^\alpha \lambda_{j\alpha} , \qquad [ij] := \tilde{\lambda}_{i\dot{\alpha}}\tilde{\lambda}_j^{\dot{\alpha}} . \tag{2.31}$$

The Mandelstam variables, in spinor helicity formalism, can be expressed in terms of these brackets as

$$s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j = \langle ij\rangle[ij] . \tag{2.32}$$

The 2-component spinor helicity variables satisfy Schouten identities. In terms of the angle or square brackets, these identities takes the following form

$$\langle 12\rangle\langle 3\lambda\rangle + \langle 23\rangle\langle 1\lambda\rangle + \langle 31\rangle\langle 2\lambda\rangle = 0 , \tag{2.33}$$

[2]For massless particles, the standard momentum remains invariant under rotations- confined in the $x-y$ plane and 2-d translations. But finite dimensional representations require all the one-particle massless momentum states to have zero eigenvalues under these translations. Therefore, only the U(1) subgroup of the ISO(2) group can be considered to be the little group for massless particles in four dimensions.

$$[12][3\lambda] + [23][1\lambda] + [31][2\lambda] = 0\,, \tag{2.34}$$

with λ being an arbitrary spinor. The momentum conservation, due to the translational invariance of amplitude is expressed in terms of spinor helicity variables as follows

$$\sum_{i=1}^{n} p_i = 0 \Leftrightarrow \sum_{i=1}^{n} \lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}} = 0 \Leftrightarrow \sum_{i=1}^{n} |i\rangle[i| = 0\,. \tag{2.35}$$

In terms of the massless spinor helicity variables, the polarization vectors can be expressed as

$$e^{+}_{\mu}(i) = \frac{\langle\zeta|\sigma_{\mu}|\tilde{\lambda}_i]}{\sqrt{2}\langle\zeta\lambda_i\rangle}\,, \quad e^{-}_{\mu}(i) = \frac{\langle\lambda_i|\sigma_{\mu}|\zeta]}{\sqrt{2}[\tilde{\lambda}_i\zeta]}\,, \tag{2.36}$$

where ζ is an arbitrary reference spinor. Finally, let us give an explicit realization of these spinor helicity variables in terms of on-shell momentum

$$\lambda^{\alpha} = \frac{1}{\sqrt{p^0+p^3}}\begin{pmatrix} p^0+p^3 \\ p^1+ip^2 \end{pmatrix}\,, \qquad \tilde{\lambda}^{\dot{\alpha}} = \frac{1}{\sqrt{p^0+p^3}}\begin{pmatrix} p^0+p^3 \\ p^1-ip^2 \end{pmatrix}\,. \tag{2.37}$$

2.2.1 Three particle amplitudes

In this section, we collect all the required massless three particle amplitudes which will be used to build four and higher point amplitudes recursively. Due to the three particle massless kinematics

$$p_1^{\mu} + p_2^{\mu} \to p_3^{\mu} \;\Rightarrow p_1\cdot p_2 = p_1\cdot p_3 = p_3\cdot p_2 = 0\,, \tag{2.38}$$

either all the angle or square brackets are vanishing

$$\langle 12\rangle[12] = \langle 13\rangle[13] = \langle 23\rangle[23] = 0\,.$$

Therefore, the three-particle amplitude involving massless particles with helicity $h_{1,2,3}$ can be expressed either in terms of angle or square brackets and the little group scaling then fixes the structure of amplitude upto an overall multiplicative constant

$$\begin{aligned}\mathcal{A}_3^{h_1 h_2 h_3}[1,2,3] &= g[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2}\,; \quad \text{with } h_1+h_2+h_3>0 \\ &= g^{'}\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_1-h_2-h_3}\,; \quad \text{with } h_1+h_2+h_3<0\,. \end{aligned} \tag{2.39}$$

Here the constraints on the sum of helicities ensure that the amplitude has a smooth vanishing limit for real momenta in Minkowski signature. We end the discussion on massless spinor helicity formalism here and proceed to introduce spinor helicity formalism for massive particles.

2.3 Massive particles

There are several attempts to develop a spinor helicity formalism for massive particles over the past few decades [26–30]. These approaches begin by expressing the SU(2) representation of massive momentum 4-vector as

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha\tilde{\lambda}_{\dot{\alpha}} - \frac{m^2}{\langle\lambda\eta\rangle[\tilde{\lambda}\tilde{\eta}]}\eta_\alpha\tilde{\eta}_{\dot{\alpha}}\,, \tag{2.40}$$

for some reference spinors η_α, $\tilde{\eta}_{\dot{\alpha}}$. The spin of the particle is projected along the lightlike direction of $\eta_\alpha\tilde{\eta}_{\dot{\alpha}}$, which was then used to label the external states. This way of introducing the massive spinor helicity variables does not manifest the Lorentz symmetry due to the choice of a lightlike direction and obscures the little group covariance of on-shell scattering amplitude from the very beginning. Therefore, this approach severely limits the scope of the program for systematically classifying and constructing on-shell amplitudes. However, recently Arkani-Hamed et al. [11] have introduced a little group covariant spinor helicity formalism for massive particles that does not involve any preferred direction for spin to label the massive external states. This formalism also allows one to take a suitable massless limit of the massive amplitude in a systematic manner. We use this formalism throughout this thesis in computing massive scattering amplitude. We now review this formalism below.

For a massive particle, the spinor representation of momentum 4-vector can be simply taken to be a linear combination of rank 1 objects [11]

$$p_{\alpha\dot{\alpha}} = \sum_{I,J=1}^{2} \epsilon_{IJ}\lambda^I_\alpha \tilde{\lambda}^J_{\dot{\alpha}}\,, \tag{2.41}$$

where (I, J) stands for SU(2) little group indices which can be lowered and raised by the Levi-Civita tensor ϵ_{IJ}. The variables $\lambda^I_\alpha, \tilde{\lambda}^J_{\dot{\alpha}}$ are called the massive spinor-helicity variables. Note that for a particular value of I, J, the bilinear $\lambda\tilde{\lambda}$ is identical to the massless representation, but due to an additional sum, the determinant of the 2×2 matrix is non vanishing. Similar to the massless case, there is no unique way to fix these spinors, as the momentum $p_{\alpha\dot{\alpha}}$ is left invariant by the following transformations

$$\lambda^I_\alpha \longrightarrow W^I{}_J \lambda^J_\alpha \qquad \tilde{\lambda}^J_{\dot{\alpha}} \longrightarrow (W^{-1})^J{}_K \tilde{\lambda}^K_{\dot{\alpha}}\,. \tag{2.42}$$

Unlike the massless case, these W matrices can generally be GL$(2,\mathbb{C})$ matrix which is not the little group for massive particles. But if we impose the reality condition on momentum vector and demand that $\det\left(\lambda^I_\alpha\right) = m$ and $\det\left(\tilde{\lambda}^J_{\dot{\alpha}}\right) = m$, it can be shown that W is indeed a SU(2) matrix. To illustrate this, let us define$M := \det\left(\lambda^I_\alpha\right)$ and $\widetilde{M} := \det\left(\tilde{\lambda}^J_{\dot{\alpha}}\right)$ which are identical to each other as m. Now we consider a GL$(2,\mathbb{C})$ transformation for the spinors, that induces the following transformation

$$M^{'} = \det(W)M\,, \qquad \tilde{M}^{'} = \det(W^{-1})\tilde{M}\,. \tag{2.43}$$

Since the mass of a particle is a physical quantity, we therefore conclude that $\det(W)^2 = 1$. We consider the case when $\det(W) = +1$ since the other choice does not form a group and find that W is an $\mathbf{SL}(2,\mathbb{C})$ element. The reality condition $p^\dagger = p$ then further restricts W to be in the SU(2) subgroup of $\mathbf{SL}(2,\mathbb{C})$. Since, the massive spinor helicity variables transform correctly under little group, we conclude that the on-shell amplitudes are Lorentz invariant functions of λ_I, $\tilde{\lambda}_I$. Let us now point out one of important difference between the massless and massive spinor helicity variables. Since both of them are on-shell variables, the massless variables satisfy the Weyl equation and therefore λ can not be traded in terms of $\tilde{\lambda}$ or vice-verse. This is not the situation in the case of massive particles as they satisfy the Dirac equation which takes the following form in terms of spinor variables

$$p_{\alpha\dot{\alpha}}\lambda_I^\alpha = -m\tilde{\lambda}_{I\dot{\alpha}}\ ; \quad p_{\alpha\dot{\alpha}}\tilde{\lambda}_I^{\dot{\alpha}} = m\lambda_{I\alpha}\,. \tag{2.44}$$

This clearly suggests that we can trade the spinor helicity variables in terms of each other and therefore we can express an amplitude solely in terms of any one of the spinor helicity variables.

To label the external massive states, it is useful to use the symmetric spin-$\frac{1}{2}$ representation of SU(2) because in the conventional spin representation for SU(2), we need to pick a preferred direction to define the S_z operator- therefore breaking the rotational invariance of the scattering amplitude. So in general, the scattering amplitude involving massive particles with spin $\{S_i\}$ and massless particles with helicity $\left\{h_j\right\}$ is represented by

$$\mathcal{A}^{\{h_j\}}_{\{I_1 I_2 \ldots I_{2S_i}\}}\left(\{p_i\}, \left\{p_j\right\}\right)\,, \tag{2.45}$$

where $\{I_i\}$ are the little group indices for the $i-$th massive particle with spin S_i and $\left\{h_j\right\}$ labels the $j-$th massless particle with helicity h_j. Then the scattering amplitude transforms as

$$\mathcal{A}^{\{h_j\}}_{\{I_1 I_2 \ldots I_{2S_i}\}}\left(\{p_i\},\{p_j\}\right) \to \left(\prod_j t^{-2h_j} \prod_i W_{i,I_1}{}^{J_1} W_{i,I_2}{}^{J_2} \cdots W_{i,I_{2S_i}}{}^{J_{2S_i}}\right) \mathcal{A}^{\{h_j\}}_{\{J_1 J_2 \ldots J_{2S_i}\}}\left(\{p_i\},\{p_j\}\right) , \tag{2.46}$$

where t^{-2h_j} is the U(1) scaling and W_i's are SU(2) matrices in the fundamental representation. To conclude this section, we give the expression of the polarization tensor of a massive spin-1 particle

$$e^a{}_{IJ} = -\frac{1}{2\sqrt{2}m}\left[\lambda_I{}^{\beta}\epsilon^{\dot{\alpha}\dot{\gamma}}(\sigma^a)_{\beta\dot{\gamma}}\tilde{\lambda}_{J\dot{\alpha}} + (I \leftrightarrow J)\right] . \tag{2.47}$$

Next, we review the construction of the three particle amplitudes with atleast one massive particle. These will serve as building blocks to set up the on-shell recursion.

2.3.1 Three particle amplitudes

Let us now review the classification of three particle amplitudes involving massive and massless particles following [11]. As mentioned earlier, all the massive external states (with spin S) are labelled by the symmetric $2S$ representation of SU(2). Since the spinor helicity variables of massive particles are related to each other by the Dirac equation (2.44), any generic amplitude can be expressed in terms of only λ^I_α variables. For instance, the three particle amplitude involving a pair of massive particles of spin $S_{1,2}$ and a massless particle of helicity h can be expressed as

$$\mathcal{A}^h_{(3)\{I_1,\ldots,I_{2S_1}\};\{J_1,\ldots,J_{2S_2}\}} = \lambda^{\alpha_1}_{I_1}\cdots\lambda^{\alpha_{2S_1}}_{I_{2S_1}}\lambda^{\beta_1}_{J_1}\cdots\lambda^{\beta_{2S_2}}_{J_{2S_2}}\mathcal{A}^h_{(3)\{\alpha_1,\ldots,\alpha_{2S_1}\};\{\beta_1,\ldots,\beta_{2S_2}\}} , \tag{2.48}$$

where the object on the RHS $\mathcal{A}^h_{(3)\{\alpha_1,\ldots,\alpha_{2S_1}\};\{\beta_1,\ldots,\beta_{2S_2}\}}$ is a Lorentz (in SL(2,$\mathbb{C}$) representation) tensor and called "stripped amplitude". The advantage of introducing the above relation is that the classification problem of three particle amplitudes reduces to finding the general structure of these stripped amplitudes. Notice that, as we are using spin $\frac{1}{2}$ representation to label the massive spin states, algebraically, the stripped amplitude is a polynomial of two linearly independent spinors $(u_\alpha\, ; v_\beta)$ in order to span the SL(2,$\mathbb{C}$) space. So we now need to find these independent spinors which depend on how many massive legs are attached to the three particle amplitude and we will analyse each case separately below. It is also useful to note that the degree of this polynomial would be $2(S_1 + S_2)$.

One massive, two massless legs

We start with the three particle amplitude involving a pair of massless legs of helicity $h_{1,2}$ and one massive leg with spin S. Since the stripped amplitude $A_{\alpha_1,..,\alpha_{2S}}$ is a polynomial of

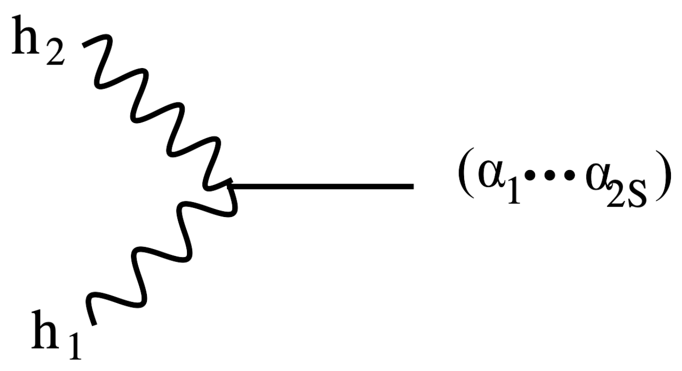


Figure 2.1: Three particle amplitude with two massless legs with helicity $h_{1,2}$ and one massive leg with spin S.

degree $2S$, we need S number of each of the linearly independent spinors $(u_\alpha\, , v_\alpha)$. Since we have two massless legs, we can use the massless spinor helicity variables, as they are independent of each other, as the candidates for $(u_\alpha\, , v_\alpha)$. Additionally, to account for the little group scaling of the massless legs, we need to introduce either angle or square bracket involving only massless spinor helicity variables. Tensorially, we can express the amplitude as

$$\mathcal{A}^{h_1 h_2}_{\{\alpha_1,..\alpha_{2S}\}} = \kappa \left(\lambda_1^a \quad \lambda_2^b\right)_{\{\alpha_1..\alpha_{2S}\}} [12]^c \, . \tag{2.49}$$

where the notation $(u^a \, , v^b)_{\{\alpha_1,...,\alpha_{2S}\}}$ represents product of the spinors u_{α_i} and v_{α_j} symmetric in the spinor indices such that $i \in [1,a]$ and $j \in [1,b]$. Here κ is a coupling constant and a, b, c are some exponents that can be fixed by little group scaling, giving the following relations

$$a - c = -2h_1 \, , \quad b - c = -2h_2 \, , \quad a + b = 2S \, , \tag{2.50}$$

where the last relation is due to the fact that the degree of the polynomial is $2S$. Taking into the account the fact that the three-point amplitude has mass dimension one, we obtain the structure of this three-point amplitude to be

$$\mathcal{A}^{h_1 h_2}_{[3]\{\alpha_1,..\alpha_{2S}\}} = \frac{g}{m^{2S+h_1+h_2-1}} \left(\lambda_1^{S-h_1+h_2} \lambda_2^{S-h_2+h_1}\right)_{\{\alpha_1..\alpha_{2S}\}} [12]^{S+h_1+h_2} \, , \tag{2.51}$$

where g is a dimensionless coupling. To elucidate our notation, we give the expression of three particle amplitude with massive particle with spin $S = 1$ and identical photons below

$$\mathcal{A}^{h_1=1,\, h_2=1}_{[3]\{\alpha_1,\alpha_2\}} = \frac{g}{m^3} \left(\lambda_{1,\alpha_1}\lambda_{2,\alpha_2} + \lambda_{1,\alpha_2}\lambda_{2,\alpha_1}\right) [12]^3 \, . \tag{2.52}$$

Notice that, we could have used $\langle 12 \rangle$ but it is related to $[12]$ as $\langle 12 \rangle = \frac{m^2}{[21]}$. The unique structure of the three particle amplitude for this specific configuration of external particle implies no-go theorems for certain types of interactions. For instance, if we consider $h_1 = h_2 = \pm h$ then the square bracket $[12]^{s \pm 2h}$ will attain an additional $(-1)^{s \pm 2h}$ factor under the exchange of massless legs. If we consider the massless legs as bosons, then the amplitude must be invariant under this exchange and we conclude that the spin of the

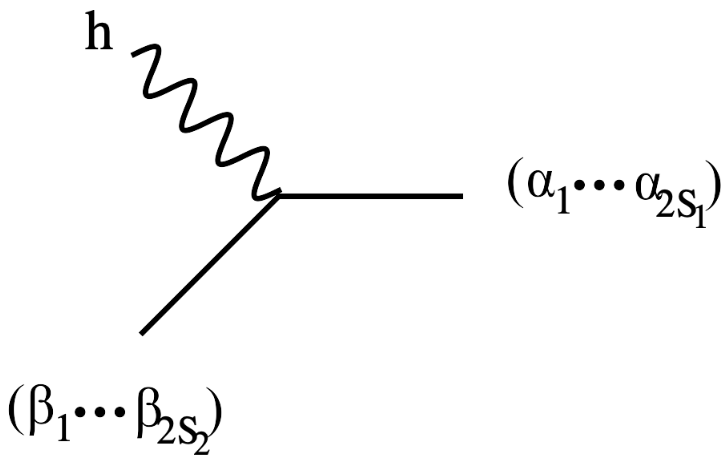


Figure 2.2: Three point amplitude with massive legs of spin $S_{1,2}$ and massless leg of helicity h.

massive particle must be $S = 2(n \pm h)$ for some integer n. Therefore, a massive particle with odd spin can not decay into two massless identical bosons. In this case, the amplitude in equation (2.52) vanishes identically.

In the case of opposite helicity massless particles, if we restrict ourselves to $S - h_1 + h_2 > 0$ and $S + h_1 - h_2 > 0$ (otherwise we end up with spurious poles), the 3-particle amplitude vanishes for $S < 2|h|$. Therefore, a massive particle with spin-1 can not decay into a pair of photons and a massive spin-3 particle can not decay into a pair of gravitons.

Three particle amplitude with a pair of massive particles and one massless particle

Next, we consider the three point amplitude with a pair of massive particles of spin $S_{1,2}$ and a massless particle of helicity h. The structure of this amplitude will heavily depend on whether the masses are equal or not, since the equal mass configuration appears precisely in the case of the minimal configuration that corresponds to the unique massless amplitude in the high energy limit[3].

[3]We discuss the concepts of minimal coupling in section 2.4 and the procedure of taking the high energy limit of an amplitude involving massive particles in Appendix A.

Let us first consider the configuration of external massive particles with different masses. In this case, we can choose the linearly independent spinors to be

$$u_\alpha = \lambda_{3\alpha}\,, \qquad v_\alpha = \frac{p_{2\alpha\dot{\alpha}}}{m}\tilde{\lambda}_3^{\dot{\alpha}}\,. \tag{2.53}$$

But unlike the previous case with one massive leg, here the structure of the amplitude is not unique since we only have two constraints: a) the little group scaling for the massless leg and b) the degree of the polynomial being $2(S_1 + S_2)$. Therefore, there is an inherent ambiguity of choosing the tensor structure in order to distribute all the $\mathbf{SL}(2,\mathbb{C})$ indices symmetrically and it can determined that there exists a total number of $N = S_1 + S_2 - |S_1 - S_2| + 1$ number [4] of structures possible. The three particle amplitude thus takes the following form

$$\mathcal{A}^h_{\{\alpha_1,\ldots,\alpha_{2S_1}\}\{\beta_1,\ldots,\beta_{2S_2}\}} = \sum_{g_i=1}^{N} g_i \left[u^{(S_1+S_2+h)}v^{(S_1+S_2-h)}\right]_{\{\alpha_1,..,\alpha_{2S_1}\}\{\beta_1,..,\beta_{2S_2}\}}\,, \tag{2.54}$$

where g_i are the coupling constants associated to different tensor structures of (u, v). To illustrate the choice of different possible tensor structures, let us consider the example with $S_1 = 1$ and $S_2 = 2$ and $h = -1$. In this case, there are three different tensor structures possible

$$(vv)_{\alpha_1\alpha_2}(vvuu)_{\beta_1,\ldots,\beta_4}\,, \quad (uu)_{\alpha_1\alpha_2}(vvvv)_{\beta_1,\ldots,\beta_4}\,, \quad (uv)_{\alpha_1\alpha_2}(uvvv)_{\beta_1,\ldots,\beta_4}\,. \tag{2.55}$$

For identical spin ($S_1 = S_2$), there are as many as $\tilde{N} = S_1 + S_2$ number of structures possible

$$S_1 = S_2 = \frac{1}{2} : (u)_{\alpha_1}(v)_{\beta_1}\,. \tag{2.56}$$

$$S_1 = S_2 = 1 : (uu)_{\alpha_1\alpha_2}(vv)_{\beta_1\beta_2}\,, \quad (uv)_{\alpha_1\alpha_2}(uv)_{\beta_1\beta_2}\,, \tag{2.57}$$

[4]This follows from the spin angular momentum addition rules in quantum mechanics.

$$S_1 = S_2 = \frac{3}{2} : (uuu)_{\alpha_1\alpha_2\alpha_3}(vvv)_{\beta_1\beta_2\beta_3}\,, \quad (uuv)_{\alpha_1\alpha_2\alpha_3}(uvv)_{\beta_1\beta_2\beta_3}\,, \quad (uvv)_{\alpha_1\alpha_2\alpha_3}(uuv)_{\beta_1\beta_2\beta_3}\,. \tag{2.58}$$

Three particle amplitude with a pair of identical mass

Note that, if the masses of the two particles are degenerate, our choice for the two linearly independent spinors in equation (2.53) does not work, since

$$p_2 + p_3 = -p_1 \Longrightarrow 2p_2 \cdot p_3 = mu^\alpha v_\alpha = 0\,. \tag{2.59}$$

Instead they are now parallel to each other and picks up a single direction in $\mathbf{SL}(2, \mathbb{C})$ space

$$xu_\alpha = v_\alpha \Rightarrow x = \frac{\langle\xi|p_2|\tilde{\lambda}_3]}{m\langle\xi\lambda_3\rangle}\,, \tag{2.60}$$

where the $x-$factor is the proportionality constant between u_α and v_α and has mass dimension 0 and little group weight -2. Sometimes, it is helpful to use the inverse expression

$$x^{-1} = \frac{\langle\lambda_3|p_2|\xi]}{m[\tilde{\lambda}_3\xi]}\,. \tag{2.61}$$

The $x-$factor plays a special role in three particle amplitude with the same mass particles. It was shown in [11] that this factor encodes an apparent non-locality, associated with amplitudes arising from minimal coupling configuration.

Since there is at least one massless leg available, we use the massless spinor variable $\lambda_{3\alpha}$ to associate the $\mathbf{SL}(2, \mathbb{C})$ indices. But with these, we can only distribute half of the $2(S_1 + S_2)$ number of $\mathbf{SL}(2, \mathbb{C})$ indices. Therefore we have to use the Levi-Civita tensors $\epsilon_{\alpha\beta}$ to carry the rest of the indices. The algebraic structure for the three particle amplitude therefore takes the form:

$$\mathcal{A}^h_{\{\alpha_1,..,\alpha_{2S_1}\}\{\beta_1,..,\beta_{2S_2}\}} = \sum_{i=|S_1-S_2|}^{S_1+S_2} g_i x^{h+i} \left(\lambda_3^{2i} \epsilon^{S_1+S_2-i}\right)_{\{\alpha_1,...,\alpha_{2S_1}\}\{\beta_1,..,\beta_{2S_2}\}} . \tag{2.62}$$

We conclude this section with the following remark: we could have used $\epsilon_{\alpha\beta}$ in the case of unequal mass, but this procedure is equivalent to the one followed here, as one can trade either u_α or v_α with the Levi-Civita tensor using

$$(u_\alpha v_\beta - u_\beta v_\alpha) = \langle uv \rangle \epsilon_{\alpha\beta} . \tag{2.63}$$

But this relation does not exist in the equal mass case as $xu_\alpha = v_\alpha$. Therefore $\epsilon_{\alpha\beta}$ can be treated as an independent tensor in the equal mass case and can be used to span the $\mathbf{SL}(2, \mathbb{C})$ space.

In this thesis, we will need the three particle amplitudes involving a pair of massive particles with same mass given in equation (2.62). This expression can be thought of as an expansion in λ_α and the term with $i = 0$, in which all the $\mathbf{SL}(2, \mathbb{C})$ indices are carried by the anti-symmetric Levi-Civita tensor and represents to a special type of interaction that is known as the minimal coupling in usual quantum field theory. Throughout this thesis, we have considered particles interacting with each other only via minimal coupling. In the following section, we discuss this concept of minimal coupling in the context of on-shell scattering amplitudes.

2.4 Minimal coupling

The notion of "minimal coupling" for massless particles simply means that the leading low energy interaction for the exchange particles such as photons, gluons and gravitons with dimensionless couplings: e, g or the gravitational coupling $1/M_{pl}$ (where M_{pl} denotes the Planck's mass) should include only massless particles with opposite helicity configurations. This can be elucidated by considering the three particle amplitude in the following equation

$$\mathcal{A}_3^{h_1h_2h_3}[1,2,3] = g[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2}\ ; \quad \text{with } h_1+h_2+h_3>0$$
$$= g^{'}\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_1-h_2-h_3}\ ; \quad \text{with } h_1+h_2+h_3<0\,, \quad (2.64)$$

in which the mass dimension of the coupling constant ($[g]_m$) is given by

$$[g]_m = [g^{'}]_m = 1-|h_1+h_2+h_3|\,, \quad (2.65)$$

which is zero for photons and gluons, and -1 for graviton [5], when the massless particles have opposite helicity. This notion of minimal coupling is familiar in textbook quantum field theory in which the minimal interactions appear with dimensionless couplings. For example,

$$\text{scalar QED}: \quad e(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi)A^\mu\,, \quad (2.66)$$

$$\text{QED}: \quad e\bar{\psi}\gamma_\mu\psi A^\mu\,, \quad (2.67)$$

$$\text{QCD}: \quad g f^{ABC}A^\mu_B A^\nu_C\partial_\mu A_{A\nu}\,, \quad (2.68)$$

$$\text{linearised gravity}: \quad \kappa hh\Box h\,. \quad (2.69)$$

In the previous section, the most general algebraic structure for a three particle amplitude involving a pair of particles with degenerate mass and a massless particle has been derived in equation (2.62) as a series expansion in the massless spinor helicity variable λ. For identical spin ($S_1 = S_2 = S$) of massive particles, this series contains a total of $(2S+1)$ number of terms. Now in the high energy limit, one of these terms reproduces the unique massless amplitude given in equation (2.39). As we explain below, this special term corresponds to $i=0$ in the expansion where the Levi-Civita tensor carries all the $\mathbf{SL}(2,\mathbb{C})$

[5] For linearised gravity, the kinetic term is $\mathcal{L}_{kin} \sim h\Box h$, where $h_{\mu\nu}$ has mass dimension 1. Therefore, the leading interaction term is $\kappa hh\Box h$ with $\kappa \sim 1/M_{PL}$.

indices. All the other terms in the expansion (2.39) correspond to the couplings to higher-dimensional operators. Since in this thesis, we have considered only minimally coupled amplitudes, we will not discuss the higher dimensional terms and contain ourselves with the following three particle stripped amplitudes associated minimal coupling:

$$\mathcal{A}^{min,h}_{[3]\{\alpha_1,..\alpha_{2S}\}\{\beta_1,..\beta_{2S}\}} = mgx^h\left(\prod_{i=1}^{2S}\epsilon_{\alpha_i\beta_i} + symm.\right), \tag{2.70}$$

where we have added all the symmetric combinations of the product of ϵ's in the $\mathbf{SL}(2,\mathbb{C})$ indices. The on-shell amplitude can be found by contracting the massive spinor helicity variables with this stripped amplitude following (2.48)

$$\mathcal{A}_3^{+h}(\mathbf{1},\mathbf{2},3^h) = gx_{12}^h\frac{\langle\mathbf{12}\rangle^{2S}}{m^{2S-1}}\,, \quad \mathcal{A}_3^{-h}(\mathbf{1},\mathbf{2},3^{-h}) = gx_{12}^{-h}\frac{[\mathbf{12}]^{2S}}{m^{2S-1}}\,, \tag{2.71}$$

where x_{12} arises due to the presence of massive particles with identical masses as discussed earlier and defined as

$$x_{12} = \frac{\langle\zeta|p_1|3]}{m\langle\zeta 3\rangle} \quad \text{or} \quad x_{12}^{-1} = \frac{\langle 3|p_1|\zeta]}{m[3\zeta]}\,. \tag{2.72}$$

Here ζ^α is a an arbitrary spinor which can be chosen appropriately. Here we have introduced bold faced notation for spinor brackets associated to massive particles. These bold faced spinor products are defined as a symmetric combination of usual spinor products carrying SU(2) indices. For example,

$$\langle\mathbf{12}\rangle^2 := \langle 1^{I_1}2^{J_1}\rangle\langle 1^{I_2}2^{J_2}\rangle + \langle 1^{I_2}2^{J_1}\rangle\langle 1^{I_1}2^{J_2}\rangle\,, \tag{2.73}$$

$$\langle 3\mathbf{2}\rangle^2 := \langle 32^{J_1}\rangle\langle 32^{J_2}\rangle\,. \tag{2.74}$$

In appendix A, we outline the procedure for taking the high energy limit of scattering amplitudes in the massive spinor helicity formalism following [11] and show that the above three particle amplitudes for $S = 1$ indeed reduce to the unique massless three particle amplitude in this high energy limit.

