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Exact solution of the Dirac equation in the presence of a gravitational instanton

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Abstract. We solve the massless Dirac equation in the presence of a Bianchi VII_0 instanton metric. In order to separate variables we apply the algebraic method of separation. We express the curved Dirac gamma matrices in a rotating tetrad and compute the corresponding spinor connections. The resulting system of equations is completely decoupled after introducing two new complex space variables. Applying a pairwise separation scheme we also succeed in separating variables in the massless Dirac equation coupled to a Eguchi-Hanson instanton metric. The spinor solutions are expressed in terms of weighted spherical spinors.

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1. Introduction

Instantons are finite-action solutions of the classical Yang-Mills equations which are localized in imaginary time. They provide the dominant contribution to the path integral in the quantization of the Yang-Mills fields. [1] It is expected that gravitational instantons should play a similar role in the path integral approach to quantum gravity. The discovery of self-dual instanton solutions to the Euclidean Yang-Mills theory suggests the possibility that analogous solutions to the Einstein Equations might be important in quantum gravity. The euclidean Taub-NUT metric is involved in many problems in theoretical Physics. The Kaluza-Klein monopole of Gross and Perry[2] was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional Kaluza-Klein theory. The Hawking's [3] suggestion that the Euclidean Taub-NUT metric might give rise to the gravitational analogue of the YangMills instanton holds true on anisotropic spaces but in this case both the metric and instanton have some anisotropically renormalized parameters being of higher dimension gravitational vacuum polarization origin. The anisotropic Euclidean Taub-NUT metric also satisfies the vacuum Einstein's equations with zero cosmological constant when the spherical symmetry is deformed, for instance, into ellipsoidal or even toroidal configuration. Such anisotropic Taub-NUT metrics can be used for generation of deformations of the space part of the line element defining an anisotropic modification of the Kaluza-Klein monopole solutions proposed by Gross and Perry [2] and Sorkin [4]. The principal class of physically interesting gravitational instantons consist of asymptotically locally Euclidean metrics with self-dual curvature. The Eguchi Hanson instanton [5] belongs to this class. Gravitational instantons which are described by hyper-Käller metrics have been studied in the framework of supergravity, M-theory as well as Seiberg-Witten theory. [6, 7] Gravitational instantons have Euclidean signature metric and self-dual curvature, which implies that they satisfy the vacuum Einstein equations. In order to compute the vacuum expectation value of the stress tensor of a gravitational instanton it is of help to compute the Green's function for massless fields, in particular for the Dirac equation. The Dirac equation plays a crucial role in case of instantons since the zero eingenvalues of the Dirac equation are related to the topological properties of the solution through the Atiyah-Singer theorem.[8] Previous studies of the Dirac equation in the background of gravitational instantons [9] have been made using the Newman-Penrose formalism [10]. In this article we solve the Dirac equation using the algebraic method of separation of variables [11, 12, 13], which has been successfully applied in the study of Dirac particles in gravitational fields.

2. The Bianchi VII_0 instanton metric

The covariant generalization of the massless Dirac equation in curved space has the form [14]

$$\gamma^\mu \nabla_\mu \Psi = \gamma^\mu (\frac{\partial}{\partial x^\mu} - \Gamma_\mu)\Psi = 0 \tag{1}$$

where the curved matrices satisfy the commutation relations

$$\{\gamma^\mu(x), \gamma^\nu(x)\}_+ = 2g^{\mu\nu},$$

and $\gamma^\mu(x)$ are related to the flat gamma matrices $\gamma^{(a)}$ via the tetrad $e^\mu_{(a)}(x)$ as follows

$$\gamma^\mu(x) = e^\mu_{(a)}(x)\gamma^{(a)}$$

The tetrad $e^\mu_{(a)}$ satisfies the relation

$$\eta_{(a)(b)} = g_{\mu\nu} e^\mu_{(a)} e^\nu_{(b)}$$

where $\eta_{(a)(b)}$ and $g_{\mu\nu}$ are the flat and curved metrics respectively.

