GENERALIZED CONDITIONAL SYMMETRIES, RELATED SOLUTIONS OF THE KLEIN–GORDON–FOCK EQUATION WITH CENTRAL SYMMETRY

RADU CONSTANTINESCU

Department of Physics, University of Craiova, A. I. Cuza 13, 200585 Craiova, Romania
E-mail: rconsta@central.ucv.ro

Received May 14, 2015

The *generalized conditional symmetry* (GCS) method is applied to a specific case of the *Klein–Gordon–Fock* (KGF) *equation* with central symmetry. We first investigate the conditions which yield the KGF equation that admits special class of second-order GCSs. The determining system for the unknown functions is solved in several special cases. New symmetry operators and related exact solutions, different in form and structure from the ones obtained using other methods, are pointed out. Several surface plots of solutions are displayed.

Key words: Generalized conditional symmetry, Klein–Gordon–Fock equation.

PACS: 11.30.Na, 47.10.ab.

1. INTRODUCTION

The concepts of symmetry, invariants and conservation laws are fundamental in the study of dynamical systems, providing a clear connection between the equations of motion and their solutions. There are many reasons for computing symmetries and conservation laws corresponding to dynamical systems described by differential equations, mainly related to finding their exact solutions. An important method to obtain the exact solutions of nonlinear partial differential equations (PDEs) is the Lie symmetry method [1, 2]. It has been applied as an established route for the reduction of (PDEs) [3, 4]. The Lie method allowed to obtain interesting class of symmetries for well-known models [5, 6], but it has been also extended, and other non-Lie methods has been formulated [7, 8]. One of these is the *generalized conditional symmetry* (GCS) method [9, 10]. This approach could be considered as a natural generalization of the non-classical method, in so far the Lie Bäcklund symmetry is a generalization of the classical method. This approach has been applied successfully for obtaining various kinds of exact solutions of nonlinear PDEs, and especially functional separable solutions.

The equation we are going to study is the Klein–Gordon–Fock (KGF) equation with central symmetry:

$$v_{2t} - v_{2r} - \frac{2}{r}v_r + \frac{b}{r^2}v = 0. \tag{1}$$

In the previous equation b is a real parameter.

By changing the dependent variable, $v(r,t) = u(r,t)/r$, the following reduced form of Eq. (1) is obtained:

$$u_{2t} - u_{2r} + \frac{b}{r^2}u = 0. \tag{2}$$

Eq. (2) belongs to the class of wave equations with time-independent potential [11]:

$$u_{2t} - u_{2r} + V(r)u = 0, \tag{3}$$

where $u(t,r) \in C^2(R^2, R^1)$ and the potential $V(r) \in C^2(R^1, R^1)$. Equations of type (3) are widely used in quantum physics and may be related to other linear and non-linear PDEs in mathematical physics. Thus, all potentials $V(r)$ allowing the separation of variables in (3) have been found in [12], where also all non-equivalent orthogonal and non-orthogonal coordinate systems providing separability of the coordinates were constructed. The structure of the KGF equation symmetry algebra on pseudo-Riemannian manifolds in the presence of an external electromagnetic field is investigated in [13]. Starting from the coadjoint orbit method and from the harmonic analysis of Lie groups, a method is proposed for integrating the equation on manifolds with simply transitive group actions .

Our main interest will be shown to the case $b \neq 0$, but the results are also transposed for the case $b = 0$, when Eq. (2) becomes d'Alembert equation. The case $b \neq 0$ may describe, for example a time evolution of transient electromagnetic fields in homogeneous media and biconical transmission lines [14, 15]. From the physical point of view, the values of the parameter $b = n(n+1)$, $n = 1, 2, 3, \cdots$ correspond to electromagnetic fields in the free space. In the case of a conical metal line, the parameter b may be equal to some arbitrary positive value. In the mentioned special case, $b = n(n+1)$, the general solution is derived in [16] and admits the expression:

$$u(r,t) = r^n \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \left(\frac{\Psi(r-t) + \Phi(r+t)}{r}\right), \tag{4}$$

where $\Psi(\cdot)$ and $\Phi(\cdot)$ are arbitrary sufficiently smooth functions.

