EIGENFUNCTION EXPANSIONS FOR DIFFERENTIAL OPERATORS WITH OPERATOR-VALUED COEFFICIENTS AND THEIR APPLICATIONS TO THE SCHRÖDINGER OPERATORS WITH LONG-RANGE POTENTIALS

Yoshimi Saitō

Department of Mathematics
Osaka City University
Sugimoto-cho, Sumiyoshi-ku, Osaka, Japan

§1. Introduction. This report is concerned with the differential operator

$$L = -\frac{d^2}{dr^2} + B(r) + C(r) \qquad (r \in (0, \infty) = I).$$

Here $B(r)$, $r \in I$, is a non-negative, self-adjoint operator in a Hilbert space X with its domain $\mathcal{D}(B(r)) = D$ which does not depend on $r \in I$. And for each $r \in I$ $C(r)$ is a bounded, self-adjoint operator on X. It should be noted that there are some partial differential operators which are converted into the operators of the form L. Let us consider, for example, the Schrödinger operator $-\Delta + Q(y)$ in R^n. Set $X = L_2(S^{n-1})$, S^{n-1} being the $(n-1)$-sphere, and define a unitary operator U from $L_2(R^n, dy)$ onto $\mathfrak{h} = L_2(I, X, dr)$ by

$$U: L_2(R^n, dy) \ni f(y) \longmapsto r^{(n-1)/2} f(r\omega) \in L_2(I, X, dr) = \mathfrak{h}$$
$$(r = |y|, \ \omega = y/r \in S^{n-1}).$$

Here and in the sequel for a real number ρ we denote by $L_2(I, X, (1+r)^\rho dr)$ the Hilbert space of all X-valued functions $u(r)$ on I such that

$$\int_I |u(r)|^2 (1+r)^\rho \, dr < \infty,$$

where $|\ \ |$ means the norm of X. The inner product $(\ ,\ )_\rho$ and norm $\|\ \ \|_\rho$ of $L_2(I, X, (1+r)^\rho dr)$ are defined by

$$(u, w)_\rho = \int_I (u(r), w(r))(1+r)^\rho \, dr$$

and

$$\|u\|_\rho = [(u, u)_\rho]^{1/2},$$

respectively, where $(\ ,\ )$ means the inner product of X. Then we have

$$\begin{cases} U(-\Delta + Q(y))U^{*} = -\dfrac{d^2}{dr^2} + B(r) + C(r) & (r\in I), \\ B(r) = r^{-2}(-\Lambda_n + (n-1)(n-3)/4), \quad C(r) = Q(r\omega)\times , & \end{cases}$$

Λ_n being the Laplace-Beltrami operator on S^{n-1} and U^* denoting the adjoint of U, and hence the Schrödinger operator $-\Delta + Q(y)$ is unitarily equivalent to the operator of the form L.

Let us impose some asymptotic conditions on the coefficients B(r) and C(r). C(r) can be decomposed as $C(r) = C_o(r) + C_1(r)$ with bounded, self-adjoint operators $C_o(r)$, $C_1(r)$ $(r\in I)$ on X. Here $C_o(r)$ and $C_1(r)$ are assumed to satisfy the following two conditions (C_o) (C_1):

$$(C_o) \qquad \begin{cases} \|C_o(r)\| \leqq c_o(1+r)^{-\varepsilon} & (r\in I), \\ \|\frac{d}{dr}C_o(r)\| \leqq c_o(1+r)^{-1-\varepsilon} & (r\in I), \end{cases}$$

$$(C_1) \qquad \|C_1(r)\| \leqq c_o(1+r)^{-1-\varepsilon} \qquad (r\in I),$$

where the norm $\| \ \|$ means the operator norm and c_o and ε are positive constants. As for B(r) let us assume

$$(B) \qquad -\frac{d}{dr}(B(r)x,\ x) \geqq \frac{\beta}{r}(B(r)x,\ x) \qquad (r \geqq R_o,\ x\in D)$$

with constants $R_o > 0$ and $\beta > 1$. In addition to these conditions we have to assume some conditions such as some local smoothness conditions on B(r) and C(r) etc. Under these assumptions we can develope an eigenfunction expansion theory (or, to be more exact, an eigenoperator expansion theory) for L. This is an extension of Jäger [2] and Saitō [3], [4], where the case that the long-range term $C_o(r)$ is identically zero has been treated.