Note that for $S > 1$, the three-particle amplitudes in (2.71) do not correspond to any minimal interaction term in a Lagrangian theory. It was shown in [40] that the term $x\langle\mathbf{12}\rangle^{2S}$ contains $(2S - 1)$ powers of momenta and therefore the amplitudes must correspond to higher derivative interactions for $S > 1$. This concept of "minimal coupling" introduced in [11] is different from earlier work on higher-spin theories [41, 42]. Conventionally, minimal coupling represents lowest derivative interaction compatible with symmetries of the theory.

Before we conclude, let us make a few comments on the equivalence between the amplitude computed using Feynman rules and the method we described in this section in the spinor helicity formalism. To make contact with the amplitude computed using Feynman diagrams, we consider an example of a three particle amplitude involving two massive scalars with mass m and a positive helicity photon. According to the momentum space Feynman rule, the amplitude is

$$\mathcal{A}_{[3]}^{min,s=1,h=+1} = -e_3^+ \cdot p_2 = \frac{\langle\xi|p_2|3]}{\langle\xi 3\rangle} = mx\,, \tag{2.75}$$

where in the last step, we have used the definition of massless polarization vector in the spinor helicity formalism in equation (2.36).

2.4.1 Three massive legs

For three particle amplitudes with all massive particles, we do not have any independent spinors available. Hence, the $\mathbf{SL}(2,\mathbb{C})$ space must be spanned by higher rank tensors $O_{\alpha\beta}$ in the spinor representation. The primary candidates for the higher rank tensors $O_{\alpha\beta}$ can be $\epsilon_{\alpha\beta}$ or ${p_1}_{\alpha}{}^{\dot\beta}{p_2}_{\beta\dot\beta}$. Therefore any of the two candidate can accommodate half of the $2(S_1 + S_2 + S_3)$ $\mathbf{SL}(2,\mathbb{C})$ indices. Now as the product can be traded for a pair of $\epsilon_{\alpha\beta}$, for

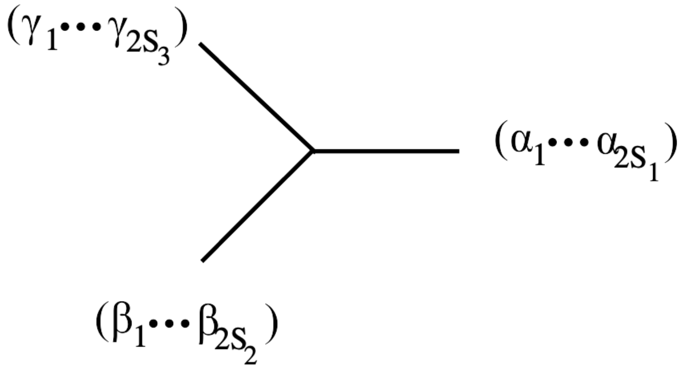


Figure 2.3: Three particle amplitude with all massive legs.

instance

$$O_{\alpha\beta}O_{\gamma\delta} - O_{\gamma\beta}O_{\alpha\delta} = \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\,, \tag{2.76}$$

for $O_{\alpha\beta} \sim \epsilon_{\alpha\beta}$ (due to "Schouten's identity") and $p_{1\alpha}{}^{\dot\beta} p_{2\beta\dot\beta}$, [6] we can use a pair of ϵ's to distribute the $\mathbf{SL}(2,\mathbb{C})$ indices. Therefore, tensorially, the amplitude can be expressed as

$$\mathcal{A}_{\{\alpha_1,..,\alpha_{2s_1}\}\{\beta_1,..,\beta_{2s_2}\}\{\gamma_1,..,\gamma_{2s_3}\}} = \sum_{i=0}^{1}\sum_{\sigma_i} g\sigma_i \left(O^{s_1+s_2+s_3-i}\epsilon^i\right)_{\{\alpha_1,..,\alpha_{2s_1}\}\{\beta_1,..,\beta_{2s_2}\}\{\gamma_1,..,\gamma_{2s_3}\}}\,, \tag{2.79}$$

where $i = 0, 1$ denotes the number of ϵ's and σ_i labels all the different ways the $\mathbf{SL}(2,\mathbb{C})$ indices can be distributed on O's.

[6] This can be simple seen by considering

$$\Xi := O_{\alpha\beta}O_{\gamma\delta} - O_{\gamma\beta}O_{\alpha\delta} \tag{2.77}$$
$$= (\lambda_\delta\eta_\beta - \eta_\delta\lambda_\beta)(\lambda_\alpha\eta_\gamma - \eta_\alpha\lambda_\gamma) \sim \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\,. \tag{2.78}$$

Chapter 3

Covariant recursion

Recursion techniques are one of the most notable and effective developments in the modern approach to derive the scattering amplitudes in gauge and gravity theory. The core idea of on-shell recursion scheme is to construct the $n-$particle scattering amplitude in terms of known lower point amplitudes. These techniques are more efficient than the traditional Feynman diagramatic methods to obtain amplitudes since the latter are obviously not recursive and require a substantial amount of knowledge of the underlying theory (such as: vertex rules, symmetry factors, etc.) at each order in the perturbation theory.

Beyond the three particle amplitudes, the principles of locality and unitarity are the central constraints for four and higher point tree level scattering amplitudes, which simply dictate that the amplitude must factorize into a product of lower point tree amplitudes when any of the internal massless or massive particle goes on-shell. For instance, the four particle amplitude can be constructed by gluing the three-point amplitudes with the internal propagator

$$\text{massless}: \quad \mathcal{A}_{(4)} \to \mathcal{A}_{(3)} \frac{1}{P^2} \mathcal{A}_{(3)}\,, \tag{3.1}$$

$$\text{massive}: \quad \mathcal{A}_{(4)} \to \mathcal{A}_{(3)} \frac{1}{P^2 - m^2} \mathcal{A}_{(3)}\,. \tag{3.2}$$

But the on-shell three particle massless amplitude vanishes in Minkowski spacetime for real momenta. Therefore the recursion method based on unitary factorization requires the off-shell three particle massless amplitude.

However, sometimes it is extremely useful to analytically continue the on-shell external momenta to the complex domain. In this case, recursion relations exploit the pole structure in the complex momentum space to recursively build higher point amplitudes from lower point amplitudes. The first step in this direction of developing a remarkably simple on-shell recursion relation to compute massless scattering amplitudes was taken by Britto-Cachazo-Feng-Witten (BCFW) [7, 8] for tree-level gluon amplitudes. Later, recursion relations were derived for general relativity [43, 44] and eventually found to be a quite general property of tree level scattering amplitudes in quantum field theories in arbitrary dimensions [19, 45].

3.1 The covariant recursion

In this section, we generalize the well-known BCFW recursion relations for scattering amplitudes involving massive particles by combining complex deformation of massive as well as massless external states. We consider $n-$particle tree amplitudes with particle configurations such that there is atleast one massless particle. In order to derive the on-shell recursion relation, a pair of external massive and massless momenta are complexified with a massless momenta (r^μ), while maintaining momentum conservation and same on-shell condition for shifted and unshifted momenta. We consider the massive p_i and massless momenta p_j are analytically continued to complex plane in the following way

$$p_i^\mu \to \widehat{p}_i^\mu = p_i^\mu - zr^\mu \, ; \qquad p_j^\mu \to \widehat{p}_j^\mu = p_j^\mu + zr^\mu . \tag{3.3}$$

Here z is called deformation parameter. The on-shell property can be maintained by imposing the following constraints

$$p_i \cdot r = p_j \cdot r = 0. \tag{3.4}$$

The $n-$point amplitude $\mathcal{A}_n(z=0)$ with real undeformed momenta can be obtained from the deformed one by using the Cauchy's theorem

$$\mathcal{A}_n = \widehat{\mathcal{A}}_n(0) = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{\widehat{\mathcal{A}}_n(z)}{z}dz = -\sum_{z_I} Res\left(\frac{\widehat{\mathcal{A}}_n(z)}{z}\right)_{z=z_I} + \mathcal{R}_n(z\to\infty)\,. \tag{3.5}$$

The contour Γ_0 encloses the pole at origin. $\mathcal{R}_n$ is the boundary term which is the contribution of the contour integral at infinity. All other simple pole locations of the amplitude are denoted by z_I.

Tree-level scattering amplitudes have well-behaved analytic structure, they can only have simple pole in kinematic space, in the form of propagator $\frac{1}{P^2-m^2}$. Simple Feynman diagram analysis indicates that when the internal propagator goes on-shell, the scattering amplitude factorizes into a pair of lower point on-shell subamplitudes. Therefore we express the amplitude with complex momenta as

$$\widehat{\mathcal{A}}_n(z) = \sum_I \widehat{\mathcal{A}}_{l+1}\frac{1}{\widehat{P}_I^2 - m^2}\widehat{\mathcal{A}}_{r+1} + \sum_I \widehat{\mathcal{A}}_{l+1}\frac{1}{\widehat{P}_I^2}\widehat{\mathcal{A}}_{r+1}\,, \tag{3.6}$$

where the sum includes different scattering channels as well as all possible polarization (helicity) states of the exchange particle and $n = l + r$. It is important to note that the constituent subamplitudes are function of complex momenta. We express the shifted propagator in terms of the physical propagator with real momenta and obtain the following simple pole in the complex $z-$plane

$$\widehat{P}_I^2|_{z_I} = m^2 \Rightarrow (P_I + z_I\, p_j)^2 = m^2 \longrightarrow \frac{1}{\widehat{P}_I^2 - m^2} = -\frac{z_I}{z - z_I}\frac{1}{P_I^{\,2} - m^2}. \tag{3.7}$$

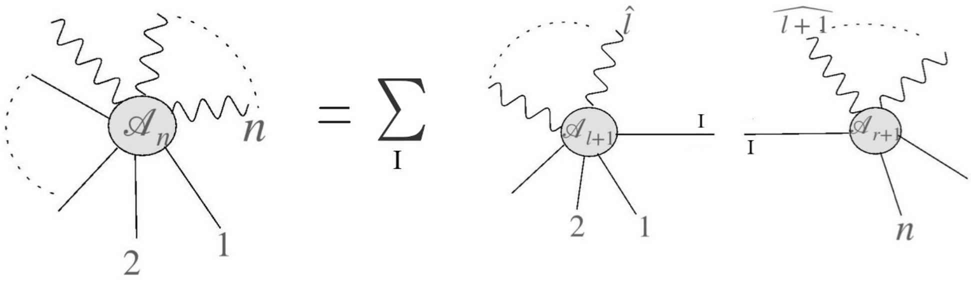


Figure 3.1: On-shell recursion scheme

The boundary term $\mathcal{R}_n$ at $z \to \infty$ can not be computed from a single recursion relation [7,8]. Therefore we assume that the boundary term vanishes for a valid massive-massless shift, that involves complexification of both massive and massless momenta, such as the one in (3.3). A stronger but practical condition to achieve this would be to restrict the allowed class of amplitude as

$$\widehat{\mathcal{A}}_n(z) \to 0\,; \qquad \text{as } z \to \infty. \tag{3.8}$$

With this assumption, we derive the covariant recursion scheme to compute the amplitude as [1]

$$\mathcal{A}_n = \sum_I \widehat{\mathcal{A}}_{l+1}(z_I)\frac{1}{P_I^2 - m^2}\widehat{\mathcal{A}}_{r+1}(z_I) + \sum_I \widehat{\mathcal{A}}_{l+1}(z_I)\frac{1}{P_I^2}\widehat{\mathcal{A}}_{r+1}(z_I)\,, \tag{3.9}$$

where the constituent subamplitudes have to be evaluated at $z = z_I$ - exactly where the shifted propagator goes on-shell. Therefore, the covariant recursion requires only on-shell three particle amplitudes unlike the unitary method.

It is important to note that the only those diagrams in which the two deformed momenta are on opposite sides of on-shell propagator contribute to the residue at $z = 0$. This simple consequence of complex deformation of external momenta enormously simplify computations as compared to the other methods such as, Feynman diagramatics, unitary

[1] In the case of massless amplitudes, only the second term contributes to the recursion known as the BCFW recursion scheme.

factorization, etc. Another point to remember while using the recursion scheme is that this procedure works only for amplitudes that can be constructed using only the three-particle amplitudes. This excludes theories like $g\phi^3 + \lambda\phi^4$, in which the four particle amplitude has a contact term without any simple pole [2].

3.1.1 The covariant massive-massless shift

On-shell recursion scheme is implemented most efficiently when the scattering amplitudes are expressed in the spinor helicity basis. Hence, the momentum shifts (14) should be appropriately realized in terms of the on-shell spinor helicity variables. As the amplitude is covariant under little group, it is paramount that the complex shift of spinor helicity variables obey little group covariance. Keeping this fact in mind and choosing the shift vector to be $r_{\alpha\dot{\alpha}} = \frac{p_{i\alpha\dot{\beta}}}{m}\tilde{\lambda}_j^{\dot{\beta}}\tilde{\lambda}_{j\dot{\alpha}}$ in (14), we propose [14] the following complex deformation of massless and massive spinor helicity variables

$$\widehat{\lambda}_{j\alpha} = \lambda_{j\alpha} + \frac{z}{m}p_{i\alpha\dot{\beta}}\tilde{\lambda}_j^{\dot{\beta}}\,; \qquad \widehat{\tilde{\lambda}}_{j\dot{\alpha}} = \tilde{\lambda}_{j\dot{\alpha}},$$
$$\widehat{\lambda}^I_{i\alpha} = \lambda^I_{i\alpha}\,; \qquad \widehat{\tilde{\lambda}}^I_{i\dot{\alpha}} = \tilde{\lambda}^I_{i\dot{\alpha}} - \frac{z}{m}\tilde{\lambda}_{j\dot{\alpha}}[i^I j]\,. \tag{3.10}$$

These complex shifts are characterize as $[\mathbf{i}\, j^h\rangle$-shift where we have bold faced the massive spinor helicity variable $\left(|\mathbf{i}] \equiv \tilde{\lambda}^I_{i\dot{\alpha}}\right)$ instead of keeping the SU(2) index and h is the helicity of the $j-$th particle. The main feature of these complex shifts is that they are manifestly little group covariant and therefore can be implemented directly into the spinor helicity representation of scattering amplitudes. We refer these kind of shifts as “covariant massive-massless shift”[3].

[2]There exist several extensions of the BCFW recursion relations to compute amplitudes for theories that do not rely on three-particle amplitudes. The most prominent example of these is the $\lambda\phi^4$ theory. Starting with the four-particle amplitude $\mathcal{A}_4 = \lambda$, it is possible to compute higher point amplitudes. In this case, the non-vanishing boundary term at ∞ is recursively constructed. Some of the other examples include-the non linear sigma model [46], or 3D Chern-Simons-matter theories [47], and multi-scalar amplitudes in supersymmetry Yang-Mills theory [48] .

[3]We have initially referred to this shift as the generalized shift and the recursion technique as generalized recursion in this thesis. However, to avoid clash with existing nomenclature in the literature we use the nomenclature “covariant massive massless shift"for the shift and “covariant recursion” for the recursion

In order to illustrate how to implement the covariant recursion, we now compute several known four and five particle amplitudes using the proposed covariant massive-massless shift. In particular, we consider the amplitudes in massive scalar QCD and spontaneously broken non-abelian gauge theory whose spectrum consists of gluons and massive vector bosons. The recursion relation can be used to calculate the n-point amplitude via the recursion relation

$$\mathcal{A}_n = \sum_I \widehat{\mathcal{A}}_{l+1}(z_I)\frac{1}{P^2 - m^2}\widehat{\mathcal{A}}_{r+1}(z_I) + \sum_J \widehat{\tilde{\mathcal{A}}}_{l+1}(z_J)\frac{1}{P^2}\widehat{\tilde{\mathcal{A}}}_{r+1}(z_J)\,. \tag{3.11}$$

We will be considering scattering amplitudes with a specific colour ordering for external particles, since the full colour dressed amplitude can be derived from the colour ordered amplitudes using well known colour decomposition rules [15–18, 49, 50].

3.2 Examples

We use the covariant recursion relation to compute several four and five point amplitudes in two models: a) scalar QCD with massive scalars scattering off gluons and b) spontaneously broken non-abelian gauge theory whose spectrum consists of gluons and massive vector bosons. The examples presented here clearly suggest that the proposed covariant massive-massless shift in equations (3.10) is a valid shift for computing amplitudes in these theories. We present an elaborate proof for this assertion in the next chapter.

3.2.1 Compton amplitude in scalar QCD

We start by considering the $2 \to 2$ scattering involving a pair of massive scalars with momentum (p_1, p_4) and positive helicity gluons with momentum (p_2, p_3). The momentum

technique, as in reference [21], in the rest of the thesis.

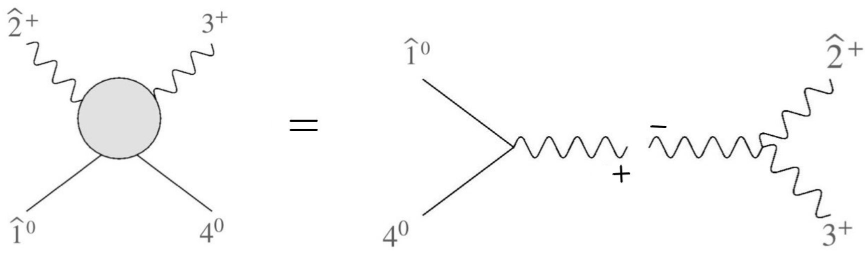


Figure 3.2: Compton amplitude in scalar QCD

shift is carried out on the momenta p_1 and p_2 as

$$\widehat{p}_1^\mu = p_1^\mu - z r^\mu \,; \quad \widehat{p}_2^\mu = p_2^\mu + z r^\mu \,. \tag{3.12}$$

In order to realize this momentum shift in spinor helicity formalism, we consider $[\mathbf{1}2\rangle$ shift introduced in (3.10). Since the scalar-gluon quartic contact term can be expressed in terms of the 3-vertices (similar to the case for only gluon), the four particle amplitude is constructible from three particle amplitudes by implementing the covariant recursion (3.11) (see Figure 3.2)

$$\mathcal{A}_4[\mathbf{1}^0, 2^+, 3^+, \mathbf{4}^0] = m g^2 \widehat{x}_{14} \frac{1}{s_{23}} \frac{[23]^3}{[2\widehat{I}][3\widehat{I}]} = m^2 g^2 \frac{[23]^3}{\langle \widehat{I}|p_4|3][\widehat{I}2] s_{23}} \,, \tag{3.13}$$

where g is a dimensionless coupling and the non-local x-factor given by

$$\widehat{x}_{14} = m \frac{[\widehat{I}3]}{\langle \widehat{I}|p_4|3]} \,, \tag{3.14}$$

and we the Mandelstam variables are defined $s_{mn} = (p_m + p_n)^2$, as usual. We consider that the massive scalars are minimally coupled to the gluons, in which case the exchange particle can be either a massive scalar or a gluon. Since a single massive scalar can not decay into a pair of gluons, only a single scattering diagram is possible.

Terms involving spinor helicity variable of the exchange particle ($\widehat{I}$) can be simplified as

$$\langle \widehat{I}|p_4|3][\widehat{I}2] = -[2|p_1 \cdot p_4|3] + m^2[23]\,, \tag{3.15}$$

using the fact that $[2\widehat{1}^I] = [21^I]$. Furthermore, the four particle kinematics allows us to simplify the following term as

$$[2|p_1 \cdot p_4|3] = (s_{12} - m^2)[23] + m^2[23] = [23]s_{12} \tag{3.16}$$

Collecting all these simplifications, we obtain the four particle amplitude

$$\mathcal{A}_4[\mathbf{1}^0, 2^+, 3^+, \mathbf{4}^0] = -m^2 g^2 \frac{[23]}{\langle 23\rangle(s_{12} - m^2)}\,. \tag{3.17}$$

This matches with the result given in [12]. This particular amplitude was derived in this reference using the standard BCFW method by complexifying a pair of massless external momenta. Nonetheless, we use this amplitude in order to check that the covariant massive-massless shift can also be used to evaluate this amplitude. A more general proof of the validity of these classes of shifts for the massive scalar QCD theory is given in the latter section.

The amplitude with opposite helicity gluons $(2^+, 3^-)$, using covariant massive-massless shift (3.10) can be computed by following almost similar methods laid in the previous example and therefore we only quote the result below

$$\mathcal{A}_4[\mathbf{1}^0, 2^+, 3^-, \mathbf{4}^0] = g^2 \frac{\langle 3|p_1|2]^2}{s_{23}(s_{12} - m^2)}\,. \tag{3.18}$$

This answer matches with the Compton amplitude derived in [11], using recursion relations based on unitarity principle. This method requires evaluation of an additional diagram in which p_1, p_2 momenta are attached to the same vertex.

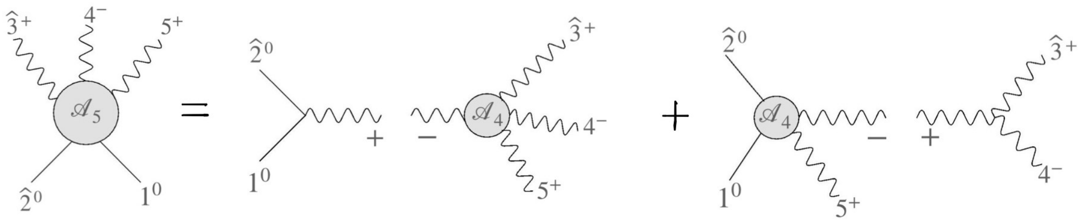


Figure 3.3: Five particle amplitude in scalar QCD

3.2.2 Scalar QCD : five particle amplitude

In order to put our proposal for the covariant recursion on a solid ground, we now consider color ordered five particle scattering amplitude comprising a pair of massive scalars and gluons with arbitrary helicity. We use the $[\mathbf{2}3^+\rangle$ massive-massless shift for this computation. The amplitude can be constructed from three and four point amplitudes via the following recursion relation

$$\mathcal{A}_5[\mathbf{1}^0, \mathbf{2}^0, 3^{h_1}, 4^{h_2}, 5^{h_3}] = \sum_I \widehat{\mathcal{A}}_3(z_I)\frac{1}{P^2 - m^2}\widehat{\mathcal{A}}_4(z_I) + \sum_J \widehat{\widetilde{\mathcal{A}}}_3(z_J)\frac{1}{P^2}\widehat{\widetilde{\mathcal{A}}}_4(z_J)\,. \tag{3.19}$$

Due to the choice of the massive-massless shift, there can only be two possible diagrams with gluon as exchange particle. The massive scalar exchange in this case is ruled out for the same reason that we mentioned in previous section. We now specialize to the case in which the gluons have the helicity configuration as specified in Figure 3.3.

First diagram :

The contribution to five particle amplitude due to the first diagram can be obtained by using the recursion relation in equation (3.19)

$$\mathcal{A}_5^{(I)}\left[\mathbf{1}^0, \mathbf{2}^0, 3^+, 4^-, 5^+\right] = \widehat{\mathcal{A}}_3\left[\mathbf{1}^0, \widehat{\mathbf{2}}^0, \widehat{I}^+\right]\frac{1}{s_{12}}\widehat{\mathcal{A}}_4\left[\widehat{I}^-, \widehat{3}^+, 4^-, 5^+\right]. \tag{3.20}$$

Note that, only the above helicity configuration of the massless internal state is contributing since for the opposite helicity configuration, the gluon amplitude vanishes. The three-particle amplitude for minimally coupled particles follows from equation (2.71) with $\widehat{x}_{12} = m\frac{[5\widehat{I}]}{\langle\widehat{I}|p_1|5]}$, while the massless four-particle pure gluon amplitude is given by the well-known Parke-Taylor formula [2]:

$$\widehat{\mathcal{A}}_3\left[\mathbf{1}^0, \widehat{\mathbf{2}}^0, \widehat{I}^+\right] = gm^2\frac{[5\widehat{I}]}{\langle\widehat{I}|p_1|5]}\,, \qquad \widehat{\mathcal{A}}_4\left[\widehat{I}^-, \widehat{3}^+, 4^-, 5^+\right] = \frac{[35]^4}{[\widehat{I}3][34][45][5\widehat{I}]}\,. \tag{3.21}$$

Substituting these lower point amplitude in the recursion relations (3.20), we obtain the contribution of first diagram

$$\mathcal{A}_5^{(I)}\left[\mathbf{1}^0, \mathbf{2}^0, 3^+, 4^-, 5^+\right] = m^2 g^3 \frac{[35]^4}{s_{12}[34][45]\left([3|\not{p}_2\cdot\not{p}_1|5] + m^2[35]\right)}\,. \tag{3.22}$$

Interestingly, the terms within parenthesis in the denominator do not correspond to any physical pole. At a first glance, one can think that these terms lead to a spurious pole. But this assertion is not true since the terms within parenthesis can not vanish. To show this, let us expand the following spinor bracket

$$[3|\not{p}_2\cdot\not{p}_1|5] = \tilde{\lambda}_{3\dot{\alpha}}\, p_{2\beta}^{\dot{\alpha}}\, p_{1\dot{\gamma}}^{\beta}\, \tilde{\lambda}_5^{\dot{\gamma}}\,. \tag{3.23}$$

Now this equals to $-m^2[35]$ only when we can set

$$p_{2\beta}^{\dot{\alpha}}\, p_{1\dot{\gamma}}^{\beta} = -m^2\delta_{\dot{\gamma}}^{\dot{\alpha}}\,, \tag{3.24}$$

i.e, the combination of terms within parenthesis in the denominator vanishes only when the massive momenta p_1 and p_2 become collinear, which is of course impossible for massive particles.

