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REGULAR PAPER

Triangulated relativistic quantum computation: a curvature-modulated unification of quantum and relativistic computing

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Abstract We introduce *Triangulated Relativistic Quantum Computation* (TRQC), a mathematically consistent framework that integrates relativistic causal constraints with quantum channel dynamics by promoting *intrinsic curvature* to a first-class control parameter. Spacetime is modeled as an oriented simplicial complex endowed with a time labeling that induces a causal partial order on events. Quantum degrees of freedom are *finite-dimensional* and attached to vertices, while local evolution along edges is given by completely positive trace-preserving (CPTP) maps generated by a *curvature-modulated Lindbladian*. Curvature on spacelike slices is estimated from vertex angle deficits of a latent triangulation—intrinsic to the induced piecewise-Euclidean metric of the chosen embedding; this estimator is O(d)-invariant under global orthogonal transformations of the latent embedding, with an explicit per-slice scale convention. We prove: (i) gauge invariance of the angle-deficit and curvature density; (ii) well-posedness and norm-continuity of curvature-modulated CPTP semigroups; (iii) causal factorization and no-signaling across spacelike-separated subcomputations via order-independence within slices; (iv) triangulation invariance under commuting-locality *with preserved per-cell generators*, and triangulation-independence in a Lie–Trotter refinement limit; (v) a discrete Gauss–Bonnet identity on closed slices; and (vi) quantum speed limits and Lindbladian perturbation bounds with explicit curvature dependence. In the flat limit, TRQC reduces to standard quantum circuits; in an entanglement-breaking limit, it reduces to classical relativistic computation. We outline algorithms for curvature evaluation on moving meshes, causal scheduling, and remeshing-robust Trotterization, and we sketch applications to relativistic quantum networking, analog simulation on curved/hyperbolic lattices, geometry-aware error correction, and transport on curved or fractal nanostructures. Beyond offering new theoretical guarantees, TRQC provides a practical semantics for designing and simulating quantum information processing in nontrivial geometries and time-dilated settings.

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1 Introduction

Quantum information science has matured into a discipline where abstract mathematical structure and practical engineering demands coevolve [1–3]. On the one hand, the standard circuit and Hamiltonian models provide the operational backbone of quantum computation; in circuit-to-Hamiltonian constructions, a dedicated *clock register* enforces a global ordering of gates, with local-clock variants also studied [4]. On the other hand, the physical carriers of quantum information—photons, trapped ions, superconducting circuits, cold atoms, and spin defects—are increasingly deployed in settings where *geometry and relativity* cannot be ignored: optical links between moving satellites [5], quantum repeaters on aircraft and high-altitude platforms [6], photonic and excitonic devices patterned on curved membranes [7], and networked processors whose clocks tick at different proper rates [8,9]. Throughout this work, the local Hilbert spaces attached to nodes are *finite-dimensional* (Assumption 1); infinite-dimensional channels are treated only under fixed finite truncations. These trends expose a conceptual gap. Today's standard models handle unitary gates, noise, and scheduling, but they do not natively encode how intrinsic curvature, holonomy, and causal structure shape the *channels* by which quantum states evolve and information flows.

This paper addresses that gap by developing *Triangulated Relativistic Quantum Computation* (TRQC), a framework that provides a curvature-modulated causal channel semantics compatible with relativistic constraints [10] and elevates *intrinsic curvature* to a first-class control parameter. The core idea is direct: represent spacetime as a causally oriented simplicial complex and compute *intrinsic* Gaussian curvature on spacelike slices via vertex angle deficits. Use that curvature to modulate both coherent and incoherent components of the local generators, Hamiltonians and Lindblad operators [11], while enforcing completely positive, trace-preserving (CPTP) [12] evolution and no-signaling across spacelike regions. This approach yields a mathematically controlled calculus with discrete analogs of foundational geometric identities (Gauss–Bonnet) [13] and with invariances that are essential for robust modeling on moving or re-meshed geometries.

The motivation is twofold, conceptual and practical. Conceptually, quantum computation and communication are constrained by relativity [14]: signals propagate within light cones; operations may be only partially ordered; clocks are local; and, in gravitational fields or accelerating frames, proper time differs across nodes. Practically, emerging architectures operate *on the move*—satellite-based quantum key distribution (QKD) [15], drone-mounted quantum sensors [16], and inter-node links subject to Doppler and gravitational shifts [17]. In condensed-matter and photonic platforms, curved or hyperbolic embeddings are no longer esoteric [18]: patterned waveguides, curved nanomembranes [19], and synthetic gauge fields [20] realize effective curvature and holonomy that alter dispersion, interference, and localization. A framework that treats curvature and causal order as fundamental, rather than as afterthoughts, is thus timely and, we argue, necessary.

Despite progress across several adjacent areas, a unified, channel-level treatment has been missing. Continuum models of quantum particles constrained to curved surfaces capture geometric potentials and curvature-induced bound states, but they do not supply a CPTP, discretized semantics that composes cleanly under partial orders or remeshing. Graph-theoretic curvatures offer valuable surrogates when smooth geometry is absent, yet they are rarely coupled to physically motivated Lindblad noise while preserving complete-positivity. Numerical solvers on curved domains can approximate transport and spectra, but they lack causal factorization guarantees and triangulation invariance. Most importantly, a few approaches treat *curvature as a control field* that modulates not only phases and couplings, but also *noise rates*—a key ingredient for realistic performance envelopes in devices where geometry impacts dephasing, loss, and cross-talk.

TRQC fills this gap with five design pillars. First, causal triangulations: spacetime is represented by an oriented simplicial complex with a time labeling that induces a partial order on events. Computations proceed by composing local channels along edges, respecting this partial order. Second, intrinsic curvature from angle deficits: curvature on a spacelike slice is computed from vertex angle deficits on a latent triangulation, using areas that are invariant under global orthogonal transformations of the latent embedding. Crucially, triangles themselves are flat in Euclidean space; curvature is encoded in the *deficit around a vertex*, ensuring that the estimator is intrinsic and gauge-invariant. Third, curvature-modulated Lindbladians: local generators depend on a scalar curvature value averaged over the relevant neighborhood. Both Hamiltonian terms and dissipators receive curvature-dependent corrections or rates, guaranteeing CPTP evolution for all curvature values through the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) structure [21]. Fourth, causal factorization and no-signaling: channels applied on spacelike-separated regions commute and tensor-factorize, making the global map independent of within-slice orderings and forbidding signaling outside the light cone. Fifth, algebraic invariance under commuting refinements: under commuting-locality conditions, and in a Lie–Trotter refinement limit, retriangulations that preserve the per-cell generators leave the global channel unchanged. This is an algebraic (non-topological) invariance that enables remeshing without altering the channel semantics.

The mathematical content supporting these pillars is developed and proved within the paper. We establish O(d)-invariance of the curvature estimator (with an explicit scale convention) under global orthogonal transforms of the latent embedding; we prove well-posedness and norm-continuity of curvature–modulated semigroups; we formalize causal factorization, monoidality, and no-signaling for spacelike-separated subcomputations; we show triangulation invariance under commuting/locality hypotheses and convergence to a triangulation-independent limit under refinement; we present a discrete Gauss–Bonnet identity validating the curvature construction on closed slices, and we derive quantum speed limits with explicit curvature dependence in the unitary case, together with a Lipschitz-type perturbation bound for general Lindbladians. Two limiting regimes anchor the framework: in the *flat* limit (zero curvature), TRQC reduces to standard quantum circuits; in an *entanglement-breaking* limit with commuting maps on disjoint factors, it reduces to a classical relativistic computation governed by a causal partial order.

The practical implications are immediate. In relativistic quantum networking, TRQC provides a scheduling semantics aligned with causal order and proper-time clocks, while geometry-aware noise models inform routing, synchronization, and key rates. In analog quantum simulation and topological photonics, curvature and holonomy enter both the coherent and dissipative parts of the dynamics, offering programmable control of phase interference and transport on curved or hyperbolic lattices. For error correction on curved codes, curvature modulates syndrome graph expansion and can be injected into decoders and noise models. For quantum transport on curved or fractal nanostructures, curvature-driven dephasing extends beyond purely Hamiltonian treatments, enabling predictions for mobility edges and anomalous diffusion that depend on geometric features.

The central research question is: *Can we construct a rigorously defined, curvature-aware, causally composable, and triangulation-invariant channel calculus for quantum information processing in relativistic and curved settings?* Our hypothesis is that vertex angle-deficit curvature on spacelike triangulations, used to modulate Lindbladian generators, yields (a) CPTP, well-posed semigroups continuous in curvature; (b) causal factorization and no-signaling across spacelike regions; (c) invariance of the global channel to within-slice orderings and to mesh refinements under commuting/locality or Trotter conditions; and (d) physically meaningful complexity bounds and speed limits that expose explicit curvature dependence.

This paper makes the following contributions:

- A curvature-modulated Lindbladian calculus on causal triangulations, with proofs of O(d)-invariance (with an explicit scale convention), well-posedness, and continuity.
- Theorems establishing causal factorization, order-independence, and no-signaling for spacelike-separated subcomputations.
- Algebraic invariance under commuting refinements when per-cell generators are preserved, and a triangulation-independent limit under Lie–Trotter refinement.
- A discrete Gauss–Bonnet identity on closed slices, validating the intrinsic curvature estimator.
- Curvature-dependent quantum speed limits (unitary case) and a contractive Duhamel-type perturbation bound for general Lindbladians.
- Recoveries of standard limits: flat-space quantum circuits and classical relativistic computation in the entanglement-breaking regime.