The Bianchi VII_0 instanton metric is [5]

$$ds^2 = \frac{1}{2}a^2 \sinh 2x (dx^2 + d\theta^2) + \frac{2}{\sinh 2x}\left[(\sinh^2 x + \sin^2\theta)dy^2 - \sin 2\theta dy dz + (\sinh^2 x + \cos^2\theta)dz^2\right]$$

Choosing the curved gamma matrices

$$\gamma^1 = \frac{\sqrt{2}}{a(\sinh 2x)^{1/2}}\tilde{\gamma}^1, \ \gamma^2 = -\frac{\sqrt{2}\sinh x \sin\theta}{(\sinh 2x)^{1/2}}\tilde{\gamma}^2 + \frac{\sqrt{2}\cosh x \cos\theta}{(\sinh 2x)^{1/2}}\tilde{\gamma}^3$$

$$\gamma^3 = \frac{\sqrt{2}\sinh x \cos\theta}{(\sinh 2x)^{1/2}}\tilde{\gamma}^2 + \frac{\sqrt{2}\cosh x \sin\theta}{(\sinh 2x)^{1/2}}\tilde{\gamma}^3, \ \gamma^0 = \frac{\sqrt{2}}{a(\sinh 2x)^{1/2}}\tilde{\gamma}^0$$

we obtain that the spinor connections defined by the relation

$$\Gamma_\mu = \frac{1}{4}\gamma_\nu \nabla_\mu \gamma^\nu$$

are

$$\Gamma_1 = 0, \ \Gamma_4 = 0,$$

$$\Gamma_2 = -\frac{1}{2}\frac{\sin\theta}{\sinh 2x \sinh x}\tilde{\gamma}^3\tilde{\gamma}^4 + \frac{1}{2}\frac{\cos\theta}{\sinh 2x \cosh x}\tilde{\gamma}^2\tilde{\gamma}^4 + \frac{\cos\theta \sinh x}{a \sinh^2 2x}\tilde{\gamma}^1\tilde{\gamma}^3 + \frac{\cosh x \sin\theta}{a\sinh^2 2x}\tilde{\gamma}^1\tilde{\gamma}^2$$

$$\Gamma_3 = \frac{1}{2}\frac{\sin\theta}{\sinh 2x \cosh x}\tilde{\gamma}^2\tilde{\gamma}^4 + \frac{1}{2}\frac{\cos\theta}{\sinh 2x \sinh x}\tilde{\gamma}^3\tilde{\gamma}^4 - \frac{\cos\theta \cosh x}{\sinh^2 2x}\tilde{\gamma}^1\tilde{\gamma}^2 + \frac{\sin\theta \sinh x}{a\sinh^2 2x}\tilde{\gamma}^1\tilde{\gamma}^3$$

Substituting the gamma matrices and the spin connections into the Dirac equation (1) we obtain two equivalent sets of coupled differential equations corresponding to particle and antiparticle states. Since we are dealing with the massless case, our problem reduces to the following system of equations:

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial \theta}+\frac{1}{2}\right)\psi_2=a\left[\cosh(x-i\theta)\frac{\partial}{\partial y}+i\sinh(x-i\theta)\frac{\partial}{\partial z}\right]\psi_1$$

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial \theta}+\frac{1}{2}\right)\psi_1=-a\left[i\sinh(x+i\theta)\frac{\partial}{\partial y}+\cosh(x+i\theta)\frac{\partial}{\partial z}\right]\psi_2$$

Introducing the transformation [9]

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\exp(-\frac{x}{2}+i(ny+mz))\begin{pmatrix}h_1\\ h_2\end{pmatrix}$$

and the angle α in terms of of n and m

$$\frac{n}{\sqrt{n^2+m^2}}=\cos\alpha,\ \ \frac{m}{\sqrt{n^2+m^2}}=\sin\alpha$$

and defining the complex variables ν and μ as

$$\nu=x+i(\theta-\alpha),\ \ \mu=x-i(\theta-\alpha)$$

we obtain the system of equations

$$\frac{\partial h_1}{\partial v}-i\frac{a}{2}\sqrt{n^2+m^2}\cosh\nu\ h_2, \tag{2}$$

$$\frac{\partial h_2}{\partial u}+i\frac{a}{2}\sqrt{n^2+m^2}\cosh u\ h_1. \tag{3}$$

