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# GENERAL FORMULAE OF LUMINOSITY FOR VARIOUS TYPES OF COLLIDING BEAM MACHINES

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Abstract

Summarized are the formulae of luminosity for proton-proton, electron-positron and electron-proton colliding beam machines. Both coasting and bunched proton beams are considered. The expressions are derived from the first principle. These formulae will be useful for the design of an intersecting storage accelerator such as TRISTAN.

## §1. Introduction

Luminosity is an important parameter characterizing the performance of a colliding beam machine. A general relation between the event rate and the cross section in an arbitrary frame of reference was discussed by Møller.[1] The formulae of luminosity appropriate for specific collision types and geometries have ever since been derived by many authors. In designing a complex intersecting storage accelerator such as TRISTAN,[2] which aims at various types of colliding beam experiments, it seems appropriate to compile the formulae scattered in various articles. The formulae are derived from the first principle. A similar work was done by Ruggiero[3] whose starting equation, however, is an approximation as pointed out in this note.

## §2. *General Expression for Luminosity*

The number of events per unit time and per unit volume $\frac{d^2N}{dtdV}$ observed in an aribitrary frame of reference is expressed[1,4] in the form

$$\frac{d^2N}{dtdV} = \sigma n_1 n_2 \sqrt{(\vec{v}_1 - \vec{v}_2)^2 - \frac{(\vec{v}_1 \times \vec{v}_2)^2}{c^2}} \quad , \tag{1}$$

where

- $\sigma$ : total cross section
- $n_1, n_2$ : densities of particles 1 and 2
- $\vec{v}_1, \vec{v}_2$ : velocity vectors of particles 1 and 2
- $c$ : velocity of light
- $dV$ : volume element of the interaction region (the region where the two beams overlap).

The derivation of eq.(1) is shown in the appendix. Since we are interested in a relativistic case, we put $|\vec{v}_1| \approx |\vec{v}_2| \approx c$ and define the angle between $\vec{v}_1$ and $\vec{v}_2$ to be a crossing angle $2\phi$. Then

$$\frac{d^2N}{dtdV} \cong \sigma n_1 n_2 \cdot 2c \cos^2\phi \; . \tag{2}$$

The luminosity $\mathcal{L}$ is defined as

$$\frac{dN}{dt} = \sigma \mathcal{L} , \tag{3}$$

and is obtained by the volume integral of eq.(2).

We use the coordinate system as shown in Fig.1. The rectangular coordinate systems $(x_i, y_i, z_i)$ $(i=1,2)$ are used for the two beams. $z_i$ denotes the direction of motion, $x_i$ denotes the horizontal axis and $y_i$ denotes the vertical axis. The origin O denotes the interaction point. We assume horizontal crossing in this note, but the formulae for vertical crossing can be obtained by interchanging the variables x and y. We define a common coordinate system (x,y,z) which is connected to $(x_i, y_i, z_i)$ through the relation

$$\begin{cases} x_1 = x\cos\phi - z\sin\phi \\ y_1 = y \\ z_1 = x\sin\phi + z\cos\phi \end{cases}$$

and

$$\begin{cases} x_2 = -x\cos\phi - z\sin\phi \\ y_2 = y \\ z_2 = x\sin\phi - z\cos\phi\ . \end{cases} \tag{4}$$

For unbunched coasting beams, the density $n_i$ is expressed as

$$n_i = \lambda_i f_i\ (x_i, y_i, z_i)\ , \tag{5}$$

where $\lambda_i$ is the line density of the beam and $f_i$ denotes a distribution function normalized such that

$$\int f_i\ (x_i, y_i, z_i)\quad dx_i dy_i = 1\ .$$

For bunched beams, $n_i$ is expressed as

$$n_i = N_i f_i\ (x_i, y_i, z_i, t)\ , \tag{6}$$

where $N_i$ is the number of particles in a bunch and $f_i$ is a distribution function normalized such that

$$\int f_i\ (x_i, y_i, z_i, t)\ dx_i\ dy_i\ dz_i = 1\ .$$

With these distribution functions, the luminosity is expressed in the following way according to the types of collisions.