The remainder of the manuscript is organized to balance formal development and practical guidance. Section 2 surveys the mathematical and physical context that motivates TRQC and clarifies the precise gap it fills. Section 3 presents the full mathematical framework: causal triangulations and slice Hilbert spaces; the latent, orthogonally gauged curvature estimator; curvature–modulated generators; and the proofs of our main structural theorems, followed by implementation details for curvature evaluation, causal Trotterization, and remeshing. Section 4 interprets the theoretical results in light of realistic architectures and identifies strengths, limitations, and open questions. Section 5 synthesizes the main insights and outlines concrete paths for extending TRQC to adaptive control, graph-curvature surrogates on fractals, and hardware-aware compilers that treat geometry and proper time as optimization variables.

Assumption 1 (*Standing scope: finite dimension and bounded generators*) Throughout the paper, all local Hilbert spaces $\mathcal{H}_v$ are finite-dimensional, and all superoperators acting on $\mathcal{B}(\mathcal{H}_v)$ are bounded. Consequently, GKSL generators generate uniformly continuous semigroups on the trace class, CPTP maps are contractions in the trace norm, and the diamond norm is well defined and finite. Infinite-dimensional channels (e.g., bosonic modes) are out of scope unless an explicit finite-dimensional truncation is fixed throughout; our continuity and Lipschitz bounds rely on boundedness.

As a guiding philosophy, the framework is designed not only to be mathematically sound but also to feel natural to practitioners: curvature and causal order are built in from the outset; complete-positivity and no-signaling are guaranteed by construction; and triangulation invariance makes mesh updates and mobility safe. In our experience, such alignment between theory and practice is essential for progress whenever quantum devices leave the laboratory bench and enter the richer, curved, and time-dilated world in which they will ultimately operate.

2 Background and related work

The operational constraints of relativity force any physically faithful model of distributed quantum information processing to abandon the fiction of a single global clock. Instead, events are partially ordered by causal reachability, and local clocks advance according to proper time. In such settings, computation is naturally described by *causal networks* in which nodes execute local transformations while respecting light-cone constraints. Classical formulations have long embraced partial orders for asynchronous systems [22]; in the quantum regime, the corresponding objects are CPTP maps [23] arranged along causal links and composed according to a *triangulated cobordism* from an initial spacelike antichain to a final one [24].

The TRQC viewpoint instantiates this semantics on an oriented simplicial complex (T, t), where t induces the causal partial order on vertices and edges [25,26]. Spacelike layers are maximal antichains; within each layer, transformations commute on disjoint tensor factors, yielding order-independence [27]. This categorical (monoidal) structure of channel composition is a structural backbone of TRQC and generalizes conventional circuit models to relativistic schedules [28].

Open quantum dynamics on finite-dimensional systems is generated, in the Markovian regime, by GKSL operators [29]. The associated semigroups $e^{t\mathcal{L}}$ are CPTP for all $t \geq 0$ [30], are stable under composition [12], and admit Trotter-type factorization when split into summands [31]. These properties make GKSL dynamics the natural substrate for a channel-level calculus that must be both physically consistent (positivity, trace preservation, and no-signaling under spacelike separation) and composable across a partial order.

TRQC adopts GKSL generators at the level of *edges* in the triangulation, with the distinctive twist that the Hamiltonian and the rates are modulated by *intrinsic curvature* computed on spacelike slices [32]. The Lie–Trotter and Duhamel expansions provide quantitative control of splitting and perturbation errors [33]; curvature continuity of the maps follows from bounded-perturbation theory [34,35].

Intrinsic curvature on a triangulated surface is captured by *vertex angle deficits*: for a vertex v, the sum of interior angles around v falls short of 2π by an amount $\delta(v)$, and the discrete Gaussian curvature density is $K(v) = \delta(v)/A_v$ for a dual area $A_v > 0$ [36]. This Regge-style discretization is intrinsic to the induced piecewise-Euclidean metric and satisfies a discrete Gauss–Bonnet identity on closed meshes; in our implementation, the metric is induced by a latent embedding, but a global embedding is not required in principle $\sum_v K(v)A_v = \sum_v \delta(v) = 2\pi\,\chi(\Sigma)$ [37].

TRQC leverages these facts in two ways. First, it uses an *orthogonally gauged latent embedding* to compute $K(v)$: latent coordinates $z_v \in \mathbb{R}^d$ determine a geometric triangulation and angles, but all physically relevant quantities are invariant under $z \mapsto zR$ for $R \in O(d) = \left\{\mathbb{R}^{d\times d} \mid R^\top R = I_d\right\}$ [38]. Second, it promotes holonomy to a programmable resource: link unitaries U_{ij} implement parallel transport, and plaquette Wilson loops $W_\Delta = \prod_{e\in\partial\Delta} U_e$ encode curvature-like phases that integrate seamlessly into coherent couplings [39].

Several mature lines of work inform TRQC and motivate its synthesis:

- *Quantum dynamics on curved substrates* [40]. Continuum models of particles constrained to curved surfaces predict curvature-induced potentials and bound states, while photonic and polaritonic platforms realize synthetic curvature and gauge fields. These approaches emphasize coherent Hamiltonian effects but rarely address CPTP, causally factorized open-system dynamics at the channel level.
- *Graph and hyperbolic lattices* [41]. Hyperbolic tilings and negatively curved graphs host unusual dispersion and topological features; tensor-network models connect such tilings to holographic encodings. Existing treatments operate coherently or at the level of abstract networks, not as curvature-modulated channel calculi with remeshing invariance.
- *Finite-element and discrete exterior calculus* [42]. Finite-element and discrete exterior calculus techniques approximate Schrödinger and Poisson equations on nonflat geometries and inform conductance predictions. These are powerful numerics but do not furnish a CPTP/no-signaling semantics for distributed quantum operations or asynchronous schedules.
- *Graph-theoretic "curvature"* [43]. Ollivier–Ricci and Forman–Ricci curvatures quantify expansion and transport properties in graphs. They provide surrogates when smooth geometry is absent (e.g., fractals), yet they are not typically tied to Lindbladian rate modulation with complete-positivity guarantees.
- *Relativistic quantum networking and QKD* [44]. Space-based and high-mobility architectures face proper-time desynchronization and light-cone constraints. Protocol design in these regimes benefits from a causal, order-independent scheduler and from geometry-aware noise budgets.
- *Quantum error correction on non-Euclidean tilings* [45]. Hyperbolic and other curved codes exploit expansion properties for improved distances and decoding; integrating curvature-aware noise models into these codes is a natural next step.

What is missing from the preceding ecosystem is a *single, end-to-end framework* that:

1. Computes an *intrinsic, gauge-invariant* curvature field on spacelike slices via angle deficits;
2. Injects curvature into *both* coherent couplings and *dissipative rates* while preserving CPTP structure and no-signaling for spacelike separation;
3. Composes channels along a *causal triangulation* with rigorous order-independence within slices and monoidality across slices;
4. Remains *triangulation-invariant* under commuting-locality and converges to a triangulation-independent limit under refinement (Lie--Trotter);
5. Provides *a priori bounds* that display explicit curvature dependence (quantum speed limits; Duhamel-type perturbation inequalities).

TRQC is designed to close this gap. It packages the geometric, algebraic, and analytic ingredients into a coherent calculus that is compatible with realistic, moving, and curved architectures while retaining the formal guarantees demanded by mathematical foundations.

3 Methodology

Let T be a finite, oriented, D–dimensional simplicial complex with vertex set V and edge set E. When defining *slice-wise Gaussian curvature*, we explicitly use the set $F = \Sigma_2(\mathsf{T}_\Sigma)$ of 2-simplices on a *2D spacelike slice* T_Σ;

this does not restrict the ambient dimension D and is standard in Regge-style discretizations. We write $\mathcal{B}(\mathcal{H})$ for the algebra of all linear operators on $\mathcal{H}$; *throughout this section and the paper we work with finite-dimensional local Hilbert spaces* (cf. Assumption 1), so with $d = \dim \mathcal{H}$, one has $\mathcal{B}(\mathcal{H}) \cong \mathrm{M}_d(\mathbb{C})$. We also write $\mathcal{D}(\mathcal{H})$ for the set of density operators on $\mathcal{H}$ (positive, trace-one). A *time labeling* is a map $t : V \to \mathbb{R}$, such that each oriented edge $e = (u \to v) \in E$ satisfies $t(u) < t(v)$. The relation $u \preceq v$ iff there exists a directed path from u to v is a partial order on V and extends to simplices by inclusion.

Definition 3.1 (*TRQC spacetime and slices*) A *TRQC spacetime* is a pair (T, t) as above. An *antichain* is a subset $\Sigma \subset V$ whose vertices are pairwise incomparable. A *spacelike (Cauchy) antichain* is a *maximal* antichain. For a slice Σ, attach a finite-dimensional Hilbert space $\mathcal{H}_v \simeq \mathbb{C}^{d_v}$ to every $v \in \Sigma$ and define the slice space $\mathcal{H}_\Sigma = \bigotimes_{v\in\Sigma} \mathcal{H}_v$. For $U \subseteq \Sigma$, we write $\mathcal{H}_U = \bigotimes_{v\in U} \mathcal{H}_v$.