The purpose of this paper is to extend the results obtained in [17], where the classical symmetries of Eq. (2) were found. We are able to do this by means of the generalized conditional symmetry (GCS) method. The approach will be similar with those applied to Grad-Shafranov model from Plasma Physics [18]. In fact, we are presenting some results concerning the structure of second-order GCSs for Eq. (2). Some physical models [19, 20] can be considered in this approach. In Section 2 we expose some basic facts on GCS approach, while in Section 3, we formulate conditions enabling Eq. (2) to admit GCSs. These conditions lead to a determining system which is solved in several specific cases. Some new interesting solutions of Eq. (2), both for $b \neq 0$ and for $b = 0$, not yet reported, are figured out. Some concluding remarks end the paper.

2. FROM CLASSICAL TO NON-CLASSICAL LIE SYMMETRIES

The *Lie (classical) symmetry method (CSM).* [1] for solving partial differential equations as it was initially formulated asks for the invariance of the equations to the action of an infinitesimal symmetry operator. Let us refer to an general m-th-order $(1+1)$-dimensional evolution equation of the form:

$$u_t = E(t,x,u,u_x,...u_{mx}), \text{with } u_{kx} = \frac{\partial^k u}{\partial x^k}, 1 \leq k \leq m. \tag{5}$$

The classical Lie operator will have the form:

$$X = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{6}$$

The n-th extension of (6) is given by:

$$X^{(n)} = X + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha}, \tag{7}$$

where

$$u_J^\alpha = \frac{\partial^J u^\alpha}{\partial x^{j_1}\partial x^{j_2}..\partial x^{j_m}};\ j_1 + j_2 + ... + j_m = J. \tag{8}$$

Other approaches which generalize CMS were formulated. The *Generalized Conditional Symmetry* (GCS) or *conditional Lie-Bäcklund symmetry* method is an example on how higher-order symmetries could help in finding new symmetries of PDEs which could generate additional new invariant solutions. The GCS method supposes Eq. (5) to be invariant under a non-Lie point group of infinitesimal transformations:

$$\begin{aligned}
\bar{u} &= u + \varepsilon\eta(t,x,u,u_x,...u_{Nx}) + O(\varepsilon^2),\\
\bar{u}_t &= u_t + \varepsilon D_t\eta(t,x,u,u_x,...u_{Nx}) + O(\varepsilon^2),\\
\bar{u}_x &= u_x + \varepsilon D_x\eta(t,x,u,u_x,...u_{Nx}) + O(\varepsilon^2),\\
&...
\end{aligned}$$

with:

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{(k+1)x}\frac{\partial}{\partial u_{kx}},\ D_x^{j+1} = D_x(D_x^j),\ D_x^0 = 1.$$

The above group is generated by a vector field V, with the characteristic η, of the form:

$$V = \sum_{k=0}^{\infty} D_x^k \eta \frac{\partial}{\partial u_{kx}}. \tag{9}$$

For understanding the connection between the Lie-Backlund symmetry and this conditional one, the following are very useful:

Definition 1: The vector (9) is said to be a Lie–Bäcklund symmetry of (5) if and only if

$$Y(u_t - E)|_L = 0,$$

where L is the set of all differential consequences of the equation, that is to say:

$$u_t - E = 0,\ D_x^j D_x^i (u_t - E) = 0,\ i,\ j = 0,1,2,\cdots.$$

Definition 2: The vector (9) is said to be a GCS of (5) if and only if

$$Y(u_t - E)|_{L\cap M} = 0, \tag{10}$$

where M denotes the set of all differential consequences for the equation $\eta = 0$ in respect to x, that is to say:

$$D_x^j \eta = 0,\ j = 0,1,2,\cdots. \tag{11}$$

Remark: If η do not depend on t explicitly, the condition for existing GCS can be expressed in the following terms:

$$\eta' E|_{L\cap M} = \lim_{\varepsilon\to 0} \frac{d}{d\varepsilon} \eta(u + \varepsilon E)|_{L\cap M} = 0, \tag{12}$$

where ”prime” denotes the Fréchet derivative of η along the E direction. Guiding center equation in drift approximation that used in plasma fusion is an example for this kind of method [21, 22].

3. KLEIN–GORDON–FOCK EQUATION

Let us start to effectively consider the Klein-Gordon-Fock equation with central symmetry in its version (1). In fact, in the remaining part of this paper we will concentrate our attention to the reduced equations (2) or (3) mentioned before.