Second diagram :

This diagram includes a four particle amplitude with pair of massive scalars interacting with opposite helicity gluons and a three gluon amplitude, both of which are already known. Therefore, the contribution for this diagram is

$$\mathcal{A}_5^{(II)}\left[\mathbf{1}^0,\mathbf{2}^0,3^+,4^-,5^+\right]=\widehat{\mathcal{A}}_4\left[\mathbf{1}^0,\mathbf{2}^0,\widehat{I}^-,5^+\right]\frac{1}{s_{34}}\widehat{\mathcal{A}}_3\left[\widehat{I}^+,\widehat{3}^+,4^-\right]. \tag{3.25}$$

Using the four-particle amplitude in (3.18) and removing the $\widehat{I}$ dependent terms, the second diagram evaluates to

$$\mathcal{A}_5^{(II)}\left[\mathbf{1}^0,\mathbf{2}^0,3^+,4^-,5^+\right]=g^3\frac{\langle 54\rangle\langle 4|p_1|5]^2}{\langle 5\widehat{3}\rangle\langle 34\rangle\widehat{s}_{12}(s_{15}-m^2)}. \tag{3.26}$$

The simple pole $z=\tilde{z}_I$ is found by simply setting the shifted propagator on-shell

$$(\widehat{p}_3+p_4)^2=0\Rightarrow\tilde{z}_I=\frac{m\langle 34\rangle}{\langle 4|p_2|3]}. \tag{3.27}$$

The deformed spinor products are evaluated at $z=\tilde{z}_I$ in terms of undeformed spinor helicity variables

$$\widehat{s}_{12}=s_{12}-\frac{\langle 34\rangle}{\langle 4|p_2|3]}[3|p_1\cdot p_2|3]\,;\quad \langle 5\widehat{3}\rangle=\frac{\langle 54\rangle\langle 3|p_2|3]}{\langle 4|p_2|3]}. \tag{3.28}$$

Assembling all these expressions, we obtain the contribution of the second diagram in Figure 3.3 to the five-point amplitude as

$$\mathcal{A}_5^{(II)}\left[\mathbf{1}^0,\mathbf{2}^0,3^+,4^-,5^+\right]=g^3\frac{\langle 4|p_1|5]^2\langle 4|p_2|3]^2}{\langle 3|p_2|3]\langle 34\rangle(s_{15}-m^2)\left(\langle 4|p_2|3]s_{12}+\langle 4|\not{p}_3\not{p}_1\not{p}_2|3]\right)}. \tag{3.29}$$

Following similar argument we made in the case of first diagram, it can be shown that the term within parenthesis in the denominator does not vanish. Summing the contributions due to the all the diagrams, we obtain the colour-ordered five-particle amplitude as

$$\begin{aligned}\mathcal{A}_5^{\text{total}}\left[\mathbf{1}^0,\mathbf{2}^0,3^+,4^-,5^+\right]&=g^3m^2\frac{[35]^4}{s_{12}[34][45]\left([3|\not{p}_2\not{p}_1|5]+m^2[35]\right)}\\&\quad+g^3\frac{\langle 4|p_1|5]^2\langle 4|p_2|3]^2}{(s_{23}-m^2)\langle 34\rangle(s_{15}-m^2)\left(\langle 4|p_2|3]s_{12}+\langle 4|\not{p}_3\not{p}_1\not{p}_2|3]\right)}.\end{aligned} \tag{3.30}$$

Scattering amplitude with identical external particle configuration that we have considered here, has been derived in [12] using different methods. In order to make contact with their result we note the following identities

$$[5|(\not{p}_3 + \not{p}_4)\not{p}_2|3] = -m^2[35] - [3|\not{p}_2\not{p}_1|5]\,, \tag{3.31}$$

$$\langle 45\rangle \left([3|\not{p}_2\not{p}_1|5] + m^2[35]\right) = -(\langle 4|\not{p}_3\not{p}_1\not{p}_2|3] + s_{12}\langle 4|p_2|3])\,. \tag{3.32}$$

Using these two identities the five-particle amplitude can be recast as (upto an overall sign)

$$\begin{aligned}\mathcal{A}_5^{\text{total}}\left[\mathbf{1}^0, \mathbf{2}^0, 3^+, 4^-, 5^+\right] = g^3 m^2 &\frac{[35]^4}{s_{12}[34][45][5|(\not{p}_3 + \not{p}_4)\not{p}_2|3]} \\ &- g^3 \frac{\langle 4|p_1|5]^2\langle 4|p_2|3]^2}{(s_{23} - m^2)\langle 34\rangle\langle 45\rangle(s_{15} - m^2)[5|(\not{p}_3 + \not{p}_4)\not{p}_2|3]}\,.\end{aligned} \tag{3.33}$$

This expression exactly matches with the result given in [12].

One can follow similar steps to derive the amplitude with all positive helicity gluons by using covariant recursion. We give the expression below for completeness

$$\mathcal{A}_5[\mathbf{1}^0, \mathbf{2}^0, 3^+, 4^+, 5^+] = m^2 g^3 \frac{[5|(\not{p}_3 + \not{p}_4)\not{p}_2|3]}{\langle 34\rangle\langle 45\rangle(s_{23} - m^2)(s_{15} - m^2)}\,, \tag{3.34}$$

which agrees with the result in [12]. Reproducing the four and five particle amplitudes using our proposed covariant massive-massless shifts strongly suggests that the amplitudes in massive scalar QCD can be obtained by using the covariant recursion relations. Indeed, this is the case and we will prove this in later section.

3.2.3 Compton amplitude in Higgsed Yang-Mills theory

Next, we focus on the scattering of massive vector bosons with gluons of arbitrary helicity. The basic ingredients that we need in this section are the three particle amplitudes

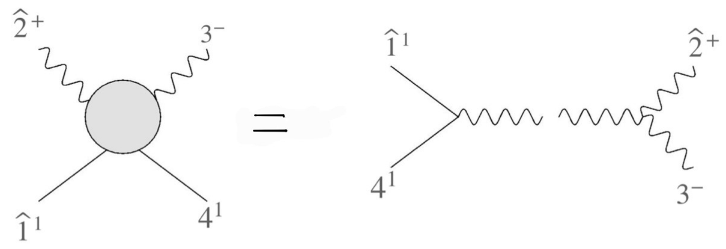


Figure 3.4: Amplitude with a pair of massive vector bosons and a pair of opposite helicity gluons.

comprising either only gluons or a pair of massive vectors bosons (of mass m) minimally coupled with a single gluon (with helicity h), given by the following expressions

$$\mathcal{A}_3^{+1}(\mathbf{1}^1, \mathbf{2}^1, 3^+) = g x_{12} \frac{\langle \mathbf{12} \rangle^2}{m} , \quad \mathcal{A}_3^{-1}(\mathbf{1}^1, \mathbf{2}^1, 3^-) = g x_{12}^{-1} \frac{[\mathbf{12}]^2}{m} . \tag{3.35}$$

We begin by considering the four particle vector boson amplitude involving opposite helicity gluons. Here, particles with momenta (p_1, p_4) are massive spin-1 particles and the particles with momenta (p_2, p_3) are gluons. We use the $[\mathbf{1}2^+\rangle$ massive-massless shift, for which we only need to compute a single scattering diagram. For convenience, let us write down the relevant complex shift in spinor helicity basis

$$\begin{aligned} \widehat{\lambda}_{2\alpha} &= \lambda_{2\alpha} + \frac{z}{m} p_{1\alpha\dot{\beta}} \tilde{\lambda}_2^{\dot{\beta}} , \qquad \widehat{\tilde{\lambda}}_{2\dot{\alpha}} = \tilde{\lambda}_{2\dot{\alpha}} , \\ \widehat{\lambda}^I_{1\alpha} &= \lambda^I_{1\alpha} \, ; \qquad \widehat{\tilde{\lambda}}^I_{1\dot{\alpha}} = \tilde{\lambda}^I_{1\dot{\alpha}} - \frac{z}{m} \tilde{\lambda}_{2\dot{\alpha}} [1^I 2] . \end{aligned} \tag{3.36}$$

By virtue of the recursion relation (3.11), the four-particle amplitude can be obtained by gluing the on-shell (complex) three particle amplitudes along with physical propagator for this channel

$$\mathcal{A}_4 \left[\mathbf{1}, 2^+, 3^-, \mathbf{4}\right] = \widehat{A}_3 \left[\widehat{\mathbf{1}}, \widehat{I}^-, \mathbf{4}\right] \frac{1}{s_{23}} \widehat{A}_3 \left[\widehat{I}^+, \widehat{2}^+, 3^-\right] . \tag{3.37}$$

The simple pole z_I for this diagram is given by

$$z_I = \frac{m\langle 23\rangle}{\langle 3|p_1|2]} \,. \tag{3.38}$$

After removing the $\widehat{I}$-dependent terms we rewrite the Compton amplitude as

$$\mathcal{A}_4\left[\mathbf{1}, 2^+, 3^-, \mathbf{4}\right] = \frac{g^2}{m^2}\frac{\langle 3|p_4|2]^2[\widehat{\mathbf{14}}]^2}{s_{23}(\widehat{s_{24}} - m^2)} \,. \tag{3.39}$$

To finish the computation we need to evaluate the "shifted" spinor products at the simple pole $z = z_I$. Let us consider the term $(\widehat{s_{24}} - m^2) = \langle\widehat{24^J}\rangle[4_J 2]$. The shifted spinor product at this simple pole is

$$\begin{aligned}
\langle\widehat{24^J}\rangle &= \langle 24^J\rangle + \frac{\langle 32\rangle\langle 4^J|p_1|2]}{\langle 3|p_1|2]} \\
&= \frac{[1_I 2]}{\langle 3|p_1|2]}\left(\langle 24^J\rangle\langle 31^I\rangle + \langle 32\rangle\langle 4^J 1^I\rangle\right) ,
\end{aligned}$$

which can be further simplified by using the Schouten identity

$$\langle 24^J\rangle\langle 31^I\rangle + \langle 23\rangle\langle 1^I 4^J\rangle + \langle 1_I 2\rangle\langle 34^J\rangle = 0 \,, \tag{3.40}$$

and we obtain

$$\langle\widehat{24^J}\rangle = \frac{(s_{12} - m^2)\langle 34^J\rangle}{\langle 3|p_1|2]} \;\Rightarrow (\widehat{s_{24}} - m^2) = -(s_{12} - m^2) \,. \tag{3.41}$$

Similarly the other shifted spinor product can be determined and we found

$$[\widehat{1^I 4^J}] = \frac{m}{\langle 3|p_1|2]}\left(\langle 31^I\rangle[24^J] + [21^I]\langle 34^J\rangle\right) . \tag{3.42}$$

In terms of bold faced notation

$$[\widehat{\mathbf{14}}]^2 = \frac{m^2}{\langle 3|p_1|2]^2}\left(\langle 3\mathbf{1}\rangle[2\mathbf{4}] + [2\mathbf{1}]\langle 3\mathbf{4}\rangle\right)^2 \,. \tag{3.43}$$

Substituting the expressions of shifted spinor products in equation (3.39), we reproduce the Compton amplitude as

$$\mathcal{A}_4\left[\mathbf{1},2^+,3^-,\mathbf{4}\right] = g^2\frac{(\langle 3\mathbf{1}\rangle[2\mathbf{4}] + [2\mathbf{1}]\langle 3\mathbf{4}\rangle)^2}{s_{23}(s_{12}-m^2)}\,. \tag{3.44}$$

This result precisely matches with the expression for four particle amplitude, obtained by recursion relations based on unitarity in [11]. Similarly we derive the amplitude for different helicity gluons and found that

$$\mathcal{A}_4\left[\mathbf{1},2^+,3^+,\mathbf{4}\right] = g^2\frac{[23]^2\langle \mathbf{14}\rangle^2}{s_{32}(s_{12}-m^2)}\,, \tag{3.45}$$

which is in agreement with the result in [12].

3.2.4 Five-particle amplitude in Higgsed Yang-Mills theory

So far we have computed results that have been previously computed in the literature but using the new recursion relations and this indicates that the new class of recursion relations is also valid for Yang-Mills theory in Higgsed phase. To put our assertion on a stronger ground, we now consider the colour-ordered five particle amplitude involving a pair of massive spin-1 particles and gluons with specific helicity configurations, using the $[\mathbf{2}3^{h_3}\rangle$ massive-massless shift. This computation leads to the first new result using the new recursion relations. Although the final expression (even for such a lower point amplitude) is rather complicated, we will verify that our result matches with the known massless result, in the high energy limit.

The recursion relation, needed for this computation is

$$\mathcal{A}_5[\mathbf{1}^1,\mathbf{2}^1,3^{h_1},4^{h_2},5^{h_3}] = \sum_I \widehat{\mathcal{A}}_3(z_I)\frac{1}{P^2-m^2}\widehat{\mathcal{A}}_4(z_I) + \sum_J \widehat{\tilde{\mathcal{A}}}_3(z_J)\frac{1}{P^2}\widehat{\tilde{\mathcal{A}}}_4(z_J)\,, \tag{3.46}$$

giving us two possible scattering diagrams in Figure 3.5, similar to the case of massive

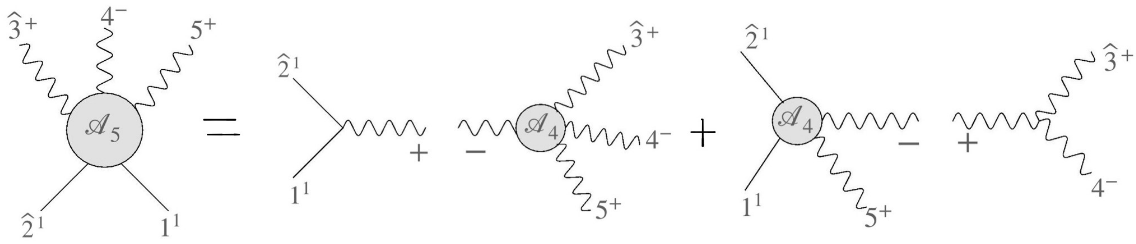


Figure 3.5: Five-particle amplitude in Higgsed Yang-Mills

scalar QCD. The diagrams with a massive propagator will again vanish due to the fact that a massive spin-1 particle can not decay into a pair of helicity-1 massless particles.

Contribution from the first diagram using the recursion relation in equation (3.46) can be obtained as

$$\mathcal{A}_5^{(I)}\left[\mathbf{1},\mathbf{2},3^+,4^-,5^+\right] = \widehat{\mathcal{A}}_3\left[\mathbf{1},\widehat{\mathbf{2}},\widehat{I}^+\right]\frac{1}{s_{12}}\widehat{\mathcal{A}}_4\left[\widehat{I}^-,\widehat{3}^+,4^-,5^+\right]. \tag{3.47}$$

The three and four particle subamplitudes have to be evaluated at $z = z_I$, which is given by

$$z_I = \frac{m^3 + m p_1 \cdot p_2}{\langle 1^I|p_2|3][31_I]}, \tag{3.48}$$

for this diagram. The lower point amplitudes are the standard three particle amplitude of [11] and the Parke-Taylor four point amplitude

$$\begin{aligned}\widehat{\mathcal{A}}_3\left[\mathbf{1},\widehat{\mathbf{2}},\widehat{I}^+\right] &= \frac{g}{m}\widehat{x}_{12}\langle\mathbf{12}\rangle^2 = g\frac{[\widehat{I}3]}{\langle\widehat{I}|p_1|3]}\langle\mathbf{12}\rangle^2,\\ \widehat{\mathcal{A}}_4\left[\widehat{I}^-,\widehat{3}^+,4^-,5^+\right] &= \frac{[35]^4}{[\widehat{I}3][34][45][5\widehat{I}]}.\end{aligned} \tag{3.49}$$

Removing the intermediate spinor helicity variable $\widehat{I}$ and evaluating the shifted spinor products at $z = z_I$, we obtain the contribution to five particle amplitude due to first diagram as

$$\mathcal{A}_5^{(I)}\left[\mathbf{1},\mathbf{2},3^+,4^-,5^+\right] = g^3 \frac{\langle \mathbf{12}\rangle^2[53]^4}{([3|\not{p}_2\not{p}_1|5]+m^2[35])[45][34]s_{12}}\,. \tag{3.50}$$

In the case of the second diagram of Figure 3.5, the recursion relation with covariant massive-massless shift is given by

$$\mathcal{A}_5^{(II)}\left[\mathbf{1},\mathbf{2},3^+,4^-,5^+\right] = \widehat{\mathcal{A}}_4\left[\mathbf{1},\mathbf{2},\widehat{I}^-,5^+\right]\frac{1}{s_{34}}\widehat{\mathcal{A}}_3\left[\widehat{I}^+,\widehat{3}^+,4^-\right]\,. \tag{3.51}$$

The simple pole in complex $z-$plane is located at

$$\tilde{z}_I = \frac{m\langle 34\rangle}{\langle 4|p_2|3]}\,. \tag{3.52}$$

This computation, although conceptually straightforward, is algebraically involved. Therefore, we will not give all the details and present the contribution due to the second diagram as

$$\mathcal{A}_5^{(II)}\left[\mathbf{1},\mathbf{2},3^+,4^-,5^+\right] = -g^3\frac{\left[\langle 4|p_2|3][\mathbf{5}\mathbf{1}]\langle 4\mathbf{2}\rangle+\langle \mathbf{1}4\rangle\left\{\langle 4|p_2|3][5\mathbf{2}]+\langle 4|p_3|5][3\mathbf{2}]\right\}\right]^2}{\langle 43\rangle\langle 45\rangle(s_{32}-m^2)(s_{15}-m^2)([3|\not{p}_2\not{p}_1|5]+m^2[35])}\,. \tag{3.53}$$

The full colour-ordered five-particle scattering amplitude is obtained by summing over the contributions from two diagrams in equations (3.50) and (3.53)

$$\begin{aligned}\mathcal{A}_5\left[\mathbf{1},\mathbf{2},3^+,4^-,5^+\right] = {} & g^3\frac{\langle \mathbf{12}\rangle^2[53]^4}{([3|\not{p}_2\not{p}_1|5]+m^2[35])[45][34]s_{12}} \\ & - g^3\frac{\left[\langle 4|p_2|3][\mathbf{5}\mathbf{1}]\langle 4\mathbf{2}\rangle+\langle \mathbf{1}4\rangle\left\{\langle 4|p_2|3][5\mathbf{2}]+\langle 4|p_3|5][3\mathbf{2}]\right\}\right]^2}{\langle 43\rangle\langle 45\rangle(s_{32}-m^2)(s_{15}-m^2)\left([3|\not{p}_2\not{p}_1|5]+m^2[35]\right)}\,.\end{aligned} \tag{3.54}$$

3.2.5 High energy limit

A rudimentary but non trivial check of our result for five particle vector boson amplitude is that, we should be able to reproduce the well-known Parke-Taylor amplitudes for gluons, by taking the high energy limit of the scattering amplitudes we derived in previous section.

The amplitude in (3.54) includes all possible helicity configurations of the massive spin-1 particle. This can be seen by expanding any spinor product involving massive spinor helicity variable using (A.6) and (A.7). For instance, consider the expansion of angle bracket $\langle 3\mathbf{1}\rangle^2$ in terms of massless spinor helicity variables (λ_1, η_1)

$$\langle 3\mathbf{1}\rangle^2 = \langle 3\lambda_1\rangle^2 \xi^{-I_1}\xi^{-I_2} + \langle 3\eta_1\rangle^2 \xi^{+I_1}\xi^{+I_2} + \langle 3\lambda_1\rangle\langle 3\eta_1\rangle \left(\xi^{-I_1}\xi^{+I_2} + \xi^{+I_1}\xi^{-I_2}\right) . \tag{3.55}$$

In this case, the $(-, +, 0)$ components of the vector boson are separately given as

$$\langle 3\lambda_1\rangle^2 \; : \; (-) \text{ helicity} \tag{3.56}$$

$$\langle 3\eta_1\rangle^2 \; : \; (+) \text{ helicity} \tag{3.57}$$

$$\langle 3\lambda_1\rangle\langle 3\eta_1\rangle \; : \; \text{longitudinal} . \tag{3.58}$$

Since η_1 scales with mass m and λ_1 is the massless spinor helicity variable corresponding to p_1, the angle bracket $\langle 3\mathbf{1}\rangle^2$ can have only $(-)$ helicity component in high energy (or massless) limit. Following similar argument, it is easy to see that the square bracket $[3\mathbf{1}]^2$ has only $(+)$ helicity component in high energy limit.

Let us now get back to the five point vector boson amplitude in equation (3.54) and take the high energy limit. Consider the helicity configuration $(1^-, 2^-)$ for massive particles. Using the procedure we just laid out, we immediately conclude that only the first diagram will contribute

$$\mathcal{A}_5^{(HE)}[1^-, 2^-, 3^+, 4^-, 5^+] = g^3 \frac{[35]^4}{[12][23][34][45][51]} . \tag{3.59}$$

It is clear from the structure of the amplitude in equation (3.54) that the $(1^+, 2^+)$ helicity configuration has vanishing massless limit- as should be the case. So we consider $(1^+, 2^-)$ configuration, in which case only the second diagram is non vanishing

$$\mathcal{A}_5^{(HE)}[1^+, 2^-, 3^+, 4^-, 5^+] = g^3 \frac{\langle 24\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} . \tag{3.60}$$

Similarly the amplitude in equation (3.54) reproduces correctly the massless amplitude with $(1^-, 2^+)$ helicity configuration. We thus find expected behaviour of the finite energy amplitude in massless limit.

In a similar spirit, we consider the amplitude with all the gluons having plus helicity. Implementing the same massive-massless shift $[\mathbf{2}3^+\rangle$ we obtain the following expression

$$\mathcal{A}_5[\mathbf{1}, \mathbf{2}, 3^+, 4^+, 5^+] = m^2 g^3 \frac{\langle \mathbf{12}\rangle^2(\langle 4|p_2|3] s_{12} + \langle 34\rangle[3|\not{p}_1 \not{p}_2|3])}{\langle 54\rangle^2\langle 34\rangle(s_{15} - m^2)(s_{23} - m^2)} . \tag{3.61}$$

The objective of this section was two fold: first, we introduce a new class of on-shell recursion relations (called "Covariant recursion") in which a combination of massive and massless complex momentum shift was used and then translated into the spinor helicity basis while maintaining the little group covariance of these spinor helicity variables.

Second, we used the covariant recursion to reproduce several four and five particle scattering amplitudes in massive scalar QCD and Higgsed Yang-Mills theory involving a pair of massive particles. The non trivial checks in turn support our claim about the validity of the new recursion for these two theories.

3.3 Large z behaviour of scattering amplitudes

On-shell recursion techniques that involve complexification of external momenta are one of the most powerful tools in the modern approach to scattering amplitudes since they

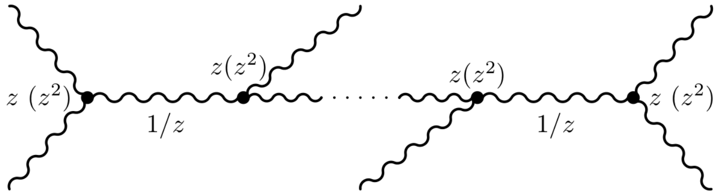


Figure 3.6: Individually all the Feynman diagrams grow as $z \to \infty$ for gauge theory and Einstein gravity where the vertices grow as z and z^2 respectively and overcompensate for the $\frac{1}{z}$ scaling behaviour of the propagators.

require only on-shell three particle amplitude as input data. However, recall that the contour derivation of recursion relations involves a residue at large z

$$\mathcal{A}_n(z=0) = \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{\widehat{\mathcal{A}}_n(z)}{z} dz = \sum_I \widehat{\mathcal{A}}_l(z_I)\frac{1}{P_I^2 - m^2}\widehat{\mathcal{A}}_r(z_I) - \mathcal{R}_n(z \to \infty)\,, \tag{3.62}$$

which can be set to zero by demanding that the deformed amplitude (involving complex momenta) vanish at large z. This is a critical ingredient for the on-shell recursion technique to work and enormously narrows down the space of allowed class of theories in which on-shell techniques can be used to compute scattering amplitudes [4]. Naively, the condition

$$\lim_{z\to\infty} \widehat{A}_n(z) = 0\,, \tag{3.63}$$

is far from obvious in the case of gauge and gravity theories as illustrated in Figure 3.6. Surprisingly, at least for some helicity combination of deformed momenta of external particle, the amplitude does vanish at large z. For gauge theories BCFW showed that [7,8]

$$\mathcal{A}^{[-+\rangle}, \mathcal{A}^{[++\rangle}, \mathcal{A}^{[--\rangle} \propto \frac{1}{z} \qquad \mathcal{A}^{[+-\rangle} \propto z^3\,, \tag{3.64}$$

[4]In some cases, one can extend the scope of recursion techniques to include theories in which the residue $\mathcal{R}_n$ does not decay at large z. An example of this is the $\lambda\phi^4$ theory, where it is possible to recursively obtain six and higher point amplitudes since the boundary term $\mathcal{R}_n$ can be computed using recursion [51]. The four particle amplitude can not be obtained in the same way as it involves a single contact diagram but it is easy to see that $\mathcal{A}_4 = -i\lambda$.

where the helicities of deformed massless particles and the kind of spinor helicity variables that need to get complexified are indicated in the superscript. Their analysis suggests that $[-+\rangle, [++\rangle, [--\rangle$ are valid shifts that can be used to construct the gluon amplitudes.

In this chapter, we present a detailed proof of the validity of covariant recursion relations by studying the behaviour of deformed amplitudes at large deformation parameter z in two classes of theories i) the Higgsed Yang-Mills theory and ii) massive scalar QCD. Since the proof is technical, we begin with a brief summary of the main concepts involved in the proof.

3.4 Higgsed Yang-Mills theory

The proof we present here, is inspired by the analogous proof in [19] for the case of massless amplitudes. However, conceptually there is a key difference that we point out next. The proof regarding the validity of the BCFW shifts for massless particles considered a set up where a highly boosted gluon was scattered off a background consisting low energy gluons. This corresponds to the familiar Eikonal scattering for real momenta. The background was referred to as a soft gluon background. In the Eikonal approximation, the conservation of helicity of the highly boosted particle was shown to be a consequence of the so-called "spin-Lorentz" symmetry, which was then used to constrain the behaviour of the amplitude at large z.

In our case, the soft background is replaced by a static background, including a collection of massive vector bosons and soft massless particles. Our set up is therefore closer to the scattering of a highly energetic gluon off a heavy scattering center surrounded by a cloud of soft gluons. As we show, the resulting outgoing states are a highly boosted massive spin-1 boson and a highly boosted gluon. At infinite boost (or large z), the dominant contribution to the amplitude is achieved when the helicity of the boosted gluon is unchanged. Thus, as in the case of massless theories, this dominant contribution is again controlled by the spin-Lorentz symmetry. We then use the Ward identity for massless gluons to constrain the sub-leading behaviour of the amplitude and show that for a particular class of covariant massive-massless shifts, the amplitude decays as $\frac{1}{z}$ for large z. We refer these classes of covariant massive-massless shifts as "valid shifts" for Yang-Mills theory in Higgsed phase.

3.4.1 Classification of covariant massive-massless shift

Since we care only about the functional dependence of the amplitude in z, all the soft physics can be included into a background, and only a single hard line (with boosted momentum) can be studied by considering the quadratic fluctuations about this background. Therefore, In order to check the validity of the covariant massive-massless shift of class $[\mathbf{m}+\rangle$ used in the covariant recursion (3.11), we consider two point amplitude $\widehat{\mathcal{A}}^h_{IJ}$ involving a highly boosted gluon with helicity h and a massive vector boson particle with little group indices (I, J). This process can be interpreted as a highly boosted gluon scattered through a static background, producing a boosted massive vector boson in the out state or vice-versa. In this case, the validity of covariant recursion requires

$$\widehat{\mathcal{A}}^h_{IJ} = 0\,, \quad \text{for } z \to \infty\,. \tag{3.65}$$

Three particle amplitudes, which is the basic building block in recursion method, can be constructed from this two point amplitude by attaching an unshifted (also soft) external momentum. Since this will also vanish at large z, any $n-$point amplitude also vanishes once the condition (3.65) is satisfied.

