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Integrability enabled computations relating to the fixed trace Laguerre ensemble*

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Abstract

Studies of density matrices for random quantum states lead naturally to the fixed trace Laguerre ensemble in random matrix theory. Previous studies have uncovered explicit rational function formulas for moments of purity statistic (trace of the squared density matrix), and also a third order linear differential equation satisfied by the eigenvalue density. We further probe the origin of these results from the viewpoint of integrability, which is taken here to mean wider classes of recursions and differential equations, and give extensions. Prominent in our study are first order linear matrix differential equations. One application given is to the derivation of the third order scalar equation for the density. Another is to obtain the explicit rational function formula for the variance of the purity statistic in the β generalised fixed trace Laguerre ensemble. In the original case ($\beta = 2$), the purity cumulants are expressed in terms of the large argument expansion of a particular σ -Painlevé IV transcendent. In a different but related direction, the exact computation of the two-point correlation for the fixed determinant circular unitary ensemble $SU(N)$ is given in the appendix.

1. Introduction

By definition, a standard Gaussian complex random matrix G has all entries independent and identically distributed as standard complex random variables $N[0, 1/\sqrt{2}] + iN[0, 1/\sqrt{2}]$ — here the adjective ‘standard’ refers to the mean of the squared modulus equalling unity. The Gaussian unitary ensemble (GUE) of complex Hermitian matrices is constructed out of the matrices $\{G\}$ by forming $\frac{1}{2}(G + G^\dagger)$. The corresponding distribution on the space of Hermitian matrices is proportional to $e^{-\text{Tr} G^2}$. Changing variables to the eigenvalues $\{\lambda_j^2\}_{j=1}^N$ (here we are assuming G is of size $N \times N$) and the eigenvectors gives for the eigenvalue probability density function (PDF)

$$\frac{1}{C_N^{(G)}} \prod_{l=1}^N e^{-\lambda_l^2} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2, \quad (1.1)$$

supported on $\lambda_l \in \mathbb{R}$ ($l = 1, \dots, N$), where the normalisation $C_N^{(G)}$ is given by

$$C_N^{(G)} = \pi^{N/2} 2^{-N(N-1)/2} \prod_{l=1}^N l!; \quad (1.2)$$

see e.g. [28, proposition 1.3.4 with $\beta = 2$].

* Dedicated to the memory of Santosh Kumar and his work on exact results in RMT and their applications. For the context of the present work in the interests and contributions of Santosh Kumar in random matrix theory (RMT), the reader is referred to the tribute and review article (Forrester 2025 *J. Phys. A: Math. Theor.* **58** 463001).

Suppose instead that the size of G is $n \times N$, $n \geq N$, also referred to as a rectangular complex Ginibre matrix [14]. The Laguerre unitary ensemble (LUE) of complex Hermitian positive definite matrices is constructed out of $\{G\}$ by forming $G^\dagger G$, and the corresponding eigenvalue PDF is given by

$$p_{N,a}^{(L)}(\lambda_1, \dots, \lambda_N) := \frac{1}{C_{N,a}^{(L)}} \prod_{l=1}^N \lambda_l^a e^{-\lambda_l} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2, \quad a := n - N, \quad (1.3)$$

supported on $\lambda_l \in \mathbb{R}^+$ ($l = 1, \dots, N$), where the normalisation $C_{N,a}^{(L)}$ is given by

$$C_{N,a}^{(L)} = N! \prod_{l=0}^{N-1} \Gamma(l+1) \Gamma(a+l+1); \quad (1.4)$$

see e.g. [28, proposition 3.2.2 with $\beta = 2$, $m \mapsto N$]. The fixed trace (chosen to equal unity) LUE (fLUE) is obtained by the same construction involving $\{G\}$, now multiplied by a scalar factor $1/\text{Tr } G^\dagger G$. The eigenvalue PDF is

$$p_{N,a}^{(\text{fL})}(\lambda_1, \dots, \lambda_N) := \frac{1}{C_{N,a}^{(\text{fL})}} \delta\left(1 - \sum_{l=1}^N \lambda_l\right) \prod_{l=1}^N \lambda_l^a \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2, \quad a := n - N, \quad (1.5)$$

supported on $\lambda_l \in (0, 1)$ ($l = 1, \dots, N$), where the normalisation $C_{N,a}^{(\text{fL})}$ is given by

$$C_{N,a}^{(\text{fL})} = \frac{N!}{\Gamma(aN + N^2)} \prod_{l=0}^{N-1} \Gamma(l+1) \Gamma(a+l+1); \quad (1.6)$$

see e.g. [28, equation (4.155) with $\beta = 2$, $m \mapsto N$]. Note that the eigenvalues must sum to unity as indicated by the Dirac delta function.

The eigenvalue PDF (1.5) is prominent in the study of the bipartite entanglement of a finite basis quantum mechanical system; see e.g. [8]. The two subsystems, denoted A and B say, are taken to have dimensions n and N and orthonormal bases $\{|a_i\rangle\}_{i=1,\dots,n}$, $\{|b_i\rangle\}_{i=1,\dots,N}$ respectively. Hence a state $|\psi\rangle$ can be written

$$|\psi\rangle = \sum_{i=1}^n \sum_{j=1}^N x_{i,j} |a_i\rangle \otimes |b_j\rangle. \quad (1.7)$$

On the other hand, according to the Schmidt decomposition, for a given $|\psi\rangle$, there exists orthonormal vectors $\{|v_i^A\rangle\}_{i=1,\dots,N}$ and $\{|v_i^B\rangle\}_{i=1,\dots,N}$ relating to the reduced density matrices of the two subsystems such that

$$|\psi\rangle = \sum_{i=1}^N \sqrt{\lambda_i} |v_i^A\rangle \otimes |v_i^B\rangle, \quad \sum_{j=1}^N \lambda_j = 1; \quad (1.8)$$

see e.g. [28, section 3.3.4]. Under the assumption that the coefficients $\{x_{i,j}\}$ in (1.7) are independent standard complex Gaussians so that $[x_{i,j}] = G$ as specified above (1.3), up to the constraint $\sum_{i=1}^n \sum_{j=1}^N |x_{ij}|^2 = \text{Tr } G^\dagger G = 1$ to ensure the normalisation $\langle \psi | \psi \rangle = 1$, the distribution of the coefficients $\{\lambda_i\}$ in (1.8) is given by (1.5).

Quantities used to quantify the bipartite entanglement include the purity, defined as $\sum_{j=1}^N \lambda_j^2$ (i.e. the trace of the squared reduced density matrix for subsystems A or B ; see e.g. [8]) and the von Neumann entropy $-\sum_{j=1}^N \lambda_j \log \lambda_j$ (i.e. minus the trace of the reduced density matrix times the logarithm of the reduced density matrix). Generalising both is the so-called quantum Tsallis entropy $(1/(1-q))(1 - \sum_{j=1}^N \lambda_j^q)$ (set $q=2$ to relate to the purity, and take the limit $q \rightarrow 1$ to obtain the von Neumann entropy). All these quantities are examples of linear statistics, being of the form

$$A = \sum_{j=1}^N a(\lambda_j) \quad (1.9)$$

for some function of a single variable $a(x)$.

Consider a random state as specified below (1.8). Let $\rho_{(1)}^{(\text{fl})}(x)$ denote the density of eigenvalues corresponding to the PDF (1.5). With respect to the random state, for the linear statistic A we have

$$\langle A \rangle = \int_I a(x) \rho_{(1)}^{(\text{fl})}(x) dx, \quad (1.10)$$

where I denotes the interval of support of the density. An expression for $\rho_{(1)}^{(\text{fl})}(x)$ is given in [53, equation (31)], which involves the difference of two single sums from 0 up to $N-1$ of Gauss hypergeometric functions. Another form, involving a sum over $\{(x/(1-x))^l\}_{l=0}^{2N-2}$, where the coefficients themselves are further sums from 0 up to N , is known from [1, 73, 75] and can also be found in [53]. With a theme based on computations relating to (1.5) enabled by integrability—specifically recursions and differential Equations—of particular interest to us is the fact, deduced in [2], that $\rho_{(1)}^{(\text{fl})}(x)$ satisfies the third order linear differential equation

$$\left(a_3(x) \frac{d^3}{dx^3} + a_2(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_0(x) \right) f(x) = 0, \quad (1.11)$$

where

$$\begin{aligned} a_0(x) &= x(2N+a)(N^2+aN-2) - a^2 \\ a_1(x) &= -x(x^2(N^2+aN-4)(N^2+aN-3) - x(2N+a)(2N^2+2aN-7) + a^2 - 2) \\ a_2(x) &= 2x^2(x^2(N^2+Na-4) - x(2N+a) + 2) \\ a_3(x) &= x^3(1-x^2). \end{aligned} \quad (1.12)$$

As an application, it was shown in [2] that this differential equation is well suited to a large N asymptotic analysis using the WKB method, resulting in asymptotic formulas which accurately reproduce the finite N graph of $\rho_{(1)}^{(\text{fl})}(x)$. We will show in section 2.1 that the differential equation (1.11) can be derived from the third order linear differential equation already known from the density of the Laguerre unitary ensemble as determined by the eigenvalue PDF (1.3) [2, 44, 70]. A viewpoint of (1.11) based on a first order 3×3 matrix differential equation is given in section 2.2. Some comparative features of $\rho_{(1)}^{(\text{l})}(x)$ and $\rho_{(1)}^{(\text{fl})}(x)$ are made in section 2.3.

The distribution of the purity for the random state specified below (1.8) is given by

$$P_{N,a}(t) = \frac{1}{C_{N,a}^{(\text{fl})}} \int_0^1 d\lambda_1 \cdots \int_0^1 d\lambda_N \delta\left(t - \sum_{j=1}^N \lambda_j^2\right) \delta\left(1 - \sum_{l=1}^N \lambda_l\right) \prod_{l=1}^N \lambda_l^a \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2. \quad (1.13)$$

For general N the exact calculation of (1.13) is from a practical viewpoint intractable, with its complexity increasing with increasing N and a . However, some exact results for small N and or a are available. Thus its evaluation is almost immediate for $N=2$, being given by [43, equation (9)]

$$P_{2,a}(t) = \frac{(2a+3)!}{2^{a+1}(a+1)!a!} (1-t)^a \sqrt{2t-1}, \quad (1.14)$$

supported on $1/2 < t < 1$. For $a=0$ and $N=3$, the distribution is piecewise real analytic, involving only elementary functions, on the intervals $1/3 < t < 1/2$ and $1/2 < t < 1$; see [43, equation (16)]. A still more complicated expression, although again involving only elementary functions, holds for $N=3$ and general $a \in \mathbb{Z}^+$; see [43, equation (31)], and for $N=4$ and $a=0$; see [43, equation (24)].

