Effective potentials in geodesic curves on surfaces

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Abstract. In this work, the equations of geodesic curves on surfaces embedded in euclidean space are obtained. By introducing a vector Lagrange multiplier, we show that the geodesic curvature of the curves are zero and the normal curvature of them can be identified with the force transmitted to the surface. We then obtain the corresponding formulas in the case of axially symmetric surfaces, where a first integral of the geodesic equations can be interpreted as a particle moving in an effective potential (being zero the total energy), and the angular momenta is conserved. The methodology developed is illustrated with some examples: the catenoid and the pseudosphere.

1. Introduction

In geometry, a geodesic curve is defined as the curve of minimum length (and less curvature) joining two given points on a given surface. Geodesics are one of the families of curves often called special lines or curves. The term comes from the word geodesy, science to measure the size and shape of the Earth. The formal study of the geodesic starts the year 1691, when John Bernoulli proposed the brachistochrone problem [1]. In 1731 A.C. Clairaut published a treatise on curves of double curvature, *Recherches sur les courbes a double curvature*, which allowed his admission to the French Academy of Sciences [2]; subsequently, C.F. Gauss (1777-1855) emphasizes the geometric nature of the mathematical analysis, with mainly two contributions: the birth of complex analysis and the differential geometry analysis [2]. In 1866 Jean Gaston Darboux in his doctoral thesis began a prolific development on geometry, some results published in two articles in the journal *Annales de l'Ecole Normale Suprieure de Paris*, Darboux was the first to characterize a curve on a surface (given its curvature and torsion) by the differential equation, this was done around 1871. Particularly a surface of revolution is a surface which is generated from the rotation of a planar curve around an axis which lies in the same plane, and is a Liouville surface.

Although the analysis of geodesic curves appear in any book of geometry, most of the analysis is done intrinsically, especially as an application of variational calculus [3]. One of the main applications of geodesic analysis is given in physics where free particles follow geodesic trajectories [4]. Recently it has been shown that, in the interaction of crystalline order on curved surfaces, there is a non-local effect that forces normal lines to be on geodesic curves [5]. If one includes the bending energy, the equilibrium configurations of elastic polymers on surfaces as spheres can be found [6]. Indeed it is even possible to include both long range forces between the curve itself, as well as between the curve and the surface [7].

2. Geodesic curves confined on surfaces

Let $\mathbf{x} = (x^1, x^2, x^3)$, be a point $\in \mathbb{R}^3$, and $\mathbf{x} = \mathbf{Y}(s)$, a curve parametrized by s, restricted to be along the surface $\mathbf{x} = \mathbf{X}(\xi^a)$, parametrized by local coordinates $\xi^a, a = 1, 2$. To take into account this requirement, we follow [6] and introduce a vector Lagrange multiplier $\boldsymbol{\lambda}$, so that the length functional is given by

$$L(\mathbf{Y}, \xi^a) = \int ds + \int \boldsymbol{\lambda} \cdot [\mathbf{Y}(s) - \mathbf{X}(\xi^a)] \, ds, \tag{1}$$

where the dot denotes the internal product in $\mathbb{R}^3$. In order to find the Euler-Lagrange equations we have to consider deformations δL as a consequence of deformations of the variables $\delta \mathbf{Y}$ and $\delta \xi^a$. We obtain

$$\delta L(\mathbf{Y}, \xi^a) = \int \frac{d}{ds}(\delta \mathbf{Y}) \cdot \mathbf{T} ds + \int \boldsymbol{\lambda} \cdot \delta \mathbf{Y}, \tag{2}$$

where $\mathbf{T} = d\mathbf{Y}/ds = \dot{\mathbf{Y}}$ is the tangent vector field to the curve. By doing one integration by parts, we have that $\delta L = 0$, implies that,

$$\dot{\mathbf{T}} = \boldsymbol{\lambda}. \tag{3}$$

Variation respect to the variable ξ^a gives $\delta L = -\int \boldsymbol{\lambda} \cdot \mathbf{e}_a \delta \xi^a$, where $\mathbf{e}_a = \partial_a \mathbf{X}$, are the two tangent vectors to the surface. We thus see that $\delta L = 0$ implies that $\boldsymbol{\lambda} = -\lambda \mathbf{n}$, being $\mathbf{n}$ the unit normal to the surface. Projection of $\dot{\mathbf{T}}$ along the Darboux basis $\{\mathbf{T}, \mathbf{n}, \mathbf{l} = \mathbf{T} \times \mathbf{n}\}$, gives us