As mentioned in chapter 2 regarding the review of background materials, the two point natural amplitude $\widehat{\mathcal{A}}^h_{IJ}$ is obtained from the amplitude $\widehat{\mathcal{A}}^{\mu\nu}$, derived from the Lagrangian of the underlying theory ([11]), by contracting the latter with the polarization tensors of gluon and massive vector boson

$$\widehat{\mathcal{A}}^{h}_{IJ} = \widehat{\mathcal{A}}^{\mu\nu} \widehat{e}^{h}_{\mu}(j) \widehat{e}_{\nu IJ}(i) \,, \tag{3.66}$$

where we have put $\widehat{\ }$ on the polarization tensors as they are now functions of complex momenta. At large z the two point natural amplitude in equation (3.66) splits into three components, depending on three modes of polarization of massive vector boson. We shall call these modes as transverse($\pm$) and longitudinal modes.

Structure of $\widehat{\mathcal{A}}^{\mu\nu}$

We find the tensor structure of field theoretic amplitude $\widehat{\mathcal{A}}_{\mu\nu}$ by considering a theory with scalar fields coupled to Yang-Mills as well as an abelian gauge field. This theory produces the three-point interaction in (2.71) between photon and massive vector bosons after Higgsing [52]. The Lagrangian of this theory is given by

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) - \frac{1}{4}\left(B_{\mu\nu}B^{\mu\nu}\right) + \frac{1}{2}\left(D_{\mu}\Phi\right)^{\dagger} D^{\mu}\Phi \,, \tag{3.67}$$

where the field strengths $F_{\mu\nu}$ and $B_{\mu\nu}$ are associated with SU(2) gauge field A^{a}_{μ} and abelian gauge field B_{μ} respectively. Next, we expand each of the gauge fields into background + fluctuation fields and then do spontaneous symmetry breaking, since the background field methods usually involves manifestly gauge invariant Lagrangian. Later, we will see that this procedure indeed gives the correct interaction term that is suitable for a constructible covariant recursion. By expanding the gauge fields as mentioned earlier

$$A^{c}_{\mu} = A^{c}_{0\mu} + a^{c}_{\mu} \,; \qquad B_{\mu} = B_{0\mu} + b_{\mu} \,, \tag{3.68}$$

we can rewrite the field strength for the non-abelian gauge field as

$$F^{c}_{\mu\nu} = F^{c}_{\mu\nu}(A_0) + D_{A[\mu}a^{c}_{\nu]} - ig\epsilon^{cde} a_{d\mu} a_{e\nu} \,. \tag{3.69}$$

Here we have introduced the background field A_0-covariant derivative as

$$D_{A\mu}a^c_\nu = \partial_\mu a^c_\nu - ig\epsilon^{cde}A_{0d\mu}a_{e\nu} \,. \tag{3.70}$$

Similarly the field strength for the abelian gauge field splits into background and fluctuation field strengths

$$B_{\mu\nu} = B_{\mu\nu}(B_0) + B_{\mu\nu}(b) \,. \tag{3.71}$$

The gauge covariant derivative appearing in the scalar kinetic term can be expanded as

$$D_\mu \Phi = \left(\partial_\mu \Phi - ig(A^m_{0\mu} + a^m_\mu)\tau_m \Phi - \frac{ig^{'}}{2}(B_{0\mu} + b_\mu)\Phi\right) \,. \tag{3.72}$$

We are interested in terms which are quadratic in a_μ as it turns out that only those terms generate field bilinear comprised of massive and massless fields after spontaneous symmetry breaking. These kind of terms are included in the kinetic term of the SU(2) gauge field

$$-\frac{1}{4}\left(F^c_{\mu\nu}\right)^2 \rightarrow -\frac{1}{2}\left(D_{A\mu}a^c_\nu D^\mu_A a^\nu_c\right) + \frac{i}{2}g\epsilon^{cde}a_{d\mu}a_{e\nu}F^{\mu\nu}_c(A_0) \,, \tag{3.73}$$

using the gauge fixing condition $D_{A\mu}a^\mu = 0$. Note that, we do not consider the kinetic term for the abelian gauge field (b_μ), because they do not lead to terms involving the massive gauge fields after Higgsing.

In order to get massive particles in the spectrum, we use the Higgs mechanism as this is the only way to generate mass for non abelian gauge fields. Since gauge invariance of the Lagrangian remains intact after Higgsing, although not manifest, we can use the Ward identities, which turns out to be crucial in determining the large z behaviour of the two point amplitude.

As a result of Higgsing, we find a pair of massive fields (w^+_μ, w^-_μ) and a massless field (u_μ), related to the massless fluctuation fields a^c_μ and b_μ as

$$a^1_\mu = \frac{1}{\sqrt{2}}(w^+_\mu + w^-_\mu)\,, \quad a^2_\mu = \frac{i}{\sqrt{2}}(w^+_\mu - w^-_\mu)\,, \quad a^3_\mu = \frac{\sqrt{g^2+g^{'2}}}{g^{'}}\left(u_\mu - \frac{g}{\sqrt{g^2+g^{'2}}}b_\mu\right). \tag{3.74}$$

We can treat these equations as defining relations for the Higgssed fields.

Recall that, we are looking at a process in which a highly boosted gluon is scattered off a static background of massive spin-1 particles, surrounded by soft gluons. Therefore, to obtain the field theoretic amplitude $\widehat{\mathcal{A}}^{\mu\nu}$, first we look for terms with $(w^-_\mu u_\nu)$ in the Lagrangian which accounts for the interaction of the massive vector boson with photon. From the kinetic term in equation (3.73) and using the definitions of massive and massless Higgsed fields in equation (3.74), we write down the relevant terms below

$$\mathcal{L}^{w^-;u} = \frac{i}{2\sqrt{2}\tilde{g}}(F^{\mu\nu}_2(A_0) - F^{\mu\nu}_1(A_0))w^-_\mu u_\nu\,, \tag{3.75}$$

where the subscripts on background field strengths $F^{\mu\nu}_{1,2}(A_0)$ refer to colour degrees of freedom and the new coupling $\tilde{g}$ is defined as

$$\tilde{g} = \frac{g^{'}}{g\sqrt{g^2+g^{'2}}}\,. \tag{3.76}$$

Now we have two massive fields $w^\pm_\mu$ after spontaneous symmetry breaking, corresponding to two different massive particles. For our current purposes, we need any one of them and of course, the final conclusion does not depend on this choice. Also note that, the Higgssed fields so far we have discussed are abelian gauge fields. But as far as the three point interaction involving gluon and massive vector bosons (with internal structure) is concerned, we can simply assign internal indices to the Higgsed fields. In this case, we can write down the interaction Lagrangian by trace out the internal degrees of freedom

$$\mathcal{L}^{w^-;u} = \frac{i}{2\sqrt{2}\tilde{g}} \mathrm{Tr}\left[(F_2^{\mu\nu}(A_0) - F_1^{\mu\nu}(A_0)) w_\mu^- u_\nu \right] . \tag{3.77}$$

So far we have considered only relevant terms in the Lagrangian that are needed for proving the validity of the massive-massless shift. However, we can also easily include the terms that are required to prove the validity of massless-massless shift in Higgsed Yang-Mills theory, by keeping track of the terms quadratic in the massless Higgsed field u_μ. There are potentially three sources for these terms: i) the kinetic term for the SU(2) gauge field: $D_{A\mu} a_{3\nu} D_A^\mu a_3^\nu$, ii) the kinetic term for the b^μ field [5]: $B_{\mu\nu}(b) B^{\mu\nu}(b)$ and iii) the kinetic term for the scalar field Φ. Taking into account all these terms, the gauge fixed Lagrangian relevant for both kinds of shifts is given by

$$\begin{aligned} \mathcal{L} = & \frac{i}{2\sqrt{2}\tilde{g}} \mathrm{Tr}\left[(F_2^{\mu\nu}(A_0) - F_1^{\mu\nu}(A_0)) w_\mu^- u_\nu \right] \\ & + \mathrm{Tr}\left[-\frac{g^2 + g'^2}{2g^2} D_{A\mu} u_\nu D_A^\mu u^\nu - \frac{g^2 + g'^2}{4g^2} \partial_\mu u_\nu \partial^\mu u^\nu + \frac{g'^2}{4g^2} (g^2 + g'^2) \phi_0^2 u_\mu u^\mu \right] , \end{aligned} \tag{3.78}$$

where we fix the gauge degrees of freedom for u^μ by setting $\partial_\mu u^\mu = 0$ and ϕ_0 is the vacuum expectation value of the scalar field Φ.

In a similar spirit, one could also consider terms that are quadratic in the massive Higgsed fields ($w_\mu^\pm$) in order to check the validity of both massive shifts. Since such shifts are outside the scope of this work, we have omitted such terms in (3.78).

Due to the introduction of background fields, the spacetime Lorentz symmetry of this Lagrangian is broken. However, following the discussion in [19] we note that the terms in the second parenthesis are such that the vector indices of the fluctuation fields are contracted with each other and therefore, exhibit an enhanced symmetry, the so called "spin Lorentz" symmetry- that acts on the spin indices of the fluctuation fields u_ν. Remember

[5]Note that after symmetry breaking, we should treat this term as non-abelian field strength of field u_μ.

that, only in the infinite momentum limit (or at large z), this symmetry emerges due to which the helicity of a highly boosted gluon with real momenta scattered through soft background, remains unchanged [19]. But in our, the gluon momenta is complex and therefore its helicity is no longer conserved, similar to the case the BCFW shifts [19].

We use this symmetry to constrain the $z-$behaviour of the two point amplitude $\widehat{\mathcal{A}}^{\mu\nu}$ at large z. In order to make this symmetry explicit, we simply re-label the the usual spacetime indices $(\mu, \nu, \ldots)$ of the fluctuation fields to spin-Lorentz indices$(a, b, \ldots)$ and rewrite the Lagrangian as

$$\begin{aligned}\mathcal{L} =& \frac{i}{2\sqrt{2}\tilde{g}} \mathrm{Tr}\left[(F_2^{cd}(A_0) - F_1^{cd}(A_0))w_c^- u_d\right] \\ &+ \mathrm{Tr}\left[-\frac{g^2+g'^2}{2g^2} D_{A\mu}u_c D_A^\mu u^c - \frac{g^2+g'^2}{4g^2}\partial_\mu u_c \partial^\mu u^c + \frac{g'^2}{4g^2}(g^2+g'^2)\phi_0^2 u_c u^c\right]. \qquad (3.79)\end{aligned}$$

The contribution to the amplitude due to the spin-Lorentz symmetric terms in the second parenthesis of the Lagrangian is dominant at large z and proportional to η_{cd}. The repeated use of these vertices will contribute to higher powers in z. The two terms in the first parenthesis that have the background fields, explicitly break spin-Lorentz symmetry and so the contribution due to a single insertion of these vertices is proportional either to the field strengths $F_{1,2}^{cd}(A_0)$ or a linear combination of these, so that the contribution is anti symmetric in spin-Lorentz indices. Further insertions of these vertices gives additional powers in $\frac{1}{z}$ multiplying general matrices. Thus by utilizing the spin Lorentz symmetry that dominates the large z behaviour, we infer the following tensor structure for the two point amplitude as

$$\widehat{\mathcal{A}}^{cd}_{\mathrm{Full}} = \eta^{cd}(a + bz + \ldots) + M^{cd} + \frac{1}{z}\left(\tilde{B}^{cd} + B^{cd}\right) + O\left(\frac{1}{z^2}\right), \qquad (3.80)$$

where M^{cd} is an anti-symmetric matrix and B^{cd} and $\tilde{B}^{cd}$ are general matrices.

Now we discuss the validity of the massive-massless and both massless shifts separately. The structure of two point amplitude with highly boosted massive and massless particles can be extracted from the above amplitude as

$$\widehat{\mathcal{A}}^{cd}_{\text{massive-massless}} = M^{cd} + \frac{1}{z}B^{cd} + \dots\,, \tag{3.81}$$

and the two point amplitude with boosted massless particles in the external state is given by

$$\widehat{\mathcal{A}}^{cd}_{\text{massless-massless}} = \eta^{cd}(a + bz + \dots) + \frac{1}{z}\tilde{B}^{cd} + \dots\,. \tag{3.82}$$

We postpone the discussion of both massless shift to section 3.5. In the following subsections, we focus on the validity of massive-massless shift and work with the two point amplitude in (3.81). To check the validity of the shift of type $[\mathbf{m}+\rangle$, we consider gluon with positive helicity. Expressing the deformed polarization vector for the positive gluon as [6]

$$\widehat{e}^{\mu}_{+}(j) = \frac{\sqrt{2}mr^{\mu}}{\langle\lambda_j|p_i|\tilde{\lambda}_j]} = \kappa r^{\mu}\,, \tag{3.83}$$

we rewrite the natural amplitude in equation (3.66) as

$$\widehat{\mathcal{A}}^{+}_{IJ} = \begin{cases} \widehat{\mathcal{A}}^{+}_{\pm} = \kappa\widehat{\mathcal{A}}^{ab} r_a \widehat{e}_{b\pm}(i) & \text{for transverse modes,} \\ \widehat{\mathcal{A}}^{+}_{0} = \kappa\widehat{\mathcal{A}}^{ab} r_a \widehat{e}_{b0}(i) & \text{for longitudinal modes.} \end{cases} \tag{3.84}$$

Since, the large z limit of the two point amplitude combined with the action of the Ward identity acts differently on longitudinal and transverse modes. On one hand, for the longitudinal mode, we use a result of [20] in order to relate the this mode with massless scalar-gluon amplitude via the Ward identity for spontaneously broken gauge theory in the large z limit. On the other hand, the transverse modes can be treated as an amplitude involving only gluons in the large z limit.

[6]For more details, see appendix B.2

Validity for transverse modes

We consider the transverse modes of the amplitude at large z. As discussed earlier, the deformed particles are highly boosted and can be thought of as being massless at large z. In this case, the on-shell deformed amplitude $\widehat{\mathcal{A}}^{ab}$ satisfies the Ward identity for Yang-Mills theory

$$\widehat{p}_{ja}\widehat{\mathcal{A}}^{ab}\widehat{e}_b^{\pm}(i) = 0\,. \tag{3.85}$$

Here we denote $\widehat{\mathcal{A}}^{ab}_{\text{Massive-massless}}$ as $\widehat{\mathcal{A}}^{ab}$ to avoid clutter. By using the Ward identity and the shift equation for p_j (3.3) we can write

$$r_a\mathcal{A}^{ab}\widehat{e}_b^{\pm}(i) = -\frac{1}{z}p_{ja}\mathcal{A}^{ab}\widehat{e}_b^{\pm}(i)\,. \tag{3.86}$$

Substituting (3.81) into the expression for transverse mode in equation (3.84) using the above identity that follows from the Ward identity, we find the transverse modes of the on-shell amplitude has the following z−expansion in the limit $z \to \infty$

$$\widehat{\mathcal{A}}^{+}_{\pm}\Big|_{z\to\infty} = -\frac{\kappa}{z}\left[M^{ab} + \frac{1}{z}B^{ab} + \dots\right]p_{ja}\widehat{e}_b^{\pm}(i)\,. \tag{3.87}$$

Next, we analyse the large z behaviour of the deformed polarization vectors $\widehat{e}_b^{\pm}(i)$. By choosing the reference spinor $\zeta^{\alpha} = \lambda_j^{\alpha}$, we can express the positive helicity polarization vector as

$$\widehat{e}_b^{+}(i) = \Sigma_{ijb} - \frac{z}{m}p_{jb}\,\Omega_{ij}\,, \tag{3.88}$$

where

$$\Sigma_{ijb} = \frac{\langle\lambda_j|\sigma_b|\tilde{\lambda}_i]}{\sqrt{2}\langle\lambda_j\lambda_i\rangle} \quad \text{and} \quad \Omega_{ij} = \frac{[\tilde{\lambda}_i\tilde{\lambda}_j]}{\sqrt{2}\langle\lambda_j\lambda_i\rangle}\,. \tag{3.89}$$

See appendix B.1 for more details. Similarly the negative helicity component of massive polarization tensor can be expressed in the following way where we choose the reference spinor $\zeta_{\dot{\alpha}}$ to be $\tilde{\lambda}_{j\dot{\alpha}}$

$$\widehat{e_b^-}(i)|_{z\to\infty} = \frac{\Sigma^*_{ijb}}{\sqrt{2}[\tilde{\lambda}_i\tilde{\lambda}_j]} = \Upsilon_{ijb}\,. \tag{3.90}$$

Substituting (3.88) and (3.90) into (3.87), we find that on-shell amplitudes have the following large z behaviour

$$\widehat{\mathcal{A}}^+_- = 0\,, \qquad \widehat{\mathcal{A}}^+_+ = \frac{k}{m}\Omega_{ij}\, M^{ab}\, p_{ja}p_{jb} = 0\,, \tag{3.91}$$

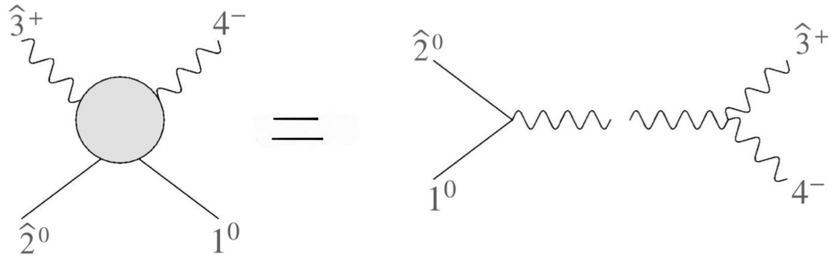
where we have used $M^{ab} = -M^{ba}$.

We thus established that both of the transverse components of natural amplitude $\widehat{\mathcal{A}}^+_\pm$ vanish in the large z limit, thereby proving the validity of the massive-massless shift for transverse modes.

Validity for longitudinal mode

We now analyse the large z behaviour of $\widehat{\mathcal{A}}^+_0$ involving the longitudinal mode of the polarization tensor of massive vector boson. This mode of the two point natural amplitude $\widehat{\mathcal{A}}^+_0$ is related to the amplitude involving massless scalars and gluons via the Ward identity for the spontaneously broken gauge theory [20]. Therefore, to find the large z behaviour of $\widehat{\mathcal{A}}^+_0$ it is sufficient to analyse the large z behaviour of the amplitude involving massless scalars and gluons.

Let us consider the four-particle colour ordered amplitude involving a pair of massless scalars and gluons



Since, we considered adjacent momentum shift in vector boson amplitude, we have to shift the adjacent scalar-gluon legs in this case. The relevant three-point interactions are given in the following Lagrangian

$$\mathcal{L}_3 = ig\left[(\partial_\mu\phi)A^\mu\phi^* - A_\mu\phi\partial^\mu\phi^* - \frac{1}{2}\mathrm{Tr}\,(\partial^\mu A^\nu)[A_\mu, A_\nu]\right]. \tag{3.92}$$

Expanding both scalar and vector fields in terms of background and fluctuation fields as

$$\phi = \phi_0 + \xi\,; \qquad A^\nu = A_0^\nu + a^\nu\,; \quad \phi^* = \phi_0^* + \xi^*\,, \tag{3.93}$$

we show that all the $O(z)$ terms in the above Lagrangian can be made $O(1)$ by using the gauge condition $\partial_\mu a^\mu = 0$. The $O(z)$ terms in the Lagrangian (3.92) are

$$\mathcal{L}_3 \supset ig((\partial_\mu\xi)\,a^\mu\phi_0^* - a_\mu\phi_0\partial^\mu\xi^* - \mathrm{Tr}\,(\partial^\mu a^\nu)[a_\mu, A_{0\nu}])\,. \tag{3.94}$$

The first two terms can be made $O(1)$ by integrating by parts and then using gauge fixing conditions $\partial_\mu a^\mu = 0$

$$\partial_\mu\xi a^\mu\phi_0^* \sim -\xi a^\mu\partial_\mu\phi_0^* \quad , \quad a_\mu\phi_0\partial^\mu\xi^* \sim -a_\mu\xi^*\partial^\mu\phi_0\,. \tag{3.95}$$

Similarly the third term can be made $O(1)$[7]. So after gauge fixing, there are no $O(z)$ vertices when a single scalar and gluon line are complexified. However, the deformed prop-

[7]Note that, when two scalar legs are shifted, the internal propagator is always a scalar and hence the $O(z)$ vertices contain terms like $(\partial_\mu\xi A^\mu\xi^* - A_\mu\xi\partial^\mu\xi^*)$. These terms cannot made $O(1)$ using gauge fixing condition. Therefore, the shifts involving only scalar external particles in Yang-Mills theory do not lead to BCFW type recursion relations.

agator scales as $\mathcal{O}(1/z)$. Therefore, the amplitude decays as $\mathcal{O}(1/z)$ for $z \to \infty$. All higher point amplitudes with adjacent scalar-gluon shift will be suppressed by additional $\frac{1}{z}$ factors due to more number of propagators. Hence, we conclude that massless scalar-gluon shift is a valid shift for the recursion to work. Note that, although we have considered only colour ordered amplitudes, this is not necessary for this proof. Incorporating this conclusion with the result of [20], we infer that the longitudinal component of natural amplitude $\widehat{\mathcal{A}}_0^+$ vanishes at large z.

In summary, we have shown that the proposed $[\mathbf{m}+\rangle$ shift is a valid shift as all three components of the natural amplitude (3.84) vanishes at large z. Following the similar line of argument,one can repeat the whole computation to show that the $[-\mathbf{m}\rangle$ shift is also a valid massive-massless shift.

In appendix C, we compare our conclusion regarding the validity of massive-massless shift with results available in reference [13].

3.4.2 Masive scalar QCD

In order to prove the validity of the massive-massless shift in massive scalar QCD, we consider the following Lagrangian that describes the theory of gluon interacting with massive scalar particles of [52] as

$$\mathcal{L}_{\text{scalar QCD}} = -\frac{1}{4}\text{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \mathcal{L}_{GF}\left(\partial_\mu A^\mu\right) - \frac{1}{2}|D_\mu\phi|^2 - \frac{1}{2}m^2|\phi|^2\,. \tag{3.96}$$

Focussing on the four particle scattering amplitude with a pair of massive scalar particles and gluons, we find this amplitude can be constructed by using the three-point scalar gluon vertex and three gluon vertex. The relevant terms in the action which account for these three point interactions are

$$\mathcal{L}_3 = ig\left[(\partial_\mu\phi)A^\mu\phi^* - A_\mu\phi\partial^\mu\phi^* - \frac{1}{2}\text{Tr}\,(\partial^\mu A^\nu)[A_\mu, A_\nu]\right]. \tag{3.97}$$

The interaction terms are identical to those we have encountered in section 3.4.1 for which we have already proven the validity of the shift. Therefore, we conclude that the massive-massless shift is valid for massive scalar QCD theory.

3.5 BCFW shift in Higgsed Yang-Mills theory

The validity of BCFW shifts was proven for various massless theories in arbitrary dimension in [19] and was utilized in deriving scattering amplitudes involving massive particles [12, 31, 52]. In this section, we prove the validity of the massless-massless (BCFW type) shift in Higgsed Yang-Mills theory. We follow similar line of arguments as we did in previous sections. Recall that, the interaction term accountable for both massless shift is given by the second line in equation (3.79):

$$\mathcal{L} \supset \mathrm{Tr}\left[-\frac{g^2+g^{'2}}{2g^2}D_{A\mu}u_c D_A^\mu u^c - \frac{g^2+g^{'2}}{4g^2}\partial_\mu u_c \partial^\mu u^c + \frac{g^{'2}}{4g^2}(g^2+g^{'2})\phi_0^2 u_c u^c\right]. \tag{3.98}$$

The structure of the two point amplitude with massless particles as external states is given in equation (3.82)

$$\widehat{\mathcal{A}}^{ab} = \eta^{ab}(a + bz + \dots) + \frac{1}{z}\tilde{B}^{ab} + \dots, \tag{3.99}$$

where we have denoted $\widehat{\mathcal{A}}^{ab}_{\text{massless-massless}}$ as $\widehat{\mathcal{A}}^{ab}$) to avoid clutter. The massless-massless shift $[ij\rangle$ is defined in terms of massless spinor helicity variables in the following way

$$|\widehat{i}] = |i] - z|j]\,; \quad |\widehat{j}\rangle = |j\rangle + z|i\rangle\,. \tag{3.100}$$

The deformed massless polarization vectors, in terms of external momenta can be expressed as

$$\widehat{e}_a^+(i) = \frac{q_a^* - zp_{ja}}{\sqrt{2}\langle ji\rangle}\,, \quad \widehat{e}_b^+(j) = \frac{q_b}{\sqrt{2}\langle ij\rangle}\,, \quad \widehat{e}_a^-(i) = \frac{q_a^*}{\sqrt{2}[ij]}\,, \quad \widehat{e}_b^-(j) = \frac{q_b^* - zp_{ib}}{\sqrt{2}[ji]}\,, \tag{3.101}$$

where $q_{\alpha\dot{\alpha}} = \lambda_{i\alpha}\tilde{\lambda}_{j\dot{\alpha}}$, is the lightlike momenta with which the external massless momenta get complexified. Using the Ward identity, we get

$$\widehat{p}_{ja}\widehat{\mathcal{A}}^{ab}\widehat{e}_b^{\pm}(i) = 0 \Rightarrow q_a\widehat{\mathcal{A}}^{ab}\widehat{e}_b^{\pm}(i) = -\frac{1}{z}p_{ja}\widehat{\mathcal{A}}^{ab}\widehat{e}_b^{\pm}(i)\,. \tag{3.102}$$

Therefore, the natural amplitude $\widehat{\mathcal{A}}_+^+$, at large z vanishes

$$\widehat{\mathcal{A}}_+^+\Big|_{z\to\infty} = -\frac{1}{2\langle ij\rangle^2}p_j^2 = 0\,. \tag{3.103}$$

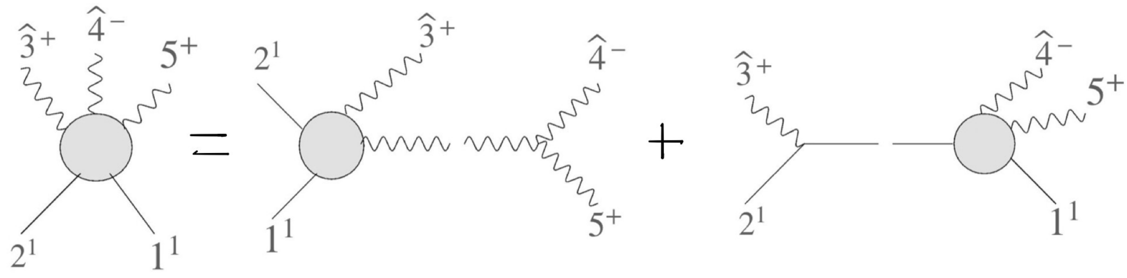
We arrive similar conclusion for $[--\rangle$ and $[-+\rangle$ massless shifts. But for $[+-\rangle$ shift, the two-particle amplitude grows as $O(z^3)$ at large z

$$\begin{aligned}\widehat{\mathcal{A}}_-^+|_{z\to\infty} &= \widehat{e}_a^+(i)\widehat{\mathcal{A}}^{ab}\widehat{e}_b^-(i)\\ &= -\frac{1}{2p_i\cdot p_j}(q_a^* - zp_{ja})\left(\eta^{ab}(a+bz+..)+\frac{1}{z}\tilde{B}^{ab}+..\right)(q_b^* - zp_{ib}) \to z^3\,.\end{aligned} \tag{3.104}$$

The results in this section proves that massless-massless shifts of type $[-+\rangle, [++\rangle, [--\rangle$ are valid, while the $[+-\rangle$ shift is invalid in Higgsed Yang-Mills theory.

