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Carlos Castro Perelman

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On Born's Reciprocal Relativity, Algebraic Extensions of the Yang and Quaplectic Algebra, and Noncommutative Curved Phase Spaces

Carlos Castro Perelman

Ronin Institute, 127 Haddon Place, Montclair, NJ 07043, USA; perelmanc@hotmail.com

Abstract: After a brief introduction of Born's reciprocal relativity theory is presented, we review the construction of the *deformed* quaplectic group that is given by the semi-direct product of $U(1,3)$ with the *deformed* (noncommutative) Weyl–Heisenberg group corresponding to *noncommutative* fiber coordinates and momenta $[X_a, X_b] \neq 0$; $[P_a, P_b] \neq 0$. This construction leads to more general algebras given by a two-parameter family of deformations of the quaplectic algebra, and to further algebraic extensions involving antisymmetric tensor coordinates and momenta of higher ranks $[X_{a_1 a_2 \cdots a_n}, X_{b_1 b_2 \cdots b_n}] \neq 0$; $[P_{a_1 a_2 \cdots a_n}, P_{b_1 b_2 \cdots b_n}] \neq 0$. We continue by examining algebraic extensions of the Yang algebra in extended noncommutative phase spaces and compare them with the above extensions of the deformed quaplectic algebra. A solution is found for the exact analytical mapping of the noncommuting x^μ, p^μ operator variables (associated to an 8D curved phase space) to the canonical Y^A, Π^A operator variables of a flat 12D phase space. We explore the geometrical implications of this mapping which provides, in the *classical* limit, the embedding functions $Y^A(x,p), \Pi^A(x,p)$ of an 8D curved phase space into a flat 12D phase space background. The latter embedding functions determine the functional forms of the base spacetime metric $g_{\mu\nu}(x,p)$, the fiber metric of the vertical space $h^{ab}(x,p)$, and the nonlinear connection $N_{a\mu}(x,p)$ associated with the 8D cotangent space of the 4D spacetime. Consequently, we find a direct link between noncommutative curved phase spaces in lower dimensions and commutative flat phase spaces in higher dimensions.

Keywords: Born Reciprocal Relativity; Yang Algebra; Phase Spaces; Finsler Geometry

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1. Introduction: Born's Reciprocal Relativity Theory

Most of the work devoted to quantum gravity has been focused on the geometry of spacetime rather than phase space per se. The first indication that phase space should play a role in quantum gravity was raised by [1]. The principle behind Born's reciprocal relativity theory [2–5] was based on the idea proposed long ago by [1] that coordinates and momenta should be unified on the same footing. Consequently, if there is a limiting speed (temporal derivative of the position coordinates) in nature, there should be a maximal force as well, since force is the temporal derivative of the momentum. The principle of maximal acceleration was advocated earlier on by [6–9]. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) [5,10].

We explored in [5,10] some novel consequences of Born's reciprocal relativity theory in a flat phase space and generalized the theory to the curved spacetime scenario. We provided, in particular, some specific results from Born's reciprocal relativity which are *not* present in special relativity. These are: a momentum-dependent time delay in the emission and detection of photons; the relativity of chronology; an energy-dependent notion of locality; a superluminal behavior; the relative rotation of photon trajectories due to the aberration of light; the invariance of area cells in the phase space; and modified dispersion relations.

The generalized velocity and force (acceleration) boosts (rotations) transformations of the *flat* 8D phase-space coordinates, where $X^i, T, E, P^i; i = 1,2,3$ are **c**-valued (classical) variables which are *all* boosted (rotated) into each other, were given by [2–4] based on the group $U(1,3)$, which is the Born version of the Lorentz group $SO(1,3)$. The $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the symplectic two-form $\Omega = - dT \wedge dE + \delta_{ij} dX^i \wedge dP^j; i,j = 1,2,3$ and also the following Born–Green line interval in the $flat$ 8D phase space

$$(d\omega)^2 = c^2(dT)^2 - (dX)^2 - (dY)^2 - (dZ)^2 + \frac{1}{b^2}\left((dE)^2 - c^2(dP_x)^2 - c^2(dP_y)^2 - c^2(dP_z)^2\right) \tag{1}$$

The maximal proper force is set to be given by b. The rotations, velocity, and force (acceleration) boosts leaving invariant the symplectic two-form and the line interval in the 8D phase space are rather elaborate; see [2–4] for details.

These transformations can be simplified drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions Y, Z, P_y, P_z intact. There is now a subgroup $U(1,1) = SU(1,1) \times U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 = c^2(dT)^2 - (dX)^2 + \frac{(dE)^2 - c^2(dP)^2}{b^2} = (d\tau)^2\left(1 + \frac{(dE/d\tau)^2 - c^2(dP/d\tau)^2}{b^2}\right) = (d\tau)^2\left(1 - \frac{F^2}{F^2_{max}}\right), P = P_x \tag{2}$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = c^2 dT^2 - dX^2$ in (2). The proper force interval $(dE/d\tau)^2 - c^2(dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $c^2(dT/d\tau)^2 - (dX/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in Equation (2) involves the ratios of two proper *forces*.

One may set the maximal proper force acting on a fundamental particle of Planck mass to be given by $F_{max} = b \equiv m_P c^2 / L_P$, where m_P is the Planck mass and L_P is the postulated minimal Planck length. Invoking a minimal/maximal length duality, one can also set $b = M_U c^2 / R_H$, where R_H is the Hubble scale and M_U is the observable mass of the universe. Equating both expressions for b leads to $M_U / m_P = R_H / L_P \sim 10^{60}$. The value of b may also be interpreted as the maximal string tension.

The $U(1,1)$ group transformation laws of the phase-space coordinates X, T, P, E which leave the interval (2) invariant are [2–4]

$$T' = T \cosh \xi + \left(\frac{\xi_v X}{c^2} + \frac{\xi_a P}{b^2}\right)\frac{\sinh \xi}{\xi} \tag{3a}$$

$$E' = E \cosh \xi + (-\xi_a X + \xi_v P)\frac{\sinh \xi}{\xi} \tag{3b}$$

$$X' = X \cosh \xi + \left(\xi_v T - \frac{\xi_a E}{b^2}\right)\frac{\sinh \xi}{\xi} \tag{4a}$$

$$P' = P \cosh \xi + \left(\frac{\xi_v E}{c^2} + \xi_a T\right)\frac{\sinh \xi}{\xi} \tag{4b}$$

where ξ_v is the velocity-boost rapidity parameter, ξ_a is the force (acceleration) boost rapidity parameter, and ξ is the net effective rapidity parameter of the primed-reference frame. These parameters ξ_a, ξ_v, ξ are defined, respectively, in terms of the velocity $v = dX/dT$ and force $f = dP/dT$ (related to acceleration) as

$$\tanh\left(\frac{\xi_v}{c}\right) = \frac{v}{c}; \quad \tanh\left(\frac{\xi_a}{b}\right) = \frac{F}{F_{max}}, \quad \xi = \sqrt{\left(\frac{\xi_v}{c}\right)^2 + \left(\frac{\xi_a}{b}\right)^2} \tag{5}$$

The $U(1,3)$ generators $Z_{ab} = \frac{1}{2}(L_{[ab]} + M_{(ab)})$ are comprised of the 6 ordinary Lorentz generators $L_{[ab]}$ and 10 force (acceleration) boost/rotation generators $M_{(ab)}$, giving a total of 16 generators.

It is straightforward to verify that the transformations (4a,b) leave invariant the phase-space interval $c^2(dT)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = c^2 dT^2 - dX^2$, nor the interval in energy–momentum space $\frac{1}{b^2}[(dE)^2 - (dP)^2]$. Only the *combination*

$$(d\omega)^2 = (d\tau)^2 \left(1 - \frac{F^2}{F_{max}^2}\right) \tag{6}$$

is truly left invariant under force (acceleration) boosts (4a,b). They also leave invariant the symplectic two-form (phase-space areas) $\Omega = -\,dT \wedge dE + dX \wedge dP$.

Some readers might note that the $U(1,3)$ algebra is usually not used in standard formulations of particle kinematics and as symmetries in the dynamical, mechanical, and field-theoretic models. For instance, Kalman [11] long ago studied the $SU(1,3)$ group (and its discrete representations) as a dynamical group for hadrons. By a dynamical group, one means in general (a noncompact one) which gives the actual energy or mass spectrum of a quantum mechanical system [12].

Low [2–4] has explained in great detail that since $U(1,3)$ is noncompact, the $U(1,3)$ infinite-dimension unitary representations contain discrete series that may be decomposed into *infinite* ladders where the rungs are finite dimensional irreducible unitary $U(3)$ representations. In particular, the rest and null frames yield the groups $SU(3)$, $SU(2)$, and $U(1)$ that appear in the Standard Model, and which is very appealing. If one has a single particle state, under force-boosts (acceleration) transformations, one would expect to transform it into a compound state that decomposes into a sum of single particle states representing the particle interactions of nonuniform velocity frames of reference.

Low [2–4] has argued that one could think of the timelike states as the rungs of the ladder and Poincare transformations transform these rungs into themselves with *no* mixing of states that are on *different* rungs; likewise with the null states. There are no Poincare transformations that take timelike states into null states. However, when one considers noninertial frames the states in different rungs of the ladder can transform into each other, and timelike and null states can mix. The reason is that due to the nonzero rates of change of the momentum, one expects the dynamical symmetry to describe transitions between these states when viewed from the interacting frames.

One should also add that these arguments presented by [2–4] bear a resemblance to the Unruh effect (the Fulling–Davies–Unruh effect) [13] which is a kinematic prediction of quantum field theory that an *accelerating* observer will observe a thermal bath, such as a blackbody radiation, whereas an inertial observer would observe none. In other words, the background appears to be warm from an accelerating reference frame. Heuristically, for a uniformly accelerating observer, the ground state of an inertial observer is seen as a *mixed* state in thermodynamic equilibrium with a nonzero temperature bath of thermal photons and whose temperature is proportional to the acceleration.

Low [2–4] also constructed the eigenvalue equations for the representation of the set of Casimir invariant operators which define the field equations of the system. The applications of the *deformed* quaplectic algebras studied in this work, in particular corresponding to the *deformed* Heisenberg algebras, to theoretical physics models remain to be studied, and in particular, within the context of quantum field theories in noncommutative spacetimes. This is beyond the scope of this work.

After this brief introduction of Born's reciprocal relativity theory, in Section 2 we review the construction of the *deformed* quaplectic group that is given by the semidirect product of $U(1,3)$ with the *deformed* (noncommutative) Weyl–Heisenberg group corresponding to *noncommutative* fiber coordinates and momenta $[X_a, X_b] \neq 0$; $[P_a, P_b] \neq 0$. This construction leads at the end of Section 2 to more general algebras given by a two-parameter family of deformations of the quaplectic algebra and to local gauge theories of gravity based on the latter deformed quaplectic algebras.