Progress for general N is possible by considering the Laplace–Fourier transform of (1.13),

$$\hat{P}_{N,a}^{(\text{fl})}(s) = \frac{1}{C_{N,a}^{(\text{fl})}} \int_0^1 d\lambda_1 \cdots \int_0^1 d\lambda_N \delta\left(1 - \sum_{l=1}^N \lambda_l\right) \prod_{l=1}^N e^{-s\lambda_l^2} \lambda_l^a \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2, \quad (1.15)$$

or equivalently the moments

$$\left\langle \left(\sum_{j=1}^N \lambda_j^2 \right)^k \right\rangle^{(\text{fl})}, \quad (1.16)$$

$k = 1, 2, \dots$; thus (1.15) is the exponential generating function for (1.16). In particular, we know from [42, equation (10), corrected by a factor of $N!$] and [43, equation (17)] that

$$\left\langle \left(\sum_{j=1}^N \lambda_j^2 \right)^k \right\rangle^{(\text{fl})} = \frac{N! (N(N+a)-1)!}{(N(N+a)+2k-1)!} \times \sum_{\substack{k_1, \dots, k_N \geq 0 \\ k_1 + \dots + k_N = k}} \frac{k!}{k_1! \dots k_N!} \prod_{i=1}^N \frac{(N+a+2k_i-i)!}{(N+a-i)! i!} \prod_{1 \leq i < j \leq N} (2k_i - i - 2k_j + j). \quad (1.17)$$

To be addressed in section 3.1 of the present paper is a relationship between the moments (1.16) and a particular σ -Painlevé IV transcendent. This in turn comes about through a relationship between the LUE version of (1.15)

$$\hat{P}_{N,a}^{(\text{L})}(s) = \frac{1}{C_{N,a}^{(\text{L})}} \int_0^\infty d\lambda_1 \dots \int_0^\infty d\lambda_N \prod_{l=1}^N e^{-\lambda_l - s\lambda_l^2} \lambda_l^a \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2, \quad (1.18)$$

and the GUE average

$$\left\langle \prod_{j=1}^N (\lambda_j - 1/(2\sqrt{s}))^a \chi_{\lambda_j > 1/(2\sqrt{s})} \right\rangle^{(\text{G})}, \quad (1.19)$$

where $\chi_A = 1$ if A is true, $\chi_A = 0$ otherwise. Thus we have [75, minor generalisation of (82)]

$$\hat{P}_{N,a}^{(\text{L})}(s) = \frac{C_N^{(\text{G})}}{C_{N,a}^{(\text{L})}} e^{N/4s} (1/\sqrt{s})^{Na+N^2} \left\langle \prod_{j=1}^N (\lambda_j - 1/(2\sqrt{s}))^a \chi_{\lambda_j > 1/(2\sqrt{s})} \right\rangle^{(\text{G})}, \quad (1.20)$$

while the average (1.19) is known to relate to a τ -function in the theory of Painlevé IV [41] (see also [28, Ch. 8]), and so satisfies a particular second order nonlinear differential equation. The large argument Laurent solution of the latter systematically generates the LUE version of the moments (1.16), and thus the moments (1.17) themselves, as they are related by a simple proportionality (see (3.7)). In section 3.2 we make use of a known integration formula involving the Schur polynomial weighted against the PDF (1.3) for purposes of generalising the formula (1.17) so that it holds for

$$\left\langle \left(\sum_{j=1}^N \lambda_j^q \right)^k \right\rangle^{(\text{fl})}, \quad (1.21)$$

where q is a non-negative integer.

First order linear matrix differential equations, introduced in section 2.2 in relation to a viewpoint on the origin of the differential equation (1.11), are derived again in section 4. This is for the purpose of obtaining recurrences for the power series expansion of (1.18) with the exponent in the product over differences generalised from the value 2 to a general parameter β . As a result we are able to explicitly determine the $k=1$ and $k=2$ moments (1.16) in the general β case. Motivated by the analogue of a fixed trace Hermitian matrix for unitary matrices being a unit determinant constraint, we provide in the appendix the exact functional form of the two-point correlation for Haar distributed $\text{SU}(N)$.

2. Differential equations for the density of the fLUE

2.1. Transforming the scalar differential equation for the LUE density

Generally, for an eigenvalue PDF $p_N(\lambda_1, \dots, \lambda_N)$, supported on $\lambda_l \in I$, ($l = 1, \dots, N$), the eigenvalue density $\rho_{(1)}(x)$ is given by

$$\rho_{(1)}(x) = N \int_I d\lambda_2 \dots \int_I d\lambda_N p_N(x, \lambda_2, \dots, \lambda_N). \quad (2.1)$$

For the Laguerre unitary ensemble we have $p_N = p_N^{(L)}$ as specified by (1.3). Previous work [2, 44, 70] has shown that the corresponding density, $\rho_{(1)}^{(L)}(x)$ say, satisfies the third order linear differential equation

$$\left(x^3 \frac{d^3}{dx^3} + 4x^2 \frac{d^2}{dx^2} - [x^2 - 2(a+2N)x + a^2 - 2] x \frac{d}{dx} + [(a+2N)x - a^2] \right) f(x) = 0. \quad (2.2)$$

Our first objective is to show that the differential equation (1.11) can be deduced as a corollary of (2.2).

To begin, generalise (1.5) by introducing a parameter t into the delta function,

$$p_{N,a}^{(fl)}(\lambda_1, \dots, \lambda_N; t) := \frac{1}{C_{N,a}^{(fl)}} \delta \left(t - \sum_{l=1}^N \lambda_l \right) \prod_{l=1}^N \lambda_l^a \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2. \quad (2.3)$$

The corresponding Laplace–Fourier transform is

$$\hat{p}_{N,a}^{(fl)}(\lambda_1, \dots, \lambda_N; s) := \frac{1}{C_{N,a}^{(fl)}} \prod_{l=1}^N e^{-s\lambda_l} \lambda_l^a \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2, \quad (2.4)$$

which up to a scaling of the eigenvalues, and the normalisation, is just the eigenvalue PDF for the Laguerre unitary ensemble (1.3).

Now set

$$\rho_{(1)}^{(fl)}(x; t) = N \int_I d\lambda_2 \cdots \int_I d\lambda_N p_{N,a}^{(fl)}(x, \lambda_2, \dots, \lambda_N; t). \quad (2.5)$$

Taking the Laplace–Fourier transform, making use of (2.4), shows

$$\begin{aligned} \hat{\rho}_{(1)}^{(fl)}(x; s) &= N \int_0^\infty d\lambda_2 \cdots \int_0^\infty d\lambda_N \hat{p}_{N,a}^{(fl)}(x, \lambda_2, \dots, \lambda_N; s) \\ &= \frac{C_{N,a}^{(L)}}{C_{N,a}^{(fl)}} \frac{1}{s^{Na+N^2-1}} \rho_{(1)}^{(L)}(sx). \end{aligned}$$

Taking the inverse transform and setting $t = 1$ then implies

$$\begin{aligned} \rho_{(1)}^{(fl)}(x) &= \frac{C_{N,a}^{(L)}}{C_{N,a}^{(fl)}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\rho_{(1)}^{(L)}(sx)}{s^{Na+N^2-1}} e^s ds, \quad c > 0 \\ &= \frac{C_{N,a}^{(L)}}{C_{N,a}^{(fl)}} x^{Na+N^2-2} \frac{1}{2\pi i} \int_{\tilde{c}-i\infty}^{\tilde{c}+i\infty} \frac{\rho_{(1)}^{(L)}(s)}{s^{Na+N^2-1}} e^{s/x} ds, \quad \tilde{c} > 0. \end{aligned} \quad (2.6)$$

We remark that the use of the Laplace–Fourier transform as outlined above can be found in nearly all³ analytic studies relating to the fixed trace Laguerre ensemble, or indeed general random matrix ensembles with a fixed trace condition; see [2, 4, 6, 7, 18, 34, 52, 53, 74] as a selection of such studies.

The relationship (2.6) allows the differential equations (1.11) and (2.2) to be linked.

Proposition 2.1. *The fact that $\rho_{(1)}^{(L)}(s)$ in (2.6) satisfies the differential equation (2.2) (with x replaced by s throughout) implies that $\rho_{(1)}^{(fl)}(x)$ satisfies (1.11).*

Proof. Begin with the differential equation (2.2), and replace x by s throughout. Multiply through by $e^{s/x}/s^{Na+N^2-1}$ and integrate over s in the complex plane from $\tilde{c} - i\infty$ to $\tilde{c} + i\infty$, $\tilde{c} > 0$. Setting

$$g_p(x) = \int_{\tilde{c}-i\infty}^{\tilde{c}+i\infty} \frac{s^p e^{s/x}}{s^{Na+N^2-1}} \rho_{(1)}^{(L)}(s) ds$$

and considering for example this procedure applied to the first term in (2.2), upon integration by parts we obtain the identity

³ An exception is [5], which uses a Gaussian approximation to the delta function.

$$\begin{aligned}
& \int_{\tilde{c}-i\infty}^{\tilde{c}+i\infty} \frac{s^3 e^{s/x}}{s^{Na+N^2-1}} \frac{d^3}{ds^3} \rho_{(1)}^{(L)}(s) ds \\
&= - (2 - aN - N^2) (3 - aN - N^2) (4 - aN - N^2) g_0(x) - \frac{1}{x^3} g_3(x) \\
&\quad - \frac{3}{x^2} (4 - aN - N^2) g_2(x) - \frac{3}{x} (3 - aN - N^2) (4 - aN - N^2) g_1(x).
\end{aligned}$$

This becomes a differential identity for $g_0(x)$ upon noting

$$g_p(x) = \left(-x^2 \frac{d}{dx} \right)^p g_0(x). \quad (2.7)$$

The above procedure applied to each of the terms in (2.2) similarly gives rise to further differential identities for $g_0(x)$. Adding them together tells us that $g_0(x)$ satisfies a particular third order linear differential equation.

Writing $g_0(x) = h(x)/x^{Na+N^2-2}$, the latter transforms to a third order linear differential equation for $h(x)$, which according to (2.6) is proportional to $\rho_{(1)}^{(N)}(x)$. Writing the transformed third order linear differential equation in the form (1.11), we see that the values of the coefficients (1.12) result (the necessary simplification is best done using computer algebra). □

2.2. Matrix differential equation viewpoint

In addition to $\rho_{(1)}^{(L)}(x)$ satisfying a third order linear scalar differential equation, it also satisfies a first order 3×3 matrix differential equation. This is a fact known explicitly for the density in the cases of the Gaussian unitary ensemble (1.1) and Jacobi unitary ensemble (up to normalisation the eigenvalue PDF of the latter is given by (1.1) with $e^{-\lambda_l^2}$ replaced by $\lambda_l^a(1-\lambda_l)^b$, $0 < \lambda_l < 1$). Here we will first make the matrix differential equation explicit. We will then show that this implies that $\rho_{(1)}^{(N)}(x)$ also satisfies a first order 3×3 matrix differential equation, and that this in turn implies the third order linear scalar differential equation (1.11).