3.5.1 Example : five-particle amplitude

As a rudimentary but useful check of our recursion relations involving either massive-massless or both massless shifts, we reproduce the five-particle amplitude in Higgsed Yang-Mills theory using both massless shift of the type $[-+\rangle$.



We complexify the massless momenta (p_3^+, p_4^-) of opposite gluons and deform the spinor helicity variables as

$$|\widehat{4}] = |4] - z|3]\,, \quad |\widehat{4}\rangle = |4\rangle\,, \tag{3.105}$$

$$|\widehat{3}\rangle = |3\rangle + z|4\rangle\,, \quad |\widehat{3}] = |3]\,. \tag{3.106}$$

Due to the adjacent shift, there are again only two possible scattering diagrams. The recursion in equation (3.11) leads to the following contributions from the each diagram

$$\mathcal{A}_5[\mathbf{1},\mathbf{2},3^+,4^-,5^+] = \mathcal{A}_4[\mathbf{1},\mathbf{2},\widehat{3}^+,\widehat{I}^+]\frac{1}{s_{45}}\mathcal{A}_3[\widehat{I}^-,\widehat{4}^-,5^+] + \mathcal{A}_4[\widehat{\mathbf{I}},\widehat{4}^-,5^+,\mathbf{1}]\frac{1}{s_{23}-m^2}\mathcal{A}_3[\mathbf{2},\widehat{3}^+,\widehat{\mathbf{I}}]\,. \tag{3.107}$$

The simple poles associated to the two diagrams are located at

$$z_I = \frac{p_4\cdot p_5}{r\cdot p_5} = \frac{[45]}{[35]} \quad \text{and} \quad \tilde{z}_I = -\frac{\langle 3|p_2|3]}{\langle 4|p_2|3]}\,. \tag{3.108}$$

Following the steps, which are now already familiar to us, we obtain the colour-ordered five point amplitude as

$$\mathcal{A}_5[\mathbf{1},\mathbf{2},3^+,4^-,5^+] = g^3\frac{\langle\mathbf{12}\rangle^2[53]^4}{([3|p_2\cdot p_1|5]+m^2[35])[45][34]s_{12}} - g^3\frac{[\langle 4|p_2|3][5\mathbf{1}]\langle 4\mathbf{2}\rangle + \langle\mathbf{14}\rangle\{\langle 4|p_2|3][5\mathbf{2}] + \langle 4|p_3|5][3\mathbf{2}]\}]^2}{\langle 43\rangle\langle 45\rangle(s_{32}-m^2)(s_{15}-m^2)([3|p_2\cdot p_1|5]+m^2[35])}\,. \tag{3.109}$$

This expression matches with the amplitude in equation (3.54), computed using the massive-massless shift.

Chapter 4

Vector boson amplitudes with arbitrary number of gluons

In [14, 36], a new set of recursion relations were derived in the massive spinor-helicity formalism for on-shell amplitudes by analytically continuing a pair of massive and massless external momenta to the complex plane. The complex shift generating the recursion involved deforming one massive and one massless momenta and was called covariant massive-massless shift (or sometimes called just massive-massless shift!) and we refer to this new recursion as covariant recursion relations. In section 3, we used these recursions to study tree-level scattering amplitudes in massive scalar QCD and amplitudes involving a pair of massive vector bosons in the Higgsed phase of Yang-Mills theory. One thus has two possible recursion relations for amplitudes involving external massive particles, namely: all massless shift or a massless-massive shift.

In this chapter, we ask the following question: how powerful and efficient is the newly introduced massive-massless shift over the massless two-line shift, which may not be available in a scattering process involving fewer than two massless particles[1]. That is, is

[1]One could ask if these recursion relations can be generalised to study elastic scattering of massive particles. However, if we do not want to introduce auxiliary null vectors, defining such recursion relations would require introducing a massive-massive shift which to the best of our knowledge has not been studied

it possible to compute certain classes of amplitudes in a more optimal way using the covariant recursion relation? These questions are closely related to the earliest applications of the BCFW recursion techniques, which were: (1) the proof of Parke-Taylor formula for n point MHV amplitude and (2) the ease with which tree-level NMHV amplitudes could be computed. Our goal was to seek similar application of the recursion relations derived in reference [14] using the so-called covariant massive-massless shift. With this goal in mind, we seek to find a class of amplitudes which were close analogues of MHV and NMHV amplitudes in the pure gluon case.

We probe these questions by studying two classes of amplitudes including an arbitrary number of gluons with specific helicity and a pair of massive vector bosons. The two classes are such that in the high energy limit, these amplitudes reduces to the maximally helicity violating (MHV) and the next-to-maximally helicity violating (NMHV) gluon amplitudes. We use the generalised recursion relations to derive the amplitudes for both of these classes. We provide an inductive proof for the first class of amplitudes and we show that for the second class of amplitudes, the covariant massive-massless shift proves to be very efficient in computing the amplitude.

The chapter is organised as follows. In section 4.1, we compute the tree level colour ordered amplitude in which a pair of (adjacent) massive vector bosons are scattering with an arbitrary number of gluons of identical helicity. To obtain this amplitude, we first use a simple relation between the amplitude involving two massive vector bosons and $(n-2)$ identical helicity gluons, and the amplitude involving two massive scalars and $(n-2)$ identical helicity gluons. This relation is a covariant version of a relation that has appeared in [22] for a particular choice of spin projection of the massive particles, which they obtain by using supersymmetric Ward identities. We then obtain this amplitude from the known scalar amplitude by using the covariant relation. We then verify the massive vector boson amplitude by using the method of induction and the covariant recursion relation. We also check consistency of this amplitude by taking the high energy limit

in the literature so far.

which matches with the pure gluon MHV scattering amplitude, as expected.

In section 4.2, we consider tree level colour ordered amplitude with two massive vector bosons, $(n-3)$ positive helicity gluons and a single gluon that is colour adjacent to any one of the massive vector boson, having negative helicity. We are interested in this amplitude because this amplitude serves as the closest massive analogue of NMHV amplitude. We obtain this amplitude using the covariant recursion relation as proposed in [14]. In this case, the covariant recursion relation involves only subamplitudes that have been previously computed. This is one particular example where one can really see the utility of the covariant recursion relation. Finally, we check the consistency of this result by taking the high energy limit, producing the n-point NMHV scattering amplitude. We find that our n-point NMHV amplitude takes an extremely simple and compact form that can be shown to match the known n-point gluon NMHV amplitude as given in [24].

4.1 Scattering of massive vector bosons with positive helicity gluons

To explore possible advantages of the covariant recursion in terms of simplicity and accessibility of computing different classes of amplitudes with massive particles, we take the first step in this direction by considering an n-point amplitude involving a pair of massive vector bosons and $(n-2)$ positive helicity gluons. We are particularly looking at this amplitude because this reproduces the MHV amplitude in the high energy limit, as we will see later in this section. We discuss how to obtain this amplitude in two different ways: firstly, we relate this amplitude to one with a pair of massive scalars and $(n-2)$ positive helicity gluons by using the covariant expression of a result that has appeared previously in [22]. Secondly, we shall verify this amplitude in detail by making use of the covariant recursion and the principle of induction.

The little group covariant relation between the n-particle amplitude involving a pair of massive bosons and positive helicity gluons and the n-particle amplitude involving pair of massive scalars and positive helicity gluons is

$$\mathcal{A}_n[\mathbf{1}, 2^+, \dots, (n-1)^+, \mathbf{n}] = \frac{\langle \mathbf{1n} \rangle^2}{m^2} \mathcal{A}_n[\mathbf{1}^0, 2^+, \dots, (n-1)^+, \mathbf{n}^0] \,. \tag{4.1}$$

This is a covariantization of a relation that has appeared previously in [22] for a particular choice of the spin projection of massive vector bosons.[2] The n-point amplitude with a pair massive scalars and $(n-2)$ positive helicity gluons is known from [23]

$$\mathcal{A}_n[\mathbf{1}^0, 2^+, \cdots, (n-1)^+, \mathbf{n}^0] = g^{n-2} \frac{m^2 [2| \prod_{k=3}^{n-2} \big((s_{1\dots k} - m^2) - \not{p}_k \not{p}_{1,k-1} \big) |n-1]}{(s_{12}-m^2)(s_{123}-m^2)\cdots(s_{12\dots(n-2)}-m^2)\langle 23\rangle\langle 34\rangle \cdots \langle (n-2)(n-1)\rangle} \,, \tag{4.2}$$

where the Mandelstam variables and $p_{1,l}$ are defined as follows

$$s_{1\dots l} := (p_1 + \cdots + p_l)^2 \,, \qquad p_{1,l} := p_1 + \cdots + p_l \,, \tag{4.3}$$

and we denote the spinor brackets appearing inside the product in the numerator as follows

$$[a|p_i p_j|b] = \tilde{\lambda}_{a\dot{\alpha}} \not{p}_i^{\dot{\alpha}\alpha} \not{p}_{j\alpha\dot{\beta}} \tilde{\lambda}_b^{\dot{\beta}} \,. \tag{4.4}$$

Note that we treat the momentum product $\not{p}_i \cdot \not{p}_j$ as SU(2) matrix valued product $p_i^{\dot{\alpha}\alpha} p_{j\alpha\dot{\beta}}$ when being contracted with spinor helicity variables. We follow this notation throughout this paper. The product appearing in the numerator of the formula (4.2) is defined as

$$[2| \prod_{k=3}^{n-2} \mathcal{B}_k |n-1] := [2|\mathcal{B}_3 \cdot \mathcal{B}_4 \cdot \ldots \cdot \mathcal{B}_{n-2}|n-1] \tag{4.5}$$

[2]In order to translate the relation (4.1) into the results obtained in [22], we use the following decomposition of little group covariant massive spinor-helicity variables of [11] $\lambda_I^\alpha = \lambda^\alpha \xi_I^+ - \eta^\alpha \xi_I^-$, here $\lambda_\alpha, \eta_\alpha$ are massless spinor-helicity variables and satisfy $\langle \lambda\eta \rangle = m$ and $\xi^{\pm I}$ are SU(2) basis vectors. Setting the particle 1 with $s_z = +1$ and particle n with $s_z = -1$ in the amplitude, we find that $\langle \mathbf{1n} \rangle_{(+,-)} \longrightarrow \langle \eta_1 \lambda_n \rangle$. Therefore, we can recast the relation (4.1) with the massive particles are being in this specific spin state as follows $\mathcal{A}_n[\mathbf{1}_+, 2^+, \dots, \mathbf{n}_-] = \left(\frac{\langle \eta_1 \lambda_n \rangle}{\langle \lambda_1 \eta_1 \rangle} \right)^2 \mathcal{A}_n[\mathbf{1}^0, 2^+, \dots, \mathbf{n}^0]$. This is the relation that appeared in [22].

Substituting the scalar amplitude in (4.1), we therefore find the following expression for the n particle amplitude with a pair massive vector bosons and $(n-2)$ positive helicity gluons (for $n > 3$)

$$\mathcal{A}_n[\mathbf{1}, 2^+, \cdots, (n-1)^+, \mathbf{n}] = g^{n-2} \frac{\langle \mathbf{1n}\rangle^2 [2| \prod_{k=3}^{n-2} \left((s_{1\ldots k} - m^2) - \not{p}_k \not{p}_{1,k-1}\right) |n-1]}{(s_{12}-m^2)(s_{123}-m^2)\cdots(s_{12\ldots(n-2)}-m^2)\langle 23\rangle\langle 34\rangle\cdots\langle (n-2)(n-1)\rangle}\,. \tag{4.6}$$

4.1.1 Inductive proof using covariant recursion

In this section, we present an inductive proof of the expression in equation (4.6) using the covariant recursion that was reviewed in chapter 3. To set up the induction, we ensure that the lower point amplitudes with $n = 4, 5$, that have been calculated previously in [14], are consistent with the general expression. We perform this check in Appendix E.1.

Given the match of the lower point amplitudes we now assume that the expression (4.6) holds for n-particle amplitude and then use this to construct $(n+1)$-particle amplitude. We use the $[\mathbf{1}2^+\rangle$ shift that corresponds to the complex deformations of the following spinor-helicity variables

$$|\widehat{1^I}] = |1^I] - \frac{z}{m}[1^I 2]|2]\,, \qquad |\widehat{2}\rangle = |2\rangle + \frac{z}{m} p_1 |2]\,. \tag{4.7}$$

With this particular shift, all possible scattering channels that contribute to $\mathcal{A}_{n+1}$ amplitude are shown in Figure 4.1. The first three diagrams do not contribute due to the following reasons: a)contribution from the first diagram vanishes since the right subamplitude involving a single massive vector boson is zero, b) the second diagram also vanishes due to the vanishing of the pure gluon subamplitude with either all positive helicity gluons or a single negative helicity gluon, c) the third diagram vanishes because a massive vector boson cannot decay into two positive helicity gluons. Thus we have to compute the contribution from the fourth diagram only, demonstrating one of the advantages of the covariant recursion.

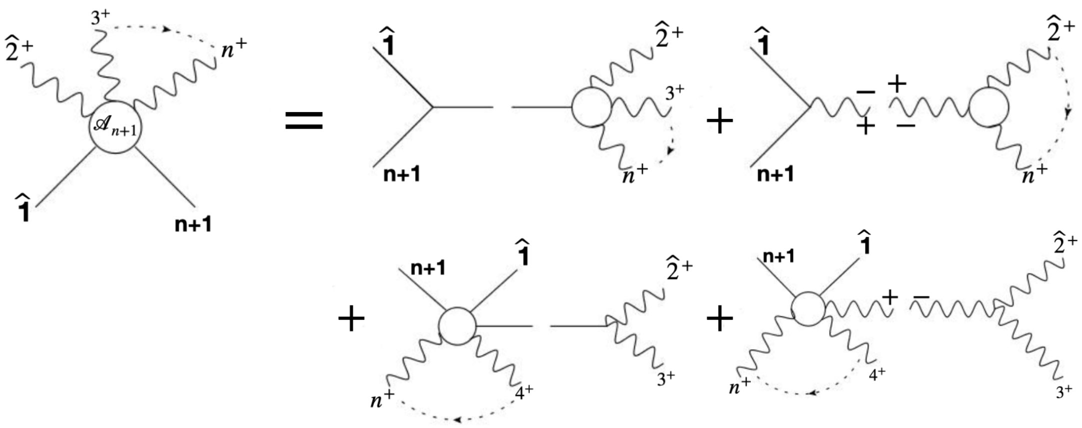


Figure 4.1: Pictorial representation of covariant recursion with $[\mathbf{1}2^+\rangle$ shift.

The simple pole $z = z_I$ for this diagram is found by setting the shifted propagator $\widehat{s}_{23}$ on-shell

$$(\widehat{p}_2 + p_3)^2 = 0 \Rightarrow z_I = \frac{m\langle 23\rangle}{\langle 3|p_1|2]} \,. \tag{4.8}$$

The $(n+1)$-particle amplitude $\mathcal{A}_{n+1}\,[\mathbf{1}, 2^+, \cdots, n^+, \mathbf{n{+}1}]$ is therefore constructed from the n-point and 3-point subamplitudes

$$\mathcal{A}_{n+1} = \mathcal{A}_n[\widehat{\mathbf{1}}, \widehat{I}^+, 4^+, \cdots, n^+, (\mathbf{n{+}1})]\frac{1}{s_{23}}\mathcal{A}_3[\widehat{I}^-, \widehat{2}^+, 3^+] \,. \tag{4.9}$$

Here we abbreviate $\mathcal{A}_{n+1}\,[\mathbf{1}, 2^+, \cdots, n^+, \mathbf{n{+}1}]$ as $\mathcal{A}_{n+1}$. The opposite helicity configuration of the internal states do not contribute to the amplitude due to the vanishing of all-positive-helicity three-particle gluon amplitude. Using the expression for n-point amplitude in equation (4.6), we get the left subamplitude but with complex spinor helicity variables

$$\mathcal{A}_n[\widehat{\mathbf{1}}^0, \widehat{I}^+, 4^+, \cdots, n^+, (\mathbf{n{+}1})^0] = g^{n-2}\frac{\langle \mathbf{1}(\mathbf{n{+}1})\rangle^2 \widehat{I}| \prod_{k=4}^{n-1}\big((\widehat{S}_{1I\ldots k} - m^2) - \not{p}_k \widehat{\not{P}}_{1,k-1}\big)|n]}{(\widehat{S}_{1I} - m^2)(\widehat{S}_{1I4} - m^2)\cdots(\widehat{S}_{1I\ldots(n-1)} - m^2)\langle \widehat{I}4\rangle\langle 45\rangle\cdots\langle (n-1)n\rangle} \,. \tag{4.10}$$

Here $\widehat{S}$ ($\widehat{P}$) are the Mandelstam (momentum) variable with shifted momenta

$$\widehat{S}_{1I\ldots r} = (\widehat{p}_1 + \widehat{p}_I + p_4 + \cdots + p_r)^2 \,, \quad \widehat{P}_{1,r} = (\widehat{p}_1 + \widehat{p}_I + \cdots + p_r) \,. \tag{4.11}$$

The internal momentum $\widehat{p}_I$ in this channel is $\widehat{p}_2 + p_3$, which we use to express shifted variables in terms of the real momenta

$$\widehat{S}_{1\ldots r} = (\widehat{p}_1 + \widehat{p}_2 + p_3 + \cdots + p_r)^2 = s_{1\ldots r}\,, \quad \widehat{P}_{1,r} = (\widehat{p}_1 + \widehat{p}_2 + p_3 + \cdots + p_r) = p_{1,r}\,.$$

Using these simplifications and gluing the three-particle gluon amplitude along with the propagator $\frac{1}{s_{23}}$ onto the left subamplitude, we obtain

$$\mathcal{A}_{n+1} = \frac{g^{n-1}\langle\mathbf{1}(\mathbf{n+1})\rangle^2[\widehat{I}|\prod_{k=4}^{n-1}\big((s_{1\ldots k}-m^2)-\not{p}_k\not{p}_{1,k-1}\big)|n]}{(s_{123}-m^2)(s_{1..4}-m^2)\cdots(s_{1\ldots(n-1)}-m^2)\langle\widehat{I}4\rangle\langle45\rangle\cdots\langle(n-1)n\rangle} \times \frac{[23]^2}{\langle23\rangle[\widehat{I}2][\widehat{I}3]}\,. \tag{4.12}$$

It remains to simplify the terms with the shifted massless spinor-helicity variable $\widehat{I}$, for which, we consider

$$\begin{aligned}\frac{[\widehat{I}|\prod_{k=4}^{n-1}\big((s_{1\ldots k}-m^2)-\not{p}_k\not{p}_{1,k-1}\big)|n]}{\langle\widehat{I}4\rangle[\widehat{I}2][\widehat{I}3]} &= \frac{[2|p_1|\widehat{I}\rangle[\widehat{I}|\prod_{k=4}^{n-1}\big((s_{1\ldots k}-m^2)-\not{p}_k\not{p}_{1,k-1}\big)|n]}{\langle4|p_3|2][21_I]\langle1^I\widehat{I}\rangle[\widehat{I}3]} \\ &= \frac{[2|\not{p}_1(\not{p}_2+\not{p}_3)\prod_{k=4}^{n-1}\big((s_{1\ldots k}-m^2)-\not{p}_k\not{p}_{1,k-1}\big)|n]}{[23]^2\langle34\rangle(s_{12}-m^2)}\,.\end{aligned} \tag{4.13}$$

We have replaced $\widehat{p}_2 \to p_2$ in the intermediate step while multiplying with $\langle\widehat{I}|p_1|2]$. This is allowed because

$$\langle1^I\widehat{2}\rangle = \langle1^I2\rangle - z_I[1^I2] \Rightarrow \langle1^I\widehat{2}\rangle[1_I2] = \langle1^I2\rangle[1_I2]\,, \tag{4.14}$$

where we have used $[1^I2][1_I2] = -m[22] = 0$. Using the following identity

$$[2|\not{p}_1(\not{p}_2 + \not{p}_3) = [2|\left\{(s_{123} - m^2) - \not{p}_3(\not{p}_1 + \not{p}_2)\right\}\,, \tag{4.15}$$

we finally obtain the $(n+1)$-point amplitude in the form

$$\mathcal{A}_{n+1}[\mathbf{1}, 2^+, \ldots, n^+, \mathbf{n+1}] = \frac{g^{n-1}\langle\mathbf{1}(\mathbf{n+1})\rangle^2[2|\prod_{k=3}^{n-1}\left((s_{1\ldots k} - m^2) - \not{p}_k\not{p}_{1,k-1}\right)|n]}{(s_{12} - m^2)(s_{123} - m^2)\cdots(s_{12\ldots(n-1)} - m^2)\langle23\rangle\langle34\rangle\cdots\langle(n-1)n\rangle}\,. \tag{4.16}$$

This completes our inductive proof of n-particle amplitude with all plus helicity gluons and a pair of massive vector bosons. Scattering amplitude with a pair of massive vector bosons and all *minus* helicity gluons can be read off from the expression in (4.6) by replacing all the angle brackets with the square brackets and vice-versa

$$\mathcal{A}_n[\mathbf{1}, 2^-, \ldots, (n-1)^-, \mathbf{n}] = g^{n-2} \frac{[\mathbf{1n}]^2 \langle 2| \prod_{k=3}^{n-2} \left((s_{1\ldots k} - m^2) - \not{p}_k \not{p}_{1,k-1} \right) |(n-1)\rangle}{(s_{12}-m^2)(s_{123}-m^2) \cdots (s_{12\ldots(n-2)}-m^2)[23][34]\cdots[(n-2)(n-1)]} \,. \tag{4.17}$$

Now we check the high energy limit of the massive vector boson amplitude (4.6). Due to the presence of angle bracket $\langle \mathbf{1n} \rangle^2$, the only non vanishing contribution can come from the component of the massive amplitude with both massive particles having negative helicity configuration in the high energy limit [11].

4.1.2 Matching the MHV amplitude in the high energy limit

We consider the high energy limit of the massive vector boson amplitude with all positive helicity gluons in this section. We show that the finite energy amplitude in equation (4.6) reproduces correct MHV amplitude in high energy limit when both of the massive particles have negative helicity configuration. The massless amplitude is given by

$$\mathcal{A}_n[1^-, 2^+, \ldots, (n-1)^+, n^-] = g^{n-2} \frac{\langle 1n \rangle^2 [2| \prod_{k=3}^{n-2} \left(s_{1\ldots k} - \not{p}_k \not{p}_{1,k-1} \right) |(n-1)]}{s_{12} s_{123} \cdots s_{12\ldots(n-2)} \langle 23 \rangle \langle 34 \rangle \cdots \langle (n-2)(n-1) \rangle} \tag{4.18}$$

$$= g^{n-2} \frac{\langle 1n \rangle^3}{\langle 12 \rangle \langle 23 \rangle \cdots \langle (n-1)n \rangle} \frac{[2| \prod_{k=3}^{n-2} \left(s_{1\ldots k} - \not{p}_k \not{p}_{1,k-1} \right) |(n-1)] \langle (n-1)n \rangle}{[21] s_{123} \cdots s_{12\ldots(n-2)} \langle 1n \rangle} \,. \tag{4.19}$$

Consider the non-trivial part of this amplitude

$$\mathcal{M}_n := \frac{[2| \prod_{k=3}^{n-2} \left(s_{1\ldots k} - \not{p}_k \not{p}_{1,k-1} \right) \not{p}_{n-1} |n\rangle}{[21] s_{123} \cdots s_{12\ldots(n-2)} \langle 1n \rangle} \,. \tag{4.20}$$

We simplify the product in numerator by using momentum conservation and identity (E.5) repeatedly. We start with the $k = n-2$ term and use momentum conservation to write[3]

$$\left(s_{1\dots n-2} - \not{p}_{n-2}\not{p}_{1,n-3}\right)\not{p}_{n-1}|n\rangle = s_{1\dots n-2}\, p_{n-1}|n\rangle + \not{p}_{n-2}\not{p}_{n}\not{p}_{n-1}|n\rangle\,. \tag{4.21}$$

Let us pause here to explain the notation that we are using here for generic momenta and spinor-helicity variables

$$(s_{ij} - \not{p}_l\not{p}_m)|r\rangle \equiv s_{ij}\lambda_{r\alpha} - p_{l\alpha\dot{\alpha}}p_m^{\dot{\alpha}\beta}\lambda_{r\beta}\,. \tag{4.22}$$

Here the Greek indices are the SL(2,$\mathbb{C}$) Lorentz indices. Going back to (4.21), we use (E.5) to express the second term as follows

$$\not{p}_{n-2}\not{p}_{n}\not{p}_{n-1}|n\rangle = (2p_{n-1}\cdot p_n)\, p_{n-2}|n\rangle\,. \tag{4.23}$$

Here we have used the fact that $p_n|n\rangle = 0$. Incorporating this with (4.21), we obtain

$$\left(s_{1\dots n-2} - \not{p}_{n-2}\not{p}_{1,n-3}\right)\not{p}_{n-1}|n\rangle = s_{1\dots n-2}(p_{n-2} + p_{n-1})|n\rangle\,. \tag{4.24}$$

Now we include the next term in the product in (4.20) and use the above result to write

$$\begin{aligned}\prod_{k=n-3}^{n-2}\left(s_{1\dots k} - \not{p}_k\not{p}_{1,k-1}\right)\cdot p_{n-1}|n\rangle &= s_{1\dots n-2}s_{1\dots n-3}(p_{n-2} + p_{n-1})|n\rangle \\ &\quad - \not{p}_{n-3}\not{p}_{1,n-4}(\not{p}_{n-2} + \not{p}_{n-1})|n\rangle\,.\end{aligned} \tag{4.25}$$

We can again simplify the second term using momentum conservation and (E.5) to get

$$-\not{p}_{n-3}\not{p}_{1,n-4}(\not{p}_{n-2} + \not{p}_{n-1})|n\rangle = (s_{1\dots n-3}s_{1\dots n-2})\, p_{n-3}|n\rangle\,. \tag{4.26}$$

[3] For a single SU(2) matrix valued momentum variable contracted to spinor-helicity variable, we omit the slash notation as in standard literature: $\not{p}_i|j\rangle \equiv p_i|j\rangle$.

Therefore, we find the following

$$\prod_{k=n-3}^{n-2} \left(s_{1\ldots k} - \not{p}_k \not{p}_{1,k-1}\right) \not{p}_{n-1} |n\rangle = s_{1\ldots n-2} s_{1\ldots n-3} (p_{n-3} + p_{n-2} + p_{n-1}) |n\rangle \,. \tag{4.27}$$

This trend continues to follow and we obtain the following identity

$$\prod_{k=3}^{n-2} \left(s_{1\ldots k} - \not{p}_k \not{p}_{1,k-1}\right) \not{p}_{n-1} |n\rangle = \prod_{k=3}^{n-2} s_{1\ldots k} (p_3 + \cdots + p_{n-2} + p_{n-1}) |n\rangle \,. \tag{4.28}$$

Using this identity, we have established that $\mathcal{M}_n = -1$. Thus the only non-vanishing high energy limit of the massive vector boson amplitude (4.16) reproduces the MHV amplitude

$$\mathcal{A}_n[1^-, 2^+, \ldots, (n-1)^+, n^-] = -g^{n-2} \frac{\langle 1n \rangle^3}{\langle 12 \rangle \langle 23 \rangle \cdots \langle (n-1)n \rangle} \,. \tag{4.29}$$

This provides a primary consistency check for the massive n-point amplitude in (4.6).