We continue in Section 2 by examining the algebraic extensions of the Yang algebra in extended noncommutative phase spaces and compare them with the extensions of the deformed quaplectic algebra involving antisymmetric tensor coordinates and momenta of higher ranks $[X_{a_1 a_2 \cdots a_n}, X_{b_1 b_2 \cdots b_n}] \neq 0$; $[P_{a_1 a_2 \cdots a_n}, P_{b_1 b_2 \cdots b_n}] \neq 0$.

In Section 2, a solution is found for the exact analytical mapping of the noncommuting x^μ, p^μ operator variables (associated with an 8D curved phase space) to the canonical Y^A, Π^A operator variables of a flat 12D phase space. We explore the geometrical implications of this mapping which provides, in the *classical* limit, the embedding functions $Y^A(x,p), \Pi^A(x,p)$ of an 8D curved phase space into a flat 12D phase-space background. The latter embedding functions determine the functional forms of the base spacetime metric $g_{\mu\nu}(x,p)$, the fiber metric of the vertical space $h^{ab}(x,p)$, and the nonlinear connection $N_{a\mu}(x,p)$ associated with the 8D cotangent space of the 4D spacetime. We finalize with some concluding remarks.

2. The Deformed Quaplectic Group and Complex Gravity

To begin this section we review the construction of the *deformed* quaplectic group given by the semidirect product of $U(1,3)$ with the deformed (noncommutative) Weyl–Heisenberg group involving noncommutative coordinates and momenta [14]. Then, we proceed to construct a two-parameter family of deformed quaplectic algebras parametrized by two complex coefficients α, β.

The (undeformed) quaplectic group is given by the semidirect product of $U(1,3)$ with the Weyl–Heisenberg group and was studied in detail by [2–4]. Physically, the quaplectic group is basically the "phase-space" version of the Poincare group (which is given by the semidirect product of the Lorenz group $SO(1,3)$ with the translation group T_4), where the translation group is replaced by the Weyl–Heisenberg group and the Lorentz group is replaced by $U(1,3)$.

The deformed Weyl–Heisenberg algebra involves the generators

$$Z_a = \frac{1}{\sqrt{2}}\left(\frac{X_a}{\lambda_l} - i\frac{P_a}{\lambda_p}\right); \quad Z_a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{X_a}{\lambda_l} + i\frac{P_a}{\lambda_p}\right); \quad a = 1,2,3,4. \tag{7}$$

Notice that we must *not* confuse the *generators* X_a, P_a (associated with the fiber coordinates of the internal space of the fiber bundle) with the ordinary base spacetime coordinates and momenta x_μ, p_μ. The local gauge theory based on the deformed quaplectic algebra was constructed in the fiber bundle over the base manifold which is a 4D curved spacetime with *commuting* coordinates $x^\mu = x^0, x^1, x^2, x^3$ [14]. The (deformed) quaplectic group acts as the automorphism group along the internal fiber coordinates. Therefore, we must *not* confuse the *deformed* complex gravitational theory constructed in [14] with the noncommutative gravity work in the literature where the spacetime coordinates x^μ are not commuting.

The four fundamental length, momentum, temporal, and energy scales are, respectively,

$$\lambda_l = \sqrt{\frac{\hbar c}{b}}; \quad \lambda_p = \sqrt{\frac{\hbar b}{c}}; \quad \lambda_t = \sqrt{\frac{\hbar}{bc}}; \quad \lambda_e = \sqrt{\hbar bc}. \tag{8}$$

where b is the *maximal* proper force associated with the Born's reciprocal relativity theory. In the natural units $\hbar = c = b = 1$, all four scales become *unity*. The gravitational coupling is given by

$$G = \frac{c^4}{\mathcal{F}_{max}} = \frac{c^4}{b}. \tag{9}$$

and the four scales then coincide with the Planck length, momentum, time, and energy, respectively. One may postulate the maximal proper force to be given by

$$\mathcal{F}_{\max} = m_P \frac{c^2}{L_P} \tag{10}$$

where L_P is the Planck scale, the Planck mass m_P is assumed to be the *maximal* mass of a *fundamental* particle, and $\frac{c^2}{L_P}$ is postulated to be its *maximal* proper acceleration. In natural units $\hbar = c = G = 1$, $\mathcal{F}_{max} \to 1$.

The generators of the $U(1,3)$ algebra given by Z_{ab} are Hermitian $(Z_{ab})^\dagger = Z_{ab}$, with $a, b = 1,2,3,4$; while the generators of the *deformed* Weyl–Heisenberg algebra $Z_a, Z_a^\dagger$ are Hermitian-conjugate pairs like $L_+ = L_x + iL_y, L_- = L_x - iL_y$ in the $SO(3)$ algebra. Note that the Hermitian-conjugate pairs of generators $Z_a, Z_a^\dagger$ in Equations (7) are not independent from each other, hence one is *not* doubling the number of physical dimensions. For instance, the complex variables $z^\mu = x^\mu + ip^\mu; \bar{z}^\mu = x^\mu - ip^\mu; \mu = 1,2,\cdots,D$ are not independent but complex-conjugate pairs. The number of physical dimensions of the 2D phase space remains the same.

The standard quaplectic group [2–4] is given by the semidirect product of the $U(1,3)$ group and the unmodified Weyl–Heisenberg $H(1,3)$ group $\mathcal{Q}(1,3) \equiv U(1,3) \otimes_s H(1,3)$ and is defined in terms of the generators $Z_{ab}, Z_a, Z_a^\dagger, \mathcal{I}$ described below with $a, b = 1,2,3,4$.

A careful analysis reveals that the generators $Z_a, Z_a^\dagger$ (comprised of Hermitian *and* anti-Hermitian pieces) of the *deformed* Weyl–Heisenberg algebra can be defined in terms of judicious linear combinations of the Hermitian $U(1,4)$ algebra generators Z_{AB}, where $A, B = 1,2,3,4,5$; $a, b = 1,2,3,4$ and $\eta_{AB} = diag\,(+,-,-,-,-)$. The linear combination is defined after introducing the following *complex*-valued coefficients as follows:

$$Z_a = (-i)^{1/2}(Z_{a5} - iZ_{5a}); \quad Z_a^\dagger = (i)^{1/2}(Z_{a5} + iZ_{5a}); \quad Z_{55} = \frac{\mathcal{I}}{2} \tag{11}$$

The reason behind this particular choice of the complex coefficients appearing in Equation (11) is explained below in Equation (20a–c). The Hermitian generators of the $U(1,4)$ algebra are given by $Z_{AB} \equiv \mathcal{E}_A^B$ and $Z_{BA} \equiv \mathcal{E}_B^A$; notice that the position of the indices is very relevant because $Z_{AB} \neq Z_{BA}$. The commutators are

$$[\mathcal{E}_a^b,\, \mathcal{E}_c^d] \;=\; -i\,\delta_c^b\,\mathcal{E}_a^d + i\,\delta_a^d\,\mathcal{E}_c^b; \quad [\mathcal{E}_c^d,\, \mathcal{E}_a^5] \;=\; -i\,\delta_a^d\,\mathcal{E}_c^5; \quad [\mathcal{E}_c^d,\, \mathcal{E}_5^a] \;=\; i\,\delta_c^a\,\mathcal{E}_5^d \tag{12}$$

and $[\mathcal{E}_5^5,\, \mathcal{E}_5^a] = -i\,\delta_5^5\,\mathcal{E}_5^a$ .... such that now, $\mathcal{I}(=2Z_{55})$ *no* longer commutes with $Z_a, Z_a^\dagger$. The generators Z_{ab} of the $U(1,3)$ algebra can be decomposed into the Lorentz subalgebra generators $L_{[ab]}$ and the "shearlike" generators $M_{(ab)}$ as

$$Z_{ab} \equiv \frac{1}{2}(M_{(ab)} + L_{[ab]}) \Rightarrow L_{ab} \equiv L_{[ab]} = (Z_{ab} - Z_{ba}); \quad M_{ab} \equiv M_{(ab)} = (Z_{ab} + Z_{ba}), \tag{13}$$

where the "shearlike" generators $M_{(ab)}$ and the Lorentz generators $L_{[ab]}$ are Hermitian. The explicit commutation relations of the M_{ab}, L_{ab} generators are given by

$$[L_{ab}, L_{cd}] = i(\eta_{bc}L_{ad} - \eta_{ac}L_{bd} - \eta_{bd}L_{ac} + \eta_{ad}L_{bc}). \tag{14a}$$
$$[M_{ab},\ M_{cd}] = -i(\eta_{bc}L_{ad} + \eta_{ac}L_{bd} + \eta_{bd}L_{ac} + \eta_{ad}L_{bc}). \tag{14b}$$
$$[L_{ab},\ M_{cd}] = i(\eta_{bc}M_{ad} - \eta_{ac}M_{bd} + \eta_{bd}M_{ac} - \eta_{ad}M_{bc}). \tag{14c}$$

Therefore, given $Z_{ab} = \frac{1}{2}(M_{ab} + L_{ab}),\ Z_{cd} = \frac{1}{2}(M_{cd} + L_{cd})$ after straightforward algebra, it leads to the $U(1,3)$ commutators

$$[Z_{ab}, Z_{cd}] = -i(\eta_{bc}Z_{ad} - \eta_{ad}Z_{cb}) \tag{14d}$$

as expected. By extension, the $U(1,4)$ commutators are[1]

$$[Z_{AB}, Z_{CD}] = -i(\eta_{BC}Z_{AD} - \eta_{AD}Z_{CB}). \tag{14e}$$

The commutators of the Lorentz boosts generators L_{ab} with the X_c, P_c generators are

$$[L_{ab}, X_c] = i(\eta_{bc}X_a - \eta_{ac}X_b); \quad [L_{ab}, P_c] = i(\eta_{bc}P_a - \eta_{ac}P_b). \tag{15}$$

The Hermitian M_{ab} generators are the "reciprocal" boosts/rotation transformations which *exchange* X for P, in addition to boosting (rotating) those variables, and one ends up with the commutators of M_{ab} with the X_c, P_c generators given by

$$[M_{ab}, \frac{X_c}{\lambda_l}] = -\frac{i}{\lambda_p}(\eta_{bc}P_a + \eta_{ac}P_b); \quad [M_{ab}, \frac{P_c}{\lambda_p}] = -\frac{i}{\lambda_l}(\eta_{bc}X_a + \eta_{ac}X_b). \tag{16}$$

The commutators in Equation (14d) and the definitions in Equation (11) lead to

$$\begin{aligned} [Z_{ab}, Z_c] &= (-i)^{3/2}(\eta_{bc}Z_{a5} + i\eta_{ac}Z_{5b}) \\ [Z_{ab}, Z_c^\dagger] &= -(i)^{1/2}(i\eta_{bc}Z_{a5} + \eta_{ac}Z_{5b}), \end{aligned} \tag{17}$$

which are consistent with the commutators in Equation (14a–c) and the definitions in Equations (11) and (13). The right-hand side of Equation (17) can be rewritten in terms of $Z_a, Z_a^\dagger, Z_b, Z_b^\dagger$ after the following replacements:

$$Z_{a5} = \frac{1}{2}[(-i)^{1/2}Z_a^\dagger + (i)^{1/2}Z_a], \quad Z_{b5} = \frac{1}{2i}[(-i)^{1/2}Z_a^\dagger - (i)^{1/2}Z_a]. \tag{18}$$