The matrix differential equation relating to $\rho_{(1)}^{(L)}(x)$ is actually a special case of a more general matrix differential equation applying to the density of the Laguerre β -ensemble, the latter characterised by having an eigenvalue PDF proportional to

$$\prod_{l=1}^N \lambda_l^a e^{-\beta \lambda_l/2} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta, \quad (2.8)$$

which each $\lambda_l \in \mathbb{R}^+$. It is known from [27] (see also [28, section 13.2.5]) that for β even

$$\rho_{N,\beta}^{(L)}(x) = N \frac{W_{2a/\beta+2,\beta,N-1}}{W_{2a/\beta,\beta,N}} x^a e^{-\beta x/2} {}_1F_1^{(\beta/2)} \left(-N+1; 2a/\beta+2; (x)^\beta \right), \quad (2.9)$$

where ${}_1F_1^{(\beta/2)}$ — technically a particular multidimensional hypergeometric function based on on Jack polynomials [28, section 13.1.1] — has the multi-dimensional integral representation

$$\begin{aligned}
& {}_1F_1^{(\beta/2)} \left(-N+1; 2a/\beta+2; (x)^\beta \right) \\
&= \frac{1}{M_\beta(2a/\beta+2/\beta-1, N-1, 2/\beta)} \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} d\theta_1 \cdots \\
&\quad \times \int_{-\pi}^{\pi} d\theta_\beta \prod_{l=1}^{\beta} e^{i\theta_l(2a/\beta+2/\beta-1)} (1 + e^{-i\theta_l})^{N+2a/\beta+2/\beta-2} e^{-x e^{i\theta_l}} \prod_{1 \leq j < k \leq \beta} |e^{i\theta_k} - e^{i\theta_j}|^{4/\beta}. \quad (2.10)
\end{aligned}$$

The quantities $W_{\alpha,\beta,n}$ and $M_n(\lambda_1, \lambda_2, \tau)$ are normalisations with known gamma function evaluations—their precise form plays no role in specifying differential equations—also $(x)^\beta$ is a multivariate notation, indicating that the argument x is repeated β times.

To specify the matrix differential equation, knowledge of a differential-difference system satisfied by (2.10) is required. As a preliminary for this purpose, let $e_p(y_1, \dots, y_N)$ denote the elementary symmetric polynomials in $\{y_j\}_{j=1}^N$, and define

$$J_{p,N,R}^{(\alpha)}(x) = \frac{1}{C_p^N} \int_R dt_1 \cdots \int_R dt_N \prod_{l=1}^N t_l^{\lambda_1} (1-t_l)^{\lambda_2} (x-t_l)^\alpha \\ \times \prod_{1 \leq j < k \leq N} |t_k - t_j|^{2\tau} e_p(x-t_1, \dots, x-t_N), \quad (2.11)$$

where R is a contour in the complex plane such that the integrand vanishes at the endpoint, e.g. $R = [0, 1]$ or $R = [x, 1]$ when the parameters $\lambda_1, \lambda_2, \alpha$ are all positive. The symbol C_p^N denotes the binomial coefficient N choose p . It is known that this family of multiple integrals satisfies the differential-difference system [26], [28, section 4.6.4]

$$(N-p)E_p J_{p+1}(x) = (A_p x + B_p) J_p(x) - x(x-1) \frac{d}{dx} J_p(x) + D_p x(x-1) J_{p-1}(x), \quad (2.12)$$

valid for $p = 0, 1, \dots, N$. Here we have abbreviated $J_{p,N,R}^{(\alpha)}(x) =: J_p(x)$, and the coefficients are specified by

$$A_p = (N-p)(\lambda_1 + \lambda_2 + 2\tau(N-p-1) + 2(\alpha+1)) \\ B_p = (p-N)(\lambda_1 + \alpha + 1 + \tau(N-p-1)) \\ D_p = p(\tau(N-p) + \alpha + 1) \\ E_p = \lambda_1 + \lambda_2 + 1 + \tau(2N-p-2) + (\alpha+1).$$

This differential-difference system is in fact equivalent to a matrix differential equation [38]; see below. The latter, in the special case $\tau = 1/2$, were first isolated in the literature in [21]; for recent developments see [36, 37, 70].

For τ a positive integer the absolute value signs in the product over differences in (2.11) can be removed. Suppose we choose R to be the unit circle in the complex plane—this is valid for $\lambda_2 > 0$ since the integrand vanishes at the (single) endpoint $t_l = 1$ —and parametrise by writing $t_l = e^{i\theta_l}$. Simple manipulation then gives that the integrand in (2.11) transforms to being proportional to

$$\prod_{l=1}^N e^{i\theta_l(\lambda_1 + 1 + \tau(N-1) + \alpha)} (1 - e^{i\theta_l})^{\lambda_2} (1 - x e^{-i\theta_l})^\alpha \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^{2\tau} e_p(x - e^{i\theta_1}, \dots, x - e^{i\theta_N}). \quad (2.13)$$

As noted in [35, appendix], direct use of integration by parts, starting with the integrand (2.13) in (2.11), shows that the recurrence (2.12) in the case of (2.13) remains valid for general $\tau > 0$. We can now specify a differential-difference recurrence for a class of multiple integrals of the type appearing in (2.10).

Proposition 2.2. *Set*

$$\tilde{J}_{p,N}(x) = \frac{1}{C_p^N} \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_N \prod_{l=1}^N e^{i\tilde{\lambda}_l \theta_l} (1 + e^{-i\theta_l})^{\lambda_2} e^{x e^{i\theta_l}} \\ \times \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^{2\tau} e_p(e^{-i\theta_1}, \dots, e^{-i\theta_N}). \quad (2.14)$$

Abbreviating $\tilde{J}_{p,N}(x) = \tilde{J}_p(x)$, for $p = 0, 1, \dots, N$ this multiple integral satisfies the differential-difference recurrence

$$(N-p)\tilde{E}_p \tilde{J}_{p+1}(x) = (N-p) \left(x + \tilde{\lambda}_1 + p\tau \right) \tilde{J}_p(x) + x \frac{d}{dx} \tilde{J}_p(x) - p x \tilde{J}_{p-1}(x), \quad (2.15)$$

where

$$\tilde{E}_p = -\tilde{\lambda}_1 + \lambda_2 + \tau(N-p-1) + 1. \quad (2.16)$$

Proof. Up to a sign independent of p , the integrand in (2.14) follows from (2.13) by taking the complex conjugate and setting

$$\theta_l \mapsto \theta_l - \pi, \quad -\tilde{\lambda}_1 = \lambda_1 + 1 + \tau(N-1) + \alpha, \quad x \mapsto x/\alpha, \quad \alpha \rightarrow \infty.$$

Applying the last three of these to (2.12) gives (2.15). $\square$

Comparing (2.14) with (2.9) in the case $\beta = 2$, we see that our immediate interest in (2.15) is for the choice of parameters and substitutions

$$N = 2, \quad \tau = 1, \quad x \mapsto -x, \quad \tilde{\lambda}_1 = a, \quad \lambda_2 = N + a - 1, \quad \hat{J}_p(x) = x^a e^{-x} \tilde{J}_p(-x),$$

when it reads

$$(2-p)(N+1-p)\hat{J}_{p+1}(x) = ((1-p)(-x+a) + (2-p)p)\hat{J}_p(x) + x \frac{d}{dx} \hat{J}_p(x) + px\hat{J}_{p-1}(x). \quad (2.17)$$

This recurrence can be written in the equivalent matrix form

$$x \frac{d}{dx} \begin{bmatrix} \hat{J}_0(x) \\ \hat{J}_1(x) \\ \hat{J}_2(x) \end{bmatrix} = \begin{bmatrix} x-a & 2(N+1) & 0 \\ -x & -1 & N \\ 0 & -2x & -x+a \end{bmatrix} \begin{bmatrix} \hat{J}_0(x) \\ \hat{J}_1(x) \\ \hat{J}_2(x) \end{bmatrix}. \quad (2.18)$$

The equation implied by the first row can be used to relate $\hat{J}_1(x)$ to $\hat{J}_0(x)$; with this knowledge, the second equation allows $\hat{J}_2(x)$ to be written in terms of $\hat{J}_0(x)$. We can check that substituting in the equation implied by the third row gives that $\hat{J}_0(x)$ satisfies the third order scalar differential equation (2.2), as is consistent with the fact that $\hat{J}_0(x)$ is proportional to $\rho_{(1)}^{(L)}(x)$.

Now in (2.18) replace x by s throughout. Multiply both sides by $e^{s/x}/s^{Na+N^2-1}$ ($0 < x < 1$) and integrate with respect to s over the contour $c - i\infty$ to $c + i\infty$. To do this, make use of integration by parts on the left hand side, and on the right hand side first decompose the matrix into a sum of one proportional to s , and one independent of s . After making use of the analogue of (2.7) in the case $p = 1$, this procedure gives a first order matrix differential equation for the column vector

$$\left[\int_{c-i\infty}^{c+i\infty} \frac{e^{s/x}}{s^{Na+N^2-1}} \hat{J}_p(s) ds \right]_{p=0,1,2}. \quad (2.19)$$

Multiply (2.19) by x^{Na+N^2-1} , and denote the resulting column vector by $[\hat{I}_p(x)]_{p=0,1,2}$. The matrix differential equation for (2.19) then transforms to read

$$x \frac{d}{dx} \begin{bmatrix} \hat{I}_0(x) \\ \hat{I}_1(x) \\ \hat{I}_2(x) \end{bmatrix} = \left((Na + N^2 - 2)x - x^2 \frac{d}{dx} \right) \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} \hat{I}_0(x) \\ \hat{I}_1(x) \\ \hat{I}_2(x) \end{bmatrix} + \begin{bmatrix} -a & 2(N+1) & 0 \\ 0 & -1 & N \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} \hat{I}_0(x) \\ \hat{I}_1(x) \\ \hat{I}_2(x) \end{bmatrix}. \quad (2.20)$$

We now follow the strategy detailed below (2.18) to deduce a differential scalar equation for $\hat{I}_0(x)$. This can be checked to be the same differential equation as (1.11). Since by definition $\hat{I}_0(x)$ is proportional to $\rho_{(1)}^{(fl)}(x)$ — recall (2.6), the definition of $\hat{I}_0(x)$ below (2.19), and the fact that $\hat{J}_0(x)$ is proportional to $\rho_{(1)}^{(L)}(x)$ — this gives an alternative derivation of the latter.