$$\begin{aligned}
\dot{\mathbf{T}} &= -K_{ab}t^a t^b \mathbf{n} + \left(\dot{t}^c + \Gamma^c_{ab} t^a t^b\right) \mathbf{e}_c, \\
&= -\kappa_n \mathbf{n} + \kappa_g \mathbf{l},
\end{aligned} \tag{4}$$

where the normal curvature $\kappa_n = K_{ab}t^a t^b$ with $t^a = d\xi^a/ds$, and the geodesic curvature $\kappa_g = \left(\dot{t}^c + \Gamma^c_{ab} t^a t^b\right) l_c$ has been introduced. One conclude from equation (3) that $\kappa_g = 0$ for geodesic curves, as it should be. We also see that $\lambda = \kappa_n$, can be interpreted as the force transmitted to the surface. Some examples about this curves and forces are given below.

3. Surfaces of revolution

A surface of revolution can be described parametrically through

$$\mathbf{X}(\varphi, v) = (\rho(v) \cos \varphi, \rho(v) \sin \varphi, z(v)). \tag{5}$$

The infinitesimal element of distance along the surface can be written as $ds^2 = \rho^2 d\varphi^2 + (\rho'^2 + z'^2)dv^2$, where $-$ ' $-$ indicates derivative respect to v. The two tangent vectors $\mathbf{e}_a$ are given by

$$\begin{aligned}
\mathbf{e}_\varphi &= (-\rho \sin \varphi, \rho \cos \varphi, 0), \\
\mathbf{e}_v &= (\rho' \cos \varphi, \rho' \sin \varphi, z'),
\end{aligned}$$

whereas the normal unit vector to the surface can be written in the form,

$$\mathbf{n} = \frac{1}{\sqrt{\rho'^2 + z'^2}} (z' \cos \varphi, z' \sin \varphi, -\rho'). \tag{6}$$

The second fundamental form K_{ab} of the surface can be calculated through its definition in the Gauss equation, $\partial_a \mathbf{e}_b = -K_{ab}\mathbf{n}$. As a consequence of the cylindrical symmetry a diagonal matrix is found for the extrinsic curvature, being $K_{\varphi\varphi} = \frac{\rho z'}{\sqrt{\rho'^2 + z'^2}}$ and $K_{vv} = \frac{\rho' z'' - \rho'' z'}{\sqrt{\rho'^2 + z'^2}}$, its nonzero

elements. Being also diagonal the matrix K_a^b, their components are the principal curvatures of the surface, $K_\varphi^\varphi = \frac{z'}{\rho\sqrt{\rho'^2+z'^2}}$ and $K_v^v = \frac{K_{vv}}{\rho'^2+z'^2}$. The trace of the second fundamental form $K = g^{ab}K_{ab}$ is given by

$$K = \frac{1}{\sqrt{\rho'^2 + z'^2}} \left(\frac{z'}{\rho} + \frac{\rho'z'' - \rho''z'}{\rho'^2 + z'^2} \right). \tag{7}$$

The curvature scalar $\mathcal{R}$ can be obtained intrinsically using the induced metric g_{ab} through the Christoffel symbols. Nevertheless, we can calculate it, through the integrability condition given by the Codazzi identity identity $\mathcal{R} = K^2 - K^{ab}K_{ab}$, so we have

$$\mathcal{R} = \frac{2z'}{\rho} \frac{\rho'z'' - \rho''z'}{(\rho'^2 + z'^2)^2}. \tag{8}$$

In addition, because the symmetry under rotations around the z axis, $\ell = \mathbf{L} \cdot \mathbf{k}$ is conserved along the curve, where $\mathbf{L} = \mathbf{x} \times \dot{\mathbf{x}}$, is the angular momenta. We find

$$\ell = \rho^2 \frac{d\varphi}{ds}. \tag{9}$$

With these elements, we can write the normal curvature along the trajectory of the particle as follows

$$\kappa_n = K_{\varphi\varphi} \left(\frac{d\varphi}{ds} \right)^2 + K_{vv} \left(\frac{dv}{ds} \right)^2. \tag{10}$$