1) coasting beam + coasting beam

$$\mathcal{L} = \lambda_1\lambda_2 \cdot 2c \cos^2\phi \int f_1(x_1,y_1,z_1) f_2(x_2,y_2,z_2)dxdydz, \quad (7)$$

2) bunched beam + bunched beam

$$\mathcal{L} = N_1N_2 \cdot 2cBf \cos^2\phi \int f_1(x_1,y_1,z_1,t)f_2(x_2,y_2,z_2,t)dxdydzdt, \quad (8)$$

3) bunched beam + coasting beam

$$\mathcal{L} = N_1\lambda_2 \cdot 2cBf \cos^2\phi \int f_1(x_1,y_1,z_1,t)f_2(x_2,y_2,z_2)dxdydzdt, \quad (9)$$

where Bf denotes the number of collisions per unit time (actually, B is the number of bunches and f is the revolution frequency). The limits of integral depend on the geometry and are usually taken from $-\infty$ to $+\infty$. The overlap integrals will be evaluated in the followings for various types of collisions.

## §3. Collision between Coasting Beams

A collision between coasting proton beams is a typical example of this case. We first consider a case where there is no low-β insertion and the variation of beam sizes along the $z_i$ axis is negligible. Then, the distribution functions fi's in eq.(7) is independent of $z_i$ and the luminosity is expressed as

$$\mathcal{L} = 2c\lambda_1\lambda_2 \cos^2\phi \int f_1(x_1,y_1)\ f_2(x_2,\ y_2)\ dxdydz.$$

By use of eq.(4), we change the variables of integration from (x,z) to $(x_1,x_2)$. Then

$$\mathcal{L} = \frac{\lambda_1\lambda_2 c}{\tan\phi} \int f_1(x_1,y)\ f_2(x_2,y)\ dx_1dx_2dy.$$

We introduce a new (vertical) distribution function $\sigma_i$ according to

$$\sigma_i(y) = \int f_i(x_i, y)\,dx_i .$$

Then, we obtain

$$\mathcal{L} = \frac{c\lambda_1\lambda_2}{h_{eff}\tan\phi} ,$$

where

$$\frac{1}{h_{eff}} = \int \sigma_1(y)\sigma_2(y)\,dy . \qquad (10)$$

This formula can be applied to CERN-ISR. The luminosity is influenced only by the vertical particle distribution and the effective height $h_{eff}$ is equal to the actual beam height if a uniform rectangular distribution is assumed for the two beams. For Gaussian beams, $h_{eff} = 2\sqrt{\pi}\sigma_y$, where $\sigma_y$ is the root-mean-square beam height.

We now consider a case where the variation of beam sizes along the $z_i$-axis is not negligible and the crossing angle is small ($\phi<<1$). We assume that the particle distribution is Gaussian both in betatron oscillations and momentum spread. Then, the rms beam size $\sigma_i$ $(z_i)$ is expressed as

$$\sigma_i^2(z_i) = \sigma_{i\beta}^{*2}\left(1 + \frac{z_i^2}{\beta_i^{*2}}\right) + x_{pi}^{*2}\left(\frac{\Delta p}{p}\right)^2 , \qquad (11)$$

where * denotes the value at the interaction point and $\beta_i$ and $x_{pi}$ are the betatron amplitude function and the off-energy dispersion function. $\sigma_{i\beta}^*$ denotes the beam size due only to betatron oscillation. Since, we assume a small crossing angle, $\sigma_i$ is a function of z, i.e. we can put $z_i = z$. The derivatives of $\beta$ and $x_p$ with respect to z are assumed to be zero at the interaction point. The distribution function is expressed as

$$f_i(x_i, y_i, z_i) = \frac{1}{2\pi\sigma_{xi}\sigma_{yi}} \exp\left[-\frac{x_i^2}{2\sigma_{xi}^2} - \frac{y_i^2}{2\sigma_{yi}^2}\right]. \qquad (12)$$

Inserting eq.(12) into eq.(7), we obtain

$$\mathcal{L} = \frac{\lambda_1\lambda_2 \cdot c \cos^2\phi}{2\pi^2} \int \frac{1}{\sigma_{x1}\sigma_{x2}\sigma_{y1}\sigma_{y2}} \exp\left[-\frac{x_1^2}{2\sigma_{x1}^2} - \frac{x_2^2}{2\sigma_{x2}^2} - \frac{y_1^2}{2\sigma_{y1}^2} - \frac{y_2^2}{2\sigma_{y2}^2}\right] dxdydz. \qquad (13)$$