2.3. A comparison of properties of $\rho_{(1)}^{(L)}(x)$ and $\rho_{(1)}^{(fl)}(x)$

Substituting (1.3) in (2.1) shows that the density of the LUE has the structure

$$\rho_{(1)}^{(L)}(x) = x^a e^{-x} \sum_{p=0}^{2(N-1)} \alpha_p x^p \quad (2.21)$$

for some coefficients $\{\alpha_p\}$. As noted in [70, remark 2.1.3] the differential equation (2.2) admits a unique solution with the $x \rightarrow \infty$ form compatible with (2.21)

$$\rho_{(1)}^{(L)}(x) = \frac{NC_{N-1,a}^{(L)}}{C_{N,a}^{(L)}} x^a e^{-x} \left(x^{2(N-1)} + O\left(x^{2(N-1)-1}\right) \right), \quad (2.22)$$

and so in particular implies a recursion for $\{\alpha_p\}$, with a unique solution upon requiring that

$$\alpha_{2(N-1)} = \frac{NC_{N-1,a}^{(L)}}{C_{N,a}^{(L)}}. \quad (2.23)$$

Substituting (2.21) in (2.6) shows that in terms of $\{\alpha_p\}$,

$$\rho_{(1)}^{(\text{fl})}(x) = \frac{C_{N,a}^{(L)}}{C_{N,a}^{(\text{fl})}} \sum_{p=0}^{2(N-1)} \Gamma((N-1)a + N^2 - 2 - p + 1) \alpha_p \left(\frac{1}{x} - 1\right)^{(N-1)a + N^2 - 2 - p}, \quad (2.24)$$

supported on $(0, 1)$. This indicates that the natural independent variable associated with (1.12) is $y = (1/x) - 1$, and that in this variable the differential equation admits a unique solution for $y \rightarrow \infty$ subject only to the specification of the leading term as made explicit by (2.23).

Consider next the moments associated with the densities $\rho_{(1)}^{(L)}(x)$ and $\rho_{(1)}^{(\text{fl})}(x)$,

$$m_p^{(L)} = \int_0^\infty x^p \rho_{(1)}^{(L)}(x) dx, \quad m_p^{(\text{fl})} = \int_0^1 x^p \rho_{(1)}^{(\text{fl})}(x) dx. \quad (2.25)$$

We know from [46] and [54] that $\{m_p^{(L)}\}$ satisfies the second order recurrence

$$(k+1)m_k^{(L)} = (2k-1)(a+2N)m_{k-1}^{(L)} + (k-2)\left((k-1)^2 - a^2\right)m_{k-2}^{(L)}, \quad (2.26)$$

which was shown in [20] to be valid for general $k > -a - 1$. One way to derive (2.26) is to multiply both sides of the differential equation (2.2) by x^{k-2} , integrate both sides from 0 to ∞ , and simplify making use of integration by parts. A formula for $m_k^{(L)}$ in terms of certain continuous Hahn polynomials of degree $N-1$ and valid for continuous k has been given in [19].

Making use of the first equality in (2.6), as well as the explicit evaluations of the normalisation (1.4) and (1.6), we can deduce [83]

$$m_p^{(\text{fl})} = \frac{1}{(N^2 + Na)_p} m_p^{(L)}, \quad (2.27)$$

where $(u)_p$ denotes the rising Pochhammer symbol. Substituting (2.27) in (2.26) gives that $\{m_p^{(\text{fl})}\}$ satisfies the second order recurrence

$$\begin{aligned} & (k+1)(N^2 + Na + k)(N^2 + Na + k - 1)m_k^{(\text{fl})} \\ &= (2k-1)(a+2N)(N^2 + Na + k - 1)m_{k-1}^{(\text{fl})} + (k-2)\left((k-1)^2 - a^2\right)m_{k-2}^{(\text{fl})}. \end{aligned} \quad (2.28)$$

Starting with the differential equation (1.12), and following the procedure noted below (2.26) gives an alternative derivation of (2.28).

We see from (1.4) that

$$\begin{aligned} C_{N,a}^{(L)} &= \int_0^\infty d\lambda_1 \cdots \int_0^\infty d\lambda_N \prod_{l=1}^N \lambda_l^a e^{-\lambda_l} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2 \\ &= t^{N^2 + Na} \int_0^\infty d\lambda_1 \cdots \int_0^\infty d\lambda_N \prod_{l=1}^N \lambda_l^a e^{-t\lambda_l} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2. \end{aligned}$$

Multiplying through by $1/(C_{N,a}^{(L)} t^{N^2 + Na})$, differentiating both sides with respect to t , and setting $t = 1$ shows that

$$m_1^{(L)} = N^2 + Na. \quad (2.29)$$

Since from the definition (1.5)

$$m_1^{(\text{fl})} = 1 \quad (2.30)$$

we see that (2.29) is consistent with (2.27).

The global scaling regime of positive definite matrices seeks a rescaling of the eigenvalues so that the first moment is of order N as $N \rightarrow \infty$ (as a normalisation we will require that the leading value be exactly N). Let us also scale the parameter a by writing $a = \alpha N$. For the LUE, it follows from (2.29) that global scaling can be achieved by replacing each λ_l by $\tilde{\lambda}_l N$; to match with the first (scaled) first moment for the fLUE we should replace each λ_l by $\tilde{\lambda}_l/(N(1+\alpha))$. Thus in (1.12) we should make the substitution $x = yN$, and in (2.2) the substitution $x = y/(N(1+\alpha))$. Doing this, and equating terms at leading order in N gives in both cases the same first order linear differential equation

$$\left(-(y^2 - 2(\alpha + 2)y + \alpha^2)y \frac{d}{dy} + (\alpha + 2)y - \alpha^2 \right) f(y) = 0. \quad (2.31)$$

The solution of this, which is to be non-negative and normalised to integrate to unity, is the well known Marchenko–Pastur law

$$f(y) = \frac{1}{2\pi y} ((y - \alpha_-)(\alpha_+ - y))^{1/2}, \quad \alpha_{\pm} = (\sqrt{\alpha + 1} \pm 1)^2, \quad (2.32)$$

supported on $y \in (\alpha_-, \alpha_+)$. This is consistent with results known from [60, 67].

3. Moments and cumulants for the purity and related statistics

3.1. Cumulants of the purity and σ -Painlevé IV

We begin this section by recalling from [41], [28, chapter 8] the precise relationship between the GUE average in (1.20) and a transcendent in the Hamiltonian formulation of Painlevé IV [63]. Thus we have

$$\frac{d}{dt} \log \left\langle \prod_{j=1}^N (\lambda_j - t)^a \chi_{\lambda_j > t} \right\rangle^{(G)} = U_N(t, a), \quad (3.1)$$

where $U_N(t, a)$, satisfies the particular σ -Painlevé IV equation, which is a second order nonlinear equation satisfied by the Hamiltonian itself,

$$(\sigma'')^2 - 4(t\sigma' - \sigma)^2 + 4\sigma'(\sigma' - 2a)(\sigma' + 2N) = 0. \quad (3.2)$$

Being a logarithmic derivative which equals a Hamiltonian in the Hamiltonian formulation of a Painlevé system, the average is said to be a τ -function; see e.g. [28, section 8.2.3].

In view of (3.1), writing in (1.20) $1/(2\sqrt{s}) = t$ we have that

$$\frac{d}{dt} \log \hat{P}_{N,a}^{(L)}(1/(4t^2)) = 2Nt + \frac{N^2 + Na}{t} + U_N(t, a). \quad (3.3)$$

Being the exponential generating function for the distribution of the linear statistic $\sum_{j=1}^N \lambda_j^2$ in the LUE, and with $\{\kappa_n^{(L)}\}$ denoting the corresponding cumulants, we have the expansion

$$\log \hat{P}_{N,a}^{(L)}(u) = \sum_{n=1}^{\infty} \frac{(-1)^n \kappa_n^{(L)}}{n!} u^n, \quad (3.4)$$

and consequently $U_N(t, a)$ permits the large- t expansion

$$U_N(t, a) = -2Nt - \frac{N^2 + Na}{t} + \frac{1}{2t^3} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa_{n+1}^{(L)}}{4^n n! t^{2n}}. \quad (3.5)$$

Proposition 3.1. *The cumulants $\{\kappa_n^{(L)}\}_{n=1}^{\infty}$ for the distribution of the LUE linear statistic $\sum_{j=1}^N \lambda_j^2$ are proportional to the coefficients in the $\{1/t^{2n+3}\}$ expansion ($n = 0, 1, \dots$) of the σ -Painlevé IV transcendent $U_N(t, a)$ according to (3.5). These coefficients follow by substituting (3.5) in (3.2) which gives for the first three cumulants*

$$\begin{aligned} \kappa_1^{(L)} &= N(N+a)(2N+a) \\ \kappa_2^{(L)} &= 2N(N+a)(1+2a^2+9aN+9N^2) \\ \kappa_3^{(L)} &= 8N(N+a)(2N+a)(10+5a^2+27aN+27N^2). \end{aligned}$$

Proof. The essential point is that the σ - PIV differential equation (3.2) implies a recurrence for $\kappa_{n+1}^{(L)}$ (linear in this quantity), depending on $\{\kappa_l^{(L)}\}_{l=1,\dots,n}$. Thus from (3.5) we have

$$\begin{aligned}\sigma'' &= -\frac{2(N^2 + Na)}{t^3} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa_{n+1}^{(L)} (2n+3)(2n+4)}{4^n n! t^{2n+5}}, \\ t\sigma' - \sigma &= \frac{2(N^2 + Na)}{t} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa_{n+1}^{(L)} (2n+4)}{4^n n! t^{2n+3}}, \\ \sigma' &= -2N + \frac{N^2 + Na}{t^2} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \kappa_{n+1}^{(L)} (2n+3)}{4^n n! t^{2n+4}}.\end{aligned}$$

Substituting in (3.2) shows that the leading order power of t^{-1} is t^{-4} , with a coefficient involving only the cumulant $\kappa_1^{(L)}$. Next we examine the coefficient of t^{-6} . This depends on $\kappa_2^{(L)}$ (linearly) as well as on $\kappa_1^{(L)}$ (which is known from the previous step). This general pattern holds for the coefficient of each successive higher powers of t^{-1} — thus the coefficient of $t^{-(2n+2)}$ is linear in $\kappa_n^{(L)}$ and involves also the already computed quantities $\{\kappa_l^{(L)}\}_{l=1,\dots,n-1}$. $\square$

The cumulants fully determine the corresponding moments according to standard formulas

$$\hat{m}_1^{(L)} = \kappa_1^{(L)}, \quad \hat{m}_2^{(L)} = \kappa_2^{(L)} + \left(\kappa_1^{(L)}\right)^2, \quad \hat{m}_3^{(L)} = \kappa_3^{(L)} + 3\kappa_2^{(L)}\kappa_1^{(L)} + \left(\kappa_1^{(L)}\right)^3. \quad (3.6)$$

The significance of knowledge of the moments $\hat{m}_n^{(L)}$ is that they relate to the moments for the purity, $\hat{m}_n^{(\text{fl})}$ say, by a formula analogous to (2.27) [75]

$$\hat{m}_p^{(\text{fl})} = \frac{1}{(N^2 + Na)_{2p}} m_p^{(L)}. \quad (3.7)$$

Inverting the relations (3.6) then allows the cumulants $\kappa_n^{(\text{fl})}$ for the purity be computed.