Note also that, using the infinitesimal element of distance we can get a first integral of the geodesic equation

$$\frac{1}{2} \left(\frac{dv}{ds} \right)^2 + U_{eff} = 0, \tag{11}$$

where

$$U_{eff} = -\frac{1}{2(\rho'^2 + z'^2)} \left(1 - \frac{\ell^2}{\rho^2} \right), \tag{12}$$

can be interpreted as an effective potential that depends on the coordinates and on the conserved angular momentum explicitly. Then we can write the normal curvature along geodesic curves as

$$\kappa_n = \frac{\ell^2}{\rho^3} \frac{z'}{\sqrt{\rho'^2 + z'^2}} + \frac{\rho'z'' - \rho''z'}{(\rho'^2 + z'^2)^{3/2}} \left(1 - \frac{\ell^2}{\rho^2} \right), \tag{13}$$

thus, the force λ is completely determined.

3.1. Catenoid
In this case we have $\mathbf{X}(\varphi, v) = (\cosh v \cos \varphi, \cosh v \sin \varphi, v)$. Then we identify $\rho(v) = \cosh v$ and $z(v) = v$. We have $K_{\varphi\varphi} = 1$ and $K_{vv} = -1$, and the trace $K = 0$ (a minimal surface), its curvature $\mathcal{R} = -2 \operatorname{sech}^4 v < 0$. The effective potential of the fictitious particle (of total energy zero) is given by

$$U_{eff}(v) = -\frac{1}{2} \operatorname{sech}^2 v \left(1 - \ell^2 \operatorname{sech}^2 v \right). \tag{14}$$

Figure (1) shows the behavior of geodesic curves: For $0 < \ell < 1$, geodesics crosses parallels along the catenoid; for $\ell > 1$, the potential barrier does not allow the geodesic going from one side to another of the catenoid. The equator corresponds to the geodesic with $\ell = 1$, the geodesic with $\ell = 0$ is a meridian. The transmitted force to the surface in this case reads as follows

$$\lambda_\ell(v) = \operatorname{sech}^2 v \left(2\ell^2 \operatorname{sech}^2 v - 1 \right). \tag{15}$$

In Fig. (2), the force λ, transmitted to the catenoid is shown. For curves crossing the catenoid, such that $\ell \geq 1/\sqrt{2}$, there exist v_0 where the force transmitted to the surface is zero, $\lambda_\ell(v_0) = 0$.
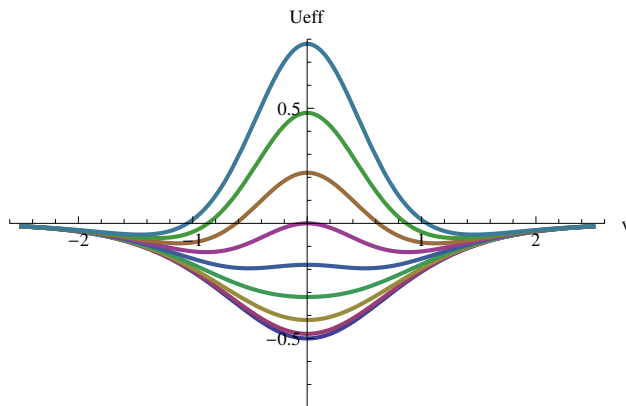
Figure 1. The effective potential (14) of the catenoid, for several values of the angular momenta ℓ: the bottom curve correspond to $\ell = 0$. After reaching $\ell = 1$, a barrier appears.
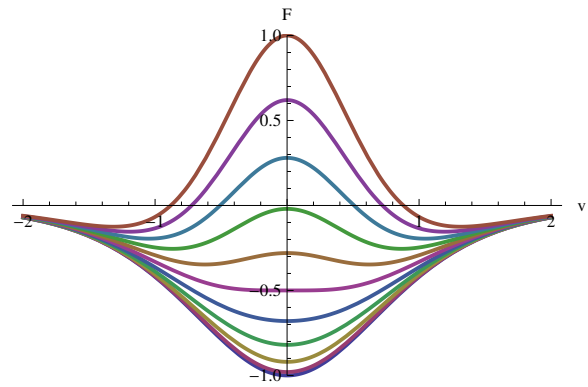
Figure 2. The transmitted force to the catenoid for several values of the angular momenta ℓ: the bottom curve correspond to $\ell = 0$, the upper one to $\ell = 1$.