Integration over x and y yields for a small crossing angle

$$\mathcal{L} = \frac{c\lambda_1\lambda_2}{\pi} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \frac{1}{\sqrt{(\sigma_{x1}^2+\sigma_{x2}^2)(\sigma_{y1}^2+\sigma_{y2}^2)}} \exp\left[-\frac{2z^2\phi^2}{\sigma_{x1}^2+\sigma_{x2}^2}\right]dz. \qquad (14)$$

Here, we assumed that the two beams are separated at $z = \pm\frac{\ell}{2}$ by bending magnets. Otherwise, $\ell$ is taken to be infinity.

Montague[5)] considered a case where the distribution in momentum is rectangular and that in betatron oscillations is Gaussian. This assumption may be more realistic for RF stacked beams. His result is as follows.

$$\mathcal{L} = \frac{c\lambda_1\lambda_2}{\sqrt{\pi}} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \sqrt{\frac{\sigma_{x1}^2+\sigma_{x2}^2}{\sigma_{y1}^2+\sigma_{y2}^2}} \frac{1}{4x_{p1}\Delta P_1 x_{p2}\Delta P_2} \times$$

$$[G(Ax_1 + B_1) - G(Ax_1 - B_1) - G(Ax_1 + B_2) + G(Ax_1 - B_2)]dz ,$$

$$A = \frac{x_{p1}}{\sqrt{2(\sigma_{x1}^2 + \sigma_{x2}^2)}} ,$$

$$B_1 = \frac{x_{01} - x_{02} + x_{p2}\Delta P_2}{\sqrt{2(\sigma_{x1}^2 + \sigma_{x2}^2)}} , \qquad (15)$$

$$B_2 = \frac{x_{01} - x_{02} - x_{p2}\Delta P_2}{\sqrt{2(\sigma_{x1}^2 + \sigma_{x2}^2)}} ,$$

$$x_1 = \Delta P_1 .$$

Here, $\Delta P_i$ is a half width of the momentum spread $\frac{\Delta p}{p}$ of the i-th beam, $x_{01}$ and $x_{02}$ are the displacements of the central orbits and equal to $\phi z$ and $-\phi z$ in the straight interaction region in Fig.1. The function G is given by

$$G(u) = u \text{ erf } (u) + \frac{1}{\sqrt{\pi}} e^{-u^2} , \qquad (16)$$

where the error function erf(u) is defined as

$$\text{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2} dt .$$

Another parameter characterizing the interaction region is a length $\ell_{int}$ of the region where the two beams overlap. We assume three standard deviations for the beam sizes and the length is determined by solving the equation

$$x_1 = 3\sigma_{x1}^* \sqrt{1 + \frac{z_1^2}{\beta_{x1}^{*2}}} \mp x_{p1}^* \left(\frac{\Delta p}{p}\right) ,$$

$$x_2 = 3\sigma_{x2}^* \sqrt{1 + \frac{z_2^2}{\beta_{x2}^2}} \pm x_{p2}^* \left(\frac{\Delta p}{p}\right) .$$

For beams of identical characteristics,

$$\ell_{int} = \frac{2\{3\sigma_x^* \sqrt{\sin^2\phi + \frac{\cos^2\phi}{\beta_x^{*2}} \{x_p^{*2}(\frac{\Delta p}{p})^2 - 9\sigma_x^{*2}\}} + x_p^* \frac{\Delta p}{p} \sin\phi}{\sin^2\phi - 9\frac{\sigma_x^{*2}}{\beta_x^{*2}} \cos^2\phi} . \qquad (17)$$

## §4. Collision between Bunched Beams

The collision between bunched beams was discussed by Smith.[6] The two bunches are assumed to have Gaussian distributions in three dimensions. Then, the distribution function $f_i$ is expressed as

$$f_i = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sigma_{xi}\sigma_{yi}\sigma_{zi}} \exp\left[-\frac{1}{2}\left\{\frac{x_i^2}{\sigma_{xi}^2} + \frac{y_1^2}{\sigma_{yi}^2} + \frac{(z_i - ct)^2}{\sigma_{zi}^2}\right\}\right] . \tag{18}$$