Definition 3.2 (*Local channels and paths*) For each $e = (u \to v) \in E$, let $\Phi_e : \mathcal{B}(\mathcal{H}_u) \to \mathcal{B}(\mathcal{H}_v)$ be CPTP. For a directed path $\gamma = e_1 \cdots e_k$ with $e_i = (v_{i-1} \to v_i)$, set $\Phi_\gamma = \Phi_{e_k} \circ \cdots \circ \Phi_{e_1}$.

A *triangulated cobordism* $W : \Sigma \rightsquigarrow \Sigma'$ is a subcomplex whose boundary is $\partial W = \Sigma \sqcup \Sigma'$ with the induced orientation; we write $E(W)$ for its internal edges. A *within-step* (or *spacelike*) set of edges is a family $\{e_\alpha = (u_\alpha \to v_\alpha)\} \subset E(W)$, such that:

(i) $u_\alpha \neq u_\beta$ and $v_\alpha \neq v_\beta$ for $\alpha \neq \beta$.
(ii) All endpoints are pairwise incomparable under the time labeling t and lie within the same step window, separated by a guard band $\delta_m > 0$, so that no directed path connects any endpoints with t-difference $< \delta_m$.
(iii) The corresponding channel factors act on *disjoint tensor legs*, i.e., they are tensored with identities outside $\mathcal{H}_{u_\alpha}$ and $\mathcal{H}_{v_\alpha}$. These conditions identify spacelike separation with the causal graph induced by t and guarantee commutation within the step.

3.1 Latent curvature estimator and orthogonal gauge

Curvature is computed intrinsically on each spacelike slice from angle deficits at vertices of an auxiliary geometric graph built on latent coordinates. This estimator is independent of any specific physical embedding of the degrees of freedom and is invariant under global orthogonal gauges.

Fix $d \in \mathbb{N}$. A *latent embedding* is a map $\varphi : \mathcal{D}(\mathcal{H}_\Sigma) \to \mathbb{R}^d$ associating with a chosen set of local features (e.g., reduced density matrices, classical labels, or external metadata) a collection of points $z_v \in \mathbb{R}^d$, $v \in \Sigma$. Write $R \in O(d) = \{R \in \mathbb{R}^{d\times d} : R^\top R = I_d\}$ for an orthogonal transform acting by $z_v \mapsto z_v R$; this is the *orthogonal gauge*.

Let $\mathcal{G}_\Sigma$ be a geometric (piecewise-Euclidean) triangulation of $\{z_v : v \in \Sigma\}$ using the ambient Euclidean metric in $\mathbb{R}^d$ (e.g., Delaunay). Each triangle is flat, and its interior angles are computed in its own plane via the law of cosines; the vertex angle deficit sums these per-face angles and is purely intrinsic to the piecewise-Euclidean metric. For a triangle $\Delta = \{i, j, k\}$ with vertices $z_i, z_j, z_k \in \mathbb{R}^d$, define interior angles $\theta_{v,\Delta}$ by the Euclidean law of cosines and let A_Δ be its Euclidean area. Let $A_v > 0$ be an orthogonally invariant dual area associated with v (e.g., mixed Voronoi/barycentric).

Because all distances and angles used by $\mathcal{G}_\Sigma$ are Euclidean in the ambient space, global orthogonal transforms $z \mapsto zR$ preserve edge lengths and per-face angles; consequently, $\delta(v)$ and $K(v)$ are $O(d)$-invariant.

Definition 3.3 (*Angle deficit and discrete Gaussian curvature*) For $v \in \Sigma$, set

$$\delta(v) = 2\pi - \sum_{\Delta\in\mathrm{star}(v)} \theta_{v,\Delta}, \qquad K(v) = \frac{\delta(v)}{A_v}. \tag{1}$$

Proposition 3.1 (O(d)-invariance of the estimator) *For any $R \in O(d)$, both $\delta(v)$ and $K(v)$ are invariant under $z \mapsto zR$. We state this explicitly as a proposition, since it is invoked repeatedly (e.g., in remeshing checks and Gauss–Bonnet diagnostics).*

Remark 1 (*Scale convention*) While $\delta(v)$ is scale-invariant, A_v scales like length^2, so $K(v) = \delta(v)/A_v$ depends on the overall scale of z. We adopt the per-slice normalization $\langle \ell \rangle_\Sigma := |E_\Sigma|^{-1} \sum_{e \in E_\Sigma} \ell_e = 1$ (rescale z to unit mean edge length), which fixes the units of K. Alternatives are to work with the dimensionless $K^\sharp(v) := \delta(v)/(A_v/\langle A \rangle_\Sigma)$, or to treat the scale as a physical choice tied to the platform and absorb it into $\gamma_{e,j}(\kappa)$.

Proof Angles are functions of inner products of edge vectors, which are preserved by $O(d)$; $A_\triangle$ and any orthogonally defined A_v are likewise invariant. Hence, $\delta(v)$ and $K(v)$ are unchanged. □

For Theorem 3.2, we assume Σ is a closed 2D simplicial complex (triangulated surface), so that each edge belongs to exactly two triangles.

Theorem 3.2 (Discrete Gauss–Bonnet on closed slices) *If Σ is a closed, oriented 2D simplicial complex (each edge belongs to exactly two faces) and the geometric triangulation $\mathcal{G}_\Sigma$ realizes the same combinatorial triangulation by Euclidean triangles, then*

$$\sum_{v \in \Sigma} \delta(v) = 2\pi\, \chi(\Sigma) \qquad \textit{and} \qquad \sum_{v \in \Sigma} K(v) A_v = 2\pi\, \chi(\Sigma). \tag{2}$$

Proof Each Δ has angle sum π, and hence, $\sum_v \sum_{\Delta \ni v} \theta_{v,\Delta} = \pi |F_\Sigma|$. Therefore

$$\sum_v \delta(v) = 2\pi |V_\Sigma| - \pi |F_\Sigma| = 2\pi(|V_\Sigma| - |E_\Sigma| + |F_\Sigma|) = 2\pi\, \chi(\Sigma), \tag{3}$$

using $3|F_\Sigma| = 2|E_\Sigma|$ for triangulations without boundary and Euler's identity. The second equality follows by definition of $K(v)$. □

Lemma 3.3 (Discrete Gauss–Bonnet with boundary) *If Σ has boundary, then*

$$\sum_{v \in \Sigma} \delta(v) \; + \sum_{e \subset \partial\Sigma} \theta_{\text{ext}}(e) \; = \; 2\pi\, \chi(\Sigma),$$

where $\theta_{\text{ext}}(e)$ are the exterior turning angles along the boundary. Equivalently, $\sum_v K(v) A_v + \sum_{e \subset \partial\Sigma} \theta_{\text{ext}}(e) = 2\pi\, \chi(\Sigma)$ for any orthogonally invariant choice of dual areas.

Remark 2 On slices with boundary, exterior turning angles appear; see the discrete boundary version stated in the lemma following Theorem 3.2.

Remark 3 (*Curvature as an external control field*) The latent embedding $\varphi : \mathcal{D}(\mathcal{H}_\Sigma) \to \mathbb{R}^d$ is introduced to describe how one *may* construct curvature fields from features of interest. All structural theorems in this paper assume that the curvature values $\{K(v)\}$ and the derived $\{\kappa_e\}$ are provided as exogenous (state-independent) controls on each step. If one lets φ depend on the evolving quantum state and feeds back $\kappa_e(\rho)$ into the generator, the evolution becomes nonlinear in ρ; complete-positivity and contractivity then require a separate analysis not undertaken here.

3.2 Curvature-modulated generators and well-posedness

For $e = (u \to v) \in E(W)$ whose application is localized near a slice Σ, define a local curvature average

$$\kappa_e = \frac{1}{|\mathcal{N}(u)|} \sum_{w \in \mathcal{N}(u)} K(w), \tag{4}$$

where $\mathcal{N}(u) \subset \Sigma$ is a finite neighborhood. *Default:* the closed star of u in $\mathcal{G}_\Sigma$, with uniform averaging. Weighted variants (e.g., inverse-area or distance weights with outlier trimming) are also admissible and remain $O(d)$-invariant. In all cases, Proposition 3.5 applies with the same structure; the Lipschitz constant scales with the chosen weights through the bound on $\sum_j \gamma'_{e,j}$ evaluated at the corresponding κ_e.

Definition 3.4 (*Curvature-modulated GKSL generator*) Fix bounded *self-adjoint* operators $H_{e,0}$, $H_{e,1}$ on $\mathcal{H}_u$ and noise operators $\{L_{e,j}\}_j$. Let $\gamma_{e,j} : \mathbb{R} \to [0, \infty)$ be *locally bounded and continuous* (and differentiable on compact sets when invoking Proposition 3.5) rate functions. For $\kappa \in \mathbb{R}$, define

$$\mathcal{L}_e^{(\kappa)}(\rho) = -i\,[H_{e,0} + \kappa H_{e,1}, \rho] + \sum_j \gamma_{e,j}(\kappa)\Big(L_{e,j}\rho L_{e,j}^\dagger - \tfrac{1}{2}\{L_{e,j}^\dagger L_{e,j}, \rho\}\Big). \tag{5}$$

Remark 4 (*Concrete norm bounds*) In finite dimension and for the $1 \to 1$ norm, one has $\|\mathrm{ad}_H\|_{1\to1} \le 2\|H\|$ and $\|\mathcal{D}_L\|_{1\to1} \le 2\|L\|^2$. Substituting these in Eq. (6) yields an explicit curvature Lipschitz constant depending only on $\|H_{e,1}\|$, $\|L_{e,j}\|$, and bounds on $\gamma'_{e,j}$ over I.