§2. The limiting absorption principle. Now let us state the limiting absorption principle for L which is our main tool.

Theorem 1 (limiting absorption principle). Let $k\in\mathbb{C}_+ = \{k\in\mathbb{C} \ / \ \mathrm{Im}\ k \geqq 0,\ \mathrm{Re}\ k \neq 0\}$ and let f(r) be an X-valued function on I with $f\in L_2(I,\ X,\ (1+r)^{2\delta}dr)$, where δ is a constant such that $1/2<\delta<(2+\varepsilon)/4$. Then there exists a unique solution $v = v(k,\ f)$ of the equation

$$(*)\qquad \begin{cases} (L - k^2)v = f, \\ v \in L_2(I,\ X,\ (1+r)^{-2\delta}dr), \\ v' - ikv \in L_2(I,\ X,\ (1+r)^{2\delta-2}dr)\ (\text{"radiation condition"}), \\ v(0) = 0. \end{cases}$$

The mapping $(k,\ f) \longmapsto v(k,\ f)$ is continuous as the mapping from $\mathbf{C}_+ \times L_2(I,\ X,\ (1+r)^{2\delta}dr)$ into $L_2(I,\ X,\ (1+r)^{-2\delta}dr)$.

We shall make some remarks on this theorem. Let Φ be all (X-valued) test functions on I, that is, the element ϕ of Φ is an X-valued, sufficiently smooth function on I with compact support in I. Since $L/_{\Phi}$, the restriction of L onto Φ, is a symmetric and lower semi-bounded operator in $\mathfrak{h} = L_2(I,\ X,\ dr)$, the Friedrichs extension T of $L/_{\Phi}$ is well-defined. T can be considered as a self-adjoint realization of L in $\mathfrak{h}$. Let $z = k^2 \pm i\alpha$, where $\alpha > 0$ and k is a real number with $k \neq 0$. Then we have

$$v(\sqrt{k^2 \pm i\alpha},\ f) = R(z)f, \qquad\qquad R(z) = (T - z)^{-1}.$$

The limiting absorption principle guarantees that $R(z)f$ can be continuously extended to the solution $v(\pm k,\ f)$ of the equation (*) as α tends to 0, i.e., we have

$$R(z)f \longrightarrow v(\pm k,\ f) \quad (\alpha \downarrow 0) \quad \text{in } L_2(I,\ X,\ (1+r)^{-2\delta}dr).$$

Let us make one more remark. Let $\mathcal{B}$ be a Hilbert space obtained by completion of Φ by the norm

$$\| u \|_{\mathcal{B}} = \Big[\int_I \{ |u'(r)|^2 + |B^{1/2}(r)u(r)|^2 + |u(r)|^2 \} dr \Big]^{1/2},$$

the norm $|\ \ |$ being the norm of X, and let ℓ belong to the conjugate space $\mathcal{B}^*$ of $\mathcal{B}$, i.e., ℓ is a continuous linear functional on $\mathcal{B}$,

$$\ell : \mathcal{B} \ni u \longmapsto \langle \ell,\ u \rangle \in \mathbf{C}.$$

Then we can replace the inhomogeneous term $f(r)$ in the equation (*) by $\ell \in \mathcal{B}^*$. In this case the first relation of (*) should be rewritten in a weak form

$$(v,\ (L - \bar{k}^2)\phi)_0 = \langle \ell,\ \phi \rangle \qquad\qquad (\phi \in \Phi),$$

where $(\ ,\ )_0$ means the inner product of $\mathfrak{h} = L_2(I,\ X,\ dr)$. It can be shown that this new equation has a unique solution $v = v(k,\ \ell)$, too, if

$$|\ell|_\delta = \sup\{|\langle \ell, (1+r)^\delta \phi\rangle| \,/\, \|\phi\|_{\mathcal{B}} = 1\} < \infty.$$

Thus we have obtained an extension of Theorem 1.