Corollary 3.1. *The first three cumulants for the purity statistic $\sum_{j=1}^N \lambda_j^2$ associated with the eigenvalue PDF (1.5) are given by*

$$\begin{aligned}\kappa_1^{(\text{fl})} &= \frac{(2N+a)}{(N(N+a)+1)} \\ \kappa_2^{(\text{fl})} &= \frac{2(N^2-1)\left((N+a)^2-1\right)}{(N(N+a)+1)^2(N(N+a)+2)(N(N+a)+3)} \\ \kappa_3^{(\text{fl})} &= \frac{8(N^2-1)\left((N+a)^2-1\right)(2N+a)(N(N+a)-5)}{(N(N+a)+1)^3(N(N+a)+2)(N(N+a)+3)(N(N+a)+4)(N(N+a)+5)}.\end{aligned}$$

The mean $\kappa_1^{(\text{fl})}$ was first calculated in [56]. This was extended to the first three cumulants in [72], and, using the sum formula (1.17) for the moments, to the first five cumulants in [42]. Although we have listed only the first three cumulants above, our method also provides an efficient and systematic way to continue this list.

Before concluding this section, it is of interest to note that in the earlier work [66], a Painlevé transcendent formalism was used for the purpose of computing the moments of the purity in the Bures ensemble statistical model of the eigenvalues of the density matrix. For an introduction to the latter, see e.g. [32, section 1.1].

3.2. The linear statistic $\sum_{k=1}^N \lambda_j^q$

The linear statistic $T_q := \sum_{k=1}^N \lambda_j^q$ is closely related to the quantum Tsallis entropy introduced in the paragraph including (1.9), with the latter specified in terms of T_q by $\tilde{T}_q := (1/(1-q))(1-T_q)$. Here we will present the analogue of the summation expression (1.17) for the moments of T_q . Note that these relate to the moments of $\tilde{T}_q$ by the simple formula

$$(\tilde{T}_q)^k = \left(\frac{1}{1-q}\right)^k \sum_{s=0}^k (-1)^s \binom{k}{s} (T_q)^s. \quad (3.8)$$

Proposition 3.2. For all $r \geq 0$, $k \in \mathbb{Z}_{\geq 0}$ and $qk + a > -1$ we have

$$\left\langle \left(\sum_{j=1}^N \lambda_j^q \right)^k \right\rangle^{(\text{fl})} = \frac{(N(N+a)-1)!}{(N(N+a)+qk-1)!} \times \sum_{\substack{k_1, \dots, k_N \geq 0 \\ k_1 + \dots + k_N = k}} \frac{k!}{k_1! \dots k_N!} \prod_{i=1}^N \frac{(N+a+qk_i-i)!}{(N+a-i)!} \prod_{1 \leq i < j \leq N} \frac{qk_i - i - qk_j + j}{-i + j}. \quad (3.9)$$

Proof. The working in [42] for the case $q = 2$ is sufficient to derive the general q result (3.9).

With

$$J_{a,N}(\mathbf{k}) := N! \int_0^1 dx_1 \dots \int_0^1 dx_N \delta \left(1 - \sum_{i=1}^N x_i \right) \Delta(\mathbf{x}) \prod_{i=1}^N x_i^{a+k_i+i-1}, \quad (3.10)$$

where $\mathbf{k} = (k_1, \dots, k_N)$, $\Delta(\mathbf{x}) = \prod_{1 \leq i < j \leq N} (x_j - x_i)$, this working relied on the evaluation formula

$$J_{a,N}(\mathbf{k}) = \frac{N! \prod_{i=1}^N \Gamma(a+k_i+i)}{\Gamma(N^2 + aN + \sum_{i=1}^N k_i - 1)!} \prod_{1 \leq i < j \leq N} (k_j - k_i + j - i). \quad (3.11)$$

To deduce (3.11), introduce the Schur polynomial $s_{\kappa_1, \dots, \kappa_N}(\mathbf{x})$ specified by

$$s_{\kappa_1, \dots, \kappa_N}(\mathbf{x}) = \det \left[x_j^{\kappa_{N+1-i}+i-1} \right]_{i,j=1}^N / \Delta(\mathbf{x}); \quad (3.12)$$

conventionally it is required that $\kappa_1 \geq \dots \geq \kappa_N \geq 0$ form a partition of non-negative integers, however (3.12) is well defined without this assumption although it will no longer necessarily be a polynomial. By symmetrising the integrand of (3.10) the final factor therein becomes a determinant, and making use of (3.12) shows

$$J_{a,N}(\mathbf{k}) := \int_0^1 dx_1 \dots \int_0^1 dx_N \delta \left(1 - \sum_{i=1}^N x_i \right) (\Delta(\mathbf{x}))^2 s_{a+k_N, \dots, a+k_1}(\mathbf{x}). \quad (3.13)$$

On the other hand, from the theory of Schur polynomials it is well known [49] (see [33, section 2.3] for another recent application in random matrix theory) that

$$\int_0^\infty dx_1 \dots \int_0^\infty dx_N \prod_{l=1}^N x_l^a e^{-x_l} (\Delta(\mathbf{x}))^2 s_{\kappa_1, \dots, \kappa_N}(\mathbf{x}) = N! \prod_{i=1}^N \Gamma(a+k_i+i) \prod_{1 \leq i < j \leq N} (\kappa_i - \kappa_j + j - i). \quad (3.14)$$

By using the same strategy as in the derivation of (2.27), we can deduce from this the evaluation (3.11).

To make use of (3.11) in the derivation of (3.9), following the working in [42] for the case $q = 2$ we begin by noting from the antisymmetry of $\Delta(\mathbf{x})$ that

$$\left\langle \left(\sum_{j=1}^N \lambda_j^q \right)^k \right\rangle^{(\text{flUE})} = \frac{N!}{C_{N,a}^{(\text{fl})}} \int_0^1 dx_1 \dots \int_0^1 dx_N \delta \left(1 - \sum_{i=1}^N x_i \right) \Delta(\mathbf{x}) \prod_{i=1}^N x_i^{a+i-1} \left(\sum_{j=1}^N x_j^q \right)^k. \quad (3.15)$$

Now substituting

$$\left(\sum_{j=1}^N x_j^q \right)^k = \sum_{\substack{k_1, \dots, k_N \geq 0 \\ k_1 + \dots + k_N = k}} \frac{k!}{k_1! \dots k_N!} x_1^{qk_1} \dots x_N^{qk_N}, \quad (3.16)$$

and taking the sum outside the integral, allows each of the integrals to be evaluated according to (3.11), and (3.9) follows. $\square$

In the case $k = 1$ the multi-index summation in (3.9) can be replaced by a sum over j , $1 \leq j \leq N$, where $k_j = 1$, $k_p = 0$ ($p \neq j$). This allows for the case $k = 1$ to be expressed in terms of a terminating ${}_3F_2$ hypergeometric function of unit argument.

Proposition 3.3. *We have*

$$\left\langle \sum_{j=1}^N \lambda_j^q \right\rangle^{(\text{fl})} = \frac{(N(N+a)-1)!}{(N(N+a)+q-1)!} \frac{\Gamma(N+a+q)}{\Gamma(N+a)} \frac{\Gamma(N+q)}{\Gamma(1+q)\Gamma(N)} \times {}_3F_2 \left(\begin{matrix} 1-N, 1-(N+a), 1-q \\ 1-(N-q), 1-(N+a+q) \end{matrix} \middle| 1 \right). \quad (3.17)$$

Proof. After parametrising the multi-index summation as indicated, the single product in the sum reduces to

$$\frac{(N+a+q-j)!}{(N+a-j)!} = \frac{\Gamma(N+a+q)}{\Gamma(N+a)} \frac{(1-(N+a))_{j-1}}{(1-(N+a+q))_{j-1}},$$

and the double product reduces to

$$\prod_{i=1}^{j-1} \frac{-i-q+j}{-i+j} \prod_{i=j+1}^N \frac{i+q-j}{i-j} = \frac{(1-q)_{j-1}}{(j-1)!} \frac{\Gamma(N+q)}{\Gamma(1+q)\Gamma(N)} \frac{(1-N)_{j-1}}{(1-(N+q))_{j-1}}.$$

The summation over j can be recognised as the series form of a particular ${}_3F_2$ function, implying (3.17). $\square$

Note that for q a positive integer less than N the hypergeometric summation in (3.17) has only q nonzero terms. In particular, for $q=1$ we can check that the result for $\kappa_1^{(\text{fl})}$ in corollary 3.1 is recovered.

Using a different formalism to our proposition 3.3, Wei in the work [77]⁴ has derived the formula

$$\left\langle \sum_{j=1}^N \lambda_j^q \right\rangle^{(\text{L})} = \frac{N\Gamma(N+a+q)}{\Gamma(N+a)} {}_3F_2 \left(\begin{matrix} 1-N, -q, 1-q \\ 2, 1-(N+a+q) \end{matrix} \middle| 1 \right). \quad (3.18)$$

In keeping with (2.27)

$$\left\langle \sum_{j=1}^N \lambda_j^q \right\rangle^{(\text{fl})} = \frac{(N(N+a)-1)!}{(N(N+a)+q-1)!} \left\langle \sum_{j=1}^N \lambda_j^q \right\rangle^{(\text{L})}$$

so after substituting (3.17) on the left hand side, and (3.18) on the right hand side, it follows that we must have

$$\frac{\Gamma(N+q)}{\Gamma(1+q)} {}_3F_2 \left(\begin{matrix} 1-N, 1-(N+a), 1-q \\ 1-(N-q), 1-(N+a+q) \end{matrix} \middle| 1 \right) = N! {}_3F_2 \left(\begin{matrix} 1-N, -q, 1-q \\ 2, 1-(N+a+q) \end{matrix} \middle| 1 \right). \quad (3.19)$$

This can be recognised as a special case of the more general identity [69, Entry (7.4.4.85)]

$${}_3F_2 \left(\begin{matrix} -n, b, c \\ d, e \end{matrix} \middle| 1 \right) = \frac{(d-b)_n}{(d)_n} {}_3F_2 \left(\begin{matrix} -n, b, e-c \\ e, b-d-n+1 \end{matrix} \middle| 1 \right), \quad (3.20)$$

valid for non-negative integer n .

In [64], Okuyama used the replica method to deduce (3.17), and noted too the alternative form provided by using (3.19). The latter was then used to obtain the further rewrite of (3.17)

$$\left\langle \sum_{j=1}^N \lambda_j^q \right\rangle^{(\text{fl})} = \frac{\Gamma(N)\Gamma(N(N+a)+1)}{\Gamma(a+N+1)\Gamma(N(N+a)+q)} \sum_{k=1}^N \mathbf{N}_{q,k} \frac{\Gamma(a+N+1+q+k)}{\Gamma(N+1-q)}, \quad (3.21)$$

where $\mathbf{N}_{q,k} := \frac{1}{q} \binom{q}{k} \binom{q}{k-1}$ are the Narayana numbers. From this form it was demonstrated that

$$\lim_{\substack{N, a \rightarrow \infty \\ N/(N+a) \rightarrow \alpha}} N^{q-1} \left\langle \sum_{j=1}^N \lambda_j^q \right\rangle^{(\text{fl})} = \sum_{k=1}^{\infty} \mathbf{N}_{q,k} \alpha^{k-1}, \quad (3.22)$$

⁴ For q, a non-negative integers, this in an equivalent form can be found in [47, theorem 2.5].

thus reclaiming a result from [51] derived using a different approach. Note that for q a positive integer, the sum on the right hand side is finite, with upper terminal q , and it specifies the integer moments of the Marchenko–Pastur law (2.32).