Within the step $\Delta\tau_e$, we take $H_{e,0}$, $H_{e,1}$ and the rates $\gamma_{e,j}(\kappa)$ as constant (time-independent). If time dependence within a step is required, replace the exponential by a time-ordered exponential; all statements continue to hold with the usual Grönwall-type bounds.

For a proper-time step $\Delta\tau_e > 0$, we implement transport via a Stinespring dilation

$$\Phi_e^{(\kappa)}(\rho_u) = \mathrm{Tr}_{E'_e}\Big[V_e\,(e^{\Delta\tau_e \mathcal{L}_{e,\mathrm{in}}^{(\kappa)}}(\rho_u \otimes |0\rangle\langle 0|_{E_e}))\,V_e^\dagger\Big],$$

where the subscript "in" indicates that $\mathcal{L}_{e,\mathrm{in}}^{(\kappa)}$ acts on $\mathcal{B}(\mathcal{H}_u \otimes \mathcal{H}_{E_e})$ (the *input* leg and its ancilla). The isometry $V_e : \mathcal{H}_u \otimes \mathcal{H}_{E_e} \to \mathcal{H}_v \otimes \mathcal{H}_{E'_e}$ types the map from the u-leg to the v-leg. (If $\dim \mathcal{H}_v \ge \dim \mathcal{H}_u$, one may equivalently choose a partial isometry $U_e : \mathcal{H}_u \to \mathcal{H}_v$ and take $E_e = E'_e = \mathbb{C}$.)

Remark 5 (*Norm conventions and contractivity*) All superoperator norms $\|\cdot\|_{1\to1}$ are taken on the trace class $(\mathcal{B}_1, \|\cdot\|_1)$; CPTP maps are contractions in $\|\cdot\|_1$ and hence have $\|\Phi\|_{1\to1} \le 1$. The diamond norm $\|\cdot\|_\diamond$ is used in finite dimension and obeys $\|\Phi\|_\diamond = 1$ for CPTP Φ.

For notational convenience, we set $\mathrm{ad}_H(\rho) := [H, \rho]$ for bounded H, and

$$\mathcal{D}_L(\rho) := L\rho L^\dagger - \tfrac{1}{2}\{L^\dagger L, \rho\}$$

for any noise operator L (the standard Lindblad dissipator).

Proposition 3.4 (CPTP well-posedness and curvature continuity) *For every $\kappa \in \mathbb{R}$, the typed channel $\Phi_e^{(\kappa)}$ in Definition 3.4 is CPTP. If $\gamma_{e,j}$ are locally bounded and continuous in κ, then $\kappa \mapsto \Phi_e^{(\kappa)}$ is norm-continuous on bounded intervals (in fact, locally Lipschitz under the hypotheses of Proposition 3.5).*

Proof The GKSL form with nonnegative rates generates a uniformly continuous quantum dynamical semigroup; thus, $t \mapsto e^{t\mathcal{L}_e^{(\kappa)}}$ is CPTP for $t \ge 0$. Continuity in κ follows from boundedness of $H_{e,1}$, continuity of $\gamma_{e,j}$, and the Trotter–Kato perturbation theory for bounded generators. □

Proposition 3.5 (Curvature Lipschitz control) *Let $\|\cdot\|_{1\to1}$ denote the induced trace-norm operator norm. For bounded $H_{e,1}$ and rates with bounded derivatives on a compact interval $I \subset \mathbb{R}$*

$$\sup_{\kappa,\kappa' \in I} \frac{\big\|\Phi_e^{(\kappa)} - \Phi_e^{(\kappa')}\big\|_{1\to1}}{|\kappa - \kappa'|} \;\le\; \Delta\tau_e \left(\|\mathrm{ad}_{H_{e,1}}\|_{1\to1} + \sum_j \sup_{\xi \in I} \gamma'_{e,j}(\xi)\,\|\mathcal{D}_{L_{e,j}}\|_{1\to1}\right) e^{\Delta\tau_e C_I}, \tag{6}$$

for a constant C_I depending only on bounds of $\mathcal{L}_e^{(\kappa)}$ over $\kappa \in I$, where $\mathrm{ad}_H(\rho) = [H, \rho]$ and $\mathcal{D}_L(\rho) = L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}$.

The bound in Eq. (6) is state-independent and hence uniform over inputs $\rho \in \mathcal{B}(\mathcal{H}_u)$ with $\|\rho\|_1 \leq 1$, which facilitates a priori step-size selection in curvature-swept simulations.

Proof Differentiate $e^{\Delta\tau_e \mathcal{L}_e^{(\kappa)}}$ in κ via the Duhamel formula and bound by submultiplicativity in $\|\cdot\|_{1\to 1}$; integrate from κ' to κ. □

3.3 Global channels, causality, and no-signaling

Fix a total order $\prec$ on $E(W)$ extending the partial order induced by t. The *global TRQC channel* is the ordered product

$$\Phi_W = \overrightarrow{\prod_{e\in E(W)}} \Phi_e^{(\kappa_e)}. \tag{7}$$

With a fixed global typing (Definition 3.5), each edge map is extended to the full slice algebra by identity on untouched legs

$$\iota_e : \mathcal{B}(\mathcal{H}_\Sigma) \to \mathcal{B}(\mathcal{H}_{\Sigma\setminus\{u\}} \otimes \mathcal{H}_v), \qquad \iota_e = \mathrm{Id}_{\Sigma\setminus\{u\}} \otimes \Phi_e^{(\kappa_e)}. \tag{8}$$

Because each edge map $\Phi_e^{(\kappa_e)}$ is typed from $\mathcal{B}(\mathcal{H}_u)$ to $\mathcal{B}(\mathcal{H}_v)$ via the fixed transport isometry (or Stinespring dilation) in Definition 3.4, the ordered product is type-consistent along the triangulated cobordism.

Definition 3.5 (*Global typing and dimension compatibility*) For each spacelike slice Σ, fix an ordered list of tensor legs $\mathcal{H}_\Sigma = \bigotimes_{v\in\Sigma} \mathcal{H}_v$. An edge $e = (u \to v)$ is typed as a CPTP map $\Phi_e : \mathcal{B}(\mathcal{H}_u) \to \mathcal{B}(\mathcal{H}_v)$, realized via a finite-dimensional Stinespring/Kraus dilation. Composition along a directed path requires that codomains/domains match on successive legs, and that extensions by identities always act on the *fixed* complement $\mathcal{H}_{\Sigma\setminus\{u\}}$ of the current slice typing. We audit trace preservation and complete-positivity via Choi positivity at each step (guaranteed here by the GKSL form).

Lemma 3.6 (Tensorial commutation on disjoint factors) *If* $\Lambda : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_{A'})$ *and* $\Gamma : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_{B'})$ *are CPTP, then*

$$(\Lambda \otimes \mathrm{Id}_{B'}) \circ (\mathrm{Id}_A \otimes \Gamma) = (\mathrm{Id}_{A'} \otimes \Gamma) \circ (\Lambda \otimes \mathrm{Id}_B) \tag{9}$$

on $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Proof Both sides act as $X \mapsto (\Lambda \otimes \Gamma)(X)$ by associativity of $\otimes$ and functoriality at the channel level. □

Theorem 3.7 (Order-independence, monoidality, and causal independence) *(i)* Φ_W *is CPTP and independent of the chosen topological ordering* $\prec$. *(ii) For cobordisms* $W_1 : \Sigma_0 \rightsquigarrow \Sigma_1$ *and* $W_2 : \Sigma_1 \rightsquigarrow \Sigma_2$, $\Phi_{W_2\circ W_1} = \Phi_{W_2} \circ \Phi_{W_1}$. *(iii) If* $W = W_A \sqcup W_B$ *with spacelike separation and* tensor-disjoint supports *at each step (no edge in* W_A *shares a vertex with an edge in* W_B *within the same spacelike layer), then* $\Phi_W = \Phi_{W_A} \otimes \Phi_{W_B}$.

Proof (i) Adjacent swaps in two linear extensions of the partial order exchange incomparable edges whose actions are on disjoint factors in that step and thus commute by Lemma 3.6. (ii) Concatenation of cobordisms corresponds to composition of channels. (iii) Spacelike separation implies actions restricted to tensor factors; iterated use of Lemma 3.6 yields the product form. □

Theorem 3.8 (No signaling across spacelike separation) *Let $W = W_A \sqcup W_B$ be spacelike separated. For any initial (possibly entangled) ρ_{AB} and any CPTP map Λ_A on A replacing the A-part of Φ_{W_A}*

$$\mathrm{Tr}_A\big[(\Lambda_A \otimes \Phi_{W_B})(\rho_{AB})\big] = \mathrm{Tr}_A\big[(\mathrm{Id}_A \otimes \Phi_{W_B})(\rho_{AB})\big]. \tag{10}$$

Proof For any observable B on $\mathcal{H}_B$

$$\mathrm{Tr}\big[B\ \mathrm{Tr}_A\left((\Lambda_A \otimes \Phi_{W_B})(\rho_{AB})\right)\big] = \mathrm{Tr}\Big[(\Lambda_A^{\dagger}(\mathrm{Id}_A) \otimes B)(\mathrm{Id}_A \otimes \Phi_{W_B})(\rho_{AB})\Big] \tag{11}$$

and $\Lambda_A^{\dagger}(\mathrm{Id}_A) = \mathrm{Id}_A$ since Λ_A is trace-preserving; because this equality of expectations holds for all $B \in \mathcal{B}(\mathcal{H}_B)$, the reduced states on B coincide. Equivalently, $\mathrm{Tr}_A[(\Lambda_A \otimes \mathrm{Id}_B) \circ (\mathrm{Id}_A \otimes \Phi_{W_B})(\rho_{AB})] = \mathrm{Tr}_A[(\mathrm{Id}_A \otimes \Phi_{W_B})(\rho_{AB})]$. □

3.4 Triangulation invariance and refinement limits

We formalize conditions under which retriangulations (e.g., Pachner moves) preserve the global channel.