Then, the integration over y and t in eq.(8) yields

$$\mathcal{L} = BfN_1N_2 \frac{\cos^2\phi}{2\pi^2\sqrt{\sigma_{z1}^2 + \sigma_{z2}^2}} \int \frac{dxdz}{\sigma_{x1}\sigma_{x2}\sqrt{\sigma_{y1}^2 + \sigma_{y2}^2}} \times$$

$$\exp\left[-\frac{1}{2}\left\{4z^2\left(\frac{\cos^2\phi}{\sigma_{z1}^2 + \sigma_{z2}^2} + \frac{\sin^2\phi}{\sigma_{x1}^2 + \sigma_{x2}^2}\right) + \left(\frac{1}{\sigma_{x1}^2} + \frac{1}{\sigma_{x2}^2}\right) \times\right.\right.$$

$$\left.\left.\left(x\cos\phi + \frac{\sigma_{x1}^2 - \sigma_{x2}^2}{\sigma_{x1}^2 + \sigma_{x2}^2} z \sin\phi\right)^2\right\}\right] . \tag{19}$$

Here, $\sigma_{zi}$ is assumed to be constant.

We first consider a case where $\beta_{yi}^* >> \sigma_{zi}$. This corresponds to an electron-positron (or electron-electron) collision where the amplitude function at the interaction point is not too small. In this case, the beam widths are considered to be constant, and the integration of eq.(19) can be performed to yield

$$\mathcal{L} = \frac{fBN_1N_2\cos\phi}{2\pi} \frac{1}{\sqrt{\sigma_{y1}^2 + \sigma_{y2}^2}\sqrt{(\sigma_{x1}^2 + \sigma_{x2}^2)\cos^2\phi + (\sigma_{z1}^2 + \sigma_{z2}^2)\sin^2\phi}} . \tag{20}$$

When $\sigma_1 = \sigma_2$ and $\phi$ is small, eq.(20) reduces to

$$\mathcal{L} = \frac{fBN_1N_2}{4\pi} \frac{1}{\sigma_y\sqrt{\sigma_x^2 + (\sigma_z\phi)^2}} . \tag{21}$$

This is a well-known expression for electron-positron collisions.[7)]

The case where, $\beta_x^* \gg \sigma_z$, but $\beta_y^*$ is small, was considered for zero crossing angle by Fischer[8)] and the result is

$$\mathcal{L} = \frac{N_1 N_2 Bf}{2\sqrt{\pi}\sqrt{\beta_x^* \beta_y^*}} \left(\frac{\beta_y^*}{\sigma_z}\right) e^{\frac{1}{2}\left(\frac{\beta_y^*}{\sigma_z}\right)^2} \left[\frac{\pi i}{2} H_0^{(1)}\left(\frac{i}{2}\frac{\beta_y^{*2}}{\sigma_z^2}\right)\right], \tag{22}$$

where $H_0^{(1)}$ is the Hankel function.

For electron-proton and proton-proton collisions, in which both beams are bunched, the bunch length of the proton beam may not be small and the general expression (19) should be used. For a small crossing angle, integration over x can be done to yield

$$\mathcal{L} = \frac{2BfN_1N_2}{(2\pi)^{\frac{3}{2}}\sqrt{\sigma_{z1}^2 + \sigma_{z2}^2}} \int \frac{dz}{\sqrt{(\sigma_{x1}^2 + \sigma_{x2}^2)(\sigma_{y1}^2 + \sigma_{y2}^2)}}$$

$$\exp\left[-2z^2\left(\frac{1}{\sigma_{z1}^2 + \sigma_{z2}^2} + \frac{\phi^2}{\sigma_{x1}^2 + \sigma_{x2}^2}\right)\right] . \tag{23}$$