Here and in the sequel let us assume that $\varepsilon > 1/2$ in the conditions (C_0) and (C_1). Then, using Theorem 1, we can show

Theorem 2 (the asymptotic behavior of $v(k, f)$). Let k be a real number with $k \neq 0$, $f(r) \in L_2(I, X, (1+r)^2 dr)$ and let $v = v(k, f)$ be a unique solution of the equation (*). Then the strong limit

$$\operatorname*{s-lim}_{n\to\infty} e^{-i\mu(r_n, k)} v(r_n)$$

exists in X, where

$$\mu(r, k) = kr - (2k)^{-1} \int_0^r C_0(t)\, dt,$$

and $\{r_n\}$ is a sequence such that $r_n \uparrow \infty$ and $v'(r_n) - ikv(r_n) \longrightarrow 0$ in X as $n \longrightarrow \infty$. The limit is independent of the choice of the sequence $\{r_n\}$.

§3. Eigenfunction expansion. Let us assume that $\varepsilon > 1/2$ in (C_0) and (C_1). Set $\bar{I} = [0, \infty)$ and let $\mathbb{C}_+$ be as in Theorem 1. First we shall define the Green kernel $G(r, s, k)$ $(r, s \in \bar{I},\ k \in \mathbb{C}_+)$ for the operator L. $G(r, s, k)$ is a bounded linear operator on X such that $v = G(\cdot, s, k)x$ is the solution of the equation (*) with $f(r)$ replaced by $\ell[s, x] \in \mathcal{B}^*$ for all $x \in X$, where $\ell[s, x]$ is a continuous linear functional over $\mathcal{B}$ defined by

$$\langle \ell[s, x], \phi \rangle = (x, \phi(s)) \qquad (\phi \in \mathcal{B}),$$

(,) being the inner product of X. The existence of $G(r, s, k)$ follows from the second remark after Theorem 1. The Green kernel $G(r, s, k)$ can be easily seen to be the resolvent kernel of the self-adjoint realization T of L, which was defined in §2, i.e., we have

$$R(z)f(r) = \int_I G(r, s, \sqrt{z}) f(s)\, ds \qquad (z \in \mathbb{C} - \mathbb{R},\ R(z) = (T - z)^{-1}).$$

Therefore, denoting by $E(\cdot)$ the spectral measure associated with T, we obtain

$$E((a,\ b))f(r) = \int_a^b d\lambda \int_I \{G(r,\ s,\ \sqrt{\lambda}) - G(r,\ s,\ -\sqrt{\lambda})\} f(s)\ ds$$

$$(0 < a < b < \infty).$$

On the other hand by the use of Theorem 2 it can be shown that there exists the strong limit

$$\eta(r,\ k) = \underset{s\to\infty}{\text{s-lim}}\ e^{-i\mu(s,\ k)} G(r,\ s,\ k) \qquad (k \in \mathbb{R} - \{0\}).$$

Here $\mu(s,\ k)$ is as in Theorem 2. For each pair $(r,\ k) \in \bar{I} \times (R - \{0\})$ $\eta(r,\ k)$ is a bounded linear operator on X and we have

$$\eta^*(0,\ k) = 0, \qquad (L - k^2)\eta^*(\cdot,\ k)x = 0 \quad (x \in X),$$

where $\eta^*(r,\ k)$ is the adjoint of $\eta(r,\ k)$. $\eta(r,\ k)$ is called the eigen-operator for L. It follows from the Green formula that

$$(\{G(r,\ s,\ k) - G(r,\ s,\ -k)\}x,\ y) = 2ik(\eta(s,\ k)x,\ \eta(r,\ k)y)$$

$$(x, y \in X,\ k \in \mathbb{R} - \{0\},\ r, s \in \bar{I}).$$

These relations are combined to give

Theorem 3 (expansion theorem). The generalized Fourier transforms $\mathcal{F}_\pm$ from $\mathfrak{h} = L_2(I,\ X,\ dr)$ into $\hat{\mathfrak{h}} = L_2((0,\ \infty),\ X,\ dk)$ are well-defined by

$$(\mathcal{F}_\pm f)(k) = \underset{N\to\infty}{\text{l.i.m.}} \int_0^N \eta_\pm(r,\ k) f(r)\ dr \qquad \text{in } \hat{\mathfrak{h}},$$

where

$$\eta_\pm(r,\ k) = \pm\sqrt{\frac{2}{\pi}}\, ik\, \eta(r,\ \pm k) \qquad (k > 0).$$

For any Borel set $B \subset (0,\ \infty)$ we have

$$\mathcal{F}_\pm^* \chi_{\sqrt{B}} \mathcal{F}_\pm = E(B),$$

$\mathcal{F}_\pm^*$ being the adjoints of $\mathcal{F}_\pm$, respectively, and $\chi_{\sqrt{B}}$ being the characteristic function of $\sqrt{B} = \{k > 0\ /\ k^2 \in B\}$.