Assumption 2 (*Locality and commutativity within spacelike steps*) There exists a partition of each step into pairwise spacelike cells $\{R_k\}$ with generators $\mathcal{L}_{R_k}$ acting nontrivially only on $\mathcal{H}_{R_k}$, such that $[\mathcal{L}_{R_k}, \mathcal{L}_{R_\ell}] = 0$ whenever R_k and R_ℓ are spacelike.

Theorem 3.9 (Algebraic invariance under commuting-refinement) *Let W and W' be triangulations of the same cobordism that differ by a finite sequence of local refinements/coarsenings preserving Assumption 2 and the multiset $\{\mathcal{L}_{R_k}\}$. Then, $\Phi_W = \Phi_{W'}$.*

Proof Within a step, $\exp\big(\Delta\tau \sum_k \mathcal{L}_{R_k}\big) = \prod_k \exp(\Delta\tau\, \mathcal{L}_{R_k})$ independently of order by commutativity. A retriangulation that repartitions spacelike cells but preserves the collection of $\mathcal{L}_{R_k}$ yields the same product per step and hence the same concatenated channel. □

Proposition 3.10 (Mesh-change bias under approximate generator preservation) *Suppose a retriangulation changes the within-step generators by $\Delta\mathcal{L}_{R_k}$, so that for some weights $w_k \geq 0$ with $\sum_k w_k \leq 1$*

$$\Big\|\sum_k \Delta\tau\ \Delta\mathcal{L}_{R_k}\Big\|_{1\to 1} \leq \varepsilon \quad \textit{and} \quad \big\|[\mathcal{L}_{R_k}, \mathcal{L}_{R_\ell}]\big\|_{1\to 1} \leq \eta$$

for spacelike R_k, R_ℓ in that step. Then, the within-step channel difference obeys

$$\Big\|\prod_k e^{\Delta\tau(\mathcal{L}_{R_k}+\Delta\mathcal{L}_{R_k})} - \prod_k e^{\Delta\tau\mathcal{L}_{R_k}}\Big\|_{1\to 1} \leq \varepsilon\, e^{\Delta\tau C} + O(\eta\, \Delta\tau^2),$$

where C bounds $\sum_k \|\mathcal{L}_{R_k}\|_{1\to 1}$ on the step. Over a fixed horizon, the global bias accumulates at most linearly in the number of affected steps.

Assumption 3 (*Generator density and refinement*) There exists a bounded generator density $\mathfrak{l}$, such that for a refinement parameter $h \downarrow 0$

$$\begin{aligned} &\sum_{e\in \mathrm{step}(W_h)} \Delta\tau_e\, \mathcal{L}_e^{(\kappa_e)} \xrightarrow[h\downarrow 0]{\|\cdot\|_{1\to 1}} \int_{\tau_m}^{\tau_{m+1}} \mathfrak{l}(\tau)\, \mathrm{d}\tau, \\ &\max_{e\neq e'\in \mathrm{step}(W_h)} \big\|\,[\mathcal{L}_e^{(\kappa_e)}, \mathcal{L}_{e'}^{(\kappa_{e'})}]\,\big\|_{1\to 1} = o(1), \end{aligned} \tag{12}$$

where $\|\cdot\|_{1\to 1}$ is the operator norm induced by the trace norm on the trace class, and $[\tau_m, \tau_{m+1}]$ denotes the proper-time window of the step.

Assumption 4 (*Curvature refinement regularity*) Let $\{\mathcal{G}_{\Sigma,h}\}$ be a shape-regular sequence of triangulations with per-slice scale normalization (unit mean edge length). Let K_h denote the piecewise-constant discrete Gaussian curvature on $\mathcal{G}_{\Sigma,h}$. Assume: (a) $K_h \to K_*$ in L^1_{loc} for some bounded field K_* on the slice; (b) for each edge e in step m, the local averages $\kappa_e(h)$ defined by (4) (with uniformly bounded averaging weights) satisfy $\sup_{e\in\mathrm{step}(m)} |\kappa_e(h) - \kappa_*(\tau_m)| \to 0$ as $h \to 0$ for some bounded profile κ_* on the step.

Corollary 3.11 (Discrete sufficient conditions for the refinement limit) *Assumption 3 holds if, for a refinement parameter $h \downarrow 0$, the following conditions are satisfied uniformly per step: (i) $\sum_e \Delta\tau_e \|\mathcal{L}_e^{(\kappa_e)}\|_{1\to 1} \le C\,\Delta\tau_{\max}$ with $\Delta\tau_{\max} = O(h)$; (ii) $\max_{e\neq e'} \|[\mathcal{L}_e^{(\kappa_e)}, \mathcal{L}_{e'}^{(\kappa_{e'})}]\|_{1\to 1} = o(1)$; (iii) the Riemann sums $\sum_e \Delta\tau_e\, \mathcal{L}_e^{(\kappa_e)}$ converge to a bounded generator density in $\|\cdot\|_{1\to 1}$. Then, the conclusion of Theorem 3.13 follows.*

Corollary 3.12 (Sufficient link from curvature refinement to Assumption 3) *If Assumption 4 holds and the rate functions $\gamma_{e,j}(\kappa)$ are locally Lipschitz and uniformly bounded on the range of $\kappa_e(h)$, then the family of generators $\{\mathcal{L}_e^{(\kappa_e(h))}\}$ satisfies Assumption 3. In particular, $\sum_e \Delta\tau_e\, \mathcal{L}_e^{(\kappa_e(h))}$ converges in $\|\cdot\|_{1\to 1}$ to a bounded generator density and $\max_{e\neq e'} \|[\mathcal{L}_e^{(\kappa_e(h))}, \mathcal{L}_{e'}^{(\kappa_{e'}(h))}]\|_{1\to 1} = o(1)$ per step.*

Remark 6 (*Standing notation*) We use $\mathcal{L}_e^{(\kappa_e)}$ for local (edge-level) GKSL generators and write $A_e := \Delta\tau_e\, \mathcal{L}_e^{(\kappa_e)}$ when forming within-step products. The symbol $\mathfrak{l}(\tau)$ denotes a bounded generator density on the trace class; $L_m = \int_{\tau_m}^{\tau_{m+1}} \mathfrak{l}(\tau)\,\mathrm{d}\tau$ is the coarse-grained generator for step m.

Theorem 3.13 (Refinement invariance (Lie–Trotter limit)) *Under Assumption 3 (and, when curvature-modulated, Assumption 4), the global channels Φ_{W_h} converge in norm to a common limit channel as $h \downarrow 0$, independent of the triangulation sequence.*

Remark 7 (*Convergence rate under weak noncommutativity*) If per step $\max_{e\neq e'} \|[\mathcal{L}_e^{(\kappa_e)}, \mathcal{L}_{e'}^{(\kappa_{e'})}]\|_{1\to 1} = O(h)$ and $\max_e \Delta\tau_e = O(h)$ as $h \downarrow 0$, then the per-step Trotter error is $O(h^2)$ and the global error over a fixed horizon is $O(h)$.

Proof We work in the Banach space $\mathcal{B}_1$ of trace-class operators with norm $\|\cdot\|_1$ and induced operator norm $\|\cdot\|_{1\to 1}$. CPTP maps are contractions in this norm. Let the mth spacelike step of W_h contain the edge set $I_m(h)$ and write

$$A_e := \Delta\tau_e\, \mathcal{L}_e^{(\kappa_e)}, \quad S_m(h) := \sum_{e\in I_m(h)} A_e, \quad P_m(h) := \overrightarrow{\prod_{e\in I_m(h)}} e^{A_e}. \tag{13}$$

By Assumption 3, $S_m(h) \xrightarrow{\|\cdot\|_{1\to 1}} L_m := \int_{\tau_m}^{\tau_{m+1}} \mathfrak{l}(\tau)\,\mathrm{d}\tau$ and $\max_{e\neq e'} \|[A_e, A_{e'}]\|_{1\to 1} = o\big((\max_e \Delta\tau_e)^2\big)$.

(i) *One-step product vs. sum.* Using a standard two-factor BCH/Duhamel bound and induction

$$\begin{aligned}\left\| \prod_{e\in I_m(h)} e^{A_e} - e^{\sum_{e\in I_m(h)} A_e} \right\|_{1\to 1} &\le \tfrac{1}{2} \sum_{e\neq e'} \|[A_e, A_{e'}]\|_{1\to 1}\, e^{\sum_e \|A_e\|_{1\to 1}} \\ &\quad + C\left(\sum_e \|A_e\|_{1\to 1}\right)^3 e^{\sum_e \|A_e\|_{1\to 1}}.\end{aligned} \tag{14}$$

Since $\sum_e \|A_e\|_{1\to 1} \le C_*\Delta\tau_m$ and $\Delta\tau_m \to 0$ uniformly, the right-hand side is $o(1)$ uniformly in m.