3.1. ALREADY KNOWN RESULTS ON KGF

Let us shortly recall the main already known results on KGF. All potentials $V(r)$ and all the non-equivalent orthogonal / non-orthogonal coordinate systems providing separability of the variables in (3) have been found in [12]. For $b = 0$ Eq. (2) becomes the d'Alembert equation. The case $b \neq 0$ was studied for describing the time evolution of transient electromagnetic fields in homogeneous media [14, 15]. For $b = n(n+1)$, $n = 1, 2, 3, \cdots$. Eq. (2) describes the electromagnetic field in the free space and it admits the general solution [16] given in (4).

The classical Lie symmetries of Eq. (2) were found in [17]. The Lie algebra of the infinitesimal symmetries for Eq. (2) contains the following basis of symmetry operators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \quad X_3 = (t^2 + r^2)\frac{\partial}{\partial t} + 2tr\frac{\partial}{\partial r},$$
$$X_4 = u\frac{\partial}{\partial u}, \quad X_5 = \gamma(r,t)\frac{\partial}{\partial u}, \tag{13}$$

where $\gamma(r,t)$ is whatever solution of Eq.(2). The structure of the symmetry algebra on pseudo-Riemannian manifolds in the presence of an external electromagnetic field was investigated in [13].

3.2. GCS FOR KGF. DETERMINING SYSTEM

Let us pass now to the computation of GCS for KGF (2). For this specific case, the operator (9) which generates the GCS group takes the form:

$$V = \eta\frac{\partial}{\partial u} + (D_r\eta)\frac{\partial}{\partial u_r} + (D_t\eta)\frac{\partial}{\partial u_t}$$
$$+ (D_{2r}\eta)\frac{\partial}{\partial u_{2r}} + (D_{2t}\eta)\frac{\partial}{\partial u_{2t}} + \cdots . \tag{14}$$

The condition (10) written for Eq. (2) is:

$$Y(u_{2t} - u_{2r} + \frac{b}{r^2}u)\,|_{L\cap M} = 0. \tag{15}$$

The previous condition is equivalent to the following relation:

$$\frac{b}{r^2}\eta - D_{2r}\eta + D_{2t}\eta\,|_{L\cap M} = 0. \tag{16}$$

If we also impose the restriction (11), it will follow that Eq. (2) admits GCSs (14) if and only if:

$$D_{2t}\eta = 0. \tag{17}$$

As (2) is second order in the radial variable, we choose the characteristic as:

$$\eta[r,u] = u_{2r} - H(u)u_r^2 - P(r,u)u_r - R(r,u). \tag{18}$$

We shall find the determining system for the unknown functions $H(u)$, $P(r,u)$, $R(r,u)$ which appear in the previous relation. Taking into account the surface condition $\eta = 0$, we may substitute the derivative u_{2r} by the expression:

$$u_{2r} = H(u)u_r^2 + P(r,u)u_r + R(r,u). \tag{19}$$

Consequently, the derivative u_{2t} from the KGF equation (2) acquires the equivalent form:

$$u_{2t} = -\frac{b}{r^2}u + H(u)u_r^2 + P(r,u)u_r + R(r,u). \tag{20}$$

Starting from the second order GCSs (18), the main condition (17) becomes:

$$\begin{aligned} u_{(2r)(2t)} - H'' u_t^2 u_r^2 - H' u_r^2 u_{2t} - 4H' u_r u_t u_{rt} - \\ 2H u_{rt}^2 - 2H u_r u_{r(2t)} - P_{2u} u_t^2 u_r - P_u u_r u_{2t} - 2P_u u_t u_{rt} - \\ P u_{r(2t)} - R_{2u} u_t^2 - R_u u_{2t} = 0. \end{aligned} \tag{21}$$