After some algebra one finds

$$\begin{aligned} [Z_{ab}, Z_c] &= -\frac{i}{2}\eta_{bc}Z_a + \frac{i}{2}\eta_{ac}Z_b - \frac{1}{2}\eta_{bc}Z_a^\dagger - \frac{1}{2}\eta_{ac}Z_b^\dagger \\ [Z_{ab}, Z_c^\dagger] &= -\frac{i}{2}\eta_{bc}Z_a^\dagger + \frac{i}{2}\eta_{ac}Z_b^\dagger + \frac{1}{2}\eta_{bc}Z_a + \frac{1}{2}\eta_{ac}Z_b. \end{aligned} \tag{19}$$

The particular *choice* of the complex coefficients appearing in Equation (11) leads to the following *deformed* Weyl–Heisenberg algebra

$$[Z_a, Z_b^\dagger] = -(\eta_{ab}\mathcal{I} + M_{ab}); \quad [Z_a, Z_b] = [Z_a^\dagger, Z_b^\dagger] = -iL_{ab} \tag{20a}$$
$$[Z_a, \mathcal{I}] = 2Z_a^\dagger; \quad [Z_a^\dagger, \mathcal{I}] = -2Z_a; \quad [Z_{ab}, \mathcal{I}] = 0. \quad \mathcal{I} = 2Z_{55} \tag{20b}$$

leading to

$$\left[\frac{X_a}{\lambda_l}, \mathcal{I}\right] = 2i\frac{P_a}{\lambda_p}; \quad \left[\frac{P_a}{\lambda_p}, \mathcal{I}\right] = 2i\frac{X_a}{\lambda_l} \tag{20c}$$

and the metric $\eta_{ab} = (+1, -1, -1, -1)$ is used to raise and lower indices. The Planck constant is given in terms of the length and momentum scales of Equation (8) as $\hbar = \lambda_l \lambda_p$. In $\hbar = 1$ units, $\lambda_l \lambda_p \to 1$.

The deformed quaplectic algebra is given explicitly by Equations (14d), (17), (19), and (20a–c) and obeys the Jacobi identities by virtue of the definitions in Equations (11) and (13). After recurring directly to the definitions in Equation (7), one finds that Equation (20a) explicitly reflects the *deformation* of the Weyl–Heisenberg algebra resulting from the non-commutative algebra of coordinates and momenta given by

$$\left[\frac{X_a}{\lambda_l}, \frac{P_b}{\lambda_p}\right] = i(\eta_{ab}\mathcal{I} + M_{ab}) \tag{21a}$$

$$[X_a, X_b] = -i(\lambda_l)^2 L_{ab}; \quad [P_a, P_b] = i(\lambda_p)^2 L_{ab}. \tag{21b}$$

One could interpret the term $\eta_{ab}\,\mathcal{I} + M_{ab}$ as a matrix-valued Planck constant $\hbar_{ab}$ (in units of $\hbar = 1$). One may also note that the generator $\mathcal{I}$ no longer commutes with $Z_a, Z_a^\dagger$, but it *exchanges* them, as one can see from Equation (20b) resulting from the definition of $\mathcal{I}$ given by $\mathcal{I} \equiv 2Z_{55} = M_{55}$.

One of the salient features of the construction of the deformed quaplectic (Weyl–Heisenberg) algebra is that by varying the values of the following complex coefficients α, β appearing in the linear combinations

$$Z_a = \alpha\, Z_{a5} + \beta Z_{5a}; \quad Z_a^\dagger = \alpha^* Z_{a5} + \beta^* Z_{5a}; \quad Z_{55} = \frac{\mathcal{I}}{2}, \tag{22}$$

it furnishes different commutation relations than the ones described by Equations (20a–c) and (21a,b). The latter commutators are found in the special case when $\alpha = (-i)^{1/2}$, $\beta = (-i)^{3/2}$, as chosen in Equation (11). For instance, if either $\alpha = 0$ or $\beta = 0$ it leads instead to vanishing commutators $[Z_a, Z_b^\dagger] = [Z_a, Z_b] = [Z_a^\dagger, Z_b^\dagger] = 0$ as a result of Equation (14e). In turn, one would have $[X_a, X_b] = [P_a, P_b] = [X_a, P_b] = 0$ instead of Equations (21a,b). Therefore, the introduction of nonvanishing complex coefficients α, β, via Equation (22), yields a two-parameter family of deformed fiber coordinates and momenta algebras parametrized by α, β. In particular, one may explicitly introduce these parameters by writing $Z_a(\alpha, \beta), Z_a^\dagger(\alpha^*, \beta^*)$.

After introducing the complex-valued vierbein $E_\mu^a = e_\mu^a + i f_\mu^a$, it leads to the complex metric

$$g_{\mu\nu} \equiv E_\mu^a (E_\nu^b)^* \eta_{ab} = g_{(\mu\nu)} + i g_{[\mu\nu]} \tag{23a}$$

with

$$g_{(\mu\nu)} = (e_\mu^a e_\nu^b + f_\mu^a f_\nu^b)\eta_{ab}, \quad i g_{[\mu\nu]} = -i(e_\mu^a f_\nu^b - e_\nu^b f_\mu^a)\eta_{ab}. \tag{23b}$$

The 4×4 complex metric $g_{\mu\nu}$ is Hermitian $g_{\mu\nu}^\dagger = g_{\mu\nu}$ as a result of $g_{\nu\mu} = (g_{\mu\nu})^*$. To verify that $g_{[\mu\nu]} = -g_{[\nu\mu]}$, one just needs to relabel the indices $a \leftrightarrow b$ in Equation (23b) and recur to $\eta_{ba} = \eta_{ab}$.

The two-parameter family of $U(1,4)$-valued Hermitian gauge fields is given by

$$\mathbf{A}_\mu = \Omega_\mu^{ab} Z_{ab} + \frac{1}{L}[E_\mu^a Z_a(\alpha, \beta) + (E_\mu^a)^* Z_a^\dagger(\alpha^*, \beta^*)] \; + \; \Omega_\mu\, \mathcal{I}, \tag{24}$$

where L is a length scale that is introduced for dimensional reasons since the physical units of $\mathbf{A}_\mu$ are $(length)^{-1}$. $\Omega_\mu^{ab} Z_{ab}$ is given by $\frac{1}{2}(\Omega_\mu^{(ab)} M_{ab} + \Omega_\mu^{[ab]} L_{ab})$, and $Z_a(\alpha, \beta), Z_a^\dagger(\alpha^*, \beta^*)$ are displayed in Equation (22).

One can rewrite the two-parameter family of $U(1,4)$-valued Hermitian gauge fields (24) as

$$\mathbf{A}_\mu = \Omega_\mu^{ab} Z_{ab} + \Omega_\mu^{(a5)} M_{a5} + \Omega_\mu^{[a5]} L_{a5} + \Omega_\mu \mathcal{I}, \quad \Omega_\mu \equiv \Omega_\mu^{55}. \tag{25}$$

After some straightforward algebra, one finds that the real-valued connection components $\Omega_\mu^{a5}, \Omega_\mu^{5a}$ are given by suitable linear combinations of the e_μ^a, f_μ^a components of the complex-valued vierbein as follows

$$\Omega_\mu^{a5} = e_\mu^a \left(\frac{\alpha + \alpha^*}{L} \right) - f_\mu^a \left(\frac{\alpha - \alpha^*}{iL} \right); \quad \Omega_\mu^{5a} = e_\mu^a \left(\frac{\beta + \beta^*}{L} \right) - f_\mu^a \left(\frac{\beta - \beta^*}{iL} \right), \tag{26a}$$

such that

$$\Omega_\mu^{(a5)} \equiv \frac{1}{2} (\Omega_\mu^{a5} + \Omega_\mu^{5a}), \quad \Omega_\mu^{[a5]} \equiv \frac{1}{2} (\Omega_\mu^{a5} - \Omega_\mu^{5a}). \tag{26b}$$

Because $\alpha \neq \beta$, one finds that $\Omega_\mu^{a5} \neq \Omega_\mu^{5a}$; consequently, $\Omega_\mu^{(a5)} \neq 0; \Omega_\mu^{[a5]} \neq 0$. Therefore, the introduction of the two distinct complex coefficients α, β is tantamount to choosing an infinite family of real-valued connection components $\Omega_\mu^{a5}, \Omega_\mu^{5a}$ given by the many different linear combinations of e_μ^a and f_μ^a. The real-valued coefficients of these linear combinations are given by the real and imaginary parts of α and β as displayed in Equation (26a). One should also emphasize that *no* zero torsion conditions were imposed in reaching the relations in Equation (26a,b) between $\Omega_\mu^{a5}, \Omega_\mu^{5a}$ and e_μ^a, f_μ^a.

The Hermitian $U(1,4)$-valued field strength is defined by

$$\mathbf{F}_{\mu\nu} \;=\; \partial_\mu \mathbf{A}_\nu \;-\; \partial_\nu \mathbf{A}_\mu \;+ i\, [\mathbf{A}_\mu,\; \mathbf{A}_\nu], \tag{27}$$

from which one can read the curvature components $R_{\mu\nu}^{(ab)}; R_{\mu\nu}^{[ab]}$, and the other components of the field strength (such as torsion), in terms of the connection components (and their derivatives) of Equation (24) from the following decomposition of the field strength

$$\mathbf{F}_{\mu\nu} = R_{\mu\nu}^{(ab)} M_{ab} + R_{\mu\nu}^{[ab]} L_{ab} + \frac{1}{L} [F_{\mu\nu}^a Z_a(\alpha, \beta) + (F_{\mu\nu}^a)^* Z_a^\dagger(\alpha^*, \beta^*)] + F_{\mu\nu} \mathcal{I}. \tag{28}$$

By proceeding as one did in [14], one may then construct the generalized actions for complex gravity after using the complex metric (vierbein) and its inverse to raise and lower indices. The simplest actions can have terms linear and quadratic in the curvature and also quadratic terms in the torsion. For further details, we refer to [14].

Alternatively, one could instead start with the $U(1,4)$-valued Hermitian gauge field in Equation (25) leading to the field strength

$$\mathbf{F}_{\mu\nu} = R_{\mu\nu}^{(ab)} M_{ab} + R_{\mu\nu}^{[ab]} L_{ab} + R_\mu^{(a5)} M_{a5} + R_\mu^{[a5]} L_{a5} + F_{\mu\nu} \mathcal{I} \tag{29}$$

and expressed in terms of $\Omega_\mu^{(ab)}, \Omega_\mu^{[ab]}, e_\mu^a, f_\mu^a, \Omega_\mu^{55} = \Omega_\mu$, and their derivatives. Note that $U(1,4)$ has 25 generators, whereas the metric affine group in 4D, given by the semidirect product of $GL(4,R)$ with the translation group T_4, has 20 generators. Therefore, the complex gravitational theory based on $U(1,4)$ and inspired from Born's reciprocal relativity theory, has more degrees of freedom than the metric affine theory of gravity in 4D. This is not surprising since one is dealing with gravity in curved phase spaces. There is also torsion in our construction.