(ii) *Replacing $S_m(h)$ by L_m.* For bounded X, Y, $\|e^X - e^Y\|_{1\to 1} \le e^{\max\{\|X\|_{1\to 1}, \|Y\|_{1\to 1}\}} \|X - Y\|_{1\to 1}$. Therefore, $\|e^{S_m(h)} - e^{L_m}\|_{1\to 1} \le e^{C_*\Delta\tau_m} \|S_m(h) - L_m\|_{1\to 1} \to 0$ uniformly in m.

(iii) *Concatenation stability.* Let $\Phi_{W_h} = P_{M(h)}(h) \circ \cdots \circ P_1(h)$ and $\widetilde{\Phi}_h = e^{L_{M(h)}} \circ \cdots \circ e^{L_1}$. Telescoping and contractivity give

$$\|\Phi_{W_h} - \widetilde{\Phi}_h\|_{1\to 1} \leq \sum_{m=1}^{M(h)} \|P_m(h) - e^{L_m}\|_{1\to 1} \xrightarrow[h\to 0]{} 0. \tag{15}$$

Finally, $\widetilde{\Phi}_h$ converges (as the mesh size $|\pi_h| = \max_m \Delta\tau_m \to 0$) to the time-ordered exponential $U(T, 0)$ solving $\dot{U}(\tau, 0) = \mathfrak{l}(\tau) \circ U(\tau, 0)$ with $U(0, 0) = \mathrm{Id}$; the limit depends only on $\mathfrak{l}$, not on the triangulation sequence. Hence, $\Phi_{W_h} \to U(T, 0)$ in $\|\cdot\|_{1\to 1}$. □

3.5 Complexity bounds and curvature dependence

Set $\hbar = 1$. Consider unitary evolution generated by $H(t, \kappa(t)) = H_0(t) + \kappa(t) H_1(t)$ on a fixed Hilbert space and a pure state $|\psi(t)\rangle$. Let the Fubini–Study angle be $\Theta(t) = \arccos \left|\langle\psi(0)\rangle\psi(t)\right|$.

Theorem 3.14 (Mandelstam–Tamm with curvature modulation) *For all $T > 0$*

$$\Theta(T) \leq \int_0^T \Delta E(t, \kappa(t))\, \mathrm{d}t, \tag{16}$$

$$\Delta E(t, \kappa) = \sqrt{\langle\psi(t)|H^2|\psi(t)\rangle - \langle\psi(t)|H|\psi(t)\rangle^2} \leq \Delta E(t, 0) + |\kappa(t)|\, \|H_1(t)\|. \tag{17}$$

For mixed states $\rho(t)$, one may use $\Delta E(t, \kappa) = \sqrt{\mathrm{Tr}[\rho(t) H^2] - \mathrm{Tr}[\rho(t) H]^2} \leq \|H - \mathrm{Tr}[\rho(t)]H\ \mathrm{Id}\,\| \leq \|H\|$. Consequently, the time $T_\perp$ to reach an orthogonal state satisfies

$$T_\perp \geq \frac{\pi/2}{\frac{1}{T_\perp}\int_0^{T_\perp} \left(\Delta E(t, 0) + |\kappa(t)|\, \|H_1(t)\|\right) \mathrm{d}t}. \tag{18}$$

Proof Differentiate Θ and apply Cauchy–Schwarz to obtain $\dot{\Theta} \leq \Delta E(t, \kappa(t))$; integrate. For pure states, the variance admits the vector-norm representation $\Delta E(A) = \|(A - \langle A\rangle)|\psi\rangle\|$, whence $\Delta E(A+B) \leq \Delta E(A) + \Delta E(B)$ and $\Delta E(A) \leq \|A\|$. (For mixed states, one may use $\Delta E(A) \leq 2\|A\|$, which preserves the stated bound up to a factor.) □

For general Lindbladians, let $\Phi_t^{(\kappa)} = e^{t\mathcal{L}^{(\kappa)}}$ with $\mathcal{L}^{(\kappa)} = \mathcal{L}^{(0)} + \kappa M$ bounded on trace class and $\Phi_t^{(\kappa)}$ CPTP.

Proposition 3.15 (Contractive Duhamel bound) *In the trace-norm induced operator norm*

$$\left\|\Phi_t^{(\kappa)} - \Phi_t^{(0)}\right\|_{1\to 1} \leq |\kappa|\, t\, \|M\|_{1\to 1}\, e^{t\|\mathcal{L}^{(0)}\|_{1\to 1}}. \tag{19}$$

Remark 8 (*Diamond-norm variant*) Since $\|\Psi\|_\diamond = 1$ for any CPTP map Ψ, the Duhamel representation and submultiplicativity yield the sharper bound

$$\|\Phi_t^{(\kappa)} - \Phi_t^{(0)}\|_\diamond \leq |\kappa|\, t\, \|M\|_\diamond. \tag{20}$$

This bound is dimension-free and directly operational.

Proof Apply the Duhamel expansion $\Phi_t^{(\kappa)} - \Phi_t^{(0)} = \int_0^t \Phi_{t-s}^{(\kappa)} \kappa M \, \Phi_s^{(0)} \, \mathrm{d}s$ and bound by submultiplicativity and uniform bounds on $\|\Phi_t^{(\cdot)}\|_{1\to 1}$ over a compact κ-range. □

3.6 Implementation details: curvature evaluation, causal Trotterization, and remeshing

This subsection translates the mathematical structure into concrete procedures; it is written to be implementation-ready and to preserve the invariants proved above.

3.6.1 Curvature evaluation on a slice

Given a slice Σ and latent points $\{z_v\}_{v\in\Sigma} \subset \mathbb{R}^d$:

1. Build a quality triangulation $\mathcal{G}_\Sigma$ on $\{z_v\}$ (e.g., 2D Delaunay after projecting onto the top two principal components of the covariance of $\{z_v\}$). For $d > 2$, work in local tangent planes obtained by PCA in neighborhoods and glue triangulations consistently; intrinsic angles are computed in these planes.
2. For every $\Delta = \{i, j, k\} \in F_\Sigma$, compute interior angles $\theta_{i,\Delta}, \theta_{j,\Delta}, \theta_{k,\Delta}$ via the Euclidean law of cosines and compute A_Δ.
3. For every vertex v, assemble $\delta(v) = 2\pi - \sum_{\Delta\in\mathrm{star}(v)} \theta_{v,\Delta}$ and choose an orthogonally invariant dual area A_v (barycentric or mixed Voronoi). Set $K(v) = \delta(v)/A_v$.
4. For each edge $e = (u \to v)$ that acts at this slice, compute κ_e via Eq. (4). Because $K(\cdot)$ and A_v are gauge-invariant, κ_e is invariant under $z \mapsto zR$ for any $R \in O(d)$.

Complexity. Building a Delaunay triangulation is $O(|\Sigma| \log |\Sigma|)$; curvature assembly is $O(|F_\Sigma|)$ with small constants.

3.6.2 Causal Trotterization with proper-time steps

Let $\{\mathrm{step}_m\}_{m=1}^M$ be a layering of $E(W)$ into spacelike sets consistent with t. For each step:

1. For every e in the step, compute the *typed* edge channel $\Phi_e^{(\kappa_e)}$ as in Definition 3.4, i.e., apply $e^{\Delta\tau_e \mathcal{L}_{e,\mathrm{in}}^{(\kappa_e)}}$ on $\mathcal{B}(\mathcal{H}_u)$ and transport to $\mathcal{B}(\mathcal{H}_v)$ via the fixed isometry (or Stinespring isometry). The proper-time increment $\Delta\tau_e$ is either prescribed or computed from an external kinematic model; only positivity is required for CPTP.
2. Form the within-step product $\Phi_{\mathrm{step}_m} = \prod_{e\in\mathrm{step}_m} \Phi_e^{(\kappa_e)}$. Under Assumption 2, the factors commute and order is immaterial; otherwise, pick any fixed ordering (e.g., lexicographic) knowing that first-order Trotter error is controlled by commutators.
3. Concatenate steps: $\Phi_W = \Phi_{\mathrm{step}_M} \circ \cdots \circ \Phi_{\mathrm{step}_1}$.

Error control. If step_m contains noncommuting generators, first-order Trotterization incurs an error bounded by

$$\Big\| \prod_{e\in\mathrm{step}_m} e^{\Delta\tau_e \mathcal{L}_e^{(\kappa_e)}} - e^{\sum_{e\in\mathrm{step}_m} \Delta\tau_e \mathcal{L}_e^{(\kappa_e)}} \Big\|_{1\to 1} \le \frac{1}{2} \sum_{e\neq e'} \Delta\tau_e \Delta\tau_{e'} \, \big\| [\mathcal{L}_e^{(\kappa_e)}, \mathcal{L}_{e'}^{(\kappa_{e'})}] \big\|_{1\to 1} + O\Big(\big(\max_{e\in\mathrm{step}_m} \Delta\tau_e\big)^3\Big). \tag{21}$$

Refinement reduces $\max_e \Delta\tau_e$ and drives the error to zero under Assumption 3.

3.6.3 Remeshing and invariance tests

During evolution, node motion or state-dependent features may necessitate remeshing of $\mathcal{G}_\Sigma$:

1. Trigger remeshing when mesh quality metrics (e.g., minimum angle, aspect ratio) fall below thresholds.
2. Recompute $K(v)$ on the new mesh; provided that Σ is closed and the geometric triangulation realizes the same combinatorial triangulation (Theorem 3.2), the slice integral $\sum_v K(v)A_v$ is invariant. When $\partial\Sigma \neq \varnothing$, include the exterior turning-angle term from the boundary version following Theorem 3.2. By Proposition 3.1, this diagnostic is O(d)-invariant.
3. If Assumption 2 holds, Theorem 3.9 guarantees that replacing the set of spacelike cells by an equivalent partition (Pachner-like move) leaves Φ_W unchanged.
4. Under refinement satisfying Assumption 3, Theorem 3.13 ensures convergence to a triangulation-independent limit channel.