Here the "prime" index denotes the derivative in respect to u, while the subscripts denote the partial derivative in respect to the indicated variables.We calculate from (20) the derivatives $u_{r(2t)}$ and $u_{(2r)(2t)}$. Then, we substitute them into (21) and we make use of (19) and (20) in order to eliminate u_{2r} and u_{2t}. Therefore, the achieved condition is verified if and only if the coefficient functions of various monomials in derivatives of u are equal to zero. These constraints lead to $H = 0$ and to $P_u(r,u) = 0$, $R_{2u}(r,u) = 0$, that is to say $P(r,u) \equiv P(r)$, $R(r,u) = Q(r)u + M(r)$. The unknown functions $P(r)$, $Q(r)$ and $M(r)$ do also satisfy the following system of ODEs:

$$\begin{aligned} \frac{4b}{r^3} + 2Q' + 2PP' + P" = 0, \\ -\frac{6b}{r^4} + 2P'Q + Q" - \frac{2b}{r^3}P = 0, \\ 2P'M + M" - \frac{b}{r^2}M = 0, \end{aligned} \tag{22}$$

with b an arbitrary constant. Here "prime" index denotes the first order derivative in respect to r. As we announced in Introduction, we shall investigate in the next two sections the proper KGF equation corresponding to $b \neq 0$ and d'Alembert Equation associated to $b = 0$.

3.3. NEW SOLUTIONS FOR THE KGF EQUATION

Solving the invariant surface conditions, the previous expressions lead to various solutions of Eq. (2), some of them expressed in terms of special functions. We are not interested here in such solutions, but in finding special cases when the solutions appear in analytic form. More precisely, we shall analyze the solutions that the system (22) may admit when the unknown functions $P(r)$ and $Q(r)$ are given by the expressions:

$$P(r) = \frac{k}{r},\ Q(r) = \frac{m}{r^2}, \tag{23}$$

with k and m arbitrary constants. For these choices, the remaining function $M(r)$ must verify the ordinary differential equation:

$$-\frac{2k}{r^2}M + M^{"} - \frac{b}{r^2}M = 0. \tag{24}$$

So, we can discuss the KGF solutions in terms of three real parameters: b from Eq. (2) and k, m from (23). For $b \neq 0$, three specific cases are investigated:

Case I: $(k-3)(k-1) = 4b$; $m \neq b$.

Case II: $m = b \neq 0; k = arbitrary.$

Case III: $m = 0; k = arbitrary.$

We shall see that they lead to new interesting solutions of the master Eq. (2) which to our best knowledge, have not been reported in literature.

3.3.1. Case I:

If we choose $(k-3)(k-1) = 4b$ with $k \neq \{1,3\}$ the system (22) does admit the solution:

$$P(r) = \frac{k}{r},\ Q(r) = -\frac{(k-1)(k+3)}{4r^2},\ M(r) = c_1 r^{-\frac{1+k}{2}} + c_2 r^{\frac{k+3}{2}}, \tag{25}$$

with c_1, c_2 arbitrary parameters, and $m = -\frac{(k-1)(k+3)}{4}$. The GCS operator takes the form:

$$V_I = \left[u_{2r} - \frac{k}{r}u_r + \frac{(k-1)(k+3)}{4r^2}u - c_1 r^{-\frac{1+k}{2}} - c_2 r^{\frac{k+3}{2}} \right] \frac{\partial}{\partial u}. \tag{26}$$

By solving the invariance surface condition $\eta = 0$, we come to the solution of the KGF equation as:

$$\begin{aligned} u(t,r) = {} & f(t) r^{(k-1)/2} + g(t) r^{(k+3)/2} + r^{(k+7)/2} + \frac{8c_1}{k(k-2)} r^{-(k-3)/2} - \\ & \frac{2[k(k+5)(k+7) - 2c_1(k-3)(k-1)]}{k(k+1)(k+3)} r^{(k+3)/2} + \\ & \frac{(k-2)(k+5)(k+7) - 4c_1(k-3)(k-1)}{(k-3)(k-1)(k-2)} r^{(k-1)/2}. \end{aligned} \tag{27}$$

Remark: This solution imposes new restrictions for k, more exactly $k \neq \{-3, -1, 0, 2\}$. Consequently, we ought to avoid for b the set of values $\{-\frac{1}{4}, 0, \frac{3}{4}, 2, 6$. By introducing the solution (27) into the main Eq. (2), we get that functions $f(t)$ and $g(t)$ must satisfy to the following differential system:

$$k^3-2(6k^2+4k+3)g^{"}+6k^2+11k+6=0,$$
$$4k(k+1)(k+3)g-2(k+1)(k+3)f^{"}-k^3+2(c_1-6)k^2-$$
$$(8c_1+35)k+6c_1=0. \tag{28}$$

The previous relations involve the second order derivative in respect to t.