Introducing the transformation $\rho=\sinh u$, the system of equations (2)-(3) becomes

$$\frac{\partial h_1}{\partial \rho^*}-i\frac{a}{2}\sqrt{n^2+m^2}h_2=0 \tag{4}$$

$$\frac{\partial h_1}{\partial \rho}+i\frac{a}{2}\sqrt{n^2+m^2}h_2=0 \tag{5}$$

whose solution are plane waves in the complex coordinates,

$$\begin{pmatrix}h_1\\ h_2\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}\exp(i\frac{a}{2}\sqrt{n^2+m^2}(\rho-\rho^*)),\ \begin{pmatrix}1\\ -1\end{pmatrix}\exp(i\frac{a}{2}\sqrt{n^2+m^2}(\rho-\rho^*)) \tag{6}$$

3. The Eguchi-Hanson instanton metric

The Eguchi-Hanson instanton metric is the solution of of Einstein equations with self-dual curvature. The metric possesses a topology $R\times S^3$ and its line element is given by the expression [5]

$$ds^2=\frac{1}{1-(\frac{a}{r})^4}dr^2+r^2(\sigma_x^2+\sigma_y^2)+r^2\left(1-(\frac{a}{r})^4\right)\sigma_z^2 \tag{7}$$

The differential forms are expressed in terms of the Euler angles θ, ϕ, and ψ as

$$\sigma_x=\frac{1}{2}(-\cos\psi d\theta-\sin\theta\sin\psi d\phi)$$

$$\sigma_y = \frac{1}{2}(\sin\psi d\theta - \sin\theta\cos\psi d\phi)$$

$$\sigma_z = \frac{1}{2}(-d\psi - \cos\theta d\phi)$$

In order to separate variables in the massless Dirac equation we choose the rotating tetrad

$$\gamma^1 = \sqrt{1-\left(\frac{a}{r}\right)^4}\tilde{\gamma}^1,\ \gamma^2 = \frac{2}{r}\tilde{\gamma}^2,\ \gamma^3 = \frac{2}{r\sin\theta}\tilde{\gamma}^3,\ \gamma^4 = -\frac{2\cot\theta}{r}\tilde{\gamma}^3 + \frac{2}{r\sqrt{1-(\frac{a}{r})^4}}\tilde{\gamma}^4 \qquad (8)$$

Substituting the tetrad (8) into the massless Dirac equation we obtain the expression

$$\frac{2\gamma^3}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \cot\theta\frac{\partial}{\partial\psi}\right]\Psi + \gamma^2\left(\frac{\cot\theta}{2} + \frac{\partial}{\partial\theta}\right)\Psi$$

$$+\gamma^1\left(\left[(1-(\frac{a}{r})^4\right]^{1/2}\frac{\partial}{\partial r} + \frac{3}{2}\frac{1}{r(1-(\frac{a}{r})^4} - \frac{1}{2}\frac{a^4}{r^5(1-(\frac{a}{r})^4}\right)\Psi$$

$$\gamma^2\gamma^3\gamma^4\left[\frac{1}{2}\frac{a^4}{r^5(1-(\frac{a}{r})^4)^{1/2}} - \frac{1}{2r(1-(\frac{a}{r})^4)^{1/2}}\right]\Psi + \frac{2\gamma^4}{r(1-(\frac{a}{r})^4)^{1/2}}\frac{\partial\Psi}{\partial\psi} = 0 \qquad (9)$$

From which we obtain the system of equations

$$\left(\sqrt{1-(\frac{a}{r})^4}\frac{\partial}{\partial r} - \frac{2}{r}\frac{m-\frac{1}{2}}{\sqrt{1-(\frac{a}{r})^4}}\right)\psi_1 + \frac{2}{r}\left[-\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + (m-\frac{1}{2})\cot\theta\right]\chi_1$$

$$\left(\sqrt{1-(\frac{a}{r})^4}\frac{\partial}{\partial r} + \frac{2}{r}\frac{m+\frac{1}{2}}{\sqrt{1-(\frac{a}{r})^4}}\right)\chi_1 + \frac{2}{r}\left[\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + (m+\frac{1}{2})\cot\theta\right]\psi_1$$

$$\left(\sqrt{1-(\frac{a}{r})^4}\frac{\partial}{\partial r} - \frac{2}{r}\frac{m+\frac{1}{2}}{\sqrt{1-(\frac{a}{r})^4}}\right)\psi_2 + \frac{2}{r}\left[-\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + (m-\frac{1}{2})\cot\theta\right]\chi_2$$