3.6.4 Numerical stability and reproducibility

To preserve complete-positivity numerically, use methods that *map GKSL generators to CPTP maps* in floating point: scaling-and-squaring with Padé for $e^{\Delta\tau\mathcal{L}}$ (applied to the Liouville superoperator), or exact/parametric Kraus factorizations when available. Avoid naive Taylor truncations and splitting schemes that can leave the CPTP cone; if splitting is used, ensure each substep is itself CPTP. Fix random seeds and record (T, t), layerings, and mesh/quality parameters to ensure reproducibility. Monitor Choi positivity and trace preservation (e.g., eigenvalues of the Choi matrix and $\mathrm{Tr}[\Phi(\rho)] = \mathrm{Tr}[\rho]$ for random ρ). Additionally, track $\|\Phi(\rho_1 - \rho_2)\|_1$ for random states $\rho_{1,2}$ to detect violations of contractivity on Hermitian inputs.

3.6.5 Limits and consistency checks

In the flat limit $K(v) \equiv 0$ and with $\gamma_{e,j}$, $H_{e,0} + \kappa H_{e,1}$ evaluated at $\kappa = 0$, Definition 3.4 reduces to a standard circuit gate $\exp(\Delta\tau_e \mathcal{L}_e^{(0)})$. In an entanglement-breaking and commuting within-step regime, the slice evolution reduces to a classical Markov update respecting the causal partial order. These limits provide straightforward unit tests for any implementation.

Remark 9 (*Optional holonomy fields*) When parallel transport of internal degrees of freedom is relevant, one may augment the coherent part by link unitaries U_{ij} and define plaquette holonomies $W_\Delta = \prod_{e\in\partial\Delta} U_e$. Such terms appear as curvature/holonomy-sensitive corrections in the Hamiltonian while leaving the CPTP/no-signaling guarantees intact.

4 Discussion

The formal results developed for triangulated relativistic quantum computation (TRQC) deliberately target the interface between mathematical consistency and hardware reality. In prospective deployments, quantum processors and channels are embedded in curved or time-dependent environments: free-space optical links between moving platforms, photonic lattices patterned on curved substrates, cold-atom simulators realizing synthetic gauge fields, and distributed sensors operating under gravitational gradients. TRQC admits these scenarios by turning curvature and causal order into explicit control variables while preserving complete-positivity and no-signaling.

TRQC enforces causal partial orders and proper-time scaling but does not assert Lorentz covariance of the channel dynamics nor foliation independence. Our results are formulated at the level of causal scheduling and CPTP composition; a covariant formulation of GKSL dynamics on 3+1D Lorentzian spacetimes is beyond the present scope.

To make the discussion concrete, consider a network on a smooth Riemannian manifold (M, g) (spatial curvature) or a Lorentzian spacetime (relativistic effects). Nodes at positions $x_v(t) \in M$ define a time-indexed triangulation $\mathsf{T}(t)$ of the point set, with edges $E(t)$ and faces $F(t)$. Geometry enters coherently through geodesic distances $\ell_e(t)$ and holonomies $U_{ij}(t)$ and dissipatively through curvature-modulated rates. A representative Hamiltonian and master equation illustrating these couplings are

$$H(t) = \sum_{e=\{i,j\}\in E(t)} \Big[J(\ell_e(t)) \big(\sigma_i^{+} U_{ij}(t)\, \sigma_j^{-} + \text{h.c.}\big) \Big] + \sum_{e\in E(t)} \alpha\, \overline{K}_e(t)\, H_e^{(1)} + \sum_{\Delta\in F(t)} \beta\, \mathfrak{Re}\,\mathrm{Tr}\; W_\Delta(t)\, H_\Delta^{(\mathrm{plaq})}, \tag{22}$$

$$\dot{\rho} \;=\; -i[H(t), \rho] \;+\; \sum_{e\in E(t)} \sum_{j} \gamma_{e,j}\big(\overline{K}_e(t)\big) \Big(L_{e,j} \rho L_{e,j}^{\dagger} - \tfrac{1}{2}\{L_{e,j}^{\dagger} L_{e,j}, \rho\}\Big), \tag{23}$$

with $W_\Delta(t) = \prod_{e\in\partial\Delta} U_e(t)$ encoding a Wilson-loop holonomy, $\overline{K}_e(t)$ a local average of vertex curvatures near e, and $\gamma_{e,j} \ge 0$ ensuring CPTP evolution. The proper time at each node v scales gate durations via $\mathrm{d}\tau_v = \sqrt{g_{\mu\nu}\dot{x}_v^{\mu}\dot{x}_v^{\nu}}\,\mathrm{d}t$, where we adopt the $(+,-,-,-)$ signature and restrict to timelike trajectories, so $g_{\mu\nu}\dot{x}_v^{\mu}\dot{x}_v^{\nu} > 0$ (in other signatures, insert the usual absolute value under the square root), and the triangulated causal order supplies an operational scheduler that respects locality. Each $L_{e,j}$ acts nontrivially only on the tensor legs associated with edge e (and possibly an environment factor in a Stinespring dilation) and as the identity elsewhere.

Three structural properties make TRQC attractive for engineering-scale modeling:

- First, physical soundness by construction. The curvature-modulated GKSL generators produce CPTP maps for all curvature values and step sizes; composition along a causal partial order yields a global channel that is independent of within-slice ordering and forbids signaling outside spacelike regions. This eliminates common ambiguities when orchestrating asynchronous controls across mobile or clock-skewed nodes.
- Second, geometric gauge- and mesh-robustness. Curvature on a slice is computed intrinsically from angle deficits and is invariant under global orthogonal changes of the latent coordinates. On closed slices, the discrete Gauss–Bonnet identity controls the total curvature budget, enabling precise sanity checks. Under commuting-locality and in Trotter refinement limits, remeshing leaves the global channel invariant, so mesh quality adaptations (e.g., local Pachner moves) do not change the physics.
- Third, direct control of coherent and incoherent pathways. Couplings depend on geodesic distances and holonomies, while noise rates depend on curvature. This dual entry point allows curvature to serve as a dial for dispersion, interference, and decoherence, which is essential when designing protocols for space-based QKD, curved-lattice simulators, or transport in nanomembranes.

The abstract calculus maps to several concrete stacks:

- *Free-space and satellite networks.* Nodes follow relativistic trajectories, inducing nonuniform proper times and Doppler/gravitational shifts. Triangulated slices provide asynchronous time steps compatible with onboard clocks. Geometry-aware rates model path loss, pointing jitter, and turbulence that correlate with effective curvature in the latent triangulation. Scheduling respects light-cone constraints, simplifying synchronization and security analyses.
- *Photonic and polaritonic lattices on curved substrates.* Patterned waveguides or microcavities realize nontrivial holonomies and curvature-dependent dispersion. The plaquette term in Eq. (22) captures flux-like phases, while curvature-driven dephasing terms in Eq. (23) model fabrication-induced roughness gradients. Remeshing tolerance is critical when the effective lattice deforms under thermal or mechanical stress.
- *Cold atoms, Rydberg arrays, and trapped ions with synthetic gauge fields.* Synthetic curvature and spin connections are programmed via laser phases and detunings. Geodesic-aware couplings $J(\ell)$ are natural when the

trap geometry is non-Euclidean or time varying. The causal product formula aligns with digital-analog hybrid control schedules.

- *Curved/fractal nanostructures and metamaterials.* In excitonic or magnonic transport, angle deficits modulate both hopping and dephasing; in fractal geometries, a graph-curvature surrogate can be substituted when smooth embeddings are unavailable, preserving the channel semantics.

Limitations and scope. The guarantees of TRQC rest on modeling choices whose scope should be explicit:

- *Latent embedding and estimator dependence.* Although the curvature estimator is orthogonally gauge-invariant, it depends on the latent embedding and neighborhood selection. PCA-based tangent planes and Delaunay triangulations are numerically convenient but can be sensitive to noise and near-degeneracies. In practice, robustification (e.g., angle clamping, aspect-ratio regularization, and outlier trimming) is advisable, and boundary terms must be handled explicitly.
- *Markovian closure.* The GKSL form assumes effective Markovianity. Long-delay optical paths, memory in reservoirs, or feedback can induce non-Markovian effects. A practical remedy is to enlarge the system with delay lines or ancilla edges, so that the enlarged dynamics is Markovian; otherwise, one must resort to time-nonlocal master equations, losing some of TRQC's simple compositionality. When $\gamma_{e,j}(\kappa)$ are inserted phenomenologically, one should check compatibility with KMS/detailed-balance if a thermal microscopic derivation is intended.
- *Calibration of curvature-rate maps.* The functions $\gamma_{e,j}(\kappa)$ encode device physics and environment. Without experimental calibration, they are phenomenological. Identifiability (recovering $\gamma_{e,j}$ from observed channels) is a nontrivial inverse problem; priors from materials science or wave optics help, but a systematic Bayesian treatment is still open.
- *Noncommuting within-slice generators.* When generators fail to commute on a spacelike step, first-order Trotterization introduces commutator errors. While refinement controls these, hardware constraints may limit time resolution. Strang or higher-order splittings reduce bias but can be costly; adaptive layerings that minimize noncommutativity are a promising compromise.
- *Topological changes and defects.* Node failures or link blockages induce local surgery on the triangulation. TRQC survives local Pachner moves, but true topological changes (e.g., component splitting/merging) require explicit boundary and interface handling, including proper reinitialization of curvature budgets and holonomy phases.