The system (28) generates the solutions:

$$f(t)=\frac{1}{24(k+3)(k+1)}\left\{k(k+1)(k+2)(k+3)t^4+8c_5k(k+1)(k+3)t^3+\right.$$
$$\left.[6(4c_6-1)k^3+12(c_1+8c_6-6)k^2+6(12c_6-8c_1-35)k+36c_1]t^2\right\}+c_3t+c_4,$$
$$g(t)=\left(1+\frac{k}{2}\right)\frac{t^2}{2}+c_5t+c_6, \tag{29}$$

with c_j, $j=\overline{3,6}$ new arbitrary constants.

We generated for the KGF equation the $6-$parameters family of solutions (27), where $f(t)$ and $g(t)$ take the general forms (29).

3.3.2. Case II:

If we choose the constraint $b=m\neq 0$, then the system (22) admits the solution:

$$P(r)=0,\ Q(r)=\frac{b}{r^2}+q_3,\ M(r)=q_1r^{\left(\frac{1+\sqrt{1+4b}}{2}\right)}+q_2r^{\left(\frac{1-\sqrt{1+4b}}{2}\right)}, \tag{30}$$

with $b\geqslant -\frac{1}{4}$ and q_1, q_2, q_3 arbitrary constants. So, in this case we have $k=0$.

It is interesting to consider the special case $b=n(n+1)$, with n rational. The GCSs are generated now by the operator:

$$V_{II}=\left[u_{2r}-\left(\frac{n(n+1)}{r^2}+q_3\right)u_r-q_1r^{(n+1)}-q_2r^{(-n)}\right]\frac{\partial}{\partial u}. \tag{31}$$

For $q_3\neq 0$, a solution expressed under the form of special functions may be derived. If we choose $q_3=0$, by solving the invariance surface condition $\eta=0$, we get for the KGF equation, the family of solutions:

$$u(r,t)=f(t)r^{(-n)}+g(t)r^{(n+1)}-$$
$$\frac{2r^2\left[-(n-\frac{1}{2})q_1r^{(n+1)}+(n+\frac{3}{2})q_2r^{(-n)}\right]}{8n^2+8n-6}. \tag{32}$$

Here we must impose $n\neq-\left\{\frac{3}{2},\ \frac{1}{2}\right\}$ or equivalently $b\neq\frac{3}{4}$.

The system (32) could admit polynomial solutions only if:

$$f(t)=\frac{q_2}{2}t^2+q_4t+q_5, g(t)=\frac{q_1}{2}t^2+q_6t+q_7. \tag{33}$$

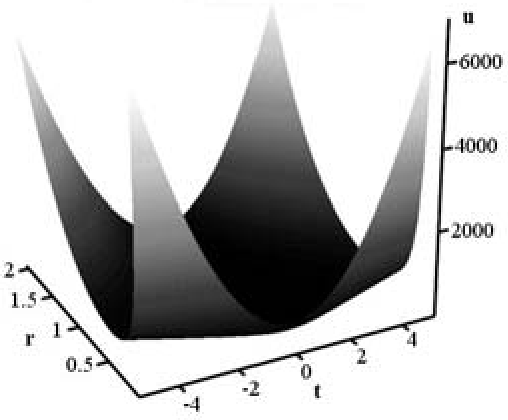

Fig. 1 – The surface plot corresponding to the solution (34) for $n = 2$, $q_1 = 60$, $q_2 = 5$, $q_4 = q_6 = -\frac{1}{10}$, $q_5 = q_7 = 5$.

where q_i, $i = \overline{1,7}$, $i \neq 3$ are parameters. Consequently, a $7-$parameters family of solutions is derived:

$$u(r,t) = \left[\frac{q_2}{2}t^2 + q_4 t + q_5\right] r^{(-n)} + \left[\frac{q_1}{2}t^2 + q_6 t + q_7\right] r^{(n+1)} - \frac{2r^2\left[-\left(n-\frac{1}{2}\right)q_1 r^{(n+1)} + \left(n+\frac{3}{2}\right)q_2 r^{(-n)}\right]}{8n^2+8n-6}. \quad (34)$$