## §5. Collision between a Bunched Beam and a Coasting Beam

A typical example of this case is a collision between a bunched electron beam and a coasting proton beam. The distribution of a bunched beam is expressed by eq.(18) and that of a coasting beam by eq.(12). Then, the luminosity is expressed as

$$\mathcal{L} = Bf\,\frac{2c\cos^2\phi N_1\lambda_2}{(2\pi)^{\frac{5}{2}}\sigma_{z1}} \int \frac{dxdydzdt}{\sigma_{x1}\sigma_{y1}\sigma_{x2}\sigma_{y2}}$$

$$\exp\left[-\frac{1}{2}\left(\frac{x_1^2}{\sigma_{x1}^2} + \frac{y_1^2}{\sigma_{y1}^2} + \frac{(z_1 - ct)^2}{\sigma_{z1}^2} + \frac{x_2^2}{\sigma_{x2}^2} + \frac{y_2^2}{\sigma_{y2}^2}\right)\right] .$$

Integration over t yields

$$\mathcal{L} = \mathrm{Bf}\,\frac{\cos^2\phi N_1\lambda_2}{2\pi^2}\int\frac{dxdydz}{\sigma_{x1}\sigma_{x2}\sigma_{y1}\sigma_{y2}}\exp\left[-\frac{1}{2}\left\{\frac{x_1^2}{\sigma_{x1}^2}+\frac{y_1^2}{\sigma_{y1}^2}+\frac{x_2^2}{\sigma_{x2}^2}+\frac{y_2^2}{\sigma_{y2}^2}\right\}\right]. \quad (24)$$

It is to be noted that putting

$$\lambda_1 = \frac{\mathrm{BfN}_1}{c},$$

eq.(24) is identical to eq.(13).

Further, intergration over y yields

$$\mathcal{L} = \mathrm{Bf}\,\frac{2\cos^2\phi\; N_1\lambda_2}{(2\pi)^{\frac{3}{2}}}\int\frac{dxdz}{\sigma_{x1}\sigma_{x2}\sqrt{\sigma_{y1}^2+\sigma_{y2}^2}}$$

$$\exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_{x1}^2}+\frac{1}{\sigma_{x2}^2}\right)\left(x\cos\phi+\frac{\sigma_{x1}^2-\sigma_{x2}^2}{\sigma_{x1}^2+\sigma_{x2}^2}z\sin\phi\right)^2\right.$$

$$\left.+\frac{4z^2\sin^2\phi}{\sigma_{x1}^2+\sigma_{x2}^2}\right\}\Big]. \quad (25)$$

Putting

$$\lambda_2 = \frac{N_2}{\sqrt{2\pi}\sqrt{\sigma_{z1}^2+\sigma_{z2}^2}},$$

and going to the limit $\sigma_{z1}, \sigma_{z2} \to \infty$, eqs.(25) and (19) are identical.

For a small crossing angle, integration over x yields

$$\mathcal{L} = \mathrm{Bf}\,\frac{N_1\lambda_2}{\pi}\int\frac{dz}{\sqrt{\sigma_{x1}^2+\sigma_{x2}^2}\sqrt{\sigma_{y1}^2+\sigma_{y2}^2}}\exp\left[-\frac{2z^2\phi^2}{\sigma_{x1}^2+\sigma_{x2}^2}\right]. \quad (26)$$

With a rectangular momentum spread in beam 2, we obtain the expression

$$\mathcal{L} = Bf \frac{N_1 \lambda_2}{\sqrt{2\pi}} \int \frac{1}{2x_{p2}\Delta p_2} \frac{dz}{\sqrt{\sigma_{x1}^2 + \sigma_{x2}^2}} [\mathrm{erf}(B_1) - \mathrm{erf}(B_2)] \; , \tag{27}$$

where the appropriate parameters are defined in §3.

The interaction length is given, by assuming that the width of a bunched beam is negligible, in the form

$$\ell_{int} = 2 \frac{x_p^* \frac{\Delta p}{p}(\phi\cos\phi + \sin\phi) + 3\sigma_x^* \sqrt{D}}{(\phi\cos\phi + \sin\phi)^2 - \frac{9\sigma_x^{*2}}{\beta_x^{*2}} (\phi\sin\phi - \cos\phi)^2}$$

$$D = (\phi\cos\phi + \sin\phi)^2 + \frac{\{x_p^{*2} (\frac{\Delta p}{p})^2 - 9\sigma_x^{*2}\} (\phi\sin\phi - \cos\phi)^2}{\beta_x^{*2}} \; . \tag{28}$$

## §6. Conclusion

The formulae summarized in this note will be used to estimate the luminosity of TRISTAN. Combined with the formulae of beam-beam tune shifts, the optimum luminosity may also be obtained in a way shown by Keil[9] and Nishikawa.[10]

## Acknowledgement

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Appendix Derivation of eq.(1)

Møller[1] derived eq.(1) from the requirement of Lorentz invariance. Middelkoop and Schoch[4] derived it in a more intuitive way. We reproduce the derivation of ref.4).