Further, we can show

Theorem 4. The generalized transforms $\mathcal{F}_\pm$ are orthogonal, or equivalently, $\mathcal{F}_\pm$ transform $\mathfrak{h}$ onto $\hat{\mathfrak{h}}$.

§4. The Schrödinger operator in R^n. As has been remarked in §1, the Schrödinger operator $-\Delta + Q(y)$ in R^n is converted into the form L by the unitary operator $U = r^{(n-1)/2}$. Let us assume that the potential Q(y) the following condition (Q):

(Q) Q(y) is a real-valued function on R^n $(n \neq 2)$. Q(y) can be decomposed as $Q(y) = Q_o(y) + Q_1(y)$ with real-valued functions $Q_o(y)$ and $Q_1(y)$ on R^n such that

$$\begin{cases} Q_o(y) = O(|y|^{-(1/2)-\alpha}), \\ \dfrac{\partial Q_o}{\partial y_j} = O(|y|^{-(3/2)-\alpha}), \\ \dfrac{\partial^2 Q_o}{\partial y_j \partial y_m} = O(|y|^{-2-\alpha}), \\ Q_1(y) = O(|y|^{-(3/2)-\alpha}) \end{cases}$$

$$(|y| \longrightarrow \infty, \quad j,m = 1,2,\cdots,n)$$

with $\alpha > 0$.

Then all the results obtained in §2~§3 can be applied to the operator $-\Delta + Q(y)$. As is well-known the restriction of $-\Delta + Q(y)$ onto C_o^∞ is essentially self-adjoint in $L_2(R^n, dy)$ with a unique self-adjoint extension H. The spectral measure associated with H will be denoted by $\tilde{E}(\cdot)$.

Theorem 5. There exist the eigenoperators $\tilde{\eta}_\pm(|y|, |\xi|)$, $y, \xi \in R^n$, which are bounded linear operators on $L_2(S^{n-1})$, and the generalized Fourier transforms $\tilde{\mathcal{F}}_\pm$ from $L_2(R^n, dy)$ onto $L_2(R^n, d\xi)$ are well-defined by

$$(\tilde{\mathcal{F}}_\pm F)(\xi) = \underset{N\to\infty}{\text{l.i.m.}} \int_{r<N} (\tilde{\eta}_\pm(r, |\xi|)F(r\cdot))(\omega')\, r^{n-1}\, dr$$

$$(F \in L_2(R^n, dy),\ \omega' = \xi/|\xi|)$$

in $L_2(R^n, d\xi)$. For any Borel set $B \subset (0, \infty)$ we have

$$\tilde{E}(B) = \tilde{\mathcal{F}}_\pm^* \chi_{\sqrt{B}} \tilde{\mathcal{F}}_\pm \,.$$

Let us consider the case of $Q(y) = 0$, i.e., the case of the Laplacian. In this case our generalized Fourier transforms become equal to the ordinary Fourier transforms in $L_2(R^n)$. Therefore the ordinary Fourier transforms are a special case of our generalized Fourier transforms.

§5. Concluding remarks. Ikebe [1] has shown that the Schrödinger operator $-\Delta + Q(y)$ can be treated directly by the use of essentially the same ideas as ours. In his treatment the condition $n \neq 2$ is unnecessary.

As for the short-range term $C_1(r)$ the condition that $\varepsilon > 1/2$ can be replaced by a weaker condition that $\varepsilon > 0$. But, since Theorem 2 does not seem to be valid in this case, we need some new devices such as the ones used in Saitō [4]. Moreover $C_1(r)$ may admit some local singularities in the interval I.

For the proof of the theorems given in this report see Saitō [5].

References

[1] T. Ikebe: Spectral representations for the Schrödinger operators with long-range potentials, I, II, Preprint, 1974.

[2] W. Jäger: Ein gewöhnlicher Differentialoperator zweiter Ordnung fur Funktionen mit Werten in einem Hilbertraum, Math. Z. 113 (1970), 68-98.

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