Having shown that the covariant recursion relations can be used to inductively prove the formula (4.6) of the massive analogue of MHV amplitude, it is worthwhile to mention that one could do the same by using the BCFW recursion relations as well. Therfore, one could ask: what new benefit that the covariant recursion relations bring to the table? To answer this question, we once again turn to the original motivation for the BCFW recursion, that is, the remarkable simplicity in deriving the tree-level NMHV amplitude using BCFW recursion.

4.2 Scattering of massive vector bosons with single flipped helicity gluons

Now we move on to the discussion of the massive analogue of NMHV amplitude. We consider a tree level colour ordered amplitude involving a pair of massive vector bosons, one minus helicity gluon adjacent to one of the massive bosons and arbitrary number of gluons with positive helicity. We will see later in this section that this particular amplitude leads to the NMHV amplitude in the high energy limit.

As mentioned in the Introduction of this chapter, this amplitude can not be computed using single BCFW recursion. For example, if we consider the $[2^-3^+\rangle$ BCFW shift to compute this amplitude, we would end up with the scattering channels involving subamplitudes that are identical to the configuration (the second diagram in figure F.1) which we intend to compute. Similar argument shows that this particular amplitude can not be obtained using a single recursion that involves two-line massive-massive shift.

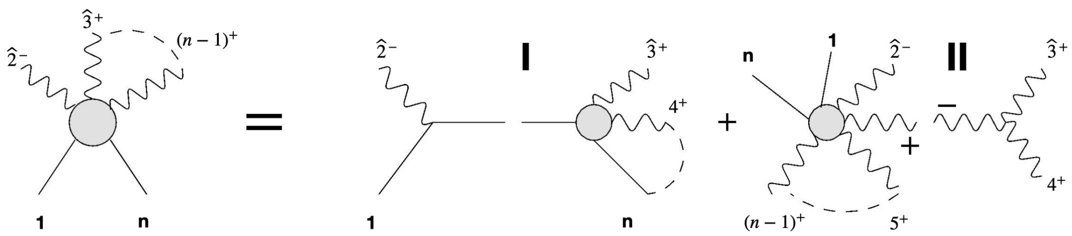


Figure 4.2: Pictorial representation of BCFW recursion with $[2^-3^+\rangle$ shift.

Therefore, we use the massive-massless shift $[2^-\mathbf{1}\rangle$ of the type $[-\mathbf{m}\rangle$, which corresponds to the following complex shift in terms of spinor helicity variables

$$|\widehat{2}] = |2] + \frac{z}{m}p_1|2\rangle\,, \qquad |\widehat{1}^I\rangle = |1^I\rangle - \frac{z}{m}\langle 21^I\rangle|2\rangle\,. \tag{4.30}$$

Due to $[2^-\mathbf{1}\rangle$ shift, all possible scattering diagrams that contribute to the $n-$particle amplitude in the covariant recursion are shown in Figure 4.3. As one can clearly see, all the constituent lower point amplitudes in these diagrams have already been computed: either they involve only pure gluon amplitudes or they involve two massiee vector bosons and all positive helicity gluons.

Since we are considering only minimal coupling while computing the amplitudes, the exchange particles can either be massive vector boson or gluon. But the exchange particle can not be a massive vector boson because a massive vector boson can not decay into two massless particles. Due to $[2^-\mathbf{1}\rangle$ shift, particles with momentum $\widehat{p}_1$ and $\widehat{p}_2$ are always

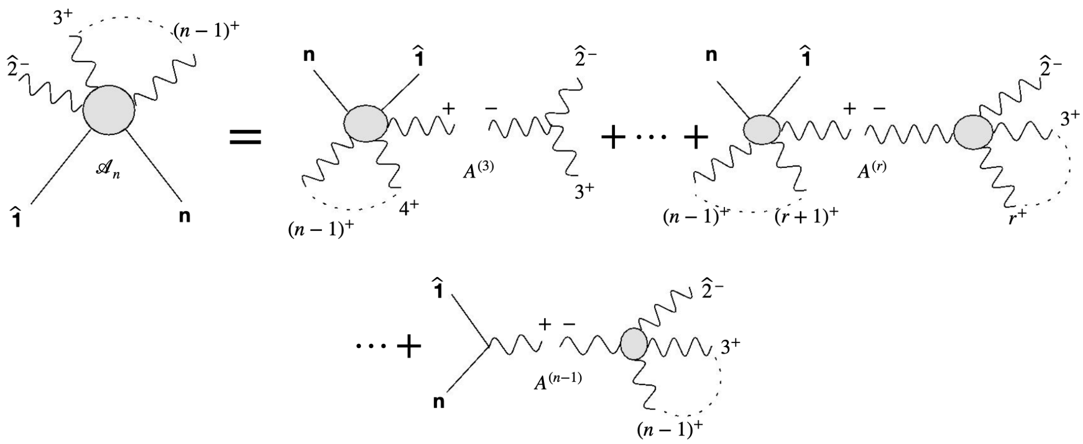


Figure 4.3: Pictorial representation of covariant recursion with $[2^-\mathbf{1}\rangle$ shift for $\mathcal{A}_n[\mathbf{1}, 2^-, 3^+, \ldots, \mathbf{n}]$.

attached to different subamplitudes in the diagramatic expansion of the colour-ordered amplitude. Note that, the subamplitudes involving external momentum $\widehat{p}_2$ always have only positive helicity gluons in the external states. But such pure gluon amplitudes with at most one opposite helicity vanishes, except for the three-particle amplitude. Therefore, the internal state attached to this subamplitude must be a negative helicity gluon. Again, due to the choice of the massive-massless shift $[2^-\mathbf{1}\rangle$, the first diagram in Figure 4.3 is non-vanishing only for the helicity configuration as indicated in the diagram.

The n-particle amplitude, obtained by summing over the all the diagrams can thus be written as follows

$$\mathcal{A}_n[\mathbf{1}, 2^-, 3^+, \ldots, (n-1)^+, \mathbf{n}] = \sum_{r=3}^{n-1} \mathcal{A}_L[\widehat{\mathbf{1}}, \widehat{I}^+, (r+1)^+, \ldots, \mathbf{n}] \frac{1}{s_{23\ldots r}} \mathcal{A}_R[\widehat{I}^-, \widehat{2}^-, 3^+, \ldots, r^+] , \tag{4.31}$$

where $s_{2\ldots r} = \left(\sum_{i=2}^{r} p_i\right)^2$. Again, we remind the reader that the subamplitudes here are all on-shell; that is, they are functions of shifted momenta and spinor-helicity variables. The right subamplitude is a pure-gluon amplitude and is given by the Parke-Taylor formula

$$\mathcal{A}_R[\widehat{I}^-, \widehat{2}^-, 3^+, \ldots, r^+] = g^{r-2} \frac{\langle \widehat{I}2\rangle^3}{\langle 23\rangle\langle 34\rangle \ldots \langle r\widehat{I}\rangle} . \tag{4.32}$$

The left subamplitude involving two massive vector bosons and all positive helicity gluons is known from previous section and is given by (see equation (4.6)):

$$\mathcal{A}_L[\widehat{\mathbf{1}}, \widehat{I}^+, (r+1)^+, \ldots, \mathbf{n}] = g^{n-r} \frac{\langle\widehat{\mathbf{1}}\mathbf{n}\rangle^2 [\widehat{I}| \prod_{k=r+1}^{n-2} \left\{(\widehat{S}_{1I\cdots k} - m^2) - \not{p}_k \widehat{\not{P}}_{1,k-1}\right\} |n-1]}{(\widehat{S}_{1I} - m^2)\ldots(\widehat{S}_{1I(r+1)\ldots(n-2)} - m^2)\langle\widehat{I}(r+1)\rangle\ldots\langle(n-2)(n-1)\rangle} \,. \tag{4.33}$$

This takes care of all but one diagram that appears in the covariant recursion. The last scattering channel in the diagramatic expansion in Figure 4.3 (which corresponds to $r = n-1$), has to be treated separately and we shall come to the evaluation of this diagram towards the end of this section.

For now, we simplify the expression in (4.33) and express it purely in terms of the undeformed spinor helicity variables of external momenta. Using $\widehat{p_I} = \widehat{p_2} + \sum_{i=3}^{r} p_i$, the shifted Mandelstam variables $(\widehat{S})$ and momenta $(\widehat{P})$ can be expressed (for $k \in \{r+1, \ldots (n-2)\}$) as

$$\begin{aligned}
\widehat{S}_{1\ldots k} &= (\widehat{p_1} + \widehat{p_I} + \cdots + p_k)^2 = (p_1 + p_2 + \cdots + p_k)^2 = s_{1\ldots k}\,, \\
\widehat{P}_{1,k-1} &= (\widehat{p_1} + \widehat{p_I} + \cdots + p_{k-1}) = (p_1 + p_2 + \cdots + p_{k-1}) = p_{1,k-1}.
\end{aligned} \tag{4.34}$$

Substituting these into the left subamplitude (4.33) and then gluing this with the pure gluon amplitude (4.32) and the physical propagator $\frac{1}{s_{2\ldots r}}$, we get the contribution to the n-particle amplitude from the r-th term in the covariant recursion (4.31)

$$A^{(r)} := g^{n-2} \frac{\langle\widehat{\mathbf{1}}\mathbf{n}\rangle^2 \langle\widehat{I}2\rangle^3 [\widehat{I}| \prod_{k=r+1}^{n-2} \left\{(s_{1\cdots k} - m^2) - \not{p}_k \not{p}_{1,k-1}\right\} |n-1]}{s_{23\ldots r}(s_{12\ldots r} - m^2)\ldots(s_{12..r(r+1)\ldots(n-2)} - m^2)\langle 23\rangle\langle 34\rangle\ldots\langle r\widehat{I}\rangle\langle\widehat{I}(r+1)\rangle\ldots\langle(n-2)(n-1)\rangle} \,. \tag{4.35}$$

where $r \in \{3, 4, \ldots (n-2)\}$. It should be noted that the product of angle brackets in the denominator, involving the massless spinor-helicity variables do not include $\langle r(r+1)\rangle$ bracket since the r- and $(r+1)$-th massless external momenta do not attach to same subamplitude in the recursion. Next, we express all the spinor products in $A^{(r)}$ involving the intermediate spinor-helicity variable $|\widehat{I}\rangle$ in terms of the spinor-helicity variables associated to external momenta. In order to do that, we collect all such terms

$$\chi_{r,I} = \frac{\langle\widehat{I}2\rangle^3[\widehat{I}|\prod_{k=r+1}^{n-2}\left\{(s_{1\cdots k}-m^2)-\not{p}_k\not{p}_{1,k-1}\right\}|n-1]}{\langle\widehat{I}(r+1)\rangle\langle r\widehat{I}\rangle}\,. \tag{4.36}$$

We use the following identities

$$\begin{gathered}\langle 2\widehat{I}\rangle[\widehat{I}|\mathcal{B}|n-1] = \langle 2|\not{p}_{3,r}\mathcal{B}|n-1]\,, \quad \mathcal{B} = \prod_{k=r+1}^{n-2}\left\{(s_{1\cdots k}-m^2)-\not{p}_k\not{p}_{1,k-1}\right\}\,, \\ \frac{\langle\widehat{I}2\rangle}{\langle\widehat{I}r\rangle} = \frac{\langle 2|\not{p}_1\not{p}_{3,r}|2\rangle}{\langle 2|\not{p}_1\not{p}_{2,r-1}|r\rangle}\,, \quad \frac{\langle\widehat{I}2\rangle}{\langle\widehat{I}(r+1)\rangle} = \frac{\langle 2|\not{p}_1\not{p}_{3,r}|2\rangle}{\langle 2|\not{p}_1\not{p}_{2,r}|r+1\rangle}\,,\end{gathered}$$

to write

$$\chi_{r,I} = \frac{\langle 2|\not{p}_1\not{p}_{3,r}|2\rangle^2\langle 2|\not{p}_{3,r}\prod_{k=r+1}^{n-2}\left\{(s_{1\cdots k}-m^2)-\not{p}_k\not{p}_{1,k-1}\right\}|n-1]}{\langle 2|\not{p}_1\not{p}_{2,r-1}|r\rangle\langle 2|\not{p}_1\not{p}_{2,r}|r+1\rangle}\,. \tag{4.37}$$

It only remains to evaluate the shifted spinor product $\langle\widehat{\mathbf{1n}}\rangle$ at the simple pole ($z_{(r)}$), associated with $r-$th scattering channel (except s_{1n}) in Figure 4.3. As usual, this is found by setting the deformed propagator $\widehat{s}_{2\cdots r}$ on-shell

$$(p_2 + z_{(r)}q + p_3 + \ldots + p_r)^2 = 0 \Rightarrow z_{(r)} = -\frac{mp_{2,r}^2}{\langle 2|\not{p}_1\not{p}_{3,r}|2\rangle}\,. \tag{4.38}$$

We then use the definition of the shifted massive spinor-helicity variable in (4.30) with $z = z_{(r)}$ to express the spinor product $\langle\widehat{1}^I n^J\rangle$ as

$$\langle\widehat{1}^I n^J\rangle = \langle 1^I n^J\rangle + \frac{p_{2,r}^2}{\langle 2|\not{p}_1\not{p}_{3,r}|2\rangle}\langle 1^I 2\rangle\langle 2n^J\rangle\,. \tag{4.39}$$

Substituting the expressions (4.37) and (4.39) in (4.35), one can rewrite $A^{(r)}$ in terms of the on-shell external variables

$$A^{(r)} = g^{n-2}\frac{\langle 2|\not{p}_{3,r}\prod_{k=r+1}^{n-2}\left\{(s_{1\cdots k}-m^2)-\not{p}_k\not{p}_{1,k-1}\right\}|n-1]\left(\langle 2|\not{p}_1\not{p}_{3,r}|2\rangle\langle\mathbf{1n}\rangle+p_{2,r}^2\langle\mathbf{1}2\rangle\langle 2\mathbf{n}\rangle\right)^2\langle r(r+1)\rangle}{s_{23\ldots r}(s_{12\ldots r}-m^2)\ldots(s_{12\ldots(n-2)}-m^2)\langle 23\rangle\langle 34\rangle\ldots\langle(n-2)(n-1)\rangle\langle 2|\not{p}_1\not{p}_{2,r-1}|r\rangle\langle 2|\not{p}_1\not{p}_{2,r}|r+1\rangle}\,. \tag{4.40}$$

Now we analyze the last diagram in Figure 4.3, which corresponds to the $r = n-1$ term in the covariant recursion. We have to treat this term separately because the left subamplitude which involve a pair of massive vector bosons and a single positive helicity gluon, cannot be read off from the general formula in equation (4.6) (we explicitly state the condition $n > 3$ in that derivation). Instead, we simply glue the three-particle amplitude with the pure gluon amplitude (4.32) for $r = n-1$ and the undeformed propagator $\frac{1}{s_{1n}}$

$$A^{(n-1)} = g^{n-2} \frac{-\langle\widehat{\mathbf{1n}}\rangle^2 \langle\widehat{I}2\rangle^2 s_{3,(n-1)}}{s_{1n}\langle 23\rangle\langle 34\rangle\cdots\langle(n-2)(n-1)\rangle\langle\widehat{I}|p_n|\widehat{2}]\langle(n-1)\widehat{I}\rangle}\,. \tag{4.41}$$

Again, we have to simplify the terms with spinor helicity variable $|\widehat{I}\rangle$ (associated with exchange particle) and evaluate all shifted spinor products at $z = z_{(n-1)}$, which is obtained by setting the deformed propagator $\widehat{s}_{1n}$ on-shell

$$z_{(n-1)} = \frac{m(p_1 + p_n)^2}{\langle 2|\not{p}_1\not{p}_n)|2\rangle}\,. \tag{4.42}$$

Firstly, by noting the following identities

$$\langle 2|\not{p}_1\widehat{\not{p}}_I\not{p}_n|\widehat{2}] = m^2\langle 2|(p_1 + p_n)|\widehat{2}]\,, \tag{4.43}$$

$$\langle 2|\not{p}_1(\widehat{\not{p}}_1 + \not{p}_n)|n-1\rangle = \langle 2|\not{p}_1\not{p}_n|n-1\rangle + m^2\langle 2(n-1)\rangle\,, \tag{4.44}$$

we get rid of the internal momentum dependence in $A^{(n-1)}$ as follows

$$\begin{aligned}\frac{-\langle\widehat{I}2\rangle^2}{\langle\widehat{I}|p_n|\widehat{2}]\langle(n-1)\widehat{I}\rangle} &= \frac{\langle 2|\not{p}_1\not{p}_n|2\rangle^2}{\langle 2|\not{p}_1\widehat{\not{p}}_I\not{p}_n|\widehat{2}]\langle 2|\not{p}_1(\widehat{\not{p}}_1 + \not{p}_n)|n-1\rangle} \\ &= \frac{\langle 2|\not{p}_1\not{p}_n|2\rangle^2}{m^2\left(\langle 2|p_1|\widehat{2}] + \langle 2|p_n|\widehat{2}]\right)\left(\langle 2|\not{p}_1\not{p}_n|n-1\rangle + m^2\langle 2(n-1)\rangle\right)}\,.\end{aligned} \tag{4.45}$$

Secondly, we calculate the shifted spinor products appearing in this expression and in

(4.41) using (4.42) and the definition of shifted spinor-helicity variables

$$\langle \widehat{1^I} n^J \rangle = \frac{m}{\langle 2|\not{p}_1 \not{p}_n|2\rangle} \Big(\langle 2|p_1|n^J]\langle 2 1^I\rangle + \langle 2|p_n|1^I]\langle 2 n^J\rangle + 2m\langle 1^I 2\rangle\langle 2 n^J\rangle \Big) \tag{4.46}$$

$$\langle 2|p_1|\widehat{2}] = \langle 2|p_1|2]\,, \qquad \langle 2|p_n|\widehat{2}] = \langle 2|p_n|2] + s_{1n}\,. \tag{4.47}$$

Finally we use the following identity

$$(\langle 2|p_1|2] + \langle 2|p_n|2] + s_{1n}) = s_{3,(n-1)}\,, \tag{4.48}$$

to derive the contribution of the last diagram $A^{(n-1)}$ as a function of only on-shell external variables

$$A^{(n-1)} = g^{(n-2)} \frac{(\langle 2|p_1|\mathbf{n}]\langle 2\mathbf{1}\rangle + \langle 2|p_n|\mathbf{1}]\langle 2\mathbf{n}\rangle + 2m\langle \mathbf{12}\rangle\langle 2\mathbf{n}\rangle)^2}{s_{1n}\langle 23\rangle\langle 34\rangle\cdots\langle (n-2)(n-1)\rangle\Big(\langle 2|\not{p}_1\not{p}_n|n-1\rangle + m^2\langle 2(n-1)\rangle\Big)}\,. \tag{4.49}$$

We combine the results in equations (4.40) and (4.49) to obtain a compact expression for the n-particle amplitude

$$\mathcal{A}_n[\mathbf{1}, 2^-, 3^+, \ldots, \mathbf{n}] = g^{n-2}\Bigg[\frac{(\langle 2|p_1|\mathbf{n}]\langle 2\mathbf{1}\rangle + \langle 2|p_n|\mathbf{1}]\langle 2\mathbf{n}\rangle + 2m\langle \mathbf{12}\rangle\langle 2\mathbf{n}\rangle)^2}{s_{1n}\langle 23\rangle\langle 34\rangle\cdots\langle (n-2)(n-1)\rangle\Big(\langle 2|\not{p}_1\not{p}_n|n-1\rangle + m^2\langle 2(n-1)\rangle\Big)}$$
$$+ \sum_{r=3}^{n-2} \frac{\langle 2|\not{p}_{3,r} \prod_{k=r+1}^{n-2}\big\{(s_{1\cdots k}-m^2) - \not{p}_k\not{p}_{1,k-1}\big\}|n-1]\Big(\langle 2|\not{p}_1\not{p}_{3,r}|2\rangle\langle \mathbf{1n}\rangle + p_{2,r}^2\langle \mathbf{12}\rangle\langle 2\mathbf{n}\rangle\Big)^2\langle r(r+1)\rangle}{s_{23\ldots r}(s_{12\ldots r}-m^2)\ldots(s_{12\ldots(n-2)}-m^2)\langle 23\rangle\langle 34\rangle\ldots\langle (n-2)(n-1)\rangle\langle 2|\not{p}_1\not{p}_{2,r-1}|r\rangle\langle 2|\not{p}_1\not{p}_{2,r}|r+1\rangle}\Bigg]. \tag{4.50}$$

This is the central result of [21] and it demonstrates that the covariant recursion relations introduced in [14] have the potential to open up new avenues to compute new classes of amplitudes that are otherwise not accessible via conventional recursion relations such as BCFW recursion relations. This computation also serves as a testament to the optimal usage [4] of the covariant massive-massless shift (4.30). Other possible two-line shifts (i.e.,

[4]In principle, one could have computed this n-point amplitude using BCFW recursion relations iteratively, building from the known three-point on-shell amplitudes. However, it would be an inefficient method to obtain the amplitude since it requires knowledge of all the amplitudes $\mathcal{A}_m$ for all $m < n$, at each step of

massless-massless or massive-massive) do not allow us to compute this amplitude as there will be scattering channels in which the negative helicity gluon is attached to the same subamplitude involving massive particles, leading to an identical particle configuration that we are trying to compute.

As a simple but non trivial check of our result, a few lower-point amplitudes are obtained by independent methods in appendix E.2 and shown to match the expected results. Additionally, in Appendix F we have checked the correctness of our result in (4.50) by taking BCFW shift on external gluon states and the method of induction.

4.2.1 Matching the NMHV amplitude in high energy limit

We now consider the high energy limit of the scattering amplitude in (4.50). This should reproduce unique massless amplitudes for different helicity configurations of massive vector bosons since we have considered only minimally coupled three particle amplitudes (2.71) as basic building blocks to construct the finite energy amplitude [11].

The procedure of taking the high energy limit of scattering amplitudes involving massive particle is discussed in [11] and further discussed in [14]. We do not repeat the procedure again but as a general rule of thumb, we show which component of massive spinor helicity variables survives in this limit below

$$|\mathbf{n}\rangle \xrightarrow{p^0>>|\vec{p}|} |n^-\rangle\,, \qquad |\mathbf{n}] \xrightarrow{p^0>>|\vec{p}|} |n^+] \quad \pm \text{ indicates helicity}\,. \tag{4.51}$$

The high energy limits of the finite energy amplitude (4.50) with opposite helicity configurations for the pair of massive particles are non-vanishing due to the presence of both angle and square brackets involving massive spinor-helicity variables and reproduces cor-

the iteration and in this case, the purpose of the on-shell recursion method would be lost ! Instead we have shown that, for this specific configuration of external particles, the covariant massive-massless shift $[2^-\mathbf{1}\rangle$ leads to an single on-shell recursion.

rect MHV amplitudes, as expected

$$\mathcal{A}_n^{\text{MHV}}[1^-, 2^-, 3^+, \ldots, n^+] = g^{n-2} \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle \cdots \langle n1\rangle}\,, \tag{4.52}$$

$$\mathcal{A}_n^{\text{MHV}}[1^+, 2^-, 3^+, \ldots, n^-] = g^{n-2} \frac{\langle 2n\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle \cdots \langle n1\rangle}\,. \tag{4.53}$$

The high energy limit of (4.50) with positive helicity configuration for both massive particles vanishes since the Parke-Taylor amplitude with single flipped helicity gluon is zero. But the negative helicity configuration for both massive particles in the high energy limit gives us the NMHV amplitude. In order to check this fact, we first obtain the following expression from (4.50)

$$\begin{aligned}&\mathcal{A}_n[1^-, 2^-, 3^+, \cdots, (n-1)^+, n^-] \\ &\quad = g^{n-2} \sum_{r=3}^{n-2} \frac{\langle 2|\not{p}_{3,r} \prod_{k=r+1}^{n-2}\left\{s_{1\cdots k} - \not{p}_k \not{p}_{1,k-1}\right\}\not{p}_{n-1}|n\rangle \left(\langle 2|\not{p}_1\not{p}_{3,r}|2\rangle\langle 1n\rangle + p_{2,r}^2\langle 12\rangle\langle 2n\rangle\right)^2 \langle r(r+1)\rangle}{s_{23\ldots r}s_{12\ldots r}\ldots s_{12\ldots(n-2)}\langle 23\rangle\langle 34\rangle\ldots\langle (n-1)n\rangle\langle 2|\not{p}_1\not{p}_{2,r-1}|r\rangle\langle 2|\not{p}_1\not{p}_{2,r}|r+1\rangle}\,. \end{aligned} \tag{4.54}$$

Then we simplify the product factor appearing in the numerator using the following identity

$$\prod_{k=r+1}^{n-2} \left\{s_{1\cdots k} - \not{p}_k \not{p}_{1,k-1}\right\} \not{p}_{n-1}|n\rangle = \left(\prod_{k=r+1}^{n-2} s_{12\ldots k}\right) (p_{r+1} + \cdot + p_{n-1})\,|n\rangle\,. \tag{4.55}$$

This identity can be derived from the one we have proved in section 4.1.2. Furthermore, we use momentum conservation to get

$$\langle 2|\not{p}_{3,r}.\not{p}_1|n\rangle + p_{2,r}^2\langle n2\rangle = \langle 2|\not{p}_{3,r}.(\not{p}_{r+1} + \ldots + \not{p}_{n-1})|n\rangle\,. \tag{4.56}$$

Substituting this in (4.54), we obtain the NMHV amplitude as

$$\begin{aligned}&\mathcal{A}_n[1^-, 2^-, 3^+, \cdots, (n-1)^+, n^-] \\ &\quad = g^{n-2} \sum_{r=3}^{n-2} \frac{\langle r(r+1)\rangle\langle 2|\not{p}_{3,r}\left(\not{p}_{r+1} + \cdots + \not{p}_{n-1}\right)|n\rangle^3}{s_{23\ldots r}s_{12\ldots r}\langle 23\rangle\langle 34\rangle\ldots\langle (n-1)n\rangle[1|p_{2,r-1}|r\rangle[1|p_{2,r}|r+1\rangle}\,. \end{aligned} \tag{4.57}$$

We have obtained in (4.57) a compact expression for the n-point NMHV amplitude that at first glance appears to be different from the standard expression in [24]. Note that, the first term of the expression in equation (4.50) (that corresponds to the last diagram in Figure 4.3) does not contribute to the high energy limit since this involves massive spinor-helicity variables that do not survive in this limit. It can be argued that this is a consequence of the massive-massless shift $[2^-\mathbf{1}\rangle$ which we have used to derive this amplitude. In a purely massless setup, one could use the BCFW shift $[1^-2^-\rangle$, in which case the last diagram in Figure 4.3 would certainly contribute. Therefore, in this case, the covariant massive-massless shift leads to a novel representation of the n-point NMHV amplitude. In what follows, we will first take the soft limit of this amplitude to show that it obeys the Weinberg's soft theorem at leading order and subsequently we prove that the NMHV amplitude (4.57) matches with the expression in [24] for this specific ordering of external particles.

Soft expansion of NMHV amplitude

We take the limit $p_n \to 0$ in the NMHV amplitude (4.57). In order to take the limit $p_n \to 0$, we first scale the spinor-helicity variables as follows

$$\lambda_{n\alpha} \longrightarrow \sqrt{\epsilon}\lambda_{n\alpha}\,, \qquad \tilde{\lambda}_{n\dot{\alpha}} \longrightarrow \sqrt{\epsilon}\tilde{\lambda}_{n\dot{\alpha}}\,, \tag{4.58}$$

and then take $\epsilon \to 0$ limit. With this scaling, we find that the $r = n-2$ channel of the NMHV amplitude in (4.57) has the leading order contribution as $O\left(\frac{1}{\epsilon}\right)$ and the amplitude factorizes as follows

$$\lim_{p_n\to 0} \mathcal{A}_n = \frac{[(n-1)1]}{[(n-1)n][n1]} \times \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\ldots\langle (n-1)1\rangle} \tag{4.59}$$

$$= \frac{[(n-1)1]}{[(n-1)n][n1]} \times \mathcal{A}^{\text{MHV}}_{n-1}\,. \tag{4.60}$$

This follows from the Weinberg soft theorem which we will see below.