The work [77] also derived a formula for $\langle (\tilde{T}_q)^2 \rangle^{(L)}$, involving finite single and double sums over particular ${}_3F_2$ functions. Specialising to $q=2$ gives a formula [77, equation (58)] consistent with the moment $\hat{m}_2^{(L)}$ as implied by proposition 3.1 and (3.6). Moreover, as initiated in [9, 71], it was shown in [77] how knowledge of $\langle \tilde{T}_q \rangle^{(\mathbb{N})}$ and $\langle (\tilde{T}_q)^2 \rangle^{(\mathbb{N})}$, as implied by first computing $\langle T_q \rangle^{(\mathbb{N})}$ and $\langle (T_q)^2 \rangle^{(\mathbb{N})}$, then using (3.8), can be used to facilitate the calculation of the variance for von Neumann entropy [75, 76]. For recent related work on this theme see [10, 50, 55, 78–80].

We also remark that the moments $\left\langle \sum_{j=1}^N \lambda_j^q \right\rangle^{(L)}$ for q continuous have featured in the recent study [19], where they are specified in terms of a certain continuous Hahn polynomial of degree $N-1$, as noted below (2.26). The latter are known to be given in terms of a terminating ${}_3F_2$ hypergeometric function, and in fact such a formula was presented in [19, equation (4.11)]. This can also be shown to be equivalent to (3.18) upon appropriate use of (3.20).

Remark 3.1. 1. Generally in random matrix theory, the average of the linear statistic $\sum_{j=1}^N \lambda_j^q$ correspond to the moments of the spectral density, and have been studied for many decades now. Motivating these studies has been the analysis of the global large N limit [81], their combinatorial significance [11, 48], applications to fluctuation formulas (see e.g. the review [30]), as well as their integrability properties as further developed here. Additional recent works on the latter theme not already cited include [3, 12, 13, 15, 16].

2. As with the case $q=2$, Painlevé structures are also present for $q=-1$. For this value of q , from the condition $qk+a > -1$ in proposition 3.2, note that the left hand side of (3.9) is only defined for $k < a+1$. On the other hand, the Laplace–Fourier transform of the linear statistic $\sum_{j=1}^N \lambda_j^{-1}$,

$$\hat{Q}_{N,a}^{(L)}(s) := \frac{1}{C_{N,a}^{(L)}} \int_0^\infty d\lambda_1 \cdots \int_0^\infty d\lambda_N \prod_{l=1}^N e^{-\lambda_l - \frac{s}{N} \lambda_l^a} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2, \quad (3.23)$$

is well defined for all $\text{Re}(s) \geq 0$ (although is not analytic at $s=0$). It has been shown in multiple works [17, 22, 58, 65] from various viewpoints that $s \frac{d}{ds} \log \hat{Q}_{N,a}^{(L)}(s)$ is a particular σ -Painlevé PIII' transcendent. An application given in [22, 82] shows that $\hat{Q}_{N,a}^{(L)}(s/N)$, and thus the cumulants κ_k/N^k ($k < a+1$), have a well defined $N \rightarrow \infty$ limit, with the former given as the solution of a particular third order nonlinear equation.

4. Purity moments for general β

The eigenvalue PDFs (1.3) and (1.5) admit natural β generalisations,

$$p_{N,a,\beta}^{(L_\beta)}(\lambda_1, \dots, \lambda_N) := \frac{1}{C_{N,a,\beta}^{(L_\beta)}} \prod_{l=1}^N \lambda_l^a e^{-\lambda_l} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta \quad (4.1)$$

(this is (2.8) up to scaling of the λ_l) and

$$p_{N,a,\beta}^{(\mathbb{N}_\beta)}(\lambda_1, \dots, \lambda_N) := \frac{1}{C_{N,a,\beta}^{(\mathbb{N}_\beta)}} \delta \left(1 - \sum_{l=1}^N \lambda_l \right) \prod_{l=1}^N \lambda_l^a \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta, \quad (4.2)$$

which are well established in random matrix theory [28]. The parameter β is often referred to as the Dyson index. In this section we will take up the task of computing the positive integer moments of the statistic $T_2 = \sum_{j=1}^N \lambda_j^2$ (i.e. the purity in the context of quantum information) with respect to (4.1) ⁵ The same moments with respect to (4.2) then follow from the analogue of (2.27),

$$m_p^{(\mathbb{N}_\beta)} = \frac{1}{(\beta N(N-1)/2 + N(a+1))_{2p}} m_p^{(L_\beta)}. \quad (4.3)$$

⁵ Earlier in this paper, in section 2.2, we have made note of a known β -dimensional integral formula for the density corresponding to (a simple scaling of) (4.1), applicable for β even. The reason for the latter restriction is an intermediate step (not shown) which expresses the $(N-1)$ -dimensional integral defining the density as β -dimensional integral. This is an example of a duality formula (see the recent review [31]) which requires that the former average be a polynomial in the variable x . This is special to β even. In contrast, in this section no use is made of duality formula, and our results are valid for general $\beta > 0$.

An explicit formula for the mean $m_1^{(L_\beta)}$ can be read off from a result in [59, equation (A.9b)]

$$\begin{aligned} m_1^{(L_\beta)} &= N(a^2 + 3a - 4\tau - 3a\tau + 2 + 2\tau^2) + N^2(3a\tau - 4\tau^2 + 4\tau) + 2N^3\tau^2 \\ &= N(\tau(N-1) + (1+a))(2\tau(N-1) + (2+a)), \end{aligned} \quad (4.4)$$

where $\tau = \beta/2$. Notice that for $\beta = 2$ this reduces to the result for $\kappa_1^{(L)}$ given in proposition 3.1.

The positive integer moments for general β can be studied by considering the Laplace–Fourier transform

$$\hat{P}_{N,a,\beta}^{(L_\beta)}(s) = \frac{1}{C_{N,a,\beta}^{(L_\beta)}} \int_0^\infty d\lambda_1 \cdots \int_0^\infty d\lambda_N \prod_{l=1}^N e^{-\lambda_l - s\lambda_l^2} \lambda_l^a \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta \quad (4.5)$$

(cf (1.18)). With the β generalisation of the GUE eigenvalue PDF (1.1) specified by

$$P_{N,\beta}^{(G_\beta)}(\lambda_1, \dots, \lambda_N) := \frac{1}{C_{N,\beta}^{(G_\beta)}} \prod_{l=1}^N e^{-\lambda_l^2} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta, \quad (4.6)$$

our analysis proceeds by first changing variables $\lambda_l \mapsto \lambda_l/\sqrt{s}$ in (4.5) and completing the square in the exponent to deduce the β generalisation of (1.20)

$$\hat{P}_{N,a,\beta}^{(L_\beta)}(s) = \frac{C_{N,\beta}^{(G_\beta)}}{C_{N,a,\beta}^{(L_\beta)}} e^{N/4s} (1/\sqrt{s})^{N(a+1) + \beta N(N-1)/2} \left\langle \prod_{j=1}^N (\lambda_j - 1/(2\sqrt{s}))^a \chi_{\lambda_j > 1/(2\sqrt{s})} \right\rangle^{(G_\beta)}. \quad (4.7)$$

The utility of (4.7) stems from the fact that a generalisation of the multidimensional integral corresponding to the right hand side,

$$\begin{aligned} I_{p,N,\tau}^{(\alpha)}(x) &= \frac{(-1)^p}{C_p^N} \int_x^\infty dt_1 \cdots \int_x^\infty dt_N \prod_{l=1}^N e^{-t_l^2} (t_l - x)^\alpha \\ &\quad \times \prod_{1 \leq j < k \leq N} |t_k - t_j|^{2\tau} e_p(t_1 - x, \dots, t_N - x), \end{aligned} \quad (4.8)$$

(cf (2.13)) satisfies the differential-difference recurrence [25, for the case $\tau = 1/2$], [40]

$$(N-p)I_{p+1}(x) = (N-p)xI_p(x) + \frac{1}{2} \frac{d}{dx} I_p(x) - \frac{p(\tau(N-p) + \alpha + 1)}{2} I_{p-1}(x), \quad (4.9)$$

valid for $p = 0, \dots, N$, where we have abbreviated $I_{p,N}^{(\alpha)}(x) = I_p(x)$. We can use (4.9) to deduce a matrix differential equation (recall section 2.2) for a family of multiple integrals generalising (4.5)

$$\begin{aligned} H_p(s) &= H_{p,N,a,\tau}(s) = \frac{(-1)^p}{C_p^N} \int_0^\infty dt_1 \cdots \int_0^\infty dt_N \prod_{l=1}^N t_l^a e^{-t_l} e^{-st_l^2} \\ &\quad \times \prod_{1 \leq j < k \leq N} |t_k - t_j|^{2\tau} e_p(t_1, \dots, t_N). \end{aligned} \quad (4.10)$$

Proposition 4.1. Introduce the column vector $\mathbf{H}(s) = [H_p(s)]_{p=0}^N$, and the bidiagonal matrices

$$\begin{aligned} \mathbf{A} &= -\text{diag}(N(a+1) + \tau N(N-1) + p)_{p=0}^N - \text{diag}^+(N-p)_{p=0}^{N-1} \\ \mathbf{B} &= -\text{diag}\left(\frac{p}{2}\right)_{p=0}^N - \text{diag}^-\left(\frac{p(\tau(N-p) + a + 1)}{2}\right)_{p=1}^N. \end{aligned} \quad (4.11)$$

Here diag^+ (diag^-) refers to the diagonal directly above (below) the main diagonal. We have

$$2s^2 \frac{d}{ds} \mathbf{H}(s) = s\mathbf{A}\mathbf{H}(s) + \mathbf{B}\mathbf{H}(s). \quad (4.12)$$

Also, denote by $\mathbf{B}_q$ the matrix which has the first row in $\mathbf{B}$, labelled by $p = 0$ and which consists of all zeros, replaced by the first row in $-\mathbf{A} + 2q\mathbf{I}_{N+1}$. Introducing the power series expansion

$$\mathbf{H}(s) = \sum_{q=0}^{\infty} \mathbf{h}_q s^q, \quad (4.13)$$

we have the recurrence

$$\mathbf{h}_{q+1} = B_{q+1}^{-1} (-\mathbf{A} + 2q\mathbf{l}_{N+1}) \mathbf{h}_q, \quad (4.14)$$

subject to the requirement that

$$\mathbf{B}\mathbf{h}_0 = \mathbf{0}. \quad (4.15)$$

Proof. From the definitions

$$I_{p,N}^{(a)}(1/(2\sqrt{s})) = (\sqrt{s})^{N(a+1)+\tau N(N-1)+p} e^{-N/4s} H_p(s). \quad (4.16)$$

Thus after changing variables $x = 1/(2\sqrt{s})$ in (4.9), then substituting according to (4.16), we have that $\{H_p(s)\}_{p=0}^N$ satisfies the differential-difference equation

$$(N-p)sH_{p+1}(s) = (N-p)\frac{1}{2}H_p(s) - \left(s(N(a+1) + \tau N(N-1) + p) + \frac{N}{2}\right)H_p(s) - 2s^2 \frac{d}{ds}H_p(s) - \frac{p(\tau(N-p) + a + 1)}{2}H_{p-1}(s),$$

valid for $p = 0, \dots, N$. Writing this in matrix form gives (4.12).