3.3.3. Case III

Let consider now $m = 0$, that it to say $Q(r) = 0$. The determining system (22) leads to a fixed value $b = 6$, while for the remaining functions, $P(r)$ and $M(r)$, the simple expressions:

$$P(r) = -\frac{3}{r},\ M(r) = sr + p,$$

with s, p arbitrary constants. The GCS operator becomes:

$$V_{III} = \left[u_{2r} + \frac{3}{r}u_r - sr - p\right]\frac{\partial}{\partial u}. \quad (35)$$

The condition $\eta = 0$ does generate for Eq. (2) the solution:

$$u(r,t) = \frac{s}{15}r^3 + \frac{p}{8}r^2 - \frac{\sigma(t)}{2r^2} + \rho(t). \quad (36)$$

We get another new solution of Eq. (2) which depends on 6 parameters and admit the form:

$$u(r,t) = \frac{s}{15}r^3 + \frac{p}{8}r^2 - \frac{-\frac{p}{4}t^4 + 2k_3t^3 + 6k_4t^2 + k_1t + k_2}{2r^2} - \frac{p}{4}t^2 + k_3t + k_4. \quad (37)$$

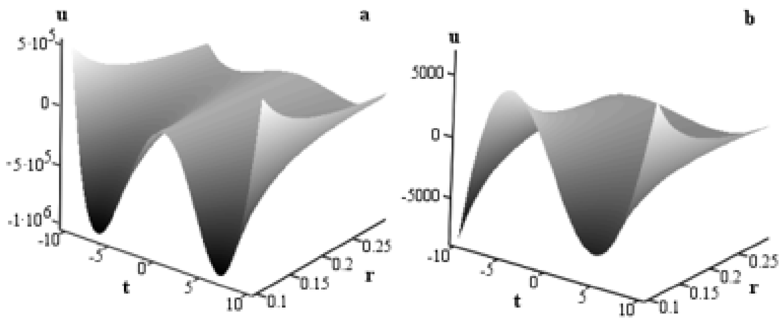

Fig. 2 – The surface plots corresponding to the solution (37) for two sets of parameters' values: a) $s = k_3 = 0$, $p = 40$, $k_1 = -\frac{1}{5}$, $k_2 = -50$, $k_4 = 150$; b) $p = k_4 = 0$, $s = 200$, $k_1 = 35$, $k_2 = 20$, $k_3 = -\frac{1}{4}$.

In Figure 2 we choose to display its surface representations for two sets of parameter values: a) $s = k_3 = 0, p = 40, k_1 = -\frac{1}{5}, k_2 = -50, k_4 = 150$; b) $p = k_4 = 0, s = 200, k_1 = 35, k_2 = 20, k_3 = -\frac{1}{4}$.

3.4. D'ALEMBERT EQUATION

As we already mentioned, the case $b = 0$ corresponds to the d'Alambert equation. The determining system (22), gives the solution:

$$P(r) = \gamma,\ Q(r) = \mu,\ M(r) = \alpha r + \beta, \tag{38}$$

which involves the parameters γ, μ, α, β. The generalized conditional symmetry is determined by the operator:

$$V_{II} = [u_{2r} - \gamma u_r - \mu u - \alpha r - \beta]\, \frac{\partial}{\partial u}. \tag{39}$$

Following the same procedure as in the previous three cases, we may associate to (39) the following invariant solution:

$$u(r,t) = h(t) \exp\left(\frac{\gamma + \sqrt{\gamma^2 + 4\mu}}{2} r\right) + \lambda(t) \exp\left(\frac{\gamma - \sqrt{\gamma^2 + 4\mu}}{2} r\right) + \frac{(-\beta - \alpha r)\mu + \alpha\gamma}{\mu^2}, \tag{40}$$

where $h(t)$ and $\lambda(t)$ take the forms:

$$h(t) = a_1 \exp\left(\frac{\sqrt{2\gamma}\sqrt{\gamma^2+2\mu+\gamma\sqrt{\gamma^2+4\mu}}}{2}t\right)+$$

$$a_2 \exp\left(-\frac{\sqrt{2\gamma}\sqrt{\gamma^2+2\mu+\gamma\sqrt{\gamma^2+4\mu}}}{2}t\right),$$

$$\lambda(t) = a_3 \exp\left(\frac{\sqrt{2\gamma}\sqrt{\gamma^2+2\mu-\gamma\sqrt{\gamma^2+4\mu}}}{2}t\right)+$$

$$a_4 \exp\left(-\frac{\sqrt{2\gamma}\sqrt{\gamma^2+2\mu-\gamma\sqrt{\gamma^2+4\mu}}}{2}t\right),$$

with a_j, $j=\overline{1,4}$ arbitrary constants.