We start from the well-known expression for the number of events per unit time $\frac{dN}{dt}$ observed in the laboratory frame of reference,

$$\frac{dN}{dt} = n_1|\vec{v}_1| \times \sigma \times n_2 dV, \tag{A1}$$

where

$n_1$: density of projectile particles

$n_2$: density of target particles

$\vec{v}_1$: speed of projectile particles

$\sigma$: total cross section

$dV$: volume of target.

We transform eq.(A1) into

$$\frac{d^2N}{dtdV} = \sigma|\vec{v}_1|n_1 n_2 \ . \tag{A2}$$

Since the four-dimensional volume dtdV is Lorentz invariant, the both sides of eq.(A2) are Lorentz invariant.

First, eq.(A2) is transformed to a system (*) moving parallel to $\vec{v}$ with a velocity $\beta_0 c$ (the center-of-mass system is a special system of this kind). The Lorentz transformations of relevant quantities are expressed as

$$\begin{aligned} pc &= \gamma_0(p^*c + \beta_0 E^*) \\ E &= \gamma_0(\beta_0 p^*c + E^*) \\ n &= \gamma_0(\beta_0 n^* \frac{v^*}{c} + n^*). \end{aligned} \tag{A3}$$

Note that n transforms as the fourth component of a four-vector $(n\vec{v}, icn)$. Taking into account the relation

$$v_1 = \frac{p_1 c}{E_1} c \ ,$$

eq. (A2) is expressed as

$$\frac{d^2N}{dtdV} = \frac{d^2N}{dt^*dV^*} = \sigma c\gamma_0(\beta_0 n_1^* \frac{v_1^*}{c} + n_1^*)\gamma_0(\beta_0 n_2^* \frac{v_2^*}{c} + n_2^*)\times$$

$$\left| \frac{p_1^* c + \beta_0 E_1^*}{\beta_0 p_1^* c + E_1^*} \right| . \tag{A4}$$

Since originally

$$p_2 c = \gamma_0(p_2^* c + \beta_0 E_2^*) = 0 ,$$

it follows that

$$\beta_0 = -\frac{p_2^* c}{E_2^*} \text{ and } \gamma_0^2 = \frac{1}{1-(\frac{p_2^* c}{E_2^*})^2} . \tag{A5}$$

Then,

$$\frac{d^2N}{dV^*dt^*} = \sigma|\vec{v}_1^* - \vec{v}_2^*|\, n_1^* n_2^* . \tag{A6}$$

Next, we consider a new laboratory system in which the (*) system is moving perpendicular to $\vec{v}_1^*$ and $\vec{v}_2^*$ with a velocity $\bar{\beta}c$. Starting from given $\vec{P}_1$ and $\vec{P}_2$ in the new laboratory system, the (*) system is obtained by moving perpendicularly to the relative velocity vector $\vec{v}_1 - \vec{v}_2$ with a velocity $v_{/\!/} = \bar{\beta}c$ such that

$$\vec{v}_{1\perp} + \vec{v}_{/\!/} = \vec{v}_1 \quad \text{and } \vec{v}_2 + \vec{v}_{/\!/} = \vec{v}_2 .$$

Then

$$P_1^* = P_{1\perp} \qquad , P_2^* = P_{2\perp}$$

$$n_1^* = \frac{n_1}{\bar{\gamma}} \qquad , n_2^* = \frac{n_2}{\bar{\gamma}} \tag{A7}$$

$$E_1^* = \frac{E_1}{\bar{\gamma}} \qquad , E_2^* = \frac{E_2}{\bar{\gamma}} .$$

Inserting eq.(A7) into eq.(A6), we obtain

$$\frac{d^2N}{dtdV} = \frac{d^2N}{dt^*dV^*} = \sigma \frac{n_1 n_2}{\bar{\gamma}^2} \left| \frac{P_{1\perp}c}{E_1} - \frac{P_{2\perp}c}{E_2} \right| \bar{\gamma}$$

$$= \sigma \frac{n_1 n_2}{\bar{\gamma}} |\vec{v}_{1\perp} - \vec{v}_{2\perp}|$$

$$= \sigma \frac{n_1 n_2}{\bar{\gamma}} |\vec{v}_1 - \vec{v}_2| \; . \qquad \text{(A8)}$$