Because the framework comes with invariants, it supports rigorous test suites:

- *Geometric checks.* On closed slices whose geometric triangulation realizes the same combinatorial triangulation, verify $\sum_v K(v)A_v = 2\pi\,\chi(\Sigma)$; with boundary include the exterior-angle term. Track compliance with these hypotheses under remeshing.
- *CPTP and no-signaling audits.* Numerically confirm that channels preserve trace and positivity for random inputs; test spacelike factorization by swapping within-slice orderings and comparing reduced states.
- *Perturbation scalings.* Vary curvature by a small $\delta\kappa$ and verify that channel deviations scale linearly in $t\,\delta\kappa$ consistent with Duhamel-type bounds.
- *Convergence under refinement.* Plot channel distance between successive refinements; exponential or algebraic decay consistent with commutator norms indicates that the Trotter limit is reached.

Several research directions emerge naturally:

- *Continuum limits.* Under what conditions does TRQC converge to a well-defined GKSL evolution on a curved manifold as mesh size and step-size vanish in tandem? Can one quantify the dependence on curvature gradients and holonomy strength?
- *Inverse geometry.* Given process-tomography data across a slice, to what extent can one reconstruct κ or a graph-curvature surrogate, and what are fundamental identifiability limits in the presence of noise?
- *Curvature as a computational resource.* Do curvature and holonomy expand or contract complexity classes for distributed tasks (e.g., clock synchronization, consensus, and secure randomness expansion)? Can speed limits with curvature dependence translate into lower bounds on communication or gate counts?

- *Fault tolerance on curved substrates.* How do curvature-modulated noise rates impact thresholds and decoder performance for homological and LDPC codes on non-Euclidean tilings? Is there an optimal curvature profile for a given code family?
- *Non-Markovian extensions.* Can one preserve causal factorization and a form of triangulation invariance for time-nonlocal master equations, perhaps via auxiliary memory edges or kernel factorizations?
- *Holonomy control and gauge design.* What systematic methods select U_{ij} and plaquette targets to synthesize desired band topology or transport features while respecting device constraints and minimizing dephasing?

TRQC elevates curvature, holonomy, and causal structure to programmable resources while holding fast to quantum information's compositional laws. Its mathematical guarantees—CPTP evolution, no-signaling across spacelike regions, O(d)-invariance (with explicit scale convention) and commuting-refinement invariance under stated hypotheses, and explicit curvature-dependent bounds—translate into actionable practices for modeling and control on moving and curved platforms. The path forward is clear: calibrate curvature-rate maps on real hardware, integrate adaptive meshing and higher-order splittings into compilers, and explore curvature-aware error correction and metrology. If successful, these efforts will turn the geometry of the world from a nuisance into an ally for robust quantum technologies.

5 Conclusion and future work

TRQC elevates curvature, holonomy, and causal order to first-class citizens in quantum computation and communication. The framework rests on five pillars: (i) a causal, triangulated scaffold for channel composition; (ii) an intrinsic, orthogonally gauged curvature estimator from vertex angle deficits, backed by discrete Gauss–Bonnet; (iii) curvature-modulated GKSL generators that guarantee CPTP evolution for all curvature values; (iv) causal factorization and order-independence within spacelike steps, implying no-signaling across spacelike regions; and (v) triangulation invariance under commuting-locality together with refinement invariance in a Lie–Trotter limit. Beyond structural results, TRQC supplies curvature-dependent quantum speed limits and contractive perturbation bounds within a curvature-modulated causal channel semantics, providing design-level guidance for schedules and tolerances.

A priority is to *close the loop* between geometry and control. Three complementary thrusts are natural:

- *Adaptive meshing and scheduling.* Online quality metrics (e.g., minimum angle, aspect ratio, and commutator norms within a slice) trigger local retriangulation and re-layering. Order-independence and commuting-locality ensure that such updates do not alter the global channel, while Trotter bounds quantify residual errors when noncommuting terms persist.
- *Curvature-aware optimal control.* Incorporate curvature fields $\{K(v)\}$ and holonomies $\{U_{ij}\}$ into objective functionals for gate synthesis, routing, and entanglement distribution. Constraints include CPTP preservation, proper-time budgets, and bounds from curvature-dependent speed limits.
- *Calibration of curvature-rate maps.* Treat $\gamma_{e,j}(\kappa)$ as control-informed parameters learned from experiments. Bayesian or variational identification loops can balance hardware priors with TRQC's Lipschitz and positivity constraints to recover rate–curvature laws in situ.

Where no smooth manifold exists (fractals, disordered networks), substitute *graph-curvature proxies* for K:

- *Ollivier/Forman drivers.* Map edge- or vertex-based Ricci-like curvatures to TRQC via $\overline{K}_e \leftarrow \text{Avg}\{K(\text{endpoints/faces})\}$, preserving channel semantics. Investigate convergence to angle-deficit curvature in random geometric graphs and the impact on transport exponents.
- *Hybrid surrogates.* Combine local embedding-based deficits (when available) with graph curvature elsewhere, yielding robust fields on mixed-quality data without sacrificing gauge invariance where an embedding is trustworthy.

- *Benchmark phenomena.* Predict and verify curvature-controlled crossovers in mean-square displacement, participation ratios, and mobility edges on Sierpiński-type and hyperbolic designer lattices, using TRQC's curvature-aware noise and holonomy controls.

Geometry and relativistic timing should be first-class optimization variables in compilers:

- *Cost models.* Extend standard gate-time and error budgets by adding curvature penalties, holonomy targets, and proper-time schedules. Use commutator-based Trotter surrogates as splitting-cost proxies for within-slice noncommutativity.
- *Scheduling under light-cone constraints.* Optimize layerings that minimize spacelike commutators and curvature-induced dephasing while meeting latency and synchronization goals in moving-node networks.
- *CPTP-preserving numerics.* Integrate scaling-and-squaring for GKSL exponentials and exact Kraus realizations where available, ensuring positivity by design during compilation and simulation.

Two theoretical frontiers invite careful development:

- *Memoryful dynamics.* Represent delays and reservoirs as explicit ancilla edges and nodes to lift non-Markovian processes into a Markovian TRQC supergraph, retaining causal factorization. Analyze limits in which coarse-graining reproduces effective time-nonlocal kernels while preserving no-signaling.
- *Continuum limits.* Establish rigorous convergence from triangulated TRQC evolutions to GKSL-type dynamics on curved manifolds as mesh size and step-size vanish together. Quantify dependence on curvature magnitude and gradients, and identify counterexamples that require higher-order geometric corrections.

Curvature-aware noise models invite domain-specific advances:

- *QEC on curved codes.* Study thresholds and decoder performance for homological and LDPC codes on hyperbolic or mixed-curvature tilings with curvature-modulated error rates. Seek curvature profiles that maximize code capacity under fixed resources.
- *Distributed metrology.* Use $\partial H/\partial\kappa$ as the generator of parameter encoding to derive Fisher-information bounds for gravimetry and inertial sensing with moving networks, accounting for proper-time schedules.
- *Band topology via holonomy.* Design link unitaries and plaquette phases to synthesize target Chern numbers or protected edge modes while minimizing curvature-driven dephasing predicted by TRQC.

To facilitate adoption, we advocate an open benchmark suite comprising: canonical meshes with known $\chi(\Sigma)$ and closed-form K; randomized mobility traces with reproducible latent embeddings; and reference channels for flat and entanglement-breaking limits. Each artifact should include diagnostics for CPTP preservation, Gauss–Bonnet consistency, refinement convergence, and spacelike factorization. Ethical guidance should accompany releases, noting dual-use implications of geometry-aware scheduling and control in critical infrastructure.

By welding intrinsic discrete geometry, relativistic causality, and open-system quantum dynamics into a single calculus, TRQC provides a principled route to modeling and controlling quantum information in the world as it is: curved, moving, and asynchronous. The next phase is inherently collaborative, combining mathematical analysis, compiler engineering, and hardware calibration. In turning curvature and proper time from constraints into resources, we anticipate not only more faithful simulations but also qualitatively new protocols that exploit geometry for robustness, efficiency, and discovery.

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During the preparation of this work, the author(s) used Writefull to edit and polish the grammar. After using this tool/service, the author(s) reviewed and edited the content as needed and take(s) full responsibility for the content of the publication.
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Data Availability A numerical *proof-of-concept (PoC)* accompanies this article as Supplementary Material. All details are explained there, including the GitHub public repository [46] as well as the interpretation for the provided folders. It implements the TRQC framework on representative graph families (*sphere*, *geometric2d*, *Erd[Pleaseinsertintopreamble]s–Rényi*, and *scale-free*) for $N \in \{6, 8, 10, 12\}$ and rounds $\{2, 3\}$, reporting final-step and peak-over-steps arrival probabilities, participation ratios, and runtimes. The PoC confirms the theoretical invariants—complete-positivity, order-independence, remeshing robustness, and curvature-driven transport behavior—and reproduces the expected $O(4^N)$ scaling.

Declarations

Conflict of interest The authors declare that there are no conflict of interest, financial or otherwise, that could be perceived to influence the results or interpretation of this work. No competing financial interests, personal relationships, or affiliations have affected the conduct of the research, the preparation of the manuscript, or its potential publication. All funding sources and institutional supports are fully acknowledged within the manuscript.

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