4. CONCLUDING REMARKS

The issue of constructing invariant solutions of wave equations with time-independent potential (3) is still open. Based on this fact, we investigated in this paper the Klein–Gordon–Fock equation (2) and we obtained new solutions using the generalized conditional symmetry method. We restrict ourselves to the existence of a special class of second order GCSs with the characteristic (18) which involves 3 arbitrary functions. The main outcome of this investigation consists in finding new solutions of Eq. (2). More precisely, we provide three distinct generalized conditional symmetry operators (26), (31), (35). By solving the invariance surface condition for each of them, three new families of solutions for Eq. (2), not yet reported, have been highlighted. They depend on different number of parameters. Namely, the solutions of type (27) and (37) involve 6 parameters, while those described by (34) suppose 7 parameters. Their common feature is that they are relatively separable in respect to the radial space coordinate r and to time t. All of them can be decomposed in monomials with separable forms. For some sets of parameters, graphical representations of solutions (34) and (37) are presented. In the case $b=0$, which corresponds to d'Alembert equation, the generalized conditional symmetry operator (39) and the associated solution (40) are also derived. At large distances, they have interesting behavior which deserves to be studied.

Acknowledgements. The author is grateful for the financial support offered by the Romanian 231 Space Agency, in the frame of the Programme "STAR 2013", as well as by ICTP - SEENET-MTP Project PRJ-09 "Cosmology and Strings".

REFERENCES

1. P. J. Olver, "*Application of Lie Groups to Differential Equations*" (Springer-Verlag, New-York 1986).
2. G. W. Bluman, S. Kumei, "*Symmetries and Differential Equations*" (Springer-Verlag, New York, 1989).
3. F. Kangalgil, F. Ayaz, Phys. Lett. A **372**, 1831–1835 (2008).
4. R. Cimpoiasu, Rom. J. Phys. **59**, 617–624 (2014).
5. R. Cimpoiasu, Pramana J. Phys. **84**, 543–553 (2014).
6. R. Cimpoiasu, Rom. J. Phys. **58**, 519–528 (2013).
7. G. W. Bluman, J. D. Cole, J. Math. Mech. **18**, 1025–1042 (1969).
8. P. A. Clarkson, M. D. Kruskal, J. Math. Phys. **30**, 2201–2213 (1989).
9. R. Z. Zhdanov, J. Phys. A: Math. Gen. **28**, 3841–3850 (1995).
10. Q. M. Liu, A. S. Fokas, J. Math. Phys. **37**, 324–345 (1996).
11. R. Z. Zhdanov, I. V. Revenko, J. Phys. A: Math. Gen. **26**, 5959–5972 (1993).
12. R. Z. Zhdanov, I. V. Revenko, W. I. Fushchych, Ukr. Math. J. **46**, 1480–1503 (1994).
13. A. Magazev, Theor. Math. Phys. **173**, 1654–1667 (2012).
14. A. Y. Butrym, B. A. Kochetov, Prog. Electromagn. Res. B **19**, 151–176 (2010).
15. B. A. Kochetov, A. Y. Butrym, Prog. Electromagn. Res. B **48**, 375–394 (2013).
16. A. D. Polyanin, "*Handbook of linear partial differential equation for engineers and scientists*" (Boca Raton: Chapman Hall/CRC, 2002).
17. B. A. Kochetov, Commun. Nonlin. Sci. Numer. Simulat. **19**, 1723–1728 (2014).
18. R. Cimpoiasu, Phys. Plasmas **21**, 042118–042125 (2014).
19. M. Negrea, I. Petrisor, Phys. AUC **11** 49–55 (2001).
20. M. Negrea, I. Petrisor, Phys. AUC **16**, 28–43 (2006).
21. R. Balescu, I. Petrisor, M. Negrea, Plasma Phys. Contr. Fusion **47**, 2145–2159 (2005).
22. I. Petrisor, M. Negrea, B. Weyssow, Phys. Scripta **75**, 1–12 (2007).