Since $v_{/\!/} |\vec{v}_1 - \vec{v}_2| = |\vec{v}_1 \times \vec{v}_2|$ (which is equal to twice the area of the triangle formed by $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_1 - \vec{v}_2$ ),

$$\frac{1}{\bar{\gamma}} = (1 - \frac{v_{/\!/}^2}{c^2})^{\frac{1}{2}} = (1 - \frac{|\vec{v}_1 \times \vec{v}_2|^2}{|\vec{v}_1 - \vec{v}_2|^2 c^2})^{\frac{1}{2}} \; .$$

Then, eq.(A8) is expressed as

$$\frac{d^2N}{dtdV} = \sigma n_1 n_2 \sqrt{(\vec{v}_1 - \vec{v}_2)^2 - \frac{(\vec{v}_1 \times \vec{v}_2)^2}{c^2}} \quad , \qquad \text{(A9)}$$

which is the desired expression eq.(1). Some authors[3,6] start from eq.(A6) instead of eq.(A9) and their formulae are valid only for small crossing angles.

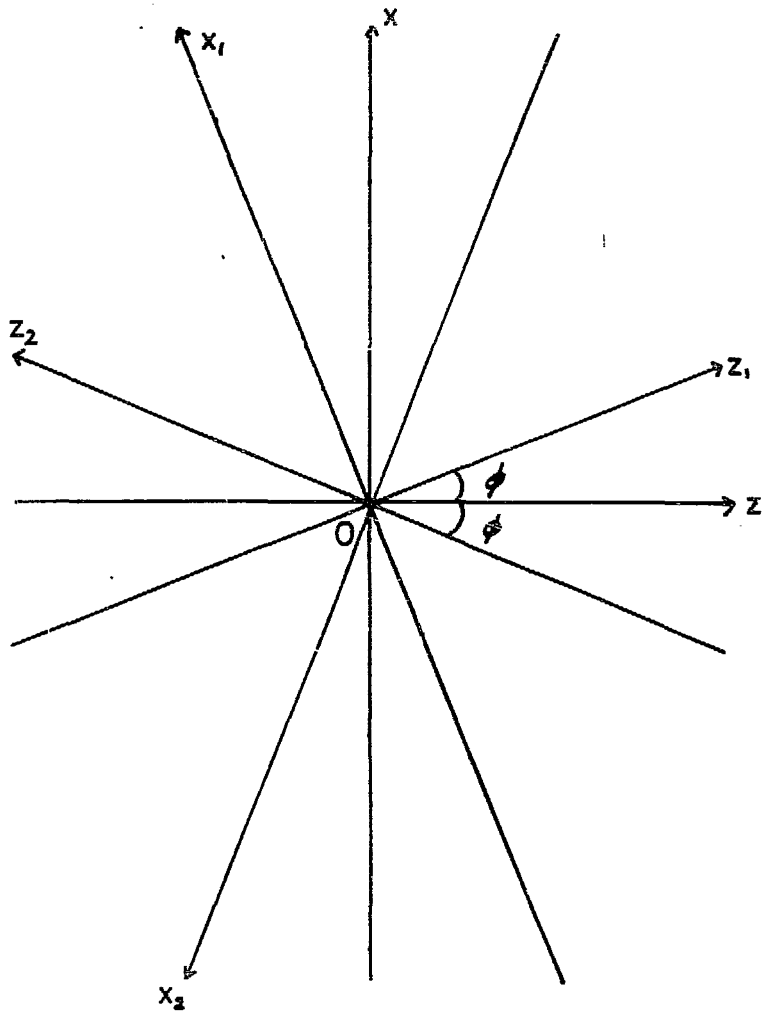


FIG. 1 GEOMETRY OF THE CROSSING REGION