Substituting (4.13) in (4.12) shows

$$\sum_{q=0}^{\infty} (-A + 2q\mathbf{l}_{N+1}) \mathbf{h}_q s^{q+1} = \sum_{q=0}^{\infty} \mathbf{B}\mathbf{h}_q s^q. \quad (4.17)$$

Equating the term independent of s on both sides gives (4.15). Equating powers of s^{q+1} for $q \geq 0$ gives

$$(-A + 2q\mathbf{l}_{N+1}) \mathbf{h}_q = \mathbf{B}\mathbf{h}_{q+1}. \quad (4.18)$$

The first row in $\mathbf{B}$, labelled by $p = 0$, has all entries equal to zero. Thus the first entry in $(-A + 2q\mathbf{l}_{N+1})$ is equal to zero. This implies that the matrix $\mathbf{B}$ as it appears in (4.18) can have its first row replaced by the first row in $-A + 2(q+1)\mathbf{l}_{N+1}$. The resulting matrix, denoted $\mathbf{B}_{q+1}$, is invertible and (4.14) follows. $\square$

Writing $\mathbf{h}_q = (h_q^{(p)})_{p=0}^N$ we see from (4.13), (4.5) and (4.10) that $h_0^{(0)} = C_{N,a,\beta}^{(L_\beta)}$. Moreover, it follows from (4.15) that

$$h_0^{(p)} = h_0^{(0)} (-1)^p \prod_{l=1}^p (\tau(N-l) + a + 1). \quad (4.19)$$

As a check, we note from the definitions that $h_0^{(N)} = (-1)^N C_{N,a+1,\beta}^{(L_\beta)}$. This substituted in (4.19) implies

$$\frac{C_{N,a+1,\beta}^{(L_\beta)}}{C_{N,a,\beta}^{(L_\beta)}} = \prod_{l=1}^N (\tau(N-l) + a + 1), \quad (4.20)$$

which is indeed true as $C_{N,a,\beta}^{(L_\beta)}$ has the product of gamma function evaluation (see e.g. [28, proposition 4.7.3])

$$C_{N,a,\beta}^{(L_\beta)} = \prod_{j=0}^{N-1} \frac{\Gamma(1 + (j+1)\tau) \Gamma(a+1+j\tau)}{\Gamma(1+\tau)}. \quad (4.21)$$

Denote the first entry in $\mathbf{h}_q$ by $h_q^{(0)}$ so that

$$H_0(s) = h_0^{(0)} \sum_{q=0}^{\infty} \frac{(-1)^q m_q^{(L_\beta)}}{q!} s^q = \sum_{q=0}^{\infty} h_q^{(0)} s^q. \quad (4.22)$$

Here the first equality follows by comparing (4.10) and (4.5), and performing a power series expansion in s of the latter, recalling too that $m_q^{(L_\beta)}$ refers to the moments of the linear statistic $T_2 = \sum_{j=1}^N \lambda_j^2$ with respect to (4.1). Thus, in light of (4.14) and (4.19), for a given N we have a computational scheme to evaluate $\{m_q^{(L_\beta)}\}_{q=1,2,\dots}$ which involves only multiplication of matrices of size $(N+1) \times (N+1)$.

Moreover, we can use the fact that each $m_q^{(L_\beta)}$ is a polynomial in N of degree $3q$ (i.e. the same degree as $(m_1^{(L_\beta)})^q$) to deduce the dependence on N from a table of $m_q^{(L_\beta)}$ (or more) different values of N . Proceeding in this way, extending the result (4.4) for $m_1^{(L_\beta)}$ we can deduce the explicit form of $m_2^{(L_\beta)}$. This simplifies upon introducing the corresponding cumulant $\kappa_2^{(L_\beta)} = m_2^{(L_\beta)} - (m_1^{(L_\beta)})^2$.

Proposition 4.2. *Let $m_1^{(L_\beta)}$ be given by (4.4), where we recall $\tau = \beta/2$. We have*

$$m_2^{(L_\beta)} = \kappa_2^{(L_\beta)} + \left(m_1^{(L_\beta)}\right)^2, \quad (4.23)$$

where

$$\kappa_2^{(L_\beta)} = 2N(1+a+\tau(-1+N))(10+2a^2+9a(1+\tau(-1+N))+\tau(-1+N)(19+\tau(-10+9N))). \quad (4.24)$$

Remark 4.1. For general non-negative integer q , the moments $m_q^{(L_\beta)}$ are unchanged, up to a factor of $(-\tau)^q$, by the mapping of parameters [24]

$$(N, \tau, a) \mapsto (-\tau N, 1/\tau, -a/\tau). \quad (4.25)$$

Note that the explicit formula (4.4) exhibits this invariance. Recently [29] it has been established that this property carries over to all cumulants, as indeed is exhibited by (4.24) as one example.

Corollary 4.1. *For the fixed trace ensemble fL_β , the cumulant $\kappa_2^{(\text{fL}_\beta)}$ is given by*

$$\begin{aligned} \kappa_2^{(\text{fL}_\beta)} = & \frac{2(10+2a^2+9a(1+\tau(-1+N))+\tau(-1+N)(19+\tau(-10+9N)))}{(\tau N(N-1)+N(a+1)+1)_3} \\ & + \frac{N(\tau(N-1)+a+1)(2\tau(N-1)+a+2)^2}{(\tau N(N-1)+N(a+1)+1)_3} - \left(\frac{2\tau(N-1)+a+2}{\tau N(N-1)+N(a+1)+1} \right)^2. \end{aligned} \quad (4.26)$$

Proof. It follows from (4.3) that

$$\kappa_2^{(\text{fL}_\beta)} = \frac{m_2^{(L_\beta)}}{(\tau N(N-1)+N(a+1))_4} - \left(\frac{m_1^{(L_\beta)}}{(\tau N(N-1)+N(a+1))_2} \right)^2.$$

Substituting (4.4) and (4.23) as appropriate gives (4.26). □

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Appendix

In the historical development of random matrix theory, the introduction of random unitary matrices preceded that of random Hermitian matrices—see the account in [23]. The natural analogue of a fixed trace Hermitian matrix for a random unitary matrix is to impose a unit determinant constraint. Then, for example, the matrix group $U(N)$ becomes the subgroup $SU(N)$. With motivations in lattice gauge theories [45], the recent work by one of the present authors [61] took up the question of computing the exact one-point correlation function for $SU(N)$, as well as the unit determinant versions of Dyson's circular orthogonal and symplectic ensembles; for an introduction to the latter see [28, chapter 2]. In the case of $SU(N)$ it was found, by the use of the Selberg integral, that

$$\rho_{(1)}^{SU(N)}(\theta) = \frac{N}{2\pi} \left(1 - (-1)^N \frac{2}{N} \cos N\theta \right). \quad (\text{A.1})$$

As an application, consider the linear statistic $\text{Tr}(U^p) = \sum_{j=1}^N e^{ip\theta_j}$, $p \in \mathbb{Z}$. It follows by integrating this against (A.1) that

$$\langle \text{Tr}(U^p) \rangle^{SU(N)} = \begin{cases} N, & p = 0 \\ (-1)^{N-1}, & p = \pm N \\ 0, & \text{otherwise;} \end{cases} \quad (\text{A.2})$$

for specific interest in this see [68]. Note that the corresponding average over $U(N)$ gives zero in all cases except $p = 0$.

At the next level of complexity is the two-point correlation. In the case of $U(N)$ one has the classical result [57]

$$\rho_{(2)}^{U(N)}(\theta, \theta') = \left(\frac{N}{2\pi} \right)^2 - \left(\frac{\sin(N(\theta - \theta')/2)}{2\pi \sin((\theta - \theta')/2)} \right)^2 = \rho_{(1)}^{U(N)}(\theta) \rho_{(1)}^{U(N)}(\theta') + \rho_{(2)}^{U(N),T}(\theta, \theta'). \quad (\text{A.3})$$

In the second equality $\rho_{(1)}^{U(N)}(\phi) = N/(2\pi)$ independent of ϕ , while $\rho_{(2)}^{U(N),T}(\theta, \theta')$ (referred to as the truncated two-point correlation) is the second term in the first equality of (A.3) and has the feature of being of order unity for $\theta \neq \theta'$ fixed. Generally the truncated two-point correlation is of interest as the kernel in the fluctuation formula for the covariance of two linear statistics

$$\text{Cov} \left(\sum_{j=1}^N f(\theta_j), \sum_{j=1}^N g(\theta_j) \right) = -\frac{1}{2} \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\theta' (f(\theta) - g(\theta')) (g(\theta) - f(\theta')) \rho_{(2)}^T(\theta, \theta'); \quad (\text{A.4})$$

see e.g. [30, proposition 2.1].

Here we provide the exact evaluation of the two-point correlation for $SU(N)$. For this we will use the functional differentiation method presented in [28, chapter 5.2.1]. This relies on introducing a ‘probe’ one-point function $a(\theta)$ and introducing the generalised partition function by $Z_N(a) = \langle \prod_{j=1}^N a(\theta_j) \rangle^{SU(N)}$. Then the p -point correlation functions are expressed as functional derivatives of $Z_N[a]$,

$$\rho_{(1)}^{SU(N)}(\theta) = \frac{\delta Z_N[a]}{\delta a(\theta)} \Big|_{a=1}, \quad \rho_{(2)}^{SU(N)}(\theta, \theta') = \frac{\delta^2 Z_N[a]}{\delta a(\theta) \delta a(\theta')} \Big|_{a=1}, \quad (\text{A.5})$$

etc. Our ability to carry out the required functional derivatives relies on an explicit (infinite) sum of Toeplitz determinants form of the generalised partition function.