A curved phase-space action associated with the geometry of the cotangent bundle of spacetime and based on Lagrange–Finsler and Hamilton–Cartan geometry [15–18] can be found in [19–21]. To conclude this section, there are two different approaches to construct generalized gravitational theories in curved phase spaces: (i) via the $U(1,4)$ local gauge theory construction presented here, or (ii) via Finsler's geometric methods.

3. The Yang Algebra versus the Deformed Quaplectic Algebra

This section is devoted to an extensive analysis of the Yang and the deformed quaplectic algebras associated with noncommutative phase spaces. Secondly, we present extensions of such algebras involving antisymmetric tensor coordinates and momenta of different ranks.

3.1. The Yang Algebra and Its Extension via Generalized Angular Momentum Operators in Higher Dimensions

Given a flat 6D spacetime with coordinates $Y^M = \{Y^1, Y^2, Y^3, Y^4, Y^5, Y^6\}$ and a metric $\eta_{MN} = diag(-1, +1, +1, \ldots, +1)$[2], the Yang algebra [22,23], which is an extension of the Snyder algebra [24], can be derived in terms of the $SO(5,1)$ Lorentz algebra generators described by the angular momentum/boost operators[3]

$$J^{MN} = -(Y^M\Pi^N - Y^N\Pi^M) = iY^M\frac{\partial}{\partial Y_N} - iY^N\frac{\partial}{\partial Y_M}, \tag{30}$$

where $\Pi^M = -i(\partial/\partial Y_A)$ is the canonical conjugate momentum variable to Y^M. Their commutators are

$$[Y^M, Y^N] = 0, [\Pi^M, \Pi^N] = 0, \quad [Y^M, \Pi^N] = i\eta^{MN}, \quad M, N = 1,2,3,4,5,6. \tag{31}$$

The coordinates Y^M commute. The momenta Π^M also commute, and the canonical conjugate variables Y^M, Π^N obey the Weyl–Heisenberg algebra in 6D.

Adopting the units $\hbar = c = 1$, the correspondence among the noncommuting 4D spacetime coordinates x^μ, the noncommuting momenta p^μ, and the Lorentz $SO(5,1)$ algebra generators leading to the Yang algebra [22,23] is given by

$$x^\mu \leftrightarrow L_P J^{\mu 5} = -L_P(Y^\mu\Pi^5 - Y^5\Pi^\mu) \tag{32a}$$

$$p^\mu \leftrightarrow \frac{1}{\mathcal{L}}J^{\mu 6} = -\frac{1}{\mathcal{L}}(Y^\mu\Pi^6 - Y^6\Pi^\mu), \quad \mu, \nu = 1,2,3,4, \tag{32b}$$

which requires the introduction of an ultraviolet cutoff scale L_P given by the Planck scale, and an infrared cutoff scale $\mathcal{L}$ that can be set equal to the Hubble scale R_H (which determines the cosmological constant). It is very important to emphasize that despite the introduction of two length scales $L_P, \mathcal{L}$, the Lorentz symmetry is not lost. This is one of the most salient features of the Snyder [24] and Yang [22,23] algebras[4].

The other generators are given by

$$\mathcal{N} \equiv J^{56} = -(Y^5\,\Pi^6 - Y^6\,\Pi^5), \quad J^{\mu\nu} = -(Y^\mu\,\Pi^\nu - Y^\nu\,\Pi^\mu), \quad \mu, \nu = 1,2,3,4 \tag{33}$$

One can then verify that the Yang algebra is recovered after imposing the correspondence in Equations (32a,b) and (33)

$$[x^\mu, x^\nu] = -iL_P^2 J^{\mu\nu}, \quad [p^\mu, p^\nu] = -i(\frac{1}{\mathcal{L}})^2 J^{\mu\nu}, \quad \eta^{55} = \eta^{66} = 1 \tag{34}$$

$$[x^\mu, J^{\nu\rho}] = i(\eta^{\mu\rho}x^\nu - \eta^{\mu\nu}x^\rho) \tag{35}$$

$$[p^\mu, J^{\nu\rho}] = i(\eta^{\mu\rho}p^\nu - \eta^{\mu\nu}p^\rho) \tag{36}$$

$$[x^\mu, p^\nu] = -i\eta^{\mu\nu}\frac{L_P}{\mathcal{L}}\mathcal{N}, \quad [J^{\mu\nu}, \mathcal{N}] = 0 \tag{37}$$

$$[x^\mu, \mathcal{N}] = iL_P\mathcal{L}p^\mu, \quad [p^\mu, \mathcal{N}] = -i\frac{1}{L_P\mathcal{L}}x^\mu \tag{38}$$

where the $[J^{\mu\nu}, J^{\rho\sigma}]$ commutators are the same as in the $SO(3,1)$ Lorentz algebra in 4D. They are of the form

$$[J^{\mu_1\mu_2}, J^{\nu_1\nu_2}] = -i\eta^{\mu_1\nu_1}J^{\mu_2\nu_2} + i\eta^{\mu_1\nu_2}J^{\mu_2\nu_1} + \\ i\eta^{\mu_2\nu_1}J^{\mu_1\nu_2} - i\eta^{\mu_2\nu_2}J^{\mu_1\nu_1}, \quad \hbar = c = 1 \tag{39}$$

The generators are assigned to be Hermitian so there are i factors in the right-hand side of Equation (39) since the commutator of two Hermitian operators is anti-Hermitian. The 4D spacetime metric is $\eta_{\mu\nu} = diag(-1,1,1,1)$.

Before continuing, it is important to point out the differences/similarities between the $U(1,4)$ algebra and the Yang algebra which is based on $SO(4,2)$ (or $SO(5,1)$). Firstly, $U(1,4)$ has 25 generators while $SO(4,2)$ has 15. Secondly, the modified Weyl–Heisenberg algebra in Equation (37) *differs* from the one displayed by Equation (21a). Equation (34) is similar to Equation (21b); Equation (38) is similar to Equation (20c); and Equations (35) and (36) are trivially similar to Equation (15). Thirdly, there is *no* analog in the Yang algebra of the Hermitian M_{ab} generators which act as the "reciprocal" boosts/rotation transformations which *exchange* X for P, in addition to boosting (rotating) those variables, and leading to the commutators of M_{ab} with the X_c, P_c generators given by Equation (16).

Another difference between the Yang and the deformed quaplectic algebra is that in the Yang algebra case, one adds two additional coordinates and momenta Y^5, Y^6, Π^5, and Π^6 in order to construct the $SO(4,2), SO(5,1)$ algebras with 15 generators. Whereas in the (deformed) quaplectic algebra case, one adds one additional coordinate and momentum Y^5, Π^5, and the extra generators $M_{ab}, M_{a5}, M_{55} = \mathcal{I}$ in order to construct the $U(1,4)$ algebra with 25 generators. Furthermore, the construction of the Yang algebra requires the two length scales $L_P, \mathcal{L}$, whereas in the (deformed) quaplectic algebra, one has the length scale λ_l and the momentum scale λ_p.

One may also clarify that quantum phase spaces can be described by real or complex phase space coordinates. A typical example of the use of complex coordinates is in the description of the coherent state $|z\rangle$ that is defined to be the unique eigenstate of the (bosonic) annihilation operator $\hat{a}|z\rangle = z|z\rangle$ [25]. The formal solution of this eigenvalue equation is the vacuum state displaced to a location z in phase space, and it is obtained by letting the unitary displacement operator $D(z)$ operate on the vacuum $|z\rangle = e^{z\hat{a}^\dagger - z^*\hat{a}}|0\rangle = D(z)|0\rangle$, where the annihilation operator $\hat{a} = \hat{X} + i\hat{P}$ and creation operator $\hat{a}^\dagger = \hat{X} - i\hat{P}$ are expressed in terms of the phase-space coordinates associated with the quantum harmonic oscillator.

Using the representation of the coherent state in the basis of Fock states, one finds $|z\rangle = e^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle = e^{-\frac{|z|^2}{2}}e^{z\hat{a}^\dagger}e^{-z^*\hat{a}}|0\rangle$, where $|n\rangle$ are the energy (number) eigenvectors of the quantum harmonic oscillator Hamiltonian $H = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$ [25].

Pertaining the role of the $U(1,4)$ symmetry, one should point that there is a standard procedure to obtain the $U(N)$ generators $E_{jk} = a_j^\dagger a_k$ in terms of the complex $Cl(2N, C)$ algebra generators via the creation and annihilation *fermionic* oscillators defined as follows: $a_j = \frac{1}{2}(\Gamma_{2j} + i\,\Gamma_{2j-1})$; $a_j^\dagger = \frac{1}{2}(\Gamma_{2j} - i\,\Gamma_{2j-1})$; $j = 1,2,\cdots,N$. One can verify that the following *anticommutators* $\{a_j, a_k^\dagger\} = \delta_{jk}$; $\{a_j, a_k\} = 0$; $\{a_j^\dagger, a_k^\dagger\} = 0$ lead to the $U(N)$ commutation relations $[E_{jk}, E_{lm}] = \delta_{kl}E_{jm} - \delta_{jm}E_{lk}$. This construction is just a reflection of the fact that $U(N) \subset SO(2N)$. In particular, $U(4) \subset SO(8)$.

After this detour, given the above correspondence (9), we can *extend* it further to the higher-grade polyvector-valued coordinates and momenta operators in noncommutative Clifford phase spaces [26,27]. Given a Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$, a polyvector-valued coordinate is defined as $\mathbf{X} = X_M\Gamma^M$ and admits the following expansion in terms of the Clifford algebra generators in D-dimensions, $\mathbf{1}, \gamma^{\mu}, \gamma^{\mu_1} \wedge \gamma^{\mu_2}, \cdots, \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \cdots \wedge \gamma^{\mu_D}$, as follows:

$$\mathbf{X} = X\mathbf{1} + X_{\mu}\gamma^{\mu} + X_{\mu_1\mu_2}\gamma^{\mu_1} \wedge \gamma^{\mu_2} + X_{\mu_1\mu_2\mu_3}\gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} + ...+ \\ X_{\mu_1\mu_2\mu_3......\mu_D}\gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \ \gamma^{\mu_3} .... \wedge \gamma^{\mu_D}. \tag{40a}$$

The numerical combinatorial factors can be omitted by imposing the ordering prescription $\mu_1 < \mu_2 < \mu_3 \cdots < \mu_D$. In order to match physical units in each term of (17), a length scale parameter must be suitably introduced in the expansion in Equation (17). In [28,29], we introduced the Planck scale as the expansion parameter in (17), which was set to unity, when one adopted the units $\hbar = c = G = 1$.