3.2. Pseudosphere

A parametrization of the pseudosphere is given by $\mathbf{X}(\varphi, v) = (\mathrm{sech}\, v \cos \varphi, \mathrm{sech}\, v \sin \varphi, v - \tanh v)$. We identify then $\rho(v) = \mathrm{sech}\, v$, and $z(v) = v - \tanh v$. The non-zero components of the second fundamental are given by $K_{\varphi\varphi} = \mathrm{sech}\, v \tanh v$ and $K_{vv} = -K_{\varphi\varphi}$. The trace is given by

$$K = \frac{(\cosh 2v - 3)}{2 \sinh v}. \tag{16}$$

As it is well known, the pseudosphere is a surface of negative curvature with $\mathcal{R} = -2$. The effective potential is now given by

$$U_{eff}(v) = -\frac{1}{2 \tanh^2 v} \left(1 - \frac{\ell^2}{\mathrm{sech}^2 v} \right). \tag{17}$$

Figure (3) shows the behavior of pseudosphere's geodesic curves: For $0 < \ell < 1$, geodesics remain on one side of the surface; for $\ell \geq 1$, the potential barrier does not allow the geodesic to exist. Meridians are geodesic curves ($\varphi = \mathrm{const.}$) along that $\ell = 0$.

The force transmitted to the surface can be written in the form

$$\lambda_\ell(v) = \frac{\ell^2 \cosh^3 v - \mathrm{sech}\, v}{\tanh v}. \tag{18}$$

4. Conclusions

By introducing a vector Lagrange multiplier into the length functional, we obtain the geodesic curves on surfaces embedded in euclidian space, through the obtention of the Euler-Lagrange equations. We found that the geodesic curvature vanishes for geodesic curves and the normal curvature is directly related with the transmitted force to the surface. We studied two axially symmetric surfaces, namely, the catenoid and the pseudosphere, where a first integral of the geodesic equation can be thought as a particle (with vanishing energy) moving in an effective potential U_{eff}. In such a cases the force or the normal curvature are completely determined and turn out to be a function of the conserved angular momenta.
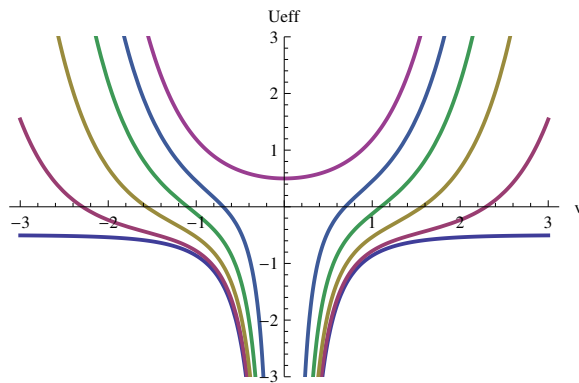
Figure 3. The effective potential (14) of the pseudosphere for several values of the angular momenta ℓ: the bottom curve corresponds to $\ell = 0$. The upper one to $\ell = 1$.
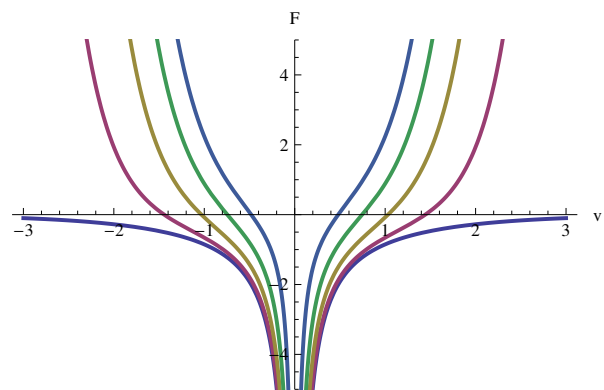

Figure 4. The transmitted force to the catenoid for several values of the angular momenta ℓ, the bottom curve corresponds to $\ell = 0$.

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