Proposition A.1. *We have*

$$Z_N(a) = \sum_{n=-\infty}^{\infty} \det \left[\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} a(\theta) e^{i(n+m-\ell)\theta} \right]_{\ell, m=1}^N. \quad (\text{A.6})$$

Proof. The Haar measure on $U(N)$, when written in terms of its eigenvalues (see e.g. [28, exercises 2.2 q.1]), leads via the additional constraint $\sum_j \theta_j = 0 \bmod 2\pi$ to the joint distribution function of the eigenphases for $SU(N)$

$$\delta \left(\sum_{j=1}^N \theta_j \bmod 2\pi \right) \cdot \frac{1}{(2\pi)^{N-1}} \frac{1}{N!} |\Delta_N(\theta)|^2 = \frac{1}{(2\pi)^N} \frac{1}{N!} \sum_{n=-\infty}^{\infty} e^{in(\theta_1 + \dots + \theta_N)} |\Delta_N(\theta)|^2, \quad (\text{A.7})$$

$$\Delta_N(\theta) := \prod_{1 \leq j < k \leq N} (e^{i\theta_k} - e^{i\theta_j}) = \det [e^{ik\theta_j}]_{\substack{j=1, \dots, N \\ k=0, \dots, N-1}}.$$

To obtain the first equality in (A.7), the Fourier sum form of the 2π -periodic Dirac delta function has been used. Clearly, the $n = 0$ term in (A.7) corresponds to the unconstrained $U(N)$ case.

By expressing the Vandermonde determinant $\Delta_N(\theta)$ and its conjugate as sums of permutations $P, Q \in \mathfrak{S}_N$, we can deduce that each (n th) component of $Z_N[a]$ takes the form of the Toeplitz determinant,

$$\begin{aligned} Z_N[a] &= \sum_{n=-\infty}^{\infty} \frac{1}{N!} \sum_{P, Q \in \mathfrak{S}_N} \text{sgn}(P) \text{sgn}(Q) \prod_{j=1}^N \int_{-\pi}^{\pi} \frac{d\theta_j}{2\pi} a(\theta_j) e^{in\theta_j} e^{iP(j)\theta_j} e^{-iQ(j)\theta_j} \\ &= \sum_{n=-\infty}^{\infty} \sum_{R \in \mathfrak{S}_N} \text{sgn}(R) \prod_{j=1}^N \int_{-\pi}^{\pi} \frac{d\theta_j}{2\pi} a(\theta_j) e^{i(n+\ell-R(\ell))\theta_j} \quad (\ell = P(j), R = Q \circ P^{-1}) \\ &= \sum_{n=-\infty}^{\infty} \det \left[\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} a(\theta) e^{i(n+m-\ell)\theta} \right]_{\ell, m=1}^N. \end{aligned}$$

Here to obtain the final equality, and thus our sought result, the general fact that the determinant of a matrix is equal to the determinant of the transpose of the matrix has been used. $\square$

Before using (A.5) to compute the two-point correlation, it is instructive to first show how the first relation in (A.5) can be used to re-derive (A.1).

We have remarked that the $n = 0$ term in (A.7) corresponds to the unconstrained $U(N)$ case. This implies that the functional derivative operation in the first relation in (A.5) applied to the $n = 0$ term in (A.6) yields $\rho_{(1)}^{U(N)}(\theta) = \frac{N}{2\pi}$, as can be checked directly. Consider now the $n = 1$ term of (A.6). By applying the functional differentiation $\frac{\delta}{\delta a(\theta)}$ at $a = 1$ to each column of the determinant (it is generally true that a single functional derivative of a determinant can be carried out column-by-column), and using the simple fact that for $\ell, m, n \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i(n+\ell-m)\theta} = \delta_{m-\ell, n}, \quad (\text{A.8})$$

one sees that a non-zero contribution comes from the final column ($m = N$),

$$\begin{vmatrix} 0 & 0 & \dots & e^{i(1+N-1)\theta}/2\pi \\ 1 & 0 & \dots & e^{i(1+N-2)\theta}/2\pi \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & e^{i(1+0)\theta}/2\pi \end{vmatrix} = (-1)^{N-1} \frac{e^{iN\theta}}{2\pi}. \quad (\text{A.9})$$

By the same token, the $n = -1$ term of (A.6) contributes the complex conjugate of (A.9) to $\rho_{(1)}^{SU(N)}(\theta)$. On the other hand terms with $|n| > 1$ do not contribute as they contain columns with entirely zero entries due to (A.8). Thus for the one-point correlation function we obtain

$$\rho_{(1)}^{SU(N)}(\theta) = \frac{N}{2\pi} - (-1)^N \frac{e^{iN\theta} + e^{-iN\theta}}{2\pi}, \quad (\text{A.10})$$

which is indeed consistent with (A.1).

With this preparation, we are now ready to deduce the explicit function form of the two-point function.

Proposition A.2. *Let U be an $SU(N)$ matrix distributed according to the Haar measure. The two-point correlation function of eigenphases θ, θ' of U is given by*

$$\begin{aligned} \rho_{(2)}^{\text{SU}(N)}(\theta, \theta') &= \rho_{(2)}^{U(N)}(\theta, \theta') + \frac{2}{\pi^2} \cos N(\theta + \theta') \sin^2 \frac{\theta - \theta'}{2} \\ &\quad + (-1)^N \frac{1}{\pi^2} \cos \frac{N(\theta + \theta')}{2} \left(\sin \frac{N(\theta - \theta')}{2} \cot \frac{\theta - \theta'}{2} - N \cos \frac{N(\theta - \theta')}{2} \right). \end{aligned} \quad (\text{A.11})$$

Proof. According to (the second relation of A.5) two functional differentiation operations are required in (A.6). From the determinant structure, this is equivalent to taking the functional derivative of any two distinct columns, and summing over all choices. One observes that the determinants with $|n| > 2$ do not contribute due to (A.8). As the contribution from $n = 0$ is guaranteed to give the $U(N)$ result (A.3), one only needs to evaluate contributions from the terms with $|n| = 1$ and 2. Easier is the $n = 2$ contribution, in which $\frac{\delta}{\delta a(\theta)}$ and $\frac{\delta}{\delta a(\theta')}$ must act on the final two columns ($m = N - 1, N$) to yield

$$\begin{vmatrix} 0 & 0 & \dots & e^{i(2+N-2)\theta}/2\pi & e^{i(2+N-1)\theta'}/2\pi \\ 0 & 0 & \dots & e^{i(2+N-3)\theta}/2\pi & e^{i(2+N-2)\theta'}/2\pi \\ 1 & 0 & \dots & e^{i(2+N-4)\theta}/2\pi & e^{i(2+N-3)\theta'}/2\pi \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & e^{i(2-1)\theta}/2\pi & e^{i(2+0)\theta'}/2\pi \end{vmatrix} = \frac{e^{iN(\theta+\theta')}(1 - e^{i(\theta'-\theta)})}{(2\pi)^2} \quad (\text{A.12})$$

and the one with θ and θ' exchanged.

For the $n = 1$ term, one of $\frac{\delta}{\delta a(\theta)}$ and $\frac{\delta}{\delta a(\theta')}$ must act on the final column ($m = N$) but the other can act on any of the remaining columns ($m = 1, \dots, N - 1$). The contribution from choosing the m th column is

$$\begin{vmatrix} 0 & 0 & \dots & e^{i(1+m-1)\theta}/2\pi & \dots & 0 & e^{i(1+N-1)\theta'}/2\pi \\ 1 & 0 & \dots & e^{i(1+m-2)\theta}/2\pi & \dots & 0 & e^{i(1+N-2)\theta'}/2\pi \\ 0 & 1 & \dots & e^{i(1+m-3)\theta}/2\pi & \dots & 0 & e^{i(1+N-3)\theta'}/2\pi \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1/2\pi & \dots & 0 & e^{i(1+N-m-1)\theta'}/2\pi \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e^{i(1+m-N)\theta}/2\pi & \dots & 1 & e^{i(1+0)\theta'}/2\pi \end{vmatrix} = (-1)^N \frac{e^{iN\theta'}(e^{im(\theta-\theta')} - 1)}{(2\pi)^2} \quad (\text{A.13})$$

and the one with θ and θ' exchanged. By summing up contributions from $m = 1, \dots, N - 1$, and including complex-conjugated contributions from $n = -2$ and -1 , we finally obtain (A.11) from (A.12) and (A.13). There the contribution from $n = \pm 2$ is expressed in a concise form with help of the summation formula

$$\sum_{m=1}^{N-1} \sin mx \sin(N-m)x = \frac{1}{2} (\sin Nx \cot x - N \cos Nx).$$

□

After use of an elementary trigonometric identity, and recalling (A.1) and (A.3), we obtain an alternative way to write (A.11):

Corollary A.2. *The truncated two-point correlation function $\rho_{(2)}^{\text{SU}(N),T}(\theta, \theta')$ of eigenphases of Haar-distributed $SU(N)$ matrices, defined by*

$$\rho_{(2)}^{\text{SU}(N)}(\theta, \theta') = \rho_{(1)}^{\text{SU}(N)}(\theta) \rho_{(1)}^{\text{SU}(N)}(\theta') + \rho_{(2)}^{\text{SU}(N),T}(\theta, \theta'), \quad (\text{A.14})$$

is expressed as

$$\begin{aligned} \rho_{(2)}^{\text{SU}(N),T}(\theta, \theta') &= \rho_{(2)}^{U(N),T}(\theta, \theta') + \frac{2}{\pi^2} \cos N(\theta + \theta') \sin^2 \frac{\theta - \theta'}{2} \\ &\quad + (-1)^N \frac{1}{\pi^2} \cos \frac{N(\theta + \theta')}{2} \sin \frac{N(\theta - \theta')}{2} \cot \frac{\theta - \theta'}{2} - \frac{1}{\pi^2} \cos N\theta \cos N\theta'. \end{aligned} \quad (\text{A.15})$$

Note that in (A.15), in distinction to (A.11), all the correction terms to the $U(N)$ result are of order unity. In relation to the use of (A.15) in (A.4), a strategy is to write the given f, g as Fourier series, and similarly make use of the Fourier series expansion of $\rho_{(2)}^{\text{SU}(N), T}$, as implied by the intermediate working in deriving (A.15).

Of interest is the large N expansion of (A.15) with bulk scaling. Such large N expansions are attracting recent attention for their special functional forms in the case of the β generalisation of Dyson's circular ensembles [39], and for application to spacing ratio distributions in the $U(N)$ case [62]. Bulk scaling corresponds to the unfolding $\theta = 2\pi x/N$, $\theta' = 2\pi x'/N$ so that the mean eigenvalue spacing is unity. Recalling too (A.3), we compute from (A.15) that

$$\begin{aligned} & \left(\frac{2\pi}{N}\right)^2 \rho_{(2)}^{\text{SU}(N), T} \left(\frac{2\pi x}{N}, \frac{2\pi x'}{N}\right) \\ &= - \left(\frac{\sin \pi(x-x')}{\pi(x-x')}\right)^2 + \frac{4(-1)^N}{N} \cos \pi(x+x') \frac{\sin \pi(x-x')}{\pi(x-x')} + \mathcal{O}\left(\frac{1}{N^2}\right). \end{aligned} \quad (\text{A.16})$$

It is notable that the unit determinant constraint alters the leading correction to the universal sine-kernel limit (the first term of (A.16)) from of $\mathcal{O}(N^{-2})$ to $\mathcal{O}(N^{-1})$, which moreover decays as $1/(x-x')$ for a large separation of two eigenvalues.

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