Using Weinberg soft theorem, we find that in the soft limit of the n-th gluon momentum, the n-particle NMHV amplitude factorizes into a soft factor times a $(n-1)$-particle MHV amplitude

$$\lim_{p_n \to 0} \mathcal{A}_n \left[1^-, 2^-, 3^+, .., (n-1)^+, n^-\right] = S^{(0)}(n^-, (n-1)^+, 1^-) \mathcal{A}_{n-1} \left[1^-, 2^-, 3^+, .., (n-1)^+\right] , \tag{4.61}$$

where the soft factor at leading order is given by [53–55]

$$S^{(0)}(n^-, (n-1)^+, 1^-) = \left(\frac{\varepsilon_n^- \cdot p_{n-1}}{p_n \cdot p_{n-1}} - \frac{\varepsilon_n^- \cdot p_1}{p_n \cdot p_1} \right) . \tag{4.62}$$

Expressing the massless polarization vector in the spinor-helicity formalism as

$$\varepsilon_n^{-\mu} := \frac{\langle n|\sigma^\mu|q]}{[nq]} , \tag{4.63}$$

and chosing the reference spinor $|q] = |1]$, we get the soft factor as follows

$$S^{(0)}(n^-, (n-1)^+, 1^-) = \frac{[(n-1)1]}{[(n-1)n][n1]} . \tag{4.64}$$

Therefore, we have

$$\mathcal{A}_n \left[1^-, 2^-, 3^+, \cdots, (n-1)^+, n^-\right] = \frac{[(n-1)1]}{[(n-1)n][n1]} \mathcal{A}_{n-1} \left[1^-, 2^-, 3^+, \cdots, (n-1)^+\right] . \tag{4.65}$$

This is matches with the expression that we obtain by taking the soft limit of the NMHV amplitude in equation (4.57).

Matching the NMHV amplitude

Now that the preliminary check regarding the soft limit has been verified, we now show that the result in (4.57) matches exactly with the NMHV amplitude computed by Dixon et al. in [24] for the specific ordering of negative helicity gluons that we have considered here. The result of [24] is of course more general in the sense that the positions of the two negative helicity gluons are completely arbitrary.

In order to compare with our result (4.57), we begin with the expression given in [24] and fix the positions of the two negative helicity particles as 1^-, 2^-. The position of the third negative helicity particle is fixed to be n^- in both of the expressions. Keeping this in mind, the n-particle amplitude $\mathcal{A}_n[1^-, 2^-, 3^+, \ldots, (n-1)^+, n^-]$ (abbreviated as $\mathcal{A}_n^{\text{NMHV}}[1^-, 2^-, n^-]$) from [24] is given by

$$\mathcal{A}_n^{\text{NMHV}}[1^-, 2^-, n^-] = \frac{1}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle} \sum_{t=4}^{n-1} \mathcal{R}[n; 2; t](\langle n1\rangle\langle n\, t\, 2 \,|\, 1\rangle)^4 \,. \tag{4.66}$$

The objects $\mathcal{R}[n; s; t]$ are defined to be

$$\mathcal{R}[n; s; t] := \frac{1}{x_{st}^2} \frac{\langle s(s-1)\rangle}{\langle n\, t\, s \,|\, s\rangle\langle n\, t\, s \,|\, s-1\rangle} \frac{\langle t(t-1)\rangle}{\langle n\, s\, t \,|\, t\rangle\langle n\, s\, t \,|\, t-1\rangle} \tag{4.67}$$

with $\mathcal{R}[n; s; t] := 0$ for $t = s + 1$ or $s = t + 1$. The spinor products are defined as

$$\langle n\, t\, s \,|\, s\rangle := \langle n \,| x_{nt}\, x_{ts} \,| s\rangle \tag{4.68}$$

where

$$x_{st}^{\alpha\dot{\alpha}} := (p_s + p_{s+1} + \ldots + p_{t-1})^{\alpha\dot{\alpha}} \tag{4.69}$$

for $s < t$, $x_{ss} = 0$ and $x_{st} = -x_{ts}$ for $s > t$. So we have

$$\mathcal{R}[n;2;t] := \frac{1}{x_{2t}^2}\frac{\langle 21\rangle}{\langle n\,t\,2\,|\,2\rangle\langle n\,t\,2\,|\,1\rangle}\frac{\langle t(t-1)\rangle}{\langle n\,2\,t\,|\,t\rangle\langle n\,2\,t\,|\,t-1\rangle} \tag{4.70}$$

with $x_{2t}^2 = (p_2 + p_3 + \ldots + p_{t-1})^2 = s_{2(t-1)}$. Therefore, the n-point NMHV gluon amplitude can be written as

$$\mathcal{A}_n^{\text{NMHV}}[1^-, 2^-, n^-] = \frac{\langle n1\rangle^3}{\langle 23\rangle \ldots \langle n1\rangle}\sum_{t=4}^{n-1}\frac{1}{s_{2(t-1)}}\frac{\langle t(t-1)\rangle\langle n\,t\,2\,|\,2\rangle^3}{\langle n\,t\,2\,|\,1\rangle\langle n\,2\,t\,|\,t\rangle\langle n\,2\,t\,|\,t-1\rangle}\,. \tag{4.71}$$

Now by writing $t = r + 1$, we can get

$$\begin{aligned}\mathcal{A}_n^{\text{NMHV}}[1^-, 2^-, n^-] = \sum_{r=3}^{n-2} & \frac{\langle n1\rangle^3}{s_{23\ldots r}\langle 23\rangle \ldots \langle (r-1)r\rangle\langle (r+1)(r+2)\rangle \ldots \langle (n-1)n\rangle} \\ & \times \frac{\langle n\,(r+1)\,2\,|\,2\rangle^3}{\langle n\,(r+1)\,2\,|\,1\rangle\langle n\,2\,(r+1)\,|\,(r+1)\rangle\langle n\,2\,(r+1)\,|\,r\rangle}\,.\end{aligned} \tag{4.72}$$

Let us now consider one of the following spinor products and simplify as follows

$$\begin{aligned}\langle n\,(r+1)\,2\,|\,1\rangle &= \langle n\,|x_{n(r+1)}\,x_{(r+1)2}\,|1\rangle \\ &= \langle n\,|x_{(r+1)n}\,x_{2(r+1)}\,|1\rangle \\ &= \langle n\,|(\not{p}_{r+1} + \not{p}_{r+2} + \ldots + \not{p}_{n-1})\,(\not{p}_2 + \not{p}_3 + \ldots + \not{p}_r)\,|1\rangle \\ &= \langle n\,|(\not{p}_{r+1} + \not{p}_{r+2} + \ldots + \not{p}_{n-1} + \not{p}_n)\,(\not{p}_1 + \not{p}_2 + \ldots + \not{p}_r)\,|1\rangle \\ &= -\langle n\,|(p_1 + p_2 + \ldots + p_r)^2\,|1\rangle \\ &= -s_{12\ldots r}\langle n1\rangle\end{aligned} \tag{4.73}$$

$$\langle n\,2\,(r+1)\,|\,r\rangle = \langle n1\rangle[1|p_{2,(r-1)}|r\rangle\,, \tag{4.74}$$

$$\langle n\,2\,(r+1)\,|\,(r+1)\rangle = \langle n1\rangle[1|p_{2,r}|(r+1)\rangle\,, \tag{4.75}$$

$$\langle n\,(r+1)\,2\,|\,2\rangle = \langle 2|\not{p}_{3,r}(\not{p}_{r+1} + \ldots + \not{p}_{n-1})|n\rangle = \langle 2|\not{p}_{3,r}\not{p}_1|n\rangle + p_{2,r}^2\langle n2\rangle. \tag{4.76}$$

Assembling all these, we can rewrite the NMHV amplitude as follows

$$\mathcal{A}_n^{\text{NMHV}}[1^-, 2^-, n^-] = \sum_{r=3}^{n-2} \frac{\langle 2|\not{p}_{3,r}(\not{p}_{r+1}+\cdots+\not{p}_{n-1})|n\rangle^3}{s_{23\ldots r}s_{12\ldots r}\langle 23\rangle\langle 34\rangle\ldots\langle (r-1)\,r\rangle\langle (r+1)(r+2)\rangle\ldots\langle (n-1)n\rangle[1|p_{2,r-1}|r\rangle[1|p_{2,r}|r+1\rangle} \,. \quad (4.77)$$

This expression matches with the NMHV amplitude in equation (4.57). Therefore, we conclude that the massive vector boson amplitude we derived has the correct high energy limit.

4.2.2 Spurious poles

Although the covariant recursion relations allows us to determine the n-particle amplitude in a compact form, in the case of $n \geq 6$ the expression in (4.50) contains spurious poles which are not associated to any physical propagator going on-shell. These poles are arising when terms of the form $\langle 2|\not{p}_1\not{p}_{2,r-1}|r\rangle$ and $\langle 2|\not{p}_1\not{p}_{2,r}|r+1\rangle$ in the denominator of the expression in (4.50), becomes zero. Any on-shell recursion scheme that involves complexification of external momenta, are generically infected with such spurious poles as the manifest locality is sacrificed at the altar of staying on-shell. In the case of BCFW recursion relations for massless theories such as non-Abelian gauge theory, the spurious poles have been analysed extensively. These poles are not physical and the residue at these poles shown to vanish [56] in the case of six point gluon amplitude.

We expect that the same should be true in the massive case as the theories under considerations are local. However, as is well known, proving that spurious poles are indeed spurious is no easy task even for scattering amplitudes of massless particles and the proofs usually involve rewriting the amplitudes in terms of some other basis such as momentum twistors [56]. We do not pursue this important question in the present work but give an evidence that the poles which arise in (4.50) do not correspond to any on-shell propagators and therefore are indeed spurious.

We consider the six-point amplitude and evaluate it using $[\mathbf{6}\mathbf{5}^+\rangle$ shift, which leads to the following expression for the six-point amplitude

$$\begin{aligned}\mathcal{A}_6[\mathbf{1},2^-,3^+,4^+,5^+,\mathbf{6}] = g^4\Bigg[& \frac{(\langle\mathbf{6}|p_2|3]\langle 2\mathbf{1}\rangle+\langle 2|p_1|3]\langle\mathbf{61}\rangle)^2[34]\langle 4|p_6|5]}{[23]\langle 54\rangle(s_{12}-m^2)(s_{123}-m^2)(s_{56}-m^2)\big(\langle 2|\not{p}_1\not{p}_6|4\rangle+m^2\langle 24\rangle\big)} \\ & + \frac{\big(\langle 4|p_6|5]\{\langle 2\mathbf{1}\rangle\langle 2|p_1|\mathbf{6}]+\langle 2|p_6|\mathbf{1}]\langle 2\mathbf{6}\rangle+2m\langle 2\mathbf{1}\rangle\langle\mathbf{6}2\rangle\}+\langle 2\mathbf{1}\rangle[\mathbf{6}5]\langle 4|\not{p}_5\not{p}_1|2\rangle+\langle 4|p_5|\mathbf{1}]\langle 2|p_6|5]\langle 2\mathbf{6}\rangle\big)^2}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\big(\langle 4|\not{p}_5\not{p}_1\not{p}_6|5]+s_{16}\langle 4|p_6|5]\big)(s_{56}-m^2)\big(\langle 2|\not{p}_1(\not{p}_5+\not{p}_6)|4\rangle+m^2\langle 24\rangle\big)} \\ & + \frac{\big(\langle 2|p_1+p_6|2][5|\not{p}_1\not{p}_6|5]-s_{16}[5|\not{p}_2\not{p}_6|5]\big)\langle\mathbf{16}\rangle^2\langle 2|p_1+p_6|5]^2}{s_{156}s_{16}\langle 23\rangle\langle 34\rangle\big([5|\not{p}_6\not{p}_1|2]+m^2[52]\big)\big(\langle 4|\not{p}_5\not{p}_1\not{p}_6|5]+s_{16}\langle 4|p_6|5]\big)}\Bigg]. \end{aligned} \tag{4.78}$$

Here the spurious pole is given by the following condition

$$\langle 4|\not{p}_5\not{p}_1\not{p}_6|5] + s_{16}\langle 4|p_6|5] = 0\,. \tag{4.79}$$

With the massive-massless shift $[2^-\mathbf{1}\rangle$, the 6-point amplitude can be written using (4.50) as

$$\begin{aligned}\mathcal{A}_6[\mathbf{1},2^-,3^+,4^+,5^+,\mathbf{6}] = g^4\Bigg[& \frac{\big(\langle 2|\not{p}_1\not{p}_3|2\rangle\langle\mathbf{16}\rangle+p_{2,3}^2\langle\mathbf{1}2\rangle\langle 2\mathbf{6}\rangle\big)^2\langle 2|\not{p}_3\big((s_{56}-m^2)-\not{p}_4\not{p}_{1,3}\big)|5]}{s_{23}(s_{123}-m^2)(s_{56}-m^2)\langle 23\rangle\langle 45\rangle\langle 2|\not{p}_1\not{p}_2|3\rangle\langle 2|\not{p}_1\not{p}_{2,3}|4\rangle} \\ & + \frac{\big(\langle 2|\not{p}_1\not{p}_{3,4}|2\rangle\langle\mathbf{16}\rangle+p_{2,4}^2\langle\mathbf{1}2\rangle\langle 2\mathbf{6}\rangle\big)^2\langle 2|p_{3,4}|5]}{s_{234}(s_{56}-m^2)\langle 23\rangle\langle 34\rangle\langle 2|\not{p}_1\not{p}_{2,3}|4\rangle\langle 2|\not{p}_1\not{p}_{2,4}|5\rangle} \\ & + \frac{(\langle 2\mathbf{1}\rangle\langle 2|p_1|\mathbf{6}]+\langle 2|p_6|\mathbf{1}]\langle 2\mathbf{6}\rangle+2m\langle 2\mathbf{1}\rangle\langle\mathbf{6}2\rangle)^2}{s_{16}\langle 23\rangle\langle 34\rangle\langle 45\rangle\big(\langle 2|\not{p}_1\not{p}_6|5\rangle+m^2\langle 25\rangle\big)}\Bigg]. \end{aligned} \tag{4.80}$$

The spurious pole in the above amplitude is given by the following condition

$$\langle 2|\not{p}_1(\not{p}_2 + \not{p}_3)|4\rangle = 0\,. \tag{4.81}$$

It is easy to check that both expressions for the six-point amplitude contain the same set of physical poles. However we see that they have different spurious poles structures. In particular, when the spurious pole condition is satisfied for one expression, the other one is finite. Since, both are representations of the same scattering amplitude, we conclude that at least in this simple example, the residues of the spurious poles must be zero [57].

Chapter 5

Conclusions

In this thesis, we have derived a specific generalization of the well known BCFW recursion relations which include a combined complex momentum shift of a pair of particles, one massless and one massive, by making use of the recently proposed little group covariant spinor helicity formalism for massive particles in [11]. We gave a complete classification of this class of two-line shifts for massive scalar QCD and Yang-Mills theory in the Higgsed phase [14]. Later, we have used the new set of recursion relations to compute scattering amplitudes which hitherto were not known in the literature [21].

We proved the validity of the recursion relations for massive scalar QCD and Higgsed Yang-Mills theories by suitably adapting the background field methods of Arkani-Hamed and Kaplan [19] to include massive particles. As an explicit check of the new recursion relations, we computed several four and five particle amplitudes in these theories and found perfect agreement with the results already known in the literature by other methods. In this process, we derived the five particle vector boson amplitudes for different helicity configurations of gluons as a new result of this formalism, which we substantiated with several consistency checks [14]. Using the background field method, we showed that the massive-massless shifts $[\mathbf{m}+\rangle, [-\mathbf{m}\rangle$ are indeed valid shifts, whereas the $[\mathbf{m}-\rangle, [+\mathbf{m}\rangle$ shifts fail to recursively construct amplitudes in massive scalar QCD and the Higgsed

Yang-Mills theories.

In chapter 4, we explored the consequences of the covariant recursion relations introduce in [14] by considering the massive analogues of NMHV and MHV amplitudes in Yang-Mills theory. In the high energy limit, we showed that these two classes indeed reduce to the MHV and NMHV amplitudes respectively. The massive analogue of the NMHV amplitude comprised of two massive vector bosons, one negative helicity gluon (that is colour adjacent to the massive bosons) and an arbitrary number of gluons with positive helicity. We showed that for this class of amplitudes, the massive-massless shift leads to a remarkably simple computation and we could generate rather compact little group covariant expression for the final amplitude. If one had used the usual BCFW shift to compute this amplitude, one would end up with subamplitudes involving the same configuration as the one we set out to compute (i.e. involving two massive vector bosons and helicity flipped gluons). As a result one would need some additional input in order to proceed further. The massive-massless shift appears to be more convenient in this case as the recursion gave rise to simpler subamplitudes that were already known. Interestingly we have shown that given this final form for the amplitude derived using the covariant recursion, one can check that our result indeed satisfies the BCFW recursion relation. This is shown in detail in Appendix F.

Although a complete theory independent analysis comparing the two recursion schemes using massless-massless and massive-massless shifts, whenever the both may be applicable is yet to be done, we showed that for a particular class of amplitudes, the covariant massive-massless shift leads to a computational advantage while preserving the little group covariance guaranteed by the massive spinor-helicity formalism.

Apart from the rudimentary but non trivial check of the high energy limits of these amplitudes, we have used several other independent methods to validate our result. In the case of vector boson amplitude with all plus helicity gluons, we directly matched with the result expected from the covariant relation (4.1), given the scalar amplitude in (4.2).

But in the flipped helicity case, since this is a new result, we checked the consistency by carrying out the soft momentum expansion of the NMHV amplitude that leads to correct universal leading soft factor (see section 4.2.1). Interestingly, our representation of the NMHV amplitude obtained in the high energy limit is not identical to the one obtained previously in [24]. We showed that the two expressions are equal and it will be interesting to study the representation for NMHV amplitude that we obtained in more detail in its own right.

There are several directions that follow from the works presented in this thesis. We highlight some of them below.

- We have restricted ourselves to the calculation of tree level amplitudes using the new recursion relation and an obvious question would be to extend these results to loop amplitudes.

- In the same spirit of massive-massless shift it would be worthwhile to find a shift involving two massive momenta. This is relevant for the computation of all massive amplitudes. Preliminary analysis seems to indicate that the little group covariance of the deformation of massive spinor helicity variables is difficult to maintain, which makes the computation of amplitudes using massive spinor helicity formalism less tractable.

- Regarding the computation of $n-$particle amplitudes, we considered a particular configuration of external particles in which the position of the negative helicity gluon is adjacent to the massive vector bosons. But in fact it is possible to make the position of the negative helicity gluon completely arbitrary and use the covariant recursion or the BCFW shift in combination with the amplitudes calculated in this work to derive these scattering amplitudes. One could also add more negative helicity gluons and systematically proceed to calculate the resulting scattering amplitudes. However in order to compute amplitudes with more than two massive particles using the covariant recursion, one would possibly require knowledge of a wider class of amplitudes.

It would be interesting to explore all these quantities in more detail and we defer this to future works.

Appendix A

The high energy limit

In this appendix, we discuss the relation between the massive and massless spinor helicity variables and the procedure to take the high-energy limit of scattering amplitudes involving massive particles. The basic objective of this procedure is to show how massive amplitudes for particles with spin decompose into the different helicity components in this limit.

We begin by expanding the massive spinor helicity variables λ_α^I and $\tilde{\lambda}_{I\dot{\alpha}}$ into explicit $\mathbf{SL}(2,\mathbb{C})$ basis. In order to do this, we consider momentum 4-vector in the spherical polar coordinates

$$p^\mu \equiv \left(p^0, |\vec{p}|\sin\theta\cos\phi, |\vec{p}|\sin\theta\sin\phi, |\vec{p}|\cos\theta\right), \tag{A.1}$$

which takes the following matrix form in SU(2) representation

$$p_{\alpha\dot{\alpha}} = \begin{pmatrix} p^0 + |\vec{p}|\cos\theta & |\vec{p}|\sin\theta(\cos\phi - i\sin\phi) \\ |\vec{p}|\sin\theta(\cos\phi + i\sin\phi) & p^0 - |\vec{p}|\cos\theta \end{pmatrix}. \tag{A.2}$$

For massless particles, we set $p^0 = |\vec{p}|$ and the matrix form is given by

$$p_{\alpha\dot{\alpha}} = 2p^0 \begin{pmatrix} \cos^2\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)e^{-i\phi} \\ \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)e^{i\phi} & \sin^2\left(\frac{\theta}{2}\right) \end{pmatrix} = 2p^0 \begin{pmatrix} c \\ s \end{pmatrix} \times \begin{pmatrix} c & s^\star \end{pmatrix}, \tag{A.3}$$

where we denote $c \equiv cos(\theta/2)\,, s \equiv sin(\theta/2)e^{i\phi}$. Therefore, we obtain the following representation for massless spinor helicity variables

$$\lambda_\alpha = \sqrt{2p^0}\begin{pmatrix} c \\ s \end{pmatrix}, \qquad \tilde{\lambda}_{\dot{\alpha}} = \sqrt{2p^0}\begin{pmatrix} c \\ s^\star \end{pmatrix}. \tag{A.4}$$

One can repeat the same procedure to find the momentum representation for massive spinor helicity formalism as

$$\lambda_\alpha^I = \begin{bmatrix} ac & -bs^* \\ as & bc \end{bmatrix}, \quad \tilde{\lambda}_{I\dot{\alpha}} = \begin{bmatrix} ac & -bs \\ as^* & bc \end{bmatrix}, \tag{A.5}$$

where $a = \sqrt{p^0 + |\vec{p}|}\,, b = \sqrt{p^0 - |\vec{p}|}$ and $\det(\lambda_\alpha^I) = \det(\tilde{\lambda}_{I\dot{\alpha}}) = m$. This matrices allows us to expand the massive spinor helicity variables in SU(2) space in terms massless spinor helicity variables

$$\lambda_I^\alpha = \lambda^\alpha \xi_I^+ - \eta^\alpha \xi_I^-\,, \tag{A.6}$$

$$\tilde{\lambda}_J^{\dot{\alpha}} = -\tilde{\lambda}^{\dot{\alpha}} \xi_J^- + \tilde{\eta}^{\dot{\alpha}} \xi_J^+\,, \tag{A.7}$$

where the SU(2) basis vectors are chosen to be $\xi^{-I} = (1 \quad 0)$ and $\xi^{+I} = (0 \quad 1)$. The massless spinor helicity variables are given by

$$\lambda_\alpha = a\begin{pmatrix} c \\ s \end{pmatrix}, \eta_\alpha = b\begin{pmatrix} -s^* \\ c \end{pmatrix}, \tag{A.8}$$

$$\tilde{\lambda}_{\dot{\alpha}} = a\begin{pmatrix} c \\ s^* \end{pmatrix}, \tilde{\eta}_{\dot{\alpha}} = b\begin{pmatrix} -s \\ c \end{pmatrix}. \tag{A.9}$$

As usual, the SU(2) indices can be raised and lowered by the anti symmetric tensor ϵ_{IJ} and the basis vectors follow $\epsilon_{IJ}\xi^{+I}\xi^{-J} = -1$. From explicit calculation we find that the massless spinors obey the following relations

$$[\tilde{\lambda}\tilde{\eta}] = m = \langle\lambda\eta\rangle\,. \tag{A.10}$$

In the high energy limit $p^0 >> |\vec{p}|$, the massless spinors can be expressed as

$$\lambda_\alpha \to \sqrt{2p^0}\begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)e^{i\phi}\end{pmatrix}, \quad \eta_\alpha \to \frac{m}{\sqrt{2p^0}}\begin{pmatrix}-\sin(\theta/2)e^{-i\phi}\\ \cos(\theta/2)\end{pmatrix}. \tag{A.11}$$

This shows that η_α is proportional to the mass m and vanishes in this limit. A similar result holds for $\tilde{\eta}_{\dot{\alpha}}$. Therefore, in the high energy limit both $(\eta_\alpha, \tilde{\eta}_{\dot{\alpha}})$ vanish.

As a general rule of thumb, the massive spinors behaves in the massless limit as follows

$$|\mathbf{n}\rangle \xrightarrow{p^0>>|\vec{p}|} |n^-\rangle\,, \qquad |\mathbf{n}] \xrightarrow{p^0>>|\vec{p}|} |n^+]\,. \quad \pm \text{ indicates helicity}\,. \tag{A.12}$$

Here, a particular helicity is picked up since the non vanishing component λ_α comes with opposite SU(2) basis vectors, which we use the identify the helicity component in this limit.

Let us now illustrate the connection between the minimally coupled three particle massive amplitude (2.71) (for $S = 1$) in the high energy limit and the massless amplitude (2.39). Recall that, the massive amplitude in consideration are given by

$$\mathcal{A}_3^{+h}(\mathbf{1},\mathbf{2},3^h) = g x_{12}^h \frac{\langle\mathbf{12}\rangle^2}{m}\,, \quad \mathcal{A}_3^{-h}(\mathbf{1},\mathbf{2},3^{-h}) = g x_{12}^{-h} \frac{[\mathbf{12}]^2}{m}\,. \tag{A.13}$$

Due to the structure of the amplitudes, only the identical helicity ($h_1 = h_2$) configurations of massive particles are allowed in high energy limit. Moreover, since we are looking at minimal coupling, we must consider the massless particle (with momentum p_3) to have opposite helicity as opposed to the other two particles. Now we consider the case of photon or gluons, in which case we set $h_1 = h_2 = -1$ and $h_3 = +1$

$$\mathcal{A}_3^{\text{high energy limit}}(1^{-1}, 2^{-1}, 3^{+1}) = g\frac{m[3\xi]}{\langle 32\rangle[2\xi]}\frac{\langle 12\rangle^2}{m} = g\frac{\langle 12\rangle^3}{\langle 31\rangle\langle 23\rangle}\,, \tag{A.14}$$

where we have used the representation in equation (2.61) for the x_{12} factor. Note that, we have recovered the correct massless amplitude and the coupling is dimensionless in this case, as promised. One can repeat this exercise to check the correspondence regarding minimal coupling for opposite helicity configurations and in the case involving gravitons. Thus the massive three-particle amplitudes coincides with the conventional notion of minimal coupling for $S \leq 1$ [11].