Similarly, the polyvector-valued momentum $\mathbf{P} = P_M \Gamma^M$ admits the following expansion in terms of the Clifford algebra generators in D-dimensions

$$\mathbf{P} = P\mathbf{1} + P_{\mu}\gamma^{\mu} + P_{\mu_1\mu_2}\gamma^{\mu_1} \wedge \gamma^{\mu_2} + P_{\mu_1\mu_2\mu_3}\gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} + ...+ \\ P_{\mu_1\mu_2\mu_3......\mu_D}\gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} .... \wedge \gamma^{\mu_D} \tag{40b}$$

The scalar, vectorial, antisymmetric tensorial coordinates $X, X_{\mu}, X_{\mu_1\mu_2} = -X_{\mu_2\mu_1}, \cdots,$ $X_{\mu_1\mu_2\cdots\mu_D}$ are the scalar, vector, bivector, trivector, etc., components of the polyvector-valued coordinates. The $X_{\mu_1\mu_2}$ bivector (antisymmetric tensor of rank two) corresponds to an oriented area element. The trivector $X_{\mu_1\mu_2\mu_3}$ (antisymmetric tensor of rank three) corresponds to an oriented volume element, and so forth.

Similarly, the scalar, vectorial, antisymmetric tensorial coordinates $P, P_{\mu}, P_{\mu_1\mu_2} = -P_{\mu_2\mu_1}, \cdots, P_{\mu_1\mu_2\cdots\mu_D}$ are the scalar, vector, bivector, trivector, etc., components of the polyvector-valued momentum coordinates. The $P_{\mu_1\mu_2}$ bivector (antisymmetric tensor of rank two) corresponds to an oriented areal-momentum element. The trivector $P_{\mu_1\mu_2\mu_3}$ (antisymmetric tensor of rank three) corresponds to an oriented volume–momentum element, and so forth.

We constructed in [26,27] the corresponding nonvanishing commutators among the *noncommutative* antisymmetric tensors $X^{\mu_1\mu_2}, X^{\mu_1\mu_2\mu_3}, \cdots; P^{\mu_1\mu_2}, P^{\mu_1\mu_2\mu_3}, \cdots$ of different ranks. We coined such *extension* of the Yang algebra the Clifford–Yang algebra, since it involves polyvector-valued coordinates and momenta associated with a Clifford algebra. The *noncommuting* bivector coordinates obey

$$[X^{\mu_1\mu_2}, X^{\nu_1\nu_2}] \sim iL_P^4 \eta^{55} J^{\mu_1\mu_2|\nu_1\nu_2}, \quad J^{\mu_1\mu_2|\nu_1\nu_2} \equiv -(Y^{\mu_1\mu_2}\Pi^{\nu_1\nu_2} - Y^{\nu_1\nu_2}\Pi^{\mu_1\mu_2} \tag{41a}$$

$Y^{\mu_1\mu_2}$ is a bivector coordinate associated with the $Cl(5,1)$ algebra of the 6D flat spacetime. $\Pi^{\mu_1\mu_2} = -i(\partial/\partial Y_{\mu_1\mu_2})$ is the corresponding bivector canonical momentum conjugate. Their commutators are

$$[Y^{\mu_1\mu_2}, Y^{\nu_1\nu_2}] = 0, \quad [\Pi^{\mu_1\mu_2}, \Pi^{\nu_1\nu_2}] = 0, \quad [Y^{\mu_1\mu_2}, P^{\nu_1\nu_2}] = i\eta^{\mu_1\mu_2|\nu_1\nu_2}, \tag{41b}$$

where the generalized metric involving bivector indices is defined as

$$\eta^{\mu_1\mu_2|\nu_1\nu_2} = \eta^{\nu_1\nu_2|\mu_1\mu_2} = \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} - \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1} \tag{41c}$$

The *noncommuting* bivector momenta obey

$$[P^{\mu_1\mu_2}, P^{\nu_1\nu_2}] \sim i\mathcal{L}^{-4}\eta^{66} J^{\mu_1\mu_2|\nu_1\nu_2}, \tag{41d}$$

and so forth. All the commutators have the same structural form of a generalized angular momentum algebra as follows

$$[J^{A(r_1)|B(r_2)}, J^{C(s_1)|D(s_2)}] = -i\eta^{A(r_1)|C(s_1)} J^{B(r_2)|D(s_2)} + i\eta^{A(r_1)|D(s_2)} J^{B(r_2)|C(s_1)} + i\eta^{B(r_2)|C(s_1)} J^{A(r_1)|D(s_2)} - i\eta^{B(r_2)|D(s_2)} J^{A(r_1)|C(s_1)}, \quad \hbar = c = 1, \tag{41e}$$

where the grades of the polyvector indices $A(r_1)B(r_2), C(s_1)$, and $D(s_2)$ appearing in the generators are r_1, r_2, s_1, and s_2, respectively. The shorthand notation for $J^{a_1a_2\cdots a_{r_1}|b_1b_2\cdots b_{r_2}}$ is $J^{A(r_1)|B(r_2)}, \cdots$. The generalized metric tensor $\eta^{A|C} = 0$ if the grade of A is *not* equal to the grade of C. Similarly, $\eta^{A|D} = 0$ if the grade of A is *not* equal to the grade of $D, \cdots$. Moreover, $\eta^{\mu 5} = \eta^{\mu 6} = 0$ since the 6D metric is diagonal. The commutators (41e) ensure that the Jacobi identities are satisfied. In addition, we found the spectrum of the quantum harmonic oscillator in noncommutative spaces in terms of the eigenvalues of the generalized angular momentum operators in higher dimensions and discussed how to extend these results to higher-grade polyvector-valued coordinates and momenta. For full details, we refer the reader to [26,27].

3.2. Realization of the Deformed Quaplectic Algebra and its Extensions

We saw above how the *noncommutative* coordinates and momenta of the Yang algebra in 4D can be realized in terms of the angular momentum operators in 6D, which, in turn, are expressed in terms of the canonical-conjugate variables Y^M, Π^N in 6D shown in Equations (32a,b) and (33) and obeying the standard commutation relations displayed in Equations (31). Inspired by this procedure, next, we find a realization of the *deformed* quaplectic algebra generators in terms of the canonical coordinate and momentum variables Y_a, Π_b, Y_5, Π_5 as follows:

$$M_{ab} = M_{ba} = \frac{1}{2}(Y_a\Pi_b + \Pi_b Y_a) + \frac{1}{2}(Y_b\Pi_a + \Pi_a Y_b) \tag{42a}$$

$$M_{a5} = M_{5a} = \frac{1}{2}(Y_a\Pi_5 + \Pi_5 Y_a) + \frac{1}{2}(Y_5\Pi_a + \Pi_a Y_5), \quad M_{55} = (Y_5\Pi_5 + \Pi_5 Y_5) \tag{42b}$$

$$L_{ab} = -L_{ba} = \frac{1}{2}(Y_a\Pi_b + \Pi_b Y_a) - \frac{1}{2}(Y_b\Pi_a + \Pi_a Y_b) \tag{42c}$$

$$L_{a5} = -L_{5a} = \frac{1}{2}(Y_a\Pi_5 + \Pi_5 Y_a) - \frac{1}{2}(Y_5\Pi_a + \Pi_a Y_5) \tag{42d}$$

From Equations (41a-e) and (42a–d), one then finds an explicit realization of the generators $Z_{AB} = \frac{1}{2}(M_{AB} + L_{AB})$ of the deformed quaplectic algebra, with $A, B = 1, 2, 3, 4, 5$, directly in terms of the canonical coordinate and momentum variables Y_a, Π_b, Y_5, Π_5, and obeying the following commutation relations:

$$[Y_a, Y_b] = 0, \quad [Y_a, Y_5] = 0, \quad [\Pi_a, \Pi_b] = 0 \tag{43a}$$

$$[\Pi_a, \Pi_5] = 0, \quad [Y_a, \Pi_b] = i\eta_{ab}, \quad [Y_5, \Pi_5] = i\eta_{55}. \tag{43b}$$

From Equation (43a,b), one learns that when $a \neq b$, the generator M_{ab} reduces to $Y_a\Pi_b + Y_b\Pi_a$, and when $a = b$, $M_{aa} = Y_a\Pi_a + \Pi_a Y_a$, while the generator $L_{ab} = Y_a\Pi_b - Y_b\Pi_a$. Similarly, M_{a5} reduces to $Y_a\Pi_5 + Y_5\Pi_a$, $M_{55} = Y_5\Pi_5 + \Pi_5 Y_5$, and $L_{a5} = Y_a\Pi_5 - Y_5\Pi_a$

The antisymmetric rank-two tensor coordinates and momenta operators' extensions of the expressions in Equations (41a–e) and (42a–d) are given by:

$$M_{a_1a_2|b_1b_2} = \frac{1}{2}(Y_{a_1a_2}\Pi_{b_1b_2} + \Pi_{b_1b_2}Y_{a_1a_2}) + \frac{1}{2}(Y_{b_1b_2}\Pi_{a_1a_2} + \Pi_{a_1a_2}Y_{b_1b_2}) \tag{44a}$$

$$L_{a_1a_2|b_1b_2} = \frac{1}{2}(Y_{a_1a_2}\Pi_{b_1b_2} + \Pi_{b_1b_2}Y_{a_1a_2}) - \frac{1}{2}(Y_{b_1b_2}\Pi_{a_1a_2} + \Pi_{a_1a_2}Y_{b_1b_2}) \tag{44b}$$

where

$$M_{a_1a_2|b_1b_2} = -M_{a_2a_1|b_1b_2} = -M_{a_1a_2|b_2b_1} = M_{b_1b_2|a_1a_2} \tag{45a}$$

$$L_{a_1a_2|b_1b_2} = -L_{a_2a_1|b_1b_2} = -L_{a_1a_2|b_2b_1} = -L_{b_1b_2|a_1a_2} \tag{45b}$$

Given $M_{a_1a_2|b_1b_2}, L_{a_1a_2|b_1b_2}$ the generalization of the operator Z_{ab} is

$$Z_{a_1a_2|b_1b_2} \equiv \frac{1}{2}(M_{a_1a_2|b_1b_2} + L_{a_1a_2|b_1b_2}) \tag{45c}$$

The generalization of the commutators in Equations (14a–c) corresponding to the $M_{a_1a_2|b_1b_2}, L_{a_1a_2|b_1b_2}$ generators is given by

$$\begin{aligned}[L_{a_1a_2|b_1b_2}, L_{c_1c_2|d_1d_2}] = i\eta_{b_1b_2|c_1c_2}L_{a_1a_2|d_1d_2} - i\eta_{a_1a_2|c_1c_2}L_{b_1b_2|d_1d_2} - \\ i\eta_{b_1b_2|d_1d_2}L_{a_1a_2|c_1c_2} + i\eta_{a_1a_2|d_1d_2}L_{b_1b_2|c_1c_2}\end{aligned} \tag{46}$$

$$\begin{aligned}[M_{ab}, M_{cd}] = -i\eta_{b_1b_2|c_1c_2}L_{a_1a_2|d_1d_2} - i\eta_{a_1a_2|c_1c_2}L_{b_1b_2|d_1d_2} - \\ i\eta_{b_1b_2|d_1d_2}L_{a_1a_2|c_1c_2} - i\eta_{a_1a_2|d_1d_2}L_{b_1b_2|c_1c_2}\end{aligned} \tag{47}$$

$$\begin{aligned}[L_{ab}, M_{cd}] = i\eta_{b_1b_2|c_1c_2}M_{a_1a_2|d_1d_2} - i\eta_{a_1a_2|c_1c_2}M_{b_1b_2|d_1d_2} + \\ i\eta_{b_1b_2|d_1d_2}M_{a_1a_2|c_1c_2} - i\eta_{a_1a_2|d_1d_2}M_{b_1b_2|c_1c_2}\end{aligned} \tag{48}$$

where

$$\eta^{a_1a_2|b_1b_2} \equiv \eta^{a_1b_1}\eta^{a_2b_2} - \eta^{a_1b_2}\eta^{a_2b_1} \tag{49}$$