$$\left(\sqrt{1-(\frac{a}{r})^4}\frac{\partial}{\partial r} - \frac{2}{r}\frac{m-\frac{1}{2}}{\sqrt{1-(\frac{a}{r})^4}}\right)\chi_2 + \frac{2}{r}\left[\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + (m+\frac{1}{2})\cot\theta\right]\psi_2$$

where we have written the upper and lower components of the Dirac spinor as

$$\begin{pmatrix}\Theta_1\\ \Theta_2\end{pmatrix} = \begin{pmatrix} e^{i(m+\frac{1}{2})\psi}\psi_1(r,\theta,\phi)\\ e^{i(m-\frac{1}{2})\psi}\chi_1(r,\theta,\phi)\end{pmatrix} \qquad (10)$$

and

$$\begin{pmatrix}\Omega_1\\ \Omega_2\end{pmatrix} = \begin{pmatrix} e^{i(m+\frac{1}{2})\psi}\psi_2(r,\theta,\phi)\\ e^{i(m-\frac{1}{2})\psi}\chi_2(r,\theta,\phi)\end{pmatrix} \qquad (11)$$

introducing the change of variables

$$\left(\frac{r}{a}\right)^2 = \cosh x \qquad (12)$$

we obtain that the angular dependence of the Dirac spinor can be expressed in terms of the weighted spherical harmonics as

$$\begin{pmatrix} \psi_1 \\ \chi_1 \end{pmatrix} = e^{in\phi} \begin{pmatrix} (\sinh x)^{\left|m-\frac{1}{2}\right|} \, {}_2F_1(\alpha_+, \alpha_-, \gamma_+; -\sinh^2 x) d^j_{n,m+\frac{1}{2}}(\theta) \\ (\sinh x)^{\left|m+\frac{1}{2}\right|} \, {}_2F_1(\beta_+, \beta_-, \gamma_-; -\sinh^2 x) d^j_{n,m-\frac{1}{2}}(\theta) \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} \psi_2 \\ \chi_2 \end{pmatrix} = e^{in\phi} \begin{pmatrix} (\sinh x)^{\left|m+\frac{1}{2}\right|} \, {}_2F_1(\beta_+, \beta_-, \gamma_-; -\sinh^2 x) d^j_{n,m+\frac{1}{2}}(\theta) \\ (\sinh x)^{\left|m-\frac{1}{2}\right|} \, {}_2F_1(\alpha_+, \alpha_-, \gamma_+; -\sinh^2 x) d^j_{n,m-\frac{1}{2}}(\theta) \end{pmatrix} \tag{14}$$

where n is half-integer, m is integer, and

$$j = -\frac{1}{2} \pm \sqrt{\lambda^2 + \frac{1}{2} - m^2}$$

$$\alpha_\pm = \frac{1}{4} + \frac{1}{2}\left|m - \frac{1}{2}\right| \pm \frac{1}{2}\sqrt{\lambda^2 + m^2}, \ \beta_\pm = \frac{1}{4} + \frac{1}{2}\left|m + \frac{1}{2}\right| \pm \frac{1}{2}\sqrt{\lambda^2 + m^2}$$

$$\gamma_\pm = \left|m + \frac{1}{2}\right| + \frac{1}{2}$$

The irreducible representations of the rotation group are given by:

$$d^j_{n,m}(\theta) = \frac{(-1)^{j-m}}{n+m}\sqrt{\frac{(j+n)!(j+m)!}{(j-n)!(j-m)!}}(\cot\frac{\theta}{2})^{n+m}(\sin\frac{\theta}{2})^{2j} . {}_2F_1(n-j; n+m+1; -\cot^2\frac{\theta}{2}) \tag{15}$$

4. Concluding remarks

In this article we have solved the massless Dirac equation in the presence of a Bianchi VII_0 and the Eguchi-Hanson instanton metrics. Using rotating no-null tetrads we have completely separated variables in the massless Dirac equation in the two Euclidean instanton metrics. The Bianchi VII_0 instanton metric is a particular example of a metric where a complete separation of variables is not possible using the standard algebraic scheme. The introduction of complex variables permits one to decouple the resulting system of differential equations. The Dirac equation in the Eguchi-Hanson metric was completely separated using a pairwise scheme. In this case the separation procedure is analogous to the Schwarzschild case [14]. The results obtained in this article encourage us to continue working on a complete scheme of separability for the Dirac in Euclidean metrics.

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