Appendix B

Reduction of Massive Polarization

In this appendix, we recover a pair of transverse and a longitudinal mode of the polarization tensor of massive vector boson corresponding to $s_z = \pm 1; 0$ spin components at large-z. This plays an important role in our classification of covariant massive-massless shifts. In the spinor-helicity formalism, the polarization tensor for a massive spin-1 particle is given by [58,59]:

$$e^{\mu}_{IJ}(i) = \frac{1}{2\sqrt{2}m}\left[\langle\lambda_{iI}|\sigma^{\mu}|\tilde{\lambda}_{iJ}] + (I \leftrightarrow J)\right] . \tag{B.1}$$

Using the expansions in equation (A.6) and (A.7), the momentum matrix $p_{i\alpha\dot{\alpha}} = p_{i,\mu}\sigma^{\mu}_{\alpha\dot{\alpha}}$ for massive particle can be written as

$$p_{i\alpha\dot{\alpha}} = -\lambda_{i\alpha}\tilde{\lambda}_{i\dot{\alpha}} + \eta_{i\alpha}\tilde{\eta}_{i\dot{\alpha}} . \tag{B.2}$$

Furthermore, the Dirac equation in spinor helicity formalism takes the following form

$$p_{\alpha\dot{\alpha}}\lambda^{\alpha}_{I} = -m\tilde{\lambda}_{I\dot{\alpha}} , \quad p_{\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_{I} = m\lambda_{I\alpha} . \tag{B.3}$$

Using the Dirac equation, it is easy to show that the polarization tensor for massive particle is orthogonal to momentum 4-vector, as expected

$$p_{i\mu}e^{\mu}_{IJ}(i) = 0\,. \tag{B.4}$$

We adopt identical momentum representation as given in [11] for the spinor helicity variables $(\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}})$, $(\eta_\alpha, \tilde{\eta}_{\dot{\alpha}})$ variables such that, they satisfy $\langle\lambda_i\eta_i\rangle = [\tilde{\lambda}_i\tilde{\eta}_i] = m$. We then use these relations to replace the mass m present in the denominator of the spinor helicity representation of $e^{\mu}_{IJ}(i)$ in (B.1). With the expansions in equations (A.6) and (A.7), the polarization tensor for massive spin 1 particle can be expressed in terms of massless spinors as

$$e^{\mu}_{IJ}(i) = \left[\frac{\langle\lambda_i|\sigma^\mu|\tilde{\eta}_i]}{\sqrt{2}[\tilde{\lambda}_i\tilde{\eta}_i]}\xi^+_I\xi^+_J - \frac{\langle\eta_i|\sigma^\mu|\tilde{\lambda}_i]}{\sqrt{2}\langle\eta_i\lambda_i\rangle}\xi^-_I\xi^-_J\right] - \frac{1}{2\sqrt{2}m}\left(\langle\lambda_i|\sigma^\mu|\tilde{\lambda}_i] + \langle\eta_i|\sigma^\mu|\tilde{\eta}_i]\right)\xi^+_{(I}\xi^-_{J)}\,. \tag{B.5}$$

Now we can read off the transverse and longitudinal polarization vectors from the coefficients of the ξ_I-bilinears. The polarization vectors associated to the transverse modes are

$$e^{\mu}_{+}(i) = \frac{\langle\eta_i|\sigma^\mu|\tilde{\lambda}_i]}{\sqrt{2}\langle\eta_i\lambda_i\rangle}\,, \qquad e^{\mu}_{-}(i) = \frac{\langle\lambda_i|\sigma^\mu|\tilde{\eta}_i]}{\sqrt{2}[\tilde{\lambda}_i\tilde{\eta}_i]}\,, \tag{B.6}$$

whereas the polarization vector associated to the longitudinal mode is given by

$$e^{\mu}_{0}(i) = \frac{1}{2\sqrt{2}m}\left(\langle\lambda_i|\sigma^\mu|\tilde{\lambda}_i] + \langle\eta_i|\sigma^\mu|\tilde{\eta}_i]\right)\,. \tag{B.7}$$

In the following subsection, we discuss the large z behaviour of transverse modes the massive polarization tensor.

B.1 Deformed massive polarization vector at large z

The polarization tensor for a massive particle, in terms of massive spinor helicity variables, is given in (B.1). An important component in our analysis of the two-line shifts is the behaviour of the polarization tensor at large-z. In this limit, the transverse and longitudinal modes get decoupled from each other. From (B.6) the massless polarization vectors are function of massless spinors (λ_i, η_i) and $(\tilde{\lambda}_i, \tilde{\eta}_i)$. In the large-$z$ limit we treat $(\eta_i^\alpha, \tilde{\eta}_i^{\dot{\alpha}})$ as the usual reference spinors $(\zeta^\alpha, \xi^{\dot{\alpha}})$ that appear in the massless case (and which are chosen to be some of the external momenta in a given scattering amplitude calculation). So the expressions for the deformed polarization vectors, which we denote as $\widehat{e_\mu^+}(i)$ can be written as follows:

$$\widehat{e_\mu^+}(i) = \frac{\langle\zeta|\sigma_\mu|\widehat{\tilde{\lambda}_i}]}{\sqrt{2}\langle\zeta\lambda_i\rangle}\,, \quad \widehat{e_\mu^-}(i) = \frac{\langle\lambda_i|\sigma_\mu|\xi]}{\sqrt{2}[\widehat{\tilde{\lambda}_i}\xi]}\,. \tag{B.8}$$

The shift for the massless spinor helicity variable $\tilde{\lambda}_{i\dot{\alpha}}$ in (A.7) can be obtained from the shift of massive spinor helicity variable $\widehat{\tilde{\lambda}}{}^I_{i\dot{\alpha}}$ as

$$\widehat{\tilde{\lambda}}_{i\dot{\alpha}} = \tilde{\lambda}_{i\dot{\alpha}} - \frac{z}{m}\tilde{\lambda}_{j\dot{\alpha}}[\tilde{\lambda}_i\tilde{\lambda}_j]. \tag{B.9}$$

By choosing reference spinors $\zeta^\alpha \equiv \lambda_j^\alpha$ and $\xi_{\dot{\alpha}} \equiv \tilde{\lambda}_{j\dot{\alpha}}$, we can express

$$\widehat{e_b^+}(i) = \Sigma_{ijb} - \frac{z}{m}p_{jb}\,\Omega_{ij}\,, \quad \widehat{e_b^-}(i) = \frac{\Sigma^*_{ijb}}{\sqrt{2}[\tilde{\lambda}_i\tilde{\lambda}_j]}\,. \tag{B.10}$$

where we define

$$\Sigma_{ijb} = \frac{\langle\lambda_j|\sigma_b|\tilde{\lambda}_i]}{\sqrt{2}\langle\lambda_j\lambda_i\rangle} \quad \text{and} \quad \Omega_{ij} = \frac{[\tilde{\lambda}_i\tilde{\lambda}_j]}{\sqrt{2}\langle\lambda_j\lambda_i\rangle}\,. \tag{B.11}$$

These relations have been used in (3.88) and (3.90).

B.2 Deformed massless polarization vector

In order to derive the limiting behaviour of $\widehat{\mathcal{A}}^h_{IJ}$ at large z, we expressed the shifted massless polarization $\widehat{e}^\mu_+(j)$ in terms of the shift momentum r^μ. In this appendix we show how to derive (3.83). We start with the massless polarization vector

$$\widehat{e}^\mu_+(j) = \frac{\langle\xi|\sigma^\mu|\widehat{\tilde{\lambda}}_j]}{\sqrt{2}\langle\xi\widehat{\lambda}_j\rangle} = \frac{\langle\xi|\sigma^\mu|\tilde{\lambda}_j]}{\sqrt{2}\langle\xi\widehat{\lambda}_j\rangle}\,. \tag{B.12}$$

Here ξ_α is a reference spinor, which we choose to be

$$\xi^\alpha = \frac{p_i^{\dot{\beta}\alpha}}{m}\tilde{\lambda}_{j\dot{\beta}}\,. \tag{B.13}$$

Then the massless polarization vector can be written as

$$\widehat{e}^\mu_+(j) = \frac{1}{\sqrt{2}\langle\lambda_j|p_i|\tilde{\lambda}_j]}p_i^{\dot{\beta}\alpha}\tilde{\lambda}_{j\dot{\beta}}\tilde{\lambda}_j^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\,. \tag{B.14}$$

Recall the expression for the shift momentum r^μ in terms of spinor helicity variables

$$r^\mu = -\frac{1}{2m}\bar{\sigma}^{\mu\dot{\alpha}\alpha}p_{i\alpha\dot{\beta}}\tilde{\lambda}_{j\dot{\alpha}}\tilde{\lambda}_j^{\dot{\beta}} = \frac{1}{2m}p_i^{\dot{\beta}\alpha}\tilde{\lambda}_{j\dot{\beta}}\tilde{\lambda}_j^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\,. \tag{B.15}$$

Thus we can write massless polarization vector as follows

$$\widehat{e}^\mu_+(j) = \frac{\sqrt{2}mr^\mu}{\langle\lambda_j|p_i|\tilde{\lambda}_j]} \equiv \kappa\, r^\mu\,. \tag{B.16}$$

We have used this result in (3.83).

Appendix C

Comparison with previous results

In [13], the authors derived conditions to check the validity of multi line shifts in the context of BCFW type recursion relations. The authors studied large z behaviour of n-point scattering amplitudes ($\widehat{\mathcal{A}}(z) \sim z^{\gamma}$) with multi-line shifts and obtained a bound on γ that would lead to valid shifts. For the two-line shifts discussed in this paper, the relevant constraint is given by:

$$\gamma \leq 1 - [\tilde{g}] - s_{\lambda} + s_{\tilde{\lambda}} \tag{C.1}$$

where $[\tilde{g}]$ is the mass dimension of coupling, s_{λ} and $s_{\tilde{\lambda}}$ are the spin projection (s_z) of the particles which are shifted by the spinor-helicity variables λ and $\tilde{\lambda}$ respectively. For the massive-massless shift we have considered, the λ-variable of the positive helicity gluon is shifted and the $\tilde{\lambda}$-variable of the massive spin-1 particle is shifted. Thus $s_{\lambda} = 1$ and $s_{\tilde{\lambda}} = -1$ (min. of s_z for massive spin-1).

For example consider the four-point amplitude in section 3.2.3,

$$\widehat{\mathcal{A}}_4 \left[\widehat{\mathbf{1}}, \widehat{2}^{+}, 3^{-}, \mathbf{4}\right] = \frac{g^2}{m^2} \frac{\langle 3|p_4|2]^2 [\widehat{\mathbf{14}}]^2}{\widehat{s}_{23}(\widehat{s}_{24} - m^2)} \, . \tag{C.2}$$

In this case, we find the condition for valid shifts as $\gamma \leq 1$ where we have used $[\tilde{g}] = -2$.

On the other hand, from (C.2) we get the larg z behaviour of the amplitude as $\mathcal{A}_4(z) \sim z^0$. So this satisfies the above validity condition of the shifts. Similarly one can check that for all the amplitudes computed in this paper, the constraints found in [13] are satisfied.

Appendix D

Examples of Invalid Shifts

In this appendix, we show that the massive-massless shifts of the type [**m**−⟩ and [+**m**⟩ do not lead to a recursion relation as the deformed amplitude grows at large z. Let us consider the [**m**−⟩ shift. The massless polarization $\widehat{e_b^-}(j)$ takes the form given below in spinor helicity formalism

$$\widehat{e_b^-}(j) = \frac{\langle\widehat{j}|\sigma_b|\eta]}{\sqrt{2}[j\eta]} = \frac{\langle j|\sigma_b|\eta] + \frac{z}{m}[i_I j]\langle i^I|\sigma_b|\eta]}{\sqrt{2}[j\eta]} \,. \tag{D.1}$$

In the last step we have used the shift equation (3.10). By choosing the reference spinor to be

$$\tilde{\eta}_{\dot{\alpha}} = \frac{p_{i\alpha\dot{\alpha}}}{m}\lambda_j^{\alpha} \,, \tag{D.2}$$

the negative helicity polarization vector in (D.1) is rewritten as

$$\widehat{e_b^-}(j) = \frac{2mr_b^* + \frac{z}{m}\left[(p_i \cdot p_j)p_{ib} - 2p_{jb}\right]}{\sqrt{2}\langle j|p_i|j]} \,. \tag{D.3}$$

The transverse polarizations of the massive particle remain identical as in (B.10). Therefore, the natural amplitudes for the transverse modes have the following large-z behaviour

$$\widehat{\mathcal{A}}_{-}^{-} = \frac{1}{\sqrt{2}\langle j|p_i|j]}\left[M^{ab} + \frac{1}{z}B^{ab} + ...\right] d_{ijb}\left(2mr_b^* + \frac{z}{m}\left[(p_i \cdot p_j)p_{ib} - 2p_{jb}\right]\right),$$
$$\sim \mathcal{O}(z)\,, \tag{D.4}$$

$$\text{and} \qquad \widehat{\mathcal{A}}_{+}^{-} \sim \mathcal{O}(z^2)\,, \tag{D.5}$$

thereby proving that the $[\mathbf{m}-\rangle$ shift is an invalid one. Following similar steps, it can be shown that the $[+\mathbf{m}\rangle$ shift is also invalid.

Appendix E

Lower-point amplitudes

In this appendix, we show that the four- and five-point amplitudes involving massive vector bosons computed previously in [14] using covariant recursion, are consistent with the general formula (4.6) and (4.50) derived in this work.

E.1 Lower-point amplitudes with identical gluons

As mentioned at the beginning of section 4.1.1, the relevant amplitudes needed to set up the method of induction are given as follows [11, 14]

$$\mathcal{A}_4\left[\mathbf{1}, 2^+, 3^+, \mathbf{4}\right] = g^2 \frac{[23]\langle\mathbf{14}\rangle^2}{\langle 23\rangle(s_{12}-m^2)}\,. \tag{E.1}$$

$$\mathcal{A}_5\left[\mathbf{1}, 2^+, 3^+, 4^+, \mathbf{5}\right] = g^3 \frac{\langle\mathbf{15}\rangle^2[2|\not{p}_1(\not{p}_2+\not{p}_3)|4]}{\langle 23\rangle\langle 34\rangle(s_{12}-m^2)(s_{45}-m^2)}\,. \tag{E.2}$$

Although, the four-particle amplitude matches straightforwardly with the expression that we obtain from the general formula (4.6) with $n = 4$, but the five-particle amplitude (E.2) does not identically match with the expression that we get from (4.6). In order to match these two expressions, we now prove the following identity:

$$[2|\not{p}_1(\not{p}_2 + \not{p}_3)|4] = [2|\left\{(s_{123} - m^2) - \not{p}_3(\not{p}_1 + \not{p}_2)\right\}|4]\,. \tag{E.3}$$

We rewrite the R.H.S in the following way

$$[2|\left\{(s_{123} - m^2) - \not{p}_3(\not{p}_1 + \not{p}_2)\right\}|4] = 2p_3 \cdot p_{1,2}[24] - [2|\not{p}_3\not{p}_{1,2}|4] + 2p_1 \cdot p_2[24]\,. \tag{E.4}$$

Using the following identity satisfied by the Pauli matrices (and identity matrix)

$$\left(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu\right)_{\dot{\alpha}}{}^{\dot{\beta}} = 2\eta^{\mu\nu}\delta_{\dot{\alpha}}{}^{\dot{\beta}}\,, \qquad \text{where} \quad \bar{\sigma}_\mu^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma_{\mu\beta\dot{\beta}}\,; \tag{E.5}$$

we get

$$2p_3 \cdot p_{1,2}[24] = [2|\not{p}_3\not{p}_{1,2}|4] + [2|\not{p}_{1,2}\not{p}_3|4]\,, \qquad 2p_1 \cdot p_2[24] = [2|\not{p}_1\not{p}_2|4]\,. \tag{E.6}$$

In the last equality, we use $p_2|2] = 0$. Substituting these results in (E.4), we easily obtain the identity (E.3). This completes the check of the formula (4.6) for n-particle amplitude involving a pair of massive vector bosons and positive helicity gluons for lower-point amplitudes.

E.2 Lower-point amplitudes with helicity flip

In this section, we verify the formula for the n-particle amplitude (4.50) involving a pair of massive vector bosons, one minus helicity gluon which is colour adjacent to the massive particles and $(n-3)$ positive helicity gluons for $n = 4$ and 5. First we write down the four- and five-point amplitudes directly by using (4.50) and then compare with the amplitudes computed using other techniques like unitarity and recursion involving massless-massless shift.

Four-point amplitude

Let us start with the four-particle amplitude for which only the first term in (4.50) contributes

$$\mathcal{A}_4\,[\mathbf{1},2^-,3^+,\mathbf{4}] = g^2\frac{(\langle 2|p_1|\mathbf{4}]\langle 2\mathbf{1}\rangle + \langle 2|p_4|\mathbf{1}]\langle 2\mathbf{4}\rangle + 2m\langle \mathbf{12}\rangle\langle 2\mathbf{4}\rangle)^2}{s_{14}\langle 23\rangle\left(\langle 2|\not{p}_1\not{p}_4|3\rangle + m^2\langle 23\rangle\right)}\,. \tag{E.7}$$

We simplify the following terms using momentum conservation

$$\langle 2|p_1|\mathbf{4}] = -\langle 2|p_3|\mathbf{4}] - m\langle 2\mathbf{4}\rangle\,,\quad \langle 2|p_4|\mathbf{1}] = -\langle 2|p_3|\mathbf{1}] - m\langle 2\mathbf{1}\rangle\,, \tag{E.8}$$

$$\langle 2|\not{p}_1\not{p}_4|3\rangle + m^2\langle 23\rangle = -(s_{12} - m^2)\langle 23\rangle\,, \tag{E.9}$$

and express the four-particle amplitude in the following form

$$\mathcal{A}_4\,[\mathbf{1},2^-,3^+,\mathbf{4}] = g^2\frac{([3\mathbf{4}]\langle 2\mathbf{1}\rangle + [3\mathbf{1}]\langle 2\mathbf{4}\rangle)^2}{s_{23}\,(s_{12} - m^2)}\,. \tag{E.10}$$

This result matches exactly with amplitudes computed in [11, 14]. Next we move to five-particle amplitude which we compute using massless-massless shift.

Five-point amplitude

We use the $[2^-3^+\rangle$ massless-massless shift to calculate the colour-ordered five-particle amplitude. The scattering channels to evaluate this amplitude using the particular shift are given in Figure E.1. We consider the following shift for massless spinor-helicity variables

$$[\widehat{2}| = [2| - z[3|\,,\qquad |\widehat{3}\rangle = |3\rangle + z|2\rangle\,. \tag{E.11}$$

The contribution to five-particle amplitude from the first diagram is obtained by gluing the four-particle amplitude alongwith the three-particle amplitude (2.71) for negative helicity

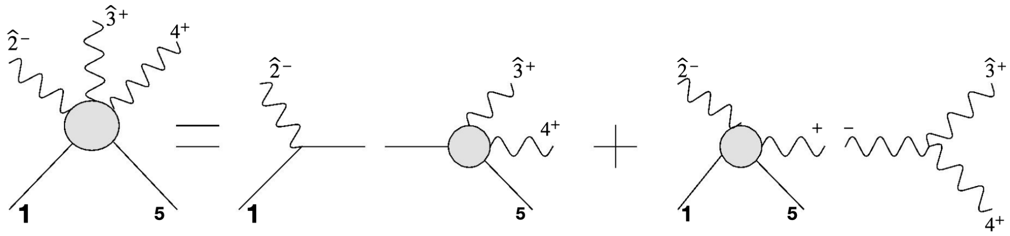


Figure E.1: Scattering channels to compute $\mathcal{A}_5\,[\mathbf{1}, 2^-, 3^+, 4^+, \mathbf{5}]$ with $[2^-3^+\rangle$ shift

gluon and unshifted propagator $\frac{1}{s_{12}-m^2}$

$$\mathcal{A}_5^I\,[\mathbf{1}, 2^-, 3^+, 4^+, \mathbf{5}] = \frac{g^3}{m^2}\frac{\langle 2|p_1|3]}{[23]}\frac{[34]\langle \mathbf{5}|\widehat{I}|\mathbf{1}]^2}{[23]\langle\widehat{34}\rangle(s_{12}-m^2)(s_{45}-m^2)}\,. \tag{E.12}$$

We get the pole $z_{(12)}$ for first diagram by setting the shifted propagator $\frac{1}{\widehat{s}_{12}-m^2}$ on-shell

$$z_{(12)} = \frac{\langle 2|p_1|2]}{\langle 2|p_1|3]}\,. \tag{E.13}$$

Using momentum conservation and definition of shifted massless spinor-helicity variables of (E.11), we get get rid of the dependence on internal momentum and evaluate remaining shifted spinor products at this pole. We obtain the contribution of first diagram

$$\begin{aligned}\mathcal{A}_5^I\,[\mathbf{1}, 2^-, 3^+, 4^+, \mathbf{5}] &= g^3\frac{\left(\langle \mathbf{5}|p_2|3]\langle 2\mathbf{1}\rangle + \langle 2|p_1|3]\langle \mathbf{51}\rangle\right)^2[34]}{[23](s_{12}-m^2)(s_{45}-m^2)\left(\langle 2|\not{p}_1\not{p}_5|4\rangle + m^2\langle 24\rangle\right)} \\ &= g^3\frac{\left(\langle 2|\not{p}_1\not{p}_3|2\rangle\langle \mathbf{15}\rangle + p_{23}^2\langle \mathbf{12}\rangle\langle 2\mathbf{5}\rangle\right)^2\langle 2|p_3|4]}{s_{23}(s_{123}-m^2)\langle 23\rangle\langle 2|\not{p}_1\not{p}_2|3\rangle\langle 2|\not{p}_1\not{p}_{2,3}|4\rangle}\,.\end{aligned} \tag{E.14}$$

According to the formula (4.50), there exists two scattering channels contributing to the five-particle amplitude. For $n = 5$, the sum in the second term of (4.50) becomes a single term which matches exactly with above expression.

The contribution from the second diagram in Figure E.1 is obtained by gluing the two subamplitudes along with unshifted propagator $\frac{1}{s_{34}}$. After evaluating the shifted spinor products at $z_{(34)} = \frac{\langle 34\rangle}{\langle 24\rangle}$, we get the contribution from this diagram as follows

$$\mathcal{A}_5^{II}[\mathbf{1}, 2^-, 3^+, 4^+, \mathbf{5}] = g^3 \frac{\left(\langle 2\mathbf{1}\rangle\langle 2|p_1|5] + \langle 2|p_5|\mathbf{1}]\langle 2\mathbf{5}\rangle + 2m\langle 2\mathbf{1}\rangle\langle \mathbf{5}2\rangle\right)^2}{\langle 23\rangle\langle 34\rangle s_{15}\left(\langle 2|\not{p}_1\not{p}_5|4\rangle + m^2\langle 24\rangle\right)}\,. \tag{E.15}$$

This expression matches with the first term in (4.50) with $n = 5$.

Appendix F

Flip helicity amplitude from BCFW recursion

In this appendix, we present an inductive proof of the formula in (4.50) using the BCFW recursion. To set up the induction, we first of all ensure that the four and five point amplitudes that have been derived previously in Appendix E.2 using BCFW recursion are consistent with the general expression.

Given the match of the lower-point amplitudes, we now assume that the expression (4.6) is true for $(n-1)$-particle amplitude and use this to derive n-particle amplitude. We use $[2^-3^+\rangle$ BCFW shift that corresponds to shifting the massless spinor-helicity variables as

$$|\widehat{2}] = |2] + z|3]\,, \qquad |\widehat{3}\rangle = |3\rangle - z|2\rangle\,. \tag{F.1}$$

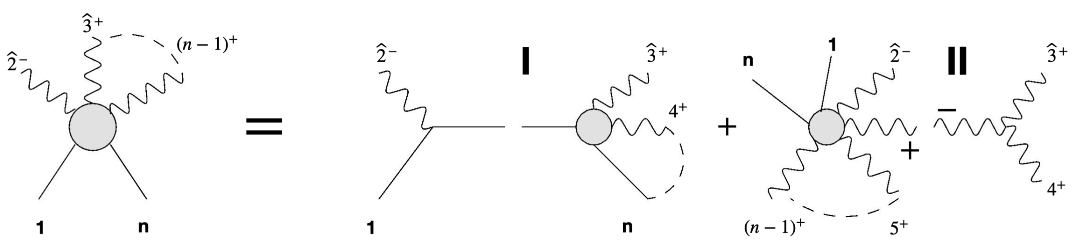


Figure F.1: Pictorial representation of BCFW recursion with $[2^-3^+\rangle$ shift.

Due to this shift, all possible scattering channels that have a non-zero contribution to n-point amplitude are shown in Figure F.1. The BCFW recursion for the first diagram is

$$\mathcal{A}_n^I = \mathcal{A}_L[\mathbf{1}, \widehat{2}^-, \widehat{\mathbf{I}}] \frac{1}{s_{12} - m^2} \mathcal{A}_R[\widehat{\mathbf{I}}, \widehat{3}^+, \ldots, (n-1)^+, \mathbf{n}] \,. \tag{F.2}$$

Substituting the 3-point amplitude as given in (2.71) and the result for the right subamplitude from (4.6) and then evaluating shifted spinor products at the simple pole for this diagram

$$z_{(12)} = -\frac{\langle 2|p_1|2]}{\langle 2|p_1|3]} \,, \tag{F.3}$$

we obtain

$$\mathcal{A}_n^I = \frac{\langle 2|\not{p}_3 \prod_{k=4}^{n-2}\left\{(s_{1\cdots k}-m^2)-\not{p}_k\not{p}_{1,k-1}\right\}|n-1]\left(\langle 2|\not{p}_1\not{p}_3|2\rangle\langle \mathbf{1n}\rangle+p_{2,3}^2\langle \mathbf{1}2\rangle\langle 2\mathbf{n}\rangle\right)^2\langle 34\rangle}{s_{23}(s_{123}-m^2)\ldots(s_{12\ldots(n-2)}-m^2)\langle 23\rangle\langle 34\rangle\ldots\langle (n-2)(n-1)\rangle\langle 2|\not{p}_1\not{p}_2|3\rangle\langle 2|\not{p}_1\not{p}_{2,3}|4\rangle} \,. \tag{F.4}$$

The BCFW recursion for the second diagram is the following

$$\mathcal{A}_n^{II} = \mathcal{A}_L[\mathbf{1}, \widehat{2}^-, \widehat{I}^+, 5^+, \ldots, (n-1)^+, \mathbf{n}] \frac{1}{s_{34}} \mathcal{A}_R[\widehat{I}^-, \widehat{3}^+, 4^+] \tag{F.5}$$

We substitute the left subamplitude from the expression in (4.6) by assuming that it holds for $(n-1)$-point amplitude. The right subamplitude is a pure gluon amplitude and is given by the Parke-Taylor formula. Using these expressions and simplifying further we get

$$\begin{aligned}\mathcal{A}_n^{II} = g^{n-2}\Bigg[& \frac{(\langle 2|p_1|\mathbf{n}]\langle 2\mathbf{1}\rangle+\langle 2|p_n|\mathbf{1}]\langle 2\mathbf{n}\rangle+2m\langle \mathbf{1}2\rangle\langle 2\mathbf{n}\rangle)^2}{s_{1n}\langle 23\rangle\langle 34\rangle\cdots\langle (n-2)(n-1)\rangle\left(\langle 2|\not{p}_1\not{p}_n|n-1\rangle+m^2\langle 2(n-1)\rangle\right)} \\ & + \sum_{r=4}^{n-2} \frac{\langle 2|\not{p}_{3,r} \prod_{k=r+1}^{n-2}\left\{(s_{1\cdots k}-m^2)-\not{p}_k\not{p}_{1,k-1}\right\}|n-1]\left(\langle 2|p_1.p_{3,r}|2\rangle\langle \mathbf{1n}\rangle+p_{2,r}^2\langle \mathbf{1}2\rangle\langle 2\mathbf{n}\rangle\right)^2\langle r(r+1)\rangle}{s_{23\ldots r}(s_{12\ldots r}-m^2)\ldots(s_{12\ldots(n-2)}-m^2)\langle 23\rangle\langle 34\rangle\ldots\langle (n-2)(n-1)\rangle\langle 2|\not{p}_1\not{p}_{2,r-1}|r\rangle\langle 2|\not{p}_1\not{p}_{2,r}|r+1\rangle}\Bigg] \,.\end{aligned} \tag{F.6}$$

Combining the contributions from two diagrams (F.4) and (F.6), we obtain the n-particle amplitude $\mathcal{A}_n[\mathbf{1}, 2^-, 3^+, \ldots, \mathbf{n}]$ which exactly matches with (4.50). This completes alternative check of (4.50) using BCFW recursion relations.

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