From Equations (45c) and (46)–(49), one finds that

$$[Z_{a_1a_2|b_1b_2}, Z_{c_1c_2|d_1d_2}] = -i(\eta_{b_1b_2|c_1c_2}Z_{a_1a_2|d_1d_2} - \eta_{a_1a_2|d_1d_2}Z_{c_1c_2|b_1b_2}). \tag{50}$$

This is a result of the canonical antisymmetric rank-two tensor coordinates and momenta variables $Y_{a_1a_2}, \Pi_{b_1b_2}$ obeying the following commutation relations (the generalization of Equation (43a,b)

$$[Y_{a_1a_2}, Y_{b_1b_2}] = 0, \quad [\Pi_{a_1a_2}, \Pi_{b_1b_2}] = 0, \quad [Y_{a_1a_2}, \Pi_{b_1b_2}] = i\eta_{a_1a_2|b_1b_2} \tag{51}$$

The other *dimensionless* generators are[5]

$$\begin{aligned} M_{a_1a_2|5} &= \frac{Y_{a_1a_2}}{\lambda_l^2}\frac{\Pi_5}{\lambda_p} + \frac{Y_5}{\lambda_l}\frac{\Pi_{a_1a_2}}{\lambda_p^2}, \\ M_{5|a_1a_2} &= \frac{Y_5}{\lambda_l}\frac{\Pi_{a_1a_2}}{\lambda_p^2} + \frac{Y_{a_1a_2}}{\lambda_l^2}\frac{\Pi_5}{\lambda_p} \\ L_{a_1a_2|5} &= \frac{Y_{a_1a_2}}{\lambda_l^2}\frac{\Pi_5}{\lambda_p} - \frac{Y_5}{\lambda_l}\frac{\Pi_{a_1a_2}}{\lambda_p^2}, \end{aligned} \tag{52}$$

$$L_{5|a_1a_2} = \frac{Y_5}{\lambda_l}\frac{\Pi_{a_1a_2}}{\lambda_p^2} - \frac{Y_{a_1a_2}}{\lambda_l^2}\frac{\Pi_5}{\lambda_p} \tag{53}$$

such that

$$Z_{a_1a_2|5} = \frac{1}{2}(M_{a_1a_2|5} + L_{a_1a_2|5}), \quad Z_{5|a_1a_2} = \frac{1}{2}(M_{5|a_1a_2} + L_{5|a_1a_2}) \tag{54}$$

and leading to the following generators

$$Z_{[a_1a_2]} \equiv \frac{1}{\sqrt{2}}\left(\frac{X_{a_1a_2}}{\lambda_l^2} - i\frac{P_{a_1a_2}}{\lambda_p^2}\right) = \alpha Z_{a_1a_2|5} + \beta Z_{5|a_1a_2}, \tag{55a}$$

$$Z^{\dagger}_{[a_1a_2]} \equiv \frac{1}{\sqrt{2}}\left(\frac{X_{a_1a_2}}{\lambda_l^2} + i\frac{P_{a_1a_2}}{\lambda_p^2}\right) = \alpha^* Z_{a_1a_2|5} + \beta^* Z_{5|a_1a_2} \tag{55b}$$

where α, β are suitable complex-valued coefficients chosen so that[6]

$$[Z_{[a_1a_2]}, Z^{\dagger}_{[b_1b_2]}] = -(\eta_{a_1a_2|b_1b_2}\mathcal{I} + M_{a_1a_2|b_1b_2}) \tag{56}$$

$$[Z_{[a_1a_2]}, Z_{[b_1b_2]}] = [Z^{\dagger}_{[a_1a_2]}, Z^{\dagger}_{[b_1b_2]}] = -iL_{a_1a_2|b_1b_2}. \tag{57}$$

Finally, from Equations (55a,b)–(57), one arrives at the desired result

$$\left[\frac{X_{a_1a_2}}{\lambda_l^2}, \frac{P_{b_1b_2}}{\lambda_p^2}\right] = i(\eta_{a_1a_2|b_1b_2}\mathcal{I} + M_{a_1a_2|b_1b_2}) \tag{58}$$

$$[X_{a_1a_2}, X_{b_1b_2}] = -i(\lambda_l)^4 L_{a_1a_2|b_1b_2}; \quad [P_{a_1a_2}, P_{b_1b_2}] = i(\lambda_p)^4 L_{a_1a_2|b_1b_2}; \tag{59}$$

The above construction can be *extended* to higher-rank antisymmetric tensor coordinates and momenta $Y_{a_1a_2,a_3}, \Pi_{a_1a_2a_3}, \cdots$ leading to the generators $Z_{a_1a_2a_3|b_1b_2b_3}; Z_{a_1a_2a_3|5}; Z_{5|a_1a_2a_3}, \ldots,$ and whose commutators are the extensions of the equations above. The end result is

$$\left[\frac{X_{a_1a_2\cdots a_n}}{\lambda_l^n}, \frac{P_{b_1b_2\cdots b_n}}{\lambda_p^n}\right] = i(\eta_{a_1a_2\cdots a_n|b_1b_2\cdots b_n}\mathcal{I} + M_{a_1a_2\cdots a_n|b_1b_2\cdots b_n}) \tag{60}$$

$$[X_{a_1a_2\cdots a_n}, X_{b_1b_2\cdots b_n}] = -i(\lambda_l)^{2n} L_{a_1a_2\cdots a_n|b_1b_2\cdots b_n} \tag{61a}$$

$$[P_{a_1a_2\cdots a_n}, P_{b_1b_2\cdots b_n}] = i(\lambda_p)^{2n} L_{a_1a_2\cdots a_n|b_1b_2\cdots b_n}, \tag{61b}$$

where $\eta_{a_1a_2\cdots a_n|b_1b_2\cdots b_n}$ can be written as the determinant of the $n \times n$ matrix whose entries are $\eta^{a_ib_j}$ with $i,j = 1,2,\cdots,n$. The same occurs with $\delta^{a_1a_2\cdots a_n}_{b_1b_2\cdots b_n}$ where the entries are $\delta^{a_i}_{b_j}$. One finds that Equations (60) and (61a,b) do not differ too much from those corresponding equations of the Clifford–Yang algebra [26,27]. In the latter algebra, $\mathcal{I} = 2Z_{55} = M_{55}$ is replaced by $\mathcal{N} \equiv J^{56}$; there are no $M_{a_1a_2\cdots a_n|b_1b_2\cdots b_n}$ terms, and λ_l, λ_p are replaced by $L_P, \mathcal{L}^{-1}$, respectively, where $L_P, \mathcal{L}$ are the lower and upper length scales.

To sum up, all the commutation relations can be obtained from

$$\begin{aligned}&[Z_{a_1a_2\cdots a_n|b_1b_2\cdots b_n}, Z_{c_1c_2\cdots c_n|d_1d_2\cdots d_n}] = \\ &-i(\eta_{b_1b_2\cdots b_n|c_1c_2\cdots c_n}Z_{a_1a_2\cdots a_n|d_1d_2\cdots d_n} - \eta_{a_1a_2\cdots a_n|d_1d_2\cdots d_n}Z_{c_1c_2\cdots c_n|b_1b_2\cdots b_n}).\end{aligned} \tag{62a}$$

$$[Z_{a_1a_2\cdots a_n|5}, Z_{5|b_1b_2\cdots b_n}] = -i(\eta_{55}Z_{a_1a_2\cdots a_n|b_1b_2\cdots b_n} - \eta_{a_1a_2\cdots a_n|b_1b_2\cdots b_n}Z_{55}), \quad \cdots \tag{62b}$$

We finalize this section by pointing out that Meljanac and collaborators introduced also the tensorial canonical Heisenberg algebras as a tool to provide the solution, e.g., of the Snyder models describing noncommutative quantum spacetime coordinates. In particular, the Yang model and its generalizations were discussed very recently [30] .

4. Curved Phase Space Due to Noncommutative Coordinates and Momenta

Noncommuting momentum operators are a reflection of the spacetime curvature after invoking the QM prescription $p_\mu \leftrightarrow -i\hbar\nabla_\mu$. By Born's reciprocity, noncommuting coordinates are a reflection of the momentum space curvature after invoking $x_\mu \leftrightarrow i\hbar\tilde{\nabla}_\mu$, where the tilde derivatives represent derivatives with respect to the momentum variables.

Having reviewed the basics of the Yang algebra of noncommutative phase spaces, Born's reciprocal relativity, and the extended Yang and (deformed) quaplectic algebras, in this section, we provide a solution for the exact analytical mapping of the noncommuting x^μ, p^μ operator variables (associated to an 8D curved phase space) into the canonical Y^A, Π^A operator variables of a flat 12D phase space. We explore the geometrical implications of this mapping which provides, in the *classical* limit, the embedding functions $Y^A(x,p), \Pi^A(x,p)$ of an 8D curved phase space into a flat 12D phase space background. The latter embedding functions determine the functional forms of the base spacetime metric $g_{\mu\nu}(x,p)$, the fiber metric of the vertical space $h^{ab}(x,p)$, and the nonlinear connection $N_{a\mu}(x,p)$ associated with the 8D cotangent space of the 4D spacetime.

Instead of working with the above *canonical* coordinates Y^A and momenta Π^A in a flat 12D phase space ($A = 1,2,\cdots,5,6$), the authors [31] were interested in finding Hermitian realizations of the above Yang algebra in an 8D phase space, and given in terms of the *canonical* variables $\tilde{x}_\mu, \tilde{p}_\mu$ satisfying $[\tilde{x}_\mu, \tilde{x}_\nu] = [\tilde{p}_\mu, \tilde{p}_\nu] = 0$, and $[\tilde{x}_\mu, \tilde{p}_\nu] = i\eta_{\mu\nu}$, with $\mu,\nu = 1,2,3,4$.

The Yang model studied by [31] was characterized by the choice of the commutator $[x_\mu, p_\nu] = i\gamma_{\mu\nu}(x,p)$, and where the rank-two tensor $\gamma_{\mu\nu}(x,p)$ is of the form

$$\gamma_{\mu\nu} = h(x^2, p^2, x\cdot p + p\cdot x)\eta_{\mu\nu} \tag{63}$$

with h a judicious function of the Lorentz scalars $x^2, p^2, x\cdot p + p\cdot x$, which is determined by solving the Jacobi identities. The rank-two tensor $\gamma_{\mu\nu}(x,p)$ is what leads to the generalized uncertainty relations. The triple special relativity model [32], an extension of [33,34], was characterized by a different choice of $\gamma_{\mu\nu}(x,p)$. The Lorentz generators were represented as

$$\mathcal{J}_{\mu\nu} = \frac{1}{2}(x_\mu p_\nu - x_\nu p_\mu + p_\nu x_\mu - p_\mu x_\nu) \tag{64}$$

In particular, the authors [31] looked for representations where the generators $\mathcal{J}_{\mu\nu}$ and the tensor $\gamma_{\mu\nu}$ could be written in terms of the canonical variables $\tilde{x}_\mu$ and $\tilde{p}_\nu$. This required the arduous task of finding the nontrivial map among the *noncanonical* variables x_μ, p_μ and the canonical ones $\tilde{x}_\mu, \tilde{p}_\mu : x_\mu = x_\mu(\tilde{x}, \tilde{p}); p_\mu = p_\mu(\tilde{x}, \tilde{p})$. The map was found iteratively in powers of $\tilde{x}, \tilde{p}$. The explicit technical details of this map can be found in [31].

4.1. Mapping of x^μ, p^μ to the Y^A, Π^A Variables in Flat Phase Space

The Y^5, Y^6, Π^5, and Π^6 canonical coordinates and momenta (operators) in the flat 12D phase space are scalars from the point of view of the 8D curved phase space parametrized by the noncanonical coordinates x^μ and momenta p^μ. Therefore, Y^5, Y^6, Π^5, and Π^6 must be functions of the Lorentz scalars

$$x^2 = \eta_{\mu\nu}x^\mu x^\nu, \quad p^2 = \eta_{\mu\nu}p^\mu p^\nu, \quad x\cdot p = \eta_{\mu\nu}x^\mu p^\mu, \quad p\cdot x = \eta_{\mu\nu}p^\mu x^\nu, \quad \mu,\nu = 1,2,3,4 \tag{65}$$

Setting $\alpha = \mathcal{L}^{-1}, \beta = L_P$, due to the Born reciprocity principle, one must have functions $f(z_1, z_2, z_3)$ of the arguments z_1, z_2, and z_3 given by the following combination of Hermitian variables (operators)

$$z_1 \equiv (\alpha^2 x^2 + \beta^2 p^2), \quad z_2 \equiv (x \cdot p + p \cdot x), \quad z_3 \equiv i(x \cdot p - p \cdot x), \quad \alpha = \mathcal{L}^{-1}, \quad \beta = L_P \quad (66)$$

The arguments z_1, z_2, and z_3 are invariant under $\alpha \leftrightarrow \beta$, $x \leftrightarrow p$, and $i \leftrightarrow -i$, if one wishes to implement Born's reciprocity symmetry. Therefore, one must have functions of the form

$$Y^5 = Y^5(z_1, z_2, z_3), \quad Y^6 = Y^6(z_1, z_2, z_3), \quad \Pi^5 = \Pi^5(z_1, z_2, z_3), \quad \Pi^6 = \Pi^6(z_1, z_2, z_3) \quad (67)$$

For instance, one could have functions linear in z_1, z_2, and z_3 defined as follows

$$Y^5(x,p) = a_1(\alpha^2 x^2 + \beta^2 p^2) + b_1(x \cdot p) + b_1^*(p \cdot x) + c_1 \quad (68a)$$
$$Y^6(x,p) = a_2(\alpha^2 x^2 + \beta^2 p^2) + b_2(x \cdot p) + b_2^*(p \cdot x) + c_2 \quad (68b)$$
$$\Pi^5(x,p) = a_3(\alpha^2 x^2 + \beta^2 p^2) + b_3(x \cdot p) + b_3^*(p \cdot x) + c_3 \quad (68c)$$
$$\Pi^6(x,p) = a_4(\alpha^2 x^2 + \beta^2 p^2) + b_4(x \cdot p) + b_4^*(p \cdot x) + c_4. \quad (68d)$$

where a_i, b_i, c_i $(i = 1, 2, 3, 4)$ are judicious numerical (dimensionful) coefficients. The units of the coefficients in Equations (68a,b) are those of length, while those in Equations (68c,d) are those of mass. Note that the b_i coefficients in Equations (68a–e) are complex-valued: $b_i = \gamma_i + i\delta_i$. The reason is that the combination

$$b_i(x \cdot p) + b_i^*(p \cdot x) = \gamma_i(x \cdot p + p \cdot x) + i\delta_i(x \cdot p - p \cdot x) = \gamma_i z_2 + \delta_i z_3, \quad i = 1, 2, 3, 4 \quad (68e)$$

ensures that Equation (68e) is Hermitian by construction. Equation (68e) is also invariant under Born's reciprocity $x \leftrightarrow p$ and $i \leftrightarrow -i$. We show that Equations (68a–e) should, in principle, provide satisfactory solutions to the embedding problem defined below.

The $[x^\mu, p^\nu]$ commutator is defined as

$$[x^\mu, p^\nu] \;=\; x^\mu\, p^\nu \;-\; p^\nu\, x^\mu \;=\; i\,\gamma^{\mu\nu}(x,p), \quad (69)$$

where $\gamma^{\mu\nu}(x,p)$ is a second-rank tensor, not necessarily symmetric, that we refrain from identifying as a metric tensor. The above commutator can also be expressed in terms of the 6D angular momenta variables displayed by Equations (32a,b) and (33) as

$$[x^\mu, p^\nu] = i\gamma^{\mu\nu}(x,p) = -i\alpha\beta J^{56}(x,p)\eta^{\mu\nu} =$$
$$i\alpha\beta[Y^5(x,p)\Pi^6(x,p) - Y^6(x,p)\Pi^5(x,p)]\eta^{\mu\nu}, \quad \alpha = \mathcal{L}^{-1}, \beta = L_P \quad (70)$$

Therefore, from Equations (69) and (70), one arrives at the following relation, after contracting both equations with $\eta_{\mu\nu}$,

$$\frac{1}{4i}\eta_{\mu\nu}(x^\mu p^\nu - p^\nu x^\mu) = \frac{1}{4i}(x \cdot p - p \cdot x) =$$
$$\alpha\beta(Y^5(x,p)\Pi^6(x,p) - Y^6(x,p)\Pi^5(x,p))) = -\alpha\beta\mathcal{N} \quad (71)$$

Therefore, in this particular case, one finds that the tensor is symmetric $\gamma^{\mu\nu}(x,p) = \Phi(x,p)\eta^{\mu\nu}$ and such that the conformal factor $\Phi(x,p)$ is Hermitian and given by the left-hand side of Equation (71). The right-hand side of (71) is Hermitian because J^{56} is Hermitian due to the canonical and Hermiticity nature of the 6D variables: $(Y^5\Pi^6)^\dagger = \Pi^6 Y^5 = Y^5\Pi^6$, and $(Y^6\Pi^5)^\dagger = \Pi^5 Y^6 = Y^6\Pi^5$ resulting from the commutators of the 6D canonical variables given by Equation (31).

From Equations (32a,b), one learns that the 4D operators x^μ, p^μ admitted a 6D angular momentum realization of the form

$$x^\mu = \beta J^{\mu 5} = -\beta(Y^\mu \Pi^5 - Y^5 \Pi^\mu), \quad \beta = L_P \tag{72}$$

$$p^\mu = \alpha J^{\mu 6} = -\alpha(Y^\mu \Pi^6 - Y^6 \Pi^\mu), \quad \alpha = \mathcal{L}^{-1} \tag{73}$$

From Equations (72) and (73), one can deduce the relation

$$\mathcal{J}^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu = \alpha\beta J^{56}(Y^\mu \Pi^\nu - Y^\nu \Pi^\mu), \tag{74}$$

where $J^{56} \equiv \mathcal{N}$ and $J^{\mu\nu}$ are given by Equation (33) explicitly in terms of the 6D canonical variables Y^A, Π^B.

One can *invert* the relations in Equations (72) and (73) as follows. After multiplying Equations (72) and (73) on the *right* by Π^6 and Π^5, respectively, and subtracting the top equation from the bottom one, it yields

$$\beta^{-1} x^\mu \Pi^6 - \alpha^{-1} p^\mu \Pi^5 = \Pi^\mu \mathcal{N} = \mathcal{N} \Pi^\mu \tag{75a}$$

due to the canonical nature of the 6D variables Y^A and Π^A described by the commutators in Equation (31) and which allows us to reorder the relevant factors due to the commutativity.

Moreover, multiplying Equations (72) and (73) on the *right* by Y^6 and Y^5, respectively, and subtracting the top equation from the bottom one yield

$$\beta^{-1} x^\mu Y^6 - \alpha^{-1} p^\mu Y^5 = Y^\mu \mathcal{N} = \mathcal{N} Y^\mu \tag{75b}$$

Next, we see that the functional forms of $Y^5(x,p), Y^6(x,p), \Pi^5(x,p)$, and $\Pi^6(x,p)$ provided by Equations (68a–e) lead to solutions to Equation (71), which, in turn, yields automatically the solutions to Equation (75a,b). In doing so, one finds the solutions to the embedding problem $Y^\mu = Y^\mu(x,p); \Pi^\mu = \Pi^\mu(x,p)$, with $\mathcal{N}(x,p) \equiv J^{56}(x,p) = -(Y^5\Pi^6 - Y^6\Pi^5)(x,p)$, where $[\mathcal{N}, Y^\mu] = [\mathcal{N}, \Pi^\mu] = 0$. The operator $\mathcal{N}$ appearing in the right-hand side of Equation (75a,b) can be moved to the left-hand side via the inverse $\mathcal{N}^{-1}$ operator, and that can be defined as a formal power series as follows: $[1-(1-\mathcal{N})]^{-1} = 1 + (1-\mathcal{N}) + (1-\mathcal{N})^2 + \cdots$.

Thus, from Equations (71) and (75a,b) one can then construct the maps from the x^μ, p^μ noncanonical (operator) variables in 4D to the canonical (operator) variables Y^A, Π^A in 6D. After a laborious but straightforward procedure we find the following family of solutions

$$Y^5(x,p) = \kappa_1\beta z_1 + \kappa_2\beta z_2 + \kappa_3\beta z_3 + \kappa_4\beta \tag{76a}$$

$$Y^6(x,p) = \kappa_1\beta z_1 + \kappa_2\beta z_2 + \kappa_3\beta z_3 + (\kappa_4+1)\beta \tag{76b}$$

$$\Pi^5(x,p) = \kappa_1\beta^{-1} z_1 + \kappa_2\beta^{-1} z_2 + \frac{5}{4}\kappa_3\beta^{-1} z_3 + \kappa_4\beta^{-1} \tag{76c}$$

$$\Pi^6(x,p) = \kappa_1\beta^{-1} z_1 + \kappa_2\beta^{-1} z_2 + \frac{5}{4}\kappa_3\beta^{-1} z_3 + (\kappa_4+1)\beta^{-1} \tag{76d}$$

where $\kappa_3 = (\alpha\beta)^{-1}$ and κ_1, κ_2, and κ_4 are three arbitrary parameters. This is due to the nonlinearity of the equations that one is solving. These solutions (76a–d) have the form $Y^6 = Y^5 + \beta; \Pi^5 = \Pi^6 - \beta^{-1}$ such that $\alpha\beta\ Y^{[5}\ \Pi^{6]} = -\frac{z_3}{4} = -\alpha\beta\mathcal{N}$ as required by Equation (71).

When one takes the classical limit, upon restoring $\hbar$ which was set to unity in the terms $\gamma_i z_2 \to \frac{\gamma_i}{\hbar} z_2$ of Equations (68e), in order to match units, one can see that these terms are *singular* in the $\hbar \to 0$ limit, whereas the terms $\frac{\delta_i}{\hbar} z_3 \to -4\delta_i$ are well-behaved and yield constants.

For these reasons, we just adhere to the following prescription when finding the *classical* limit of the embedding functions $Y^A(x,p), \Pi^A(x,p)$. We could simply drop the *singular* $\frac{1}{\hbar} z_2$ terms in Equation (76a–d) by setting the arbitrary constant κ_2 to zero $\kappa_2 = 0$ and set the $\frac{1}{\hbar} z_3$ terms to constants that can be reabsorbed into a redefinition of the κ_4 parameter in the explicit solutions for Y^5, Y^6, Π^5, and Π^6 given by Equation (76a–d). In doing so, one ends up with the following expressions in the *classical* limit

$$Y^5(z_1) = \kappa_1 \beta z_1 + \beta(\kappa_4 - 4(\alpha\beta)^{-1}) \tag{77a}$$

$$Y^6(z_1) = \kappa_1 \beta z_1 + \beta(\kappa_4 + 1 - 4(\alpha\beta)^{-1}) \tag{77b}$$

$$\Pi^5(z_1) = \kappa_1 \beta^{-1} z_1 + \beta^{-1}(\kappa_4 - 5(\alpha\beta)^{-1}) \tag{77c}$$

$$\Pi^6(z_1) = \kappa_1 \beta^{-1} z_1 + \beta^{-1}(\kappa_4 + 1 - 5(\alpha\beta)^{-1}) \tag{77d}$$

To conclude, one can finally obtain the explicit solutions for $Y^\mu, (z_1, x^\mu, p^\mu); \Pi^\mu(z_1, x^\mu, p^\mu)$, in the classical limit, given in terms of the functions $Y^5(z_1), Y^6(z_1), \Pi^5(z_1)$, and $\Pi^6(z_1)$ in Equation (77a–d) (and x^μ, p^μ) as follows:

$$\alpha x^\mu \Pi^6(z_1) - \beta p^\mu \Pi^5(z_1) = -\Pi^\mu(z_1, x^\mu, p^\mu) \tag{78a}$$

$$\alpha x^\mu Y^6(z_1) - \beta p^\mu Y^5(z_1) = -Y^\mu(z_1, x^\mu, p^\mu) \tag{78b}$$

where $z_1 \equiv \alpha^2 x^2 + \beta^2 p^2$, $\alpha = \mathcal{L}^{-1}; \beta = L_P$. Next, we study the geometrical implications of the (classical) embedding solutions found in this section and provided by Equations (77a–d) and (78a,b).

4.2. Embedding an 8D Curved Phase Space into a 12D Flat Phase Space

The previous section involved the use of coordinates and momenta *operators*. In this section, we shall deal with *classical* variables (**c**-numbers) x, p. A more rigorous notation in the previous section would have been to assign "hats" to operators $\hat{x}^\mu, \hat{p}^\mu; \hat{Y}^A, \hat{\Pi}^A$. For the sake of simplicity, we avoided it. The geometry of the cotangent bundle of spacetime (phase space) can be best-explored within the context of Lagrange–Finsler, Hamilton–Cartan geometry [15–18]. The line element in the 8D curved phase space is

$$(ds)^2 = g_{\mu\nu}(x,p) dx^\mu dx^\nu + h^{ab}(x,p)(dp_a + N_{a\mu}(x,p)dx^\mu)(dp_b + N_{b\nu}(x,p)dx^\nu) \tag{79}$$

where $g_{\mu\nu}(x,p), h^{ab}(x,p)$ are the base spacetime and internal space metrics, respectively, with $a, b = 1,2,3,4$, $\mu, \nu = 1,2,3,4$, and $N_{a\mu}(x,p)$ is the nonlinear connection.

One should note that the metric tensor $g_{\mu\nu}$ is *not* the vertical Hessian of the square of a Finsler function, and h^{ab} is *not* the inverse of $g_{\mu\nu}$. h^{ab} represents, physically, the cotangent bundle's internal-space metric tensor which is independent from the base-spacetime metric tensor $g_{\mu\nu}$. The number of total components of $g_{\mu\nu}, h^{ab}, N_{a\mu}$ is $10 + 10 + 16 = 36 = (8 \times 9)/2$).

The generalized (vacuum) gravitational field equations associated with the geometry of the 8D cotangent bundle differ considerably from the standard (vacuum) Einstein field equations in 8D based on Riemannian geometry. Thus, for instance, by using a base-spacetime $g_{\mu\nu}$ metric to be *independent* from the internal-space metric h_{ab} and a nonlinear connection $N_{\mu a}$, it might avoid the reduction of the solutions of the generalized gravitational field equations to the standard Schwarzschild (Tangherlini) solutions when radial symmetry is imposed.

For example, in [19] we further studied a scalar-gravity model in curved phase spaces proposed by [20,21]. After a very laborious procedure, the variation of the action S with respect to the fundamental fields

$$\frac{\delta\mathcal{S}}{\delta g_{\mu\nu}} = 0, \quad \frac{\delta\mathcal{S}}{\delta h_{ab}} = 0, \quad \frac{\delta\mathcal{S}}{\delta N_{\mu a}} = 0, \quad \frac{\delta\mathcal{S}}{\delta\Phi} = 0 \tag{80}$$

led to very *complicated* field equations which differed considerably from the Einstein field equations. Exact nontrivial analytical solutions for the base-spacetime $g_{\mu\nu}$, the internal-space metric h_{ab} components, the nonlinear connection N_{ia}, and the scalar field Φ were found that obeyed the generalized gravitational field equations, in addition to satisfying the *zero*-torsion conditions for *all* of the torsion components. See [19] for details.

The embedding of the 8D curved phase space into the 12D flat phase space is described by equating the 8D line interval ds^2 in (79) with the 12D one $ds^2 = \eta_{AB}dZ^AdZ^B$. After doing so, given $Z^A \equiv (Y^A, \Pi^A)$ one learns that

$$g_{\mu\nu} + h^{ab}N_{a\mu}N_{b\nu} = \eta_{AB}\frac{\partial Z^A}{\partial x^\mu}\frac{\partial Z^A}{\partial x^\nu} \tag{81}$$

$$h^{ab} = \eta_{AB}\frac{\partial Z^A}{\partial p_a}\frac{\partial Z^A}{\partial p_b} \tag{82}$$

$$h^{ab}N_{b\nu} = \eta_{AB}\frac{\partial Z^A}{\partial p_a}\frac{\partial Z^A}{\partial x^\nu} \quad A, B = 1, 2, \cdots, 5, 6 \tag{83}$$

Equations (81)–(83) *determine* the functional form of $g_{\mu\nu}, h^{ab}, N_{a\mu}$ after one inserts the functional forms of the embedding functions $Z^A(x,p) = Y^A(x,p), \Pi^A(x,p)$ found in the previous section. However, there is a subtlety: to match indices with the ones appearing in Equations (77a–d) and (78a,b) it is necessary to make the following key *replacements* (index adjustments) $p_a \to p^\sigma$, $p_b \to p^\tau$, $h^{ab} \to h_{\sigma\tau}$, $N_{a\mu} \to N^\sigma_\mu$, $N_{b\nu} \to N^\tau_\nu$ in Equations (79) and (81)–(83).

To sum up, the (classical) embedding functions $Z^A(x,p) = Y^A(x,p), \Pi^A(x,p)$ obtained in the previous section in Equations (77a–d) and (78a,b) determine the functional form of $g_{\mu\nu}, h^{ab}, N_{a\mu}$ in Equations (81)–(83), after adjusting the indices. The key question is whether or not the solutions found for $g_{\mu\nu}, h^{ab}, N_{a\mu}$ *also* solve the vacuum field equations. If not, can one find the appropriate field/matter sources which are consistent with these solutions? It is natural to assume that quantum matter/fields could be the source of the noncommutativity of the spacetime coordinates and momenta. After all, quantum fields live in spacetime. If this were not the case, what then is the source of this phase-space noncommutativity? Is it spacetime foam, dark matter, dark energy? If one expects to have a space–time–matter unification in the quantum gravity program, then, if matter curves spacetime, spacetime, in turn, could backreact on matter curving momentum space, "curving matter". To conclude, to find solutions of Equations (81)–(83) for $g_{\mu\nu}, h^{ab}, N_{a\mu}$ (after adjusting indices) is a highly nontrivial task, and so is to verify that they also solve the field equations in [19–21].

5. Concluding Remarks

After a review of Born's reciprocal relativity and its physical implications, this work was mainly devoted to the Yang and the deformed quaplectic algebras associated with noncommutative phase spaces, and to their extensions involving antisymmetric tensor coordinates and momenta of different ranks. Our approach to construct extended Yang algebras *differs* from the study by [35]. We finalized with an analysis of the embedding of an 8D curved phase space into a 12D flat phase space which provided a direct link between noncommutative curved phase spaces in lower dimensions to commutative flat phase spaces in higher dimensions. Left from our discussion was the role of quantum groups, Hopf algebras, κ-deformed Poincare algebras, and of the deformed special relativity [32–34,36–40]. This will be the subject of future investigations.

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Notes

1 Strictly speaking, $U(1,4)$ is a pseudo-unitary group. After performing the Weyl unitary "trick" via an analytical continuation $U(1,4) \to U(5)$, one obtains the unitary group $U(5)$ comprised of 5×5 unitary matrices obeying $U^\dagger = U^{-1}$. A unitary matrix can be written as $U = e^A$, where A is an anti-Hermitian matrix $A^\dagger = -A$, and any anti-Hermitian matrix A can be written as $A = \pm iH$, where H is Hermitian; therefore, all group elements can be written in the form $U = e^{\pm i\theta^{AB} Z_{AB}}$, where θ^{AB} are the corresponding parameters associated to every generator.

2 We choose a different signature than the one in the Introduction.

3 Our choice differs by a minus sign from the conventional definition.

4 A simple inspection reveals that a correspondence of the form $\frac{x^\mu}{L_P} = a_1 J^{\mu 5} + b_1 J^{\mu 6}; \mathcal{L} p^\mu = a_2 J^{\mu 5} + b_2 J^{\mu 6}$ will automatically lead to $b_1 = 0, a_2 = 0$; or $b_2 = 0, a_1 = 0$ resulting from the antisymmetry of the commutators $[x^\mu, x^\nu], [p^\mu, p^\nu]$

5 Since $\lambda_l \lambda_p = 1$, in units of $\hbar = 1$, the powers of λ_l, λ_p decouple explicitly from Equation (44a,b)

6 Note that one must not confuse $Z_{ab} \equiv \frac{1}{2}(M_{ab} + L_{ab})$ with $Z_{[a_1 a_2]}$ defined by Equation (55a)

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